MODELLING AND ANALYSIS OF AN INVERSE PARAMETER IDENTIFICATION PROBLEM IN PIEZOELECTRICITY

RAPHAEL KUESS[a], DANIEL WALTER[b], AND ANDREA WALTHER[a]

[a] *Humboldt-Universität zu Berlin, Institut für Mathematik, 10099 Berlin, Germany*
[b] *Johannes Kepler Universität Linz, Institut für Numerische Mathematik, 4040 Linz, Austria*

ABSTRACT. Piezoelectric material behavior is mathematically described by coupled hyperbolic-elliptic partial differential equations (PDEs) governing mechanical displacement and electrical potential. This paper presents advancements in the theory of identifying material parameters in piezoelectric PDEs. We focus on modeling and analyzing the inverse problem assuming matrix-valued Sobolev-Bochner parameters to encompass a time and space-dependent setting and thus external physical influences. This is followed by results regarding the existence, uniqueness and improved regularity of solutions to the piezoelectric PDE. Based on these findings, well-definedness and regularity of the parameter-to-state map and Fréchet differentiability of the observation operator are proven. Finally, the inverse problem is formulated using a minimization approach, where weak lower semi-continuity of the objective functional, first-order optimality conditions and the derivation and analysis of the adjoint PDE are presented.

Keywords: Existence, uniqueness and regularity; operator analysis; inverse parameter identification; piezoelectricity; adjoint PDEs
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1. INTRODUCTION

Piezoelectric materials are extensively utilized in numerous electrical devices nowadays, being prevalent not only in households but also in industrial and medical settings. The versatility of piezoelectric materials extends across a diverse range of products, including microphones and headphones as well as ultrasound imaging devices and power generation systems. The underlying piezoelectric effect, which is the fundamental property of these applications, describes a coupling phenomenon between electrical and mechanical fields, where mechanical pressure generates an electric potential and vice versa. Thoroughly understanding the behavior of these materials is essential, especially given their time and space dependent characteristics. Their temporal and spatial dependence can occur implicitly via the influence of external physical quantities such as temperature, which appears as temporally and spatially varying functions, on the material parameters. Simplistically, the piezoelectric material is described by a system of coupled PDEs for the mechanical displacement and the electrical potential. The material behavior depends significantly on the material parameters occurring in this PDE system. Hence, the inverse problem aims at identifying material parameters from observations $C(p,z)$ of the state and the parameters. As the observed data is usually contaminated with noise, we have given noisy measurements y^δ. Employing the reduced method, i.e., the model is eliminated by introducing a parameter-to-state map S, in an optimization approach results in solving

$$\min_{p\in X} \quad \frac{1}{2}\left\|F(p)-y^\delta\right\|^2+\mathcal{R}_\alpha(p), \tag{1.1}$$

where $F : X \to Y$, $F(p) = C(p, S(p))$ is called the forward operator. If the preimage space X is a Sobolev space of higher order, implementing regularization methods on this preimage space becomes particularly challenging and, in many cases, impractical. To address this issue, it is beneficial to leverage the physical behavior of material parameters by parameterizing them in terms of a relevant physical quantity. We consider, for example, the dependence of the parameters on a known temperature function θ, which is a function of space and time. It is reasonable to assume a polynomial or Hadamrd exponential structure

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Corresponding author: Raphael Kuess, raphael.kuess@hu-berlin.de.

of the material parameters with respect to the temperature function, see [10]. The coefficients of these polynomials or exponential functions are constant parameters. This means that we can reconstruct these parameters by reconstructing constant matrices of appropriate size for each parameter, which transfers an optimization problem in higher order Sobolev spaces to an optimization problem in a real-valued finite dimensional vector space. As this surrogate modeling approach is a linear transformation into higher order Sobolev spaces, the analysis of the individual components of the inverse problem must be conducted in the framework of an infinite-dimensional function space, including the analysis of the underlying PDE and the associated adjoint PDE, in order to preserve generality.

Related Work. Existence, uniqueness and regularity of solutions of the piezoelectric dynamical system have been studied in [1], [3], [4], [11], [13], [16], [19], [18], [20], [21] and [23], among others and the references therein. For example, in [11] and [20] existence and uniqueness of solutions of the undamped inhomogeneous piezoelectric PDE is discussed, where the material parameters are constant. In [1] the material parameters are spatially dependent $L^\infty(\Omega)$ functions and inhomogeneities are included. An optimal control problem for the electrical flux (boundary control problem) is studied in [3], where an existence and uniqueness result is given for solutions of the undamped piezoelectric PDE and the corresponding adjoint differential PDE with spatially dependent $L^\infty(\Omega)$ material parameters. The papers [23] and [21] consider a boundary control problem, where existence and uniqueness results for solutions of the undamped homogeneous piezoelectric PDE are discussed, where [21] deals with constant material parameters and [23] focuses on an elasticity parameter comprised of spatially dependent $C^2(\Omega)$ functions, a permittivity parameter comprised of spatially dependent $L^\infty(\Omega)$ functions and a constant piezoelectric coupling parameter. In [4], an existence and uniqueness result is given for solutions of the undamped piezoelectric PDE coupled to a parabolic temperature equation and the magnetic field in the form of an elliptic equation, similar to the electrical equation of the classical piezoelectric system, where the parameters have $C^{0,1}(\Omega)$ regularity or $L^\infty(\Omega)$ regularity in the space. In [18], respectively in [19], a shape optimization problem is studied, where an existence and uniqueness result for solutions of the undamped inhomogeneous piezoelectric PDE is given for time/space-constant parameters and the corresponding adjoint PDE is presented. Furthermore, [13] gives a result heavily based on [7], [16], [1], and [20], on existence and uniqueness of solutions for the Rayleigh damped homogeneous piezoelectric PDE, where the material parameters are constant.

Contribution. The aim of this paper is to model and analyze the inverse problem. We establish the well-definedness, existence, and regularity of the parameter-to-state map. For this purpose, we have to extend and generalize previous existence and uniqueness results for piezoelectric PDEs, by considering matrix valued Sobolev-Bochner functions as material parameters and also Sobolev-Bochner density- and damping functions. Moreover the Rayleigh damped piezoelectric system is extended by a further damping term and Sobolev-Bochner inhomogeneities, which allows the application of the contributed existence and uniqueness theorem not only to the state equation but also to the adjoint PDE. Subsequently, we define the observation operator, demonstrating that its well-definedness requires higher regularity of the state. For this an a-priori energy estimate was established, which has not yet been treated in this general setting. Consequently, we provide a rigorous Dirichlet lift Ansatz and contribute a result that provides arbitrary Sobolev regularity in space, for sufficiently regular boundary data and right-hand sides, to satisfy the well-definedness requirement. We prove Fréchet differentiability of the observation operator, leading to the definition of the forward operator, which inherits the properties of both the observation operator and the parameter-to-state map and prove its weak-to-strong continuity. We model the inverse problem as a minimization problem of an objective functional and prove the existence of a minimizer. Furthermore, we formulate the necessary first-order optimality conditions. Motivated by this, we derive the adjoint PDE and analyze it with respect to the existence and uniqueness of solutions by employing the main existence and uniqueness result of this article. Additionally we give insights in the structure of the derivative of the objective functional.
To the best of our knowledge, this generalized problem has not been previously addressed in the literature.

Structure of the paper. The structure of this paper is as follows: The second section addresses the modeling of the underlying PDE to the inverse problem, as well as notations and the introduction of necessary definitions and lemmata essential for the paper's objectives. The third section proves existence, uniqueness and regularity of weak solutions for the generalized damped piezoelectric PDEs. Furthermore, results on an a-priori energy estimate and arbitrary Sobolev regularity in space for sufficient regular boundary data and right-hand sides are proven. The fourth section models and analyses the forward operator, including the well-definedness and Fréchet differentiability of the observation operator and the existence and regularity of the parameter to state map. Then, the inverse problem is formulated as minimization problem and first-order optimality conditions are derived. The derivation and analysis of the adjoint PDE concludes this section. The fifth section discusses a numerical example and the final section briefly summarizes the contributions of this paper.

2. Modeling

Let $T > 0$ be the end time of the observed time period $(0, T)$ and denote the geometry of the considered piezoceramic, i.e., the domain, with $\Omega \subset \mathbb{R}^3$. For the latter we assume that its boundary can be represented as the disjoint union $\partial\Omega := \Gamma_e \dot{\cup} \Gamma_0 \dot{\cup} \Gamma_n$. Thereby Γ_e describes the boundary segment which is excited electrically with a known excitation signal ϕ^e and Γ_0 refers to the boundary segment which is grounded. This setting can be modeled in the system of PDEs as mixed Dirichlet boundary conditions. For the regularity of Ω and ϕ^e we will employ the following assumptions:

Assumptions 2.1.

- D1 Ω is a bounded Lipschitz domain and $\phi^e \in H^1\left(0, T; H^{\frac{1}{2}}(\partial\Omega, \mathbb{R})\right)$.
- D2 Ω is a bounded $C^{m,1}$-domain and $\phi^e \in H^1\left(0, T; H^{m+\frac{3}{2}}(\partial\Omega, \mathbb{R})\right)$, for some $m \geq 2,\ m \in \mathbb{N}$.
- D3 Ω is a bounded Lipschitz domain and $\phi^e \in H^1(0, T)$ is spatially constant on Γ_e.

The boundary segment Γ_n is included in the PDE by Neumann boundary conditions. We denote the non-empty mixed Dirichlet boundary with $\Gamma_d := \Gamma_e \dot{\cup} \Gamma_0$, i.e., $\partial\Omega := \Gamma_d \dot{\cup} \Gamma_n$ and we assume that every boundary part has a positive two-dimensional Hausdorff measure. We will denote time derivatives of a function f with $\dot{f}$, and spatial derivatives with ∇f. Furthermore, $n = (n_x, n_y, n_z)$ is the three-dimensional normal vector corresponding to the normal derivative with respect to ∇, $\mathcal{B}f$ is the symmetric gradient of a function f in Voigt notation and $\mathcal{N}$ is the corresponding normal matrix with respect to $\mathcal{B}$, where

$$\mathcal{B} = \begin{pmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{pmatrix}, \qquad \mathcal{N} = \begin{pmatrix} n_x & 0 & 0 \\ 0 & n_y & 0 \\ 0 & 0 & n_z \\ 0 & n_z & n_y \\ n_z & 0 & n_x \\ n_y & n_x & 0 \end{pmatrix}.$$

Since all derivatives are understood in the distributional sense, we use the standard Sobolev space $H^1(\Omega, \mathbb{R})$ associated with ∇, and

$$H^1_{\mathcal{B}}(\Omega, \mathbb{R}^3) := \left\{ f \in L^2(\Omega, \mathbb{R}^3) \,:\, \mathcal{B}f \in L^2(\Omega, \mathbb{R}^6) \right\}$$

equipped with the norm

$$\|f\|_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} := \left(\|f\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|\mathcal{B}f\|^2_{L^2(\Omega,\mathbb{R}^6)} \right)^{1/2}.$$

as H^1-Sobolev space associated with the spatial differential operator $\mathcal{B}$. Since Ω is at least Lipschitz, due to Assumption 2.1, Korn's inequality implies that $H^1_{\mathcal{B}}(\Omega, \mathbb{R}^3)$ and $H^1(\Omega, \mathbb{R}^3)$ coincide as Hilbert spaces with equivalent norms. The H^1-Sobolev space whose functions vanish only on Γ_d is defined by

$$H^1_{0,\Gamma_d}(\Omega, \mathbb{R}) = \left\{ f \in H^1(\Omega, \mathbb{R}) |\; f|_{\Gamma_d} = 0 \right\}.$$

Furthermore, we denote the dual of a Hilbert space H with H^*. Then, the three-dimensional mechanical displacement $u(t, x)$ and the one-dimensional electrical potential $\phi(t, x)$ of a piezoceramic specimen can be described by the following piezoelectric dynamical system

$$
\begin{aligned}
\rho\ddot{u} + \alpha\rho\dot{u} - \mathcal{B}^T\left(c^E\mathcal{B}u + \beta c^E\mathcal{B}\dot{u} + e^T\nabla\phi\right) &= 0 \quad \text{in } \Omega \times (0,T),\\
-\nabla \cdot (e\mathcal{B}u - \epsilon\nabla\phi) &= 0 \quad \text{in } \Omega \times (0,T),\\
\phi &= 0 \quad \text{on } \Gamma_0 \times (0,T),\\
\phi &= \phi^e \quad \text{on } \Gamma_e \times (0,T),\\
n \cdot (e\mathcal{B}u - \epsilon\nabla\phi) &= 0 \quad \text{on } \Gamma_n \times (0,T),\\
\mathcal{N}^T\left(c^E\mathcal{B}u + \beta c^E\mathcal{B}\dot{u} + e^T\nabla\phi\right) &= 0 \quad \text{on } \partial\Omega \times (0,T),\\
u(t=0) &= u_0 \quad \text{in } \Omega,\\
\dot{u}(t=0) &= u_1 \quad \text{in } \Omega.
\end{aligned}
$$

where ρ is the mass density, c^E, e and ϵ are the material parameters describing the given piezoceramic and α, β are damping parameters. Since we consider the same time interval throughout the paper, we skip it when referring to Bochner spaces.

Assumptions 2.2.

A1 The damping parameters $\alpha \in H^1\big(L^\infty(\Omega,\mathbb{R})\big)$, $\beta \in H^1\big(L^\infty(\Omega,\mathbb{R})\big)$, are non-negative and uniformly bounded, and there exists $\beta^\star > 0$ such that

$$\beta(t,x) \geq \beta^\star \qquad \text{for a.e. } (t,x) \in (0,T) \times \Omega.$$

A2 The density $\rho \in H^2\big(L^\infty(\Omega,\mathbb{R})\big)$ is uniformly bounded, and there exists $\rho^\star > 0$ such that

$$\rho(t,x) \geq \rho^\star \qquad \text{for a.e. } (t,x) \in (0,T) \times \Omega.$$

A3 The elasticity parameter is of the form

$$
c^E(t,x) := \begin{pmatrix}
c^E_{11}(t,x) & c^E_{12}(t,x) & c^E_{13}(t,x) & 0 & 0 & 0\\
c^E_{12}(t,x) & c^E_{11}(t,x) & c^E_{13}(t,x) & 0 & 0 & 0\\
c^E_{13}(t,x) & c^E_{13}(t,x) & c^E_{33}(t,x) & 0 & 0 & 0\\
0 & 0 & 0 & c^E_{44}(t,x) & 0 & 0\\
0 & 0 & 0 & 0 & c^E_{44}(t,x) & 0\\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}\big(c^E_{11}(t,x) - c^E_{12}(t,x)\big)
\end{pmatrix},
$$

where $c^E_{11}, c^E_{12}, c^E_{13}, c^E_{33}, c^E_{44} \in H^2\big(L^\infty(\Omega,\mathbb{R})\big)$, i.e., $c^E \in H^2\big(L^\infty(\Omega,\mathbb{R}^{6\times 6})\big)$, and is uniformly positive definite, meaning that there exists $c_\star > 0$ such that

$$\xi^T c^E(t,x)\xi \geq c_\star|\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^6$$

and for a.e. $(t,x) \in (0,T) \times \Omega$.

A4 The piezoelectric coupling parameter is of the form

$$
e(t,x) := \begin{pmatrix}
0 & 0 & 0 & 0 & e_{15}(t,x) & 0\\
0 & 0 & 0 & e_{15}(t,x) & 0 & 0\\
e_{31}(t,x) & e_{31}(t,x) & e_{33}(t,x) & 0 & 0 & 0
\end{pmatrix},
$$

where $e_{15}, e_{31}, e_{33} \in H^2\big(L^\infty(\Omega,\mathbb{R})\big)$, i.e., $e \in H^2\big(L^\infty(\Omega,\mathbb{R}^{3\times 6})\big)$.

A5 The permittivity parameter is of the form

$$
\epsilon(t,x) := \begin{pmatrix}
\epsilon_{11}(t,x) & 0 & 0\\
0 & \epsilon_{11}(t,x) & 0\\
0 & 0 & \epsilon_{33}(t,x)
\end{pmatrix},
$$

where $\epsilon_{11}, \epsilon_{33} \in H^2\big(L^\infty(\Omega,\mathbb{R})\big)$, i.e., $\epsilon \in H^2\big(L^\infty(\Omega,\mathbb{R}^{3\times 3})\big)$, and is uniformly positive definite, meaning that there exists $\epsilon_\star > 0$ such that

$$\eta^T \epsilon(t,x)\eta \geq \epsilon_\star|\eta|^2 \qquad \text{for all } \eta \in \mathbb{R}^3$$

and for a.e. $(t,x) \in (0,T) \times \Omega$.

