MODELLING AND ANALYSIS OF AN INVERSE PARAMETER IDENTIFICATION PROBLEM IN PIEZOELECTRICITY

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ABSTRACT. Piezoelectric material behavior is mathematically described by coupled hyperbolic-elliptic partial differential equations (PDEs) governing mechanical displacement and electrical potential. This paper introduces advancements within the theoretical framework of identifying material parameters in piezoelectric PDEs. We focus on modeling and analyzing the inverse problem, where matrix-valued Sobolev-Bochner parameters encompass external physical influences. In this setting results concerning the existence, uniqueness, and increased regularity of solutions to the piezoelectric PDE are established. Based on these findings, well-definedness and regularity of the parameter-to-state map and Fréchet differentiability of the observation operator are proven. Finally, the inverse problem is formulated using a minimization approach, where weak lower semi-continuity of the objective functional, first-order optimality conditions and the derivation and analysis of the adjoint PDE are presented.

Keywords: Existence, uniqueness and regularity; operator analysis; inverse parameter identification; piezoelectricity; adjoint PDEs
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1. INTRODUCTION

Piezoelectric materials are extensively utilized in numerous electrical devices nowadays, being prevalent not only in households but also in industrial and medical applications. Their versatility extends across a diverse range of products, including microphones and headphones as well as ultrasound imaging devices and power generation systems. The underlying piezoelectric effect, which is the fundamental property of these applications, describes a coupling phenomenon between electrical and mechanical fields, where mechanical pressure generates an electric potential and vice versa. Thoroughly understanding the behavior of these materials is essential, especially given their time and space dependent characteristics. Their temporal and spatial dependence can occur implicitly via the influence of external physical quantities such as temperature, which appears as temporally and spatially varying functions, on the material parameters. Simplistically, the piezoelectric material is described by the system of coupled PDEs,

$$\begin{aligned} \rho u_{tt} + \alpha \rho u_t - \mathcal{B}^T \left(c^E \mathcal{B} u + \beta c^E \mathcal{B} u_t + e^T \nabla \phi \right) &= 0, \\ -\nabla \cdot \left(e \mathcal{B} u - \epsilon \nabla \phi \right) &= 0, \end{aligned} \tag{1.1}$$

for the mechanical displacement u and the electrical potential ϕ with appropriate boundary conditions, where $\mathcal{B}$ is the differential operator corresponding to the symmetric gradient and α, β are damping functions, see Section 2. The material behavior depends significantly on the material parameters c^E, e, and ϵ, which vary substantially for different temperatures, see [5, 9, 11, 30]. Hence, the inverse problem aims at identifying material parameters from observations of the total charge, see Section 4. As the observed data is usually contaminated with noise, we have given noisy measurements y^δ. Employing the reduced method, abbreviating the parameters with p, and using an optimization approach results in solving

$$\min_{p \in X} \quad \frac{1}{2} \left\| F(p) - y^\delta \right\|^2 + \mathcal{R}_\alpha(p), \tag{1.2}$$

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where $F : X \to Y$ is the forward operator. If the preimage space X is a Sobolev space of higher order, implementing regularization methods on this preimage space becomes particularly challenging and, in many cases, impractical. To address this issue, it is beneficial to leverage the physical behavior of material parameters by parameterizing them in terms of a relevant physical quantity. We consider, for example, the dependence of the parameters on a known temperature function θ, which is a function of space and time. It is reasonable to assume a polynomial or Hadamard exponential structure of the material parameters with respect to the temperature function, see [14]. The coefficients of these polynomials or exponential functions are constant parameters. This means that we can reconstruct these parameters by reconstructing constant matrices of appropriate size for each parameter, which transfers an optimization problem in higher order Sobolev spaces to an optimization problem in a real-valued finite dimensional vector space. As this surrogate modeling approach is a transformation into higher order Sobolev spaces, the analysis of the individual components of the inverse problem must be conducted in the framework of an infinite-dimensional function space, including the analysis of the underlying PDE and the associated adjoint PDE, in order to preserve generality.

Related Work. Existence, uniqueness and regularity of solutions of the piezoelectric dynamical system have been studied in [1], [3], [4], [15], [17], [20], [23], [22], [24], [25] and [27], among others and the references therein. For example, in [15] and [24] existence and uniqueness of solutions of the undamped inhomogeneous piezoelectric PDE is discussed, where the material parameters are constant. In [1] the material parameters are spatially dependent $L^\infty(\Omega)$ functions and inhomogeneities are included. An optimal control problem for the electrical flux (boundary control problem) is studied in [3], where an existence and uniqueness result is given for solutions of the undamped piezoelectric PDE and the corresponding adjoint differential PDE with spatially dependent $L^\infty(\Omega)$ material parameters. The papers [27] and [25] consider a boundary control problem, where existence and uniqueness results for solutions of the undamped homogeneous piezoelectric PDE are discussed, where [25] deals with constant material parameters and [27] focuses on an elasticity parameter comprised of spatially dependent $C^2(\Omega)$ functions, a permittivity parameter comprised of spatially dependent $L^\infty(\Omega)$ functions and a constant piezoelectric coupling parameter. In [4], an existence and uniqueness result is given for solutions of the undamped piezoelectric PDE coupled to a parabolic temperature equation and the magnetic field in the form of an elliptic equation, similar to the electrical equation of the classical piezoelectric system, where the parameters have $C^{0,1}(\Omega)$ regularity or $L^\infty(\Omega)$ regularity in the space. In [22], respectively in [23], a shape optimization problem is studied, where an existence and uniqueness result for solutions of the undamped inhomogeneous piezoelectric PDE is given for time/space-constant parameters and the corresponding adjoint PDE is presented. Furthermore, [17] gives a result heavily based on [10], [20], [1], and [24], on existence and uniqueness of solutions for the Rayleigh damped homogeneous piezoelectric PDE, where the material parameters are constant.

Contribution. The aim of this paper is to model and analyze the inverse problem. We demonstrate existence, well-definedness and continuous Fréchet differentiability of the parameter-to-state map. To accomplish this, we have to extend and generalize previous existence and uniqueness results for piezoelectric PDEs, by considering matrix valued Sobolev-Bochner functions as material parameters and also Sobolev-Bochner density- and damping functions. Additionally, we have added a further damping term to the Rayleigh-damped piezoelectric system and included additional Sobolev-Bochner inhomogeneities, which allows the application of our contributed existence and uniqueness theorem not only to the state equation but also to the adjoint PDE. Subsequently, we define the observation operator, demonstrating that its well-definedness requires higher regularity of the state. For this an a-priori energy estimate was established, which has not yet been treated in this general setting. Consequently, we provide a rigorous Dirichlet lift Ansatz and present a result that provides arbitrary Sobolev regularity in space of solutions to the piezoelectric PDE for sufficiently regular boundary data, material parameters and right-hand sides. In addition, we demonstrate Frechet differentiability of the observation operator, leading to the definition of the forward operator, which inherits the properties of both the observation operator and the parameter-to-state map and prove its weak-to-strong continuity. We model the inverse problem as a minimization problem of an objective functional and prove the existence of a minimizer. Furthermore, we formulate the necessary first-order optimality conditions. Motivated by this, we derive the adjoint PDE and analyze it with respect to the existence and uniqueness of solutions by employing the main existence and uniqueness result of this article. Additionally we give insights in the structure of the derivative of the objective functional.

Structure of the paper. The structure of this paper is as follows: The second section addresses the modeling of the underlying PDE to the inverse problem, as well as notations and the introduction of necessary definitions and lemmata essential for the paper's objectives. The third section proves existence, uniqueness and regularity of weak solutions for the generalized damped piezoelectric PDEs. Furthermore, results on an a-priori energy estimate and arbitrary Sobolev regularity in space for sufficient regular boundary data and right-hand sides are proven. The fourth section models and analyses the forward operator, including the well-definedness and Fréchet differentiability of the observation operator and the existence and regularity of the parameter to state map. Then, the inverse problem is formulated as minimization problem and first-order optimality conditions are derived. The derivation and analysis of the adjoint PDE concludes this section. The fifth section discusses a numerical example and the final section briefly summarizes the contributions of this paper.

2. Modeling

Let $T > 0$ be the end time of the observed time period $(0, T)$ and denote the geometry of the considered piezoceramic, i.e., the domain, with $\Omega \subset \mathbb{R}^3$. For the latter we assume that its boundary can be represented as the disjoint union $\partial\Omega := \Gamma_e \dot{\cup} \Gamma_0 \dot{\cup} \Gamma_n$. Thereby Γ_e describes the boundary segment which is excited electrically with a known excitation signal ϕ^e and Γ_0 refers to the boundary segment which is grounded. This setting can be modeled in the system of PDEs as mixed Dirichlet boundary conditions. The boundary segment Γ_n is included in the PDE by Neumann boundary conditions. We denote the non-empty mixed Dirichlet boundary with $\Gamma_d := \Gamma_e \dot{\cup} \Gamma_0$, i.e., $\partial\Omega := \Gamma_d \dot{\cup} \Gamma_n$ and we assume that every boundary part has a positive two-dimensional Hausdorff measure. For the regularity of Ω and ϕ^e we will employ the following assumptions:

Assumptions 2.1.

D1 Ω is a bounded Lipschitz domain and $\phi^e \in H^1\left(0, T; H^{\frac{1}{2}}(\partial\Omega, \mathbb{R})\right)$.

D2 Ω is a bounded $C^{m,1}$-domain and $\phi^e \in H^1\left(0, T; H^{m+\frac{3}{2}}(\partial\Omega, \mathbb{R})\right)$, for some $m \geq 2,\ m \in \mathbb{N}$.

D3 Ω is a bounded Lipschitz domain and $\phi^e \in H^1(0, T)$ is spatially constant on Γ_e.

Note that Assumption D3 is covered by Assumption D1. Since $\phi^e \in H^1(0, T)$ is the most relevant excitation behavior in praxis (fully conductive electrodes, see [28]), and higher order Sobolev regularity requires either Assumption D2 or Assumption D3, see Section 3), the latter is kept separately.
First and second order time derivatives of an arbitrary function f are f are denoted by $\dot{f}$ and $\ddot{f}$, respectively. Furthermore, we denote the normal vector $n = (n_x, n_y, n_z)$ as the outward unit normal vector on $\partial\Omega$. Therefore we can write for the corresponding normal derivative $\partial_n = n \cdot \nabla$. Next, $\mathcal{B}f$ denotes the symmetric gradient in Voigt notation. The matrix $\mathcal{N}$ denotes the normal matrix associated with $\mathcal{B}$, obtained by replacing the partial derivatives in $\mathcal{B}$ by the components of the normal vector n, i.e.,

$$\mathcal{B} = \begin{pmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{pmatrix}, \qquad \mathcal{N} = \begin{pmatrix} n_x & 0 & 0 \\ 0 & n_y & 0 \\ 0 & 0 & n_z \\ 0 & n_z & n_y \\ n_z & 0 & n_x \\ n_y & n_x & 0 \end{pmatrix}. \tag{2.1}$$

We denote the standard Sobolev space $H^1(\Omega, \mathbb{R})$ associated with ∇, and

$$H^1_{\mathcal{B}}(\Omega, \mathbb{R}^3) := \left\{ f \in L^2(\Omega, \mathbb{R}^3) \, : \, \mathcal{B}f \in L^2(\Omega, \mathbb{R}^6) \right\}$$

equipped with the norm

$$\|f\|_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} := \left(\|f\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|\mathcal{B}f\|^2_{L^2(\Omega,\mathbb{R}^6)} \right)^{1/2}.$$

as H^1-Sobolev space associated with the spatial differential operator $\mathcal{B}$. Since Ω is at least Lipschitz, due to Assumption 2.1, Korn's inequality implies that $H^1_{\mathcal{B}}(\Omega, \mathbb{R}^3)$ and $H^1(\Omega, \mathbb{R}^3)$ coincide as sets with equivalent norms. The H^1-Sobolev space whose functions vanish only on Γ_d is defined by

$$H^1_{0,\Gamma_d}(\Omega,\mathbb{R}) = \left\{ f \in H^1(\Omega,\mathbb{R}) |\ f|_{\Gamma_d} = 0 \right\}.$$

Furthermore, we denote the dual of a Hilbert space H with H^*. Then, the three-dimensional mechanical displacement $u(t,x)$ and the one-dimensional electrical potential $\phi(t,x)$ of a piezoceramic specimen can be described by the following piezoelectric dynamical system

$$\begin{aligned}
\rho\ddot{u} + \alpha\rho\dot{u} - \mathcal{B}^T\left(c^E\mathcal{B}u + \beta c^E \mathcal{B}\dot{u} + e^T\nabla\phi\right) &= 0 \quad \text{in } \Omega \times (0,T),\\
-\nabla\cdot(e\mathcal{B}u - \epsilon\nabla\phi) &= 0 \quad \text{in } \Omega \times (0,T),\\
\phi &= 0 \quad \text{on } \Gamma_0 \times (0,T),\\
\phi &= \phi^e \quad \text{on } \Gamma_e \times (0,T),\\
n\cdot(e\mathcal{B}u - \epsilon\nabla\phi) &= 0 \quad \text{on } \Gamma_n \times (0,T),\\
\mathcal{N}^T\left(c^E\mathcal{B}u + \beta c^E \mathcal{B}\dot{u} + e^T\nabla\phi\right) &= 0 \quad \text{on } \partial\Omega \times (0,T),\\
u(t=0) &= u_0 \quad \text{in } \Omega,\\
\dot{u}(t=0) &= u_1 \quad \text{in } \Omega.
\end{aligned}$$

where ρ is the mass density, c^E, e and ϵ are the material parameters describing the given piezoceramic and α,β are damping parameters. This system is linear in the state (u,ϕ) and the material parameters, but non-autonomous through the temperature-dependency of the latter, where non-linearity might occur. Since we consider the same time interval throughout the paper, we skip it when referring to Bochner spaces.

Assumptions 2.2.

A1 The Rayleigh damping parameters $\alpha \in H^1\big(L^\infty(\Omega,\mathbb{R})\big),\ \beta \in H^1\big(L^\infty(\Omega,\mathbb{R})\big)$ are non-negative and uniformly bounded, and there exists $\beta^\star > 0$ such that

$$\beta(t,x) \geq \beta^\star \qquad \text{for a.e. } (t,x) \in (0,T)\times\Omega.$$

A2 The mass density $\rho \in H^2\big(L^\infty(\Omega,\mathbb{R})\big)$ is uniformly bounded, and there exists $\rho^\star > 0$ such that

$$\rho(t,x) \geq \rho^\star \qquad \text{for a.e. } (t,x) \in (0,T)\times\Omega.$$

A3 The elasticity parameter is of the form

$$c^E(t,x) := \begin{pmatrix}
c^E_{11}(t,x) & c^E_{12}(t,x) & c^E_{13}(t,x) & 0 & 0 & 0\\
c^E_{12}(t,x) & c^E_{11}(t,x) & c^E_{13}(t,x) & 0 & 0 & 0\\
c^E_{13}(t,x) & c^E_{13}(t,x) & c^E_{33}(t,x) & 0 & 0 & 0\\
0 & 0 & 0 & c^E_{44}(t,x) & 0 & 0\\
0 & 0 & 0 & 0 & c^E_{44}(t,x) & 0\\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}\big(c^E_{11}(t,x) - c^E_{12}(t,x)\big)
\end{pmatrix},$$

where $c^E_{11}, c^E_{12}, c^E_{13}, c^E_{33}, c^E_{44} \in H^2\big(L^\infty(\Omega,\mathbb{R})\big)$, i.e., $c^E \in H^2\big(L^\infty(\Omega,\mathbb{R}^{6\times 6})\big)$, and is uniformly positive definite, meaning that there exists $c_\star > 0$ such that

$$\xi^T c^E(t,x)\xi \geq c_\star|\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^6$$

and for a.e. $(t,x) \in (0,T)\times\Omega$.

