MODELLING AND ANALYSIS OF AN INVERSE PARAMETER IDENTIFICATION PROBLEM IN PIEZOELECTRICITY

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ABSTRACT. Piezoelectric material behavior is mathematically described by coupled hyperbolic-elliptic partial differential equations (PDEs) governing mechanical displacement and electrical potential. This paper presents advancements in the theory of identifying material parameters in piezoelectric PDEs. We focus on modeling and analyzing the inverse problem assuming matrix-valued Sobolev-Bochner parameters to encompass a time and space-dependent setting and thus external physical influences. This is followed by results regarding the existence, uniqueness and improved regularity of solutions to the piezoelectric PDE. Based on these findings, well-definedness and regularity of the parameter-to-state map and Fréchet differentiability of the observation operator are proven. Finally, the inverse problem is formulated using a minimization approach, where weak lower semi-continuity of the objective functional, first-order optimality conditions and the derivation and analysis of the adjoint PDE are presented.

Keywords: Existence, uniqueness and regularity; operator analysis; inverse parameter identification; piezoelectricity; adjoint PDEs
MSC 2020: 74H20, 74H25, 74H30, 74J25, 47B02, 35R30

1. INTRODUCTION

Piezoelectric materials are extensively utilized in numerous electrical devices nowadays, being prevalent not only in households but also in industrial and medical settings. The versatility of piezoelectric materials extends across a diverse range of products, including microphones and headphones as well as ultrasound imaging devices and power generation systems. The underlying piezoelectric effect, which is the fundamental property of these applications, describes a coupling phenomenon between electrical and mechanical fields, where mechanical pressure generates an electric potential and vice versa. Thoroughly understanding the behavior of these materials is essential, especially given their time and space dependent characteristics. Their temporal and spatial dependence can occur implicitly via the influence of external physical quantities such as temperature, which appears as temporally and spatially varying functions, on the material parameters. Simplistically, the piezoelectric material is described by a system of coupled PDEs for the mechanical displacement and the electrical potential. The material behavior depends significantly on the material parameters occurring in this PDE system. Hence, the inverse problem aims at identifying material parameters from observations $C(p, z)$ of the state and the parameters. As the observed data is usually contaminated with noise, we have given noisy measurements y^δ. Employing the reduced method, i.e., the model is eliminated by introducing a parameter-to-state map S, in an optimization approach results in solving

$$\min_{p\in X} \quad \frac{1}{2}\left\|F(p) - y^\delta\right\|^2 + \mathcal{R}_\alpha(p), \tag{1.1}$$

where $F : X \to Y,\ F(p) = C(p, S(p))$ is called the forward operator. If the preimage space X is a Sobolev space of higher order, implementing regularization methods on this preimage space becomes particularly challenging and, in many cases, impractical. To address this issue, it is beneficial to leverage the physical behavior of material parameters by parameterizing them in terms of a relevant physical quantity. We consider, for example, the dependence of the parameters on a known temperature function θ, which is a function of space and time. It is reasonable to assume a polynomial or Hadamrd exponential structure

Date: February 23, 2026.
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of the material parameters with respect to the temperature function, see [10]. The coefficients of these polynomials or exponential functions are constant parameters. This means that we can reconstruct these parameters by reconstructing constant matrices of appropriate size for each parameter, which transfers an optimization problem in higher order Sobolev spaces to an optimization problem in a real-valued finite dimensional vector space. As this surrogate modeling approach is a linear transformation into higher order Sobolev spaces, the analysis of the individual components of the inverse problem must be conducted in the framework of an infinite-dimensional function space, including the analysis of the underlying PDE and the associated adjoint PDE, in order to preserve generality.

Related Work. Existence, uniqueness and regularity of solutions of the piezoelectric dynamical system have been studied in [1], [3], [4], [11], [13], [16], [19], [18], [20], [21] and [23], among others and the references therein. For example, in [11] and [20] existence and uniqueness of solutions of the undamped inhomogeneous piezoelectric PDE is discussed, where the material parameters are constant. In [1] the material parameters are spatially dependent $L^\infty(\Omega)$ functions and inhomogeneities are included. An optimal control problem for the electrical flux (boundary control problem) is studied in [3], where an existence and uniqueness result is given for solutions of the undamped piezoelectric PDE and the corresponding adjoint differential PDE with spatially dependent $L^\infty(\Omega)$ material parameters. The papers [23] and [21] consider a boundary control problem, where existence and uniqueness results for solutions of the undamped homogeneous piezoelectric PDE are discussed, where [21] deals with constant material parameters and [23] focuses on an elasticity parameter comprised of spatially dependent $C^2(\Omega)$ functions, a permittivity parameter comprised of spatially dependent $L^\infty(\Omega)$ functions and a constant piezoelectric coupling parameter. In [4], an existence and uniqueness result is given for solutions of the undamped piezoelectric PDE coupled to a parabolic temperature equation and the magnetic field in the form of an elliptic equation, similar to the electrical equation of the classical piezoelectric system, where the parameters have $C^{0,1}(\Omega)$ regularity or $L^\infty(\Omega)$ regularity in the space. In [18], respectively in [19], a shape optimization problem is studied, where an existence and uniqueness result for solutions of the undamped inhomogeneous piezoelectric PDE is given for time/space-constant parameters and the corresponding adjoint PDE is presented. Furthermore, [13] gives a result heavily based on [7], [16], [1], and [20], on existence and uniqueness of solutions for the Rayleigh damped homogeneous piezoelectric PDE, where the material parameters are constant.

Contribution. The aim of this paper is to model and analyze the inverse problem. We establish the well-definedness, existence, and regularity of the parameter-to-state map. For this purpose, we have to extend and generalize previous existence and uniqueness results for piezoelectric PDEs, by considering matrix valued Sobolev-Bochner functions as material parameters and also Sobolev-Bochner density- and damping functions. Moreover the Rayleigh damped piezoelectric system is extended by a further damping term and Sobolev-Bochner inhomogeneities, which allows the application of the contributed existence and uniqueness theorem not only to the state equation but also to the adjoint PDE. Subsequently, we define the observation operator, demonstrating that its well-definedness requires higher regularity of the state. For this an a-priori energy estimate was established, which has not yet been treated in this general setting. Consequently, we provide a rigorous Dirichlet lift Ansatz and contribute a result that provides arbitrary Sobolev regularity in space, for sufficiently regular boundary data and right-hand sides, to satisfy the well-definedness requirement. We prove Fréchet differentiability of the observation operator, leading to the definition of the forward operator, which inherits the properties of both the observation operator and the parameter-to-state map. We model the inverse problem as a minimization problem of an objective functional and prove its weakly lower semi-continuity. Furthermore, we formulate the necessary first-order optimality conditions. Motivated by this, we derive the adjoint PDE and analyze it with respect to the existence and uniqueness of solutions by employing the main existence and uniqueness result of this article. Additionally we give insights in the structure of the derivative of the objective functional.
To the best of our knowledge, this generalized problem has not been previously addressed in the literature.

Structure of the paper. The structure of this paper is as follows: The second section addresses the modeling of the underlying PDE to the inverse problem, as well as notations and the introduction of necessary definitions and lemmata essential for the paper's objectives. The third section proves existence, uniqueness and regularity of weak solutions for the generalized damped piezoelectric PDEs. Furthermore, results on an a-priori energy estimate and arbitrary Sobolev regularity in space for sufficient regular boundary data and right-hand sides are proven. The fourth section models and analyses the forward operator, including the well-definedness and Fréchet differentiability of the observation operator and the existence and regularity of the parameter to state map. Then, the inverse problem is formulated as minimization problem and first-order optimality conditions are derived. The derivation and analysis of the adjoint PDE concludes this section. The fifth section discusses a numerical example and the final section briefly summarizes the contributions of this paper.

2. Modeling

Let $T > 0$ be the end time of the observed time period $(0, T)$ and denote the geometry of the considered piezoceramic, i.e., the domain, with $\Omega \subset \mathbb{R}^3$. For the latter we assume that its boundary can be represented as the disjoint union $\partial\Omega := \Gamma_e \dot\cup \Gamma_0 \dot\cup \Gamma_n$. Thereby Γ_e describes the boundary segment which is excited electrically with a known excitation signal ϕ^e and Γ_0 refers to the boundary segment which is grounded. This setting can be modeled in the system of PDEs as mixed Dirichlet boundary conditions. For the regularity of Ω and ϕ^e we will employ the following assumptions:

Assumptions 2.1.

D1 Ω is a Lipschitz domain and $\phi^e \in H^1\left(0, T; H^{\frac{1}{2}}(\partial\Omega, \mathbb{R})\right)$.

D2 Ω is a $C^{m,1}$-domain and $\phi^e \in H^1\left(H^{m+\frac{3}{2}}(\partial\Omega, \mathbb{R})\right)$ with some $m \geq 2,\ m \in \mathbb{N}$.

D3 Ω is a Lipschitz domain and $\phi^e \in H^1(0, T)$.

The boundary segment Γ_n is included in the PDE by Neumann boundary conditions. We denote the non-empty mixed Dirichlet boundary with $\Gamma_d := \Gamma_e \dot\cup \Gamma_0$, i.e., $\partial\Omega := \Gamma_d \dot\cup \Gamma_n$ and we assume that every boundary part has a positive two-dimensional Hausdorff measure. We will denote time derivatives of a function f with $\dot f$, and spatial derivatives with ∇f. Furthermore, $n = (n_x, n_y, n_z)$ is the three-dimensional normal vector corresponding to the normal derivative with respect to ∇, $\mathcal{B}f$ is the symmetrical gradient of a function f in Voigt notation and $\mathcal{N}$ is the corresponding normal matrix with respect to $\mathcal{B}$, where

$$\mathcal{B} = \begin{pmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{pmatrix}, \qquad \mathcal{N} = \begin{pmatrix} n_x & 0 & 0 \\ 0 & n_y & 0 \\ 0 & 0 & n_z \\ 0 & n_z & n_y \\ n_z & 0 & n_x \\ n_y & n_x & 0 \end{pmatrix}.$$

Since we consider all derivatives in a distributional sense, we use the standard Sobolev space $H^1(\Omega, \mathbb{R})$ associated with ∇ and

$$H^1_{\mathcal{B}}(\Omega, \mathbb{R}^3) = \left\{ f \in L^2(\Omega, \mathbb{R}^3) |\ \|f\|_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} := \|f\|_{L^2(\Omega,\mathbb{R}^3)} + \|\mathcal{B}f\|_{L^2(\Omega,\mathbb{R}^3)} \right\}$$

as H^1-Sobolev space associated with the spatial differential operator $\mathcal{B}$. Note that we can identify $H^1_{\mathcal{B}}(\Omega, \mathbb{R}^3)$ as $H^1(\Omega, \mathbb{R}^3)$. Due to Assumption 2.1, Ω is at least a Lipschitz domain, since Ω is bounded and Korn's inequality holds. The H^1-Sobolev space whose functions vanish only on Γ_d is given by

$$H^1_{0,\Gamma_d}(\Omega, \mathbb{R}) = \left\{ f \in H^1(\Omega, \mathbb{R}) |\ f|_{\Gamma_d} = 0 \right\}.$$

Furthermore, we denote the dual of a Hilbert space H with H^*. Then, the three-dimensional mechanical displacement $u(t, x)$ and the one-dimensional electrical potential $\phi(t, x)$ of a mechanically unclamped piezoceramic can be described by the following piezoelectric dynamical system

$$
\begin{aligned}
\rho\ddot{u} + \alpha\rho\dot{u} - \mathcal{B}^T\left(c^E\mathcal{B}u + \beta c^E\mathcal{B}\dot{u} + e^T\nabla\phi\right) &= 0 \ \text{ in } \ \Omega\times(0,T)\\
-\nabla\cdot(e\mathcal{B}u - \epsilon\nabla\phi) &= 0 \ \text{ in } \ \Omega\times(0,T)\\
\phi &= 0 \ \text{ on } \ \Gamma_0\times(0,T)\\
\phi &= \phi^e \quad \text{on } \ \Gamma_e\times(0,T)\\
n\cdot(e\mathcal{B}u - \epsilon\nabla\phi) &= 0 \ \text{ on } \ \Gamma_n\times(0,T)\\
\mathcal{N}^T\left(c^E\mathcal{B}u + \beta c^E\mathcal{B}\dot{u} + e^T\nabla\phi\right) &= 0 \ \text{ on } \ \partial\Omega\times(0,T)\\
u(t=0) &= u_0 \quad \text{in } \ \Omega\\
\dot{u}(t=0) &= u_1 \quad \text{in } \ \Omega,
\end{aligned}
$$

where ρ is the mass density, c^E, e and ϵ are the material parameters describing of the given piezoceramic and α, β are damping parameters. Since we consider the same time interval throughout the paper, we skip it when referring to Bochner spaces.

Assumptions 2.2.

A1 $\alpha, \beta \in H^1(L^\infty(\Omega, \mathbb{R}))$ are non-negative and uniformly bounded.

A2 $\rho \in H^2(L^\infty(\Omega, \mathbb{R}))$ is positive and uniformly bounded.

A3 The elasticity parameter is a matrix-valued function

$$
c^E(t,x) := \begin{pmatrix}
c^E_{11}(t,x) & c^E_{12}(t,x) & c^E_{13}(t,x) & 0 & 0 & 0\\
c^E_{12}(t,x) & c^E_{11}(t,x) & c^E_{13}(t,x) & 0 & 0 & 0\\
c^E_{13}(t,x) & c^E_{13}(t,x) & c^E_{33}(t,x) & 0 & 0 & 0\\
0 & 0 & 0 & c^E_{44}(t,x) & 0 & 0\\
0 & 0 & 0 & 0 & c^E_{44}(t,x) & 0\\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}\left(c^E_{11}(t,x) - c^E_{12}(t,x)\right)
\end{pmatrix}
$$

with $c^E_{11}, c^E_{12}, c^E_{13}, c^E_{33}, c^E_{44} \in H^2(L^\infty(\Omega, \mathbb{R}))$, i.e. $c^E \in H^2(L^\infty(\Omega, \mathbb{R}^{6\times 6}))$, is uniformly positive definite, where all eigenvalues are uniformly bounded away from zero.

A4 The piezoelectric coupling parameter is a matrix-valued function

$$
e(t,x) := \begin{pmatrix}
0 & 0 & 0 & 0 & e_{15}(t,x) & 0\\
0 & 0 & 0 & e^E_{15}(t,x) & 0 & 0\\
e_{31}(t,x) & e_{31}(t,x) & e_{33}(t,x) & 0 & 0 & 0
\end{pmatrix}
$$

with $e_{15}, e_{31}, e_{33} \in H^2(L^\infty(\Omega, \mathbb{R}))$, i.e. $e \in H^2(L^\infty(\Omega, \mathbb{R}^{3\times 6}))$.

A5 The permittivity parameter is a matrix-valued function

$$
\epsilon(t,x) := \begin{pmatrix}
\epsilon_{11}(t,x) & 0 & 0\\
0 & \epsilon_{11}(t,x) & 0\\
0 & 0 & \epsilon_{33}(t,x)
\end{pmatrix}
$$

with $\epsilon_{11}, \epsilon_{33} \in H^1(L^\infty(\Omega, \mathbb{R}))$, i.e. $\epsilon \in H^1(L^\infty(\Omega, \mathbb{R}^{3\times 3}))$, is uniformly positive definite, where all eigenvalues are uniformly bounded away from zero.