In particular, Assumptions A3 and A5 imply that $c^E(t,x)$ and $\epsilon(t,x)$ are invertible for almost all $(t,x) \in (0,T) \times \Omega$, and that their inverses are uniformly bounded. If the parameters depend on e.g., a known temperature function θ, we have that $(c^E(x,t), e(x,t), \epsilon(x,t)) = p(x,t) = \tilde{p}(\theta(x,t))$ and the following parametrizations can be proposed

$$(c^E(x,t), e(x,t), \epsilon(x,t)) = \left(\sum_{j=0}^{n} a_j \theta(x,t)^j, \sum_{j=0}^{n} b_j \theta(x,t)^j, \sum_{j=0}^{n} k_j \theta(x,t)^j \right), \tag{2.1}$$

$$(a_j, b_j, k_j) \in \mathbb{R}^{6\times 6} \times \mathbb{R}^{3\times 6} \times \mathbb{R}^{3\times 3}, \qquad 0 \le j \le n,$$

$$(c^E(x,t), e(x,t), \epsilon(x,t)) = \left(e^{c_1\theta(x,t)} + c_0,\, e^{l_1\theta(x,t)} + l_0,\, e^{m_1\theta(x,t)} + m_0 \right), \tag{2.2}$$

$$(c_j, l_j, m_j) \in \mathbb{R}^{6\times 6} \times \mathbb{R}^{3\times 6} \times \mathbb{R}^{3\times 3}, \qquad j = 0, 1,$$

where in (2.2) the exponential is understood component-wise, i.e. as the Hadamard exponential. The coefficient matrices in these parameterizations are to be chosen such that the structural symmetries and the uniform positive definiteness of c^E and ϵ are preserved for all relevant temperature values.
Finally, α and β are the so-called Rayleigh damping parameters. If $\alpha \equiv 0$, then β is called the Kelvin-Voigt damping parameter and can be interpreted as a relaxation parameter. Hence, the Rayleigh damping model generalizes the Kelvin-Voigt damping model.

3. Existence, Uniqueness and Regularity

In order To discuss weak solvability, we homogenize the mixed Dirichlet boundary conditions using a Dirichlet lift Ansatz. Therefore, let Assumption D1 hold. As $\Gamma_e \cap \Gamma_0 = \emptyset$ and Ω is Lipschitz, the trace operator $\mathrm{Tr} : H^1(\Omega,\mathbb{R}) \to H^{1/2}(\Gamma_d)$ is continuous and surjective, and admits a continuous right inverse. Hence there exists $\chi \in H^1\big(H^1(\Omega,\mathbb{R})\big)$ with the property that

$$\mathrm{Tr}(\chi(t)) = \begin{cases} \phi_e(t) & \text{on } \Gamma_e \\ 0 & \text{on } \Gamma_0 \end{cases} \quad \text{a.e. in time.} \tag{3.1}$$

Hence, with $\phi_0(t) \in H^1_{0,\Gamma_d}(\Omega,\mathbb{R})$, we rewrite ϕ as $\phi(t) = \phi_0(t) + \chi(t)$ a.e. in time. Plugging this representation in our piezoelectric dynamical system leads to

$$\rho\ddot{u} + \alpha\rho\dot{u} - \mathcal{B}^T\left(c^E\mathcal{B}u + \beta c^E\mathcal{B}\dot{u} + e^T\nabla\phi_0\right) = \mathcal{B}^T e^T \nabla\chi \qquad \text{in } \Omega \times (0,T) \tag{3.2}$$

$$-\nabla\cdot(e\mathcal{B}u - \epsilon\nabla\phi_0) = -\nabla\cdot\epsilon\nabla\chi \qquad \text{in } \Omega\times(0,T) \tag{3.3}$$

$$n\cdot(e\mathcal{B}u - \epsilon\nabla\phi_0) = n\cdot\epsilon\nabla\chi \qquad \text{on } \Gamma_n\times(0,T) \tag{3.4}$$

$$\mathcal{N}^T\left(c^E\mathcal{B}u + \beta c^E\mathcal{B}\dot{u} + e^T\nabla\phi_0\right) = -\mathcal{N}^T e^T\nabla\chi \qquad \text{on } \partial\Omega\times(0,T) \tag{3.5}$$

$$u(t=0) = u_0 \qquad \text{in } \Omega \tag{3.6}$$

$$\dot{u}(t=0) = u_1 \qquad \text{on } \Omega. \tag{3.7}$$

We now derive the weak formulation of the system above. For this purpose, we consider the weak form of (3.2) and (3.3) separately and include the corresponding boundary conditions (3.5) and (3.4). Note that by simply adding both forms, we obtain the weak form of the whole system, since we have to use different test functions for (3.2) and (3.3). We start deriving the weak form of (3.2) for almost all $t \in (0,T)$ by testing the system (3.2)-(3.7) with $(v,0)$, where $v \in H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)$, i.e.,

$$\int_\Omega \rho\ddot{u}^T v + \alpha\rho\dot{u}^T v + \left(c^E\mathcal{B}u + \beta c^E\mathcal{B}\dot{u} + e^T\nabla\phi_0\right)^T \mathcal{B}v \,\mathrm{d}\Omega$$
$$= -\int_\Omega (e^T\nabla\chi)^T\mathcal{B}v\,\mathrm{d}\Omega + \int_{\partial\Omega}\mathcal{N}^T(e^T\nabla\chi)^T v\,\mathrm{d}\Gamma + \int_{\partial\Omega}\underbrace{\mathcal{N}^T\left(c^E\mathcal{B}u + \beta c^E\mathcal{B}\dot{u} + e^T\nabla\phi_0\right)^T}_{\underset{(3.5)}{=} -\mathcal{N}^T(e^T\nabla\chi)^T} v\,\mathrm{d}\Gamma$$

$$\Leftrightarrow \int_\Omega \rho\ddot{u}^T v + \alpha\rho\dot{u}^T v + \left(c^E\mathcal{B}u + \beta c^E\mathcal{B}\dot{u} + e^T\nabla\phi_0\right)^T\mathcal{B}v\,\mathrm{d}\Omega = -\int_\Omega (e^T\nabla\chi)^T\mathcal{B}v\,\mathrm{d}\Omega\,. \tag{3.8}$$

To derive the weak form of (3.3) for almost all $t \in (0,T)$, we test the system (3.2)-(3.7) with $(0,w)$, where $w \in H^1_{0,\Gamma_d}(\Omega,\mathbb{R})$, i.e.,

$$\begin{aligned}
&\int_\Omega \left(e\mathcal{B}u - \epsilon\nabla\phi_0\right)^T \nabla w \,\mathrm{d}\Omega - \int_{\Gamma_n} \underbrace{n\cdot\left(e\mathcal{B}u - \epsilon\nabla\phi_0\right)}_{\underset{(3.4)}{=} n\cdot\epsilon\nabla\chi} w \,\mathrm{d}\Gamma - \underbrace{\int_{\Gamma_d} n\cdot\left(e\mathcal{B}u - \epsilon\nabla\phi_0\right) w \,\mathrm{d}\Gamma}_{\underset{w\in H^1_{0,\Gamma_d}(\Omega,\mathbb{R})}{=} 0} \\
&= \int_\Omega (\epsilon\nabla\chi)^T \nabla w \,\mathrm{d}\Omega - \int_{\Gamma_n} n\cdot(\epsilon\nabla\chi) w \,\mathrm{d}\Gamma - \underbrace{\int_{\Gamma_d} n\cdot(\epsilon\nabla\chi) w \,\mathrm{d}\Gamma}_{\underset{w\in H^1_{0,\Gamma_d}(\Omega,\mathbb{R})}{=} 0}
\end{aligned}$$

$$\Leftrightarrow \int_\Omega \left(e\mathcal{B}u - \epsilon\nabla\phi_0\right)^T \nabla w \,\mathrm{d}\Omega = \int_\Omega (\epsilon\nabla\chi)^T \nabla w \,\mathrm{d}\Omega\,. \tag{3.9}$$

As we used different test functions for (3.8) and (3.9), we can define the time-dependent variational identities corresponding to the PDE system (3.2)-(3.7) by

$$B((u,\phi_0),(v,w)) :=$$

$$\int_\Omega \rho\ddot{u}^T v + \alpha\rho\dot{u}^T v + \left(c^E\mathcal{B}u + \beta c^E\mathcal{B}\dot{u} + e^T\nabla\phi_0\right)^T \mathcal{B}v + \left(e\mathcal{B}u - \epsilon\nabla\phi_0\right)^T \nabla w \,\mathrm{d}\Omega \tag{3.10}$$

$$L(v,w) = \int_\Omega -(e^T\nabla\chi)^T\mathcal{B}v + (\epsilon\nabla\chi)^T\nabla w \,\mathrm{d}\Omega\,. \tag{3.11}$$

To prove existence and uniqueness of solutions to the piezoelectric system (3.2)-(3.7), we use Chapter XVIII in [5], especially Theorem 1 in Paragraph 5 and Remark 9 in Paragraph 6. Using the following lemma, we will exploit that c^E and e are continuously differentiable in time.

Lemma 3.1. *Let $(0,T)\subset\mathbb{R}$, $k,k_1,k_2\in\mathbb{N}$ and Assumption D1 hold. Then, one has*

$$H^k(L^\infty(\Omega)) \hookrightarrow C^{k-1}([0,T];L^\infty(\Omega)) \tag{3.12}$$

$$H^{k_1}(H^{k_2}(\Omega)) \hookrightarrow C^{k_1-1}([0,T];C^{k_2-2}(\bar\Omega)). \tag{3.13}$$

Proof. Let $U\subset\mathbb{R}$. Due to Theorem 6, Chapter 5.6.3 in [7] one obtains for $k>\frac{1}{2}$ the existence of a continuous embedding $\iota_1 : H^k(U)\hookrightarrow C^{k-1,\frac{1}{2}}(\bar U)$ with $\iota_1(f)\equiv f$ almost everywhere. Since $C^{k-1,\frac{1}{2}}(\bar U)\hookrightarrow C^{k-1}(\bar U)$, one has $H^k(U)\hookrightarrow C^{k-1,\frac{1}{2}}(\bar U)\hookrightarrow C^{k-1}(\bar U)$, meaning that there is a constant $C^U_{H^k\hookrightarrow C^{k-1}}>0$ such that

$$\forall f\in H^k(U):\ \|f\|_{C^{k-1}(\bar U)} \le C^U_{H^k\hookrightarrow C^{k-1}}\|f\|_{H^k(U)}.$$

Analogously, due to Theorem 6, Chapter 5.6.3 in [7] one obtains for $k>\frac{3}{2}$ that there exists a continuous embedding $\iota_3 : H^k(\Omega)\hookrightarrow C^{k-2,\frac{1}{2}}(\bar\Omega)$ with $\iota_3(f)\equiv f$ almost everywhere. Furthermore, it holds that $C^{k-2,\frac{1}{2}}(\bar\Omega)\hookrightarrow C^{k-2}(\bar\Omega)$, which yields $H^k(\Omega)\hookrightarrow C^{k-2,\frac{1}{2}}(\bar\Omega)\hookrightarrow C^{k-2}(\bar\Omega)$, i.e., for a constant $C^\Omega_{H^k\hookrightarrow C^{k-2}}>0$,

$$\forall f\in H^k(\Omega):\ \|f\|_{C^{k-2}(\bar\Omega)} \le C^\Omega_{H^k\hookrightarrow C^{k-2}}\|f\|_{H^k(\Omega)}$$

is valid. Hence, there exists a constant $C^{(0,T)}_{H^k\hookrightarrow C^{k-1}}>0$ such that

$$\forall f\in H^k(L^\infty(\Omega)):\ \|f\|_{C^{k-1}([0,T];L^\infty(\Omega))} \le C^{(0,T)}_{H^k\hookrightarrow C^{k-1}}\|f\|_{H^k(L^\infty(\Omega))},$$

which proves that $H^k(L^\infty(\Omega))\hookrightarrow C^{k-1}([0,T];L^\infty(\Omega))$ continuously.
By the same argument we conclude that $H^{k_1}(H^{k_2}(\Omega))\hookrightarrow C^{k_1-1}([0,T];H^{k_2}(\Omega))$, i.e., there is a constant $C^{(0,T)}_{H^{k_1}\hookrightarrow C^{k_1-1}}>0$ such that

$$\forall f\in H^{k_1}(H^{k_2}(\Omega)):\ \|f\|_{C^{k_1-1}([0,T];H^{k_2}(\Omega))} \le C^{(0,T)}_{H^{k_1}\hookrightarrow C^{k_1-1}}\|f\|_{H^{k_1}(H^{k_2}(\Omega))}. \tag{3.14}$$

Furthermore, there exists a constant $C^\Omega_{H^{k_2}\hookrightarrow C^{k_2-2}}>0$ such that for all $f\in C^{k_1-1}([0,T];H^{k_2}(\Omega))$

$$\|f\|_{C^{k_1-1}([0,T];C^{k_2-2}(\bar\Omega))} \le C^\Omega_{H^{k_2}\hookrightarrow C^{k_2-2}}\|f\|_{C^{k_1-1}([0,T];H^{k_2}(\Omega))} \tag{3.15}$$

holds. Together, the inequalities (3.14) and (3.15) yield the existence of a constant

$$C := C^{\Omega}_{H^{k_2} \hookrightarrow C^{k_2-2}} C^{(0,T)}_{H^{k_1} \hookrightarrow C^{k_1-1}} > 0$$

such that

$$\forall f \in H^{k_1}(H^{k_2}(\Omega)) : \ \|f\|_{C^{k_1-1}([0,T];C^{k_2-2}(\bar{\Omega}))} \leq C \|f\|_{H^{k_1}(H^{k_2}(\Omega))}, \tag{3.16}$$

which proves that $H^{k_1}(H^{k_2}(\Omega)) \hookrightarrow C^{k_1-1}(C^{k_2-2}(\bar{\Omega}))$ continuously. □

As we also want to use the following result for the analysis of the adjoint system, we introduce an additional damping function $a \in H^2(L^\infty(\Omega, \mathbb{R}))$ and inhomogeneities $(f, g) \in L^2(H^1_{\mathcal{B}}(\Omega, \mathbb{R}^3)^*) \times L^2(L^2(\Omega, \mathbb{R}))$ to the piezoelectric system (3.2)-(3.7).