A4 The piezoelectric coupling parameter is of the form

$$e(t,x) := \begin{pmatrix}
0 & 0 & 0 & 0 & e_{15}(t,x) & 0\\
0 & 0 & 0 & e_{15}(t,x) & 0 & 0\\
e_{31}(t,x) & e_{31}(t,x) & e_{33}(t,x) & 0 & 0 & 0
\end{pmatrix},$$

where $e_{15}, e_{31}, e_{33} \in H^2\big(L^\infty(\Omega,\mathbb{R})\big)$, i.e., $e \in H^2\big(L^\infty(\Omega,\mathbb{R}^{3\times 6})\big)$.

A5 The permittivity parameter is of the form

$$\epsilon(t,x) := \begin{pmatrix}
\epsilon_{11}(t,x) & 0 & 0\\
0 & \epsilon_{11}(t,x) & 0\\
0 & 0 & \epsilon_{33}(t,x)
\end{pmatrix},$$

where $\epsilon_{11}, \epsilon_{33} \in H^2\big(L^\infty(\Omega,\mathbb{R})\big)$, i.e., $\epsilon \in H^2\big(L^\infty(\Omega,\mathbb{R}^{3\times 3})\big)$, and is uniformly positive definite, meaning that there exists $\epsilon_\star > 0$ such that

$$\eta^T\epsilon(t,x)\eta \geq \epsilon_\star|\eta|^2 \qquad \text{for all } \eta\in\mathbb{R}^3$$

and for a.e. $(t,x)\in(0,T)\times\Omega$.

In particular, Assumptions A3 and A5 imply that $c^E(t,x)$ and $\epsilon(t,x)$ are invertible for almost all $(t,x)\in(0,T)\times\Omega$, and that their inverses are uniformly bounded. If the parameters depend on e.g., a known temperature function θ, we have that $(c^E(x,t), e(x,t), \epsilon(x,t)) = p(x,t) = \tilde{p}(\theta(x,t))$ and the following parameterization can be proposed

$$(c^E(x,t), e(x,t), \epsilon(x,t)) = \left(\sum_{j=0}^n a_j\theta(x,t)^j,\ \sum_{j=0}^n b_j\theta(x,t)^j,\ \sum_{j=0}^n k_j\theta(x,t)^j\right), \tag{2.2}$$
$$(a_j,b_j,k_j)\in\mathbb{R}^{6\times 6}\times\mathbb{R}^{3\times 6}\times\mathbb{R}^{3\times 3}, \qquad 0\leq j\leq n,$$

see [11]. Alternatively, parametrizations based on Hadamard exponentials, [14], can be considered. The coefficient matrices in these parameterizations are to be chosen such that the structural symmetries and the uniform positive definiteness of c^E and ϵ are preserved for all relevant temperature values.

3. Existence, Uniqueness and Regularity

To discuss weak solvability, the mixed Dirichlet boundary conditions are homogenized again by the Dirichlet lift. Since $\Gamma_e\cap\Gamma_0=\emptyset$ and Ω is Lipschitz, the trace operator $\mathrm{Tr}: H^1(\Omega,\mathbb{R})\to H^{1/2}(\Gamma_d)$ is continuous and surjective and admits a continuous right inverse. Hence there exists $\chi\in H^1\big(H^1(\Omega,\mathbb{R})\big)$ such that

$$\mathrm{Tr}(\chi(t)) = \begin{cases}\phi^e(t) & \text{on } \Gamma_e,\\ 0 & \text{on } \Gamma_0\end{cases} \qquad \text{for a.e. in time.} \tag{3.1}$$

Hence the reformulation $\phi(t)=\phi_0(t)+\chi(t)$ a.e. in time, where $\phi_0(t)\in H^1_{0,\Gamma_d}(\Omega,\mathbb{R})$, can be made. Substituting this representation into the piezoelectric dynamical system above leads to

$$\rho\ddot{u}+\alpha\rho\dot{u}-\mathcal{B}^T\left(c^E\mathcal{B}u+\beta c^E\mathcal{B}\dot{u}+e^T\nabla\phi_0\right) = \mathcal{B}^Te^T\nabla\chi \qquad \text{in } \Omega\times(0,T) \tag{3.2}$$
$$-\nabla\cdot(e\mathcal{B}u-\epsilon\nabla\phi_0) = -\nabla\cdot\epsilon\nabla\chi \qquad \text{in } \Omega\times(0,T) \tag{3.3}$$
$$n\cdot(e\mathcal{B}u-\epsilon\nabla\phi_0) = n\cdot\epsilon\nabla\chi \qquad \text{on } \Gamma_n\times(0,T) \tag{3.4}$$
$$\mathcal{N}^T\left(c^E\mathcal{B}u+\beta c^E\mathcal{B}\dot{u}+e^T\nabla\phi_0\right) = -\mathcal{N}^Te^T\nabla\chi \qquad \text{on } \partial\Omega\times(0,T) \tag{3.5}$$
$$u(t=0)=u_0 \qquad \text{in } \Omega \tag{3.6}$$
$$\dot{u}(t=0)=u_1 \qquad \text{on } \Omega. \tag{3.7}$$

We now derive the weak formulation of the system above. For this purpose, we consider the weak form of (3.2) and (3.3) separately and include the corresponding boundary conditions (3.5) and (3.4). Note that by simply adding both forms, we obtain the weak form of the whole system, since we have to use different test functions for (3.2) and (3.3). We start deriving the weak form of (3.2) for almost all $t\in(0,T)$ by testing the system (3.2)-(3.7) with $(v,0)$, where $v\in H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)$, i.e.,

$$\int_\Omega \rho\ddot{u}^Tv+\alpha\rho\dot{u}^Tv+\left(c^E\mathcal{B}u+\beta c^E\mathcal{B}\dot{u}+e^T\nabla\phi_0\right)^T\mathcal{B}v\,\mathrm{d}\Omega$$
$$= -\int_\Omega (e^T\nabla\chi)^T\mathcal{B}v\,\mathrm{d}\Omega + \int_{\partial\Omega}\mathcal{N}^T(e^T\nabla\chi)^Tv\,\mathrm{d}\Gamma + \int_{\partial\Omega}\underbrace{\mathcal{N}^T\left(c^E\mathcal{B}u+\beta c^E\mathcal{B}\dot{u}+e^T\nabla\phi_0\right)^T}_{\underset{(3.5)}{=}-\mathcal{N}^T(e^T\nabla\chi)^T}v\,\mathrm{d}\Gamma$$

$$\Leftrightarrow \int_\Omega \rho\ddot{u}^Tv+\alpha\rho\dot{u}^Tv+\left(c^E\mathcal{B}u+\beta c^E\mathcal{B}\dot{u}+e^T\nabla\phi_0\right)^T\mathcal{B}v\,\mathrm{d}\Omega = -\int_\Omega (e^T\nabla\chi)^T\mathcal{B}v\,\mathrm{d}\Omega\,. \tag{3.8}$$

To derive the weak form of (3.3) for almost all $t \in (0,T)$, we test the system (3.2)-(3.7) with $(0,w)$, where $w \in H^1_{0,\Gamma_d}(\Omega,\mathbb{R})$, i.e.,

$$\int_\Omega (e\mathcal{B}u - \epsilon\nabla\phi_0)^T \nabla w \,\mathrm{d}\Omega - \int_{\Gamma_n} \underbrace{n\cdot(e\mathcal{B}u - \epsilon\nabla\phi_0)}_{\underset{(3.4)}{=} n\cdot\epsilon\nabla\chi} w \,\mathrm{d}\Gamma - \underbrace{\int_{\Gamma_d} n\cdot(e\mathcal{B}u - \epsilon\nabla\phi_0)\, w \,\mathrm{d}\Gamma}_{\underset{w\in H^1_{0,\Gamma_d}(\Omega,\mathbb{R})}{=} 0}$$

$$= \int_\Omega (\epsilon\nabla\chi)^T \nabla w \,\mathrm{d}\Omega - \int_{\Gamma_n} n\cdot(\epsilon\nabla\chi) w \,\mathrm{d}\Gamma - \underbrace{\int_{\Gamma_d} n\cdot(\epsilon\nabla\chi) w \,\mathrm{d}\Gamma}_{\underset{w\in H^1_{0,\Gamma_d}(\Omega,\mathbb{R})}{=} 0}$$

$$\Leftrightarrow \int_\Omega (e\mathcal{B}u - \epsilon\nabla\phi_0)^T \nabla w \,\mathrm{d}\Omega = \int_\Omega (\epsilon\nabla\chi)^T \nabla w \,\mathrm{d}\Omega\,. \tag{3.9}$$

Consequently, we can define the time-dependent variational identities corresponding to the PDE system (3.2)-(3.7) by

$$B((u,\phi_0),(v,w)) :=$$
$$\int_\Omega \rho\ddot{u}^T v + \alpha\rho\dot{u}^T v + \left(c^E\mathcal{B}u + \beta c^E\mathcal{B}\dot{u} + e^T\nabla\phi_0\right)^T \mathcal{B}v + (e\mathcal{B}u - \epsilon\nabla\phi_0)^T \nabla w \,\mathrm{d}\Omega \tag{3.10}$$

$$L(v,w) = \int_\Omega -(e^T\nabla\chi)^T\mathcal{B}v + (\epsilon\nabla\chi)^T\nabla w \,\mathrm{d}\Omega\,. \tag{3.11}$$

As the subsequent result will be used later for the adjoint system as well, a slightly more general inhomogeneous problem will be considered with inhomogeneities $(f,g) \in L^2\big(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*\big) \times L^2\big(L^2(\Omega,\mathbb{R})\big)$. For the same purpose an additional damping function $a \in H^2\big(L^\infty(\Omega,\mathbb{R})\big)$, which is non-negative and uniformly bounded, is introduced. This results in the system

$$\rho\ddot{u} + \alpha\rho\dot{u} + au - \mathcal{B}^T\Big(c^E\mathcal{B}u + \beta c^E\mathcal{B}\dot{u} + c^T\nabla\phi_0\Big) = f + \mathcal{B}^T c^T\nabla\chi \qquad \text{in } \Omega\times(0,T) \tag{3.12}$$

$$-\nabla\cdot\Big(c\mathcal{B}u - \epsilon\nabla\phi_0\Big) = g - \nabla\cdot\epsilon\nabla\chi \qquad \text{in } \Omega\times(0,T), \tag{3.13}$$

$$n\cdot\Big(c\mathcal{B}u - \epsilon\nabla\phi_0\Big) = n\cdot\epsilon\nabla\chi \qquad \text{on } \Gamma_n\times(0,T), \tag{3.14}$$

$$\mathcal{N}^T\Big(c^E\mathcal{B}u + \beta c^E\mathcal{B}\dot{u} + c^T\nabla\phi_0\Big) = -\mathcal{N}^T c^T\nabla\chi \qquad \text{on } \partial\Omega\times(0,T), \tag{3.15}$$

$$u(0) = u_0, \quad \dot{u}(0) = u_1 \qquad \text{in } \Omega. \tag{3.16}$$

To prove existence and uniqueness of solutions to the piezoelectric system (3.12)-(3.16), we use Chapter XVIII in [7], especially Theorem 1 in Paragraph 5 and Remark 9 in Paragraph 6. For this purpose, we eliminate the electrostatic equation (3.13), and hence the variable ϕ^0.

Proposition 3.1 (Electrostatic solution operator)**.** Let Assumption D1 or D3 and Assumptions 2.2 hold as well as $g \in L^2(L^2(\Omega,\mathbb{R}))$ be a given inhomogeneity. Then, for any ϕ^e according to Assumption D1 or (D3) with $\chi \in H^1\big(0,T;H^1(\Omega,\mathbb{R})\big)$ and $\mathrm{Tr}(\chi(t))$ defined as in (3.1), there exists a linear and continuous operator

$$\zeta^t : H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3) \to H^1_{0,\Gamma_d}(\Omega,\mathbb{R}),\ u(t) \mapsto \phi_0^0(t), \tag{3.17}$$

for almost all $t \in (0,T)$ such that for every $u \in L^2(H^1(\Omega))$, the function $\zeta^t(u) \in L^2(H^1_0(\Omega))$ is unique and

$$\int_\Omega (\epsilon\nabla\zeta^t(u))^T\nabla w \,\mathrm{d}\Omega = \int_\Omega (e\mathcal{B}u)^T\nabla w \quad \forall w \in H^1_{0,\Gamma_d}(\Omega,\mathbb{R}). \tag{3.18}$$

Moreover, there exists a unique $\phi_0^r \in L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}))$ such that $\phi_0(t) = \zeta^t(u(t)) + \phi_0^r(t)$ solves the electrostatic equation (3.13) uniquely. Additionally, there exists a constant $C > 0$ such that

$$\|\zeta(u)\|_{L^2(H^1_0(\Omega))} \le C\Big(\|u\|_{L^2(H^1(\Omega))} + \|\chi\|_{L^2(H^1(\Omega))}\Big). \tag{3.19}$$

Proof. Set $V := H^1_{\mathcal{B}}(\Omega, \mathbb{R}^3)$ and $H := L^2(\Omega, \mathbb{R}^3)$. Similarly to (3.9) we obtain the affine linear (in ϕ_0) mapping,

$$\int_\Omega (\epsilon \nabla \phi_0)^T \nabla w \,\mathrm{d}\Omega = \int_\Omega (e\mathcal{B}u)^T \nabla w - (\epsilon \nabla \chi)^T \nabla w - gw \,\mathrm{d}\Omega \,. \tag{3.20}$$

Therefore, we define for almost all $t \in (0,T)$ the linear operator (3.17) where $\phi_0(t) = \phi_0^0(t) + \phi_0^r(t) = \zeta^t(u(t)) + \phi_0^r(t)$ by satisfying identity (3.18). By the lemma of Lax-Milgram together with coercivity of $\epsilon(t)$ and boundedness of $\epsilon(t)$, this determines $\zeta^t(u) \in H^1_{0,\Gamma_d}(\Omega, \mathbb{R})$ uniquely. Choosing $w = \zeta^t(u)$ and using coercivity together with the Poincaré inequality on $H^1_{0,\Gamma_d}(\Omega, \mathbb{R})$, we obtain with some $C_1 > 0$ that

$$\|\zeta^t(u)\|_{H^1_{0,\Gamma_d}(\Omega,\mathbb{R})} \le C_1 \|u(t)\|_V$$

for almost every $t \in (0, T)$. Hence,

$$\|\zeta^t(u)\|_{L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}))} \le C_1 \|u\|_{L^2(V)}.$$

Moreover, we define $\phi_0^r \in L^2(H^1_{0,\Gamma_d}(\Omega, \mathbb{R}))$ by

$$\int_\Omega (\epsilon(t) \nabla \phi_0^r(t))^T \nabla w \,\mathrm{d}\Omega = -\int_\Omega (\epsilon(t) \nabla \chi(t))^T \nabla w \mathrm{d}\Omega - \int_\Omega g(t) w \mathrm{d}\Omega \tag{3.21}$$

for all $w \in H^1_{0,\Gamma_d}(\Omega, \mathbb{R})$ and almost all $t \in (0,T)$. Using the same arguments as above results in the existence of a unique $\phi_0^r(t) \in H^1_{0,\Gamma_d}(\Omega, \mathbb{R})$ for almost all $t \in (0,T)$ and a constant $C_2 > 0$, such that by integration over time we obtain

$$\|\phi_0^r\|_{L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}))} \le C_2 \Big(\|\chi\|_{L^2(H^1(\Omega,\mathbb{R}))} + \|g\|_{L^2(L^2(\Omega,\mathbb{R}))} \Big). \tag{3.22}$$

