

Tight semidefinite programming relaxations for sparse box-constrained quadratic programs

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Abstract

We introduce a new class of semidefinite programming (SDP) relaxations for sparse box-constrained quadratic programs, obtained by a novel integration of the Reformulation Linearization Technique into standard SDP relaxations while explicitly exploiting the sparsity of the problem. The resulting relaxations are not implied by the existing LP and SDP relaxations for this class of optimization problems. We establish a sufficient condition under which the convex hull of the feasible region of the lifted quadratic program is SDP-representable; the proof is constructive and yields an explicit extended formulation. Although the resulting SDP may be of exponential size in general, we further identify additional structural conditions on the sparsity of the optimization problem that guarantee the existence of a polynomial-size SDP-representable formulation, which can be constructed in polynomial time.

Keywords: nonconvex quadratic programming, sparsity, convex hull, Reformulation-Linearization Technique, semidefinite programming relaxation, polynomial-size extended formulation.

1 Introduction

We consider a nonconvex box-constrained quadratic program:

$$\begin{aligned} \min \quad & x^\top Qx + c^\top x \\ \text{s.t.} \quad & x \in [0, 1]^n, \end{aligned} \tag{QP}$$

where $c \in \mathbb{R}^n$ and $Q \in \mathbb{R}^{n \times n}$ is a symmetric matrix that is not positive semidefinite. Following a common practice in nonconvex optimization, we linearize the objective function of Problem QP by introducing new variables $Y := xx^\top$, thus, obtaining a reformulation of this problem in a lifted space of variables:

$$\begin{aligned} \min \quad & \langle Q, Y \rangle + c^\top x \\ \text{s.t.} \quad & Y = xx^\top \\ & x \in [0, 1]^n, \end{aligned} \tag{\ell QP}$$

where $\langle Q, Y \rangle$ denotes the matrix inner product. Given a set \mathcal{S} , we denote by $\text{conv}(\mathcal{S})$ the convex hull of the set \mathcal{S} . We then define

$$\text{QP}_n := \text{conv} \left\{ (x, Y) \in \mathbb{R}^{n+\frac{n(n+1)}{2}} : Y = xx^\top, x \in [0, 1]^n \right\}.$$

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Constructing strong and cheaply computable convex relaxations for QP_n has been the subject of extensive research over the last three decades [31, 36, 5, 10, 2, 4]. The most popular convex relaxations of Problem ℓQP are semidefinite programming (SDP) relaxations [34]. The key idea is to replace the nonconvex set defined by the constraint $Y = xx^\top$ by its convex hull defined by $Y \succeq xx^\top$, to obtain the following convex relaxation for QP_n :

$$\mathcal{C}_n^{\text{SDP}} := \left\{ (x, Y) \in \mathbb{R}^{n+\frac{n(n+1)}{2}} : \begin{bmatrix} 1 & x^\top \\ x & Y \end{bmatrix} \succeq 0, \text{diag}(Y) \leq x, x \in [0, 1]^n \right\},$$

where $\text{diag}(Y)$ denotes the vector in \mathbb{R}^n containing the diagonal entries of Y . In [10], the authors investigated some fundamental properties of QP_n . In particular, they studied the relationship between QP_n , and its well-known discrete counterpart; namely, the *Boolean quadric polytope* [27]:

$$\text{BQP}_n := \text{conv} \left\{ (x, Y) \in \mathbb{R}^{n+\frac{n(n-1)}{2}} : Y_{ij} = x_i x_j, \forall 1 \leq i < j \leq n, x \in \{0, 1\}^n \right\}.$$

The Boolean quadric polytope is a well-studied polytope in the context of binary quadratic programming, and the max-cut problem and its facial structure has been thoroughly investigated in the literature [27, 19]. The authors of [10] obtained a sufficient condition under which a facet-defining inequality for BQP_n defines a facet of QP_n as well. Their result implies that the *McCormick inequalities* [25] given by:

$$Y_{ij} \geq 0, \quad Y_{ij} \geq x_i + x_j - 1, \quad Y_{ij} \leq x_i, \quad Y_{ij} \leq x_j, \quad \forall 1 \leq i < j \leq n, \quad (1)$$

and the *triangle inequalities* [27] given by:

$$\begin{aligned} Y_{ij} + Y_{ik} &\leq x_i + Y_{jk} \\ Y_{ij} + Y_{jk} &\leq x_j + Y_{ik} \\ Y_{ik} + Y_{jk} &\leq x_k + Y_{ij} \quad \forall 1 \leq i < j < k \leq n, \\ x_i + x_j + x_k - Y_{ij} - Y_{ik} - Y_{jk} &\leq 1 \end{aligned} \quad (2)$$

define facets of QP_n . We then define two stronger convex relaxations of QP_n , the first one is obtained by adding McCormick inequalities to $\mathcal{C}_n^{\text{SDP}}$:

$$\mathcal{C}_n^{\text{SDP+MC}} := \left\{ (x, Y) \in \mathbb{R}^{n+\frac{n(n+1)}{2}} : (x, Y) \in \mathcal{C}_n^{\text{SDP}}, (x, Y) \text{ satisfy inequalities (1)} \right\}, \quad (3)$$

while the second one is obtained by adding triangle inequalities to $\mathcal{C}_n^{\text{SDP+MC}}$:

$$\mathcal{C}_n^{\text{SDP+MC+Tri}} := \left\{ (x, Y) \in \mathbb{R}^{n+\frac{n(n+1)}{2}} : (x, Y) \in \mathcal{C}_n^{\text{SDP}}, (x, Y) \text{ satisfy inequalities (1) -- (2)} \right\}. \quad (4)$$

In [5], building upon the results in [9], the authors proved that if $n = 2$, then $\text{QP}_n = \mathcal{C}_n^{\text{SDP+MC}}$. Moreover, they showed that if $n = 3$, then $\text{QP}_n \subsetneq \mathcal{C}_n^{\text{SDP+MC+Tri}}$. In fact, to date, obtaining an explicit algebraic description for QP_3 remains an open question. Since Problem QP is \mathcal{NP} -hard in general, unless $\mathcal{P} = \mathcal{NP}$, one cannot construct in polynomial time a polynomial-size extended formulation for QP_n . Recall that, given a convex set $\mathcal{C} \subseteq \mathbb{R}^n$, an *extended formulation* for \mathcal{C} is a convex set $\mathcal{Q} \subseteq \mathbb{R}^{n+r}$, for some $r > 0$, such that $\mathcal{C} = \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^r \text{ such that } (x, y) \in \mathcal{Q}\}$. It can be shown that if \mathcal{C} admits a polynomial-size extended formulation that can be constructed in polynomial time, then optimizing a linear function over \mathcal{C} can also be performed in polynomial time. However, the key observation is that many quadratic programs are *sparse*; i.e., $q_{ij} = 0$ for many

pairs i, j . Motivated by the same observation in the context of binary quadratic programming, Padberg [27] introduced the Boolean quadric polytope of a sparse graph $G = (V, E)$:

$$\text{BQP}(G) := \text{conv} \left\{ (x, Y) \in \{0, 1\}^{V \cup E} : Y_{ij} = x_i x_j, \forall \{i, j\} \in E \right\}.$$

In [24, 20, 8], the authors proved that for a graph $G = (V, E)$ with treewidth $\text{tw}(G) = \kappa$, the polytope $\text{BQP}(G)$ has a linear extended formulation with $O(2^\kappa |V|)$ variables and inequalities. This result implies that if $\kappa \in O(\log |V|)$, then $\text{BQP}(G)$ admits a polynomial-size linear extended formulation. Moreover, from [11, 1] it follows that the linear extension complexity of $\text{BQP}(G)$ grows exponentially in the treewidth of G . Recall that given a polytope \mathcal{P} , the *linear extension complexity* of \mathcal{P} is the minimum number of linear inequalities and equalities in a linear extended formulation of \mathcal{P} . Hence, a bounded treewidth for G is a necessary and sufficient condition for the existence of a polynomial-size linear extended formulation for $\text{BQP}(G)$.

Motivated by these ground-breaking results for sparse binary quadratic programs, in [18], the authors considered sparse box-constrained quadratic programs. They then introduced a graph representation for this class of optimization problems, which we define next. Consider a graph $G = (V, E, L)$, where V denotes the node set of G , E denotes the edge set of G in which each $\{i, j\} \in E$ is an edge connecting two distinct nodes $i, j \in V$, and L denotes the loop set of G in which each $\{i, i\} \in L$ is a loop connecting some node $i \in V$ to itself. We then associate a graph G with Problem ℓQP (or with Problem QP), where we define a node i for each independent variable x_i , for all $i \in [n] := \{1, \dots, n\}$, two distinct nodes i and j are adjacent if the coefficient q_{ij} is nonzero, and there is a loop $\{i, i\}$ for some $i \in [n]$, if the coefficient q_{ii} is nonzero. Moreover, we say that a loop $\{i, i\}$ is a *plus loop*, if $q_{ii} > 0$ and is a *minus loop*, if $q_{ii} < 0$. We denote the set of plus loops and minus loops by L^+ and L^- , respectively. Henceforth, for notational simplicity, we denote variables x_i by z_i , for all $i \in [n]$, and we denote variables Y_{ij} by z_{ij} , for all $\{i, j\} \in E \cup L$. Similarly to [18], we consider the following reformulation of Problem ℓQP :

$$\begin{aligned} \min \quad & \sum_{\{i, i\} \in L} q_{ii} z_{ii} + 2 \sum_{\{i, j\} \in E} q_{ij} z_{ij} + \sum_{i \in V} c_i z_i & (\ell\text{QPG}) \\ \text{s.t.} \quad & z_{ii} \geq z_i^2, \forall \{i, i\} \in L^+ \\ & z_{ii} \leq z_i^2, \forall \{i, i\} \in L^- \\ & z_{ij} = z_i z_j, \forall \{i, j\} \in E \\ & z_i \in [0, 1], \forall i \in V. \end{aligned}$$

Furthermore, we define:

$$\begin{aligned} \text{QP}(G) := \text{conv} \left\{ z \in \mathbb{R}^{V \cup E \cup L} : z_{ii} \geq z_i^2, \forall \{i, i\} \in L^+, z_{ii} \leq z_i^2, \forall \{i, i\} \in L^-, z_{ij} = z_i z_j, \right. \\ \left. \forall \{i, j\} \in E, z_i \in [0, 1], \forall i \in V \right\}. \end{aligned}$$

In [18], the authors proposed a new second-order cone (SOC) representable relaxation for $\text{QP}(G)$. They then examined the tightness of the proposed relaxation; namely, they proved that if the set of nodes of G with plus loops forms a stable set of G , then $\text{QP}(G)$ is SOC-representable. While the proposed SOC relaxation may be of exponential size in general, they proved if G admits a tree decomposition whose width is bounded and the *spreads* of nodes with plus loops are also bounded, then the proposed SOC formulation is polynomial size.