Theorem 3.2 (Existence and Uniqueness)**.** *Let Assumption D1 and Assumptions 2.2 hold. Let $a \in H^2(L^\infty(\Omega, \mathbb{R}))$ be non-negative and uniformly bounded, and let $(f, g) \in L^2\big(H^1_{\mathcal{B}}(\Omega, \mathbb{R}^3)^*\big) \times L^2(L^2(\Omega, \mathbb{R}))$ be inhomogeneities. Then, for any initial values*

$$u_0 \in H^1_{\mathcal{B}}(\Omega, \mathbb{R}^3), \qquad u_1 \in L^2(\Omega, \mathbb{R}^3),$$

and any ϕ^e according to Assumption D1 with $\chi \in H^1\big(0, T; H^1(\Omega, \mathbb{R})\big)$ and $\mathrm{Tr}(\chi(t))$ defined as in (3.1)*, there exists a unique weak solution*

$$(u, \phi_0) \in L^2\big(H^1_{\mathcal{B}}(\Omega, \mathbb{R}^3)\big) \times L^2\big(H^1_{0,\Gamma_d}(\Omega, \mathbb{R})\big)$$

with

$$\dot{u} \in L^2\big(H^1_{\mathcal{B}}(\Omega, \mathbb{R}^3)\big), \qquad \ddot{u} \in L^2\big(H^1_{\mathcal{B}}(\Omega, \mathbb{R}^3)^*\big)$$

to the system

$$\begin{aligned}
\rho\ddot{u} + \alpha\rho\dot{u} + au - \mathcal{B}^T \left(c^E \mathcal{B} u + \beta c^E \mathcal{B} \dot{u} + e^T \nabla \phi_0\right) &= f + \mathcal{B}^T e^T \nabla \chi \quad \text{in } \Omega \times (0,T), &(3.17)\\
-\nabla \cdot (e\mathcal{B}u - \epsilon \nabla \phi_0) &= g - \nabla \cdot \epsilon \nabla \chi \quad \text{in } \Omega \times (0,T),\\
n \cdot (e\mathcal{B}u - \epsilon \nabla \phi_0) &= n \cdot \epsilon \nabla \chi \quad \text{on } \Gamma_n \times (0,T),\\
\mathcal{N}^T \left(c^E \mathcal{B} u + \beta c^E \mathcal{B} \dot{u} + e^T \nabla \phi_0\right) &= -\mathcal{N}^T e^T \nabla \chi \quad \text{on } \partial\Omega \times (0,T),\\
u(t=0) &= u_0 \quad \text{in } \Omega,\\
\dot{u}(t=0) &= u_1 \quad \text{in } \Omega. &(3.18)
\end{aligned}$$

Proof. Set $V := H^1_{\mathcal{B}}(\Omega, \mathbb{R}^3)$ and $H := L^2(\Omega, \mathbb{R}^3)$. By Korn's inequality, $V = H^1(\Omega, \mathbb{R}^3)$ as Hilbert spaces with equivalent norms. Similarly to (3.9) we obtain an affine linear mapping in ϕ_0,

$$\int_\Omega (\epsilon \nabla \phi_0)^T \nabla w \,\mathrm{d}\Omega = \int_\Omega (e\mathcal{B}u)^T \nabla w - (\epsilon \nabla \chi)^T \nabla w - gw \,\mathrm{d}\Omega \,. \tag{3.19}$$

Therefore, we define for almost all $t \in (0,T)$ the linear operator

$$\zeta^t : H^1_{\mathcal{B}}(\Omega, \mathbb{R}^3) \to H^1_{0,\Gamma_d}(\Omega, \mathbb{R}), \ u(t) \mapsto \phi^0_0(t), \tag{3.20}$$

where $\phi_0(t) = \phi_0^0(t) + \phi_0^r(t) = \zeta^t(u(t)) + \phi_0^r(t)$ by satisfying

$$\int_\Omega (\epsilon \nabla \zeta^t(u))^T \nabla w \,\mathrm{d}\Omega = \int_\Omega (e\mathcal{B}u)^T \nabla w. \tag{3.21}$$

By the lemma of Lax-Milgram together with coercivity of $\epsilon(t)$ and boundedness of $\epsilon(t)$, this determines $\zeta^t(u) \in H^1_{0,\Gamma_d}(\Omega, \mathbb{R})$ uniquely. Choosing $w = \zeta^t(u)$ and using coercivity together with the Poincaré inequality on $H^1_{0,\Gamma_d}(\Omega, \mathbb{R})$, we obtain with some $C_1 > 0$ that

$$\|\zeta^t(u)\|_{H^1_{0,\Gamma_d}(\Omega,\mathbb{R})} \leq C_1 \|u(t)\|_V$$

for almost every $t \in (0,T)$. Hence,

$$\|\zeta^t(u)\|_{L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}))} \leq C_1 \|u\|_{L^2(V)}.$$

Moreover, we define $\phi_0^r \in L^2(H^1_{0,\Gamma_d}(\Omega, \mathbb{R}))$ by

$$\int_\Omega (\epsilon(t) \nabla \phi_0^r(t))^T \nabla w \,\mathrm{d}\Omega = -\int_\Omega (\epsilon(t) \nabla \chi(t))^T \nabla w \mathrm{d}\Omega - \int_\Omega g(t) w \mathrm{d}\Omega \tag{3.22}$$

for all $w \in H^1_{0,\Gamma_d}(\Omega,\mathbb{R})$ and almost all $t \in (0,T)$. Using the same arguments as above results in the existence of a unique $\phi_0^r(t) \in H^1_{0,\Gamma_d}(\Omega,\mathbb{R})$ for almost all $t \in (0,T)$ and a constant $C_2 > 0$, such that by integration over time we obtain

$$\|\phi_0^r\|_{L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}))} \leq C_2\Big(\|\chi\|_{L^2(H^1(\Omega,\mathbb{R}))} + \|g\|_{L^2(L^2(\Omega,\mathbb{R}))}\Big). \tag{3.23}$$

Since $H^1_{0,\Gamma_d}(\Omega,\mathbb{R}) \hookrightarrow H^1(\Omega,\mathbb{R})$ continuously, we also have $\phi_0^r \in L^2(H^1(\Omega,\mathbb{R}))$ and thus $\phi_0(t) = \zeta^t(u(t)) + \phi_0^r(t)$. Hence, we obtain an equivalent weak form of the inhomogeneous piezoelectric PDE including the Dirichlet lift similarly to (3.8) through the following weak form

$$\begin{aligned}\int_\Omega \rho\ddot{u}^T v + \alpha\rho\dot{u}^T v + a u^T v + &\left(c^E\mathcal{B}u + \beta c^E\mathcal{B}\dot{u} + e^T\nabla\zeta^t(u)\right)^T\mathcal{B}v \,\mathrm{d}\Omega\\ &= \int_\Omega fv - (e^T\nabla(\chi+\phi_0^r))^T\mathcal{B}v\,\mathrm{d}\Omega\,.\end{aligned} \tag{3.24}$$

We now define the operators for almost all $t \in (0,T)$

$$\begin{aligned}
a_0(t,u,v) &:= \int_\Omega u^T v + \left(c^E(t)\mathcal{B}u\right)^T\mathcal{B}v + \left(e(t)^T\nabla\zeta^t(u)\right)^T\mathcal{B}v\,\mathrm{d}\Omega\,,\\
a_1(t,u,v) &:= \int_\Omega (a(t)-1)u^T v\,\mathrm{d}\Omega\,,\\
b_0(t,u,v) &:= \int_\Omega u^T v + \left(\beta(t)c^E(t)\mathcal{B}u\right)^T\mathcal{B}v\,\mathrm{d}\Omega\,,\\
b_1(t,u,v) &:= \int_\Omega (\alpha(t)\rho(t) - \dot\rho(t) - 1)u^T v\,\mathrm{d}\Omega\,,\\
c(t,u,v) &:= \int_\Omega \rho(t)u^T v\,\mathrm{d}\Omega\,,\\
\langle\tilde f(t), v\rangle &:= \int_\Omega f(t)v - (e(t)^T\nabla(\chi+\phi_0^r)(t))^T\mathcal{B}v\,\mathrm{d}\Omega\,,
\end{aligned} \tag{3.25}$$

where $a_{01} = a_0 + a_1$ and $b_{01} = b_0 + b_1$. Using these operators, we have to prove existence and uniqueness of $u \in L^2(V)$ with

$$\frac{\mathrm{d}}{\mathrm{d}t}c(t;\dot u(t),v) + b_{01}(t;\dot u(t),v) + a_{01}(t;u(t),v) = \langle\tilde f(t),v\rangle \tag{3.26}$$

for all $v \in V$, together with $u(0) = u_0 \in V,\ \dot u(0) = u_1 \in H$ and $\tilde f \in L^2(V^*)$. To show that $\tilde f \in L^2(V^*)$, we denote the unit ball with $B^1_{H^1(\Omega,\mathbb{R}^3)}$, yielding

$$\begin{aligned}
\left\|\tilde f\right\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)} &= \int_0^T \left\|\tilde f(t)\right\|^2_{H^1(\Omega,\mathbb{R}^3)^*}\mathrm{d}t\\
&\leq \int_0^T \sup_{v\in\partial B^1_{H^1(\Omega,\mathbb{R}^3)}} |(f,v)_{L^2(\Omega,\mathbb{R}^3)} - (e(t)^T\nabla(\chi(t)+\phi_0^r(t)),\mathcal{B}v)_{L^2(\Omega,\mathbb{R}^3)}|^2\,\mathrm{d}t\\
&\leq \int_0^T \sup_{v\in\partial B^1_{H^1(\Omega,\mathbb{R}^3)}} \left\|-e(t)^T\nabla(\chi(t)+\phi_0^r(t))\right\|^2_{L^2(\Omega,\mathbb{R}^3)}\|v\|^2_{H^1(\Omega,\mathbb{R}^3)}\,\mathrm{d}t\\
&\quad + \int_0^T \sup_{v\in\partial B^1_{H^1(\Omega,\mathbb{R}^3)}} \|f(t)\|^2_{H^1(\Omega,\mathbb{R}^3)^*}\|v\|^2_{H^1(\Omega,\mathbb{R}^3)}\,\mathrm{d}t\\
&\leq \|e\|^2_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 6}))}\left(\|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))}\right) + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)} < \infty.
\end{aligned} \tag{3.27}$$

As our function space setting are real Hilbert spaces and due to

$$\begin{aligned}\int_\Omega \left(e^T\nabla\zeta^t(u)\right)^T \mathcal{B}v\,\mathrm{d}\Omega &= \int_\Omega \left(\nabla\zeta^t(u)\right)^T e\mathcal{B}v\,\mathrm{d}\Omega = \int_\Omega (e\mathcal{B}v)^T\,\nabla\zeta^t(u)\,\mathrm{d}\Omega \\ &\underset{(3.21)}{=} \int_\Omega \left(\epsilon\nabla\zeta^t(v)\right)^T \nabla\zeta^t(u)\,\mathrm{d}\Omega \underset{\epsilon\text{ diag.}}{=} \int_\Omega \left(\epsilon\nabla\zeta^t(u)\right)^T \nabla\zeta^t(v)\,\mathrm{d}\Omega \\ &\underset{(3.21)}{=} \int_\Omega (e\mathcal{B}u)^T\,\nabla\zeta^t(v)\,\mathrm{d}\Omega = \int_\Omega \left(e^T\nabla\zeta^t(v)\right)^T \mathcal{B}u\,\mathrm{d}\Omega\,,\end{aligned} \tag{3.28}$$

we immediately conclude that a_0 and b_0 are Hermitian. Furthermore, note that by assumption A3 it holds for almost all $t \in (0, T)$ that

$$(c^E(t)\mathcal{B}u(t), \mathcal{B}v(t))_{L^2(\Omega,\mathbb{R}^3)} \geq c_\star(\mathcal{B}u(t), \mathcal{B}v(t))_{L^2(\Omega,\mathbb{R}^3)}.$$

by positive definiteness of c^E. With this we conclude that there exists some constant $\sigma \in \mathbb{R}^+$ with

$$\sigma := \min\{1, c_\star\} > 0, \tag{3.29}$$

such that

$$\begin{aligned} a_0(t,u,u) &= \int_\Omega u^T u\mathrm{d}\Omega + \int_\Omega \left(c^E(t)\mathcal{B}u\right)^T \mathcal{B}u\mathrm{d}\Omega + \int_\Omega \left(e(t)^T\nabla\zeta^t(u)\right)^T \mathcal{B}u\mathrm{d}\Omega \\ &\underset{(3.28)}{=} \|u\|^2_{L^2(\Omega,\mathbb{R}^3)} + \int_\Omega \left(c^E(t)\mathcal{B}u\right)^T \mathcal{B}u\mathrm{d}\Omega + \int_\Omega \left(\epsilon(t)\nabla\zeta^t(u)\right)^T \nabla\zeta^t(u)\mathrm{d}\Omega \\ &\geq \|u\|^2_{L^2(\Omega,\mathbb{R}^3)} + c_\star\|\mathcal{B}u\|^2_{L^2(\Omega,\mathbb{R}^6)} + \|(\epsilon)^{-1}\|^{-1}_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 3}))}\|\nabla\zeta^t(u)\|^2_{L^2(\Omega,\mathbb{R}^3)} \\ &\geq \sigma\|u\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)}. \end{aligned} \tag{3.30}$$

To prove that the mapping $t \mapsto a_0(t, u, v)$ is one time continuously differentiable, we use the regularities of the material parameters together with Lemma 3.1. For the continuous differentiability of the mapping $t \mapsto \zeta^t(u) \in H^1_{0,\Gamma_d}(\Omega, \mathbb{R})$ we obtain for almost every $t \in (0, T)$ and all $w \in H^1_{0,\Gamma_d}(\Omega, \mathbb{R})$ that

$$\int_\Omega \epsilon(t)\frac{\mathrm{d}}{\mathrm{d}t}(\nabla\zeta^t(u))^T\nabla w\ \mathrm{d}\Omega = \int_\Omega \Big(\dot{e}(t)\mathcal{B}u - \dot{\epsilon}(t)\nabla\zeta^t(u)\Big)^T\nabla w\ \mathrm{d}\Omega\,.$$

Choosing $w = \frac{\mathrm{d}}{\mathrm{d}t}(\nabla\zeta^t(u))$ and using coercivity, boundedness of the coefficients, and the previous estimate for $\zeta^t(u)$, we deduce that there exists C_3

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}(\zeta^t(u))\right\|_{H^1_{0,\Gamma_d}(\Omega,\mathbb{R})} = \left\|\dot{\zeta}^t(u)\right\|_{H^1_{0,\Gamma_d}(\Omega,\mathbb{R})} \leq C_3\Big(\|\dot{u}(t)\|_V + \|u\|_V\Big)$$

for almost every $t \in (0, T)$. Integrating over $(0, T)$ yields

$$\left\|\dot{\zeta}^t(u)\right\|_{L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}))} \leq C_3\Big(\|\dot{u}(t)\|_{L^2(V)} + \|u\|_{L^2(V)}\Big)$$

Combining the last estimate with the L^2-bound for $\zeta(u)$ proves that there exists a constant $C_4 > 0$ such that

$$\left\|\dot{\zeta}^t(u)\right\|_{H^1(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}))} \leq C_4\|u\|_{H^1(V))}. \tag{3.31}$$

Consequently, the mapping $t \mapsto a_0(t, u, v)$ is one time continuously differentiable with derivative

$$\dot{a}_0(t,u,v) = \int_\Omega \dot{a}(t)u^T v + \left(\dot{c}^E(t)\mathcal{B}u\right)^T \mathcal{B}v + \Big(\dot{e}(t)^T\nabla\zeta^t(u) + e(t)^T\nabla\dot{\zeta}^t(u)\Big)^T \mathcal{B}v\,\mathrm{d}\Omega\,. \tag{3.32}$$

Moreover, the mapping $t \mapsto a_1(t, u, v)$ is continuous, and

$$|a_1(t,u,v)| \leq \|a(t) - 1\|_{L^\infty(\Omega,\mathbb{R})}\|u\|_H\|v\|_H \leq C_a\big(1 + \|a\|_{L^\infty(L^\infty(\Omega,\mathbb{R}))}\big)\|u\|_V\|v\|_V. \tag{3.33}$$

Next, with

$$\beta_0 := \min\{1, \beta^\star c_\star\} > 0, \tag{3.34}$$

we have

$$b_0(t,u,u) = \int_\Omega u^T u \mathrm{d}\Omega + \int_\Omega \left(\beta(t)c^E(t)\mathcal{B}u\right)^T \mathcal{B}u\mathrm{d}\Omega$$
$$\geq \|u\|^2_{L^2(\Omega,\mathbb{R}^3)} + \beta^\star c_\star \|\mathcal{B}u\|^2_{L^2(\Omega,\mathbb{R}^6)} \geq \beta_0 \|u\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)}. \tag{3.35}$$

the mapping $t \mapsto b_1(t,u,v)$ is continuous, and

$$|b_1(t,u,v)| \leq \|\alpha(t)\rho(t) - \dot\rho(t) - 1\|_{L^\infty(\Omega,\mathbb{R})}\|u\|_H\|v\|_H \leq C_{b_1}\|u\|_V\|v\|_V \tag{3.36}$$

with $C_{b_1} := C\Big(1 + \|\alpha\rho - \dot\rho\|_{L^\infty(L^\infty(\Omega,\mathbb{R}))}\Big)$. Finally, the mapping $t \mapsto c(t,u,v)$ is one time continuously differentiable and

$$\dot c(t,u,v) = \int_\Omega \dot\rho(t) u^T v \mathrm{d}\Omega. \tag{3.37}$$

Moreover,

$$c(t,v,v) \geq \rho^\star \|v\|^2_{L^2(\Omega,\mathbb{R}^3)} \qquad \text{for all } v \in H. \tag{3.38}$$

Thus all assumptions of Theorem 1, Paragraph 5, Chapter XVIII in [5] are satisfied, while further conditions on c are ensured by Remark 9 in Paragraph 6 of the same chapter. Therefore, (3.26) admits a unique solution u with the asserted regularity. Setting

$$\phi_0(t) = \zeta^t(u(t)) + \phi_0^r(t)$$

yields the unique weak solution (u, ϕ_0) of (3.17)-(3.18). □

To prove boundedness of the operators in Section 4, the following theorem is beneficial.