Note, that c^E and ϵ are invertible with bounded inverse due to Assumption A3 and A5. If the parameters depend on e.g., a known temperature function θ, we have that $(c^E(x,t), e(x,t), \epsilon(x,t)) = p(x,t) = \tilde{p}(\theta(x,t))$ and the following parametrizations can be proposed

$$
(c^E(x,t), e(x,t), \epsilon(x,t)) = \left(\sum_{j=0}^{n} a_j\theta(x,t)^j, \sum_{j=0}^{n} b_j\theta(x,t)^j, \sum_{j=0}^{n} k_j\theta(x,t)^j\right), \tag{2.1}
$$

$$
(a_j, b_j, k_j) \in \mathbb{R}^{6\times 6}\times\mathbb{R}^{3\times 6}\times\mathbb{R}^{3\times 3}, \ \text{for} \ 0 \le j \le n,
$$

$$
(c^E(x,t), e(x,t), \epsilon(x,t)) = \left(e^{c_1\theta(x,t)} + c_0, e^{l_1\theta(x,t)} + l_0, e^{m_1\theta(x,t)} + m_0\right), \tag{2.2}
$$

$$
(c_j, l_j, m_j) \in \mathbb{R}^{6\times 6}\times\mathbb{R}^{3\times 6}\times\mathbb{R}^{3\times 3}, \ \text{for} \ j = 0, 1,
$$

where in (2.2) we used the Hadarmard exponential, i.e., element-wise application of the the exponential function. Note that α, β are the so-called Rayleigh damping parameters. If $\alpha \equiv 0$, then β is called Kelvin-Voigt damping parameter and can be understood as relaxation. Hence, the Rayleigh damping model is a generalization of the Kelvin-Voigt damping model.

3. Existence, Uniqueness and Regularity

In order to discuss weak solvability, we homogenize the mixed Dirichlet boundary conditions using a Dirichlet lift Ansatz. Therefore, let Assumption D1 hold. As $\Gamma_e \cap \Gamma_0 = \emptyset$, there exists a $\chi(t) \in H^1(\Omega, \mathbb{R})$ with the property that

$$\mathrm{Tr}(\chi(t)) = \begin{cases} \phi_e(t) & \text{on } \Gamma_e \\ 0 & \text{on } \Gamma_0 \end{cases} \quad \text{a.e. in time.} \tag{3.1}$$

Hence, with $\phi_0(t) \in H^1_{0,\Gamma_d}(\Omega, \mathbb{R})$, we rewrite ϕ as $\phi(t) = \phi_0(t) + \chi(t)$ a.e. in time. Plugging this representation in our piezoelectric dynamical system leads to

$$\rho\ddot{u} + \alpha\rho\dot{u} - \mathcal{B}^T\left(c^E\mathcal{B}u + \beta c^E\mathcal{B}\dot{u} + e^T\nabla\phi_0\right) = \mathcal{B}^T e^T\nabla\chi \qquad \text{in } \Omega\times(0,T) \tag{3.2}$$

$$-\nabla\cdot(e\mathcal{B}u - \epsilon\nabla\phi_0) = -\nabla\cdot\epsilon\nabla\chi \qquad \text{in } \Omega\times(0,T) \tag{3.3}$$

$$n\cdot(e\mathcal{B}u - \epsilon\nabla\phi_0) = n\cdot\epsilon\nabla\chi \qquad \text{on } \Gamma_n\times(0,T) \tag{3.4}$$

$$\mathcal{N}^T\left(c^E\mathcal{B}u + \beta c^E\mathcal{B}\dot{u} + e^T\nabla\phi_0\right) = -\mathcal{N}^T e^T\nabla\chi \qquad \text{on } \partial\Omega\times(0,T) \tag{3.5}$$

$$u(t=0) = u_0 \qquad \text{in } \Omega \tag{3.6}$$

$$\dot{u}(t=0) = u_1 \qquad \text{on } \Omega. \tag{3.7}$$

In order to derive the weak form of the system above, we consider the weak form of (3.2) and (3.3) separately and include the corresponding boundary conditions (3.5) and (3.4). Note that by simply adding both forms, we obtain the weak form of the whole system, since we have to use different test functions for (3.2) and (3.3). We start deriving the weak form of (3.2) for almost all $t \in (0,T)$ by testing the system (3.2)-(3.7) with $(v,0)$, where $v \in H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)$, i.e.,

$$\begin{aligned}
&\int_\Omega \rho\ddot{u}^T v + \alpha\rho\dot{u}^T v + \left(c^E\mathcal{B}u + \beta c^E\mathcal{B}\dot{u} + e^T\nabla\phi_0\right)^T\mathcal{B}v\,\mathrm{d}\Omega \\
&= -\int_\Omega (e^T\nabla\chi)^T\mathcal{B}v\,\mathrm{d}\Omega + \int_{\partial\Omega}\mathcal{N}^T(e^T\nabla\chi)^T v\,\mathrm{d}\Gamma + \int_{\partial\Omega}\underbrace{\mathcal{N}^T\left(c^E\mathcal{B}u + \beta c^E\mathcal{B}\dot{u} + e^T\nabla\phi_0\right)^T}_{\underset{(3.5)}{=}-\mathcal{N}^T(e^T\nabla\chi)^T} v\,\mathrm{d}\Gamma
\end{aligned}$$

$$\Leftrightarrow \int_\Omega \rho\ddot{u}^T v + \alpha\rho\dot{u}^T v + \left(c^E\mathcal{B}u + \beta c^E\mathcal{B}\dot{u} + e^T\nabla\phi_0\right)^T\mathcal{B}v\,\mathrm{d}\Omega = -\int_\Omega (e^T\nabla\chi)^T\mathcal{B}v\,\mathrm{d}\Omega\,. \tag{3.8}$$

In order to derive the weak form of (3.3) for almost all $t \in (0,T)$, we test the system (3.2)-(3.7) with $(0,w)$, where $w \in H^1_{0,\Gamma_d}(\Omega,\mathbb{R})$, i.e.,

$$\begin{aligned}
&\int_\Omega (e\mathcal{B}u - \epsilon\nabla\phi_0)^T\nabla w\,\mathrm{d}\Omega - \int_{\Gamma_n}\underbrace{n\cdot(e\mathcal{B}u - \epsilon\nabla\phi_0)}_{\underset{(3.4)}{=}n\cdot\epsilon\nabla\chi} w\,\mathrm{d}\Gamma - \underbrace{\int_{\Gamma_d} n\cdot(e\mathcal{B}u - \epsilon\nabla\phi_0)\,w\,\mathrm{d}\Gamma}_{\underset{w\in H^1_{0,\Gamma_d}(\Omega,\mathbb{R})}{=}0} \\
&= \int_\Omega (\epsilon\nabla\chi)^T\nabla w\,\mathrm{d}\Omega - \int_{\Gamma_n} n\cdot(\epsilon\nabla\chi)w\,\mathrm{d}\Gamma - \underbrace{\int_{\Gamma_d} n\cdot(\epsilon\nabla\chi)w\,\mathrm{d}\Gamma}_{\underset{w\in H^1_{0,\Gamma_d}(\Omega,\mathbb{R})}{=}0}
\end{aligned}$$

$$\Leftrightarrow \int_\Omega (e\mathcal{B}u - \epsilon\nabla\phi_0)^T\nabla w\,\mathrm{d}\Omega = \int_\Omega (\epsilon\nabla\chi)^T\nabla w\,\mathrm{d}\Omega\,. \tag{3.9}$$

As we used different test functions for (3.8) and (3.9), we can define the bilinear form B corresponding to the PDE system (3.2)-(3.7) and the respective right-hand side operator L by

$$B((u,\phi_0),(v,w)) := \int_\Omega \rho \ddot{u}^T v + \alpha\rho\dot{u}^T v + \left(c^E \mathcal{B}u + \beta c^E \mathcal{B}\dot{u} + e^T\nabla\phi_0\right)^T \mathcal{B}v + \left(e\mathcal{B}u - \epsilon\nabla\phi_0\right)^T \nabla w \,\mathrm{d}\Omega \tag{3.10}$$

$$L(v,w) = \int_\Omega -(e^T\nabla\chi)^T\mathcal{B}v + (\epsilon\nabla\chi)^T\nabla w\,\mathrm{d}\Omega\,. \tag{3.11}$$

To prove existence and uniqueness of solutions to the piezoelectric system (3.2)-(3.7), we use Chapter XVIII in [5], especially Theorem 1 in Paragraph 5 and Remark 9 in Paragraph 6. Using the following lemma, we will exploit that c^E and e are continuously differentiable in time.

Lemma 3.1. *Let $(0,T)\subset\mathbb{R}$, $k,k_1,k_2\in\mathbb{N}$ and Assumption D1 hold. Then, one has*

$$H^k(L^\infty(\Omega)) \hookrightarrow C^{k-1}([0,T];L^\infty(\Omega)) \tag{3.12}$$

$$H^{k_1}(H^{k_2}(\Omega)) \hookrightarrow C^{k_1-1}([0,T];C^{k_2-2}(\bar{\Omega})). \tag{3.13}$$

Proof. Let $U\subset\mathbb{R}$. Due to Theorem 6, Chapter 5.6.3 in [7] one obtains for $k>\frac{1}{2}$ the existence of a continuous embedding $\iota_1 : H^k(U)\hookrightarrow C^{k-1,\frac{1}{2}}(\bar{U})$ with $\iota_1(f)\equiv f$ almost everywhere. Since $C^{k-1,\frac{1}{2}}(\bar{U})\hookrightarrow C^{k-1}(\bar{U})$, one has $H^k(U)\hookrightarrow C^{k-1,\frac{1}{2}}(\bar{U})\hookrightarrow C^{k-1}(\bar{U})$, meaning that there is a constant $C^U_{H^k\hookrightarrow C^{k-1}}>0$ such that

$$\forall f\in H^k(U):\ \|f\|_{C^{k-1}(\bar{U})}\le C^U_{H^k\hookrightarrow C^{k-1}}\|f\|_{H^k(U)}.$$

Analogously, due to Theorem 6, Chapter 5.6.3 in [7] one obtains for $k>\frac{3}{2}$ that there exists a continuous embedding $\iota_3 : H^k(\Omega)\hookrightarrow C^{k-2,\frac{1}{2}}(\bar{\Omega})$ with $\iota_3(f)\equiv f$ almost everywhere. Furthermore, it holds that $C^{k-2,\frac{1}{2}}(\bar{\Omega})\hookrightarrow C^{k-2}(\bar{\Omega})$, which yields $H^k(\Omega)\hookrightarrow C^{k-2,\frac{1}{2}}(\bar{\Omega})\hookrightarrow C^{k-2}(\bar{\Omega})$, i.e., for a constant $C^\Omega_{H^k\hookrightarrow C^{k-2}}>0$,

$$\forall f\in H^k(\Omega):\ \|f\|_{C^{k-2}(\bar{\Omega})}\le C^\Omega_{H^k\hookrightarrow C^{k-2}}\|f\|_{H^k(\Omega)}$$

is valid. Hence, there exists a constant $C^{(0,T)}_{H^k\hookrightarrow C^{k-1}}>0$ such that

$$\forall f\in H^k(L^\infty(\Omega)):\ \|f\|_{C^{k-1}([0,T];L^\infty(\Omega))}\le C^{(0,T)}_{H^k\hookrightarrow C^{k-1}}\|f\|_{H^k(L^\infty(\Omega))},$$

which proves that $H^k(L^\infty(\Omega))\hookrightarrow C^{k-1}([0,T];L^\infty(\Omega))$ continuously.
By the same argument we conclude that $H^{k_1}(H^{k_2}(\Omega))\hookrightarrow C^{k_1-1}([0,T];H^{k_2}(\Omega))$, i.e., there is a constant $C^{(0,T)}_{H^{k_1}\hookrightarrow C^{k_1-1}}>0$ such that

$$\forall f\in H^{k_1}(H^{k_2}(\Omega)):\ \|f\|_{C^{k_1-1}([0,T];H^{k_2}(\Omega))}\le C^{(0,T)}_{H^{k_1}\hookrightarrow C^{k_1-1}}\|f\|_{H^{k_1}(H^{k_2}(\Omega))}. \tag{3.14}$$

Furthermore, there exists a constant $C^\Omega_{H^{k_2}\hookrightarrow C^{k_2-2}}>0$ such that for all $f\in C^{k_1-1}([0,T];H^{k_2}(\Omega))$

$$\|f\|_{C^{k_1-1}([0,T];C^{k_2-2}(\bar{\Omega}))}\le C^\Omega_{H^{k_2}\hookrightarrow C^{k_2-2}}\|f\|_{C^{k_1-1}([0,T];H^{k_2}(\Omega))} \tag{3.15}$$

holds. Together, the inequalities (3.14) and (3.15) yield the existence of a constant

$$C := C^\Omega_{H^{k_2}\hookrightarrow C^{k_2-2}} C^{(0,T)}_{H^{k_1}\hookrightarrow C^{k_1-1}} > 0$$

such that

$$\forall f\in H^{k_1}(H^{k_2}(\Omega)):\ \|f\|_{C^{k_1-1}([0,T];C^{k_2-2}(\bar{\Omega}))}\le C\|f\|_{H^{k_1}(H^{k_2}(\Omega))}, \tag{3.16}$$

which proves that $H^{k_1}(H^{k_2}(\Omega))\hookrightarrow C^{k_1-1}(C^{k_2-2}(\bar{\Omega}))$ continuously. □

As we also want to use the following result for the analysis of the adjoint system, we introduce an additional damping function $a\in H^2(L^\infty(\Omega,\mathbb{R}))$ and inhomogeneities $(f,g)\in L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*)\times L^2(L^2(\Omega,\mathbb{R}))$ to the piezoelectric system (3.2)-(3.7).

Theorem 3.2 (Existence and Uniqueness)**.** *Let Assumption D1 and Assumptions 2.2 hold. Let* $a \in H^2(L^\infty(\Omega,\mathbb{R}))$ *be non-negative and uniformly bounded and* $(f,g) \in L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*) \times L^2(L^2(\Omega,\mathbb{R}))$ *be inhomogeneities. Then for any initial values*

$$u_0 \in H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3),\ u_1 \in H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*$$

and ϕ^e *according to Assumption D1 with* $\chi \in H^1\left(H^1(\Omega,\mathbb{R})\right)$ *and* $\mathrm{Tr}(\chi(t))$ *defined as in* (3.1)*, there exists a unique weak solution*

$$(u,\phi_0) \in L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)) \times L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R})) \tag{3.17}$$

with

$$\dot{u} \in L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)) \text{ and } \ddot{u} \in L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*) \tag{3.18}$$

to

$$\rho\ddot{u} + \alpha\rho\dot{u} + au - \mathcal{B}^T\left(c^E\mathcal{B}u + \beta c^E\mathcal{B}\dot{u} + e^T\nabla\phi_0\right) = f + \mathcal{B}^Te^T\nabla\chi \quad \text{in } \Omega\times(0,T) \tag{3.19}$$
$$-\nabla\cdot(e\mathcal{B}u - \epsilon\nabla\phi_0) = g - \nabla\cdot\epsilon\nabla\chi \quad \text{in } \Omega\times(0,T) \tag{3.20}$$
$$n\cdot(e\mathcal{B}u - \epsilon\nabla\phi_0) = n\cdot\epsilon\nabla\chi \quad \text{on } \Gamma_n\times(0,T) \tag{3.21}$$
$$\mathcal{N}^T\left(c^E\mathcal{B}u + \beta c^E\mathcal{B}\dot{u} + e^T\nabla\phi_0\right) = -\mathcal{N}^Te^T\nabla\chi \quad \text{on } \partial\Omega\times(0,T) \tag{3.22}$$
$$u(t=0) = u_0 \quad \text{in } \Omega \tag{3.23}$$
$$\dot{u}(t=0) = u_1 \quad \text{on } \Omega. \tag{3.24}$$

Proof. Similarly to (3.9) we obtain an affine linear mapping in ϕ_0

$$\int_\Omega (\epsilon\nabla\phi_0)^T\nabla w\,\mathrm{d}\Omega = \int_\Omega (e\mathcal{B}u)^T\nabla w - (\epsilon\nabla\chi)^T\nabla w - gw\,\mathrm{d}\Omega\,. \tag{3.25}$$