In this paper, we introduce a new class of SDP relaxations for box-constrained quadratic programs. Our proposed relaxations are obtained via a novel incorporation of the Reformulation-Linearization Technique (RLT) [32] into the existing SDP relaxations and are provably stronger than the existing LP and SDP relaxations. The main contributions of this paper are as follows:

- (i) We introduce a new type of SDP relaxations for box-constrained quadratic programs by combining the existing RLT framework and SDP relaxations, while exploiting the sparsity of the problem to control the size of the resulting relaxation. We prove that the SOC relaxation of [18] is implied by the proposed SDP relaxation. We further show that the proposed relaxations can be interpreted as a special case of Schmüdgen hierarchy, where the multipliers and factors have a specific structure.
- (ii) We examine the tightness of the proposed SDP relaxation, and obtain a sufficient condition under which $QP(G)$ is SDP-representable. Namely, we prove that if G does not have any three nodes with plus loops such that at least two pairs among them are adjacent, then $QP(G)$ is SDP-representable. Our proof is constructive and yields a simple algorithm for explicitly constructing the corresponding SDP formulation.
- (iii) In general, the proposed SDP formulation may be of exponential size. We then identify sufficient conditions based on the structure of graph G , under which $QP(G)$ admits a polynomial-size SDP-representable formulation that can be constructed in polynomial time. Our sufficient condition essentially states that if the treewidth of G and the degrees of nodes with plus loops are both bounded, then the SDP formulation is guaranteed to be of polynomial size.

Our theoretical results significantly generalize those of [18]. However, this generalization comes at a cost: the proposed SDP relaxations are computationally more expensive to solve than the SOC relaxations in [18].

The remainder of the paper is structured as follows. In Section 2, we review the preliminary material that we need to develop our new relaxations. In Section 3, we introduce a new class of SDP relaxations for box-constrained quadratic programs. In Section 4, we examine the tightness of the proposed SDP relaxations and obtain a sufficient condition for the SDP-representability of $QP(G)$. In Section 5, we derive sufficient conditions under which $QP(G)$ admits a polynomial-size SDP-representable formulation that can be constructed in polynomial time. Finally, in Section 6, we detail on the connections between our proposed relaxations and the existing SDP hierarchies.

2 Preliminaries

In this section, we review the preliminary material that we will use to develop and analyze our proposed SDP relaxations in the subsequent sections.

2.1 Higher-order extended formulations and decomposability of $QP(G)$

In this paper, we propose new SDP relaxations for sparse box-constrained quadratic programs via a novel incorporation of the RLT framework into existing SDP relaxations. The proposed relaxations lie in an extended space where some extended variables represent products of more than two original variables. In this section, we present the necessary background to construct these extended formulations.

A *hypergraph* G is a pair (V, E) , where V is a finite set of nodes and E is a set of subsets of V of cardinality at least two, called the edges of G . In this paper, we consider hypergraphs with loops; i.e., $G = (V, E, L)$, where V, E are the same as those of ordinary hypergraphs and L denotes the set of loops of G . As for graphs with loops, we partition the loop set of hypergraphs as $L = L^- \cup L^+$, where L^- and L^+ denote the sets of minus loops and plus loops of G , respectively. With any hypergraph $G = (V, E, L)$, we associate the convex set $PP(G)$ defined as

$$PP(G) := \text{conv} \left\{ z \in \mathbb{R}^{V \cup E \cup L} : z_{ii} \geq z_i^2, \forall \{i, i\} \in L^+, z_{ii} \leq z_i^2, \forall \{i, i\} \in L^-, z_e = \prod_{i \in e} z_i, \forall e \in E, \right\}$$

$$z_i \in [0, 1], \forall i \in V \Big\}. \quad (5)$$

In the special case where G is a graph, $\text{PP}(G)$ simplifies to $\text{QP}(G)$. Moreover, if the hypergraph G does not have any loops, then $\text{PP}(G)$ coincides with the well-studied multilinear polytope $\text{MP}(G)$ in the context of binary polynomial optimization [13, 14, 16]. In the next four lemmata, we review some basic properties of $\text{PP}(G)$ that we will later use to prove our results. These results were originally proved in [18].

Lemma 1 (lemma 2 and lemma 3 in [18]). *Let $G = (V, E, L)$ be a hypergraph and consider the convex set $\text{PP}(G)$ as defined by (5). Then:*

- (i) $\text{PP}(G)$ is a closed set.
- (ii) The set of extreme points of $\text{PP}(G)$ is given by:

$$\begin{aligned} \text{ext}(\text{PP}(G)) = & \Big\{ z \in \mathbb{R}^{V \cup E \cup L} : z_i \in [0, 1], \forall i \in V : \{i, i\} \in L^+, z_i \in \{0, 1\}, \forall i \in V : \{i, i\} \notin L^+, \\ & z_p = \prod_{i \in p} z_i, \forall p \in E \cup L \Big\}, \end{aligned}$$

- (iii) The recession code of $\text{PP}(G)$ is given by:

$$\text{rec}(\text{PP}(G)) = \Big\{ z \in \mathbb{R}^{V \cup E \cup L} : z_{ii} \geq 0, \forall \{i, i\} \in L^+, z_{ii} \leq 0, \forall \{i, i\} \in L^-, z_p = 0, \forall p \in V \cup E \Big\}.$$

Now, let $G = (V, E, L)$ be a graph and let $G' = (V, E', L)$ be a hypergraph such that $E \subseteq E'$. The next lemma indicates that a formulation for $\text{PP}(G')$ serves as an extended formulation for $\text{QP}(G)$.

Lemma 2 (lemma 4 in [18]). *Consider two hypergraphs $G_1 = (V, E_1, L)$ and $G_2 = (V, E_2, L)$ such that $E_1 \subseteq E_2$. Then a formulation for $\text{PP}(G_2)$ is an extended formulation for $\text{PP}(G_1)$.*

The next result implies that, to construct $\text{QP}(G)$, one can essentially ignore the minus loops.

Lemma 3 (lemma 5 in [18]). *Consider a hypergraph $G = (V, E, L)$, where $L = L^- \cup L^+$. Define $G' = (V, E, L^+)$. Then a formulation for $\text{PP}(G)$ is obtained by putting together a formulation for $\text{PP}(G')$ together with the following linear inequalities:*

$$z_{ii} \leq z_i, z_i \in [0, 1], \forall \{i, i\} \in L^-. \quad (6)$$

We say that a hypergraph $G = (V, E, L)$ is a *complete* hypergraph if E contains all subsets of V of cardinality at least two. Given a hypergraph $G = (V, E, L)$, and $V' \subseteq V$, the *section hypergraph* of G induced by V' is the hypergraph $G' = (V', E', L')$, where $E' = \{e \in E : e \subseteq V'\}$ and $L' = \{\{i, i\} \in L : i \in V'\}$. Moreover, if a loop is a plus (resp. minus) loop in G , it is also a plus (resp. minus) loop in G' . Given hypergraphs $G_1 = (V_1, E_1, L_1)$ and $G_2 = (V_2, E_2, L_2)$, we denote by $G_1 \cap G_2$ the hypergraph $(V_1 \cap V_2, E_1 \cap E_2, L_1 \cap L_2)$, and we denote by $G_1 \cup G_2$ the hypergraph $(V_1 \cup V_2, E_1 \cup E_2, L_1 \cup L_2)$. In the following, we consider a hypergraph G , and two distinct section hypergraphs of G , denoted by G_1 and G_2 , such that $G_1 \cup G_2 = G$. We say that $\text{PP}(G)$ is *decomposable* into $\text{PP}(G_1)$ and $\text{PP}(G_2)$ if the system comprising a description of $\text{PP}(G_1)$ and a description of $\text{PP}(G_2)$ is a description of $\text{PP}(G)$. The next proposition provides a sufficient condition for the decomposability of $\text{PP}(G)$. This result is a consequence of a result in [15] regarding the decomposability of the multilinear polytope (see theorem 1 in [15]).

Lemma 4 (corollary 1 in [18]). *Let $G = (V, E, L)$ be a hypergraph. Let G_1, G_2 be section hypergraphs of G such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2$ is a complete hypergraph. Suppose that the hypergraph $G_1 \cap G_2$ has no plus loops. Then, $PP(G)$ is decomposable into $PP(G_1)$ and $PP(G_2)$.*

Consider the graph $G = (V, E, L)$. By Lemma 3, if G has no plus loops, then $QP(G)$ is a polyhedron. Now suppose that $L^+ \neq \emptyset$. Define

$$V^+ := \{i \in V : \{i, i\} \in L^+\}. \quad (7)$$

Denote by G_{V^+} the subgraph of G induced by V^+ and denote by V_c , $c \in \mathcal{C}$ the connected components of G_{V^+} . The next result indicates that there exists an extended formulation for $QP(G)$ whose structure depends on the “size” of the connected components V_c , $c \in \mathcal{C}$.

Proposition 1. *Let $G = (V, E, L)$ be a graph. Denote by V_c , $c \in \mathcal{C}$ the connected components of G_{V^+} , where G_{V^+} is the subgraph of G induced by V^+ defined by (7). For each $c \in \mathcal{C}$, define*

$$N(V_c) := \{i \in V \setminus V_c : \{i, j\} \in E, \text{ for some } j \in V_c\}. \quad (8)$$

Then $QP(G)$ admits an extended formulation obtained by putting together formulations for $PP(G'_c)$, $c \in \mathcal{C} \cup \{0\}$, where G'_c , $c \in \mathcal{C}$ is a hypergraph with $|V_c| + |N(V_c)|$ nodes and $|V_c|$ plus loops, and G'_0 is a hypergraph with $|V| - \sum_{c \in \mathcal{C}} |V_c|$ nodes and no plus loops.