Thus, $\phi_0(t) = \zeta^t(u(t)) + \phi_0^r(t)$. □

Theorem 3.2 (Existence and Uniqueness). *Let Assumption D1 or D3 and Assumptions 2.2 hold. Let $a \in H^2(L^\infty(\Omega, \mathbb{R}))$ be non-negative and uniformly bounded, and let $(f, g) \in L^2\big(H^1_{\mathcal{B}}(\Omega, \mathbb{R}^3)^*\big) \times L^2(L^2(\Omega, \mathbb{R}))$ be inhomogeneities. Then, for any initial values*

$$u_0 \in H^1_{\mathcal{B}}(\Omega, \mathbb{R}^3), \qquad u_1 \in L^2(\Omega, \mathbb{R}^3),$$

and any ϕ^e according to Assumption D1 or (D3) *with $\chi \in H^1\big(0, T; H^1(\Omega, \mathbb{R})\big)$ and $\mathrm{Tr}(\chi(t))$ defined as in* (3.1)*, there exists a unique weak solution*

$$(u, \phi_0) \in L^2\big(H^1_{\mathcal{B}}(\Omega, \mathbb{R}^3)\big) \times L^2\big(H^1_{0,\Gamma_d}(\Omega, \mathbb{R})\big)$$

with

$$\dot{u} \in L^2\big(H^1_{\mathcal{B}}(\Omega, \mathbb{R}^3)\big), \qquad \ddot{u} \in L^2\big(H^1_{\mathcal{B}}(\Omega, \mathbb{R}^3)^*\big)$$

to the system (3.12)-(3.16)*.*

Proof. Set $V := H^1_{\mathcal{B}}(\Omega, \mathbb{R}^3)$ and $H := L^2(\Omega, \mathbb{R}^3)$. By Korn's inequality, the V-norm and the H^1-norm are equivalent. Proposition 3.1 induces an equivalent weak form of the inhomogeneous piezoelectric PDE including the Dirichlet lift similarly to (3.8),

$$\begin{aligned} \int_\Omega \rho \ddot{u}^T v + \alpha \rho \dot{u}^T v + a u^T v + &\left(c^E \mathcal{B}u + \beta c^E \mathcal{B}\dot{u} + e^T \nabla \zeta^t(u) \right)^T \mathcal{B}v \,\mathrm{d}\Omega \\ &= \int_\Omega fv - (e^T \nabla(\chi + \phi_0^r))^T \mathcal{B}v \,\mathrm{d}\Omega \quad \forall v \in V. \end{aligned} \tag{3.23}$$

To prove existence and uniqueness of solutions to the piezoelectric system (3.12)-(3.16), Chapter XVIII in [7], especially Theorem 1 in Paragraph 5 and Remark 9 in Paragraph 6, is used. Furthermore, the standard embedding (see Theorem 6, Chapter 5.6.3 in [10])

$$H^k(L^\infty(\Omega)) \hookrightarrow C^{k-1}([0,T]; L^\infty(\Omega)) \tag{3.24}$$

for some $k \in \mathbb{N}$ will be used to deduce that $c^E,\ e$ and ϵ are continuously differentiable in time. For this purpose, the operators

$$
\begin{aligned}
a_0(t,u,v) &:= \int_\Omega u^T v + \left(c^E(t)\mathcal{B}u\right)^T \mathcal{B}v + \left(e(t)^T \nabla\zeta^t(u)\right)^T \mathcal{B}v \,\mathrm{d}\Omega\,,\\
a_1(t,u,v) &:= \int_\Omega (a(t)-1)u^T v \,\mathrm{d}\Omega\,,\\
b_0(t,u,v) &:= \int_\Omega u^T v + \left(\beta(t)c^E(t)\mathcal{B}u\right)^T \mathcal{B}v \,\mathrm{d}\Omega\,,\\
b_1(t,u,v) &:= \int_\Omega (\alpha(t)\rho(t) - \dot\rho(t) - 1)u^T v \,\mathrm{d}\Omega\,,\\
c(t,u,v) &:= \int_\Omega \rho(t) u^T v \,\mathrm{d}\Omega\,,\\
\langle \tilde f(t), v\rangle &:= \int_\Omega f(t)v - (e(t)^T\nabla(\chi + \phi_0^r)(t))^T \mathcal{B}v \,\mathrm{d}\Omega\,,
\end{aligned}
\tag{3.25}
$$

where $a_{01} = a_0 + a_1$ and $b_{01} = b_0 + b_1$, are defined for all $v \in V$ almost all $t \in (0,T)$. Using Chapter XVIII in [7], proving existence and uniqueness of $u \in L^2(V)$ solving

$$
\frac{\mathrm{d}}{\mathrm{d}t} c(t;\dot u(t), v) + b_{01}(t;\dot u(t), v) + a_{01}(t; u(t), v) = \langle \tilde f(t), v\rangle
\tag{3.26}
$$

for all $v \in V$, together with $u(0) = u_0 \in V,\ \dot u(0) = u_1 \in H$ and $\tilde f \in L^2(V^*)$ is the goal. To show that $\tilde f \in L^2(V^*)$, we denote the unit ball in V with B_V^1 and observe

$$
\begin{aligned}
\left\|\tilde f\right\|^2_{L^2(V^*)} &= \int_0^T \left\|\tilde f(t)\right\|^2_{H^1(\Omega,\mathbb{R}^3)^*} \mathrm{d}t\\
&\le \int_0^T \sup_{v\in\partial B_V^1} |(f,v)_{L^2(\Omega,\mathbb{R}^3)} - (e(t)^T\nabla(\chi(t)+\phi_0^r(t)), \mathcal{B}v)_{L^2(\Omega,\mathbb{R}^3)}|^2 \,\mathrm{d}t\\
&\le \int_0^T \sup_{v\in\partial B_V^1} \left\|-e(t)^T\nabla(\chi(t)+\phi_0^r(t))\right\|^2_{L^2(\Omega,\mathbb{R}^3)} \|v\|^2_{H^1(\Omega,\mathbb{R}^3)} \,\mathrm{d}t\\
&\quad + \int_0^T \sup_{v\in\partial B_V^1} \|f(t)\|^2_{H^1(\Omega,\mathbb{R}^3)^*} \|v\|^2_{H^1(\Omega,\mathbb{R}^3)} \,\mathrm{d}t\\
&\le \|e\|^2_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 6}))} \left(\|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))}\right) + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)} < \infty.
\end{aligned}
\tag{3.27}
$$

As our function space setting are real Hilbert spaces and due to

$$
\begin{aligned}
\int_\Omega \left(e^T\nabla\zeta^t(u)\right)^T \mathcal{B}v \,\mathrm{d}\Omega &= \int_\Omega \left(\nabla\zeta^t(u)\right)^T e\mathcal{B}v \,\mathrm{d}\Omega = \int_\Omega (e\mathcal{B}v)^T \nabla\zeta^t(u) \,\mathrm{d}\Omega\\
&\underset{(3.18)}{=} \int_\Omega \left(\epsilon\nabla\zeta^t(v)\right)^T \nabla\zeta^t(u)\,\mathrm{d}\Omega \underset{\epsilon\text{ diag.}}{=} \int_\Omega \left(\epsilon\nabla\zeta^t(u)\right)^T \nabla\zeta^t(v)\,\mathrm{d}\Omega\\
&\underset{(3.18)}{=} \int_\Omega (e\mathcal{B}u)^T \nabla\zeta^t(v)\,\mathrm{d}\Omega = \int_\Omega \left(e^T\nabla\zeta^t(v)\right)^T \mathcal{B}u \,\mathrm{d}\Omega\,,
\end{aligned}
\tag{3.28}
$$

we immediately conclude that a_0 and b_0 are Hermitian. By assumption A3 and A5 it holds that

$$
\begin{aligned}
(c^E(t)\mathcal{B}u(t), \mathcal{B}u(t))_{L^2(\Omega,\mathbb{R}^6)} &\ge c_\star(\mathcal{B}u(t), \mathcal{B}u(t))_{L^2(\Omega,\mathbb{R}^6)}\\
(\epsilon(t)\nabla\zeta^t(u(t)), \nabla\zeta^t(u(t)))_{L^2(\Omega,\mathbb{R}^3)} &\ge \epsilon_\star(\nabla\zeta^t(u(t)), \nabla\zeta^t(u(t)))_{L^2(\Omega,\mathbb{R}^3)}
\end{aligned}
$$

for almost all $t \in (0,T)$, which induces the existence of some constant $\sigma \in \mathbb{R}^+$ with

$$
\sigma := \min\{1, c_\star\} > 0,
\tag{3.29}
$$

such that

$$\begin{aligned} a_0(t,u,u) &= \int_\Omega u^T u \mathrm{d}\Omega + \int_\Omega \left(c^E(t)\mathcal{B}u\right)^T \mathcal{B}u \mathrm{d}\Omega + \int_\Omega \left(e(t)^T \nabla \zeta^t(u)\right)^T \mathcal{B}u \mathrm{d}\Omega \\ &\underset{(3.28)}{=} \|u\|^2_{L^2(\Omega,\mathbb{R}^3)} + \int_\Omega \left(c^E(t)\mathcal{B}u\right)^T \mathcal{B}u \mathrm{d}\Omega + \int_\Omega \left(\epsilon(t)\nabla\zeta^t(u)\right)^T \nabla\zeta^t(u) \mathrm{d}\Omega \\ &\geq \|u\|^2_{L^2(\Omega,\mathbb{R}^3)} + c_\star \|\mathcal{B}u\|^2_{L^2(\Omega,\mathbb{R}^6)} + \epsilon_\star \|\nabla\zeta^t(u)\|^2_{L^2(\Omega,\mathbb{R}^3)} \\ &\geq \sigma \|u\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)}. \end{aligned} \tag{3.30}$$

To prove that the mapping $t \mapsto a_0(t,u,v)$ is one time continuously differentiable, the regularities of the material parameters are used. For the continuous differentiability of the mapping $t \mapsto \zeta^t(u) \in H^1_{0,\Gamma_d}(\Omega,\mathbb{R})$ we obtain for almost every $t \in (0,T)$ and all $w \in H^1_{0,\Gamma_d}(\Omega,\mathbb{R})$ that

$$\int_\Omega \epsilon(t) \frac{\mathrm{d}}{\mathrm{d}t} (\nabla \zeta^t(u))^T \nabla w \ \mathrm{d}\Omega = \int_\Omega \Big(\dot{e}(t)\mathcal{B}u + e(t)\mathcal{B}\dot{u} - \dot{\epsilon}(t)\nabla\zeta^t(u)\Big)^T \nabla w \ \mathrm{d}\Omega \,.$$

Thus, by the regularity of the material and damping parameters, the mapping $t \mapsto a_0(t,u,v)$ is continuously differentiable with derivative

$$\dot{a}_0(t,u,v) = \int_\Omega \dot{a}(t) u^T v + \left(\dot{c}^E(t)\mathcal{B}u\right)^T \mathcal{B}v + \Big(\dot{e}(t)^T \nabla\zeta^t(u) + e(t)^T \nabla\dot{\zeta}^t(u)\Big)^T \mathcal{B}v \, \mathrm{d}\Omega \,. \tag{3.31}$$

Moreover, the mapping $t \mapsto a_1(t,u,v)$ is continuous, and

$$|a_1(t,u,v)| \leq \|a(t)-1\|_{L^\infty(\Omega,\mathbb{R})} \|u\|_H \|v\|_H \leq C_a \big(1 + \|a\|_{L^\infty(L^\infty(\Omega,\mathbb{R}))}\big) \|u\|_V \|v\|_V. \tag{3.32}$$

Next, with

$$\beta_0 := \min\{1, \beta^\star c_\star\} > 0, \tag{3.33}$$

we have

$$\begin{aligned} b_0(t,u,u) &= \int_\Omega u^T u \mathrm{d}\Omega + \int_\Omega \left(\beta(t) c^E(t)\mathcal{B}u\right)^T \mathcal{B}u \mathrm{d}\Omega \\ &\geq \|u\|^2_{L^2(\Omega,\mathbb{R}^3)} + \beta^\star c_\star \|\mathcal{B}u\|^2_{L^2(\Omega,\mathbb{R}^6)} \geq \beta_0 \|u\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)}. \end{aligned} \tag{3.34}$$

The mapping $t \mapsto b_1(t,u,v)$ is continuous and

$$|b_1(t,u,v)| \leq \|\alpha(t)\rho(t) - \dot{\rho}(t) - 1\|_{L^\infty(\Omega,\mathbb{R})} \|u\|_H \|v\|_H \leq C_{b_1} \|u\|_V \|v\|_V \tag{3.35}$$

with $C_{b_1} := C\Big(1 + \|\alpha\rho - \dot{\rho}\|_{L^\infty(L^\infty(\Omega,\mathbb{R}))}\Big)$. Finally, the mapping $t \mapsto c(t,u,v)$ is one time continuously differentiable and

$$\dot{c}(t,u,v) = \int_\Omega \dot{\rho}(t) u^T v \mathrm{d}\Omega. \tag{3.36}$$

Moreover,

$$c(t,v,v) \geq \rho^\star \|v\|^2_{L^2(\Omega,\mathbb{R}^3)} \qquad \text{for all } v \in H. \tag{3.37}$$

Thus, all assumptions of Theorem 1, Paragraph 5, Chapter XVIII in [7] are satisfied, while further conditions on c are ensured by Remark 9 in Paragraph 6 of the same chapter. Therefore, (3.26) admits a unique solution u with the asserted regularity. Setting

$$\phi_0(t) = \zeta^t(u(t)) + \phi_0^r(t)$$

yields the unique weak solution (u,ϕ_0) of (3.12)-(3.16). □

To prove boundedness of the operators in Section 4, the following theorem is beneficial.

Theorem 3.3 (Energy estimates)**.** *Let the assumptions of Theorem 3.2 hold. Then there exists a constant $\widetilde{K} > 0$ such that*

$$\|\ddot{u}\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*)} + \|\dot{u}\|^2_{L^\infty(L^2(\Omega,\mathbb{R}^3))} + \|u\|^2_{L^\infty(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\dot{u}\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\phi_0\|^2_{L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}))}$$
$$\leq \widetilde{K}\Big(\|u_1\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u_0\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))}$$
$$+ \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|f\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*)} + \|g\|^2_{L^2(L^2(\Omega,\mathbb{R}))}\Big).$$

Proof. Set $V := H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)$ and define

$$\begin{aligned}
X(\tau) &:= c(\tau,\dot{u}(\tau),\dot{u}(\tau)) + a_0(\tau,u(\tau),u(\tau)) + 2\int_0^\tau b_0(s,\dot{u}(s),\dot{u}(s))\mathrm{d}s \\
&= \int_\Omega \rho(\tau)\dot{u}(\tau)^T\dot{u}(\tau)\mathrm{d}\Omega + \int_\Omega u(\tau)^T u(\tau)\mathrm{d}\Omega + \int_\Omega \left(c^E(\tau)\mathcal{B}u(\tau)\right)^T \mathcal{B}u(\tau)\mathrm{d}\Omega \\
&+ \int_\Omega \left(e(\tau)^T\nabla\zeta^\tau(u(\tau))\right)^T \mathcal{B}u(\tau)\mathrm{d}\Omega + 2\int_0^\tau\int_\Omega \dot{u}(s)^T\dot{u}(s)\mathrm{d}\Omega\mathrm{d}s \\
&+ 2\int_0^\tau\int_\Omega \left(\beta(s)c^E(s)\mathcal{B}\dot{u}(s)\right)^T\mathcal{B}\dot{u}(s)\mathrm{d}\Omega\mathrm{d}s.
\end{aligned}$$

By (3.28), (3.30), and (3.34), with

$$k := \min\{\rho^\star, \sigma, \epsilon_\star, \beta_0\} > 0,$$

we obtain

$$\begin{aligned}
X(\tau) &\geq \rho^\star\|\dot{u}(\tau)\|^2_{L^2(\Omega,\mathbb{R}^3)} + \sigma\|u(\tau)\|^2_V + \int_\Omega \left(\epsilon(\tau)\nabla\zeta^\tau(u(\tau))\right)^T\nabla\zeta^\tau(u(\tau))\mathrm{d}\Omega \\
&+ \beta_0\int_0^\tau \|\dot{u}(s)\|^2_V\mathrm{d}s \\
&\geq k\Big(\|\dot{u}(\tau)\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u(\tau)\|^2_V + \|\nabla\zeta^\tau(u(\tau))\|^2_{L^2(\Omega,\mathbb{R}^3)} + \int_0^\tau\|\dot{u}(s)\|^2_V\mathrm{d}s\Big).
\end{aligned} \tag{3.38}$$

