A Structural Equivalence of Symmetric TSP to a Constrained Group Steiner Tree Problem

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Abstract

We present a brief structural equivalence between the symmetric TSP and a constrained Group Steiner Tree Problem (cGSTP) defined on a simplicial incidence graph. Given the complete weighted graph on the city set V, we form the bipartite incidence graph between triangles and edges. Selecting an admissible, disk-like set of triangles induces a unique boundary cycle. With global connectivity and local regularity constraints, maximizing net weight in the cGSTP is exactly equivalent to minimizing the TSP tour length.

Keywords: Traveling Salesman Problem, Steiner Trees, Computational Topology, Simplicial Complex, Euler Characteristic

1. Introduction and Geometric View

Let V be a set of $n \geq 3$ cities and $G = (V, E)$ be the complete undirected graph with symmetric edge lengths $L_e > 0$. The *symmetric Traveling Salesman Problem (TSP)* asks for a minimum-length Hamiltonian cycle on V [1].

In this note, we describe an exact structural equivalence of symmetric TSP to a constrained Group Steiner Tree Problem (GSTP) [2]. The construction utilizes elementary combinatorial topology [3] to view a tour not as a one-dimensional cycle, but as the *boundary* of a two-dimensional simplicial surface. By selecting a set of triangles forming a topological disk, edges shared by two selected triangles (internal edges) cancel out, leaving a unique boundary cycle. Figure 1 shows the same surface–boundary viewpoint on a small symmetric instance: a connected selection of triangles behaves as an abstract disk, and its induced boundary is a single cycle (the tour).

2. The cGSTP Equivalence

We define the cGSTP on the bipartite *incidence graph* $B = (U \cup W, A)$ (see Fig. 2). $U = \binom{V}{3}$ is the set of triangle nodes (circles), $W = E$ is the set of primal edge nodes (squares), and $(t, e) \in A$ denotes incidence. For each city $v \in V$, we define a group $U(v) = \{t \in U : v \in t\}$.

Triangle nodes act as *group terminals* (cost 0), while edge nodes and incidences carry weights defined to enforce boundary cancellation:

1. Each edge node $e \in W$ has cost $c(e) = 2L_e$.
2. Each incidence arc $(t, e) \in A$ has profit $p(t, e) = L_e$.

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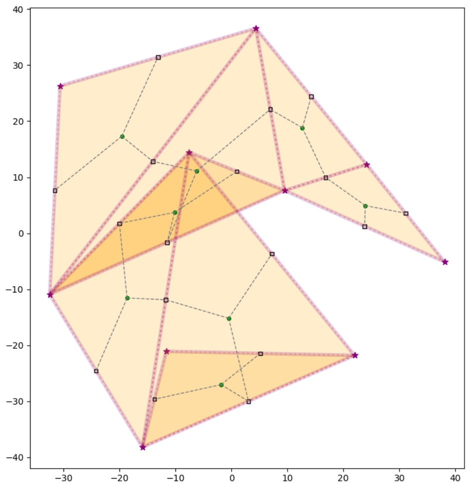

Figure 1: A small symmetric instance ($n = 10$) shown with an arbitrary planar embedding for visualization. A connected set of selected triangles (shaded) forms an abstract disk; internal edges cancel, and the induced boundary is a single simple cycle (the tour).

Let $x_t, y_e, z_{t,e} \in \{0, 1\}$ indicate the selection of triangles, edges, and incidences respectively. We select an "admissible" subgraph B' to maximize the net weight:

$$W(B') := \sum_{(t,e)\in A} z_{t,e} L_e \; - \; \sum_{e\in W} y_e 2L_e.$$

Admissibility is defined by the following constraints:

(C1) Incidence linking (Terminal Node Degrees). Every active triangle must use all three of its edges:

$$z_{t,e} \le y_e$$

$$z_{t,e} \le x_t$$

$$\sum_{e\subset t} z_{t,e} = 3x_t$$

(C2) Manifold regularity (Steiner Node Degrees). Every primal edge is incident to at most two selected triangles (Fig. 3A):

$$y_e \le \sum_{t\supset e} z_{t,e} \le 2y_e$$

(C3) Global Euler counts (Node Cardinalities). These match the cardinality of an abstract triangulated disk with n vertices:

$$\sum x_t = n - 2$$

$$\sum y_e = 2n - 3$$

(C4) Global connectivity (Tree). The active subgraph B' must be a tree.