Proof. Let $G' = (V, E', L)$ be a hypergraph with $E' \supseteq E$ such that for each edge $e \in E' \setminus E$ we have $e \subseteq N(V_c)$ for some $c \in \mathcal{C}$. We prove that $PP(G')$ admits an extended formulation obtained by putting together formulations for $PP(G'_c)$, $c \in \mathcal{C} \cup \{0\}$, where G'_c , $c \in \mathcal{C}$ is a hypergraph with $|V_c| + |N(V_c)|$ nodes and $|V_c|$ plus loops, and G'_0 is a hypergraph with $|V| - \sum_{c \in \mathcal{C}} |V_c|$ nodes and no plus loops. Then by Lemma 2, the statement follows for $QP(G)$. Notice that since by assumption, for each edge $e \in E' \setminus E$ we have $e \subseteq N(V_c)$ for some $c \in \mathcal{C}$, the section hypergraph of G' induced by V^+ coincides with the subgraph of G induced by V^+ ; i.e., the graph G_{V^+} .

We prove by induction on the number of connected components of the section hypergraph of G' induced by nodes with plus loops; i.e., $|\mathcal{C}|$. In the base case we have $|\mathcal{C}| = 0$; in this case, $PP(G')$ is a polyhedron and therefore by defining $G'_0 := G'$ the statement follows. Hence, suppose that $|\mathcal{C}| \geq 1$. Consider a connected component $V_{\tilde{c}}$ for some $\tilde{c} \in \mathcal{C}$. Define the hypergraph $\bar{G} = (V, \bar{E}, L)$, where $\bar{E} := E' \cup \{e \subseteq N(V_{\tilde{c}}) : |e| \geq 2\}$. By Lemma 2, a formulation for $PP(\bar{G})$ serves as an extended formulation for $PP(G')$. Denote by $\bar{G}_{\tilde{c}}$ the section hypergraph of \bar{G} induced by $V_{\tilde{c}} \cup N(V_{\tilde{c}})$ and denote by H the section hypergraph of \bar{G} induced by $V \setminus V_{\tilde{c}}$. Notice that $\bar{G} = \bar{G}_{\tilde{c}} \cup H$ and that $\bar{G}_{\tilde{c}} \cap H$ is a complete hypergraph with node set $N(V_{\tilde{c}})$. Moreover, since $V_{\tilde{c}}$ is a connected component of G_{V^+} , we have $V^+ \cap N(V_{\tilde{c}}) = \emptyset$; i.e., the hypergraph $\bar{G}_{\tilde{c}} \cap H$ does not have any plus loops. Therefore, all assumptions of Lemma 4 are satisfied and $PP(\bar{G})$ is decomposable into $PP(G'_{\tilde{c}})$ and $PP(H)$. By construction, the hypergraph $\bar{G}_{\tilde{c}}$ has $|V_{\tilde{c}}| + |N(V_{\tilde{c}})|$ nodes and $|V_{\tilde{c}}|$ plus loops. Moreover, the connected components of the section hypergraph of H induced by nodes with plus loops are V_c , $c \in \mathcal{C} \setminus \{\tilde{c}\}$. Therefore, by the induction hypothesis, $PP(H)$ admits an extended formulation obtained by putting together formulations for $PP(G'_c)$, $c \in \mathcal{C} \setminus \{\tilde{c}\} \cup \{0\}$, where G'_c , $c \in \mathcal{C} \setminus \{\tilde{c}\}$ is a hypergraph with $|V_c| + |N(V_c)|$ nodes and $|V_c|$ plus loops, and G'_0 is a hypergraph with $|V| - \sum_{c \in \mathcal{C}} |V_c|$ nodes and no plus loops and this completes the proof. \square

In the special case where $|V_c| = 1$ for all $c \in \mathcal{C}$; i.e., the set V^+ defined by (7) is a stable set of the graph G , from Proposition 1 it follows that $QP(G)$ has an extended formulation obtained by putting together formulations for $PP(G'_v)$, $v \in V^+ \cup \{0\}$, where each G'_v , $v \in V^+$ is a hypergraph with one plus loop and $\deg(v) + 1$ nodes, where $\deg(v)$ denotes the degree of node v , and where G'_0 is a hypergraph with $|V| - |V^+|$ nodes with no plus loops. In [18], the authors use this key property to show that in this case $QP(G)$ is SOC-representable. In this paper, we will use Proposition 1 to obtain a more general sufficient condition under which $QP(G)$ is SDP-representable.

2.2 The RLT hierarchy for polynomial optimization

The RLT is a systematic procedure for constructing a hierarchy of increasingly stronger LP relaxations for mixed-integer polynomial optimization problems [32]. In the pure binary setting, the n th-level relaxation, where n is the number of variables, coincides with the convex hull of the feasible set. Although RLT was originally developed for 0-1 problems, several extensions have been proposed for continuous optimization problems [31, 33, 3, 7]. These approaches yield strong relaxations for general nonconvex problems, but they typically do not characterize the convex hull. In contrast, in [18], the authors present an extension of RLT for box-constrained quadratic programs and obtain sufficient conditions under which the proposed SOC-representable relaxations characterize QP(G). Their construction combines classical RLT constraints with convex quadratic inequalities of the form $z_{ii} \geq z_i^2$. In this paper, we significantly generalize this convexification framework by developing a novel way to integrate RLT constraints into existing SDP relaxations.

In the following, we provide a brief overview of the RLT terminology and results that we will use to develop our convexification technique. Consider any $\mathcal{S} \subseteq \{0, 1\}^n$, let $z \in \mathcal{S}$. In a similar vein to [32], for some $d \in [n]$, we define a *polynomial factor* as

$$f_d(J_1, J_2) := \prod_{i \in J_1} z_i \prod_{i \in J_2} (1 - z_i), \quad J_1, J_2 \subseteq [n], \quad J_1 \cap J_2 = \emptyset, \quad |J_1 \cup J_2| = d, \quad (9)$$

where we define $z_\emptyset = 1$. Since by assumption $z \in \{0, 1\}^n$, it follows that $f_d(J_1, J_2) \geq 0$ for all $d \in [n]$ and all J_1, J_2 satisfying the above conditions. We then expand these polynomial factors and rewrite them as

$$f_d(J_1, J_2) = \sum_{t: t \subseteq J_2} (-1)^{|t|} \prod_{i \in J_1} z_i \prod_{i \in t} z_i.$$

Next, we linearize the polynomial factors by introducing new variables

$$z_{J_1 \cup t} := \prod_{i \in J_1} z_i \prod_{i \in t} z_i, \quad \forall t \subseteq J_2, \quad (10)$$

to obtain the following system of valid linear inequalities, in an extended space, for the set \mathcal{S} :

$$\ell_d(J_1, J_2) := \sum_{t: t \subseteq J_2} (-1)^{|t|} z_{J_1 \cup t} \geq 0, \quad J_1, J_2 \subseteq [n], \quad J_1 \cap J_2 = \emptyset, \quad |J_1 \cup J_2| = d. \quad (11)$$

Observation 1. In [32], the authors proved that for any $1 \leq d < n$, the system of inequalities $\ell_d(J_1, J_2) \geq 0$ for all $J_1, J_2 \subseteq [n]$ satisfying $J_1 \cap J_2 = \emptyset$ and $|J_1 \cup J_2| = d$ is implied by the system of inequalities $\ell_{d+1}(J_1, J_2) \geq 0$ for all $J_1, J_2 \subseteq [n]$ satisfying $J_1 \cap J_2 = \emptyset$ and $|J_1 \cup J_2| = d+1$. The proof uses the fact that for any $k \in [n] \setminus (J_1 \cup J_2)$ we have

$$f_d(J_1, J_2) = f_{d+1}(J_1 \cup \{k\}, J_2) + f_{d+1}(J_1, J_2 \cup \{k\}),$$

implying that the nonnegativity of $\ell_d(J_1, J_2)$ follows from the nonnegativity of $\ell_{d+1}(J_1 \cup \{k\}, J_2)$ and $\ell_{d+1}(J_1, J_2 \cup \{k\})$ (see Lemma 1 [32]).

We should remark that the McCormick inequalities (1) are obtained by letting $d = 2$ in (11), while the triangle inequalities (2) are implied by inequalities (11) with $d = 3$. For binary polynomial optimization, the explicit description for the multilinear polytope of a complete hypergraph can be obtained using the RLT framework.

Theorem 1 ([32]). *Let $G = (V, E)$ be a complete hypergraph with n nodes. Then the multilinear polytope $MP(G)$ is given by*

$$\ell_n(J, V \setminus J) \geq 0, \quad \forall J \subseteq V, \quad (12)$$

where $\ell_n(J, V \setminus J)$ is defined by (11).

To obtain extended formulations for $QP(G)$, by Lemma 2 and Lemma 3, it suffices to generalize Theorem 1 to the case where the hypergraph G has plus loops. In [18], the authors address the special case in which G has one plus loop.

Theorem 2 (theorem 1 in [18]). *Let $G = (V, E, L)$ be a complete hypergraph with n nodes. Suppose that $L = L^+ = \{j, j\}$ for some $j \in V$. Then $PP(G)$ is given by:*

$$\begin{aligned} z_{jj} &\geq \sum_{J \subseteq V: J \ni \{j\}} \frac{(\ell_n(J, V \setminus J))^2}{\ell_{n-1}(J \setminus \{j\}, V \setminus J)} \\ \ell_n(J, V \setminus J) &\geq 0, \quad \forall J \subseteq V. \end{aligned} \quad (13)$$

In this paper, we consider the case where the complete hypergraph G has two plus loops and obtain an SDP-representable extended formulation for $PP(G)$.