Moreover, with

$$c_0 := \max\left\{\|\rho\|_{L^\infty(L^\infty(\Omega,\mathbb{R}))}, 1, \|c^E\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{6\times6}))}, \|\epsilon\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times3}))}\right\},$$

we have

$$X(0) \leq c_0\Big(\|u_1\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u_0\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|\mathcal{B}u_0\|^2_{L^2(\Omega,\mathbb{R}^6)} + \|\nabla\zeta^0(u_0)\|^2_{L^2(\Omega,\mathbb{R}^3)}\Big). \tag{3.39}$$

From our previous results and [7], Chapter XVIII, Paragraph 5, section 4.1, we know that

$$\begin{aligned}
X(\tau) &= X(0) + 2\int_0^\tau \dot{a}_0(s,u(s),u(s))\mathrm{d}s - 2\int_0^\tau b_1(s,\dot{u}(s),\dot{u}(s))\mathrm{d}s \\
&- \int_0^\tau \dot{c}(s,\dot{u}(s),\dot{u}(s))\mathrm{d}s + 2\int_0^\tau\langle\tilde{f}(s),\dot{u}(s)\rangle\mathrm{d}s
\end{aligned} \tag{3.40}$$

holds. Using (3.31), Proposition 3.1, and Cauchy-Schwarz, there exists $K_0 > 0$ such that

$$\int_0^\tau \dot{a}_0(s,u(s),u(s))\mathrm{d}s \leq K_0\left(\|u\|^2_{L^2(V)} + \|\nabla(\zeta^t(u))\|^2_{L^2(L^2(\Omega,\mathbb{R}^3))}\right). \tag{3.41}$$

Furthermore, by (3.35) and (3.36),

$$-2\int_0^\tau b_1(s,\dot{u}(s),\dot{u}(s))\mathrm{d}s - \int_0^\tau \dot{c}(s,\dot{u}(s),\dot{u}(s))\mathrm{d}s \leq K_1\|\dot{u}\|^2_{L^2(V)} \tag{3.42}$$

for some constant $K_1 > 0$. By (3.27) and Young's inequality,

$$\begin{aligned}
2\int_0^\tau\langle\tilde{f}(s),\dot{u}(s)\rangle\mathrm{d}s &\leq \|\tilde{f}\|^2_{L^2(V^*)} + \|\dot{u}\|^2_{L^2(V)} \\
&\leq C\Big(\|f\|^2_{L^2(V^*)} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))}\Big) + \|\dot{u}\|^2_{L^2(V)}.
\end{aligned} \tag{3.43}$$

Combining (3.40), (3.41), (3.42), and (3.43), we obtain a constant $K_2 > 0$ such that

$$X(\tau) \le X(0) + K_2 \int_0^\tau \Big(\|u(s)\|_V^2 + \|\nabla\zeta^s(u(s))\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|\dot u(s)\|_V^2 \Big) \mathrm{d}s$$

$$+ K_2 \Big(\|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|f\|^2_{L^2(V^*)} \Big). \tag{3.44}$$

Now (3.38) and (3.44) imply

$$k\Big(\|\dot u(\tau)\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u(\tau)\|_V^2 + \|\nabla\zeta^\tau(u(\tau))\|^2_{L^2(\Omega,\mathbb{R}^3)} + \int_0^\tau \|\dot u(s)\|_V^2 \mathrm{d}s \Big) \le X(\tau) \tag{3.45}$$

$$\le X(0) + K_2 \int_0^\tau \Big(\|u(s)\|_V^2 + \|\nabla\zeta^s(u(s))\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|\dot u(s)\|_V^2 \Big) \mathrm{d}s$$

$$+ K_2 \Big(\|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|f\|^2_{L^2(V^*)} \Big). \tag{3.46}$$

By Gronwall's lemma, there exists $C > 0$ such that

$$\|\dot u\|^2_{L^\infty(L^2(\Omega,\mathbb{R}^3))} + \|u\|^2_{L^\infty(V)} + \|\dot u\|^2_{L^2(V)} + \|\nabla(\zeta(u))\|^2_{L^\infty(L^2(\Omega,\mathbb{R}^3))}$$

$$\le C\Big(\|u_1\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u_0\|_V^2 + \|\nabla\zeta^0(u_0)\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))}$$

$$+ \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|f\|^2_{L^2(V^*)} \Big). \tag{3.47}$$

Testing (3.18) with $\zeta^\tau(u(\tau))$, we obtain

$$\|\nabla\zeta^\tau(u(\tau))\|_{L^2(\Omega,\mathbb{R}^3)} \le \|e\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times6}))} \|(\epsilon)^{-1}\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times3}))} \|\mathcal{B}u(\tau)\|_{L^2(\Omega,\mathbb{R}^6)}. \tag{3.48}$$

In particular,

$$\|\nabla\zeta^0(u_0)\|_{L^2(\Omega,\mathbb{R}^3)} \le \|e\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times6}))} \|(\epsilon)^{-1}\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times3}))} \|\mathcal{B}u_0\|_{L^2(\Omega,\mathbb{R}^6)}. \tag{3.49}$$

Moreover, there exists some $C_r > 0$ such that

$$\begin{aligned}
&\epsilon_\star \|\nabla\phi_0^r(\tau)\|^2_{L^2(\Omega,\mathbb{R}^3)} \le \int_\Omega (\epsilon\nabla\phi_0^r(\tau))^T \nabla\phi_0^r \,\mathrm{d}\Omega \\
&\le \|\epsilon\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times3}))} \|\nabla\chi(\tau)\|_{L^2(\Omega,\mathbb{R}^3)} \|\nabla\phi_0^r(\tau)\|_{L^2(\Omega,\mathbb{R}^3)} + \|g(\tau)\|_{L^2(\Omega,\mathbb{R})} \|\phi_0^r(\tau)\|_{L^2(\Omega,\mathbb{R}^3)} \\
&\le \|\epsilon\|^2_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times3}))} \left(\|\nabla\chi(\tau)\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|\nabla\phi_0^r(\tau)\|^2_{L^2(\Omega,\mathbb{R}^3)} \right) + \|g\|^2_{L^2(L^2(\Omega,\mathbb{R}))} + \|\phi_0^r(\tau)\|^2_{L^2(\Omega,\mathbb{R}^3)} \\
&\le C_r \left(\|\nabla\chi\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|\phi_0^r(\tau)\|^2_{H^1(\Omega,\mathbb{R}^3)} + \|g\|^2_{L^2(L^2(\Omega,\mathbb{R}))} \right).
\end{aligned}$$

Thus, there exists some constant $\tilde C > 0$ such that for almost all $\tau \in (0,T)$

$$\|\dot u(\tau)\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u(\tau)\|_V^2 + \|\dot u\|^2_{L^2((0,\tau);V)} \le \tilde C \left(\|u_1\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u_0\|_V^2 \right.$$

$$\left. + \|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)} + \|g\|^2_{L^2(L^2(\Omega,\mathbb{R}))} \right)$$

and therefore

$$\|\dot u\|^2_{L^\infty(L^2(\Omega,\mathbb{R}^3))} + \|u\|^2_{L^\infty(V)} + \|\dot u\|^2_{L^2(V)} \le \tilde C \left(\|u_1\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u_0\|_V^2 \right.$$

$$\left. + \|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)} + \|g\|^2_{L^2(L^2(\Omega,\mathbb{R}))} \right). \tag{3.50}$$

It remains to estimate $\ddot u$. Testing (3.23) with $v \in V$, we obtain

$$\begin{aligned}
(\rho(\tau)\ddot u(\tau), v)_{L^2(\Omega,\mathbb{R}^3)} = {}& \langle \tilde f(\tau), v\rangle_{V^*,V} - (\alpha(\tau)\rho(\tau)\dot u(\tau), v)_{L^2(\Omega,\mathbb{R}^3)} - (a(\tau)u(\tau), v)_{L^2(\Omega,\mathbb{R}^3)} \\
&- (c^E(\tau)\mathcal{B}u(\tau), \mathcal{B}v)_{L^2(\Omega,\mathbb{R}^6)} - (\beta(\tau)c^E(\tau)\mathcal{B}\dot u(\tau), \mathcal{B}v)_{L^2(\Omega,\mathbb{R}^6)} \\
&- (e(\tau)^T\nabla\zeta^\tau(u(\tau)), \mathcal{B}v)_{L^2(\Omega,\mathbb{R}^6)}.
\end{aligned}$$

Hence, using (3.48) and $\rho(\tau) \ge \rho^\star$, implies that there exists $G > 0$ such that

$$|(\ddot u(\tau), v)_{L^2(\Omega,\mathbb{R}^3)}| \le G\Big(\|\tilde f(\tau)\|_{V^*} + \|\dot u(\tau)\|_V + \|u(\tau)\|_V \Big) \|v\|_V.$$

Taking the supremum over $v \in B_V^1$, squaring, and integrating over $(0,T)$ and repeatedly applying Young's inequality yields

$$\|\ddot{u}(\tau)\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*} \leq 3G^2 \left(\left\|\tilde{f}(\tau)\right\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*} + \|\dot{u}(\tau)\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|u(\tau)\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} \right).$$

Integration over $(0,T)$, employing (3.27) and (3.50) as well as using

$$\tilde{C}_{\ddot{u}} = 3G^2 \max\left\{ \|e\|^2_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 6}))} + \frac{\tilde{C}}{\min\{1,T\}}, 1 + \frac{\tilde{C}}{\min\{1,T\}} \right\}$$

results in

$$\begin{aligned} \|\ddot{u}\|^2_{L^2(V^*)} \leq \widetilde{C}_{\ddot{u}} \Big(&\|u_1\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u_0\|^2_V + \|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} \\ &+ \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|f\|^2_{L^2(V^*)} + \|g\|^2_{L^2(L^2(\Omega,\mathbb{R}))} \Big). \end{aligned} \tag{3.51}$$

Finally, testing (3.20) with $\phi_0(\tau)$, using coercivity of $\epsilon(\tau)$, Cauchy-Schwarz, and Poincaré's inequality, we obtain

$$\begin{aligned} \epsilon_\star \|\nabla\phi_0\|^2_{L^2(\Omega,\mathbb{R}^3)} &\leq \left| \int_\Omega (\epsilon\nabla\phi_0)^T \nabla\phi_0 \, \mathrm{d}\Omega \right| \\ &= \left| \int_\Omega (e\mathcal{B}u)^T \nabla\phi_0 - (\epsilon\nabla\chi)^T \nabla\phi_0 - g\phi_0 \, \mathrm{d}\Omega \right| \\ &\leq \Big(\|e\mathcal{B}u\|_{L^2(\Omega,\mathbb{R}^3)} + \|\epsilon\nabla\chi\|_{L^2(\Omega,\mathbb{R}^3)} \Big) \|\nabla\phi_0\|_{L^2(\Omega,\mathbb{R}^3)} + \|g\|_{L^2(\Omega,\mathbb{R})} \|\phi_0\|_{L^2(\Omega,\mathbb{R})} \\ &\leq \Big(\|e\mathcal{B}u\|_{L^2(\Omega,\mathbb{R}^3)} + \|\epsilon\nabla\chi\|_{L^2(\Omega,\mathbb{R}^3)} \Big) \|\nabla\phi_0\|_{L^2(\Omega,\mathbb{R}^3)} + C_P \|g\|_{L^2(\Omega,\mathbb{R})} \|\nabla\phi_0\|_{L^2(\Omega,\mathbb{R})}. \end{aligned} \tag{3.52}$$

After dividing by $\|\nabla\phi_0\|_{L^2(\Omega,\mathbb{R}^3)} > 0$ for almost all $t \in (0,T)$), ($\|\nabla\phi_0\|_{L^2(\Omega,\mathbb{R}^3)} = 0$ is a trivial case), using the norm equivalence of the H^1_{0,Γ_d}- norm and the L^2-norm of the gradient, integrating over time and applying Hölder's inequality there exists a constant C_ϕ such that

$$\|\phi_0\|_{L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}))} \leq C_\phi \left(\|u\|_{L^\infty(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\chi\|_{L^2(H^1(\Omega,\mathbb{R}))} + \|g\|_{L^2(L^2(\Omega,\mathbb{R}))} \right). \tag{3.53}$$

Combining (3.50), (3.51), and (3.53) proves the claim. □

Remark 3.4. Note that if $g \in L^\infty(L^2(\Omega,\mathbb{R}))$ and $\chi \in H^1\left(H^1(\Omega,\mathbb{R})\right) \cap L^\infty\left(H^1(\Omega,\mathbb{R})\right)$ then we do not necessarily have to integrate (3.52) over time after dividing by $\|\nabla\phi_0\|_{L^2(\Omega,\mathbb{R}^3)} > 0$, since by taking the essential supremum, there exists a constant $\tilde{C}_\phi$ such that

$$\|\phi_0\|_{L^\infty(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}))} \leq \tilde{C}_\phi \left(\|u\|_{L^\infty(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\chi\|_{L^\infty(H^1(\Omega,\mathbb{R}))} + \|g\|_{L^\infty(L^2(\Omega,\mathbb{R}))} \right).$$

4. Analysis of the forward operator

Before we define the model operator, we have to specify the parameter space. We abbreviate $p = (c^E, e, \epsilon)$ and $z = (u, \phi_0)$ from now on.

Definition 4.1 (Parameter space)**.** We define the Hilbert space

$$\mathcal{X} := H^3\big(H^3(\Omega,\mathbb{R}^{6\times 6})\big) \times H^3\big(H^3(\Omega,\mathbb{R}^{3\times 6})\big) \times H^3\big(H^3(\Omega,\mathbb{R}^{3\times 3})\big),$$

equipped with the natural norm. The parameter space $X \subset \mathcal{X}$ is given by

$$X := \Big\{ p \in \mathcal{X} : c^E, e, \epsilon \text{ are of structure as in Assumptions A3-A5} \Big\}.$$

and we assume that there exists a constant $M > 0$ such that

$$\|p\|_X = \|c^E\|_{H^3(H^3(\Omega,\mathbb{R}^{6\times 6}))} + \|e\|_{H^3(H^3(\Omega,\mathbb{R}^{3\times 6}))} + \|\epsilon\|_{H^3(H^3(\Omega,\mathbb{R}^{3\times 3}))} \leq M$$

for all $(c^E, e, \epsilon) \in X$.

Uniform boundedness of X is a physically reasonable assumption, since otherwise the corresponding material parameters would attain unrealistically large values and the model would cease to describe the underlying system in a meaningful way.

Definition 4.2 (State space)**.** The state space W is defined as

$$W := \Big\{(u,\phi_0) \in H^1\big(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)\big) \times L^2\big(H^1_{0,\Gamma_d}(\Omega,\mathbb{R})\big) : \dot{u} \in L^\infty\big(L^2(\Omega,\mathbb{R}^3)\big),\ \ddot{u} \in L^2\big(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*\big)\Big\}.$$

We now define the model operator corresponding to the system (3.2)-(3.7).

Definition 4.3 (Model operator)**.** We identify the piezoelectric model operator $\mathcal{A} : X \times W \to W^*$ in the dual pairing using (3.10) and (3.11) by

$$\begin{aligned}\langle \mathcal{A}(p,z),(v,w)\rangle_{W^*,W} := \int_0^T \int_\Omega & \rho\ddot{u}^T v + \alpha\rho\dot{u}^T v + \left(c^E\mathcal{B}u + \beta c^E \mathcal{B}\dot{u} + e^T\nabla\phi_0\right)^T \mathcal{B}v \\ & + \left(e\mathcal{B}u - \epsilon\nabla\phi_0\right)^T \nabla w + \left(e^T\nabla\chi\right)^T \mathcal{B}v - \left(\epsilon\nabla\chi\right)^T \nabla w \,\mathrm{d}\Omega\,\mathrm{d}t\,. \end{aligned} \tag{4.1}$$

Note that the model operator incorporates all boundary conditions. For the reduced approach, the parameter-to-state map is required.