Therefore, we define for almost all $t \in (0,T)$ the linear operator

$$\zeta : H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3) \to H^1_{0,\Gamma_d}(\Omega,\mathbb{R}),\ u(t) \mapsto \phi_0^0(t),$$

where $\phi_0(t) = \phi_0^0(t) + \phi_0^r(t) = \zeta(u(t)) + \phi_0^r(t)$ by satisfying

$$\int_\Omega (\epsilon\nabla\zeta(u))^T\nabla w\,\mathrm{d}\Omega = \int_\Omega (e\mathcal{B}u)^T\nabla w. \tag{3.26}$$

As $H^1_{0,\Gamma_d}(\Omega,\mathbb{R}) \subset H^1(\Omega,\mathbb{R})$, we define $\phi_0^r \in L^2(H^1(\Omega,\mathbb{R}))$ by satisfying

$$\int_\Omega (\epsilon\nabla\phi_0^r)^T\nabla w\,\mathrm{d}\Omega = -\int_\Omega (\epsilon\nabla\chi)^T\nabla w + gw\,\mathrm{d}\Omega\,. \tag{3.27}$$

Hence, we obtain a simplified weak form of the inhomogeneous piezoelectric PDE including the Dirichlet lift similarly to (3.8) through the following bilinear form

$$\int_\Omega \rho\ddot{u}^Tv + \alpha\rho\dot{u}^Tv + au^Tv + \left(c^E\mathcal{B}u + \beta c^E\mathcal{B}\dot{u} + e^T\nabla\zeta(u)\right)^T\mathcal{B}v\,\mathrm{d}\Omega$$
$$= \int_\Omega fv - (e^T\nabla(\chi+\phi_0^r))^T\mathcal{B}v\,\mathrm{d}\Omega\,. \tag{3.28}$$

We now define the operators

$$a_0(t,u,v) := \int_\Omega a(t)u^T v + \left(c^E(t)\mathcal{B}u\right)^T \mathcal{B}v + \left(e(t)^T\nabla\zeta(u)\right)^T \mathcal{B}v \,\mathrm{d}\Omega\,, \tag{3.29}$$

$$a_1(t,u,v) := 0, \tag{3.30}$$

$$b_0(t,u,v) := \int_\Omega \alpha(t)\rho(t)u^T v + \left(\beta(t)c^E(t)\mathcal{B}u\right)^T \mathcal{B}v \,\mathrm{d}\Omega\,, \tag{3.31}$$

$$b_1(t,u,v) := \int_\Omega -\dot\rho(t)u^T v \,\mathrm{d}\Omega\,, \tag{3.32}$$

$$c(t,u,v) := \int_\Omega \rho(t)u^T v \,\mathrm{d}\Omega\,, \tag{3.33}$$

$$\langle \tilde f(t), v\rangle := \int_\Omega f(t)v - (e(t)^T\nabla(\chi+\phi_0^r)(t))^T\mathcal{B}v\,\mathrm{d}\Omega\,, \tag{3.34}$$

where $a_{01} = a_0 + a_1$ and $b_{01} = b_0 + b_1$. Due to Korn's inequality and the regularity and boundedness of Ω, we can identify $H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)$ as $H^1(\Omega,\mathbb{R}^3)$. We denote the unit ball with $B^1_{H^1(\Omega,\mathbb{R}^3)}$, yielding

$$\begin{aligned}
\left\|\tilde f\right\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)} &= \int_0^T \left\|\tilde f(t)\right\|^2_{H^1(\Omega,\mathbb{R}^3)^*}\mathrm{d}t \\
&= \int_0^T \sup_{v\in\partial B^1_{H^1(\Omega,\mathbb{R}^3)}} |\langle \tilde f(t), v\rangle_{H^1(\Omega,\mathbb{R}^3)^*,H^1(\Omega,\mathbb{R}^3)}|^2\,\mathrm{d}t \\
&\le \int_0^T \sup_{v\in\partial B^1_{H^1(\Omega,\mathbb{R}^3)}} |(f,v)_{L^2(\Omega,\mathbb{R}^3)} - (e(t)^T\nabla(\chi(t)+\phi_0^r(t)),\mathcal{B}v)_{L^2(\Omega,\mathbb{R}^3)}|^2\,\mathrm{d}t \\
&\le \int_0^T \sup_{v\in\partial B^1_{H^1(\Omega,\mathbb{R}^3)}} \left\|-e(t)^T\nabla(\chi(t)+\phi_0^r(t))\right\|^2_{L^2(\Omega,\mathbb{R}^3)}\|v\|^2_{H^1(\Omega,\mathbb{R}^3)}\,\mathrm{d}t \\
&\quad + \int_0^T \sup_{v\in\partial B^1_{H^1(\Omega,\mathbb{R}^3)}} \|f(t)\|^2_{H^1(\Omega,\mathbb{R}^3)^*}\|v\|^2_{H^1(\Omega,\mathbb{R}^3)}\,\mathrm{d}t \\
&\le \|e\|^2_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 6}))}\left(\|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))}\right) + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)} < \infty.
\end{aligned} \tag{3.35}$$

Consequently, we have to prove the existence and uniqueness of $u\in L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))$ satisfying

$$\dot c(t;\dot u(t),v) + b_{01}(t;\dot u(t),v) + a_{01}(t;u(t),v) = \langle \tilde f(t), v\rangle, \tag{3.36}$$

where $\tilde f\in L^2\left(H^1(\Omega,\mathbb{R}^3)^*\right)$, $u(0)=u_0\in H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)$ and $\dot u(0) = u_1\in H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*$.
As our function space setting are real Hilbert spaces and due to

$$\begin{aligned}
\int_\Omega \left(e^T\nabla\zeta(u)\right)^T\mathcal{B}v\,\mathrm{d}\Omega &= \int_\Omega (\nabla\zeta(u))^T e\mathcal{B}v\,\mathrm{d}\Omega = \int_\Omega (e\mathcal{B}v)^T\nabla\zeta(u)\,\mathrm{d}\Omega \\
&\underset{(3.26)}{=} \int_\Omega (\epsilon\nabla\zeta(v))^T\nabla\zeta(u)\,\mathrm{d}\Omega \underset{\epsilon\text{ diag.}}{=} \int_\Omega (\epsilon\nabla\zeta(u))^T\nabla\zeta(v)\,\mathrm{d}\Omega \\
&\underset{(3.26)}{=} \int_\Omega (e\mathcal{B}u)^T\nabla\zeta(v)\,\mathrm{d}\Omega = \int_\Omega \left(e^T\nabla\zeta(v)\right)^T\mathcal{B}u\,\mathrm{d}\Omega\,,
\end{aligned} \tag{3.37}$$

we immediately conclude that a_0 and b_0 are Hermitian. Furthermore, let the superscript * indicate the non-negative minimum of any real-valued, non-negative or positive, uniformly bounded, one dimensional function and note that for almost all $t\in(0,T)$ it holds that

$$\begin{aligned}
(\mathcal{B}u(t),\mathcal{B}v(t))_{L^2(\Omega,\mathbb{R}^3)} &= ((c^E(t))^{-1}c^E(t)\mathcal{B}u(t),\mathcal{B}v(t))_{L^2(\Omega,\mathbb{R}^3)} \\
&\le \left\|(c^E(t))^{-1}\right\|_{L^\infty(\Omega,\mathbb{R}^{6\times 6})}(c^E(t)\mathcal{B}u(t),\mathcal{B}v(t))_{L^2(\Omega,\mathbb{R}^3)} \\
&\le \left\|(c^E)^{-1}\right\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{6\times 6}))}(c^E(t)\mathcal{B}u(t),\mathcal{B}v(t))_{L^2(\Omega,\mathbb{R}^3)} \\
\Leftrightarrow\ (c^E(t)\mathcal{B}u(t),\mathcal{B}v(t))_{L^2(\Omega,\mathbb{R}^3)} &\ge \left\|(c^E)^{-1}\right\|^{-1}_{L^\infty(L^\infty(\Omega,\mathbb{R}^{6\times 6}))}(\mathcal{B}u(t),\mathcal{B}v(t))_{L^2(\Omega,\mathbb{R}^3)}.
\end{aligned}$$

With this we conclude that there exists some constant $\sigma \in \mathbb{R}^+$ with

$$\sigma := \min\left\{a^\star, \left\|(c^E)^{-1}\right\|^{-1}_{L^\infty(L^\infty(\Omega,\mathbb{R}^{6\times 6}))}\right\} \tag{3.38}$$

such that

$$\begin{aligned}
a_0(t,u,u) &= \int_\Omega a(t)u^T u + \left(c^E(t)\mathcal{B}u\right)^T \mathcal{B}u + \left(e(t)^T\nabla\zeta(u)\right)^T \mathcal{B}v \,\mathrm{d}\Omega \\
&\underset{(3.37)}{\geq} a^\star \int_\Omega u^T u \,\mathrm{d}\Omega + \left\|(c^E)^{-1}\right\|^{-1}_{L^\infty(L^\infty(\Omega,\mathbb{R}^{6\times 6}))} \int_\Omega (\mathcal{B}u)^T \mathcal{B}u \,\mathrm{d}\Omega + \int_\Omega (\epsilon\nabla\zeta(u))^T \nabla\zeta(v)\,\mathrm{d}\Omega \\
&\geq a^\star \|u\|^2_{L^2(\Omega,\mathbb{R}^3)} + \left\|(c^E)^{-1}\right\|^{-1}_{L^\infty(L^\infty(\Omega,\mathbb{R}^{6\times 6}))} \|\mathcal{B}u\|^2_{L^2(\Omega,\mathbb{R}^3)} \\
&\quad + \underbrace{\left\|(\epsilon)^{-1}\right\|^{-1}_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 3}))} \|\nabla\zeta(u)\|^2_{L^2(\Omega,\mathbb{R}^3)}}_{\geq 0} \\
&\geq \sigma \|u\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)}\,. \qquad (3.39)
\end{aligned}$$

Additionally the mapping $t \mapsto a_0(t,u,v)$ is one time continuously differentiable with derivative

$$\dot{a}_0(t,u,v) = \int_\Omega \dot{a}(t)u^T v + \left(\dot{c}^E(t)\mathcal{B}u\right)^T \mathcal{B}v + \left(\dot{e}(t)^T\nabla\zeta(u)\right)^T \mathcal{B}v \,\mathrm{d}\Omega\,, \tag{3.40}$$

due to the regularity of the material parameters and Lemma 3.1. Moreover, with

$$\beta_0 = \min\left\{\alpha^\star\rho^\star, \beta^\star\left\|(c^E)^{-1}\right\|^{-1}_{L^\infty(L^\infty(\Omega,\mathbb{R}^{6\times 6}))}\right\} \in \mathbb{R}^+ \tag{3.41}$$

it holds that

$$\begin{aligned}
b_0(t,u,u) &= \int_\Omega \alpha(t)\rho(t)u^T u + \left(\beta(t)c^E(t)\mathcal{B}u\right)^T \mathcal{B}u \,\mathrm{d}\Omega \\
&\geq \alpha^\star\rho^\star \int_\Omega u^T u\,\mathrm{d}\Omega + \beta^\star\left\|(c^E)^{-1}\right\|^{-1}_{L^\infty(L^\infty(\Omega,\mathbb{R}^{6\times 6}))} \int_\Omega (\mathcal{B}u)^T \mathcal{B}u\,\mathrm{d}\Omega \geq \beta_0 \|u\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)}
\end{aligned}$$

and with some $\tilde{c} > 0$ we obtain

$$\begin{aligned}
|b_1(t,u,v)|^2 &= \left|\int_\Omega -\dot{\rho}u^T v\,\mathrm{d}\Omega\right|^2 \leq \|\dot{\rho}\|^2_{L^\infty(L^\infty(\Omega,\mathbb{R}))} \|u\|^2_{L^2(\Omega,\mathbb{R}^3)} \|v\|^2_{L^2(\Omega,\mathbb{R}^3)} \\
&\leq \tilde{c}\|\dot{\rho}\|^2_{L^\infty(L^\infty(\Omega,\mathbb{R}))} \|u\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} \|v\|^2_{H^1(\Omega,\mathbb{R}^3)^*}\,. \qquad (3.42)
\end{aligned}$$

Using these results and Lemma 3.1, all introduced operators satisfy the conditions of [5], Chapter XVIII, Paragraph 5, Theorem 1, as c satisfies the conditions due to [5], Chapter XVIII, Paragraph 6, Remark 9. □

In order to prove boundedness of the operators in in Section 4, the following theorem is beneficial.

Theorem 3.3 (Energy estimates)**.** *Let the assumptions of Theorem 3.2 hold. Then for some* $\tilde{K} > 0$ *the following a-priori estimate holds*

$$\begin{aligned}
&\|\ddot{u}\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*)} + \|\dot{u}\|^2_{L^\infty(L^2(\Omega,\mathbb{R}^3))} + \|u\|^2_{L^\infty(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\dot{u}\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\phi_0\|^2_{L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}))} \\
&\qquad \leq \tilde{K}\Big(\|u_1\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u_0\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} \\
&\qquad\qquad + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)} + \|g\|^2_{L^2(L^2(\Omega,\mathbb{R}))}\Big)\,.
\end{aligned}$$

Proof. Consider

$$\begin{aligned}
X(t) &= c(t,\dot{u}(t),\dot{u}(t)) + a_0(t,u(t),u(t)) + 2\int_0^t b_0(s,\dot{u}(s),\dot{u}(s))\,\mathrm{d}s \\
&= \int_\Omega \rho(t)\dot{u}(t)^T\dot{u}(t) + a(t)u(t)^T u(t) + \left(c^E(t)\mathcal{B}u(t)\right)^T \mathcal{B}u(t) + \left(e(t)^T\nabla\zeta(u(t))\right)^T \mathcal{B}u(t)\,\mathrm{d}\Omega \\
&\quad + 2\int_0^t \int_\Omega \alpha(s)\rho(s)\dot{u}(s)^T\dot{u}(s) + \left(\beta(s)c^E(s)\mathcal{B}\dot{u}(s)\right)^T \mathcal{B}\dot{u}(s)\,\mathrm{d}\Omega\,\mathrm{d}s\,.
\end{aligned}$$

With $k := \min\left\{\rho^\star, \sigma, \left\|(\epsilon)^{-1}\right\|^{-1}_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 3}))}\right\}$, it holds

$$\begin{aligned}
X(t) &\geq \rho^\star \|\dot u(t)\|^2_{L^2(\Omega,\mathbb{R}^3)} + \sigma \|u(t)\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \int_\Omega \left(e(t)^T \nabla\zeta(u(t))\right)^T \mathcal{B}u(t)\,\mathrm{d}\Omega \\
&\quad + 2\beta_0 \int_0^t \int_\Omega \dot u(s)^T \dot u(s) + \left(\mathcal{B}\dot u(s)\right)^T \mathcal{B}\dot u(s)\,\mathrm{d}\Omega\,\mathrm{d}s \\
&\underset{(3.37)}{=} \rho^\star \|\dot u(t)\|^2_{L^2(\Omega,\mathbb{R}^3)} + \sigma \|u(t)\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \int_\Omega \left(\epsilon \nabla\zeta(u(t))\right)^T \nabla\zeta(u(t))\,\mathrm{d}\Omega + \underbrace{2\beta_0 \|\dot u\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))}}_{\geq 0} \\
&\geq \rho^\star \|\dot u(t)\|^2_{L^2(\Omega,\mathbb{R}^3)} + \sigma \|u(t)\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \left\|(\epsilon)^{-1}\right\|^{-1}_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 3}))} \|\nabla\zeta(u(t))\|^2_{L^2(\Omega,\mathbb{R}^3)} \\
&\geq k\left(\|\dot u(t)\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u(t)\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|\nabla\zeta(u(t))\|^2_{L^2(\Omega,\mathbb{R}^3)}\right).
\end{aligned} \tag{3.43}$$