3 New SDP relaxations

In this section, we introduce a new class of SDP relaxations for box-constrained quadratic programs. Our proposed relaxations are obtained by a novel incorporation of the RLT framework into existing SDP relaxations. Moreover, our SDP relaxations exploit the sparsity of the optimization problem. In the remainder of the paper, thanks to Lemma 3, for simplicity of presentation and without loss of generality, we assume that the graph G has no minus loops.

To extend the RLT scheme to continuous quadratic programs, we introduce additional polynomial factors that contain a square term z_i^2 , for some $i \in [n]$. In the remainder of this paper, for notational simplicity, instead of $f_d(J_1 \cup J_2)$ (resp. $\ell_d(J_1 \cup J_2)$), we write $f(J_1 \cup J_2)$ (resp. $\ell(J_1 \cup J_2)$), since d is uniquely determined by J_1, J_2 ; i.e., $d = |J_1 \cup J_2|$. In addition, unlike the original definition, we include the case where $J_1 = J_2 = \emptyset$, for which we have $d = 0$ and $f(\emptyset, \emptyset) = \ell(\emptyset, \emptyset) = 1$. Now, for any $J_1, J_2 \subseteq [n]$ satisfying $J_1 \cap J_2 = \emptyset$, consider the polynomial factor $f(J_1, J_2)$ as defined by (9). For any $i \in [n] \setminus (J_1 \cup J_2)$, we define a new polynomial factor:

$$g(i, J_1, J_2) := z_i^2 f(J_1, J_2). \quad (14)$$

We further expand and linearize $g(i, J_1, J_2)$ by introducing new variables

$$z_{ii}^{J_1 \cup t} := z_i^2 \prod_{j \in J_1 \cup t} z_j, \quad \forall t \subseteq J_2, \quad (15)$$

where we define $z_{ii}^\emptyset := z_{ii} = z_i^2$. Hence, we obtain the following linear relationships in an extended space:

$$\rho(i, J_1, J_2) := \sum_{t: t \subseteq J_2} (-1)^{|t|} z_{ii}^{J_1 \cup t}, \quad \forall J_1, J_2 \subseteq [n] : J_1 \cap J_2 = \emptyset, \quad \forall i \in [n] \setminus (J_1 \cup J_2). \quad (16)$$

Henceforth, given two convex sets $\mathcal{S} \subseteq \mathbb{R}^n$ and $\mathcal{C} \subseteq \mathbb{R}^{n+d}$ for some $d \geq 0$, we say that \mathcal{C} is a *convex relaxation* of \mathcal{S} if the projection of \mathcal{C} onto the space of \mathcal{S} contains \mathcal{S} ; i.e., $\{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^d, (x, y) \in \mathcal{C}\} \supseteq \mathcal{S}$.

$\mathcal{C} \supseteq \mathcal{S}$. Moreover, whenever we give an explicit description for \mathcal{C} in which some of the variables in \mathcal{S} are not explicitly present, we imply that those variables can take any value in \mathbb{R} and we still say that the explicit description defines a convex relaxation for \mathcal{S} .

Consider a graph $G = (V, E, L)$. Denote by $P \subseteq V$ a set of nodes with plus loops and denote by $M \subseteq V$ a set of nodes such that $M \cap P = \emptyset$. Moreover, if $M \neq \emptyset$, then suppose that each node in M is adjacent to at least one node in P . The next proposition presents a new type of SDP relaxations for Problem ℓQP .

Theorem 3. *Let $G = (V, E, L)$ be a graph and let P, M be (possibly empty) subsets of V as defined above. Define*

$$M_R := (M \cup P) \setminus R, \quad \forall R \subseteq P$$

Then the following defines a convex relaxation of $\text{QP}(G)$:

$$\begin{bmatrix} \ell(J, M_R \setminus J) & \ell(J \cup \{i_1\}, M_R \setminus J) & \ell(J \cup \{i_2\}, M_R \setminus J) & \cdots & \ell(J \cup \{i_p\}, M_R \setminus J) \\ \ell(J \cup \{i_1\}, M_R \setminus J) & \rho(i_1, J, M_R \setminus J) & \ell(J \cup \{i_1, i_2\}, M_R \setminus J) & \cdots & \ell(J \cup \{i_1, i_p\}, M_R \setminus J) \\ \ell(J \cup \{i_2\}, M_R \setminus J) & \ell(J \cup \{i_1, i_2\}, M_R \setminus J) & \rho(i_2, J, M_R \setminus J) & \cdots & \ell(J \cup \{i_2, i_p\}, M_R \setminus J) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \ell(J \cup \{i_p\}, M_R \setminus J) & \ell(J \cup \{i_1, i_p\}, M_R \setminus J) & \ell(J \cup \{i_2, i_p\}, M_R \setminus J) & \cdots & \rho(i_p, J, M_R \setminus J) \end{bmatrix} \succeq 0 \quad (17)$$

$$\forall R := \{i_1, \dots, i_p\} \subseteq P, \quad \forall J \subseteq M_R.$$

Proof. We first establish the convexity of the set defined by constraints (17). Two cases arise: if $R = \emptyset$, we have $M_R = M \cup P$; it then follows that inequalities (17) simplify to level- $|M| + |P|$ RLT inequalities, which are linear; i.e., $\ell(J, (M \cup P) \setminus J) \geq 0$ for all $J \subseteq M \cup P$. If $R \neq \emptyset$, then by definition of $\ell(\cdot, \cdot)$ and $\rho(\cdot, \cdot, \cdot)$, each constraint in (17) is the inverse image of the positive semidefinite cone under an affine mapping, hence, defining a convex set.

Next, we show that the set defined by (17) is a convex relaxation of $\text{QP}(G)$. Letting $R = \emptyset$, as we described above, we obtain level- $|M| + |P|$ RLT inequalities which are valid inequalities for $\text{QP}(G)$. Now let $R = \{i_1, \dots, i_p\}$ for some $1 \leq p \leq |P|$. Let G_R denote the subgraph of G induced by R . Then the following linear matrix inequality (LMI) defines a convex relaxation for $\text{QP}(G_R)$ and therefore it is a convex relaxation for $\text{QP}(G)$:

$$\begin{bmatrix} 1 & z_{i_1} & \cdots & z_{i_p} \\ z_{i_1} & z_{i_1 i_1} & \cdots & z_{i_1 i_p} \\ \vdots & \vdots & \ddots & \vdots \\ z_{i_p} & z_{i_1 i_p} & \cdots & z_{i_p i_p} \end{bmatrix} \succeq 0, \quad (18)$$

where as before we define $z_{ij} = z_i z_j$ for all $1 \leq i \leq j \leq n$. Now, for any $J \subseteq M_R$, consider the polynomial factor $f(J, M_R \setminus J)$ as defined by (9). Recall that $f(J, M_R \setminus J) \geq 0$ over the unit hypercube. Multiplying the LMI (18) by $f(J, M_R \setminus J)$, using (9) and (14) to deduce the following relations

$$\begin{aligned} f(J \cup \{i\}, M_R \setminus J) &= z_i f(J, M_R \setminus J), \quad \forall i \in R \\ f(J \cup \{i, j\}, M_R \setminus J) &= z_{ij} f(J, M_R \setminus J), \quad \forall i \neq j \in R \\ g(i, J, M_R \setminus J) &= z_{ii} f(J, M_R \setminus J), \quad \forall i \in R, \end{aligned}$$

and linearizing the resulting monomials using (10) and (15), we obtain (17). Therefore, the set defined by the collection of LMIs (17) defines a convex relaxation of $\text{QP}(G)$. \square

In Theorem 3, letting $P = \emptyset$, LMIs (17) simplify precisely to the level- $|M|$ RLT inequalities. Otherwise, if $|P| \geq 1$, letting $R = \emptyset$ in (17), we obtain level- $|M|$ RLT inequalities. Moreover, if $M = \emptyset$, then LMI (17) corresponding to $R = P$ (i.e., corresponding to $M_R = \emptyset$), simplifies to LMI (18), which is used in existing SDP relaxations. However, as we will prove in this section, for any $M_R \neq \emptyset$, the corresponding systems of LMI (17) imply LMI (18). That is, our proposed convexification technique combines and strengthens existing LP and SDP relaxations, while exploiting the sparsity of box-constrained QPs.

Now, consider a pair P, M corresponding to LMIs (17); suppose that there exists some node $k \in M$ such that $\{k, k\} \in L^+$; i.e., node k has a plus loop. Define $P' := P \cup \{k\}$ and $M' = M \setminus \{k\}$. Then all LMIs (17) corresponding to P, M are present in the LMIs (17) corresponding to P', M' . That is, the convex relaxation corresponding to P', M' is stronger than the convex relaxation corresponding to P, M . Henceforth, we assume that the nodes in M do not have plus loops.

Observation 2. The convexification technique of [18] can be obtained as a special case of our proposed convexification technique by letting $|P| = 1$. To see this, without loss of generality, let $P = \{1\}$ and let $M \subseteq [n] \setminus \{1\}$. Then system (17) simplifies to:

$$\ell(J, M \cup \{1\} \setminus J) \geq 0, \forall J \subseteq M \cup \{1\}. \quad (19)$$

$$\begin{bmatrix} \ell(J, M \setminus J) & \ell(J \cup \{1\}, M \setminus J) \\ \ell(J \cup \{1\}, M \setminus J) & \rho(1, J, M \setminus J) \end{bmatrix} \succeq 0, \forall J \subseteq M. \quad (20)$$