Definition 4.4 (Parameter-to-state map)**.** The parameter-to-state map $S : X \to W,\ p \mapsto z$ is defined by satisfying the model equation

$$\forall p \in X : \quad \mathcal{A}(p, S(p)) = 0, \tag{4.2}$$

with the model operator from Definition 4.3, such that

$$\forall z \in W : \ [(p,z) \in X \times W \ \wedge\ \mathcal{A}(p,z) = 0] \implies z = S(p).$$

Thus, well-definedness of the forward operator requires existence of the parameter-to-state map S, where the condition

$$\exists C_A\ \forall (p,z) \in X \times W : \ \mathcal{A}'_z(p,z)^{-1} \text{ exists and } \left\|\mathcal{A}'_z(p,z)^{-1}\right\| \le C_A \tag{4.3}$$

is required. The following corollary guarantees bijectivity of the model operator on W.

Corollary 4.5 (Bijectivity of $\mathcal{A}$ on W)**.** Let Assumption D1 or D3 and Assumption 2.2 hold. Then model operator from Definition 4.3 with $\rho \in H^4(L^\infty(\Omega,\mathbb{R}))$ and $\alpha,\ \beta \in H^3(L^\infty(\Omega,\mathbb{R}))$ as well as material parameters $p = (c^E, e, \epsilon) \in X$ is bijective on W for fixed $p \in X$, i.e., for the piezoelectric dynamical system (3.2)-(3.7) there exists a unique weak solution $(u,\phi_0) \in W$ of the system (3.2)-(3.7) with

$$\dot{u} \in L^2\big(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)\big) \cap L^\infty\big(L^2(\Omega,\mathbb{R}^3)\big) \quad \text{and} \quad \ddot{u} \in L^2\big(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*\big).$$

Moreover, there exists a constant C_p such that

$$\begin{aligned} &\|\ddot{u}\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*)} + \|\dot{u}\|^2_{L^\infty(L^2(\Omega,\mathbb{R}^3))} + \|u\|^2_{L^\infty(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\dot{u}\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\phi_0\|^2_{L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}))} \\ &\le C_p\Big(\|u_1\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u_0\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} \\ &\qquad + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)} + \|g\|^2_{L^2(L^2(\Omega,\mathbb{R}))}\Big). \end{aligned}$$

Proof. For the piezoelectric dynamical system the system (3.2)-(3.7) with $\rho \in H^4\big(L^\infty(\Omega,\mathbb{R})\big)$ being positive and uniformly bounded, $\alpha,\ \beta \in H^3\big(L^\infty(\Omega,\mathbb{R})\big)$ being non-negative and uniformly bounded, and material parameters $p = (c^E, e, \epsilon) \in X$ from Definition 4.1, we obtain the same, or partially even higher, regularity of damping and material parameters than in Theorem 3.2. Furthermore, $a \equiv f \equiv g \equiv 0$. Therefore, all assumptions of Theorem 3.2 are satisfied. Finally, due to Theorem 3.3, the second claim holds. □

For an arbitrary direction $\xi = (\mu,\nu) \in W$, the Gâteaux derivative $\mathcal{A}'_z(p,z)\xi$ with respect to the state can be identified as

$$\begin{aligned}\langle \mathcal{A}'_z(p,z)\xi,(v,w)\rangle_{W^*,W} := \int_0^T \int_\Omega & \rho\ddot{\mu}^T v + \alpha\rho\dot{\mu}^T v + \left(c^E\mathcal{B}\mu + \beta c^E \mathcal{B}\dot{\mu} + e^T\nabla\nu\right)^T \mathcal{B}v \\ & + \left(e\mathcal{B}\mu - \epsilon\nabla\nu\right)^T \nabla w \,\mathrm{d}\Omega\,\mathrm{d}t\,, \end{aligned} \tag{4.4}$$

which is again bijective on W for fixed $p \in X$ due to Corollary 4.5. Furthermore, for an arbitrary parameter direction $h = (h_{c^E}, h_e, h_\epsilon) \in X$, the derivative with respect to the parameter can be identified as

$$\begin{aligned}\langle \mathcal{A}'_p(p,z)h,(v,w)\rangle_{W^*,W} := \int_0^T \int_\Omega &\Big(h_{c^E}\mathcal{B}u + \beta h_{c^E}\mathcal{B}\dot{u} + h_e^T \nabla\phi_0\Big)^T \mathcal{B}v \\ &+ \Big(h_e \mathcal{B}u - h_\epsilon \nabla\phi_0\Big)^T \nabla w + \Big(h_e^T \nabla\chi\Big)^T \mathcal{B}v - \Big(h_\epsilon \nabla\chi\Big)^T \nabla w \,\mathrm{d}\Omega\,\mathrm{d}t.\end{aligned} \tag{4.5}$$

Theorem 4.6 (Existence and regularity of S)**.** *The parameter-to-state map S from Definition 4.4 exists and satisfies $S \in C^1(X,W)$.*

Proof. To apply the Implicit Function Theorem, we first prove Fréchet differentiability of $\mathcal{A}$ with respect to the state. Consider

$$\langle \mathcal{A}(p,z+\xi),(v,w)\rangle_{W^*,W} - \langle \mathcal{A}(p,z),(v,w)\rangle_{W^*,W} - \langle \mathcal{A}'_z(p,z)\xi,(v,w)\rangle_{W^*,W}.$$

Due to the affine linearity of $\mathcal{A}$ with respect to the state, it follows that

$$\begin{aligned}&\langle \mathcal{A}(p,z+\xi),(v,w)\rangle_{W^*,W} - \langle \mathcal{A}(p,z),(v,w)\rangle_{W^*,W} - \langle \mathcal{A}'_z(p,z)\xi,(v,w)\rangle_{W^*,W} \\ &\quad= \langle \mathcal{A}(p,z),(v,w)\rangle_{W^*,W} + \int_0^T \int_\Omega \rho\ddot{\mu}^T v + \alpha\rho\dot{\mu}^T v + \left(c^E \mathcal{B}\mu + \beta c^E \mathcal{B}\dot{\mu} + e^T \nabla\nu\right)^T \mathcal{B}v \\ &\qquad+ \left(e\mathcal{B}\mu - \epsilon\nabla\nu\right)^T \nabla w \,\mathrm{d}\Omega\,\mathrm{d}t - \langle \mathcal{A}(p,z),(v,w)\rangle_{W^*,W} - \langle \mathcal{A}'_z(p,z)\xi,(v,w)\rangle_{W^*,W} \\ &\underset{(4.1),(4.4)}{=} 0.\end{aligned}$$

Hence $\mathcal{A}$ is Fréchet differentiable with respect to the state. Since $\mathcal{A}$ is also affine linear in the material parameters, the same argument yields Fréchet differentiability with respect to the material parameters. Furthermore, the Fréchet derivative of $\mathcal{A}$ is continuous, since it is linear in the respective variable and bounded due to Corollary 4.5. Therefore,

$$\mathcal{A} \in C^1(X \times W, W^*).$$

Moreover, by Corollary 4.5, the operator $\mathcal{A}'_z(p,z)$ is bijective on W for fixed $p \in X$. Hence, by the Bounded Inverse Theorem,

$$\exists C_A\ \forall (p,z) \in X \times W,\ \forall f \in W^* :\ \mathcal{A}'_z(p,z)^{-1} f \ \text{ exists and } \ \left\|\mathcal{A}'_z(p,z)^{-1} f\right\|_W \le C_A \|f\|_{W^*}.$$

Taking the operator norm yields $\mathcal{A}'_z(p,z)^{-1} \in \mathcal{L}(W^*,W)$. Thus, all assumptions of the Implicit Function Theorem are satisfied, yielding the existence of $S \in C^1(X,W)$ as in Definition 4.4. ☐

Due to Theorem 4.6, the parameter-to-state map $S \in C^1(X,W)$ from Definition 4.4 is well-defined, since for a fixed $p \in X$ there cannot exist more than one state $z \in W$. Furthermore, S is nonlinear, which follows directly from the structure of the model operator $\mathcal{A}$ used to define S.

To state and solve an inverse problem we need additional observations. In our case we obtain a measured charge pulse. Therefore, the observation operator reads as

$$\widetilde{\mathcal{C}}(p,z) := \int_{\Gamma_e} \Big(e\mathcal{B}u - \epsilon\nabla\phi_0 - \epsilon\nabla\chi\Big) \cdot n \,\mathrm{d}\Gamma \tag{4.6}$$

which means that the electrodes are conductive and thus the charge is distributed equally on the loaded electrode. As $(u,\phi_0) \in W$ due to Theorem 3.3 and $\chi \in H^1(H^1(\Omega,\mathbb{R}))$, it holds that

$$e\mathcal{B}u \in L^2\big(L^2(\Omega,\mathbb{R}^3)\big), \tag{4.7}$$

$$\epsilon\nabla\phi_0 \in L^2\big(L^2(\Omega,\mathbb{R}^3)\big), \tag{4.8}$$

$$\epsilon\nabla\chi \in H^1\big(L^2(\Omega,\mathbb{R}^3)\big). \tag{4.9}$$

Thus, $\widetilde{\mathcal{C}}$ is not well-defined, as we have a boundary integral with L^2-functions, which in general cannot be evaluated. However, this boundary integral can be converted into a volume integral on an open neighborhood of the boundary Γ_e, denoted by $U_\gamma(\Gamma_e)$ with $\gamma > 0$, if one lacks regularity of the state. This requires the continuous extension of the normal vector in this neighborhood, which is obtained by solving the eikonal equation.

Definition 4.7 (Approximated observation operator)**.** Let $Y := L^2(0,T)$ and $\gamma > 0$ be fixed and sufficiently small. Let b solve

$$\|\nabla b\|_{L^2(U_\gamma(\Gamma_e))} = 1 \qquad \text{in } U_\gamma(\Gamma_e), \tag{4.10}$$

$$b = 0 \qquad \text{on } \partial U_\gamma(\Gamma_e). \tag{4.11}$$

Then the approximated observation operator $\mathcal{C}^\gamma : X \times W \to Y$ is defined by

$$\mathcal{C}^\gamma(p,z) := |\Gamma_e|\,|U_\gamma(\Gamma_e)|^{-1} \int_{U_\gamma(\Gamma_e)} \Big(e\mathcal{B}u - \epsilon\nabla\phi_0 - \epsilon\nabla\chi\Big) \cdot \nabla b \,\mathrm{d}\Omega. \tag{4.12}$$

Proposition 4.8 (Well-definedness and differentiability of the approximated observation)**.** For fixed and sufficiently small $\gamma > 0$, the operator $\mathcal{C}^\gamma : X \times W \to Y$ is well-defined, bounded and continuously Fréchet differentiable with respect to both the state and the parameter. Moreover,

$$\mathcal{C}^\gamma \longrightarrow \widetilde{\mathcal{C}} \qquad \text{as } \gamma \to 0$$

whenever $\widetilde{\mathcal{C}}$ is well-defined.

Proof. Due to [8], the eikonal equation admits a classical solution. Thus, $\nabla b \in L^\infty(U_\gamma(\Gamma_e), \mathbb{R}^3)$. Using and Hölder's inequality, integration over time and Corollary 4.5 together with (4.7)-(4.9) yields that there exists $C_O > 0$ such that

$$\|\mathcal{C}^\gamma(p,z)\|_Y^2 \le C_O\Big(\|u_1\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u_0\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))}\Big). \tag{4.13}$$

Hence $\mathcal{C}^\gamma$ is well-defined and bounded. Next, let $\xi = (\mu,\nu) \in W$. Then, the derivative with respect to the state is given by

$$\mathcal{C}^\gamma_z(p,z)\xi = |\Gamma_e|\,|U_\gamma(\Gamma_e)|^{-1} \int_{U_\gamma(\Gamma_e)} \Big(e\mathcal{B}\mu - \epsilon\nabla\nu\Big) \cdot \nabla b \,\mathrm{d}\Omega. \tag{4.14}$$

Since

$$\mathcal{C}^\gamma(p,z+\xi) - \mathcal{C}^\gamma(p,z) - \mathcal{C}^\gamma_z(p,z)\xi = 0, \tag{4.15}$$

$\mathcal{C}^\gamma$ is Fréchet differentiable with respect to the state. Similarly, for a parameter direction $h = (h_{c^E}, h_e, h_\epsilon) \in X$, one obtains

$$\mathcal{C}^\gamma(p+h,z) - \mathcal{C}^\gamma(p,z) = |\Gamma_e|\,|U_\gamma(\Gamma_e)|^{-1} \int_{U_\gamma(\Gamma_e)} \Big(h_e\mathcal{B}u - h_\epsilon\nabla\phi_0 - h_\epsilon\nabla\chi\Big) \cdot \nabla b \,\mathrm{d}\Omega,$$

implying that $\mathcal{C}^\gamma$ is also Fréchet differentiable with respect to the parameter. Since the operator is affine linear in both variables, continuity of the derivatives follows immediately from inequality (4.13). Therefore $\mathcal{C}^\gamma \in C^1(X \times W, Y)$. Due to [8], the system (4.10)-(4.11) admits a classical solution that provides the required extension of the normal vector in $U_\gamma(\Gamma_e)$. Consequently, $\mathcal{C}^\gamma$ converges to $\widetilde{\mathcal{C}}$ as $\gamma \to 0$, whenever $\widetilde{\mathcal{C}}$ is well-defined. □

Next the requirements for higher order spatial regularity will be analyzed to define the observation operator according to (4.6).

Remark 4.9 (Higher-order Dirichlet lift). If solutions with higher regularity in space are aimed for, we have to perform the Dirichlet lift in higher order Sobolev spaces, where we distinguish the following cases.

1. Assumption D2 holds: we have $\chi \in H^1\left(H^{m+2}(\Omega,\mathbb{R})\right)$, see the trace theorem in [21, Chapter 3].
2. Assumption D3 holds: according to Theorem 4.12 in [21], we have $\chi \in H^1\left(H^m(\Omega,\mathbb{R})\right)$). As ϕ^e is constant in space the compatibility condition is fulfilled.

In both cases $\mathrm{Tr}(\chi(t))$ is defined as in (3.1).