Additionally, with

$$c_0 = \max\left\{\|\rho\|_{L^\infty(L^\infty(\Omega,\mathbb{R}))}, \|a\|_{L^\infty(L^\infty(\Omega,\mathbb{R}))}, \left\|c^E\right\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{6\times 6}))}, \|\epsilon\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 3}))}\right\},$$

it holds

$$\begin{aligned}
X(0) &\underset{(3.37)}{\leq} \|\rho\|_{L^\infty(L^\infty(\Omega,\mathbb{R}))} \|\dot u(0)\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|a\|_{L^\infty(L^\infty(\Omega,\mathbb{R}))} \|u(0)\|^2_{L^2(\Omega,\mathbb{R}^3)} \\
&\quad + \left\|c^E\right\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{6\times 6}))} \|\mathcal{B}u(0)\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|\epsilon\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 3}))} \|\nabla\zeta u(0)\|^2_{L^2(\Omega,\mathbb{R}^3)} \\
&\leq c_0 \left(\|u_1\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u_0\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|\mathcal{B}u_0\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|\nabla\zeta u_0\|^2_{L^2(\Omega,\mathbb{R}^3)}\right).
\end{aligned}$$

From our previous results and [5], Chapter XVIII, Paragraph 5, section 4.1, we know that

$$\begin{aligned}
X(t) = X(0) &+ 2\int_0^t \dot a_0(s, u(s), u(s))\,\mathrm{d}s - 2\int_0^t b_1(s, \dot u(s), \dot u(s))\,\mathrm{d}s \\
&- \int_0^t c(s, \dot u(s), \dot u(s))\,\mathrm{d}s + 2\int_0^t \langle \tilde f(s), \dot u(s)\rangle\,\mathrm{d}s
\end{aligned} \tag{3.44}$$

holds. With $K_0 = \max\left\{\|\dot a\|_{L^\infty(L^\infty(\Omega,\mathbb{R}))}, \left\|\dot c^E\right\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{6\times 6}))}, \|\dot\epsilon\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 3}))}\right\}$ we obtain

$$\begin{aligned}
&\int_0^t \dot a_0(s, u(s), u(s))\,\mathrm{d}s \\
&\quad = \int_0^t \int_\Omega \dot a(s) u(s)^T u(s) + \left(\dot c^E(s)\mathcal{B}u(s)\right)^T \mathcal{B}u(s) + \left(\dot e(s)^T \nabla\zeta(u(s))\right)^T \mathcal{B}u(s)\,\mathrm{d}\Omega\,\mathrm{d}s \\
&\quad \leq \|\dot a\|_{L^\infty(L^\infty(\Omega,\mathbb{R}))} \|u\|^2_{L^2(L^2(\Omega,\mathbb{R}^3))} + \left\|\dot c^E\right\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{6\times 6}))} \|\mathcal{B}u\|^2_{L^2(L^2(\Omega,\mathbb{R}^3))} \\
&\quad\quad + \|\dot\epsilon\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 3}))} \|\nabla\zeta u\|^2_{L^2(L^2(\Omega,\mathbb{R}^3))} \\
&\quad \leq K_0\left(\|u\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\nabla\zeta u\|^2_{L^2(L^2(\Omega,\mathbb{R}^3))}\right)
\end{aligned} \tag{3.45}$$

and

$$\begin{aligned}
&-2\int_0^t b_1(s, \dot u(s), \dot u(s))\,\mathrm{d}s - \int_0^t c(s, \dot u(s), \dot u(s))\,\mathrm{d}s \\
&\quad \leq 2\|\dot\rho\|_{L^\infty(L^\infty(\Omega,\mathbb{R}))} \|\dot u\|^2_{L^2(L^2(\Omega,\mathbb{R}^3))} - \|\rho\|_{L^\infty(L^\infty(\Omega,\mathbb{R}))} \|\dot u\|^2_{L^2(L^2(\Omega,\mathbb{R}^3))}.
\end{aligned} \tag{3.46}$$

Due to (3.35) and Young's inequality it holds that

$$\begin{aligned}
2\int_0^t \langle \tilde f(s), \dot u(s)\rangle\,\mathrm{d}s \leq \|e\|^2_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 6}))} &\left(\|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))}\right) + \|\dot u\|^2_{L^2(L^2(\Omega,\mathbb{R}^3))} \\
&+ \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)} + \|\dot u\|^2_{L^2(L^2(\Omega,\mathbb{R}^3))}.
\end{aligned} \tag{3.47}$$

Consequently, using $K_2 = \max\left\{1, 2(\|\dot{\rho}\|_{L^\infty(L^\infty(\Omega,\mathbb{R}))} + 1), \|e\|^2_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 6}))}, 2K_0\right\}$, we obtain

$$\begin{aligned}
X(t) &\le X(0) + 2K_0(\|u\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\nabla\zeta u\|^2_{L^2(L^2(\Omega,\mathbb{R}^3))}) + 2\|\dot{\rho}\|_{L^\infty(L^\infty(\Omega,\mathbb{R}))}\|\dot{u}\|^2_{L^2(L^2(\Omega,\mathbb{R}^3))} \\
&\quad - \|\rho\|_{L^\infty(L^\infty(\Omega,\mathbb{R}))}\|\dot{u}\|^2_{L^2(L^2(\Omega,\mathbb{R}^3))} \\
&\quad + \|e\|^2_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 6}))}\left(\|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))}\right) \\
&\quad + \|\dot{u}\|^2_{L^2(L^2(\Omega,\mathbb{R}^3))} + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)} + \|\dot{u}\|^2_{L^2(L^2(\Omega,\mathbb{R}^3))} \\
&\le X(0) + K_2\left(\|u\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\nabla\zeta u\|^2_{L^2(L^2(\Omega,\mathbb{R}^3))} + \|\dot{u}\|^2_{L^2(L^2(\Omega,\mathbb{R}^3))}\right. \\
&\qquad \left. + \|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)}\right),
\end{aligned} \tag{3.48}$$

due to (3.45), (3.46) and (3.47). Hence,

$$\begin{aligned}
X(t) &\le X(0) + K_2\int_0^t \|u(s)\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|\nabla\zeta(u(s))\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|\dot{u}\|^2_{L^2(\Omega,\mathbb{R}^3)}\,\mathrm{d}s \\
&\quad + K_2\left(\|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)}\right).
\end{aligned} \tag{3.49}$$

Therefore, (3.43) and (3.49) yield

$$k(\|\dot{u}(t)\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u(t)\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|\nabla\zeta(u(t))\|^2_{L^2(\Omega,\mathbb{R}^3)}) \le X(t) \tag{3.50}$$

$$\begin{aligned}
&\le X(0) + K_2\int_0^t \|u(s)\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|\nabla\zeta(u(s))\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|\dot{u}\|^2_{L^2(\Omega,\mathbb{R}^3)}\,\mathrm{d}s \\
&\quad + K_2\left(\|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)}\right).
\end{aligned} \tag{3.51}$$

Applying Gronwall's Lemma results in

$$\begin{aligned}
&\|\dot{u}(t)\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u(t)\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|\dot{u}\|^2_{L^2(0,t;H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\nabla\zeta(u)\|^2_{L^2(\Omega,\mathbb{R}^3)} \\
&\le \left(\frac{X(0)}{k} + \frac{K_2}{k}\left(\|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)}\right)\right)\left(1 + \frac{K_2 T}{k}e^{\frac{K_2 T}{k}}\right) \\
&\le K\left(X(0) + \|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)}\right) \\
&\le C\left(\|u_1\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u_0\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|\mathcal{B}u_0\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|\nabla\zeta u_0\|^2_{L^2(\Omega,\mathbb{R}^3)}\right. \\
&\quad \left. + \|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)}\right) \\
&\le C\left(\|u_1\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u_0\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|\nabla\zeta u_0\|^2_{L^2(\Omega,\mathbb{R}^3)}\right. \\
&\quad \left. + \|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)}\right).
\end{aligned}$$

Now consider (3.26) with test function $\zeta(u)$ and (3.27) with test function ϕ_0^r

$$\begin{aligned}
\left\|(\epsilon)^{-1}\right\|^{-1}_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 3}))}\|\nabla\zeta(u(t))\|^2_{L^2(\Omega,\mathbb{R}^3)} &\le \int_\Omega (\epsilon\nabla\zeta(u))^T\nabla\zeta(u)\,\mathrm{d}\Omega = \int_\Omega (e\mathcal{B}u)^T\nabla\zeta(u) \\
&\le \|e\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 6}))}\|\mathcal{B}u(t)\|_{L^2(\Omega,\mathbb{R}^3)}\|\nabla\zeta(u(t))\|_{L^2(\Omega,\mathbb{R}^3)} \\
\Leftrightarrow\ \|\nabla\zeta(u(t))\|_{L^2(\Omega,\mathbb{R}^3)} &\le \|e\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 6}))}\left\|(\epsilon)^{-1}\right\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 3}))}\|\mathcal{B}u(t)\|_{L^2(\Omega,\mathbb{R}^3)},
\end{aligned} \tag{3.52}$$

yielding

$$\|\nabla\zeta u_0\|_{L^2(\Omega,\mathbb{R}^3)} \le \|e\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 6}))}\left\|(\epsilon)^{-1}\right\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 3}))}\|\mathcal{B}u_0\|_{L^2(\Omega,\mathbb{R}^3)} \tag{3.53}$$

and

$$\begin{aligned}&\left\|(\epsilon)^{-1}\right\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 3}))}^{-1}\|\nabla\phi_0^r(t)\|_{L^2(\Omega,\mathbb{R}^3)}^2 \le \int_\Omega (\epsilon\nabla\phi_0^r(t))^T\nabla\phi_0^r \,\mathrm{d}\Omega = -\int_\Omega (\epsilon\nabla\chi)^T\nabla\phi_0^r + g\phi_0^r \,\mathrm{d}\Omega\\
&\le \|\epsilon\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 3}))}\|\nabla\chi\|_{L^2(\Omega,\mathbb{R}^3)}\|\nabla\phi_0^r(t)\|_{L^2(\Omega,\mathbb{R}^3)} + \|g\|_{L^2(L^2(\Omega,\mathbb{R}))}\|\phi_0^r(t)\|_{L^2(\Omega,\mathbb{R}^3)}\\
&\le \|\epsilon\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 3}))}^2\|\nabla\chi\|_{L^2(\Omega,\mathbb{R}^3)}^2 + \|\nabla\phi_0^r(t)\|_{L^2(\Omega,\mathbb{R}^3)}^2 + \|g\|_{L^2(L^2(\Omega,\mathbb{R}))}^2 + \|\phi_0^r(t)\|_{L^2(\Omega,\mathbb{R}^3)}^2\\
&= \|\epsilon\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 3}))}^2\|\nabla\chi\|_{L^2(\Omega,\mathbb{R}^3)}^2 + \|\phi_0^r(t)\|_{H^1(\Omega,\mathbb{R}^3)}^2 + \|g\|_{L^2(L^2(\Omega,\mathbb{R}))}^2.\end{aligned}$$

Thus, there exists some constant $\tilde{C} > 0$ such that for almost all $t \in (0,T)$

$$\begin{aligned}&\|\dot{u}(t)\|_{L^2(\Omega,\mathbb{R}^3)}^2 + \|u(t)\|_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)}^2 + \|\dot{u}\|_{L^2(0,t;H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))}^2 \le \tilde{C}\Big(\|u_1\|_{L^2(\Omega,\mathbb{R}^3)}^2 + \|u_0\|_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)}^2\\
&+\|\phi_0^r\|_{L^2(H^1(\Omega,\mathbb{R}))}^2 + \|\chi\|_{L^2(H^1(\Omega,\mathbb{R}))}^2 + \|f\|_{L^2(H^1(\Omega,\mathbb{R}^3)^*)}^2 + \|g\|_{L^2(L^2(\Omega,\mathbb{R}))}^2\Big)\end{aligned}$$

and therefore

$$\begin{aligned}&\|\dot{u}\|_{L^\infty(L^2(\Omega,\mathbb{R}^3))}^2 + \|u\|_{L^\infty(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))}^2 + \|\dot{u}\|_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))}^2 \le \tilde{C}\Big(\|u_1\|_{L^2(\Omega,\mathbb{R}^3)}^2 + \|u_0\|_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)}^2\\
&+\|\phi_0^r\|_{L^2(H^1(\Omega,\mathbb{R}))}^2 + \|\chi\|_{L^2(H^1(\Omega,\mathbb{R}))}^2 + \|f\|_{L^2(H^1(\Omega,\mathbb{R}^3)^*)}^2 + \|g\|_{L^2(L^2(\Omega,\mathbb{R}))}^2\Big).\end{aligned} \tag{3.54}$$

Now we have to estimate $\ddot{u}$. Thus, with the same $\tilde{f}$ as in (3.34), we test the weak form with $(v,0)$, where $v \in B_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)}$ and obtain

$$\begin{aligned}(\rho(t)\ddot{u}(t),v)_{L^2(\Omega,\mathbb{R}^3)} &= \langle\tilde{f}(t),v\rangle_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*,H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} - (\alpha(t)\rho(t)\dot{u}(t),v)_{L^2(\Omega,\mathbb{R}^3)}\\
&\quad - (a(t)u(t),v)_{L^2(\Omega,\mathbb{R}^3)} - (c^E(t)\mathcal{B}u(t),\mathcal{B}v)_{L^2(\Omega,\mathbb{R}^3)}\\
&\quad - (\beta(t)c^E(t)\mathcal{B}\dot{u}(t),\mathcal{B}v)_{L^2(\Omega,\mathbb{R}^3)} - (e^T\nabla\zeta(u),\mathcal{B}v)_{L^2(\Omega,\mathbb{R}^3)}.\end{aligned}$$

With Cauchy-Schwarz inequality it holds that

$$\begin{aligned}|(\rho(t)\ddot{u}(t),v)_{L^2(\Omega,\mathbb{R}^3)}| &\le \langle\tilde{f}(t),v\rangle_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*,H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|\alpha(t)\|_{L^\infty(\Omega,\mathbb{R})}\|\rho(t)\|_{L^\infty(\Omega,\mathbb{R})}\|\dot{u}(t)\|_{L^2(\Omega,\mathbb{R}^3)}\\
&\quad + \|a(t)\|_{L^\infty(\Omega,\mathbb{R})}\|u(t)\|_{L^2(\Omega,\mathbb{R}^3)} + \left\|c^E(t)\right\|_{L^\infty(\Omega,\mathbb{R}^{6\times 6})}\|\mathcal{B}u(t)\|_{L^2(\Omega,\mathbb{R}^3)}\\
&\quad + \|\beta(t)\|_{L^\infty(\Omega,\mathbb{R})}\left\|c^E(t)\right\|_{L^\infty(\Omega,\mathbb{R}^{6\times 6})}\|\mathcal{B}\dot{u}(t)\|_{L^2(\Omega,\mathbb{R}^3)} + \|e(t)\|_{L^\infty_F(\Omega,\mathbb{R}^{3\times 6})}\|\nabla\zeta(u)\|_{L^2(\Omega,\mathbb{R}^3)}.\end{aligned}$$