Inequalities (19) are obtained by letting $R = \emptyset$ in (17), while LMIs (20) are obtained by letting $R = P = \{1\}$ in (17). From RLT inequalities (19) and Observation 1 it follows that $\ell(J, M \setminus J) \geq 0$ for all $J \subseteq M$. Therefore, constraints (20) can be equivalently written as

$$\rho(1, J, M \setminus J) \geq \frac{(\ell(J \cup \{1\}, M \setminus J))^2}{\ell(J, M \setminus J)}, \forall J \subseteq M, \quad (21)$$

where we define $\frac{0}{0} := 0$. We next prove that by projecting out variables z_{11}^J , for all nonempty $J \subseteq M$ from system (21), we obtain

$$z_{11} \geq \sum_{J \subseteq M} \frac{(\ell(J \cup \{1\}, M \setminus J))^2}{\ell(J, M \setminus J)}, \quad (22)$$

which together with inequalities (19) coincide with the convex relaxation proposed in [18] (see proposition 5 in [18]). To see this, we first prove that inequality (22) is implied by inequalities (21). From (16) it follows that

$$\begin{aligned} \sum_{J \subseteq M} \rho(1, J, M \setminus J) &= \sum_{J \subseteq M} \sum_{t: t \subseteq M \setminus J} (-1)^{|t|} z_{11}^{J \cup t} \\ &= \sum_{J \subseteq M} \sum_{S \subseteq M: S \supseteq J} (-1)^{|S| - |J|} z_{11}^S \\ &= \sum_{S \subseteq M} z_{11}^S \sum_{J \subseteq S} (-1)^{|S| - |J|} \\ &= z_{11}^{\emptyset} - \sum_{S \subseteq M, S \neq \emptyset} z_{11}^S \sum_{k=0}^{|S|} \binom{|S|}{k} (-1)^{|S| - k} \\ &= z_{11} - \sum_{S \subseteq M, S \neq \emptyset} z_{11}^S (1 - 1)^{|S|} \end{aligned}$$

$$= z_{11}.$$

Recall that by definition $z_{11}^\emptyset = z_{11}$. Therefore, summing up the inequalities (21) we obtain the inequality (22). To complete the proof, we next show that for any z satisfying inequality (22), we can find z_{11}^S for all nonempty $S \subseteq M$ such that inequalities (21) are satisfied. For notational simplicity for each $J \subseteq M$, denote by u_J the right-hand side of inequality (21). For any z satisfying the inequality (22), let

$$z_{11}^S = \sum_{J \supseteq S} u_J, \quad \forall S \subseteq M : S \neq \emptyset. \quad (23)$$

Substituting (23) in inequalities (21) we obtain:

$$\begin{aligned} \rho(1, J, M \setminus J) &= \sum_{S \subseteq M : S \supseteq J} (-1)^{|S|-|J|} z_{11}^S \\ &\geq \sum_{S \subseteq M : S \supseteq J} (-1)^{|S|-|J|} \sum_{K \supseteq S} u_K \\ &= \sum_{K \supseteq J} u_K \sum_{S : J \subseteq S \subseteq K} (-1)^{|S|-|J|} \\ &= u_J, \end{aligned}$$

where the inequality follows from (22)- (23) and $z_{11}^\emptyset = z_{11}$. Moreover, the last equality follows since

$$\sum_{S : J \subseteq S \subseteq K} (-1)^{|S|-|J|} = \sum_{i=0}^{|K|-|J|} \binom{|K|-|J|}{i} (-1)^i = (1-1)^{|K|-|J|} = \begin{cases} 1, & \text{if } K = J, \\ 0, & \text{if } K \neq J. \end{cases}$$

Therefore, inequalities (21) are satisfied. Hence, we have proved that the projection of the set defined by the system (19)- (20) onto the z space is given by inequalities (19) and (22), which is identical to the convex relaxation of [18]. While in the special case with $|P| = 1$, it is possible to project out the auxiliary variables z_{11}^J , $J \subseteq M$, such a simplification is not possible for $|P| > 1$. Moreover, as detailed in [18], in the special case with $|P| = 1$, the proposed convex relaxation is SOCP-representable. For $|P| > 1$, however, which is the focus of this paper, our proposed convex relaxation is SDP-representable. Therefore, for $|P| = 1$, the formulation of [18] is preferable, whereas for $|P| > 1$, one should rely on our SDP-representable extended formulations. \diamond

For a graph $G = (V, E, L)$ there are exponentially many choices for subsets P and M in Theorem 3. As before, denote by G_{V^+} the subgraph of G induced by V^+ as defined by (7). Again, denote by V_c , $c \in \mathcal{C}$ the connected components of G_{V^+} . From the proof of Proposition 1 it follows that an extended formulation for $\text{QP}(G)$ is obtained by putting together formulations for $\text{PP}(G'_c)$, $c \in \mathcal{C} \cup \{0\}$, where the node set of each hypergraph G'_c , $c \in \mathcal{C}$ consists of V_c and all nodes in $V \setminus V_c$ that are adjacent to at least one node in V_c . Notice that the latter does not have any plus loops since V_c is a connected component of G_{V^+} . Moreover, the hypergraph G'_0 has no plus loops. As our proposed SDP relaxations are closely related to the extended formulation constructed in the proof of Proposition 1, henceforth, we assume that each $P \subseteq V_c$ for some $c \in \mathcal{C}$ and that each node in M is adjacent to at least some node in P . In the remainder of this paper, we refer to any such P and M as a *plus set*, and a *minus set*, respectively. Note that $\text{PP}(G'_0)$ coincides with the multilinear polytope $\text{MP}(G'_0)$, whose facial structure has been extensively studied in the literature [13, 14, 16, 17]. The following proposition establishes a simple criterion for selecting plus and minus sets that result in stronger relaxations.

Proposition 2. Let $G = (V, E, L)$ be a graph such that $V^+ \neq \emptyset$, where V^+ is defined by (7). Then we have the following:

- (i) Let P and P' be two plus sets, and let M be a corresponding minus set. If $P \subseteq P'$, then LMIs (17) corresponding to P, M are implied by LMIs (17) corresponding to P', M .
- (ii) Let M and M' be two minus sets, and let P be a corresponding plus set. If $M \subseteq M'$, then LMIs (17) corresponding to P, M are implied by LMIs (17) corresponding to P, M' .

Proof. Part (i) follows since all principal sub-matrices of a positive semidefinite matrix are positive semidefinite as well. Now consider Part (ii); without loss of generality, let $M' = M \cup \{k\}$ for some $k \notin M \cup P$. As before, for any $R \subseteq P$, define $M_R := (M \cup P) \setminus R$ and $M'_R := (M' \cup P) \setminus R = M_R \cup \{k\}$. Without loss of generality, let $R := \{1, \dots, p\}$ for some $p \geq 0$. Then for any $J \subseteq M_R$, we have:

$$\begin{bmatrix} f(J, M_R \setminus J) & f(J \cup \{1\}, M_R \setminus J) & \cdots & f(J \cup \{p\}, M_R \setminus J) \\ f(J \cup \{1\}, M_R \setminus J) & g(1, J, M_R \setminus J) & \cdots & f(J \cup \{1, p\}, M_R \setminus J) \\ \vdots & \vdots & \ddots & \vdots \\ f(J \cup \{p\}, M_R \setminus J) & f(J \cup \{1, p\}, M_R \setminus J) & \cdots & g(p, J, M_R \setminus J) \end{bmatrix} =$$

$$+ \begin{bmatrix} f(J \cup \{k\}, M'_R \setminus (J \cup \{k\})) & f(J \cup \{1, k\}, M'_R \setminus (J \cup \{k\})) & \cdots & f(J \cup \{p, k\}, M'_R \setminus (J \cup k)) \\ f(J \cup \{1, k\}, M'_R \setminus (J \cup k)) & g(1, J \cup \{k\}, M'_R \setminus (J \cup \{k\})) & \cdots & f(J \cup \{1, p, k\}, M'_R \setminus (J \cup k)) \\ \vdots & \vdots & \ddots & \vdots \\ f(J \cup \{p, k\}, M'_R \setminus (J \cup k)) & f(J \cup \{1, p, k\}, M'_R \setminus (J \cup k)) & \cdots & g(p, J \cup \{k\}, M'_R \setminus (J \cup \{k\})) \end{bmatrix} +$$

$$\begin{bmatrix} f(J, M'_R \setminus J) & f(J \cup \{1\}, M'_R \setminus J) & \cdots & f(J \cup \{p\}, M'_R \setminus J) \\ f(J \cup \{1\}, M'_R \setminus J) & g(1, J, M'_R \setminus J) & \cdots & f(J \cup \{1, p\}, M'_R \setminus J) \\ \vdots & \vdots & \ddots & \vdots \\ f(J \cup \{p\}, M'_R \setminus J) & f(J \cup \{1, p\}, M'_R \setminus J) & \cdots & g(p, J, M'_R \setminus J) \end{bmatrix}.$$

Using (11) and (16) to linearize the polynomial factors, we obtain:

$$\begin{bmatrix} \ell(J, M_R \setminus J) & \ell(J \cup \{1\}, M_R \setminus J) & \cdots & \ell(J \cup \{p\}, M_R \setminus J) \\ \ell(J \cup \{1\}, M_R \setminus J) & \rho(1, J, M_R \setminus J) & \cdots & \ell(J \cup \{1, p\}, M_R \setminus J) \\ \vdots & \vdots & \ddots & \vdots \\ \ell(J \cup \{p\}, M_R \setminus J) & \ell(J \cup \{1, p\}, M_R \setminus J) & \cdots & \rho(p, J, M_R \setminus J) \end{bmatrix} =$$

$$+ \begin{bmatrix} \ell(J \cup \{k\}, M'_R \setminus (J \cup \{k\})) & \ell(J \cup \{1, k\}, M'_R \setminus (J \cup \{k\})) & \cdots & \ell(J \cup \{p, k\}, M'_R \setminus (J \cup k)) \\ \ell(J \cup \{1, k\}, M'_R \setminus (J \cup k)) & \rho(1, J \cup \{k\}, M'_R \setminus (J \cup \{k\})) & \cdots & \ell(J \cup \{1, p, k\}, M'_R \setminus (J \cup k)) \\ \vdots & \vdots & \ddots & \vdots \\ \ell(J \cup \{p, k\}, M'_R \setminus (J \cup k)) & \ell(J \cup \{1, p, k\}, M'_R \setminus (J \cup k)) & \cdots & \rho(p, J \cup \{k\}, M'_R \setminus (J \cup \{k\})) \end{bmatrix} +$$

$$\begin{bmatrix} \ell(J, M'_R \setminus J) & \ell(J \cup \{1\}, M'_R \setminus J) & \cdots & \ell(J \cup \{p\}, M'_R \setminus J) \\ \ell(J \cup \{1\}, M'_R \setminus J) & \rho(1, J, M'_R \setminus J) & \cdots & \ell(J \cup \{1, p\}, M'_R \setminus J) \\ \vdots & \vdots & \ddots & \vdots \\ \ell(J \cup \{p\}, M'_R \setminus J) & \ell(J \cup \{1, p\}, M'_R \setminus J) & \cdots & \rho(p, J, M'_R \setminus J) \end{bmatrix}.$$