Theorem 4.10 (Higher-order spatial regularity)**.** *Let $m \in \mathbb{N}$, $m \ge 2$ and consider the system* (3.12)-(3.16)*. Suppose that the assumptions of Theorem 3.2 are satisfied and that*

- *either Assumption D2 or Assumption D3 hold, with χ as in Remark 4.9,*
- *e and ϵ are spatially constant and satisfy $e \in H^1\big((0,T),\mathbb{R}^{3\times 6}\big)$, $\epsilon \in H^1\big((0,T),\mathbb{R}^{3\times 3}\big)$,*
- *the initial data satisfy $u_0, u_1 \in H^m(\Omega,\mathbb{R}^3)$,*
- *and the inhomogeneities satisfy $f \in L^2\big(H^m(\Omega,\mathbb{R}^3)^*\big)$, $g \in L^2\big(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}) \cap H^m(\Omega,\mathbb{R})\big)$.*

Then, the unique weak solution (u, ϕ_0) *of the system* (3.12)-(3.16) *satisfies*

$$u \in L^2\big(H^m(\Omega,\mathbb{R}^3)\big), \qquad \dot{u} \in L^2\big(H^m(\Omega,\mathbb{R}^3)\big),$$

$$\ddot{u} \in L^2\big(H^m(\Omega,\mathbb{R}^3)^*\big), \qquad \phi_0 \in L^2\big(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}) \cap H^m(\Omega,\mathbb{R})\big).$$

Proof. We follow the proof of Proposition (3.1) and Theorem 3.2. For this purpose, we introduce $V_m := H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3)$ and equip it with the norm

$$\|u\|^2_{V_m} := \|u\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \sum_{1\le|\lambda|\le m-1} \|\partial^\lambda \mathcal{B}u\|^2_{L^2(\Omega,\mathbb{R}^6)}.$$

Furthermore, we equip $H^1_{0,\Gamma_d}(\Omega,\mathbb{R}) \cap H^m(\Omega,\mathbb{R})$ with the norm

$$\|\phi_0\|^2_{H^1_{0,\Gamma_d}(\Omega,\mathbb{R})\cap H^m(\Omega,\mathbb{R})} := \|\phi_0\|^2_{H^1_{0,\Gamma_d}(\Omega,\mathbb{R})} + \sum_{1\le|\lambda|\le m-1} \|\partial^\lambda \nabla\phi_0\|^2_{L^2(\Omega,\mathbb{R}^3)}.$$

Since Ω is Lipschitz, Korn's inequality implies that $\|\cdot\|_{V_m}$ is equivalent to the standard $H^m(\Omega,\mathbb{R}^3)$-norm. Now consider the linear mapping defined in (3.17). For every multi-index λ with $1 \le |\lambda| \le m-1$, the function $\zeta^t(\partial^\lambda u) \in H^1_{0,\Gamma_d}(\Omega,\mathbb{R})$ is well-defined whenever $\partial^\lambda u \in H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)$. Since e and ϵ are constant in space, spatial derivatives commute with the electrostatic equation. Hence, identity (3.18) yields

$$\int_\Omega (\epsilon\nabla\zeta^t(\partial^\lambda u))^T \nabla w \,\mathrm{d}\Omega = \int_\Omega (e\partial^\lambda \mathcal{B}u)^T \nabla w \,\mathrm{d}\Omega \qquad \forall\, w \in H^1_{0,\Gamma_d}(\Omega,\mathbb{R}). \tag{4.16}$$

Testing identity (4.16) with $\zeta^t(\partial^\lambda u)$ and using coercivity of ϵ, boundedness of e, and Poincaré's inequality, we obtain for almost all $t \in (0,T)$ that $\zeta^t(u) \in H^1_{0,\Gamma_d}(\Omega,\mathbb{R}) \cap H^m(\Omega,\mathbb{R})$ together with a constant $C_\zeta > 0$ independent of t such that

$$\|\zeta^t_m(u)\|_{H^1_{0,\Gamma_d}(\Omega,\mathbb{R})\cap H^m(\Omega,\mathbb{R})} \le C_\zeta \|u\|_{V_m}.$$

Thus we define, for almost all $t \in (0,T)$,

$$\zeta^t_m : H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3) \to H^1_{0,\Gamma_d}(\Omega,\mathbb{R}) \cap H^m(\Omega,\mathbb{R}), \qquad u(t) \mapsto \phi^0_0(t),$$

where $\phi_0(t) = \phi^0_0(t) + \phi^r_0(t) = \zeta^t_m(u(t)) + \phi^r_0(t)$. Since $H^1_{0,\Gamma_d}(\Omega,\mathbb{R}) \cap H^m(\Omega,\mathbb{R}) \subset H^m(\Omega,\mathbb{R})$, we use the higher spatial regularity of χ and g and define $\phi^r_0 \in L^2(H^m(\Omega,\mathbb{R}))$ analogously to the proof of Proposition (3.1), now by solving (3.21) in the higher-order setting. Hence, there exists $C_r > 0$ such that

$$\|\phi^r_0\|_{L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R})\cap H^m(\Omega,\mathbb{R}))} \le C_r\Big(\|\chi\|_{L^2(H^m(\Omega,\mathbb{R}))} + \|g\|_{L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R})\cap H^m(\Omega,\mathbb{R}))}\Big).$$

As we obtain the same weak form (3.23), we define the operators

$$\hat{a}_0(t,u,v) := a_0(t,u,v) + \sum_{1\le|\lambda|\le m-1}\left[\int_\Omega \big(c^E_I \mathcal{B}(\partial^\lambda u)\big)^T \mathcal{B}(\partial^\lambda v)\,\mathrm{d}\Omega + \int_\Omega \big((e_I)^T \nabla\zeta^t(\partial^\lambda u)\big)^T \mathcal{B}(\partial^\lambda v)\,\mathrm{d}\Omega\right],$$

$$\hat{a}_1(t,u,v) := a_1(t,u,v) - \sum_{1\le|\lambda|\le m-1}\left[\int_\Omega \big(c^E_I \mathcal{B}(\partial^\lambda u)\big)^T \mathcal{B}(\partial^\lambda v)\,\mathrm{d}\Omega + \int_\Omega \big((e_I)^T \nabla\zeta^t(\partial^\lambda u)\big)^T \mathcal{B}(\partial^\lambda v)\,\mathrm{d}\Omega\right],$$

$$\hat{b}_0(t,u,v) := b_0(t,u,v) + \sum_{1\le|\lambda|\le m-1}\int_\Omega \big(c^E_I \mathcal{B}(\partial^\lambda u)\big)^T \mathcal{B}(\partial^\lambda v)\,\mathrm{d}\Omega,$$

$$\hat{b}_1(t,u,v) := b_1(t,u,v) - \sum_{1\le|\lambda|\le m-1}\int_\Omega \big(c^E_I \mathcal{B}(\partial^\lambda u)\big)^T \mathcal{B}(\partial^\lambda v)\,\mathrm{d}\Omega,$$

$$\hat{c}(t,u,v) := c(t,u,v), \tag{4.17}$$

where $c^E_I \in \mathbb{R}^{6\times 6}$ and $e_I \in \mathbb{R}^{3\times 6}$ have the same structural form as the material parameters, with all non-zero entries are set to 1. Then $\hat{a}_{01} := \hat{a}_0 + \hat{a}_1$ and $\hat{b}_{01} := \hat{b}_0 + \hat{b}_1$ have the same form as in the proof of

Theorem 3.2. Analogously to the proof of Theorem 3.2 it follows that $\tilde{f} \in L^2(V_m^*)$. Furthermore,

$$\begin{aligned}\int_\Omega \big((e(t))^T \nabla\zeta^t(\partial^\lambda u)\big)^T \mathcal{B}(\partial^\lambda v)\,\mathrm{d}\Omega &= \int_\Omega \big(\epsilon(t)\nabla\zeta^t(\partial^\lambda u)\big)^T \nabla\zeta^t(\partial^\lambda v)\,\mathrm{d}\Omega \\ &= \int_\Omega \big((e(t))^T \nabla\zeta^t(\partial^\lambda v)\big)^T \mathcal{B}(\partial^\lambda u)\,\mathrm{d}\Omega \end{aligned} \tag{4.18}$$

and a_0 and b_0 are Hermitian implies that $\hat{a}_0$ and $\hat{b}_0$ are Hermitian as well. With the same $\sigma \in \mathbb{R}^+$ defined in (3.30) we obtain analogously

$$\begin{aligned}\hat{a}_0(t,u,u) &= a_0(t,u,u) \\ &\quad + \sum_{1\le|\lambda|\le m-1}\left[\int_\Omega \big(c_I^E \mathcal{B}(\partial^\lambda u)\big)^T \mathcal{B}(\partial^\lambda u)\,\mathrm{d}\Omega + \int_\Omega \big((e_I)^T \nabla\zeta^t(\partial^\lambda u)\big)^T \mathcal{B}(\partial^\lambda u)\,\mathrm{d}\Omega\right] \\ &\quad \ge \sigma\|u\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \sigma \sum_{1\le|\lambda|\le m-1} \|\mathcal{B}(\partial^\lambda u)\|^2_{L^2(\Omega,\mathbb{R}^6)} \ge \sigma\|u\|^2_{V_m}.\end{aligned}$$

Thus $\hat{a}_0$ is coercive on V_m. In the same way, using (3.34), we obtain

$$\hat{b}_0(t,u,u) \ge \beta_0\|u\|^2_{V_m} \qquad \forall\, u \in V_m.$$

Continuous differentiability of the mappings $t \mapsto \hat{a}_0(t,u,v)$ and $t \mapsto \hat{c}(t,u,v)$, boundedness $\hat{a}_1$ and $\hat{b}_1$ on $V_m \times V_m$, as well as continuity of the mappings $t \mapsto \hat{a}_1(t,u,v)$ and $t \mapsto \hat{b}_1(t,u,v)$ follow from the proof of Theorem 3.2. Finally, $\hat{c} = c$ has the same properties as in the proof of Theorem 3.2. Therefore all assumptions of Theorem 1, Paragraph 5, Chapter XVIII in [7] are satisfied on the space V_m. Hence there exists a unique solution (u, ϕ_0) to the system (3.12)-(3.16) satisfying

$$\begin{gathered} u \in L^2\big(H^m(\Omega,\mathbb{R}^3)\big), \quad \dot{u} \in L^2\big(H^m(\Omega,\mathbb{R}^3)\big), \quad \ddot{u} \in L^2\big(H^m(\Omega,\mathbb{R}^3)^*\big) \\ \phi_0 \in L^2\big(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}) \cap H^m(\Omega,\mathbb{R})\big)\end{gathered}$$

□

Corollary 4.11 (Observation operator)**.** Let $Y := L^2(0,T)$ and the assumptions of Theorem 4.10 hold. Then, the observation operator

$$\widetilde{\mathcal{C}} : X \times W_m \to Y, \qquad \widetilde{\mathcal{C}}(p,z) := \int_{\Gamma_e} \Big(e\mathcal{B}u - \epsilon\nabla\phi_0 - \epsilon\nabla\chi\Big)\cdot n\,\mathrm{d}\Gamma,$$

is well-defined and bounded. Moreover,

$$\widetilde{\mathcal{C}} \in C^1(X \times W_m, Y).$$

Proof. Since $u \in L^2\big(H^m(\Omega,\mathbb{R}^3)\big)$, $\phi_0 \in L^2\big(H^m(\Omega,\mathbb{R})\big)$ and $\chi \in L^2\big(H^m(\Omega,\mathbb{R})\big)$. it follows that

$$\mathcal{B}u,\ \nabla\phi_0,\ \nabla\chi \in L^2\big(H^{m-1}(\Omega)\big).$$

This proves well-definedness. Boundedness and continuous Fréchet differentiability of $\mathcal{C}$ follows analogously as for $\mathcal{C}^\gamma$. □

In real world application the assumptions of Theorem 4.10 with at least $m = 2$ are usually fulfilled. From now on, if the assumptions of Theorem 4.10 with at least $m = 2$ are not fulfilled, we fix a sufficiently small $\gamma > 0$ and abbreviate $\mathcal{C} = \mathcal{C}^\gamma$, otherwise we use $\mathcal{C} = \widetilde{\mathcal{C}}$ and adapt X and W according to the given higher regularities.

Definition 4.12 (Forward operator)**.** The forward operator $F : X \to Y$ is defined by

$$F(p) := \mathcal{C}\big(p, S(p)\big).$$

This reduces the inverse problem to a single operator equation for the unknown parameter p. Hence the forward operator F acts directly on p and returns the data y.

Theorem 4.13 (Properties of the forward operator)**.** *Let $p_n = (c_n^E, e_n, \epsilon_n) \in X$. Assume either*

(i) $\mathcal{C} = \mathcal{C}^\gamma$ for some fixed and sufficiently small $\gamma > 0$, or

(ii) $\mathcal{C} = \widetilde{\mathcal{C}}$, when the assumptions of Corollary 4.11 are satisfied.

Then the forward operator of Definition 4.12 is weak-to-strong continuous, i.e., if $p_n \rightharpoonup p$ weakly in X, then $F(p_n) \to F(p)$ strongly in Y. Furthermore, $F \in C^1(X,Y)$ is well-defined and nonlinear.

Proof. Let $p_n \rightharpoonup p = (c^E, e, \epsilon)$ weakly in X. Since $\{p_n\}$ is bounded in X, compactness of the embedding $H^3(\Omega) \hookrightarrow C^1(\bar{\Omega})$ and Corollary 4 in [29] implies

$$c_n^E \to c^E, \qquad e_n \to e, \qquad \epsilon_n \to \epsilon$$

in the respective C^1-space. Now set $z_n := S(p_n) = (u_n, \phi_{0,n}),\ z := S(p) = (u, \phi_0)$. Since $\mathcal{A}(p_n, z_n) = 0$ and $\mathcal{A}(p,z) = 0$, as well as $\mathcal{A}$ is affine linear with respect to the state variable, one obtains

$$\mathcal{A}'_z(p_n, z_n)(z_n - z) = -(\mathcal{A}(p_n, z) - \mathcal{A}(p,z)).$$

Applying condition (4.3) yields

$$\|z_n - z\|_W \le C_A \|\mathcal{A}(p_n, z) - \mathcal{A}(p,z)\|_{W^*}.$$

Let $(v,w) \in W$ be arbitrary. Using Definition 4.3 we deduce

$$\begin{aligned}
&\left|\langle \mathcal{A}(p_n, z) - \mathcal{A}(p,z), (v,w)\rangle_{W^*,W}\right| \\
&\le \int_0^T \int_\Omega \left| \Big((c_n^E - c^E)\mathcal{B}u + \beta(c_n^E - c^E)\mathcal{B}\dot{u} + ((e_n - e))^T \nabla\phi_0 \Big)^T \mathcal{B}v \right| \,\mathrm{d}\Omega\,\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \left| \Big((e_n - e)\mathcal{B}u - (\epsilon_n - \epsilon)\nabla\phi_0 - (\epsilon_n - \epsilon)\nabla\chi \Big)^T \nabla w \right| \,\mathrm{d}\Omega\,\mathrm{d}t.
\end{aligned}$$

By Hölder's inequality and the strong convergence of p_n to p in the respective L^∞-space, we obtain

$$\|\mathcal{A}(p_n, z) - \mathcal{A}(p,z)\|_{W^*} \to 0,$$

implying $z_n \to z$ strongly in W. Furthermore, observe

$$F(p_n) - F(p) = \mathcal{C}(p_n, z_n) - \mathcal{C}(p,z) = \big(\mathcal{C}(p_n, z_n) - \mathcal{C}(p_n, z)\big) + \big(\mathcal{C}(p_n, z) - \mathcal{C}(p,z)\big).$$

For the first term, boundedness of $\mathcal{C}_z$ yields

$$\|\mathcal{C}(p_n, z_n) - \mathcal{C}(p_n, z)\|_Y \le c_1 \|z_n - z\|_W \to 0$$

for some constant $c_1 > 0$. For the second term, if $\mathcal{C} = \mathcal{C}^\gamma$, then Proposition 4.8 and the strong convergence of p_n imply

$$\|\mathcal{C}^\gamma(p_n, z) - \mathcal{C}^\gamma(p,z)\|_Y \to 0,$$

whereas if $\mathcal{C} = \widetilde{\mathcal{C}}$, then Corollary 4.11 and the strong convergence of p_n implies

$$\|\widetilde{\mathcal{C}}(p_n, z) - \widetilde{\mathcal{C}}(p,z)\|_Y \to 0.$$

Thus in either case, $F(p_n) \to F(p)$ strongly in Y. The last claim follows from the chain rule on Banach spaces and Theorem 4.6, Proposition 4.8, Corollary 4.11. □

By denoting the noisy measurements with y^δ and introducing a weakly lower semi-continuous regularizer $\mathcal{R}_\mu : X \to \mathbb{R}$ with an regularization parameter $\mu > 0$, we now define the regularized target functional.

Definition 4.14 (Regularized target functional)**.** The regularized target functional $\mathcal{J} : X \to \mathbb{R}$ is defined by

$$\mathcal{J}(p) := \frac{1}{2}\left\|F(p) - y^\delta\right\|^2_{L^2(0,T)} + \mathcal{R}_\mu(p). \tag{4.19}$$

The inverse problem then consists in finding a minimizer of

$$\min_{p \in X} \mathcal{J}(p). \tag{4.20}$$

Corollary 4.15 (Existence of minimizers)**.** Let the assumptions of Theorem 4.13 are satisfied. If $\mathcal{R}_\mu$ is weakly lower semi-continuous on X, then the minimization problem (4.20) admits at least one minimizer.