Theorem 3.3 (Energy estimates)**.** *Let the assumptions of Theorem 3.2 hold. Then there exists a constant* $\widetilde{K} > 0$ *such that*

$$\|\ddot u\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*)} + \|\dot u\|^2_{L^\infty(L^2(\Omega,\mathbb{R}^3))} + \|u\|^2_{L^\infty(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\dot u\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\phi_0\|^2_{L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}))}$$
$$\leq \widetilde{K}\Big(\|u_1\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u_0\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))}$$
$$+ \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)} + \|g\|^2_{L^2(L^2(\Omega,\mathbb{R}))}\Big).$$

Proof. Define

$$X(\tau) := c(\tau,\dot u(\tau),\dot u(\tau)) + a_0(\tau,u(\tau),u(\tau)) + 2\int_0^\tau b_0(s,\dot u(s),\dot u(s))\mathrm{d}s$$
$$= \int_\Omega \rho(\tau)\dot u(\tau)^T\dot u(\tau)\mathrm{d}\Omega + \int_\Omega u(\tau)^T u(\tau)\mathrm{d}\Omega + \int_\Omega \left(c^E(\tau)\mathcal{B}u(\tau)\right)^T \mathcal{B}u(\tau)\mathrm{d}\Omega$$
$$+ \int_\Omega \left(e(\tau)^T\nabla\zeta^\tau(u(\tau))\right)^T \mathcal{B}u(\tau)\mathrm{d}\Omega + 2\int_0^\tau\int_\Omega \dot u(s)^T\dot u(s)\mathrm{d}\Omega\mathrm{d}s$$
$$+ 2\int_0^\tau\int_\Omega \left(\beta(s)c^E(s)\mathcal{B}\dot u(s)\right)^T \mathcal{B}\dot u(s)\mathrm{d}\Omega\mathrm{d}s.$$

By (3.28), (3.30), and (3.35), with

$$k := \min\left\{\rho^\star, \sigma, \|(\epsilon)^{-1}\|^{-1}_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times3}))}, \beta_0\right\} > 0,$$

we obtain

$$X(\tau) \geq \rho^\star\|\dot u(\tau)\|^2_{L^2(\Omega,\mathbb{R}^3)} + \sigma\|u(\tau)\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \int_\Omega \left(\epsilon(\tau)\nabla\zeta^\tau(u(\tau))\right)^T\nabla\zeta^\tau(u(\tau))\mathrm{d}\Omega$$
$$+ \beta_0\int_0^\tau \|\dot u(s)\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)}\mathrm{d}s$$
$$\geq k\Big(\|\dot u(\tau)\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u(\tau)\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|\nabla\zeta^\tau(u(\tau))\|^2_{L^2(\Omega,\mathbb{R}^3)} + \int_0^\tau \|\dot u(s)\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)}\mathrm{d}s\Big). \tag{3.39}$$

Moreover, with

$$c_0 := \max\left\{\|\rho\|_{L^\infty(L^\infty(\Omega,\mathbb{R}))}, 1, \|c^E\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{6\times 6}))}, \|\epsilon\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 3}))}\right\},$$

we have

$$X(0) \le c_0\Big(\|u_1\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u_0\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|\mathcal{B}u_0\|^2_{L^2(\Omega,\mathbb{R}^6)} + \|\nabla\zeta^0(u_0)\|^2_{L^2(\Omega,\mathbb{R}^3)}\Big). \tag{3.40}$$

From our previous results and [5], Chapter XVIII, Paragraph 5, section 4.1, we know that

$$\begin{aligned} X(\tau) = X(0) &+ 2\int_0^\tau \dot{a}_0(s,u(s),u(s))\mathrm{d}s - 2\int_0^\tau b_1(s,\dot{u}(s),\dot{u}(s))\mathrm{d}s \\ &- \int_0^\tau \dot{c}(s,\dot{u}(s),\dot{u}(s))\mathrm{d}s + 2\int_0^\tau \langle \tilde{f}(s),\dot{u}(s)\rangle \mathrm{d}s \end{aligned} \tag{3.41}$$

holds. Using (3.32), (3.31), and Cauchy-Schwarz, there exists $K_0 > 0$ such that

$$\int_0^\tau \dot{a}_0(s,u(s),u(s))\mathrm{d}s \le K_0\left(\|u\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\nabla(\zeta^t(u))\|^2_{L^2(L^2(\Omega,\mathbb{R}^3))}\right). \tag{3.42}$$

Furthermore, by (3.36) and (3.37),

$$-2\int_0^\tau b_1(s,\dot{u}(s),\dot{u}(s))\mathrm{d}s - \int_0^\tau \dot{c}(s,\dot{u}(s),\dot{u}(s))\mathrm{d}s \le K_1\|\dot{u}\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} \tag{3.43}$$

for some constant $K_1 > 0$. By (3.27) and Young's inequality,

$$\begin{aligned} 2\int_0^\tau \langle \tilde{f}(s),\dot{u}(s)\rangle\mathrm{d}s &\le \|\tilde{f}\|^2_{L^2(V^*)} + \|\dot{u}\|^2_{L^2(V)} \\ &\le C\Big(\|f\|^2_{L^2(V^*)} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))}\Big) + \|\dot{u}\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))}. \end{aligned} \tag{3.44}$$

Combining (3.41), (3.42), (3.43), and (3.44), we obtain a constant $K_2 > 0$ such that

$$\begin{aligned} X(\tau) \le X(0) &+ K_2\int_0^\tau \Big(\|u(s)\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|\nabla\zeta^s(u(s))\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|\dot{u}(s)\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)}\Big)\mathrm{d}s \\ &+ K_2\Big(\|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)}\Big). \end{aligned} \tag{3.45}$$

Now (3.39) and (3.45) imply

$$k\Big(\|\dot{u}(\tau)\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u(\tau)\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|\nabla\zeta^\tau(u(\tau))\|^2_{L^2(\Omega,\mathbb{R}^3)} + \int_0^\tau \|\dot{u}(s)\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)}\mathrm{d}s\Big) \le X(\tau) \tag{3.46}$$

$$\begin{aligned} &\le X(0) + K_2\int_0^\tau \Big(\|u(s)\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|\nabla\zeta^s(u(s))\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|\dot{u}(s)\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)}\Big)\mathrm{d}s \\ &+ K_2\Big(\|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)}\Big). \end{aligned} \tag{3.47}$$

By Gronwall's lemma, there exists $C > 0$ such that

$$\begin{aligned} &\|\dot{u}\|^2_{L^\infty(L^2(\Omega,\mathbb{R}^3))} + \|u\|^2_{L^\infty(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\dot{u}\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\nabla(\zeta(u))\|^2_{L^\infty(L^2(\Omega,\mathbb{R}^3))} \\ &\le C\Big(\|u_1\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u_0\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|\nabla\zeta^0(u_0)\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} \\ &\qquad + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)}\Big). \end{aligned} \tag{3.48}$$

Testing (3.21) with $\zeta^\tau(u(\tau))$, we obtain

$$\|\nabla\zeta^\tau(u(\tau))\|_{L^2(\Omega,\mathbb{R}^3)} \le \|e\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 6}))}\|(\epsilon)^{-1}\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 3}))}\|\mathcal{B}u(\tau)\|_{L^2(\Omega,\mathbb{R}^6)}. \tag{3.49}$$

In particular,

$$\|\nabla\zeta^0(u_0)\|_{L^2(\Omega,\mathbb{R}^3)} \le \|e\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 6}))}\|(\epsilon)^{-1}\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 3}))}\|\mathcal{B}u_0\|_{L^2(\Omega,\mathbb{R}^6)}. \tag{3.50}$$

Moreover, there exists some $C_r > 0$ such that

$$\begin{aligned}
&\left\|(\epsilon)^{-1}\right\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 3}))}^{-1}\|\nabla\phi_0^r(\tau)\|_{L^2(\Omega,\mathbb{R}^3)}^2 \le \int_\Omega (\epsilon\nabla\phi_0^r(\tau))^T\nabla\phi_0^r\,\mathrm{d}\Omega \\
&\le \|\epsilon\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 3}))}\|\nabla\chi(\tau)\|_{L^2(\Omega,\mathbb{R}^3)}\|\nabla\phi_0^r(\tau)\|_{L^2(\Omega,\mathbb{R}^3)} + \|g(\tau)\|_{L^2(\Omega,\mathbb{R})}\|\phi_0^r(\tau)\|_{L^2(\Omega,\mathbb{R}^3)} \\
&\le \|\epsilon\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 3}))}^2\left(\|\nabla\chi(\tau)\|_{L^2(\Omega,\mathbb{R}^3)}^2 + \|\nabla\phi_0^r(\tau)\|_{L^2(\Omega,\mathbb{R}^3)}^2\right) + \|g\|_{L^2(L^2(\Omega,\mathbb{R}))}^2 + \|\phi_0^r(\tau)\|_{L^2(\Omega,\mathbb{R}^3)}^2 \\
&\le C_r\left(\|\nabla\chi\|_{L^2(\Omega,\mathbb{R}^3)}^2 + \|\phi_0^r(\tau)\|_{H^1(\Omega,\mathbb{R}^3)}^2 + \|g\|_{L^2(L^2(\Omega,\mathbb{R}))}^2\right).
\end{aligned}$$

Thus, there exists some constant $\tilde{C} > 0$ such that for almost all $\tau \in (0,T)$

$$\begin{aligned}
&\|\dot{u}(\tau)\|_{L^2(\Omega,\mathbb{R}^3)}^2 + \|u(\tau)\|_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)}^2 + \|\dot{u}\|_{L^2(0,\tau;H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))}^2 \le \tilde{C}\Big(\|u_1\|_{L^2(\Omega,\mathbb{R}^3)}^2 + \|u_0\|_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)}^2 \\
&+\|\phi_0^r\|_{L^2(H^1(\Omega,\mathbb{R}))}^2 + \|\chi\|_{L^2(H^1(\Omega,\mathbb{R}))}^2 + \|f\|_{L^2(H^1(\Omega,\mathbb{R}^3)^*)}^2 + \|g\|_{L^2(L^2(\Omega,\mathbb{R}))}^2\Big)
\end{aligned}$$

and therefore

$$\begin{aligned}
&\|\dot{u}\|_{L^\infty(L^2(\Omega,\mathbb{R}^3))}^2 + \|u\|_{L^\infty(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))}^2 + \|\dot{u}\|_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))}^2 \le \tilde{C}\Big(\|u_1\|_{L^2(\Omega,\mathbb{R}^3)}^2 + \|u_0\|_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)}^2 \\
&+\|\phi_0^r\|_{L^2(H^1(\Omega,\mathbb{R}))}^2 + \|\chi\|_{L^2(H^1(\Omega,\mathbb{R}))}^2 + \|f\|_{L^2(H^1(\Omega,\mathbb{R}^3)^*)}^2 + \|g\|_{L^2(L^2(\Omega,\mathbb{R}))}^2\Big). \qquad (3.51)
\end{aligned}$$

It remains to estimate $\ddot{u}$. Testing (3.24) with $v \in V$, we obtain

$$\begin{aligned}
(\rho(\tau)\ddot{u}(\tau), v)_{L^2(\Omega,\mathbb{R}^3)} = {}& \langle\tilde{f}(\tau), v\rangle_{V^*,V} - (\alpha(\tau)\rho(\tau)\dot{u}(\tau), v)_{L^2(\Omega,\mathbb{R}^3)} - (a(\tau)u(\tau), v)_{L^2(\Omega,\mathbb{R}^3)} \\
&- (c^E(\tau)\mathcal{B}u(\tau), \mathcal{B}v)_{L^2(\Omega,\mathbb{R}^6)} - (\beta(\tau)c^E(\tau)\mathcal{B}\dot{u}(\tau), \mathcal{B}v)_{L^2(\Omega,\mathbb{R}^6)} \\
&- (e(\tau)^T\nabla\zeta^\tau(u(\tau)), \mathcal{B}v)_{L^2(\Omega,\mathbb{R}^6)}.
\end{aligned}$$

Hence, using Cauchy-Schwarz, (3.49), and $\rho(\tau) \ge \rho^\star$, there exists $G > 0$ such that

$$|(\ddot{u}(\tau), v)_{L^2(\Omega,\mathbb{R}^3)}| \le G\Big(\|\tilde{f}(\tau)\|_{V^*} + \|\dot{u}(\tau)\|_V + \|u(\tau)\|_V\Big)\|v\|_V.$$

Taking the supremum over $v \in B_V^1$, squaring, and integrating over $(0,T)$ and repeatedly applying Young's inequality yields

$$\|\ddot{u}(\tau)\|_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*}^2 \le 3G^2\left(\left\|\tilde{f}(\tau)\right\|_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*}^2 + \|\dot{u}(\tau)\|_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)}^2 + \|u(\tau)\|_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)}^2\right).$$

Integration over $(0,T)$, employing (3.27) and (3.51) as well as using

$$\tilde{C}_{\ddot{u}} = 3G^2\max\left\{\|e\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 6}))}^2 + \frac{\tilde{C}}{\min\{1,T\}}, 1 + \frac{\tilde{C}}{\min\{1,T\}}\right\}$$

results in

$$\begin{aligned}
\|\ddot{u}\|_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*)}^2 \le \widetilde{C}_{\ddot{u}}\Big(&\|u_1\|_{L^2(\Omega,\mathbb{R}^3)}^2 + \|u_0\|_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)}^2 + \|\phi_0^r\|_{L^2(H^1(\Omega,\mathbb{R}))}^2 \\
&+\|\chi\|_{L^2(H^1(\Omega,\mathbb{R}))}^2 + \|f\|_{L^2(H^1(\Omega,\mathbb{R}^3)^*)}^2 + \|g\|_{L^2(L^2(\Omega,\mathbb{R}))}^2\Big). \qquad (3.52)
\end{aligned}$$

Finally, testing (3.19) with $\phi_0(\tau)$, using coercivity of $\epsilon(\tau)$, Cauchy-Schwarz, and Poincaré's inequality, we obtain

$$\begin{aligned}
&\left\|(\epsilon)^{-1}\right\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 3}))}^{-1}\|\nabla\phi_0\|_{L^2(\Omega,\mathbb{R}^3)}^2 \le \left|\int_\Omega (\epsilon\nabla\phi_0)^T\nabla\phi_0\,\mathrm{d}\Omega\right| \\
&= \left|\int_\Omega (e\mathcal{B}u)^T\nabla\phi_0 - (\epsilon\nabla\chi)^T\nabla\phi_0 - g\phi_0\,\mathrm{d}\Omega\right| \\
&\le \left(\|e\mathcal{B}u\|_{L^2(\Omega,\mathbb{R}^3)} + \|\epsilon\nabla\chi\|_{L^2(\Omega,\mathbb{R}^3)}\right)\|\nabla\phi_0\|_{L^2(\Omega,\mathbb{R}^3)} + \|g\|_{L^2(\Omega,\mathbb{R})}\|\phi_0\|_{L^2(\Omega,\mathbb{R})} \\
&\le \left(\|e\mathcal{B}u\|_{L^2(\Omega,\mathbb{R}^3)} + \|\epsilon\nabla\chi\|_{L^2(\Omega,\mathbb{R}^3)}\right)\|\nabla\phi_0\|_{L^2(\Omega,\mathbb{R}^3)} + C_P\|g\|_{L^2(\Omega,\mathbb{R})}\|\nabla\phi_0\|_{L^2(\Omega,\mathbb{R})}. \qquad (3.53)
\end{aligned}$$

After dividing by $\|\nabla\phi_0\|_{L^2(\Omega,\mathbb{R}^3)} > 0$, ($\|\nabla\phi_0\|_{L^2(\Omega,\mathbb{R}^3)} = 0$ is a trivial case), using the norm equivalence of the H^1_{0,Γ_d}- norm and the L^2-norm of the gradient, integrating over time and applying Hölder's inequality there exists a constant C_ϕ such that

$$\|\phi_0\|_{L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}))} \leq C_\phi \left(\|u\|_{L^\infty(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\chi\|_{L^2(H^1(\Omega,\mathbb{R}))} + \|g\|_{L^2(L^2(\Omega,\mathbb{R}))} \right). \tag{3.54}$$

Combining (3.51), (3.52), and (3.54) proves the claim. □

Remark 3.4. Note that if $g \in L^\infty(L^2(\Omega,\mathbb{R}))$ and $\chi \in H^1\left(H^1(\Omega,\mathbb{R})\right) \cap L^\infty\left(H^1(\Omega,\mathbb{R})\right)$ then we do not necessarily have to integrate (3.53) over time after dividing by $\|\nabla\phi_0\|_{L^2(\Omega,\mathbb{R}^3)} > 0$, since by taking the essential supremum, there exists a constant $\tilde{C}_\phi$ such that

$$\|\phi_0\|_{L^\infty(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}))} \leq \tilde{C}_\phi \left(\|u\|_{L^\infty(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\chi\|_{L^\infty(H^1(\Omega,\mathbb{R}))} + \|g\|_{L^\infty(L^2(\Omega,\mathbb{R}))} \right).$$

This motivates the definition of the state space W.

Definition 3.5. The state space W is defined as

$$W := \left\{(u,\phi_0) \in H^1(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)) \times L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R})) :\ \dot{u} \in L^\infty(L^2_{\mathcal{B}}(\Omega,\mathbb{R}^3)),\ \ddot{u} \in L^2(H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*)\right\}.$$

We aim for improving the regularity in space, in order to prove well-definedness of the observation operator in Section 4. For this purpose, the following remark is beneficial.