It then follows that the LMI (17) for some $J \subseteq M_R$ is implied by two LMIs of the form (17), one for $J \subseteq M'_R$ and one for $J \cup \{k\} \subseteq M'_R$. Therefore, the statement of Part (ii) follows. \square

By Proposition 2, the strongest SDP relaxation is obtained by choosing $|\mathcal{C}|$ plus and minus sets, where for each $c \in \mathcal{C}$, the plus set P is the connected component V_c , and the minus set M is $N(V_c)$, as defined by (8); namely, the set of all nodes without plus loops such that each node is adjacent to some node in P . However, it is important to note that, for a given choice of P, M , the system (17) consists of $(\frac{|P|}{2} + 1)2^{|M|+|P|}$ variables, $2^{|M|+|P|}$ linear inequalities, and $2^{|M|}(3^{|P|} - 2^{|P|})$ LMIs. The extended variables in this formulation are z_p for all $p \subseteq M \cup P$ such that $|p| \geq 2$ and $p \notin E$ and z_{ii}^J for all $J \subseteq M \cup P \setminus \{i\}$ and for all $i \in P$. Therefore, to obtain a polynomial-size SDP relaxation, we must choose

$$|M| \in O(\log_2(|V|)) \quad \text{and} \quad |P| \in O(\log_3(|V|)).$$

This mirrors the situation in the RLT hierarchy, where lower-level inequalities are implied by higher-level ones and the level- d LP relaxation contains 2^d variables and inequalities.

The following examples illustrate the application and strength of the proposed SDP relaxations for box-constrained QPs. In these examples, given a graph $G = (V, E)$ for notational simplicity, instead of $z_{\{ij\cdots k\}}$ for some $\{ij \cdots k\} \subseteq V$, we write $z_{ij\cdots k}$.

Example 1. Consider the graph $G = (V, E, L)$ with $V = \{1, 2\}$, $E = \{\{1, 2\}\}$, and $L = L^+ = \{\{1, 1\}, \{2, 2\}\}$. In this case, we have $P = V$ and $M = \emptyset$ and the proposed SDP relaxation for $\text{QP}(G)$ is given by:

$$\begin{aligned} & \begin{bmatrix} 1 & z_1 & z_2 \\ z_1 & z_{11} & z_{12} \\ z_2 & z_{12} & z_{22} \end{bmatrix} \succeq 0 \\ & \begin{bmatrix} z_2 & z_{12} \\ z_{12} & z_{11}^{\{2\}} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} 1 - z_2 & z_1 - z_{12} \\ z_1 - z_{12} & z_{11} - z_{11}^{\{2\}} \end{bmatrix} \succeq 0 \\ & \begin{bmatrix} z_1 & z_{12} \\ z_{12} & z_{22}^{\{1\}} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} 1 - z_1 & z_2 - z_{12} \\ z_2 - z_{12} & z_{22} - z_{22}^{\{1\}} \end{bmatrix} \succeq 0 \\ & z_{12} \geq 0, \quad z_1 - z_{12} \geq 0, \quad z_2 - z_{12} \geq 0, \quad 1 - z_1 - z_2 + z_{12} \geq 0. \end{aligned} \tag{24}$$

We will prove in the next section that, as a direct consequence of the work by Anstreicher and Burer [5], the above relaxation is tight; i.e., it is an extended formulation for $\text{QP}(G)$ (see Lemma 5). Notice that this formulation contains two extra variables $z_{11}^{\{2\}}, z_{22}^{\{1\}}$. \diamond

Example 2. Consider the graph $G = (V, E, L)$ with $V = \{1, 2, 3\}$, $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, and $L = L^+ = \{\{1, 1\}, \{2, 2\}\}$. In this case, we have $P = \{1, 2\}$ and $M = \{3\}$, and the proposed SDP relaxation for $\text{QP}(G)$ is given by:

$$\begin{aligned} & \begin{bmatrix} z_3 & z_{13} & z_{23} \\ z_{13} & z_{11}^{\{3\}} & z_{123} \\ z_{23} & z_{123} & z_{22}^{\{3\}} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} 1 - z_3 & z_1 - z_{13} & z_2 - z_{23} \\ z_1 - z_{13} & z_{11} - z_{11}^{\{3\}} & z_{12} - z_{123} \\ z_2 - z_{23} & z_{12} - z_{123} & z_{22} - z_{22}^{\{3\}} \end{bmatrix} \succeq 0 \\ & \begin{bmatrix} z_{23} & z_{123} \\ z_{123} & z_{11}^{\{2,3\}} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} z_2 - z_{23} & z_{12} - z_{123} \\ z_{12} - z_{123} & z_{11}^{\{2\}} - z_{11}^{\{2,3\}} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} z_3 - z_{23} & z_{13} - z_{123} \\ z_{13} - z_{123} & z_{11}^{\{3\}} - z_{11}^{\{2,3\}} \end{bmatrix} \succeq 0, \\ & \begin{bmatrix} 1 - z_2 - z_3 + z_{23} & z_1 - z_{12} - z_{13} + z_{123} \\ z_1 - z_{12} - z_{13} + z_{123} & z_{11} - z_{11}^{\{2\}} - z_{11}^{\{3\}} + z_{11}^{\{2,3\}} \end{bmatrix} \succeq 0, \end{aligned}$$

$$\begin{aligned}
\begin{bmatrix} z_{13} & z_{123} \\ z_{123} & z_{22}^{\{13\}} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} z_1 - z_{13} & z_{12} - z_{123} \\ z_{12} - z_{123} & z_{22}^{\{1\}} - z_{22}^{\{1,3\}} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} z_3 - z_{13} & z_{23} - z_{123} \\ z_{23} - z_{123} & z_{22}^{\{3\}} - z_{22}^{\{1,3\}} \end{bmatrix} \succeq 0, \\
\begin{bmatrix} 1 - z_1 - z_3 + z_{13} & z_2 - z_{12} - z_{23} + z_{123} \\ z_2 - z_{12} - z_{23} + z_{123} & z_{22} - z_{22}^{\{1\}} - z_{22}^{\{3\}} + z_{22}^{\{1,3\}} \end{bmatrix} \succeq 0, \\
z_{123} \geq 0, \quad z_{12} - z_{123} \geq 0, \quad z_{13} - z_{123} \geq 0, \quad z_{23} - z_{123} \geq 0, \quad z_1 - z_{12} - z_{13} + z_{123} \geq 0 \\
z_2 - z_{12} - z_{23} + z_{123} \geq 0, \quad z_3 - z_{13} - z_{23} + z_{123} \geq 0, \quad 1 - z_1 - z_2 - z_3 + z_{12} + z_{13} + z_{23} - z_{123} \geq 0. \tag{25}
\end{aligned}$$

We will prove in the next section that the above formulation is tight; i.e., it is an extended formulation for $\text{QP}(G)$ (see Theorem 4). To demonstrate the strength of the proposed relaxation compared to existing techniques, let us consider an instance of Problem QP with $n = 3$, defined by

$$\begin{aligned}
q_{11} &= 5080, \quad q_{12} = -5849, \quad q_{13} = 5767, \quad q_{22} = 5, \quad q_{23} = -1824, \quad q_{33} = -40, \\
c_1 &= -254, \quad c_2 = 1824, \quad c_3 = 37.
\end{aligned}$$

Utilizing the above relaxation together with the inequality $z_{33} \leq z_3$, we find that the optimal solution of Problem QP is given by $x_1 = \frac{3}{5}$, $x_2 = 1.0$, $x_3 = 0.0$ and its optimal value is -4.0 . If instead we use $\mathcal{C}_n^{\text{SDP+MC+Tri}}$ (4) to convexify the problem, the optimal value of the resulting SDP is -177.36 ; that is, we obtain a very weak lower bound. Alternatively, we can use the relaxation proposed in [18], which for this example, by Observation 2, can be equivalently written as

$$\begin{aligned}
\begin{bmatrix} z_{23} & z_{123} \\ z_{123} & z_{11}^{\{2,3\}} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} z_2 - z_{23} & z_{12} - z_{123} \\ z_{12} - z_{123} & z_{11}^{\{2\}} - z_{11}^{\{2,3\}} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} z_3 - z_{23} & z_{13} - z_{123} \\ z_{13} - z_{123} & z_{11}^{\{3\}} - z_{11}^{\{2,3\}} \end{bmatrix} \succeq 0, \\
\begin{bmatrix} 1 - z_2 - z_3 + z_{23} & z_1 - z_{12} - z_{13} + z_{123} \\ z_1 - z_{12} - z_{13} + z_{123} & z_{11} - z_{11}^{\{2\}} - z_{11}^{\{3\}} + z_{11}^{\{2,3\}} \end{bmatrix} \succeq 0, \\
\begin{bmatrix} z_{13} & z_{123} \\ z_{123} & z_{22}^{\{13\}} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} z_1 - z_{13} & z_{12} - z_{123} \\ z_{12} - z_{123} & z_{22}^{\{1\}} - z_{22}^{\{1,3\}} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} z_3 - z_{13} & z_{23} - z_{123} \\ z_{23} - z_{123} & z_{22}^{\{3\}} - z_{22}^{\{1,3\}} \end{bmatrix} \succeq 0, \\
\begin{bmatrix} 1 - z_1 - z_3 + z_{13} & z_2 - z_{12} - z_{23} + z_{123} \\ z_2 - z_{12} - z_{23} + z_{123} & z_{22} - z_{22}^{\{1\}} - z_{22}^{\{3\}} + z_{22}^{\{1,3\}} \end{bmatrix} \succeq 0, \\
z_{123} \geq 0, \quad z_{12} - z_{123} \geq 0, \quad z_{13} - z_{123} \geq 0, \quad z_{23} - z_{123} \geq 0, \quad z_1 - z_{12} - z_{13} + z_{123} \geq 0 \\
z_2 - z_{12} - z_{23} + z_{123} \geq 0, \quad z_3 - z_{13} - z_{23} + z_{123} \geq 0, \quad 1 - z_1 - z_2 - z_3 + z_{12} + z_{13} + z_{23} - z_{123} \geq 0.
\end{aligned}$$