As $\rho^\star|(\ddot{u}(t),v)_{L^2(\Omega,\mathbb{R}^3)}| \le |(\rho(t)\ddot{u}(t),v)_{L^2(\Omega,\mathbb{R}^3)}|$, using (3.52) and $G = \frac{G_r}{\rho^\star}$ with

$$\begin{aligned}G_r = \max\Big\{&1, \|\alpha\|_{L^\infty(L^\infty(\Omega,\mathbb{R}))}\|\rho\|_{L^\infty(L^\infty(\Omega,\mathbb{R}))}, \|\beta\|_{L^\infty(L^\infty(\Omega,\mathbb{R}))}\left\|c^E\right\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{6\times 6}))},\\
&\left\|c^E\right\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{6\times 6}))} + \|e\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 6}))}^2\left\|(\epsilon)^{-1}\right\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 3}))}, \|a(t)\|_{L^\infty(\Omega,\mathbb{R})}\Big\},\end{aligned}$$

we obtain that

$$\begin{aligned}&\rho^\star|(\ddot{u}(t),v)_{L^2(\Omega,\mathbb{R}^3)}|\\
&\quad\le G_r\left(\langle\tilde{f}(t),v\rangle_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*,H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|\dot{u}(t)\|_{L^2(\Omega,\mathbb{R}^3)} + \|u(t)\|_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|\mathcal{B}\dot{u}(t)\|_{L^2(\Omega,\mathbb{R}^3)}\right)\\
&\Leftrightarrow |(\ddot{u}(t),v)_{L^2(\Omega,\mathbb{R}^3)}| \le G\left(\langle\tilde{f}(t),v\rangle_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*,H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|\dot{u}(t)\|_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|u(t)\|_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)}\right).\end{aligned}$$

Squaring and taking the supremum over all $v \in B^1_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)}$ and repeatedly applying Young's inequality yields

$$\|\ddot{u}(t)\|_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*}^2 \le 3G^2\left(\left\|\tilde{f}(t)\right\|_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*}^2 + \|\dot{u}(t)\|_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)}^2 + \|u(t)\|_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)}^2\right).$$

Integration over $(0,T)$, employing (3.35) and (3.54) as well as using

$$\tilde{C}_{\ddot{u}} = 3G^2\max\left\{\|e\|_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 6}))}^2 + \frac{\tilde{C}}{\min\{1,T\}}, 1 + \frac{\tilde{C}}{\min\{1,T\}}\right\}$$

results in

$$
\begin{aligned}
&\|\ddot{u}\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*)} \le 3G^2 \left(\left\|\tilde{f}\right\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*)} + \|\dot{u}\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + T\|u\|^2_{L^\infty(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} \right) \\
&\le 3G^2 \left(\|e\|^2_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 6}))} \left(\|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} \right) \right. \\
&\quad + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)} + \frac{\tilde{C}}{\min\{1,T\}} \left(\|u_1\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u_0\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} \right. \\
&\quad \left.\left. + \|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)} + \|g\|^2_{L^2(L^2(\Omega,\mathbb{R}))} \right) \right) \\
&\le \tilde{C}_{\ddot{u}} \left(\|u_1\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u_0\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} \right. \\
&\quad \left. + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)} + \|g\|^2_{L^2(L^2(\Omega,\mathbb{R}))} \right).
\end{aligned}
$$

Furthermore, by (3.25) tested with ϕ_0 we obtain

$$
\begin{aligned}
&\left\|(\epsilon)^{-1}\right\|^{-1}_{L^\infty(L^\infty(\Omega,\mathbb{R}^{3\times 3}))} \|\nabla\phi_0\|^2_{L^2(\Omega,\mathbb{R}^3)} \le \left| \int_\Omega (\epsilon\nabla\phi_0)^T \nabla\phi_0 \,\mathrm{d}\Omega \right| \\
&= \left| \int_\Omega (e\mathcal{B}u)^T \nabla\phi_0 - (\epsilon\nabla\chi)^T\nabla\phi_0 - g\phi_0 \,\mathrm{d}\Omega \right| \\
&\underset{\text{C-S ineq.}}{\le} \left(\|e\mathcal{B}u\|_{L^2(\Omega,\mathbb{R}^3)} + \|\epsilon\nabla\chi\|_{L^2(\Omega,\mathbb{R}^3)} \right) \|\nabla\phi_0\|_{L^2(\Omega,\mathbb{R}^3)} + \|g\|_{L^2(\Omega,\mathbb{R})}\|\phi_0\|_{L^2(\Omega,\mathbb{R})}
\end{aligned}
$$

$$
\underset{\text{Poincarè ineq.}}{\le} \left(\|e\mathcal{B}u\|_{L^2(\Omega,\mathbb{R}^3)} + \|\epsilon\nabla\chi\|_{L^2(\Omega,\mathbb{R}^3)} \right) \|\nabla\phi_0\|_{L^2(\Omega,\mathbb{R}^3)} + C_P\|g\|_{L^2(\Omega,\mathbb{R})}\|\nabla\phi_0\|_{L^2(\Omega,\mathbb{R})}. \tag{3.55}
$$

After dividing by $\|\nabla\phi_0\|_{L^2(\Omega,\mathbb{R}^3)} > 0$, ($\|\nabla\phi_0\|_{L^2(\Omega,\mathbb{R}^3)} = 0$ is a trivial case), using the norm equivalence of the H^1_{0,Γ_d}- norm and the L^2-norm of the gradient, integrating over time and applying Hölder's inequality there exists a constant C_ϕ such that

$$
\|\phi_0\|_{L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}))} \le C_\phi \left(\|u\|_{L^\infty(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\chi\|_{L^2(H^1(\Omega,\mathbb{R}))} + \|g\|_{L^2(L^2(\Omega,\mathbb{R}))} \right). \tag{3.56}
$$

Thus, due to (3.54) and (3.56) there exists some constant $\tilde{K} > 0$, such that

$$
\begin{aligned}
&\|\ddot{u}\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*)} + \|\dot{u}\|^2_{L^\infty(L^2(\Omega,\mathbb{R}^3))} + \|u\|^2_{L^\infty(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\dot{u}\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\phi_0\|^2_{L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}))} \\
&\le \tilde{K} \left(\|u_1\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u_0\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} \right. \\
&\quad \left. + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)} + \|g\|^2_{L^2(L^2(\Omega,\mathbb{R}))} \right).
\end{aligned}
$$

□

Remark 3.4. Note that if $g \in L^\infty(L^2(\Omega,\mathbb{R}))$ and $\chi \in H^1\left(H^1(\Omega,\mathbb{R})\right) \cap L^\infty\left(H^1(\Omega,\mathbb{R})\right)$ then we do not necessarily have to integrate (3.55) over time after dividing by $\|\nabla\phi_0\|_{L^2(\Omega,\mathbb{R}^3)} > 0$, since by taking the essential supremum, there exists a constant $\tilde{C}_\phi$ such that

$$
\|\phi_0\|_{L^\infty(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}))} \le \tilde{C}_\phi \left(\|u\|_{L^\infty(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\chi\|_{L^\infty(H^1(\Omega,\mathbb{R}))} + \|g\|_{L^\infty(L^2(\Omega,\mathbb{R}))} \right).
$$

This motivates the definition of the state space W.

Definition 3.5. The state space W is defined as

$$
W := \left\{ z = (u,\phi_0) \in L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)) \times L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R})) : \ \|u\|_{L^\infty(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} < \infty \right\}
$$

and we define for almost all $t \in (0,T)$

$$
\widetilde{W} := H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3) \times H^1_{0,\Gamma_d}(\Omega,\mathbb{R}).
$$

We aim for improving the regularity in space, in order to prove well-definedness of the observation operator in Section 4. For this purpose, the following remark is beneficial.

Remark 3.6. If solutions in spaces with higher regularity are aimed for, we have to perform the Dirichlet lift in higher order Sobolev spaces, where we distinguish the following cases.

1. Assumption D2 holds: according to the Trace Theorem in [17] Chapter 3, we have $\chi \in H^1\left(H^{m+2}(\Omega,\mathbb{R})\right)$.
2. Assumption D3 holds: according to Theorem 4.12 in [17], we have $\chi \in H^1\left(H^{m}(\Omega,\mathbb{R})\right))$. As ϕ^e is constant in space the compatibility condition is fulfilled.

In both cases $\mathrm{Tr}(\chi(t))$ is defined as in (3.1).

Corollary 3.7 (Regularity)**.** Let $m \in \mathbb{N},\ m \geq 2$ and the assumptions of Theorem 3.2 be satisfied with $e \in H^1(L^\infty(\Omega,\mathbb{R}^{3\times 6}) \cap H^{m+1}(\Omega,\mathbb{R}^{3\times 6}))$ and $(f,g) \in L^2(H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3)) \times L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}) \cap H^m(\Omega,\mathbb{R}))$. Suppose either Assumption D2 or Assumption D3 with χ according to Remark 3.6 hold, then for any $u_0 \in H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3),\ u_1 \in H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3)$ there exists a unique solution

$$(u,\phi_0) \in L^2(H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3)) \times L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}) \cap H^m(\Omega,\mathbb{R})) \tag{3.57}$$

with

$$\dot{u} \in L^2(H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3)) \ \text{ and } \ \ddot{u} \in L^2(H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*) \tag{3.58}$$

to the system (3.19)-(3.24).

Proof. Similarly to the proof of Theorem 3.2 we employ (3.25) and define for almost all $t \in (0,T)$

$$\zeta : H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3) \to H^1_{0,\Gamma_d}(\Omega,\mathbb{R}) \cap H^m(\Omega,\mathbb{R}),\ u(t) \mapsto \phi^0_0(t),$$

where $\phi_0(t) = \phi^0_0(t) + \phi^r_0(t) = \zeta(u(t)) + \phi^r_0(t)$ by satisfying equation (3.26) in this setting. As $H^1_{0,\Gamma_d}(\Omega,\mathbb{R})^1 \cap H^m(\Omega,\mathbb{R}) \subset H^m(\Omega,\mathbb{R})$ we define $\phi^r_0 \in L^2(H^m(\Omega,\mathbb{R}))$ by satisfying equation (3.27). We obtain the same weak form of the inhomogeneous piezoelectric PDE including the Dirichlet lift and thus the bilinear form (3.28), which we employ to define the operators

$$\hat{a}_0(t,u,v) := a_0(t,u,v) + \sum_{l=2}^{m} \int_\Omega (\mathcal{B}^l u)^T \mathcal{B}^l v, \tag{3.59}$$

$$\hat{a}_1(t,u,v) := \sum_{l=2}^{m} \int_\Omega -(\mathcal{B}^l u)^T \mathcal{B}^l v, \tag{3.60}$$

$$\hat{b}_0(t,u,v) := b_0(t,u,v) + \sum_{l=2}^{m} \int_\Omega (\mathcal{B}^l u)^T \mathcal{B}^l v, \tag{3.61}$$

$$\hat{b}_1(t,u,v) := b_1(t,u,v) + \sum_{l=2}^{m} \int_\Omega -(\mathcal{B}^l u)^T \mathcal{B}^l v, \tag{3.62}$$

$$\hat{c}(t,u,v) := c(t,u,v), \tag{3.63}$$

where $a_{01} = \hat{a}_0 + \hat{a}_1$ and $b_{01} = \hat{b}_0 + \hat{b}_1$ have the same form as in proof of Theorem 3.2. Note that the right hand side of equation (3.19) is $f + \mathcal{B}^T e^T \nabla\chi =: \tilde{f} \in L^2(H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3))$. Therefore, the requirements of Theorem 1, Paragraph 5, Chapter XVIII of [5] for the right hand side are satisfied. Following the same structure as in proof of Theorem 3.2 now with the higher regularity assumptions we see, as a_0 and b_0 are Hermitian, that $\hat{a}_0$ and $\hat{b}_0$ are Hermitian. Therefore, with the same $\sigma \in \mathbb{R}^+$ defined in (3.38) we conclude with (3.39) and $\hat{\sigma} := \min\{\sigma, 1\}$ that

$$\hat{a}(t,u,u) = a_0(t,u,u) + \sum_{l=2}^{m} \int_\Omega (\mathcal{B}^l u)^T \mathcal{B}^l u \geq \sigma \|u\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \sum_{l=2}^{m} \left\|\mathcal{B}^l u\right\|^2_{L^2(\Omega,\mathbb{R}^3)} \geq \hat{\sigma}\|u\|^2_{H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3)}.$$

Furthermore, the mapping $t \mapsto \hat{a}_0(t,u,v)$ is one time continuously differentiable with the same derivative (3.40). The mapping $t \mapsto \hat{a}_1(t,u,v)$ is one time continuous differentiable as it is fixed in time. Next we prove boundedness of $\hat{a}_1(t,u,v)$ by

$$|\hat{a}_1(t,u,v)|^2 \leq \sum_{l=2}^{m} \int_\Omega \left|(\mathcal{B}^l u)^T \mathcal{B}^l v\right|^2 \leq \sum_{l=2}^{m} \left\|\mathcal{B}^l u\right\|^2_{L^2(\Omega,\mathbb{R}^3)} \left\|\mathcal{B}^l v\right\|^2_{L^2(\Omega,\mathbb{R}^3)} \leq \|u\|^2_{H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3)} \|v\|^2_{H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3)}.$$

Moreover, with the same β_0 defined in (3.41) and (3.42) as well as $\hat{\beta}_0 = \min\{\beta_0, 1\}$ it holds that

$$\hat{b}_0(t,u,u) = b_0(t,u,u) + \sum_{l=2}^{m} \int_\Omega (\mathcal{B}^l u)^T \mathcal{B}^l v \geq \beta_0 \|u\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \sum_{l=2}^{m} \left\|\mathcal{B}^l u\right\|^2_{L^2(\Omega,\mathbb{R}^3)} \geq \hat{\beta}_0 \|u\|^2_{H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3)}.$$

Due to (A4) it holds with $\hat{h} = \max\{1, \|\dot{\rho}\|^2_{L^\infty(L^\infty(\Omega,\mathbb{R}))}\}$ that

$$\begin{aligned}
\left|\hat{b}_1(t,u,v)\right|^2 &\leq |b_1(t,u,v)|^2 + \sum_{l=2}^{m} \int_\Omega \left|(\mathcal{B}^l u)^T \mathcal{B}^l v\right|^2 \\
&\leq \|\dot{\rho}\|^2_{L^\infty(L^\infty(\Omega,\mathbb{R}))} \|u\|^2_{L^2(\Omega,\mathbb{R}^3)} \|v\|^2_{L^2(\Omega,\mathbb{R}^3)} + \sum_{l=2}^{m} \left\|\mathcal{B}^l u\right\|^2_{L^2(\Omega,\mathbb{R}^3)} \left\|\mathcal{B}^l v\right\|^2_{L^2(\Omega,\mathbb{R}^3)} \\
&\leq \hat{h} \|u\|^2_{H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3)} \left(\|v\|^2_{L^2(\Omega,\mathbb{R}^3)} + \sum_{l=2}^{m} \left\|\mathcal{B}^l v\right\|^2_{L^2(\Omega,\mathbb{R}^3)} \right) \leq \hat{h} \|u\|^2_{H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3)} \|v\|^2_{H^m_{\mathcal{B}}(\Omega,\mathbb{R}^3)}.
\end{aligned}$$

As $\hat{c} = c$ stayed the same as in proof of Theorem 3.2 and all introduced operators satisfy the conditions of [5], Chapter XVIII, Paragraph 5, Theorem 1, we conclude the proof. ☐

4. Analysis of the forward operator

Before we define the model operator, we have to specify the parameter space.

Definition 4.1. The parameter space X is defined as

$$\begin{aligned}
X := \big\{ & c^E, e, \epsilon \in H^3(H^3(\Omega, \mathbb{R}^{6\times 6})) \times H^2(H^3(\Omega, \mathbb{R}^{3\times 6})) \times H^2(H^3(\Omega, \mathbb{R}^{3\times 3})) : \\
& c^E, e, \epsilon \ \text{ of structure as in Assumptions } A3 - A5 \big\}
\end{aligned}$$

We now define the model operator corresponding to (3.2)-(3.7).