Remark 3.6. If solutions in spaces with higher regularity are aimed for, we have to perform the Dirichlet lift in higher order Sobolev spaces, where we distinguish the following cases.

1. Assumption D2 holds: we have $\chi \in H^1\left(H^{m+2}(\Omega,\mathbb{R})\right)$, according to the trace theorem in [17, Chapter 3].
2. Assumption D3 holds: according to Theorem 4.12 in [17], we have $\chi \in H^1\left(H^m(\Omega,\mathbb{R})\right))$. As ϕ^e is constant in space the compatibility condition is fulfilled.

In both cases $\mathrm{Tr}(\chi(t))$ is defined as in (3.1).

Corollary 3.7 (Regularity)**.** Let $m \in \mathbb{N}$, $m \geq 2$ and the assumptions of Theorem 3.2 be satisfied with $e, \epsilon \in H^1(0,T)$ constant in space and $(f,g) \in L^2(H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3)) \times L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}) \cap H^m(\Omega,\mathbb{R}))$. Suppose either Assumption D2 or Assumption D3 with χ according to Remark 3.6 hold, then for any $u_0 \in H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3),\ u_1 \in H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3)$ there exists a unique solution

$$(u,\phi_0) \in L^2(H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3)) \times L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}) \cap H^m(\Omega,\mathbb{R})) \tag{3.55}$$

with

$$\dot{u} \in L^2(H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3)) \ \text{ and } \ \ddot{u} \in L^2(H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*) \tag{3.56}$$

to the system (3.17)-(3.18).

Proof. We follow the proof of Theorem 3.2, but now on the higher-order space $V_m := H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3)$. We equip V_m with the norm

$$\|u\|^2_{V_m} := \|u\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \sum_{1\leq|\lambda|\leq m-1} \|\partial^\lambda \mathcal{B}u\|^2_{L^2(\Omega,\mathbb{R}^6)}.$$

Furthermore, we equip $H^1_{0,\Gamma_d}(\Omega,\mathbb{R}) \cap H^m(\Omega,\mathbb{R})$ with the norm

$$\|\phi_0\|^2_{H^1_{0,\Gamma_d}(\Omega,\mathbb{R})\cap H^m(\Omega,\mathbb{R})} := \|\phi_0\|^2_{H^1_{0,\Gamma_d}(\Omega,\mathbb{R})} + \sum_{1\leq|\lambda|\leq m-1} \|\partial^\lambda \nabla\phi_0\|^2_{L^2(\Omega,\mathbb{R}^6)}.$$

Since Ω is Lipschitz, Korn's inequality implies that $\|\cdot\|_{V_m}$ is equivalent to the standard $H^m(\Omega,\mathbb{R}^3)$-norm. Now consider the linear mapping defined in (3.20) and observe that for every multi-index λ with $1 \leq |\lambda| \leq m-1$, $\zeta^t(\partial^\lambda u) \in H^1_{0,\Gamma_d}(\Omega,\mathbb{R})$ is well-defined for $\partial^\lambda u \in H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)$. Moreover, using (3.21) yields

$$\begin{aligned}\int_\Omega (\epsilon\nabla\zeta^t(\partial^\lambda u))^T \nabla w \,\mathrm{d}\Omega &= \int_\Omega (\epsilon\partial^\lambda\nabla\zeta^t(u))^T \nabla w \,\mathrm{d}\Omega = \int_\Omega (\epsilon\partial^\lambda\nabla(\phi_0^0)^T \nabla w \,\mathrm{d}\Omega \\ &= \int_\Omega (e\partial^\lambda \mathcal{B}u)^T \nabla w,\end{aligned} \tag{3.57}$$

which yields by testing (3.57) with $\zeta^t(\partial^\lambda u)$ for almost all $t \in (0,T)$ that $\phi_0^0 \in H^1_{0,\Gamma_d}(\Omega,\mathbb{R}) \cap H^m(\Omega,\mathbb{R})$. Thus we define, similarly to the proof of Theorem 3.2, for almost all $t \in (0,T)$

$$\zeta_m^t : H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3) \to H^1_{0,\Gamma_d}(\Omega,\mathbb{R}) \cap H^m(\Omega,\mathbb{R}),\ u(t) \mapsto \phi_0^0(t),$$

where $\phi_0(t) = \phi_0^0(t) + \phi_0^r(t) = \zeta_m^t(u(t)) + \phi_0^r(t)$ by satisfying equation (3.57). As $H^1_{0,\Gamma_d}(\Omega,\mathbb{R})^1 \cap H^m(\Omega,\mathbb{R}) \subset H^m(\Omega,\mathbb{R})$ we use the higher spatial regularity of χ and g define $\phi_0^r \in L^2(H^m(\Omega,\mathbb{R}))$ in a similar manner as above by satisfying equation (3.22) in this setting. We obtain the same weak form of the inhomogeneous piezoelectric PDE including the Dirichlet lift and thus the bilinear form (3.24), which we employ to define the operators

$$\begin{aligned}
\hat{a}_0(t,u,v) &:= a_0(t,u,v) + \sum_{1\le|\lambda|\le m-1}\left[\int_\Omega \left(c_I^E\mathcal{B}(\partial^\lambda u)\right)^T\mathcal{B}(\partial^\lambda v)\mathrm{d}\Omega + \int_\Omega \left(e_I^T\nabla\zeta^t(\partial^\lambda u)\right)^T\mathcal{B}(\partial^\lambda v)\mathrm{d}\Omega\right],\\
\hat{a}_1(t,u,v) &:= a_1(t,u,v) - \sum_{1\le|\lambda|\le m-1}\left[\int_\Omega \left(c_I^E\mathcal{B}(\partial^\lambda u)\right)^T\mathcal{B}(\partial^\lambda v)\mathrm{d}\Omega + \int_\Omega \left(e_I^T\nabla\zeta^t(\partial^\lambda u)\right)^T\mathcal{B}(\partial^\lambda v)\mathrm{d}\Omega\right],\\
\hat{b}_0(t,u,v) &:= b_0(t,u,v) + \sum_{1\le|\lambda|\le m-1}\left[\int_\Omega \left(c_I^E\mathcal{B}(\partial^\lambda u)\right)^T\mathcal{B}(\partial^\lambda v)\mathrm{d}\Omega\right],\\
\hat{b}_1(t,u,v) &:= b_1(t,u,v) - \sum_{1\le|\lambda|\le m-1}\left[\int_\Omega \left(c_I^E\mathcal{B}(\partial^\lambda u)\right)^T\mathcal{B}(\partial^\lambda v)\mathrm{d}\Omega\right],
\end{aligned}$$

(3.58)

$$\hat{c}(t,u,v) := c(t,u,v)$$

where c_I^E and $e_I \in \mathbb{R}^{3\times 6}$ are of the same form as the material parameters, where the non-zero entries are kept constant 1. Then, $a_{01} = \hat{a}_0 + \hat{a}_1$ and $b_{01} = \hat{b}_0 + \hat{b}_1$ have the same form as in proof of Theorem 3.2, so the evolution equation itself is unchanged. Note that the right hand side of equation (3.17) is $f + \mathcal{B}^T e^T\nabla\chi =: \tilde{f} \in L^2(H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3))$. Therefore, the requirements of Theorem 1, Paragraph 5, Chapter XVIII of [5] for the right hand side are satisfied. Following the same structure as in proof of Theorem 3.2 now with the higher regularity assumptions we see, by (3.28), applied with u and v replaced by $\partial^\lambda u$ and $\partial^\lambda v$, that

$$\begin{aligned}
\int_\Omega \left(e(t)^T\nabla\zeta^t(\partial^\lambda u)\right)^T\mathcal{B}(\partial^\lambda v)\mathrm{d}\Omega &= \int_\Omega \left(\epsilon(t)\nabla\zeta^t(\partial^\lambda u)\right)^T\nabla\zeta^t(\partial^\lambda v)\mathrm{d}\Omega\\
&= \int_\Omega \left(e(t)^T\nabla\zeta^t(\partial^\lambda v)\right)^T\mathcal{B}(\partial^\lambda u)\mathrm{d}\Omega
\end{aligned} \tag{3.59}$$

holds. As a_0 and b_0 are Hermitian, both $\hat{a}_0$ and $\hat{b}_0$ are Hermitian as well. Therefore, with the same $\sigma \in \mathbb{R}^+$ defined in (3.29) we use (3.30) and obtain for every $\partial^\lambda u$ that

$$\begin{aligned}
\hat{a}_0(t,u,u) = a_0(t,u,u) &+ \sum_{1\le|\lambda|\le m-1}\left[\int_\Omega \left(c_I^E\mathcal{B}(\partial^\lambda u)\right)^T\mathcal{B}(\partial^\lambda u)\mathrm{d}\Omega + \int_\Omega \left(e_I^T\nabla\zeta^t(\partial^\lambda u)\right)^T\mathcal{B}(\partial^\lambda u)\mathrm{d}\Omega\right]\\
&\ge \sigma\|u\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \sigma\sum_{1\le|\lambda|\le m-1}\left(\|\mathcal{B}(\partial^\lambda u)\|^2_{L^2(\Omega,\mathbb{R}^6)}\right) \ge \sigma\|u\|^2_{V_m}.
\end{aligned}$$

Thus $\hat{a}_0$ is coercive on V_m. In the same way, using (3.35), we obtain

$$\hat{b}_0(t,u,u) \ge \beta_0\|u\|^2_{V_m} \qquad \forall\, u \in V_m.$$

Furthermore, continuous differentiability of the mappings $t \mapsto \hat{a}_0(t,u,v)$ and $t \mapsto \hat{a}_1(t,u,v)$ follow immediately from Theorem 3.2. Moreover, the correction terms in $\hat{a}_1$ and $\hat{b}_1$ are bounded on $V_m \times V_m$. Hence there exists $C > 0$ such that

$$|\hat{a}_1(t,u,v)| + |\hat{b}_1(t,u,v)| \le C\|u\|_{V_m}\|v\|_{V_m} \qquad \forall\, u,v \in V_m.$$

Finally, $\hat{c} = c$ has the same properties as in the proof of Theorem 3.2. Therefore all assumptions of Theorem 1, Paragraph 5, Chapter XVIII in [5] are satisfied on the space V_m. Hence,

$$u \in L^2\big(H^m(\Omega,\mathbb{R}^3)\big), \qquad \dot{u} \in L^2\big(H^m(\Omega,\mathbb{R}^3)\big), \qquad \ddot{u} \in L^2\big(H^m(\Omega,\mathbb{R}^3)^*\big).$$

and with $\phi_0 = \zeta_m^t(u) + \phi_0^r$ we obtain

$$\phi_0 \in L^2\big(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}) \cap H^m(\Omega,\mathbb{R})\big).$$

□

4. Analysis of the forward operator

Before we define the model operator, we have to specify the parameter space.

Definition 4.1. The parameter space X is defined as

$$X := \Big\{(c^E, e, \epsilon) \in H^3(H^3(\Omega,\mathbb{R}^{6\times 6})) \times H^3(H^3(\Omega,\mathbb{R}^{3\times 6})) \times H^3(H^3(\Omega,\mathbb{R}^{3\times 3})) : \\ c^E, e, \epsilon \text{ are of structure as in Assumptions A3-A5}\Big\},$$

and we assume that there exists a constant $M > 0$ such that

$$\|c^E\|_{H^3(H^3(\Omega,\mathbb{R}^{6\times6}))} + \|e\|_{H^3(H^3(\Omega,\mathbb{R}^{3\times6}))} + \|\epsilon\|_{H^3(H^3(\Omega,\mathbb{R}^{3\times3}))} \le M$$

for all $(c^E, e, \epsilon) \in X$.

Uniform boundedness of X is a physically reasonable assumption, since otherwise the corresponding material parameters would attain unrealistically large values and the model would cease to describe the underlying system in a meaningful way. We now define the model operator corresponding to (3.2)-(3.7).

Definition 4.2 (Model operator)**.** We abbreviate $p = (c^E, e, \epsilon)$ as well as $z = (u, \phi_0)$ and identify the piezoelectric model operator $A : X \times W \to W^*$ in the dual pairing using (3.10) and (3.11) by

$$\begin{aligned}\langle A(p,z),(v,w)\rangle_{W^*,W} := \int_0^T \int_\Omega & \rho\ddot{u}^T v + \alpha\rho\dot{u}^T v + \left(c^E\mathcal{B}u + \beta c^E\mathcal{B}\dot{u} + e^T\nabla\phi_0\right)^T \mathcal{B}v \\ & + \left(e\mathcal{B}u - \epsilon\nabla\phi_0\right)^T\nabla w + (e^T\nabla\chi)^T\mathcal{B}v - (\epsilon\nabla\chi)^T\nabla w \,\mathrm{d}\Omega\,\mathrm{d}t\,.\end{aligned} \tag{4.1}$$

For the classic reduced approach, we need the parameter-to-state map, motivating the following definition.

Definition 4.3 (Parameter-to-state map)**.** We define the parameter-to-state map

$$\begin{aligned} S : X &\to W, \\ p &\mapsto z, \end{aligned}$$

via satisfying the model

$$\forall p \in X : \quad A(p, S(p)) = 0, \tag{4.2}$$

such that

$$\forall z \in W : \ [(p,z) \in X \times W \ \wedge \ A(p,z) = 0] \implies z = S(p),$$

with the model operator defined in Definition 4.2.

Thus, well-definedness of the forward operator needs the existence of the parameter-to-state map S. This is achieved by exploiting the Implicit Function Theorem, i.e., we employ the condition

$$\exists C_A \ \forall (p,z) \in X \times W : \ A'_z(p,z)^{-1} \text{ exists and } \left\|A'_z(p,z)^{-1}\right\| \le C_A. \tag{4.3}$$

The following remark guarantees existence and uniqueness of our piezoelectric dynamical system under consideration (3.2)-(3.7).

Remark 4.4 (Existence and Uniqueness of (3.2)-(3.7)). Let Assumption D1 hold. For the piezoelectric dynamical system (3.2)-(3.7) with $\rho \in H^4(L^\infty(\Omega,\mathbb{R}))$ being positive and uniformly bounded, $\alpha,\ \beta \in H^3(L^\infty(\Omega,\mathbb{R}))$ being non-negative and uniformly bounded, and material parameters in X of Definition 4.1, we obtain the same or partially even more regularity of damping and material parameters than in Theorem 3.2. Furthermore, $a \equiv f \equiv g \equiv 0$. Therefore, all assumptions of Theorem 3.2 are satisfied, yielding the existence of a unique weak solution $(u,\phi_0) \in W$ of (3.2)-(3.7) with $\dot{u} \in L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)) \cap L^\infty(L^2(\Omega,\mathbb{R}^3))$ and $\ddot{u} \in L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*)$ and due to Theorem 3.3 the existence of a constant C_p such that

$$
\begin{aligned}
&\|\ddot{u}\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*)} + \|\dot{u}\|^2_{L^\infty(L^2(\Omega,\mathbb{R}^3))} + \|u\|^2_{L^\infty(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\dot{u}\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\phi_0\|^2_{L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}))}\\
&\le C_p \Big(\|u_1\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u_0\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|\phi_0^\tau\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))}\\
&\quad + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)} + \|g\|^2_{L^2(L^2(\Omega,\mathbb{R}))} \Big).
\end{aligned}
$$

Note, that the model operator of Definition 4.2 includes all boundary conditions and is bijective on W for fixed $p \in X$ due to Remark 4.4. Furthermore with an arbitrary direction $\xi = (\mu,\nu) \in W$ the Gâteaux derivative $\delta_z A(p,z)\xi = A_z(p,z)\xi$ with respect to the state can be identified as

$$
\begin{aligned}
\langle A_z(p,z)\xi,(v,w)\rangle_{W^*,W} := \int_0^T \int_\Omega & \rho \ddot{\mu}^T v + \alpha\rho\dot{\mu}^T v + \left(c^E\mathcal{B}\mu + \beta c^E \mathcal{B}\dot{\mu} + e^T\nabla\nu\right)^T \mathcal{B}v\\
&+ (e\mathcal{B}\mu - \epsilon\nabla\nu)^T \nabla w \,\mathrm{d}\Omega\,\mathrm{d}t\,,
\end{aligned} \tag{4.4}
$$

which is also bijective on W for fixed $p \in X$ due to Remark 4.4.