The optimal value of the above relaxation is given by -4.53 . This relaxation can be further strengthened by adding the LMI:

$$\begin{bmatrix} 1 & z_1 & z_2 & z_3 \\ z_1 & z_{11} & z_{12} & z_{13} \\ z_2 & z_{12} & z_{22} & z_{23} \\ z_3 & z_{13} & z_{23} & z_{33} \end{bmatrix} \succeq 0. \tag{26}$$

The optimal value of the resulting SDP is -4.32 . Therefore, while for this example, incorporating the inequalities proposed in [18] leads to a significantly stronger relaxation than the popular relaxation (4), it still exhibits an 8% relative gap. However, our proposed relaxation solves this problem exactly. \diamond

Example 3. Consider the graph $G = (V, E, L)$ with $V = \{1, 2, 3\}$, $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, and $L = L^+ = \{\{1, 1\}, \{2, 2\}, \{3, 3\}\}$. In this case, we have $P = V$, $M = \emptyset$, and the proposed SDP relaxation for $\text{QP}(G)$ is given by:

$$\begin{aligned} & \begin{bmatrix} 1 & z_1 & z_2 & z_3 \\ z_1 & z_{11} & z_{12} & z_{13} \\ z_2 & z_{12} & z_{22} & z_{23} \\ z_3 & z_{13} & z_{23} & z_{33} \end{bmatrix} \succeq 0 \\ & \begin{bmatrix} z_k & z_{ik} & z_{jk} \\ z_{ik} & z_{ii}^{\{k\}} & z_{ijk} \\ z_{jk} & z_{ijk} & z_{jj}^{\{k\}} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} 1 - z_k & z_i - z_{ik} & z_j - z_{jk} \\ z_i - z_{ik} & z_{ii} - z_{ii}^{\{k\}} & z_{ij} - z_{ijk} \\ z_j - z_{jk} & z_{ij} - z_{ijk} & z_{jj} - z_{jj}^{\{k\}} \end{bmatrix} \succeq 0, \quad \forall i, j, k \in \{1, 2, 3\} : i < j, i \neq k, j \neq k \\ & \begin{bmatrix} z_{jk} & z_{ijk} \\ z_{ijk} & z_{ii}^{\{j,k\}} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} z_j - z_{jk} & z_{ij} - z_{ijk} \\ z_{ij} - z_{ijk} & z_{ii}^{\{j\}} - z_{ii}^{\{j,k\}} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} z_k - z_{jk} & z_{ik} - z_{ijk} \\ z_{ik} - z_{ijk} & z_{ii}^{\{k\}} - z_{ii}^{\{j,k\}} \end{bmatrix} \succeq 0, \\ & \begin{bmatrix} 1 - z_j - z_k + z_{jk} & z_i - z_{ij} - z_{ik} + z_{ijk} \\ z_i - z_{ij} - z_{ik} + z_{ijk} & z_{ii} - z_{ii}^{\{j\}} - z_{ii}^{\{k\}} + z_{ii}^{\{j,k\}} \end{bmatrix} \succeq 0, \quad \forall i, j, k \in \{1, 2, 3\} : j < k, i \neq j, i \neq k \\ & z_{123} \geq 0, \quad z_{12} - z_{123} \geq 0, \quad z_{13} - z_{123} \geq 0, \quad z_{23} - z_{123} \geq 0, \quad z_1 - z_{12} - z_{13} + z_{123} \geq 0 \\ & z_2 - z_{12} - z_{23} + z_{123} \geq 0, \quad z_3 - z_{13} - z_{23} + z_{123} \geq 0, \quad 1 - z_1 - z_2 - z_3 + z_{12} + z_{13} + z_{23} - z_{123} \geq 0, \end{aligned}$$

where we define $z_{ji} := z_{ij}$ for $1 \leq i < j \leq 3$ and $z_{ijk} := z_{123}$, for all $i \neq j \neq k \in \{1, 2, 3\}$. To demonstrate the strength of the proposed relaxation compared to existing techniques, let us consider an instance of Problem QP with $n = 3$, defined by

$$\begin{aligned} q_{11} &= 8, \quad q_{12} = 2732, \quad q_{13} = -4923, \quad q_{22} = 11, \quad q_{23} = -5960, \quad q_{33} = 3500, \\ c_1 &= 2, \quad c_2 = 140, \quad c_3 = 4523. \end{aligned}$$

For this example, the above formulation turns out to be tight and the optimal solution is given by $x_1 = 1$, $x_2 = 0$, $x_3 = \frac{2}{35}$ with the optimal value $-\frac{10}{7} \approx -1.43$. If instead we use $\mathcal{C}_n^{\text{SDP+MC+Tri}}$ (4) to convexify the problem, we obtain a lower bound of -173.93 , which as in the previous example is very weak. Alternatively, we can use the relaxation proposed in [18], which for this example by Observation 2, can be equivalently written as

$$\begin{aligned} & \begin{bmatrix} z_{jk} & z_{ijk} \\ z_{ijk} & z_{ii}^{\{j,k\}} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} z_j - z_{jk} & z_{ij} - z_{ijk} \\ z_{ij} - z_{ijk} & z_{ii}^{\{j\}} - z_{ii}^{\{j,k\}} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} z_k - z_{jk} & z_{ik} - z_{ijk} \\ z_{ik} - z_{ijk} & z_{ii}^{\{k\}} - z_{ii}^{\{j,k\}} \end{bmatrix} \succeq 0, \\ & \begin{bmatrix} 1 - z_j - z_k + z_{jk} & z_i - z_{ij} - z_{ik} + z_{ijk} \\ z_i - z_{ij} - z_{ik} + z_{ijk} & z_{ii} - z_{ii}^{\{j\}} - z_{ii}^{\{k\}} + z_{ii}^{\{j,k\}} \end{bmatrix} \succeq 0, \quad \forall i, j, k \in \{1, 2, 3\} : j < k, i \neq j, i \neq k \\ & z_{123} \geq 0, \quad z_{12} - z_{123} \geq 0, \quad z_{13} - z_{123} \geq 0, \quad z_{23} - z_{123} \geq 0, \quad z_1 - z_{12} - z_{13} + z_{123} \geq 0 \\ & z_2 - z_{12} - z_{23} + z_{123} \geq 0, \quad z_3 - z_{13} - z_{23} + z_{123} \geq 0, \quad 1 - z_1 - z_2 - z_3 + z_{12} + z_{13} + z_{23} - z_{123} \geq 0, \end{aligned}$$

Utilizing the above relaxation, we obtain a lower bound of -2.93 . We can further strengthen the above relaxation by adding LMI (26) to it. The resulting SDP yields a lower bound of -2.06 . Therefore, the strongest existing relaxation for this example results in about 44% gap, whereas our proposed relaxation is tight.

For a complete graph with three nodes and three plus loops, we leave it as an open question whether our proposed SDP relaxation gives an extended formulation for $\text{QP}(G)$. \diamond

In [18], the authors consider the special case with $|P| = 1$, and prove that if $|M| = 1$, then the proposed convex relaxation is implied by $\mathcal{C}_n^{\text{SDP+MC}}$ (3), while if $|M| > 1$, then the proposed relaxation is not implied by $\mathcal{C}_n^{\text{SDP+MC+Tri}}$ (4). From Proposition 2, it then follows that if $|M| > 1$ and $|P| \geq 1$, our proposed SDP relaxation is not implied by existing relaxations. Hence, it remains to analyze the case with $|M| \in \{0, 1\}$ and $|P| > 1$. The following proposition provides a complete comparison between our relaxation and the existing ones.

Proposition 3. *Consider a graph $G = (V, E, L)$ and consider the convex set defined by LMIs (17) for some plus and minus sets denoted by P and M , respectively. Denote by $\mathcal{S}_{P,M}$ the projection of this set onto the space z_l , $l \in V \cup E \cup L$. Define $p := |P|$ and $m := |M|$. We have the following cases:*

1. if $p + m \leq 2$, then $\mathcal{S}_{P,M}$ is implied by the relaxation $\mathcal{C}_n^{\text{SDP+MC}}$ defined by (3).
2. if $p \geq 1$ and $p + m \geq 3$, then $\mathcal{S}_{P,M}$ is not implied by the relaxation $\mathcal{C}_n^{\text{SDP+MC+Tri}}$ defined by (4).

Proof. Part 1 follows from theorem 2 of [5] stating that $\text{QP}_n = \mathcal{C}_n^{\text{SDP+MC}}$ for $n = 2$. Now consider Part 2; in Proposition 7 of [18], the authors proved that if $p = 1$ and $m \geq 2$, then $\mathcal{S}_{P,M}$ is not implied by the relaxation $\mathcal{C}_n^{\text{SDP+MC+Tri}}$. This result has the following implications:

- from Part (i) of Proposition 2 it follows that $\mathcal{S}_{P,M}$ is not implied by $\mathcal{C}_n^{\text{SDP+MC+Tri}}$ for $p \geq 1$ and $m \geq 2$.
- since $\mathcal{S}_{P,M}$ is not implied by $\mathcal{C}_n^{\text{SDP+MC+Tri}}$ for $p = 1$ and $m = 2$, from the definition of relaxation (17) it immediately follows that the same statement holds for $p = 2$ and $m = 1$. Therefore, by Part (ii) of Proposition 2 the same statement holds for $p > 2$ and $m = 1$.

The proof of Part (ii) then follows from the two cases above. \square

We conclude this section by remarking that if we ignore the sparsity of Problem QP, by Proposition 2, the strongest relaxation is obtained by letting $P = V$. In this case, the resulting SDP relaxation is stronger than both conventional RLT and SDP relaxations for $|V| \geq 3$. Namely, letting $R = \emptyset$, inequalities (17) simplifies to level- $|V|$ RLT inequalities $\ell(J, V \setminus J) \geq 0$ for all $J \subseteq V$, and letting $R = V$, inequalities (17) simplifies to the standard LMI $Y \succeq xx^\top$. By Proposition 3, the LMIs obtained by letting $R \subsetneq V$ are not implied by $Y \succeq xx^\top$ for $p = p + m = n + 0 \geq 3$. Therefore, in its strongest form, relaxation (17) is strictly stronger than both RLT and SDP relaxations. However, the resulting SDP relaxation is of exponential size. Therefore, we exploit the sparsity of Problem QP to choose plus and minus sets P, M carefully, hence, controlling the size of the relaxation.