Proof. First, we directly obtain that X is is non-empty, convex and closed. Second, by Theorem 4.13, F is weak-to-strong continuous. Let $p_n \rightharpoonup p$ weakly in X. Since F is weak-to-strong continuous and $y^\delta \in Y$ is fixed, we obtain

$$F(p_n) - y^\delta \to F(p) - y^\delta \qquad \text{strongly in } Y.$$

Since the norm is weakly lower semi-continuous, it holds that

$$\|F(p) - y^\delta\|_Y^2 \le \liminf_{n\to\infty} \|F(p_n) - y^\delta\|_Y^2.$$

Therefore, $\|F(p) - y^\delta\|_Y^2$ is weakly lower semi-continuous. As the sum of weakly lower semi-continuous functions is weakly lower semi-continuous, we obtain weakly lower semi-continuity of $\mathcal{J}$. Hence, existence of a minimizer to the optimization problem (4.20) is guaranteed by Tonelli's Theorem, see [6, Theorem 11.10]. ☐

In the parametrization approach (2.2), the admissible set X consists only of constant real-valued matrices with the same structural properties as before. Hence, X remains bounded, convex and closed, and is a subset of a finite-dimensional real vector space.
Next, we derive the first-order optimality conditions. For this purpose, we assume from now on that $\mathcal{R}_\mu \in C^1(X, \mathbb{R})$, and denote $p^\dagger \in X$ as a minimizer of $\mathcal{J}$. Since $\mathcal{J}$ is continuously Fréchet differentiable and $p^\dagger$ is a minimizer, we obtain the following first-order optimality conditions:

- $\mathcal{A}(p^\dagger, S(p^\dagger)) = 0$ (state equation),
- $\mathcal{A}'_z(p^\dagger, S(p^\dagger))^* q = -C'_z(p^\dagger, S(p^\dagger))^* \big(C(p^\dagger, S(p^\dagger)) - y^\delta\big)$ (adjoint equation),

where the superscript * denotes the adjoint operator, and $q = (q_1, q_2) \in W$ denotes the adjoint state. In general, the first-order condition for the parameter is

$$\mathcal{J}'(p^\dagger)(\bar{p} - p^\dagger) \ge 0 \quad \forall \bar{p} \in X,$$

which transforms to $\mathcal{J}'(p^\dagger)(\bar{p} - p^\dagger) = 0$ for all $\bar{p} \in X$, if $p^\dagger \in \operatorname{int}(X)$. Moreover, for a direction $h = \bar{p} - p^\dagger = (h_{c^E}, h_e, h_\epsilon),\ \bar{p} \in X$, we have

$$\mathcal{J}'(p^\dagger)h = \big\langle \mathcal{A}'_p(p^\dagger, S(p^\dagger))h, q\big\rangle_{W^*,W} + \big(C'_p(p^\dagger, S(p^\dagger))h, C(p^\dagger, S(p^\dagger)) - y^\delta\big)_Y + \mathcal{R}'_\mu(p^\dagger)h, \tag{4.21}$$

where

$$\begin{aligned}\big\langle \mathcal{A}'_p(p^\dagger, S(p^\dagger))h, q\big\rangle_{W^*,W} &= \int_I \int_\Omega \Big(h_{c^E}\mathcal{B}u + \beta h_{c^E}\mathcal{B}\dot{u} + h_e^T \nabla\phi_0\Big)^T \mathcal{B}q_1 \\ &\quad + \Big(h_e\mathcal{B}u - h_\epsilon\nabla\phi_0\Big)^T \nabla q_2 + \Big(h_e^T\nabla\chi\Big)^T \mathcal{B}q_1 - \Big(h_\epsilon\nabla\chi\Big)^T \nabla q_2 \,\mathrm{d}\Omega\,\mathrm{d}t\,.\end{aligned}$$

If $C = \widetilde{C}$, then

$$\begin{aligned}\big(C'_p(p^\dagger, S(p^\dagger))h,\ C(p^\dagger, S(p^\dagger)) - y^\delta\big)_Y &= \int_I \bigg(\int_{\Gamma_e} \Big(h_e\mathcal{B}u - h_\epsilon\nabla\phi_0 - h_\epsilon\nabla\chi\Big)\cdot n\,\mathrm{d}\Gamma\bigg) \\ &\qquad \cdot \bigg(\int_{\Gamma_e} \Big(e\mathcal{B}u - \epsilon\nabla\phi_0 - \epsilon\nabla\chi\Big)\cdot n\,\mathrm{d}\Gamma - y^\delta\bigg)\,\mathrm{d}t\,.\end{aligned}$$

If $C = C^\gamma$, then

$$\begin{aligned}&\big(C'_p(p^\dagger, S(p^\dagger))h,\ C(p^\dagger, S(p^\dagger)) - y^\delta\big)_Y \\ &= \int_I |\Gamma_e|^2\,|U_\gamma(\Gamma_e)|^{-2} \int_{U_\gamma(\Gamma_e)} \Big(h_e\mathcal{B}u - h_\epsilon\nabla\phi_0 - h_\epsilon\nabla\chi\Big)\cdot\nabla b\,\mathrm{d}\Omega \\ &\quad \cdot \bigg(\int_{U_\gamma(\Gamma_e)} \Big(e\mathcal{B}u - \epsilon\nabla\phi_0 - \epsilon\nabla\chi\Big)\cdot\nabla b\,\mathrm{d}\Omega - |\Gamma_e|^{-1}|U_\gamma(\Gamma_e)|\,y^\delta\bigg)\,\mathrm{d}t\,,\end{aligned}$$

where b solves (4.10)-(4.11). Hence, the adjoint state is essential, which can be seen as revealing the influence of a cause on a target functional. To derive the adjoint PDE system of the piezoelectric dynamical system (3.2)-(3.7), we differentiate the model operator, see Definition 4.3, with respect to the state. Similarly to (4.4), we consider an arbitrary direction $\kappa := (d, \psi) \in W$ with d satisfying the initial conditions

$d(0) = \dot{d}(0) = 0$. Then,

$$\langle \mathcal{A}_z'(p,z)\kappa,(v,w)\rangle_{W^*,W} = \int_I \int_\Omega \rho \ddot{d}^T v + \alpha\rho\dot{d}^T v + \left(c^E \mathcal{B} d + \beta c^E \mathcal{B}\dot{d} + e^T \nabla\psi\right)^T \mathcal{B} v + (e\mathcal{B} d - \epsilon\nabla\psi)^T \nabla w \;\mathrm{d}\Omega\;\mathrm{d}t\,.$$

Denoting the adjoint state by $q = (q_1, q_2) \in W$, we obtain

$$\langle \mathcal{A}_z'(p,z)\kappa, q\rangle_{W^*,W} = \int_I \int_\Omega \rho \ddot{d}^T q_1 + \alpha\rho\dot{d}^T q_1 + \left(c^E \mathcal{B} d + \beta c^E \mathcal{B}\dot{d} + e^T \nabla\psi\right)^T \mathcal{B} q_1 + (e\mathcal{B} d - \epsilon\nabla\psi)^T \nabla q_2 \;\mathrm{d}\Omega\;\mathrm{d}t\,. \tag{4.22}$$

We consider every single term individually, with the terminal conditions $q_1(T) = 0$ and $\dot{q}_1(T) = 0$ in Ω. Let $q_{1\rho} := \rho q_1$, i.e. $q_{1\rho}(T) = \rho q_1(T) = 0,\ \dot{q}_{1\rho}(T) = \dot{\rho}\, q_1(T) + \rho\dot{q}_1(T) = 0$. Then, for the first term in (4.22), we have

$$\int_I \int_\Omega \rho \ddot{d}^T q_1 \;\mathrm{d}\Omega\;\mathrm{d}t = \int_I \int_\Omega \ddot{d}^T q_{1\rho} \;\mathrm{d}\Omega\;\mathrm{d}t = -\int_I \int_\Omega \dot{d}^T \dot{q}_{1\rho} \;\mathrm{d}\Omega\;\mathrm{d}t = \int_I \int_\Omega d^T \ddot{q}_{1\rho} \;\mathrm{d}\Omega\;\mathrm{d}t.$$

For the second term in (4.22), define $q_{1\alpha\rho} := \alpha\rho q_1$, i.e. $q_{1\alpha\rho}(T) = \alpha\rho q_1(T) = 0$. Then

$$\int_I \int_\Omega \alpha\rho\, \dot{d}^T q_1 \;\mathrm{d}\Omega\;\mathrm{d}t = \int_I \int_\Omega \dot{d}^T q_{1\alpha\rho} \;\mathrm{d}\Omega\;\mathrm{d}t = -\int_I \int_\Omega d^T \dot{q}_{1\alpha\rho} \;\mathrm{d}\Omega\;\mathrm{d}t.$$

For the third term in (4.22), using the symmetry of c^E, we obtain

$$\int_I \int_\Omega \left(c^E \mathcal{B} d\right)^T \mathcal{B} q_1 \;\mathrm{d}\Omega\;\mathrm{d}t = -\int_I \int_\Omega \mathcal{B}^T \left(c^E \mathcal{B} q_1\right)^T d \;\mathrm{d}\Omega\;\mathrm{d}t + \int_I \int_{\partial\Omega} \mathcal{N}^T \left(c^E \mathcal{B} q_1\right)^T d \;\mathrm{d}\Gamma\;\mathrm{d}t.$$

For the fourth term in (4.22), let $q_{\mathcal{B}} := \beta c^E \mathcal{B} q_1$. Then

$$\begin{aligned}\int_I \int_\Omega \left(\beta c^E \mathcal{B}\dot{d}\right)^T \mathcal{B} q_1 \;\mathrm{d}\Omega\;\mathrm{d}t &= \int_I \int_\Omega \left(\mathcal{B}\dot{d}\right)^T q_{\mathcal{B}} \;\mathrm{d}\Omega\;\mathrm{d}t = -\int_I \int_\Omega (\mathcal{B} d)^T \dot{q}_{\mathcal{B}} \;\mathrm{d}\Omega\;\mathrm{d}t \\ &= \int_I \int_\Omega d^T \mathcal{B}^T \dot{q}_{\mathcal{B}} \;\mathrm{d}\Omega\;\mathrm{d}t - \int_I \int_{\partial\Omega} d^T \mathcal{N}^T \dot{q}_{\mathcal{B}} \;\mathrm{d}\Gamma\;\mathrm{d}t.\end{aligned}$$

Finally, the remaining three terms in (4.22) yield

$$\begin{aligned}\int_I \int_\Omega \left(e^T \nabla\psi\right)^T \mathcal{B} q_1 \;\mathrm{d}\Omega\;\mathrm{d}t &= -\int_I \int_\Omega \nabla\cdot(e\mathcal{B} q_1)\,\psi \;\mathrm{d}\Omega\;\mathrm{d}t + \int_I \int_{\partial\Omega} n\cdot(e\mathcal{B} q_1)\,\psi \;\mathrm{d}\Gamma\;\mathrm{d}t, \\ \int_I \int_\Omega (e\mathcal{B} d)^T \nabla q_2 \;\mathrm{d}\Omega\;\mathrm{d}t &= -\int_I \int_\Omega d^T \mathcal{B}^T \left(e^T \nabla q_2\right) \;\mathrm{d}\Omega\;\mathrm{d}t + \int_I \int_{\partial\Omega} d^T \mathcal{N}^T \left(e^T \nabla q_2\right) \;\mathrm{d}\Gamma\;\mathrm{d}t, \\ \int_I \int_\Omega -(\epsilon\nabla\psi)^T \nabla q_2 \;\mathrm{d}\Omega\;\mathrm{d}t &= \int_I \int_\Omega \nabla\cdot\left((\epsilon)^T \nabla q_2\right)\psi \;\mathrm{d}\Omega\;\mathrm{d}t - \int_I \int_{\partial\Omega} n\cdot\left((\epsilon)^T \nabla q_2\right)\psi \;\mathrm{d}\Gamma\;\mathrm{d}t.\end{aligned}$$

As we deal with terminal conditions, we perform the time transformation $t \mapsto T - t$ and define $\widetilde{q}(t) := q(T-t)$. Then $\dot{\widetilde{q}}(t) = -\dot{q}(T-t)$ and $\ddot{\widetilde{q}}(t) = \ddot{q}(T-t)$. Moreover, $\dot{q}_{1\alpha\rho} = (\dot{\alpha}\,\rho + \alpha\dot{\rho})\, q_1 + \alpha\rho\,\dot{q}_1,\ \ddot{q}_{1\rho} = \ddot{\rho}\, q_1 + 2\dot{\rho}\,\dot{q}_1 + \rho\,\ddot{q}_1$, and $\dot{q}_{\mathcal{B}} = \left(\dot{\beta}\, c^E + \beta\dot{c}^E\right)\mathcal{B} q_1 + \beta c^E \mathcal{B}\dot{q}_1$. Furthermore, we introduce

$$\begin{aligned}r_u(p, S(p)) &:= C_u'(p, S(p))^* \big(C(p, S(p)) - y^\delta\big), \\ r_{\phi_0}(p, S(p)) &:= C_{\phi_0}'(p, S(p))^* \big(C(p, S(p)) - y^\delta\big).\end{aligned}$$

Writing $\widetilde{p}(t) := p(T-t)$, we obtain the time-transformed adjoint PDE

$$\widetilde{\rho}\,\ddot{\widetilde{q}}_1 + \left(2\dot{\widetilde{\rho}} + \widetilde{\alpha}\widetilde{\rho}\right)\dot{\widetilde{q}}_1 + \left(\ddot{\widetilde{\rho}} + \dot{\widetilde{\alpha}}\widetilde{\rho} + \widetilde{\alpha}\dot{\widetilde{\rho}}\right)\widetilde{q}_1$$

$$-\mathcal{B}^T\left(\left(\left(\dot{\widetilde{\beta}}+1\right)\widetilde{c}^E + \widetilde{\beta}\dot{\widetilde{c}}^E\right)\mathcal{B}\widetilde{q}_1 + \widetilde{\beta}\widetilde{c}^E\mathcal{B}\dot{\widetilde{q}}_1 + (\widetilde{e})^T\nabla\widetilde{q}_2\right) = r_u(\widetilde{p}, S(\widetilde{p})) \quad \text{in } \Omega\times(0,T), \tag{4.23}$$

$$-\nabla\cdot\left(\widetilde{e}\mathcal{B}\widetilde{q}_1 - (\widetilde{\epsilon})^T\nabla\widetilde{q}_2\right) = r_{\phi_0}(\widetilde{p}, S(\widetilde{p})) \quad \text{in } \Omega\times(0,T), \tag{4.24}$$

$$n\cdot\left(\widetilde{e}\mathcal{B}\widetilde{q}_1 - (\widetilde{\epsilon})^T\nabla\widetilde{q}_2\right) = 0 \quad \text{on } \Gamma_n\times(0,T), \tag{4.25}$$

$$\mathcal{N}^T\left(\left(\left(\dot{\widetilde{\beta}}+1\right)\widetilde{c}^E + \widetilde{\beta}\dot{\widetilde{c}}^E\right)\mathcal{B}\widetilde{q}_1 + \widetilde{\beta}\widetilde{c}^E\mathcal{B}\dot{\widetilde{q}}_1 + (\widetilde{e})^T\nabla\widetilde{q}_2\right) = 0 \quad \text{on } \partial\Omega\times(0,T), \tag{4.26}$$

$$\widetilde{q}_1(0) = 0 \quad \text{in } \Omega, \tag{4.27}$$

$$\dot{\widetilde{q}}_1(0) = 0 \quad \text{in } \Omega. \tag{4.28}$$

Corollary 4.16 (Existence and uniqueness of the adjoint system)**.** Let the Assumptions 2.2 hold, but with $\rho\in H^4\big(L^\infty(\Omega,\mathbb{R})\big)$, and $\alpha,\beta\in H^3\big(L^\infty(\Omega,\mathbb{R})\big)$. Suppose that there exists $c_\star^{\mathrm{ad}}>0$ such that

$$\xi^T\big((1-\dot{\beta})c^E - \beta\dot{c}^E\big)\xi \ge c_\star^{\mathrm{ad}}|\xi|^2 \qquad \forall\xi\in\mathbb{R}^6$$

for almost all $(t,x)\in(0,T)\times\Omega$. Assume, in addition, that one of the following two conditions is satisfied:

(i) $\alpha\rho - 2\dot{\rho}\ge 0$, and $\ddot{\rho} - \dot{\alpha}\,\rho - \alpha\dot{\rho}\ge 0$ a.e. in $(0,T)\times\Omega$.
(ii) ρ is constant in time and $\dot{\alpha}\le 0$ a.e. in $(0,T)\times\Omega$.