Lemma 4.5 (Existence and Regularity of S)**.** *The parameter-to-state map S of Definition 4.3 exists and it holds that $S \in C^1(X,W)$.*

Proof. In order to apply the Implicit Function Theorem we first have to prove Fréchet differentiability of A with respect to the state. Therefore, we consider

$$
\langle A(p,z+\xi),(v,w)\rangle_{W^*,W} - \langle A(p,z),(v,w)\rangle_{W^*,W} - \langle A_z(p,z)\xi,(v,w)\rangle_{W^*,W}.
$$

Due to the affine linearity of A with respect to the state, it holds that

$$
\begin{aligned}
&\langle A(p,z+\xi),(v,w)\rangle_{W^*,W} - \langle A(p,z),(v,w)\rangle_{W^*,W} - \langle A_z(p,z)\xi,(v,w)\rangle_{W^*,W}\\
&= \langle A(p,z),(v,w)\rangle_{W^*,W} + \int_0^T \int_\Omega \rho \ddot{\mu}^T v + \alpha\rho\dot{\mu}^T v + \left(c^E\mathcal{B}\mu + \beta c^E\mathcal{B}\dot{\mu} + e^T\nabla\nu\right)^T \mathcal{B}v\\
&\quad + (e\mathcal{B}\mu - \epsilon\nabla\nu)^T \nabla w\,\mathrm{d}\Omega\,\mathrm{d}t - \langle A(p,z),(v,w)\rangle_{W^*,W} - \langle A_z(p,z)\xi,(v,w)\rangle_{W^*,W} \underset{(4.1),(4.4)}{=} 0.
\end{aligned}
$$

This yields Fréchet differentiability of A with respect to the state. As A is also affine linear in the material parameters, similar arguments yield Fréchet differentiability of A with respect to the material parameters. Furthermore, the Fréchet derivative of A is continuous as it is linear with respect to the respective variable and bounded due to Theorem 3.3. Therefore, we have that $A \in C^1(X \times W, W^*)$. Since A is affine linear in z, the derivative $A_z(p,z) = A_z(p)$ is linear and independent of z. Furthermore, due to linearity of $A_z(p,z)\xi$ in ξ, boundedness of $A_z(p,z)\xi$ as well as bijectivity of $A_z(p,z)\xi$ on W for fixed $p \in X$ it holds due to the Bounded Inverse Theorem that

$$
\exists C_A\ \forall (p,z) \in X \times W,\ \forall f \in W^*:\ A_z^{-1}(p,z)f \text{ exists and } \left\|A_z^{-1}(p,z)f\right\|_W \le C_A \|f\|_{W^*}.
$$

Taking the operator norm of $A_z^{-1}(p,z)$ yields $A_z(p)^{-1} \in \mathcal{L}(W^*,W)$.Together with assumption (4.3), all assumptions of the Implicit Function Theorem are satisfied. Therefore, there exists a uniquely determined parameter-to-state map $S \in C^1(X,W)$ as in Definition 4.3. □

Due to Remark 4.4, the parameter-to-state map $S \in C^1(X,W)$, see Definition 4.3 is well-defined, as for an arbitrary fixed $p \in X$ it is not possible to have more than one state $z \in W$. Furthermore, S is non-linear, which can be seen by the structure of the model operator, which is used to define S.

To state and solve an inverse problem we need additional observations. In our case we obtain a measured charge pulse. Therefore, the observation operator reads as

$$\tilde{C}(p,z) := \int_{\Gamma_e} (e\mathcal{B}u - \epsilon\nabla\phi_0 - \epsilon\nabla\chi)\cdot n \,\mathrm{d}\Gamma\,,$$

which means that the electrodes are conductive and thus the charge is distributed equally on the loaded electrode. As $(u,\phi_0)\in W$ and $\chi\in H^1(H^1(\Omega,\mathbb{R}))$, due to Theorem 3.3, there exists a constant $\tilde{L}>0$, such that $\|u\|_{L^\infty(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))}\leq\tilde{L}$, $\|\phi_0\|_{L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}))}\leq\tilde{L}$ and $\|\chi\|_{H^1(H^1(\Omega,\mathbb{R}))}\leq\tilde{L}$. Therefore

$$e\mathcal{B}u =: h_1 \in L^2(L^2(\Omega,\mathbb{R}^3)) \tag{4.5}$$

$$\epsilon\nabla\phi_0 =: h_2 \in L^2(L^2(\Omega,\mathbb{R}^3)) \tag{4.6}$$

$$\epsilon\nabla\chi =: h_3 \in H^1(L^2(\Omega,\mathbb{R}^3)), \tag{4.7}$$

which yields that $\tilde{C}$ is not well-defined as we have a boundary integral with L^2-functions, which in general cannot be evaluated. However, this boundary integral can be converted into a volume integral on an open neighborhood of the boundary Γ_e, denoted by $U_\gamma(\Gamma_e)$ with $\gamma>0$, if one lacks in space regularity of the state. This requires the continuous extension of the normal vector in this neighborhood, which is obtained by solving the eikonal equation.

Definition 4.6 (Observation operator)**.** Let $\gamma>0$ be fixed and small enough and $Y=L^2(0,T)$. Then, we define the observation operator as $C^\gamma : X\times W\to Y$ by

$$C^\gamma(p,z) := |\Gamma_e||U_\gamma(\Gamma_e)|^{-1}\int_{U_\gamma(\Gamma_e)} (e\mathcal{B}u - \epsilon\nabla\phi_0 - \epsilon\nabla\chi)\cdot\nabla b\,\mathrm{d}\Omega,$$

where b solves the eikonal equation

$$\|\nabla b\|_{L^2(U_\gamma(\Gamma_e))} = 1 \;\text{ in }\; U_\gamma(\Gamma_e) \tag{4.8}$$

$$b = 0 \;\text{ on }\; \partial U_\gamma(\Gamma_e). \tag{4.9}$$

This operator is well-defined and bounded, as for some fixed and small enough $\gamma>0$, we obtain

$$\begin{aligned}\|C^\gamma(p,z)\|_Y^2 &\leq C_b\left(\|h_1\|^2_{L^\infty(L^2(\Omega,\mathbb{R}^3))} + \|h_2\|^2_{L^2(L^2(\Omega,\mathbb{R}^3))} + \|h_3\|^2_{H^1(L^2(\Omega,\mathbb{R}^3))}\right)\\ &\leq C_O\Big(\|u_1\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u_0\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))}\\ &\qquad + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)} + \|g\|^2_{L^2(L^2(\Omega,\mathbb{R}))}\Big)\end{aligned} \tag{4.10}$$

for some constant $C_b, C_O>0$, due to Remark 4.4 and Definition 4.6. Note that for some fixed and small enough $\gamma>0$ the observation operator C^γ is affine linear in the state z and the parameters p. Furthermore, C^γ is continuously Fréchet differentiable with respect to the state, as for an arbitrary fixed direction $\xi=(\mu,\nu)\in W$, the Gâteaux derivative reads as

$$C^\gamma_z(p,z)\xi := |\Gamma_e||U_\gamma(\Gamma_e)|^{-1}\int_{U_\gamma(\Gamma_e)} (e\mathcal{B}\mu - \epsilon\nabla\nu)\cdot\nabla b\;\mathrm{d}\Omega\,. \tag{4.11}$$

yielding

$$C^\gamma(p,z+\xi) - C^\gamma(p,z) - C^\gamma_z(p,z)\xi = 0. \tag{4.12}$$

By similar arguments, we deduce that C^γ is continuously Fréchet differentiable with respect to the material parameters. Due to [6], the eikonal equation admits a classical solution which is the proper extension of the normal vector on Γ_e and therefore converges for $\gamma\to 0$ to the normal vector on Γ_e. This yields that

$$C^\gamma \underset{\gamma\to 0}{\longrightarrow} \tilde{C}.$$

Remark 4.7. Note that for solutions with higher regularity, i.e., solutions as in Corollary 3.7 with $m\geq 2$, the observation operator will be $\tilde{C}$, as it is well-defined and bounded due to similar arguments as in (4.10). Furthermore, we deduce in the same manner as above (see(4.11)-(4.12)), that $\tilde{C}$ is continuously Fréchet differentiable with respect to the state and the material parameters.

In real world application the assumptions of Corollary 3.7 with at least $m = 2$ are usually fulfilled. From now on, if the assumptions of Corollary 3.7 with at least $m = 2$ are not fulfilled, we fix a sufficiently small $\gamma > 0$ and abbreviate $C = C^\gamma$, otherwise we use $C = \tilde{C}$ and adapt X and W according to the given higher regularities.
Inserting $S(p)$ into the observation operator C identifies the forward operator $F : X \to Y$, i.e.,

$$C(p, S(p)) = F(p) = y.$$

This casts the problem into a single operator equation for the unknown p. Hence, the forward operator F acts directly on p and returns the data y. Note that due to the properties of the parameter-to-state map and the observation operator, we can conclude that the forward operator is well-defined, non-linear and continuously Fréchet differentiable as it inherits these properties from C and S. By denoting the noisy measurements with y^δ and introducing a weakly lower semi-continuous regulariser $\mathcal{R}_\mu : X \to \mathbb{R}$ with an regularization parameter $\mu > 0$, we now define the regularized target functional $J : X \to \mathbb{R}$ by

$$J(p) := \frac{1}{2}\left\|F(p) - y^\delta\right\|^2_{L^2(0,T)} + \mathcal{R}_\mu(p). \tag{4.13}$$

Using an optimization approach, the inverse problem aims at finding a minimizer of

$$\min_{p \in X} J(p). \tag{4.14}$$

Theorem 4.8. *Let*

$$(c_n^E, e_n, \epsilon_n) \in \widetilde{X} \subseteq X \cap \left(H^2(H^2(\Omega, \mathbb{R}^{6\times6})) \times H^2(H^2(\Omega, \mathbb{R}^{3\times6})) \times H^2(H^2(\Omega, \mathbb{R}^{3\times3}))\right).$$

Then the forward operator

$$F : X \to Y, \qquad F(p) := C(p, S(p)),$$

is weak-to-strong continuous, i.e., if $p_n \rightharpoonup p$ weakly in X, then

$$F(p_n) \to F(p) \qquad \textit{strongly in } Y.$$

Proof. Let $p_n = (c_n^E, e_n, \epsilon_n) \rightharpoonup p = (c^E, e, \epsilon)$ weakly in $\widetilde{X}$. Then (p_n) is bounded in $\widetilde{X}$. By compactness of the embedding $H^2(\Omega) \hookrightarrow^c C(\bar{\Omega})$ together with Corollary 4 in [25] implies, after passing to a subsequence, the strong convergence

$$c_n^E \to c^E, \qquad e_n \to e, \qquad \epsilon_n \to \epsilon$$

in $C([0,T] \times \bar{\Omega}))$, componentwise. By uniqueness of the weak limit, the whole sequence converges in this way. Now set

$$z_n := S(p_n) = (u_n, \phi_{0,n}), \qquad z := S(p) = (u, \phi_0).$$

Since $A(p_n, z_n) = 0$ and $A(p, z) = 0$, we have

$$A(p_n, z_n) - A(p_n, z) = -(A(p_n, z) - A(p, z)).$$

Because A is affine linear with respect to the state variable, this yields

$$A_z(p_n)(z_n - z) = -(A(p_n, z) - A(p, z)).$$

Hence, by (4.3),

$$\|z_n - z\|_W \le C_A \|A(p_n, z) - A(p, z)\|_{W^*}.$$

It therefore remains to show that $\|A(p_n, z) - A(p, z)\|_{W^*} \to 0$. Let $(v, w) \in W$. By (4.1),

$$\begin{aligned}
&\left|\langle A(p_n, z) - A(p, z), (v, w)\rangle_{W^*,W}\right| \\
&\le \int_0^T \int_\Omega \left|\left((c_n^E - c^E)\mathcal{B}u + \beta(c_n^E - c^E)\mathcal{B}\dot{u} + (e_n - e)^T \nabla\phi_0\right)^T \mathcal{B}v\right| \mathrm{d}\Omega\, \mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \left|\left((e_n - e)\mathcal{B}u - (\epsilon_n - \epsilon)\nabla\phi_0 - (\epsilon_n - \epsilon)\nabla\chi\right)^T \nabla w\right| \mathrm{d}\Omega\, \mathrm{d}t.
\end{aligned}$$

Using Hölder's inequality and the strong convergence of p_n to p in $L^\infty((0,T) \times \Omega)$, we obtain

$$\|A(p_n, z) - A(p, z)\|_{W^*} \to 0.$$

Consequently,

$$z_n \to z \qquad \text{strongly in } W.$$

We now prove strong convergence of the observations. We split

$$F(p_n) - F(p) = C(p_n, z_n) - C(p, z) = \big(C(p_n, z_n) - C(p_n, z)\big) + \big(C(p_n, z) - C(p, z)\big).$$

For the first term, by and Hölder's and Poincaré's inequality there exists $c_1 > 0$

$$\|C(p_n, z_n) - C(p_n, z)\|_Y \leq c_1 \left(\|u_n - u\|_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\phi_{0,n} - \phi_0\|_{L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}^3))} \right).$$

Since $z_n \to z$ strongly in W, this term converges to 0. For the second term, by Hölder's inequality and Theorem 3.3 there exists $c_2 > 0$

$$\|C(p_n, z) - C(p, z)\|_Y \leq c_2 \left(\|e_n - e\|_{L^\infty} \|\mathcal{B}u\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 6}))} + \|\epsilon_n - \epsilon\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 6}))} \right),$$

which converges to 0 because $p_n \to p$ strongly in $L^\infty((0,T) \times \Omega)$. □

Corollary 4.9. Let the Assumptions of Theorem 4.8 hold and let the regulariser $\mathcal{R}_\mu$ be weakly lower semi-continuous. Then there exists a minimizer of the functional $J : \widetilde{X} \to \mathbb{R}$ defined in (4.13).

Proof. First, we directly obtain that $\widetilde{X}$ is convex and closed, since all eigenvalues of c^E and ϵ are bounded away from 0. Second, we prove that $J : \widetilde{X} \to \mathbb{R}$ defined in (4.13) is weakly lower semi-continuous. By Theorem 4.8 F is weak-to-strong continuous. Let $p_n \rightharpoonup p$ weakly in $\widetilde{X}$. Since F is weak-to-strong continuous and $y^\delta \in Y$ is fixed, we obtain

$$F(p_n) - y^\delta \to F(p) - y^\delta \qquad \text{strongly in } Y.$$

Since $Y = L^2(0, T)$ is a Hilbert space, the norm is weakly lower semi-continuous. Hence,

$$\|F(p) - y^\delta\|_Y^2 \leq \liminf_{n\to\infty} \|F(p_n) - y^\delta\|_Y^2.$$

Therefore, $\|F(p) - y^\delta\|_Y^2$ is weakly lower semi-continuous. As the sum of weakly lower semi-continuous functions is weakly lower semi-continuous, we obtain weakly lower semi-continuity of J. Hence, existence of a minimizer to the optimization problem (4.14) is guaranteed by Tonelli's Theorem. □

In the parametrization approaches (2.1) and (2.2), the admissible set $\widetilde{X} = X$ consists only of constant real-valued matrices with the same structural properties as before. Hence, X remains bounded, convex and closed, and is a subset of a finite-dimensional real vector space. We now derive first-order optimality conditions in the smooth case. Assume from now on that $\mathcal{R}_\mu$ is continuously Fréchet differentiable on X, and let $p^* \in \operatorname{int}(X)$ be a minimizer of J. Since J is continuously Fréchet differentiable and p^* is an interior minimizer, the first-order necessary optimality condition reads

$$J'(p^*) = 0. \tag{4.15}$$

Thus, we obtain the following first-order optimality conditions:

- $A(p^*, S(p^*)) = 0$ (state equation),
- $A'_z(p^*, S(p^*))^* q = -C'_z(p^*, S(p^*))^* \big(C(p^*, S(p^*)) - y^\delta\big)$ (adjoint equation),

where the superscript * denotes the adjoint operator, and $q = (q_1, q_2) \in W$ denotes the adjoint state. Moreover, for an admissible parameter direction $h = (h_{c^E}, h_e, h_\epsilon)$, we have

$$J'(p^*)[h] = \big\langle A'_p(p^*, S(p^*))[h], q\big\rangle_{W^*,W} + \big(C'_p(p^*, S(p^*))[h], C(p^*, S(p^*)) - y^\delta\big)_Y + \mathcal{R}'_\mu(p^*)[h]. \tag{4.16}$$

Furthermore, using that same admissible parameter direction h, we identify

$$\begin{aligned}\big\langle A'_p(p, S(p))[h], q\big\rangle_{W^*,W} = \int_0^T \int_\Omega &\Big(h_{c^E}\mathcal{B}u + \beta h_{c^E}\mathcal{B}\dot{u} + h_e^T\nabla\phi_0\Big)^T \mathcal{B}q_1 \\ &+ \Big(h_e\mathcal{B}u - h_\epsilon\nabla\phi_0\Big)^T \nabla q_2 + \Big(h_e^T\nabla\chi\Big)^T \mathcal{B}q_1 - \Big(h_\epsilon\nabla\chi\Big)^T \nabla q_2 \,\mathrm{d}\Omega\,\mathrm{d}t\,.\end{aligned}$$