Definition 4.2 (Model operator)**.** We abbreviate $p = (c^E, e, \epsilon)$ as well as $z = (u, \phi_0)$ and identify the piezoelectric model operator $A : X \times W \to W^*$ for almost all $t \in (0,T)$ via the bilinear form (3.10) and the linear form (3.11) by

$$\begin{aligned}
\langle A(p,z), (v,w)\rangle_{\widetilde{W}^*,\widetilde{W}} &:= B((u,\phi_0),(v,w)) - L(v,w) \\
&= \int_\Omega \rho \ddot{u}^T v + \alpha\rho\dot{u}^T v + \left(c^E \mathcal{B}u + \beta c^E \mathcal{B}\dot{u} + e^T \nabla\phi_0\right)^T \mathcal{B}v \\
&\qquad + (e\mathcal{B}u - \epsilon\nabla\phi_0)^T \nabla w + (e^T\nabla\chi)^T \mathcal{B}v - (\epsilon\nabla\chi)^T \nabla w \,\mathrm{d}\Omega\,.
\end{aligned} \tag{4.1}$$

For the classic reduced approach, we need the parameter-to-state map, motivating the following definition.

Definition 4.3 (Parameter-to-state map)**.** We define the parameter-to-state map

$$\begin{aligned}
S : X &\to W, \\
p &\mapsto z,
\end{aligned}$$

via satisfying the model

$$\forall p \in X : \quad A(p, S(p)) = 0, \tag{4.2}$$

such that

$$\forall z \in W : \ [(p,z) \in X \times W \ \wedge \ A(p,z) = 0] \implies z = S(p),$$

with the model operator defined in Definition 4.2.

Thus, well-definedness of the forward operator needs the existence of the parameter-to-state map S. This is achieved by exploiting the Implicit Function Theorem, i.e., we employ the condition

$$\exists C_A \ \forall (p,z) \in X \times W : \ A'_z(p,z)^{-1} \ \text{ exists and } \ \left\|A'_z(p,z)^{-1}\right\| \leq C_A. \tag{4.3}$$

The following remark guarantees existence and uniqueness of our piezoelectric dynamical system under consideration (3.2)-(3.7).

Remark 4.4 (Existence and Uniqueness of (3.2)-(3.7)). Let Assumption D1 hold. For the piezoelectric dynamical system (3.2)-(3.7) with $\rho \in H^4(L^\infty(\Omega,\mathbb{R}))$ being positive and uniformly bounded, $\alpha,\ \beta \in H^3(L^\infty(\Omega,\mathbb{R}))$ being non-negative and uniformly bounded, and material parameters in X of Definition 4.1, we obtain the same or partially even more regularity of damping and material parameters than in Theorem 3.2. Furthermore, $a \equiv f \equiv g \equiv 0$. Therefore, all assumptions of Theorem 3.2 are satisfied, yielding the existence of a unique weak solution $(u,\phi_0) \in W$ of (3.2)-(3.7) with $\dot{u} \in L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)) \cap L^\infty(L^2(\Omega,\mathbb{R}^3))$ and $\ddot{u} \in L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*)$ and due to Theorem 3.3 the existence of a constant C_p such that

$$\begin{aligned}
&\|\ddot{u}\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*)} + \|\dot{u}\|^2_{L^\infty(L^2(\Omega,\mathbb{R}^3))} + \|u\|^2_{L^\infty(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\dot{u}\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \|\phi_0\|^2_{L^2(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}))} \\
&\leq C_p \Big(\|u_1\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u_0\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} \\
&\quad + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)} + \|g\|^2_{L^2(L^2(\Omega,\mathbb{R}))} \Big) .
\end{aligned}$$

Note, that the model operator of Definition 4.2 includes all boundary conditions and is bijective due to Remark 4.4. Furthermore with an arbitrary direction $\xi = (\mu,\nu) \in \widetilde{W}$ the Gâteaux derivative $\delta_z A(p,z)\xi = A_z(p,z)\xi$ with respect to the state can be identified for almost all $t \in (0,T)$ as

$$\begin{aligned}
\langle A_z(p,z)\xi,(v,w)\rangle_{\widetilde{W}^*,\widetilde{W}} := \int_\Omega &\rho\ddot{\mu}^T v + \alpha\rho\dot{\mu}^T v + \left(c^E\mathcal{B}\mu + \beta c^E\mathcal{B}\dot{\mu} + e^T\nabla\nu\right)^T \mathcal{B}v \\
&+ \left(e\mathcal{B}\mu - \epsilon\nabla\nu\right)^T \nabla w \,\mathrm{d}\Omega\,,
\end{aligned} \tag{4.4}$$

which is also bijective due to Remark 4.4.

Lemma 4.5 (Existence and Regularity of S)**.** *The parameter-to-state map S of Definition 4.3 exists and it holds that $S \in C^1(X,W)$.*

Proof. In order to apply the Implicit Function Theorem we first have to prove Fréchet differentiability of A with respect to the state. Therefore, we consider

$$\langle A(p,z+\xi),(v,w)\rangle_{\widetilde{W}^*,\widetilde{W}} - \langle A(p,z),(v,w)\rangle_{\widetilde{W}^*,\widetilde{W}} - \langle A_z(p,z)\xi,(v,w)\rangle_{\widetilde{W}^*,\widetilde{W}}.$$

Due to the affine linearity of A with respect to the state, it holds for almost all $t \in (0,T)$ that

$$\begin{aligned}
&\langle A(p,z+\xi),(v,w)\rangle_{\widetilde{W}^*,\widetilde{W}} - \langle A(p,z),(v,w)\rangle_{\widetilde{W}^*,\widetilde{W}} - \langle A_z(p,z)\xi,(v,w)\rangle_{\widetilde{W}^*,\widetilde{W}} \\
&= \langle A(p,z),(v,w)\rangle_{\widetilde{W}^*,\widetilde{W}} + \int_\Omega \rho\ddot{\mu}^T v + \alpha\rho\dot{\mu}^T v + \left(c^E\mathcal{B}\mu + \beta c^E\mathcal{B}\dot{\mu} + e^T\nabla\nu\right)^T \mathcal{B}v \\
&\quad + \left(e\mathcal{B}\mu - \epsilon\nabla\nu\right)^T \nabla w \,\mathrm{d}\Omega - \langle A(p,z),(v,w)\rangle_{\widetilde{W}^*,\widetilde{W}} - \langle A_z(p,z)\xi,(v,w)\rangle_{\widetilde{W}^*,\widetilde{W}} \underset{(4.1),(4.4)}{=} 0.
\end{aligned}$$

This yields Fréchet differentiability of A for almost all $t \in (0,T)$ with respect to the state. As A is also affine linear in the material parameters, similar arguments yield Fréchet differentiability of A for almost all $t \in (0,T)$ with respect to the material parameters. Furthermore, the Fréchet derivative of A is continuous as it is linear with respect to the respective variable and bounded due to Theorem 3.3. Therefore, we have that $A \in C^1(X \times W, W^*)$. Due to linearity of $A_z(p,z)\xi$ in z and ξ, boundedness of $A_z(p,z)\xi$ as well as bijectivity of $A_z(p,z)\xi$ it holds due to the Bounded Inverse Theorem that

$$\exists C_A\ \forall (p,z) \in X \times W,\ \forall \xi \in W:\ A_z^{-1}(p,z)\xi \text{ exists and } \left\|A_z^{-1}(p,z)\xi\right\|_{X\times W} \leq C_A.$$

With this, all assumptions of the Implicit Function Theorem are satisfied, thus applying it yields that there exists $S \in C^1(X,W)$ defined as in Definition 4.3. ☐

Due to Remark 4.4, the parameter-to-state map $S \in C^1(X,W)$, see Definition 4.3 is well-defined, as for an arbitrary fixed $p \in X$ it is not possible to have more than one state $z \in W$. Furthermore, S is non-linear, which can be seen by the structure of the model operator, which is used to define S.

To state and solve an inverse problem we need additional observations. In our case we obtain a measured charge pulse. Therefore, the observation operator reads as

$$\tilde{C}(p,z) := \int_{\Gamma_e} \left(e\mathcal{B}u - \epsilon\nabla\phi_0 - \epsilon\nabla\chi\right)\cdot n \,\mathrm{d}\Gamma\,,$$

which means that the electrodes are conductive and thus the charge is distributed equally on the loaded electrode. As $(u, \phi_0) \in W$ and $\chi \in H^1(H^1(\Omega, \mathbb{R}))$, due to Theorem 3.3, there exists a constant $\tilde{L} > 0$, such that $\|u\|_{L^\infty(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} \leq \tilde{L}$, $\|\phi_0\|_{L^2(H^1_{0,\Gamma_e}(\Omega,\mathbb{R}^3))} \leq \tilde{L}$ and $\|\chi\|_{H^1(H^1(\Omega,\mathbb{R}^3))} \leq \tilde{L}$. Therefore

$$e\mathcal{B}u =: h_1 \in L^2(L^2(\Omega, \mathbb{R}^3)) \tag{4.5}$$

$$\epsilon\nabla\phi_0 =: h_2 \in L^2(L^2(\Omega, \mathbb{R})) \tag{4.6}$$

$$\epsilon\nabla\chi =: h_3 \in H^1(L^2(\Omega, \mathbb{R})), \tag{4.7}$$

which yields that $\tilde{C}$ is not well-defined as we have a boundary integral with L^2-functions, which in general cannot be evaluated. However, this boundary integral can be converted into a volume integral on an open ball around the boundary Γ_e, denoted by $U_\gamma(\Gamma_e)$ with $\gamma > 0$, if one lacks in space regularity of the state. This requires the continuous extension of the normal vector in this neighborhood, which is obtained by solving the eikonal equation.

Definition 4.6 (Observation operator)**.** Let $\gamma > 0$ be fixed and small enough and $Y = L^2(0, T)$. Then, we define the observation operator as $C^\gamma : X \times W \to Y$ by

$$C^\gamma(p, z) := |U_\gamma(\Gamma_e)|^{-1} \int_{U_\gamma(\Gamma_e)} (e\mathcal{B}u - \epsilon\nabla\phi_0 - \epsilon\nabla\chi) \cdot \nabla b \,\mathrm{d}\Omega,$$

where b solves the eikonal equation

$$\|\nabla b\|_{L^2(U_\gamma(\Gamma_e))} = 1 \;\text{ in }\; U_\gamma(\Gamma_e) \tag{4.8}$$

$$b = 0 \;\text{ on }\; \partial U_\gamma(\Gamma_e). \tag{4.9}$$

This operator is well-defined and bounded, as for some fixed and small enough $\gamma > 0$, we obtain

$$\begin{aligned} \|C^\gamma(p, z)\|_Y^2 &\leq \|h_1\|^2_{L^\infty(L^2(\Omega,\mathbb{R}^3))} + \|h_2\|_{L^2(L^2(\Omega,\mathbb{R}^3))} + \|h_3\|_{H^1(L^2(\Omega,\mathbb{R}^3))} \\ &\leq C_O \Big(\|u_1\|^2_{L^2(\Omega,\mathbb{R}^3)} + \|u_0\|^2_{H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)} + \|\phi_0^r\|^2_{L^2(H^1(\Omega,\mathbb{R}))} + \|\chi\|^2_{L^2(H^1(\Omega,\mathbb{R}))} \\ &\qquad + \|f\|^2_{L^2(H^1(\Omega,\mathbb{R}^3)^*)} + \|g\|^2_{L^2(L^2(\Omega,\mathbb{R}))} \Big) \end{aligned} \tag{4.10}$$

for some constant $C_O > 0$, due to Remark 4.4 and Definition 4.6. Note that for some fixed and small enough $\gamma > 0$ the observation operator C^γ is affine linear in the state z and the parameters p. Furthermore, C^γ is continuously Fréchet differentiable with respect to the state, as for an arbitrary fixed direction $\xi = (\mu, \nu) \in \widetilde{W}$, the Gâteaux derivative reads as

$$C_z(p, z)\xi := |U_\gamma(\Gamma_e)|^{-1} \int_{U_\gamma(\Gamma_e)} (e\mathcal{B}\mu - \epsilon\nabla\nu) \cdot \nabla b \,\mathrm{d}\Omega, \tag{4.11}$$

yielding

$$C(p, z + \xi) - C(p, z) - C_z(p, z)\xi = C(p, \xi) - C_z(p, z)\xi = 0. \tag{4.12}$$

By similar arguments, we deduce that C^γ is continuously Fréchet differentiable with respect to the material parameters. Due to [6], the eikonal equation admits a classical solution which is the proper extension of the normal vector on Γ_e and therefore converges for $\gamma \to 0$ to the normal vector on Γ_e. This yields that

$$C^\gamma \underset{\gamma\to 0}{\longrightarrow} \tilde{C}.$$

Remark 4.7. Note that for solutions with higher regularity, i.e., solutions as in Corollary 3.7 with $m \geq 2$, the observation operator will be $\tilde{C}$, as it is well-defined and bounded due to similar arguments as in (4.10). Furthermore, we deduce in the same manner as above (see(4.11)-(4.12)), that $\tilde{C}$ is continuously Fréchet differentiable with respect to the state and the material parameters.

In real world application the assumptions of Corollary 3.7 with at least $m = 2$ are usually fulfilled. From now on, if the assumptions of Corollary 3.7 with at least $m = 2$ are not fulfilled, we fix a sufficiently small $\gamma > 0$ and abbreviate $C = C^\gamma$, otherwise we use $C = \tilde{C}$.
Inserting $S(p)$ into the observation operator C identifies the forward operator $F : X \to Y$, i.e.,

$$C(p, S(p)) = F(p) = y.$$

This casts the problem into a single operator equation for the unknown p. Hence, the forward operator F acts directly on p and returns the data y. Note that due to the properties of the parameter-to-state map and the observation operator, we can conclude that the forward operator is well-defined, non-linear and continuously Fréchet differentiable as it inherits these properties from C and S. By denoting the noisy measurements with y^δ and introducing a weakly lower semi-continuous regulariser $\mathcal{R}_\tau : X \to \mathbb{R}$ with an regularization parameter $\tau > 0$, we now define the regularized target functional $J : X \to \mathbb{R}$ by

$$J(p) := \frac{1}{2}\left\|F(p) - y^\delta\right\|^2_{L^2(0,T)} + \mathcal{R}_\tau(p). \tag{4.13}$$

Using an optimization approach, the inverse problem aims at finding a minimizer of

$$\min_{p \in X} J(p). \tag{4.14}$$

Proposition 4.8. Let X be defined as in Definition 4.1. and let the regulariser $\mathcal{R}_\tau$ be weakly lower semi-continuous. Then there exists a minimizer of the functional $J : X \to \mathbb{R}$ defined in (4.13).