Then there exists a unique weak solution $(\widetilde{q}_1,\widetilde{q}_2)\in W$ with

$$\dot{\widetilde{q}}_1 \in L^2\big(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)\big)\cap L^\infty\big(L^2(\Omega,\mathbb{R}^3)\big), \qquad \ddot{\widetilde{q}}_1\in L^2\big(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*\big)$$

to the system (4.23)–(4.28) with the more general right-hand side

$$(\widetilde{f},\widetilde{g})\in L^2\big(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*\big)\times L^2\big(L^2(\Omega,\mathbb{R})\big).$$

Furthermore, there exists a constant $C_a>0$ such that

$$\begin{aligned}\|\ddot{\widetilde{q}}_1\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*)} + \|\dot{\widetilde{q}}_1\|^2_{L^\infty(L^2(\Omega,\mathbb{R}^3))} + \|\widetilde{q}_1\|^2_{L^\infty(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\dot{\widetilde{q}}_1\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))}\\ + \|\widetilde{q}_2\|^2_{L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}))} \le C_a\left(\|\widetilde{f}\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*)} + \|\widetilde{g}\|^2_{L^2(L^2(\Omega,\mathbb{R}))}\right).\end{aligned} \tag{4.29}$$

Proof. We rewrite the transformed adjoint system (4.23)–(4.28) in the form

$$\rho_{\mathrm{ad}}\ddot{\widetilde{q}}_1 + \alpha_{\mathrm{ad}}\rho_{\mathrm{ad}}\dot{\widetilde{q}}_1 + a_{\mathrm{ad}}\widetilde{q}_1 - \mathcal{B}^T\left(C_{0,\mathrm{ad}}\mathcal{B}\widetilde{q}_1 + C_{1,\mathrm{ad}}\mathcal{B}\dot{\widetilde{q}}_1 + e_{\mathrm{ad}}^T\nabla\widetilde{q}_2\right) = \widetilde{f},$$
$$-\nabla\cdot\left(e_{\mathrm{ad}}\mathcal{B}\widetilde{q}_1 - \epsilon_{\mathrm{ad}}\nabla\widetilde{q}_2\right) = \widetilde{g},$$

where we set

$$\rho_{\mathrm{ad}} := \widetilde{\rho}, \qquad C_{0,\mathrm{ad}} := \left(\left(\dot{\widetilde{\beta}}+1\right)\widetilde{c}^E + \widetilde{\beta}\dot{\widetilde{c}}^E\right), \qquad C_{1,\mathrm{ad}} := \widetilde{\beta}\,\widetilde{c}^E,$$
$$e_{\mathrm{ad}} := \widetilde{e}, \qquad \epsilon_{\mathrm{ad}} := \widetilde{\epsilon}, \qquad a_{\mathrm{ad}} := \ddot{\widetilde{\rho}} + \dot{\widetilde{\alpha}}\widetilde{\rho} + \widetilde{\alpha}\dot{\widetilde{\rho}},$$

and, since $\widetilde{\rho}$ is uniformly positive,

$$\alpha_{\mathrm{ad}} := \frac{2\dot{\widetilde{\rho}} + \widetilde{\alpha}\widetilde{\rho}}{\widetilde{\rho}}.$$

Since $\rho\in H^4(L^\infty(\Omega,\mathbb{R}))$ is positive and uniformly bounded, it follows that $\rho_{\mathrm{ad}}\in H^4(L^\infty(\Omega,\mathbb{R}))$ is positive, uniformly bounded and three times continuously differentiable in time. As $\widetilde{\rho}$ is bounded away from zero α_{ad} is well-defined. Moreover, $\alpha\in H^3(L^\infty(\Omega,\mathbb{R}))$ is non-negative and uniformly bounded. Thus, $\widetilde{\alpha}\in H^3(L^\infty(\Omega,\mathbb{R}))$ is non-negative, uniformly bounded and two times continuously differentiable in time. If condition (i) holds, then

$$2\dot{\widetilde{\rho}} + \widetilde{\alpha}\widetilde{\rho} = (\alpha\rho - 2\dot{\rho})(T-t)\ge 0$$

and

$$a_{\mathrm{ad}} = \ddot{\widetilde{\rho}} + \dot{\widetilde{\alpha}}\widetilde{\rho} + \widetilde{\alpha}\dot{\widetilde{\rho}} = (\ddot{\rho} - \dot{\alpha}\,\rho - \alpha\dot{\rho})(T-t)\ge 0.$$

If condition (ii) holds, then ρ is constant in time, hence $\dot{\rho}=\ddot{\rho}=0$, and therefore

$$2\dot{\widetilde{\rho}} + \widetilde{\alpha}\widetilde{\rho} = \widetilde{\alpha}\widetilde{\rho}\ge 0.$$

Moreover,

$$a_{\mathrm{ad}} = \ddot{\widetilde{\rho}} + \dot{\widetilde{\alpha}}\widetilde{\rho} + \widetilde{\alpha}\dot{\widetilde{\rho}} = \dot{\widetilde{\alpha}}\widetilde{\rho} = -\dot{\alpha}(T-t)\rho \geq 0.$$

Thus, in either case,

$$2\dot{\widetilde{\rho}} + \widetilde{\alpha}\widetilde{\rho} \in H^3(L^\infty(\Omega,\mathbb{R}))$$

is non-negative and uniformly bounded, and $a_{\mathrm{ad}} \in H^2(L^\infty(\Omega,\mathbb{R}))$ is non-negative and uniformly bounded. Consequently,

$$\alpha_{\mathrm{ad}} = \frac{2\dot{\widetilde{\rho}} + \widetilde{\alpha}\widetilde{\rho}}{\widetilde{\rho}} \in H^3(L^\infty(\Omega,\mathbb{R}))$$

is non-negative and uniformly bounded. Next, $\widetilde{c}^E, \widetilde{e}, \widetilde{\epsilon}$ have the same regularity as c^E, e, ϵ. Thus, $C_{0,\mathrm{ad}}$ and $C_{1,\mathrm{ad}}$ are uniformly positive definite. Furthermore, $e_{\mathrm{ad}} = \widetilde{e}$ and $\epsilon_{\mathrm{ad}} = \widetilde{\epsilon}$ have the same structure as the material parameters and ϵ_{ad} is uniformly positive definite. Therefore, all coefficients of the adjoint system have the same regularity and coercivity properties as in those in Theorem 3.2. Moreover, the initial conditions are homogeneous, the right-hand side $(\widetilde{f}, \widetilde{g})$ has the same regularity as in Theorem 3.2, and the Dirichlet boundary is homogeneous, i.e. $\chi \equiv 0$. Consequently, Theorem 3.2 is applicable, yielding existence and uniqueness of a weak solution $(\widetilde{q}_1, \widetilde{q}_2) \in W$. By Theorem 3.3 then gives the stated bound. □

Our adjoint system (4.23)-(4.28) has a right-hand side with the same regularity in time as in Corollary 4.16. Hence, under the assumptions stated above, Corollary 4.16 yields a unique adjoint state for every given state $S(p)$. In this sense, the adjoint state exists uniquely for each parameter p and each state $S(p)$ and not only for a Lagrange multiplier associated to a minimizer. Nevertheless, at a local minimizer $p^\dagger$, an additional interpretation can be made. Since

$$\mathcal{A}'_z(p^\dagger, S(p^\dagger)) : W \to W^*$$

is an isomorphism, its adjoint is an isomorphism as well. Consequently, there exists a unique adjoint state $q = (q_1, q_2) \in W$ satisfying

$$\mathcal{A}'_z(p^\dagger, S(p^\dagger))^* q = -\, C'_z(p^\dagger, S(p^\dagger))^* \big(C(p^\dagger, S(p^\dagger)) - y^\delta\big). \tag{4.30}$$

5. A NUMERICAL EXAMPLE

To computationally solve the inverse problem, we follow the discretize-then-optimize approach where we discretize the problem setting, i.e., the forward operator and the corresponding spaces first and optimize afterwards. Here, easy access to the first derivative of the forward operator F is provided by algorithmic differentiation (AD), see [13]. The central concept is that the computation of a discretized operator can be decomposed into a finite sequence of elementary operations, where then the chain rule is applied systematically. The reverse mode of AD can be seen as a discrete analogue of the continuous adjoint PDE enabling an efficient gradient calculation. The analysis of the continuous adjoint system ensures that, with appropriate discretization, the discrete adjoint state converges to the continuous adjoint state as the discretization gets finer. Hence, the analysis of the continuous problem (4.23) - (4.28) is, an important prerequisite for the discretize-then-optimize approach. For the space discretization we use a classic finite element method (FEM) with an element size of $150\mu m$ and continuous Galerkin elements of polynomial order 3, implemented by the finite element tool FEniCS [2] in dolfin version 2019.2.0.dev0, using AD via the dolfin adjoint [26] library of FEniCS in version 2019.1.0. For the temporal discretization, we employ the Crank–Nicolson scheme, where Δt is the time step size, which will be chosen in the numerical realization as 10^{-6} and n denotes the current step up to $N = 1000$.

As geometry we consider a piezoelectric ring, with outer radius of 6.35 mm, inner radius of 2.6 mm and thickness of 1mm. Hence, the geometry is rotationally symmetric. To reduce the computational effort we exploit the inherent rotational symmetry and transform the ring into a rectangular domain by adopting cylindrical coordinates rather than Cartesian coordinates, where the z-axis is selected as the axis of rotation. In this coordinate system, the piezoelectric ring is assumed to be a homogeneous and transversely isotropic material. The latter is physically essential to exploit the rotational symmetry. In addition, we have converted the setting from seconds to milliseconds, which results in a better condition number of the PDE system, as the magnitudes of the material parameters differ significantly less.

As an example for the numerical realization of the inverse problem, we assume that the elasticity parameter and the permittivity parameter are constant, i.e., parameterized in a polynomial way as in identity (2.2), with

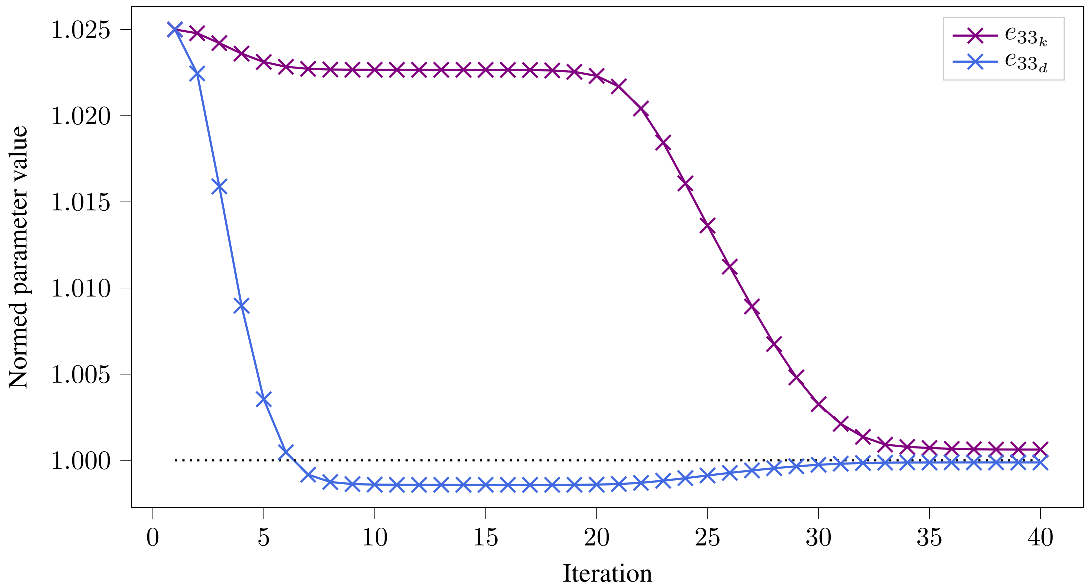

FIGURE 1. Identification of e_{33_k} and e_{33_d}.

a polynomial order 0 and the coupling parameter e is parameterized as in identity (2.2), with a polynomial order 1, where $\theta(t) := 25 + 7\sqrt{(0.01t)}$. Since the problem of identifying the material parameters is extremely challenging due to very different orders of sensitivities even in the frequency dependent case, see e.g., [16], [18], [20], [28], we want to reconstruct one entry of the coupling parameter e, namely e_{33}. To simulate the data we started with the following set of material parameters

$$\begin{aligned} &c_{11}^E = 151400, \quad c_{12}^E = 132700, \quad c_{13}^E = 83600, \quad c_{33}^E = 128800, \quad c_{44}^E = 25900, \quad \epsilon_{11} = 2700 \\ &\epsilon_{33} = 5500, \quad e_{15} = 9400, \quad e_{31} = -5200, \quad e_{33} = 24\theta(t) + 13300. \end{aligned} \tag{5.1}$$

The constant parameters are chosen according to material parameters and damping parameters presented in [12]. The polynomial parameters of $e_{33}(\theta(t))$ are chosen such that they equal e_{33} in [12] at $25C^\circ$. To generate the noisy data y^δ we contaminate the exact simulated data y, generated with the parameters defined above, additively with uniformly distributed random noise with a noise level of 1%. The excitation signal $\phi_e(t_n)$, applied via the Dirichlet boundary condition at the top surface, is defined as a discrete triangular pulse, i.e.,

$$\phi_e(t_n) = 10^{-9} \cdot \begin{cases} n & \text{for } 1 \leq n \leq 10 \\ 20 - n & \text{for } 11 \leq n \leq 19 \\ 0 & \text{for } n \geq 20 \end{cases}. \tag{5.2}$$

Note that specifying the Dirichlet lift function χ used for stating the system (3.12)-(3.16) is not necessary, as it is possible to directly implement mixed Dirichlet conditions in FEniCS. Since the forward operator is non-linear, we employ the GRSE method, see [19] with 0.5 as decay factor and 4 as growth factor of the regularization parameters. We used the the initial regularization parameter $\tau_0 = 10^{-6}$ and the scaled identity with scale 10^{-6} as initial Quasi-Newton matrix. The initial guess for polynomial parameters of the piezoelectric coupling parameter $e_{33}(\theta(t))$ are chosen with a 2.5% deviation to the ground truth in (5.1). Furthermore, we scaled the observation operator with 10^7, the first order polynomial parameter with 5 and the zero order polynomial parameter with 10^{-2}, to reach similar orders of magnitude. The numerical results for the identification of the polynomial parameters e_{33_k} and e_{33_d}, which are the first and zeroth order polynomial parameters of the piezoelectric coupling parameter $e_{33}(\theta(t))$ showed convergence to the exact parameter, as illustrated in Figure 1.

6. Conclusion

We modeled and analyzed an inverse problem governed by a piezoelectric system represented by a coupled hyperbolic-elliptic PDE with matrix-valued Sobolev-Bochner functions as parameters and Sobolev-Bochner density and damping functions. We extended the PDE with an additional term based on a Sobolev-Bochner function and the mechanical deformation as well as inhomogeneities, ensuring the applicability of our generalized existence and uniqueness theorem on the associated adjoint PDE. In addition, an a priori energy estimate and conditions for arbitrary Sobolev regularity in space were established to ensure the well-definedness of the observation operator of the inverse parameter identification problem. Then, we proved the Fréchet differentiability of the observation operator and discussed the treatment of the observation operator given that PDE solutions have lower regularity. With respect to the modeling and regularity of the forward operator of the parameter identification problem in the reduced approach, we considered the well-definedness, existence and regularity of the parameter-state map. Furthermore, we modeled the inverse problem as an optimization problem in which a target functional consisting of the forward operator, the given data and a regularizer is minimized. To provide a framework for the computation of solutions to the inverse problem, we showed that there exists a minimizer and derived first-order optimality conditions. This motivated the derivation of the adjoint PDE, where we used our existence and uniqueness results to analyze the adjoint PDE, demonstrating its utility. Finally, a numerical example was given, where the proposed parameterization approach for describing the material parameters was used.

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