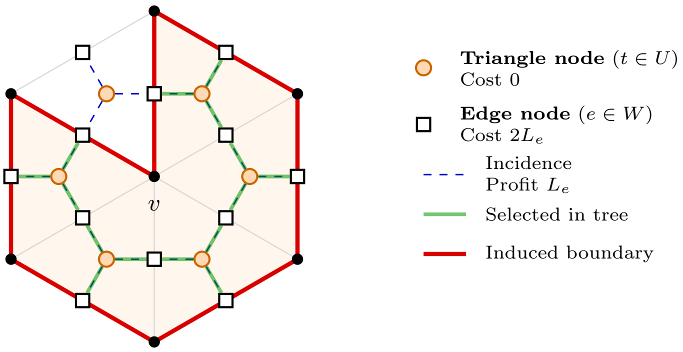

Figure 2: The incidence graph structure. Selecting a set of triangles that form a topological disk (shaded) results in cost cancellation on internal edges, leaving a net cost corresponding to the induced boundary tour (red).

Figure 3: Forbidden anomalies: (A) Edge incident to > 2 triangles; (B) Disconnected vertex link.

(C5) Local connectivity (Steiner Groups). For each city v, the active local incidence subgraph H_v must satisfy the *Euler Characteristic*:

$$\chi(H_v) = |V(H_v)| - |E(H_v)| = 1$$

Here, H_v is the subgraph of B' induced by selected triangles and edges incident to vertex v (cf. the *star* of v in Fig. 2). Given the global tree constraint, this enforces the vertex link to be a simple path (excluding bowties like Fig. 3B and cyclic vertex links, which would arise if the local fan in Fig. 2 were closed). The presence of at least one terminal per group is thus also implicitly enforced.

3. Theoretical Properties

Let $K = \{t \in U : x_t = 1\}$ be the selected triangles and define the boundary edge set

$$\partial K = \{e \in E : \sum_{t \supset e} z_{t,e} = 1\}.$$

Lemma 1 (Combinatorial boundary identity)**.** *For any selection satisfying (C1)–(C5),*

$$-W(B') = \sum_{e \in \partial K} L_e.$$

Proof. If e is incident to two selected triangles, it contributes $2L_e$ profit and incurs $2L_e$ cost, yielding net 0. If e is incident to exactly one selected triangle, it contributes L_e profit and incurs $2L_e$ cost, yielding net $-L_e$. Summing over all edges gives the claim. □

Lemma 2 (Soundness)**.** *Any admissible solution defines an abstract triangulated disk whose boundary is a single simple Hamiltonian cycle.*

Proof. By (C4), the selected triangles form a connected complex. Constraints (C2) and (C5) exclude non-manifold edges and disconnected vertex links, so the complex is a simplicial surface with boundary. By (C3), its Euler characteristic is $\chi = 1$, hence it is a disk. A disk has exactly one boundary component, so the boundary is a single simple cycle. Moreover, (C5) implies every vertex lies on the boundary, so this cycle is Hamiltonian. □

Theorem 3 (Equivalence)**.** $\mathrm{OPT}_{\mathrm{TSP}} = -\mathrm{OPT}_{\mathrm{cGSTP}}$.

Proof. **Soundness** follows from Lemma 2 and Lemma 1.

Completeness. Let $C = (v_1, v_2, \ldots, v_n)$ be any Hamiltonian cycle. Select the $n-2$ triangles

$$K := \big\{\{v_1, v_i, v_{i+1}\} : i = 2, \ldots, n-1\big\}.$$

Then $\partial K = C$. Setting $x_t = 1$ for $t \in K$, $y_e = 1$ for all edges used by K, and $z_{t,e} = 1$ for all incidences $e \subset t$ yields a feasible solution satisfying (C1)–(C5). By Lemma 1, its objective value equals $-L(C)$. Taking the optimum over C gives $\mathrm{OPT}_{\mathrm{cGSTP}} \geq -\mathrm{OPT}_{\mathrm{TSP}}$, and combining with soundness yields equality. □

4. Discussion

In this note, we established a structural equivalence between the symmetric TSP and a constrained variant of the Group Steiner Tree Problem by reformulating tours as boundaries of admissible triangle selections. The resulting model provides a unified constraint system in which global connectivity and objective cancellation arise naturally from the underlying simplicial incidence structure. A key feature is that the construction is input-decoupled: it is exact when the full complex is available, and becomes a controlled heuristic when restricted to a prescribed candidate triangle set. From a practical modeling perspective, the tour objective emerges through local cancellation of internal edges, while feasibility is enforced by a compact combination of a global tree constraint and local Euler regularity [3]. This makes it possible to use sparse geometric complexes (e.g., Delaunay or related triangulations) as black-box restrictions of the search space [4, 5, 6], while preserving exactness whenever an optimal tour is contained in the chosen complex. Algorithmic development and a systematic computational study are left for future work.

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