Proof. First, we directly obtain that X is convex and closed, since all eigenvalues of c^E and ϵ are bounded away from 0. Second, for $K := [0,T] \times \bar{\Omega} \subset \mathbb{R}^4$, which is closed and bounded and therefore compact, we have that functions $p \in X$ are continuous on K due to Lemma 3.1 and hence uniformly bounded on K, which yields boundedness of X. Third, we prove that $J : X \to \mathbb{R}$ defined in (4.13) is weakly lower semi-continuous. We know that F is continuous and so is $h : X \to Y,\ h(p) = F(p) - y^\delta$. Furthermore, the norm of any normed space is weakly lower semi-continuous. Given $h : X \to Y$ is continuous, and $\|\cdot\| : Y \to \mathbb{R}$ is weakly lower semicontinuous, we aim to show that $J(p) = \|h(p)\|_Y$ is weakly lower semi-continuous. Let $p_n \rightharpoonup p_0$ weakly in X. We want to show that $\liminf_{n\to\infty} J(p_n) \geq J(p_0)$. Since h is continuous, we have $h(p_n) \underset{n\to\infty}{\longrightarrow} h(p_0)$, which implies $h(p_n) \underset{n\to\infty}{\rightharpoonup} h(p_0)$. We denote $y_n = h(p_n)$ and $y_0 = h(p_0)$. Therefore we have $p_n \rightharpoonup p_0 \Rightarrow y_n \rightharpoonup y_0$, as $n \to \infty$. Since the norm is weakly lower semi-continuous, for any sequence $y_n \rightharpoonup y_0$ converging weakly in Y, we have

$$\liminf_{n\to\infty} \|y_n\| \geq \|y_0\|,$$

meaning that

$$p_n \rightharpoonup p_0 \Rightarrow \liminf_{n\to\infty} \|h(p_n)\| \geq \|h(p_0)\|.$$

As the sum of weakly lower semi-continuous functions is weakly lower semi-continuous, we obtain weakly lower semi-continuity of J. Hence, existence of a minimizer to the optimization problem (4.14) is guaranteed by Tonelli's Theorem. □

In the parametrization approaches (2.1) and (2.2) X only consists of constant real valued matrices with the same properties. However, X stays bounded, convex and closed as well as a subset of a finite dimensional real-valued vector space. Thus, we have the following first order optimality conditions

- $A(p, S(p)) = 0$ (state equation),
- $A'_z(p, S(p))^* q = -C'_z(p, S(p))^*(C(p, S(p)) - y^\delta)$ (adjoint equation),

where the superscript * of an operator denotes its adjoint, yielding

$$J'(p) := \langle A'_p(p, S(p)), q\rangle_{W^*,W} + \partial\mathcal{R}_\tau(p) + (C'_p(p, S(p)), C(p, S(p)) - y^\delta)_Y = 0,$$

where $q = (q_1, q_2) \in W$ refers to the adjoint state. Then

$$\begin{aligned}\langle A'_p(p, S(p)), q\rangle_{W^*,W} = \int_0^T \int_\Omega &\left(\mathbb{1}_{c^E}\mathcal{B}u + \beta\mathbb{1}_{c^E}\mathcal{B}\dot{u} + \mathbb{1}_e^T\nabla\phi_0\right)^T \mathcal{B}q_1 \\ &+ \left(\mathbb{1}_e\mathcal{B}u - \mathbb{1}_\epsilon\nabla\phi_0\right)^T \nabla q_2 + (\mathbb{1}_e^T\nabla\chi)^T\mathcal{B}q_1 - (\mathbb{1}_\epsilon\nabla\chi)^T\nabla q_2 \,\mathrm{d}\Omega\,\mathrm{d}t\end{aligned}$$

and, depending on the space regularity of the state,

$$\begin{aligned}&(C'_p(p, S(p)), C(p, S(p)) - y^\delta)_Y \\ &= \int_0^T \int_{\Gamma_e} (\mathbb{1}_e\mathcal{B}u - \mathbb{1}_\epsilon\nabla\phi_0 - \mathbb{1}_\epsilon\nabla\chi)\cdot\nabla c\,\mathrm{d}\Gamma \left(\int_{\Gamma_e} (e\mathcal{B}u - \epsilon\nabla\phi_0 - \epsilon\nabla\chi)\cdot\nabla n\,\mathrm{d}\Gamma - y^\delta\right)\mathrm{d}t\,,\end{aligned}$$

or

$$(C_p'(p,S(p)),C(p,S(p))-y^\delta)_Y=\int_0^T |U_\gamma(\Gamma_e)|^{-2}$$
$$\int_{U_\gamma(\Gamma_e)}(\mathbb{1}_e\mathcal{B}u-\mathbb{1}_\epsilon\nabla\phi_0-\mathbb{1}_\epsilon\nabla\chi)\cdot\nabla b\,\mathrm{d}\Omega\left(\int_{U_\gamma(\Gamma_e)}(e\mathcal{B}u-\epsilon\nabla\phi_0-\epsilon\nabla\chi)\cdot\nabla b\,\mathrm{d}\Omega-y^\delta\right)\mathrm{d}t\,,$$

where b solves (4.8)-(4.9), $\mathbb{1}_{c^E}, \mathbb{1}_e$ and $\mathbb{1}_\epsilon$ have the same structure as the material parameter matrices in Assumption 2.2, where the non-zero entries are constant 1. Hence, the adjoint state is essential. We now focus on the unique existence of the adjoint state, which can be seen as revealing the influence of a cause on a target functional. Therefore, it naturally arises in the context of parameter identification problems, especially in computing gradients of the regularized target functional using Lagrange formalism. To derive the adjoint PDE system of the piezoelectric dynamical system (3.2)-(3.7), we have to differentiate the model operator, see Definition 4.2, with respect to the state. Similarly to (4.4), we consider an arbitrary direction denoted by $\kappa:=(d,\psi)\in\tilde{W}$ with d satisfying the initial conditions $d(t=0)=\dot{d}(t=0)=0$, as κ can be viewed as infinitesimal perturbation of the solution. For almost all $t\in(0,T)$ we obtain

$$\begin{aligned}&\langle A_z'(p,z)\kappa,(v,w)\rangle_{\widetilde{W}^*,\widetilde{W}}\\&=\int_\Omega\rho\ddot{d}^Tv+\alpha\rho\dot{d}^Tv+\left(c^E\mathcal{B}d+\beta c^E\mathcal{B}\dot{d}+e^T\nabla\psi\right)^T\mathcal{B}v+(e\mathcal{B}d-\epsilon\nabla\psi)^T\nabla w\,\mathrm{d}\Omega\,.\end{aligned}$$

Denoting the adjoint state with $q=(q_1,q_2)\in W$ and using the inner product of W yields

$$\langle A_z'(p,z)\kappa,(q_1,q_2)\rangle_{W^*,W}=\int_0^T\langle A_z'(p,z)\kappa,(q_1,q_2)\rangle_{\widetilde{W}^*,\widetilde{W}}\,\mathrm{d}t$$
$$=\int_0^T\int_\Omega\rho\ddot{d}^Tq_1+\alpha\rho\dot{d}^Tq_1+\left(c^E\mathcal{B}d+\beta c^E\mathcal{B}\dot{d}+e^T\nabla\psi\right)^T\mathcal{B}q_1+(e\mathcal{B}d-\epsilon\nabla\psi)^T\nabla q_2\,\mathrm{d}\Omega\,\mathrm{d}t\,. \tag{4.15}$$

We consider every single term individually, with the initial conditions $q_1(T)=0$ on Ω and $\dot{q}_1(T)=0$ on Ω. Let $q_{1\rho}=\rho q_1$, i.e. $q_{1\rho}(T)=\rho q_1(T)=0$ and $\dot{q}_{1\rho}(T)=\dot{\rho}q_1(T)+\rho\dot{q}_1(T)=0$. Then, for the first term in (4.15), we have

$$\int_0^T\int_\Omega\rho\ddot{d}^Tq_1\,\mathrm{d}\Omega\,\mathrm{d}t\underset{q_{1\rho}(T)=\dot{d}(0)=0}{=}\int_0^T\int_\Omega-\dot{d}^T\dot{q}_{1\rho}\,\mathrm{d}\Omega\,\mathrm{d}t\underset{\dot{q}_{1\rho}(T)=d(0)=0}{=}\int_0^T\int_\Omega\ddot{q}_{1\rho}^Td\,\mathrm{d}\Omega\,\mathrm{d}t\,.$$

For the second term in (4.15), we define $q_{1\alpha\rho}=\alpha\rho q_1$, i.e., $q_{1\alpha\rho}(T)=\alpha\rho q_1(T)=0$, and obtain

$$\int_0^T\int_\Omega\alpha\rho\dot{d}^Tq_1\,\mathrm{d}\Omega\,\mathrm{d}t\underset{q_{1\alpha\rho}(T)=d(0)=0}{=}\int_0^T\int_\Omega-d^T\dot{q}_{1\alpha\rho}\,\mathrm{d}\Omega\,\mathrm{d}t\,.$$

For the third term in (4.15), we deduce

$$\int_0^T\int_\Omega\left(c^E\mathcal{B}d\right)^T\mathcal{B}q_1\,\mathrm{d}\Omega\,\mathrm{d}t\underset{c^E\text{sym.}}{=}\int_0^T\int_\Omega-\mathcal{B}^T\left(c^E\mathcal{B}q_1\right)^Td\,\mathrm{d}\Omega\,\mathrm{d}t+\int_0^T\int_\Gamma\mathcal{N}^T\left(c^E\mathcal{B}q_1\right)^Td\,\mathrm{d}\Gamma\,\mathrm{d}t\,.$$

With $q_\mathcal{B}=\beta c^E\mathcal{B}q_1$ and keeping the symmetry of c^E in mind, the fourth term in (4.15) yields

$$\begin{aligned}\int_0^T\int_\Omega\left(\beta c^E\mathcal{B}\dot{d}\right)^T\mathcal{B}q_1\,\mathrm{d}\Omega\,\mathrm{d}t&=-\int_0^T\int_\Omega(\mathcal{B}d)^T\dot{q}_\mathcal{B}\,\mathrm{d}\Omega\,\mathrm{d}t+\int_\Omega(\mathcal{B}d(T))^T\beta(T)c^E(T)\mathcal{B}q_1(T)\,\mathrm{d}\Omega\\&\quad+\int_\Omega\left(\mathcal{B}^Tq_\mathcal{B}(0)\right)^Td(0)\,\mathrm{d}\Omega-\int_\Gamma(\mathcal{N}^Tq_\mathcal{B}(0))^Td(0)\,\mathrm{d}\Gamma\\&=-\int_0^T\int_\Omega(\mathcal{B}d)^T\dot{q}_\mathcal{B}\,\mathrm{d}\Omega\,\mathrm{d}t-\int_\Omega\mathcal{B}^T\left(\beta(T)c^E(T)\mathcal{B}d(T)\right)q_1(T)\,\mathrm{d}\Omega\\&\quad+\int_\Gamma\mathcal{N}^T\left(\beta(T)c^E(T)\mathcal{B}d(T)\right)q_1(T)\,\mathrm{d}\Gamma\\&=\int_0^T\int_\Omega d^T\mathcal{B}^T\dot{q}_\mathcal{B}\,\mathrm{d}\Omega\,\mathrm{d}t-\int_0^T\int_\Gamma d^T\mathcal{N}^T\dot{q}_\mathcal{B}\,\mathrm{d}\Gamma\,\mathrm{d}t\,.\end{aligned}$$

Consequently, the last three terms of (4.15) can be achieved by

$$\int_0^T \int_\Omega \left(e^T \nabla \psi\right)^T \mathcal{B} q_1 \,\mathrm{d}\Omega\,\mathrm{d}t = \int_0^T \int_\Omega -\nabla\cdot(e\mathcal{B}q_1)\,\psi\,\mathrm{d}\Omega\,\mathrm{d}t + \int_0^T \int_\Gamma n\cdot(e\mathcal{B}q_1)\,\psi\,\mathrm{d}\Gamma\,\mathrm{d}t$$
$$\int_0^T \int_\Omega (e\mathcal{B}d)^T \nabla q_2 \,\mathrm{d}\Omega\,\mathrm{d}t = \int_0^T \int_\Omega -d^T\mathcal{B}^T\left(e^T\nabla q_2\right)\mathrm{d}\Omega\,\mathrm{d}t + \int_0^T \int_\Gamma d^T\mathcal{N}^T\left(e^T\nabla q_2\right)\mathrm{d}\Gamma\,\mathrm{d}t$$
$$\int_0^T \int_\Omega -(\epsilon\nabla\psi)^T \nabla q_2 \,\mathrm{d}\Omega\,\mathrm{d}t = \int_0^T \int_\Omega \nabla\cdot\left(\epsilon^T\nabla q_2\right)\psi\,\mathrm{d}\Omega\,\mathrm{d}t - \int_0^T \int_\Gamma n\cdot\left(\epsilon^T\nabla q_2\right)\psi\,\mathrm{d}\Gamma\,\mathrm{d}t\,.$$

As we deal with end conditions, we use a time transformation from t to $T-t$ such that $\tilde{q}(t) = q(T-t)$. Therefore, $\dot{\tilde{q}}(t) = -\dot{q}(T-t)$ and $\ddot{\tilde{q}}(t) = \ddot{q}(T-t)$. Note that $\dot{q}_{1\alpha\rho} = (\dot{\alpha}\rho + \alpha\dot{\rho})\,q_1 + \alpha\rho\dot{q}_1$ and $\ddot{q}_{1\rho} = \ddot{\rho}q_1 + 2\dot{\rho}\dot{q}_1 + \rho\ddot{q}_1$ as well as $\dot{q}_{\mathcal{B}} = \left(\dot{\beta}c^E + \beta\dot{c}^E\right)\mathcal{B}q_1 + \beta c^E\mathcal{B}\dot{q}_1$. Furthermore, we introduce

$$\frac{\partial J}{\partial u}(p,S(p)) = C_u'(p,S(p))^*(C(p,S(p)) - y^\delta) \in L^2(0,T),$$
$$\frac{\partial J}{\partial \phi_0}(p,S(p)) = C_\phi'(p,S(p))^*(C(p,S(p)) - y^\delta) \in L^2(0,T).$$

Then, we obtain the time transformed adjoint PDE

$$\tilde{\rho}\ddot{\tilde{q}}_1 + \left(2\dot{\tilde{\rho}} + \tilde{\alpha}\tilde{\rho}\right)\dot{\tilde{q}}_1 + \left(\ddot{\tilde{\rho}} + \dot{\tilde{\alpha}}\tilde{\rho} + \tilde{\alpha}\dot{\tilde{\rho}}\right)\tilde{q}_1$$
$$-\mathcal{B}^T\left(\left(\left(\dot{\tilde{\beta}}+1\right)\tilde{c}^E + \tilde{\beta}\dot{\tilde{c}}^E\right)\mathcal{B}\tilde{q}_1 + \tilde{\beta}\tilde{c}^E\mathcal{B}\dot{\tilde{q}}_1 + \tilde{e}^T\nabla\tilde{q}_2\right) = \frac{\partial J}{\partial u}(\tilde{p},S(\tilde{p})) \quad \text{in } \Omega\times(0,T) \tag{4.16}$$
$$-\nabla\cdot\left(\tilde{e}\mathcal{B}\tilde{q}_1 - \tilde{\epsilon}^T\nabla\tilde{q}_2\right) = \frac{\partial J}{\partial \phi_0}(\tilde{p},S(\tilde{p})) \quad \text{in } \Omega\times(0,T) \tag{4.17}$$
$$n\cdot\left(\tilde{e}\mathcal{B}\tilde{q}_1 - \epsilon^T\nabla\tilde{q}_2\right) = 0 \quad \text{on } \Gamma_n\times(0,T) \tag{4.18}$$
$$\mathcal{N}^T\left(\left(\left(\dot{\tilde{\beta}}+1\right)\tilde{c}^E + \tilde{\beta}\dot{\tilde{c}}^E\right)\mathcal{B}\tilde{q}_1 + \tilde{\beta}\tilde{c}^E\mathcal{B}\dot{\tilde{q}}_1 + e^T\nabla q_2\right) = 0 \quad \text{on } \partial\Omega\times(0,T) \tag{4.19}$$
$$\tilde{q}_1(0) = 0 \quad \text{in } \Omega \tag{4.20}$$
$$\dot{\tilde{q}}_1(0) = 0 \quad \text{in } \Omega. \tag{4.21}$$

Corollary 4.9 (Existence and Uniqueness of the adjoint system)**.** Let Assumption 2.2 hold but with $\rho \in H^4(L^\infty(\Omega,\mathbb{R}))$, $\alpha,\beta \in H^3(L^\infty(\Omega,\mathbb{R}))$ and let Assumption D1 hold. Suppose the material parameters are elements of X defined in Definition 4.1. Then there exists a unique weak solution $(\tilde{q_1},\tilde{q_2}) \in W$ with $\dot{\tilde{q}}_1 \in L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)) \cap L^\infty(L^2(\Omega,\mathbb{R}^3))$ and $\ddot{\tilde{q}}_1 \in L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*)$ to the system (4.16)-(4.21) with the even more general right-hand side

$$\frac{\partial J}{\partial(u,\phi)} \in L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*) \times L^2(L^2(\Omega,\mathbb{R})).$$

Furthermore, there exists a constant $C_a > 0$ such that

$$\left\|\ddot{\tilde{q}}_1\right\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*)} + \left\|\dot{\tilde{q}}_1\right\|^2_{L^\infty(L^2(\Omega,\mathbb{R}^3))} + \|\tilde{q}_1\|^2_{L^\infty(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))} + \left\|\dot{\tilde{q}}_1\right\|^2_{L^2(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3))}$$
$$+ \|\tilde{q}_2\|_{L^\infty(H^1_{0,\Gamma_d}(\Omega,\mathbb{R}))} \le C_a\left(\left\|\frac{\partial J}{\partial u}\right\|^2_{L^2\left(H^1_{\mathcal{B}}(\Omega,\mathbb{R}^3)^*\right)} + \left\|\frac{\partial J}{\partial \phi_0}\right\|^2_{L^2(L^2(\Omega,\mathbb{R}^3))}\right). \tag{4.22}$$

Proof. First we consider

$$\tilde{\rho}\ddot{\tilde{q}}_1 + \left(2\dot{\tilde{\rho}} + \tilde{\alpha}\tilde{\rho}\right)\dot{\tilde{q}}_1 + \left(\ddot{\tilde{\rho}} + \dot{\tilde{\alpha}}\tilde{\rho} + \tilde{\alpha}\dot{\tilde{\rho}}\right)\tilde{q}_1$$
$$-\mathcal{B}^T\left(\left(\left(\dot{\tilde{\beta}}+1\right)\tilde{c}^E + \tilde{\beta}\dot{\tilde{c}}^E\right)\mathcal{B}\tilde{q}_1 + \tilde{\beta}\tilde{c}^E\mathcal{B}\dot{\tilde{q}}_1 + \tilde{e}^T\nabla\tilde{q}_2\right) = \frac{\partial J}{\partial u}(\tilde{p},S(\tilde{p})).$$

As $\rho \in H^4(L^\infty(\Omega,\mathbb{R}))$ is positive and uniformly bounded and due to Lemma 3.1 three times continuously differentiable in time it holds that $\tilde{\rho} \in H^4(L^\infty(\Omega,\mathbb{R}))$ is positive, uniformly bounded and three times continuously differentiable in time. Furthermore, $\alpha \in H^3(L^\infty(\Omega,\mathbb{R}))$ is a non-negative and uniformly bounded damping function and due to Lemma 3.1 two times continuously differentiable in time.