Moreover, depending on the space regularity of the state, we obtain the following expressions for the derivative of the observation operator. If $C = \tilde{C}$, then

$$\begin{aligned}\big(C'_p(p, S(p))[h], C(p, S(p)) - y^\delta\big)_Y = \int_0^T &\left(\int_{\Gamma_e} \Big(h_e\mathcal{B}u - h_\epsilon\nabla\phi_0 - h_\epsilon\nabla\chi\Big)\cdot n \,\mathrm{d}\Gamma\right) \\ &\cdot \left(\int_{\Gamma_e} \Big(e\mathcal{B}u - \epsilon\nabla\phi_0 - \epsilon\nabla\chi\Big)\cdot n\,\mathrm{d}\Gamma - y^\delta\right)\mathrm{d}t\,.\end{aligned}$$

If $C = C^\gamma$, then

$$\left(C_p'(p,S(p))[h],\, C(p,S(p)) - y^\delta\right)_Y = \int_0^T |\Gamma_e|^2\, |U_\gamma(\Gamma_e)|^{-2} \int_{U_\gamma(\Gamma_e)} \Big(h_e\mathcal{B}u - h_\epsilon\nabla\phi_0 - h_\epsilon\nabla\chi\Big)\cdot\nabla b \;\mathrm{d}\Omega \cdot \left(\int_{U_\gamma(\Gamma_e)} \Big(e\mathcal{B}u - \epsilon\nabla\phi_0 - \epsilon\nabla\chi\Big)\cdot\nabla b\;\mathrm{d}\Omega - |\Gamma_e|^{-1}|U_\gamma(\Gamma_e)|\, y^\delta\right)\mathrm{d}t\,,$$

where b solves (4.8)–(4.9). Hence, the adjoint state is essential.
We now focus on the unique existence of the adjoint state, which can be seen as revealing the influence of a cause on a target functional. Therefore, it naturally arises in the context of parameter identification problems, especially in computing gradients of the regularized target functional using Lagrange formalism. To derive the adjoint PDE system of the piezoelectric dynamical system (3.2)–(3.7), we differentiate the model operator, see Definition 4.2, with respect to the state. Similarly to (4.4), we consider an arbitrary direction $\kappa := (d,\psi) \in W$ with d satisfying the initial conditions $d(0) = \dot{d}(0) = 0$, as κ can be viewed as an infinitesimal perturbation of the solution. Then

$$\langle A_z'(p,z)\kappa,(v,w)\rangle_{W^*,W} = \int_0^T \int_\Omega \rho\ddot{d}^T v + \alpha\rho\dot{d}^T v + \Big(c^E\mathcal{B}d + \beta c^E\mathcal{B}\dot{d} + e^T\nabla\psi\Big)^T\mathcal{B}v + \left(e\mathcal{B}d - \epsilon\nabla\psi\right)^T\nabla w\;\mathrm{d}\Omega\;\mathrm{d}t\,.$$

Denoting the adjoint state by $q = (q_1,q_2) \in W$, we obtain

$$\langle A_z'(p,z)\kappa,q\rangle_{W^*,W} = \int_0^T \int_\Omega \rho\ddot{d}^T q_1 + \alpha\rho\dot{d}^T q_1 + \Big(c^E\mathcal{B}d + \beta c^E\mathcal{B}\dot{d} + e^T\nabla\psi\Big)^T\mathcal{B}q_1 + \left(e\mathcal{B}d - \epsilon\nabla\psi\right)^T\nabla q_2\;\mathrm{d}\Omega\;\mathrm{d}t\,. \tag{4.17}$$

We consider every single term individually, with the terminal conditions

$$q_1(T) = 0 \qquad \text{and} \qquad \dot{q}_1(T) = 0 \quad \text{in } \Omega.$$

Let $q_{1\rho} := \rho q_1$, i.e.

$$q_{1\rho}(T) = \rho q_1(T) = 0, \qquad \dot{q}_{1\rho}(T) = \dot{\rho}\, q_1(T) + \rho\dot{q}_1(T) = 0.$$

Then, for the first term in (4.17), we have

$$\int_0^T \int_\Omega \rho\ddot{d}^T q_1\;\mathrm{d}\Omega\;\mathrm{d}t = \int_0^T \int_\Omega \ddot{d}^T q_{1\rho}\;\mathrm{d}\Omega\;\mathrm{d}t = -\int_0^T \int_\Omega \dot{d}^T \dot{q}_{1\rho}\;\mathrm{d}\Omega\;\mathrm{d}t = \int_0^T \int_\Omega d^T \ddot{q}_{1\rho}\;\mathrm{d}\Omega\;\mathrm{d}t.$$

For the second term in (4.17), define $q_{1\alpha\rho} := \alpha\rho q_1$, i.e.

$$q_{1\alpha\rho}(T) = \alpha\rho q_1(T) = 0.$$

Then

$$\int_0^T \int_\Omega \alpha\rho\,\dot{d}^T q_1\;\mathrm{d}\Omega\;\mathrm{d}t = \int_0^T \int_\Omega \dot{d}^T q_{1\alpha\rho}\;\mathrm{d}\Omega\;\mathrm{d}t = -\int_0^T \int_\Omega d^T \dot{q}_{1\alpha\rho}\;\mathrm{d}\Omega\;\mathrm{d}t.$$

For the third term in (4.17), using the symmetry of c^E, we obtain

$$\int_0^T \int_\Omega \left(c^E\mathcal{B}d\right)^T\mathcal{B}q_1\;\mathrm{d}\Omega\;\mathrm{d}t = -\int_0^T \int_\Omega \mathcal{B}^T\left(c^E\mathcal{B}q_1\right)^T d\;\mathrm{d}\Omega\;\mathrm{d}t + \int_0^T \int_{\partial\Omega} \mathcal{N}^T\left(c^E\mathcal{B}q_1\right)^T d\;\mathrm{d}\Gamma\;\mathrm{d}t.$$

For the fourth term in (4.17), let

$$q_{\mathcal{B}} := \beta c^E\mathcal{B}q_1.$$

Then

$$\begin{aligned}\int_0^T \int_\Omega \left(\beta c^E \mathcal{B}\dot{d}\right)^T \mathcal{B}q_1 \, \mathrm{d}\Omega \, \mathrm{d}t &= \int_0^T \int_\Omega \left(\mathcal{B}\dot{d}\right)^T q_{\mathcal{B}} \, \mathrm{d}\Omega \, \mathrm{d}t \\ &= -\int_0^T \int_\Omega (\mathcal{B}d)^T \dot{q}_{\mathcal{B}} \, \mathrm{d}\Omega \, \mathrm{d}t \\ &= \int_0^T \int_\Omega d^T \mathcal{B}^T \dot{q}_{\mathcal{B}} \, \mathrm{d}\Omega \, \mathrm{d}t - \int_0^T \int_{\partial\Omega} d^T \mathcal{N}^T \dot{q}_{\mathcal{B}} \, \mathrm{d}\Gamma \, \mathrm{d}t.\end{aligned}$$

Finally, the remaining three terms in (4.17) yield

$$\begin{aligned}\int_0^T \int_\Omega \left(e^T \nabla\psi\right)^T \mathcal{B}q_1 \, \mathrm{d}\Omega \, \mathrm{d}t &= -\int_0^T \int_\Omega \nabla \cdot (e\mathcal{B}q_1)\, \psi \, \mathrm{d}\Omega \, \mathrm{d}t + \int_0^T \int_{\partial\Omega} n \cdot (e\mathcal{B}q_1)\, \psi \, \mathrm{d}\Gamma \, \mathrm{d}t, \\ \int_0^T \int_\Omega (e\mathcal{B}d)^T \nabla q_2 \, \mathrm{d}\Omega \, \mathrm{d}t &= -\int_0^T \int_\Omega d^T \mathcal{B}^T \left(e^T \nabla q_2\right) \mathrm{d}\Omega \, \mathrm{d}t + \int_0^T \int_{\partial\Omega} d^T \mathcal{N}^T \left(e^T \nabla q_2\right) \mathrm{d}\Gamma \, \mathrm{d}t, \\ \int_0^T \int_\Omega -(\epsilon\nabla\psi)^T \nabla q_2 \, \mathrm{d}\Omega \, \mathrm{d}t &= \int_0^T \int_\Omega \nabla \cdot \left(\epsilon^T \nabla q_2\right) \psi \, \mathrm{d}\Omega \, \mathrm{d}t - \int_0^T \int_{\partial\Omega} n \cdot \left(\epsilon^T \nabla q_2\right) \psi \, \mathrm{d}\Gamma \, \mathrm{d}t.\end{aligned}$$

As we deal with terminal conditions, we perform the time transformation $t \mapsto T - t$ and define $\tilde{q}(t) := q(T-t)$. Then

$$\dot{\tilde{q}}(t) = -\dot{q}(T-t), \qquad \ddot{\tilde{q}}(t) = \ddot{q}(T-t).$$

Moreover,

$$\dot{q}_{1\alpha\rho} = (\dot{\alpha}\,\rho + \alpha\dot{\rho})\, q_1 + \alpha\rho\, \dot{q}_1, \qquad \ddot{q}_{1\rho} = \ddot{\rho}\, q_1 + 2\dot{\rho}\, \dot{q}_1 + \rho\, \ddot{q}_1,$$

and

$$\dot{q}_{\mathcal{B}} = \left(\dot{\beta}\, c^E + \beta\dot{c}^E\right) \mathcal{B}q_1 + \beta c^E \mathcal{B}\dot{q}_1.$$

Furthermore, we introduce

$$\begin{aligned} r_u(p, S(p)) &:= C_u'(p, S(p))^* \big(C(p, S(p)) - y^\delta\big), \\ r_{\phi_0}(p, S(p)) &:= C_{\phi_0}'(p, S(p))^* \big(C(p, S(p)) - y^\delta\big). \end{aligned}$$

Then, we obtain the time-transformed adjoint PDE

$$\begin{aligned} &\tilde{\rho}\,\ddot{\tilde{q}}_1 + \left(2\dot{\tilde{\rho}} + \tilde{\alpha}\tilde{\rho}\right)\dot{\tilde{q}}_1 + \left(\ddot{\tilde{\rho}} + \dot{\tilde{\alpha}}\tilde{\rho} + \tilde{\alpha}\dot{\tilde{\rho}}\right)\tilde{q}_1 \\ &\quad -\mathcal{B}^T\left(\left(\left(\dot{\tilde{\beta}} + 1\right)\tilde{c}^E + \tilde{\beta}\dot{\tilde{c}}^E\right)\mathcal{B}\tilde{q}_1 + \tilde{\beta}\tilde{c}^E\mathcal{B}\dot{\tilde{q}}_1 + \tilde{e}^T\nabla\tilde{q}_2\right) = r_u(\tilde{p}, S(\tilde{p})) \quad \text{in } \Omega \times (0,T), \end{aligned} \tag{4.18}$$

$$\begin{aligned} -\nabla \cdot \left(\tilde{e}\mathcal{B}\tilde{q}_1 - \tilde{\epsilon}^T\nabla\tilde{q}_2\right) &= r_{\phi_0}(\tilde{p}, S(\tilde{p})) \quad \text{in } \Omega \times (0,T), \\ n \cdot \left(\tilde{e}\mathcal{B}\tilde{q}_1 - \tilde{\epsilon}^T\nabla\tilde{q}_2\right) &= 0 \quad \text{on } \Gamma_n \times (0,T), \\ \mathcal{N}^T\left(\left(\left(\dot{\tilde{\beta}} + 1\right)\tilde{c}^E + \tilde{\beta}\dot{\tilde{c}}^E\right)\mathcal{B}\tilde{q}_1 + \tilde{\beta}\tilde{c}^E\mathcal{B}\dot{\tilde{q}}_1 + \tilde{e}^T\nabla\tilde{q}_2\right) &= 0 \quad \text{on } \partial\Omega \times (0,T), \\ \tilde{q}_1(0) &= 0 \quad \text{in } \Omega, \end{aligned}$$

$$\dot{\tilde{q}}_1(0) = 0 \quad \text{in } \Omega. \tag{4.19}$$

Corollary 4.10 (Existence and Uniqueness of the adjoint system)**.** Let Assumption 2.2 hold, but with $\rho \in H^4(L^\infty(\Omega,\mathbb{R})),\ \alpha, \beta \in H^3(L^\infty(\Omega,\mathbb{R}))$, and let Assumption D1 hold. Suppose furthermore that the material parameters are elements of X defined in Definition 4.1, and that there exists $c_\star^{\mathrm{ad}} > 0$ such that

$$\xi^T\big((1-\dot{\beta})c^E - \beta\dot{c}^E\big)\xi \geq c_\star^{\mathrm{ad}}|\xi|^2 \qquad \forall \xi \in \mathbb{R}^6$$

for almost all $(t,x) \in (0,T) \times \Omega$. Assume, in addition, that one of the following two conditions is satisfied:

(i) $\alpha\rho - 2\dot{\rho} \geq 0$, and $\ddot{\rho} - \dot{\alpha}\,\rho - \alpha\dot{\rho} \geq 0$ a.e. in $(0,T) \times \Omega$.
(ii) ρ is constant in time and $\dot{\alpha} \leq 0$ a.e. in $(0,T) \times \Omega$.

Then there exists a unique weak solution

$$(\tilde{q}_1, \tilde{q}_2) \in W$$

with

$$\dot{\tilde{q}}_1 \in L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)) \cap L^\infty(L^2(\Omega,\mathbb{R}^3)), \qquad \ddot{\tilde{q}}_1 \in L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*)$$

to the system (4.18)–(4.19) with the more general right-hand side

$$(\tilde f, \tilde g) \in L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*) \times L^2(L^2(\Omega,\mathbb{R})).$$

Furthermore, there exists a constant $C_a > 0$ such that

$$\begin{aligned} &\|\ddot{\tilde q}_1\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*)} + \|\dot{\tilde q}_1\|^2_{L^\infty(L^2(\Omega,\mathbb{R}^3))} + \|\tilde q_1\|^2_{L^\infty(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\dot{\tilde q}_1\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} \\ &\quad + \|\tilde q_2\|^2_{L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}))} \le C_a \left(\|\tilde f\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*)} + \|\tilde g\|^2_{L^2(L^2(\Omega,\mathbb{R}))} \right). \end{aligned} \tag{4.20}$$

Proof. We rewrite the transformed adjoint system (4.18)–(4.19) in the form

$$\begin{aligned} \rho_{\mathrm{ad}}\ddot{\tilde q}_1 + \alpha_{\mathrm{ad}}\rho_{\mathrm{ad}}\dot{\tilde q}_1 + a_{\mathrm{ad}}\tilde q_1 - \mathcal{B}^T\left(C_{0,\mathrm{ad}}\mathcal{B}\tilde q_1 + C_{1,\mathrm{ad}}\mathcal{B}\dot{\tilde q}_1 + e^T_{\mathrm{ad}}\nabla\tilde q_2\right) &= \tilde f, \\ -\nabla\cdot\left(e_{\mathrm{ad}}\mathcal{B}\tilde q_1 - \epsilon_{\mathrm{ad}}\nabla\tilde q_2\right) &= \tilde g, \end{aligned}$$

where we set

$$\rho_{\mathrm{ad}} := \tilde\rho, \qquad C_{0,\mathrm{ad}} := \left(\left(\dot{\tilde\beta}+1\right)\tilde c^E + \tilde\beta\dot{\tilde c}^E\right), \qquad C_{1,\mathrm{ad}} := \tilde\beta\,\tilde c^E,$$

$$e_{\mathrm{ad}} := \tilde e, \qquad \epsilon_{\mathrm{ad}} := \tilde\epsilon, \qquad a_{\mathrm{ad}} := \ddot{\tilde\rho} + \dot{\tilde\alpha}\tilde\rho + \tilde\alpha\dot{\tilde\rho},$$

and, since $\tilde\rho$ is uniformly positive,

$$\alpha_{\mathrm{ad}} := \frac{2\dot{\tilde\rho} + \tilde\alpha\tilde\rho}{\tilde\rho}.$$