Consequently, $\tilde{\alpha} \in H^3(L^\infty(\Omega, \mathbb{R}))$ is non-negative, uniformly bounded and two times continuously differentiable in time yielding

$$(2\dot{\tilde{\rho}} + \tilde{\alpha}\tilde{\rho}) \in H^3(L^\infty(\Omega, \mathbb{R}))$$

is non-negative, uniformly bounded and two times continuously differentiable in time. Additionally, it follows that

$$a = \left(\ddot{\tilde{\rho}} + \dot{\tilde{\alpha}}\tilde{\rho} + \tilde{\alpha}\dot{\tilde{\rho}}\right) \in H^2(L^\infty(\Omega, \mathbb{R}))$$

is non-negative, uniformly bounded and one time continuously differentiable in time. As $\beta \in H^3(L^\infty(\Omega, \mathbb{R}))$ and $c^E \in H^3(H^3_F(\Omega, \mathbb{R}^{6\times 6}))$, it holds that

$$\left(\left(\dot{\tilde{\beta}} + 1\right)\tilde{c}^E + \tilde{\beta}\dot{\tilde{c}}^E\right) \in H^2(L^\infty(\Omega, \mathbb{R}^{6\times 6})).$$

Therefore all parameters have the same or even higher regularity and the same characteristics as the corresponding parameters in Theorem 3.2. The initial conditions are even simpler, as they are both zero. The right-hand side of the adjoint equation has the same regularity as f and g in Theorem 3.2 and, as usual for adjoint equations, the Dirichlet boundary is homogeneous, meaning that $\chi \equiv 0$. Therefore, all assumptions of Theorem 3.2 and Theorem 3.3 are satisfied, which proves this corollary. □

Our adjoint system (4.16)-(4.21) has a right-hand side with even more regularity in space, as it does not vary in space, and the same regularity in time. Hence, all assumptions are fulfilled to apply Corollary 4.9. This result means that there is a unique adjoint state for each PDE solution and not only a Lagrangian multiplier (i.e. adjoint state) associated to a minimizer. The set of multipliers is bounded, which one gets by the Zowe-Kurcyusz condition.

5. A NUMERICAL EXAMPLE

To computationally solve the inverse problem, we follow the discretize-then-optimize approach where we discretize the problem setting, i.e., the forward operator and the corresponding spaces first and optimize afterwards. Here, easy access to the first derivative of the forward operator F is provided by algorithmic differentiation (AD), see [9]. The central concept is that the computation of a discretized operator can be decomposed into a finite sequence of elementary operations, where then the chain rule is applied systematically. The reverse mode of AD can be seen as a discrete analogue of the continuous adjoint PDE enabling an efficient gradient calculation. The analysis of the continuous adjoint system ensures that, with appropriate discretization, the discrete adjoint state converges to the continuous adjoint state as the discretization gets finer. Hence, the analysis of the continuous problem (4.16) - (4.21) is, an important prerequisite for the discretize-then-optimize approach. For the space discretization we use a classic finite element method (FEM) implemented by the finite element tool FEniCS [2] in dolfin version 2019.2.0.dev0, using AD via the dolfin adjoint [22] library of FEniCS in version 2019.1.0. For the temporal discretization, we employ the Crank–Nicolson scheme. Therefore, we set $z = \dot{u}$ and $\dot{z} = \ddot{u}$ and rewrite the weak form of system (3.19)-(3.24) for all $v \in H^1_{\mathcal{B}}(\Omega), w \in H^1_{0,\Gamma_g}(\Omega), y \in H^1_{\mathcal{B}}(\Omega)$ as

$$\begin{aligned}&\langle \rho\dot{z}, v\rangle_{L^2(\Omega)} + \alpha\langle \rho z, v\rangle_{L^2(\Omega)} + \langle c^E\mathcal{B}u, \mathcal{B}v\rangle_{L^2(\Omega)} + \langle \beta c^E\mathcal{B}z, \mathcal{B}v\rangle_{L^2(\Omega)}\\ &+\langle e^T\nabla\phi, \mathcal{B}v\rangle_{L^2(\Omega)} + \langle e\mathcal{B}u, \nabla w\rangle_{L^2(\Omega)} - \langle \epsilon\nabla\phi, \nabla w\rangle_{L^2(\Omega)} = 0\end{aligned} \tag{5.1}$$

$$\langle \dot{u}, y\rangle_{L^2(\Omega)} - \langle z, y\rangle_{L^2(\Omega)} = 0. \tag{5.2}$$

Then we obtain the Crank-Nicolson time discretized system,

$$\begin{aligned}2\langle \rho z_{n+1}, v\rangle_{L^2(\Omega)} - 2\langle \rho z_n, v\rangle_{L^2(\Omega)} + \Delta t\alpha\langle \rho z_n, v\rangle_{L^2(\Omega)} + \Delta t\alpha\langle \rho z_{n+1}, v\rangle_{L^2(\Omega)}\\ +\Delta t\langle c^E\mathcal{B}u_n, \mathcal{B}v\rangle_{L^2(\Omega)} + \Delta t\langle c^E\mathcal{B}u_{n+1}, \mathcal{B}v\rangle_{L^2(\Omega)} + \Delta t\beta\langle c^E\mathcal{B}z_n, \mathcal{B}v\rangle_{L^2(\Omega)}\\ +\Delta t\beta\langle c^E\mathcal{B}z_{n+1}, \mathcal{B}v\rangle_{L^2(\Omega)} + \Delta t\langle e^T\nabla\phi_n, \mathcal{B}v\rangle_{L^2(\Omega)} + \Delta t\langle e^T\nabla\phi_{n+1}, \mathcal{B}v\rangle_{L^2(\Omega)}\\ +\Delta t\langle e\mathcal{B}u_n, \nabla w\rangle_{L^2(\Omega)} + \Delta t\langle e\mathcal{B}u_{n+1}, \nabla w\rangle_{L^2(\Omega)} - \Delta t\langle \epsilon\nabla\phi_n, \nabla w\rangle_{L^2(\Omega)} - \Delta t\langle \epsilon\nabla\phi_{n+1}, \nabla w\rangle_{L^2(\Omega)} = 0\\ 2\langle u_{n+1}, y\rangle_{L^2(\Omega)} - 2\langle u_n, y\rangle_{L^2(\Omega)} - \Delta t\langle z_{n+1}, y\rangle_{L^2(\Omega)} - \Delta t\langle z_n, y\rangle_{L^2(\Omega)} = 0,\end{aligned}$$

where Δt is the time step size, which will be chosen in the numerical realization as 10^{-6} and n is the current step up to $N = 1000$.

As geometry we consider a piezoelectric ring, with outer radius of 6.35 mm, inner radius of 2.6 mm and thickness of 1mm. Hence, the geometry is rotationally symmetric. To reduce the computational effort we exploit the inherent rotational symmetry and transform the ring into a rectangular domain by adopting cylindrical coordinates rather than Cartesian coordinates, where the z-axis is selected as the axis of rotation. In this coordinate system, the piezoelectric ring is assumed to be a homogeneous and transversely isotropic material. The latter is physically essential to exploit the rotational symmetry. In addition, we have converted the setting from seconds to milliseconds, which results in a better condition number of the PDE system, as the magnitudes of the material parameters differ significantly less.
As an example for the numerical realization of the inverse problem, we assume that the elasticity parameter and the permittivity parameter are constant, i.e., parameterized in a polynomial way as in identity (2.1), with a polynomial order 0 and the coupling parameter e is parameterized as in identity (2.1), with an polynomial order 1, where $\theta(t) := 25 + 7\sqrt{(0.01t)}$. Since the problem of identifying the material parameters is extremely challenging due to very different orders of sensitivities even in the frequency dependent case, see e.g., [12], [14], [16], [24], we want to reconstruct one entry of the coupling parameter e, namely e_{33}. To simulate the data we started with the following set of material parameters

$$c_{11}^E = 151400, \quad c_{12}^E = 132700, \quad c_{13}^E = 83600, \quad c_{33}^E = 128800, \quad c_{44}^E = 25900, \quad \epsilon_{11} = 2700$$
$$\epsilon_{33} = 5500, \quad e_{15} = 11\theta(t) + 9125, \quad e_{31} = -7\theta(t) - 5025, \quad e_{33} = 24\theta(t) + 13300. \tag{5.3}$$

The constant parameters are chosen according to material parameters and damping parameters presented in [8]. The polynomial parameters of the entries of the piezoelectric coupling parameter are chosen such that they equal the entries of the piezoelectric coupling parameter in [8] at $25C^\circ$. To generate the noisy data y^δ we contaminate the exact simulated data y, generated with the parameters defined above, additively with uniformly distributed random noise with a noise level of 1%. The excitation signal $\phi_e(t_n)$, applied via the Dirichlet boundary condition at the top surface, is defined as a discrete triangular pulse, i.e.,

$$\phi_e(t_n) = 10^{-9} \cdot \begin{cases} n & \text{for } 1 \le n \le 10 \\ 20 - n & \text{for } 11 \le n \le 19 \\ 0 & \text{for } n \ge 20 \end{cases}. \tag{5.4}$$

Note that specifying the Dirichlet lift function χ used for stating the system (3.19)-(3.24) is not necessary, as it is possible to directly implement mixed Dirichlet conditions in FEniCS. As optimization method we use the GRSE method, see [15], which employs a Tikhonov-type regularization with 0.5 as decay factor

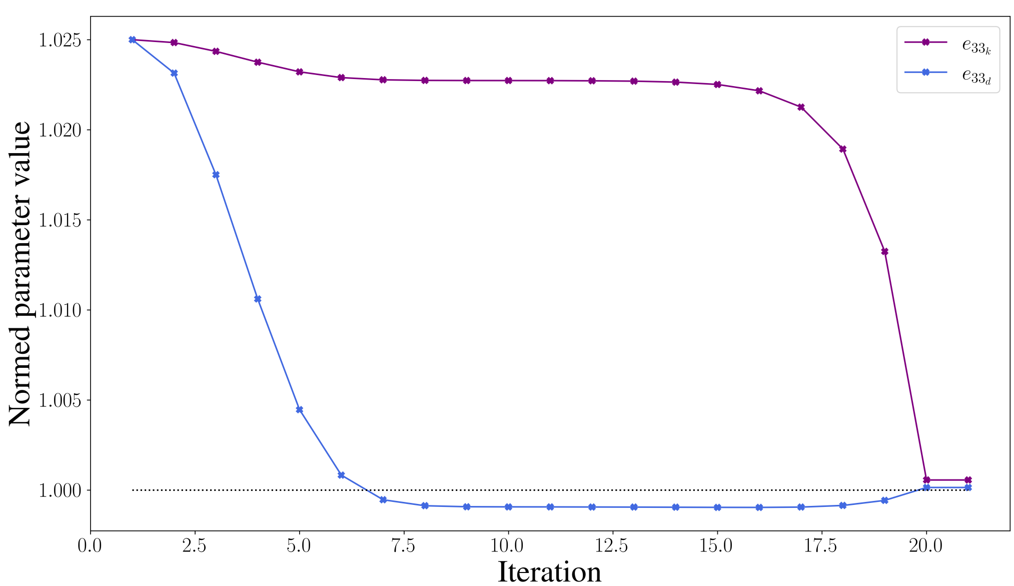

FIGURE 1. Identification of e_{33_k} and e_{33_d}.

and 4 as growth factor of the regularization parameters. We started with the initial regularization parameter $\tau_0 = 10^{-6}$. As initial Quasi-Newton matrix we used the scaled identity with scale 10^{-6}. The initial guesses for polynomial parameters of the piezoelectric coupling parameter $e_{33}(\theta(t))$ are chosen with a 2.5% deviation to the ground truth in (5.3). Furthermore, we scaled the first order polynomial parameter with 5 and the zero order polynomial parameter with 10^{-2}, to reach similar orders of magnitude. The numerical results for the identification of the polynomial parameters e_{33_k} and e_{33_d}, which are the first and zeroth order polynomial parameters of the piezoelectric coupling parameter $e_{33}(\theta(t))$ showed convergence to the exact parameter, as illustrated in Figure 1.

6. Conclusion

We modeled and analyzed an inverse problem governed by a piezoelectric system represented by a coupled hyperbolic-elliptic PDE with matrix-valued Sobolev-Bochner functions as parameters and Sobolev-Bochner density and damping functions. We extended the PDE with an additional term based on a Sobolev-Bochner function and the mechanical deformation as well as inhomogeneities, ensuring the applicability of our generalized existence and uniqueness theorem on the associated adjoint PDE. In addition, an a priori energy estimate and conditions for arbitrary Sobolev regularity in space were established to ensure the well-definedness of the observation operator of the inverse parameter identification problem. Then, we proved the Fréchet differentiability of the observation operator and discussed the treatment of the observation operator given that PDE solutions have lower regularity. With respect to the modeling and regularity of the forward operator of the parameter identification problem in the reduced approach, we considered the well-definedness, existence and regularity of the parameter-state map. Furthermore, we modeled the inverse problem as an optimization problem in which a target functional consisting of the forward operator, the given data and a regularizer is minimized. To provide a framework for the computation of solutions to the inverse problem, we showed that there exists a minimizer and derived first-order optimality conditions. This motivated the derivation of the adjoint PDE, where we used our existence and uniqueness results to analyze the adjoint PDE, demonstrating its utility. Finally, a numerical example was given, where the proposed parameterization approach for describing the material parameters was used.

Acknowledgements

The authors would like to thank the German Research Foundation (DFG) for financial support within the research group 5208 NEPTUN (444955436). The work on this paper was partly funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany´s Excellence Strategy – The Berlin Mathematics Research Center MATH+ (EXC-2046/1, EXC-2046/2, project ID: 390685689).

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