Since $\rho \in H^4(L^\infty(\Omega,\mathbb{R}))$ is positive and uniformly bounded, it follows that

$$\rho_{\mathrm{ad}} = \tilde\rho \in H^4(L^\infty(\Omega,\mathbb{R}))$$

is positive, uniformly bounded and, by Lemma 3.1, three times continuously differentiable in time. As $\tilde\rho$ is bounded away from zero, the reciprocal $1/\tilde\rho$ has the same time regularity. Hence, α_{ad} is well-defined. Moreover, $\alpha \in H^3(L^\infty(\Omega,\mathbb{R}))$ is non-negative and uniformly bounded. Thus, $\tilde\alpha \in H^3(L^\infty(\Omega,\mathbb{R}))$ is non-negative, uniformly bounded and, again by Lemma 3.1, two times continuously differentiable in time. If condition (i) holds, then

$$2\dot{\tilde\rho} + \tilde\alpha\tilde\rho = (\alpha\rho - 2\dot\rho)(T-t) \ge 0$$

and

$$a_{\mathrm{ad}} = \ddot{\tilde\rho} + \dot{\tilde\alpha}\tilde\rho + \tilde\alpha\dot{\tilde\rho} = (\ddot\rho - \dot\alpha\,\rho - \alpha\dot\rho)(T-t) \ge 0.$$

If condition (ii) holds, then ρ is constant in time, hence $\dot\rho = \ddot\rho = 0$, and therefore

$$2\dot{\tilde\rho} + \tilde\alpha\tilde\rho = \tilde\alpha\tilde\rho \ge 0.$$

Moreover,

$$a_{\mathrm{ad}} = \ddot{\tilde\rho} + \dot{\tilde\alpha}\tilde\rho + \tilde\alpha\dot{\tilde\rho} = \dot{\tilde\alpha}\tilde\rho = -\dot\alpha(T-t)\rho \ge 0.$$

Thus, in either case,

$$2\dot{\tilde\rho} + \tilde\alpha\tilde\rho \in H^3(L^\infty(\Omega,\mathbb{R}))$$

is non-negative and uniformly bounded, and $a_{\mathrm{ad}} \in H^2(L^\infty(\Omega,\mathbb{R}))$ is non-negative and uniformly bounded. Consequently,

$$\alpha_{\mathrm{ad}} = \frac{2\dot{\tilde\rho} + \tilde\alpha\tilde\rho}{\tilde\rho} \in H^3(L^\infty(\Omega,\mathbb{R}))$$

is non-negative and uniformly bounded. Next, by Definition 4.1, time reversal, and Lemma 3.1, the transformed coefficients $\tilde c^E, \tilde e, \tilde\epsilon$ have the same regularity as the original ones. In particular,

$$C_{0,\mathrm{ad}} = \left(\left(\dot{\tilde\beta}+1\right)\tilde c^E + \tilde\beta\dot{\tilde c}^E\right) = \left((1-\dot\beta)c^E - \beta\dot c^E\right)(T-t),$$

and therefore $C_{0,\mathrm{ad}}$ is uniformly positive definite by the standing assumption of the corollary. Furthermore, $C_{1,\mathrm{ad}} = \tilde\beta\,\tilde c^E$ is uniformly positive definite, since $\tilde\beta \ge \beta^\star > 0$ by Assumption A1 and $\tilde c^E$ is uniformly positive definite. Likewise, $e_{\mathrm{ad}} = \tilde e$ and $\epsilon_{\mathrm{ad}} = \tilde\epsilon$ satisfy the same structural assumptions as in Theorem 3.2, and ϵ_{ad} is uniformly positive definite.

Therefore, all coefficients of the transformed adjoint system possess the same regularity and coercivity properties as those used in the proof of Theorem 3.2. Moreover, the initial conditions are homogeneous, the right-hand side $(\tilde f, \tilde g)$ has the same regularity as (f, g) in Theorem 3.2, and the Dirichlet boundary is homogeneous, i.e. $\chi \equiv 0$. Hence, the proof of Theorem 3.2 carries over to the present system, with $C_{0,\mathrm{ad}}$

and $C_{1,\mathrm{ad}}$ replacing the coefficients c^E and βc^E, respectively. This yields the existence and uniqueness of a weak solution

$$(\tilde{q}_1, \tilde{q}_2) \in W$$

with the asserted regularity. Repeating the proof of Theorem 3.3 with the same substitutions then gives the stated estimate. □

Our adjoint system (4.18)–(4.19) has a right-hand side with the same temporal regularity as in Corollary 4.10. More precisely, for every admissible parameter $p \in X$, one has

$$\tilde{f} = \frac{\partial J}{\partial u}(p, S(p)), \qquad \tilde{g} = \frac{\partial J}{\partial \phi_0}(p, S(p)).$$

Hence, under the assumptions stated above, Corollary 4.10 applies and yields a unique adjoint state for every given state $S(p)$. In this sense, the adjoint variable is not merely a Lagrange multiplier associated with a minimizer, but a uniquely determined auxiliary state corresponding to each admissible state of the model.

At a local minimizer p^*, an additional abstract operator-theoretic interpretation can be made. Since

$$A'_z(p^*, S(p^*)) : W \to W^*$$

is an isomorphism, its adjoint

$$A'_z(p^*, S(p^*))^* : W \to W^*$$

is an isomorphism as well. Consequently, there exists a unique adjoint state $q = (q_1, q_2) \in W$ satisfying

$$A'_z(p^*, S(p^*))^* q = -\,C'_z(p^*, S(p^*))^* \big(C(p^*, S(p^*)) - y^\delta\big). \tag{4.21}$$

Thus, at a minimizer, the adjoint state is characterized both analytically, as the unique solution of the adjoint PDE, and variationally, as the unique multiplier in the reduced first-order optimality system.

5. A NUMERICAL EXAMPLE

To computationally solve the inverse problem, we follow the discretize-then-optimize approach where we discretize the problem setting, i.e., the forward operator and the corresponding spaces first and optimize afterwards. Here, easy access to the first derivative of the forward operator F is provided by algorithmic differentiation (AD), see [9]. The central concept is that the computation of a discretized operator can be decomposed into a finite sequence of elementary operations, where then the chain rule is applied systematically. The reverse mode of AD can be seen as a discrete analogue of the continuous adjoint PDE enabling an efficient gradient calculation. The analysis of the continuous adjoint system ensures that, with appropriate discretization, the discrete adjoint state converges to the continuous adjoint state as the discretization gets finer. Hence, the analysis of the continuous problem (4.18) - (4.19) is, an important prerequisite for the discretize-then-optimize approach. For the space discretization we use a classic finite element method (FEM) implemented by the finite element tool FEniCS [2] in dolfin version 2019.2.0.dev0, using AD via the dolfin adjoint [22] library of FEniCS in version 2019.1.0. For the temporal discretization, we employ the Crank–Nicolson scheme. Therefore, we set $z = \dot{u}$ and $\dot{z} = \ddot{u}$ and rewrite the weak form of system (3.17)-(3.18) for all $v \in H^1_{\mathcal{B}}(\Omega), w \in H^1_{0,\Gamma_g}(\Omega), y \in H^1_{\mathcal{B}}(\Omega)$ as

$$\begin{aligned} &\langle \rho \dot{z}, v\rangle_{L^2(\Omega)} + \alpha\langle \rho z, v\rangle_{L^2(\Omega)} + \langle c^E \mathcal{B}u, \mathcal{B}v\rangle_{L^2(\Omega)} + \langle \beta c^E \mathcal{B}z, \mathcal{B}v\rangle_{L^2(\Omega)} \\ &\quad + \langle e^T \nabla\phi, \mathcal{B}v\rangle_{L^2(\Omega)} + \langle e\mathcal{B}u, \nabla w\rangle_{L^2(\Omega)} - \langle \epsilon\nabla\phi, \nabla w\rangle_{L^2(\Omega)} = 0 \end{aligned} \tag{5.1}$$

$$\langle \dot{u}, y\rangle_{L^2(\Omega)} - \langle z, y\rangle_{L^2(\Omega)} = 0. \tag{5.2}$$

Then we obtain the Crank-Nicolson time discretized system,

$$\begin{aligned} &2\langle \rho z_{n+1}, v\rangle_{L^2(\Omega)} - 2\langle \rho z_n, v\rangle_{L^2(\Omega)} + \Delta t\alpha\langle \rho z_n, v\rangle_{L^2(\Omega)} + \Delta t\alpha\langle \rho z_{n+1}, v\rangle_{L^2(\Omega)} \\ &+ \Delta t\langle c^E \mathcal{B}u_n, \mathcal{B}v\rangle_{L^2(\Omega)} + \Delta t\langle c^E \mathcal{B}u_{n+1}, \mathcal{B}v\rangle_{L^2(\Omega)} + \Delta t\beta\langle c^E \mathcal{B}z_n, \mathcal{B}v\rangle_{L^2(\Omega)} \\ &+ \Delta t\beta\langle c^E \mathcal{B}z_{n+1}, \mathcal{B}v\rangle_{L^2(\Omega)} + \Delta t\langle e^T\nabla\phi_n, \mathcal{B}v\rangle_{L^2(\Omega)} + \Delta t\langle e^T\nabla\phi_{n+1}, \mathcal{B}v\rangle_{L^2(\Omega)} \\ &+ \Delta t\langle e\mathcal{B}u_n, \nabla w\rangle_{L^2(\Omega)} + \Delta t\langle e\mathcal{B}u_{n+1}, \nabla w\rangle_{L^2(\Omega)} - \Delta t\langle \epsilon\nabla\phi_n, \nabla w\rangle_{L^2(\Omega)} - \Delta t\langle \epsilon\nabla\phi_{n+1}, \nabla w\rangle_{L^2(\Omega)} = 0 \\ &2\langle u_{n+1}, y\rangle_{L^2(\Omega)} - 2\langle u_n, y\rangle_{L^2(\Omega)} - \Delta t\langle z_{n+1}, y\rangle_{L^2(\Omega)} - \Delta t\langle z_n, y\rangle_{L^2(\Omega)} = 0, \end{aligned}$$

where Δt is the time step size, which will be chosen in the numerical realization as 10^{-6} and n is the current step up to $N = 1000$.
As geometry we consider a piezoelectric ring, with outer radius of 6.35 mm, inner radius of 2.6 mm and thickness of 1mm. Hence, the geometry is rotationally symmetric. To reduce the computational effort we exploit the inherent rotational symmetry and transform the ring into a rectangular domain by adopting cylindrical coordinates rather than Cartesian coordinates, where the z-axis is selected as the axis of rotation. In this coordinate system, the piezoelectric ring is assumed to be a homogeneous and transversely isotropic material. The latter is physically essential to exploit the rotational symmetry. In addition, we have converted the setting from seconds to milliseconds, which results in a better condition number of the PDE system, as the magnitudes of the material parameters differ significantly less.
As an example for the numerical realization of the inverse problem, we assume that the elasticity parameter and the permittivity parameter are constant, i.e., parameterized in a polynomial way as in identity (2.1), with a polynomial order 0 and the coupling parameter e is parameterized as in identity (2.1), with an polynomial order 1, where $\theta(t) := 25 + 7\sqrt{(0.01t)}$. Since the problem of identifying the material parameters is extremely challenging due to very different orders of sensitivities even in the frequency dependent case, see e.g., [12], [14], [16], [24], we want to reconstruct one entry of the coupling parameter e, namely e_{33}. To simulate the data we started with the following set of material parameters

$$\begin{gathered} c_{11}^E = 151400, \quad c_{12}^E = 132700, \quad c_{13}^E = 83600, \quad c_{33}^E = 128800, \quad c_{44}^E = 25900, \quad \epsilon_{11} = 2700 \\ \epsilon_{33} = 5500, \quad e_{15} = 11\theta(t) + 9125, \quad e_{31} = -7\theta(t) - 5025, \quad e_{33} = 24\theta(t) + 13300. \end{gathered} \tag{5.3}$$

The constant parameters are chosen according to material parameters and damping parameters presented in [8]. The polynomial parameters of the entries of the piezoelectric coupling parameter are chosen such that they equal the entries of the piezoelectric coupling parameter in [8] at $25C^\circ$. To generate the noisy data y^δ we contaminate the exact simulated data y, generated with the parameters defined above, additively with uniformly distributed random noise with a noise level of 1%. The excitation signal $\phi_e(t_n)$, applied via the Dirichlet boundary condition at the top surface, is defined as a discrete triangular pulse, i.e.,

$$\phi_e(t_n) = 10^{-9} \cdot \begin{cases} n & \text{for } 1 \leq n \leq 10 \\ 20 - n & \text{for } 11 \leq n \leq 19 \\ 0 & \text{for } n \geq 20 \end{cases}. \tag{5.4}$$

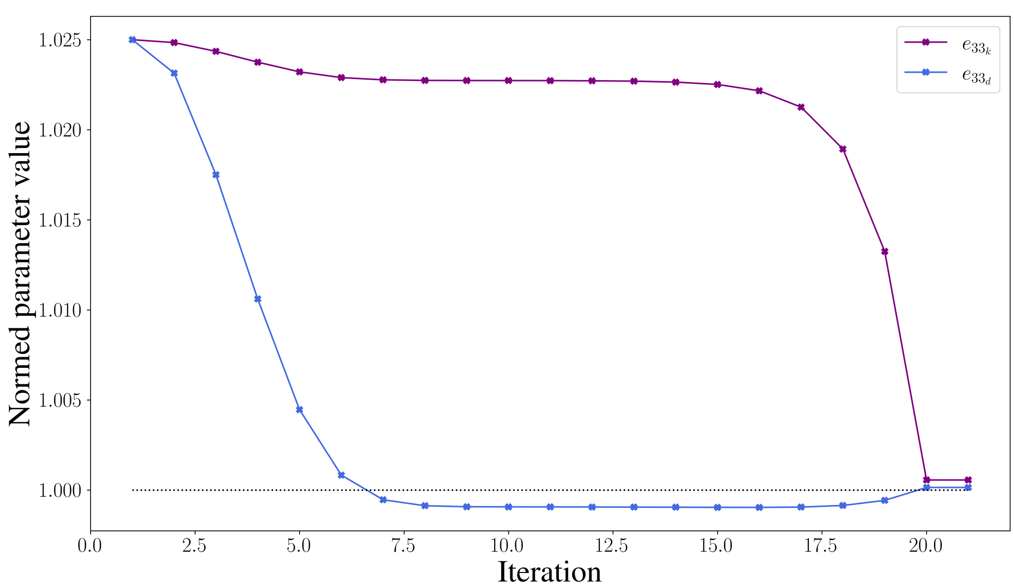

FIGURE 1. Identification of e_{33_k} and e_{33_d}.

Note that specifying the Dirichlet lift function χ used for stating the system (3.17)-(3.18) is not necessary, as it is possible to directly implement mixed Dirichlet conditions in FEniCS. As optimization method we use the GRSE method, see [15], which employs a Tikhonov-type regularization with 0.5 as decay factor and 4 as growth factor of the regularization parameters. We started with the initial regularization parameter $\tau_0 = 10^{-6}$. As initial Quasi-Newton matrix we used the scaled identity with scale 10^{-6}. The initial guesses for polynomial parameters of the piezoelectric coupling parameter $e_{33}(\theta(t))$ are chosen with a 2.5% deviation to the ground truth in (5.3). Furthermore, we scaled the first order polynomial parameter with 5 and the zero order polynomial parameter with 10^{-2}, to reach similar orders of magnitude. The numerical results for the identification of the polynomial parameters e_{33_k} and e_{33_d}, which are the first and zeroth order polynomial parameters of the piezoelectric coupling parameter $e_{33}(\theta(t))$ showed convergence to the exact parameter, as illustrated in Figure 1.

6. Conclusion

We modeled and analyzed an inverse problem governed by a piezoelectric system represented by a coupled hyperbolic-elliptic PDE with matrix-valued Sobolev-Bochner functions as parameters and Sobolev-Bochner density and damping functions. We extended the PDE with an additional term based on a Sobolev-Bochner function and the mechanical deformation as well as inhomogeneities, ensuring the applicability of our generalized existence and uniqueness theorem on the associated adjoint PDE. In addition, an a priori energy estimate and conditions for arbitrary Sobolev regularity in space were established to ensure the well-definedness of the observation operator of the inverse parameter identification problem. Then, we proved the Fréchet differentiability of the observation operator and discussed the treatment of the observation operator given that PDE solutions have lower regularity. With respect to the modeling and regularity of the forward operator of the parameter identification problem in the reduced approach, we considered the well-definedness, existence and regularity of the parameter-state map. Furthermore, we modeled the inverse problem as an optimization problem in which a target functional consisting of the forward operator, the given data and a regularizer is minimized. To provide a framework for the computation of solutions to the inverse problem, we showed that there exists a minimizer and derived first-order optimality conditions. This motivated the derivation of the adjoint PDE, where we used our existence and uniqueness results to analyze the adjoint PDE, demonstrating its utility. Finally, a numerical example was given, where the proposed parameterization approach for describing the material parameters was used.

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