4 SDP-representability of $\text{QP}(G)$

In this section, we obtain a sufficient condition in terms of the structure of the graph G under which $\text{QP}(G)$ is SDP-representable. Recall that a set is *SDP-representable* if it is the projection of a higher-dimensional set defined by LMIs. We first show that if G consists of two adjacent nodes with plus loops, then the proposed SDP relaxation is an extended formulation for $\text{QP}(G)$. This result is a consequence of the following well-known result of Burer and Anstreicher [5]:

Proposition 4 (theorem 2 in [5]). *Consider the set*

$$\mathcal{C} = \left\{ (z_1, z_2, z_{11}, z_{12}, z_{22}) : z_{11} = z_1^2, z_{12} = z_1 z_2, z_{22} = z_2^2, z_1, z_2 \in [0, 1]^2 \right\}. \quad (27)$$

Then the convex hull of \mathcal{C} is defined by the following inequalities:

$$\begin{bmatrix} 1 & z_1 & z_2 \\ z_1 & z_{11} & z_{12} \\ z_2 & z_{12} & z_{22} \end{bmatrix} \succeq 0$$

$$z_{11} \leq z_1, z_{22} \leq z_2$$

$$z_{12} \geq 0, z_1 - z_{12} \geq 0, z_2 - z_{12} \geq 0, 1 - z_1 - z_2 + z_{12} \geq 0.$$

We then have the following result:

Lemma 5. Consider the graph $G = (V, E, L)$ with $V = \{1, 2\}$, $E = \{\{1, 2\}\}$, and $L = L^+ = \{\{1, 1\}, \{2, 2\}\}$. Then $QP(G)$ is defined by the following inequalities:

$$\begin{bmatrix} 1 & z_1 & z_2 \\ z_1 & z_{11} & z_{12} \\ z_2 & z_{12} & z_{22} \end{bmatrix} \succeq 0 \quad (28)$$

$$z_{11} \geq \frac{z_{12}^2}{z_2} + \frac{(z_1 - z_{12})^2}{1 - z_2}, \quad z_{22} \geq \frac{z_{12}^2}{z_1} + \frac{(z_2 - z_{12})^2}{1 - z_1} \quad (29)$$

$$z_{12} \geq 0, z_1 - z_{12} \geq 0, z_2 - z_{12} \geq 0, 1 - z_1 - z_2 + z_{12} \geq 0. \quad (30)$$

Proof. Define the sets

$$\mathcal{C}_+ = \{(z_1, z_2, z_{11}, z_{12}, z_{22}) : z_{11} \geq z_1^2, z_{12} = z_1 z_2, z_{22} \geq z_2^2, z_1, z_2 \in [0, 1]^2\},$$

and

$$\mathcal{C}_\infty := \{(z_1, z_2, z_{11}, z_{12}, z_{22}) : z_1 = z_2 = z_{12} = 0, z_{11} \geq 0, z_{22} \geq 0\}.$$

We then have that $\mathcal{C}_+ = \mathcal{C} \oplus \mathcal{C}_\infty$, where the set \mathcal{C} is defined by (27) and \oplus denotes the Minkowski sum of sets. This, in turn, implies that $QP(G) = \text{conv}(\mathcal{C}_+) = \text{conv}(\mathcal{C}_\infty) \oplus \text{conv}(\mathcal{C})$. From Proposition 4 it follows that:

$$QP(G) = \left\{ (z_1, z_2, z_{11} + \alpha, z_{12}, z_{22} + \beta) : \begin{bmatrix} 1 & z_1 & z_2 \\ z_1 & z_{11} & z_{12} \\ z_2 & z_{12} & z_{22} \end{bmatrix} \succeq 0, z_{11} \leq z_1, z_{22} \leq z_2, z_{12} \geq 0, z_2 - z_{12} \geq 0, \right. \\ \left. z_1 - z_{12} \geq 0, 1 - z_1 - z_2 + z_{12} \geq 0, \alpha \geq 0, \beta \geq 0 \right\}.$$

Defining $z'_{11} = z_{11} + \alpha$, $z'_{22} = z_{22} + \beta$, rewriting the positive semidefiniteness condition in terms of the nonnegativity of the principal minors and using Lemma 2 in [12] to remove the redundant principal minor inequality, we conclude that $QP(G)$ is obtained by projecting out variables α and β from the following system:

$$\begin{aligned} z_{12} &\geq 0, z_1 - z_{12} \geq 0, z_2 - z_{12} \geq 0, 1 - z_1 - z_2 + z_{12} \geq 0 \\ z'_{11} - \alpha &\geq z_1^2, z'_{22} - \beta \geq z_2^2 \\ (z'_{11} - \alpha - z_1^2)(z'_{22} - \beta - z_2^2) &\geq (z_{12} - z_1 z_2)^2 \\ z'_{11} - \alpha &\leq z_1, z'_{22} - \beta \leq z_2 \\ \alpha &\geq 0, \beta \geq 0. \end{aligned}$$

Notice that the inequalities in the first line of the above system coincide with McCormick inequalities (30). To project out α, β from the remaining inequalities, first, consider the inequalities containing α :

$$z'_{11} - \alpha \geq z_1^2 \quad (31)$$

$$(z'_{11} - \alpha - z_1^2)(z'_{22} - \beta - z_2^2) \geq (z_{12} - z_1 z_2)^2 \quad (32)$$

$$z'_{11} - \alpha \leq z_1 \quad (33)$$

$$\alpha \geq 0. \quad (34)$$

Projecting out α from (31) and (33) yields $z_1^2 \leq z_1$, which is implied by the McCormick inequalities (30). Projecting out α from (31) and (34) we get

$$z'_{11} \geq z_1^2. \quad (35)$$

Projecting out α from (32) and (33) gives

$$(z_1 - z_1^2)(z'_{22} - \beta - z_2^2) \geq (z_{12} - z_1 z_2)^2,$$

while projecting out α from (32) and (34) gives

$$(z'_{11} - z_1^2)(z'_{22} - \beta - z_2^2) \geq (z_{12} - z_1 z_2)^2.$$

Next, consider the remaining inequalities containing β :

$$z'_{22} - \beta \geq z_2^2 \quad (36)$$

$$z'_{22} - \beta \leq z_2 \quad (37)$$

$$(z_1 - z_1^2)(z'_{22} - \beta - z_2^2) \geq (z_{12} - z_1 z_2)^2 \quad (38)$$

$$(z'_{11} - z_1^2)(z'_{22} - \beta - z_2^2) \geq (z_{12} - z_1 z_2)^2 \quad (39)$$

$$\beta \geq 0. \quad (40)$$

Projecting out β from (36) and (37) yields $z_2^2 \leq z_2$, which is implied by inequalities (30). Projecting out β from (36) and (40) yields

$$z'_{22} \geq z_2^2. \quad (41)$$

Projecting out β from (37) and (38) yields:

$$(z_1 - z_1^2)(z_2 - z_2^2) \geq (z_{12} - z_1 z_2)^2. \quad (42)$$

We now show that inequality (42) is implied by the McCormick inequalities (30). First, suppose that $z_{12} - z_1 z_2 \geq 0$. In this case, by $z_{12} \leq z_1$, we have $z_{12} - z_1 z_2 \leq z_1(1 - z_2)$, while by $z_{12} \leq z_2$, we have $z_{12} - z_1 z_2 \leq z_2(1 - z_1)$. Multiplying the two inequalities and using $z_{12} - z_1 z_2 \geq 0$, we obtain (42). Next, suppose that $z_{12} - z_1 z_2 \leq 0$. In this case, by $z_{12} \geq z_1 + z_2 - 1$, we have $z_1 z_2 - z_{12} \leq z_1 z_2 - z_1 - z_2 + 1$, while by $z_{12} \geq 0$, we have $z_1 z_2 - z_{12} \leq z_1 z_2$. Multiplying these two inequalities and using $z_1 z_2 - z_{12} \geq 0$, we obtain (42).

Projecting out β from (37) and (39) yields $(z'_{11} - z_1^2)(z_2 - z_2^2) \geq (z_{12} - z_1 z_2)^2$. It can be checked that, this inequality can be equivalently written as the first inequality in (29). Projecting out β from (38) and (40) gives $(z_1 - z_1^2)(z'_{22} - z_2^2) \geq (z_{12} - z_1 z_2)^2$. Again, it can be checked that, this inequality can be equivalently written as the second inequality in (29). Finally, projecting out β from (39) and (40), we obtain

$$(z'_{11} - z_1^2)(z'_{22} - z_2^2) \geq (z_{12} - z_1 z_2)^2. \quad (43)$$

By Lemma 2 in [12], the three inequalities (35), (41), and (43) are equivalent to LMI (28). Therefore, the statement follows. \square

Observation 3. It can be checked that, inequalities (28)- (30) can be obtained by projecting out variables $z_{11}^{\{2\}}, z_{22}^{\{1\}}$ from system (24). Namely, the proposed SDP relaxation is an extended formulation for $\text{QP}(G)$ in this case. Moreover, notice that inequalities (29) are not redundant. To see this, consider the point $\tilde{z}_1 = \tilde{z}_2 = \frac{1}{2}, \tilde{z}_{11} = \frac{1}{3}, \tilde{z}_{12} = 0, \tilde{z}_{22} = 1$. It is simple to check that this point satisfies (28) and (30). Now consider the first inequality in (29). Substituting \tilde{z} in this inequality we obtain: $\frac{1}{3} \not\geq 0 + \frac{(\frac{1}{2})^2}{\frac{1}{2}} = \frac{1}{2}$.

We make use of the following lemma to establish our next convex hull characterization.

Lemma 6. *Let \mathcal{C}, S^0, S^1 be three convex sets such that $\mathcal{C} = \text{conv}(S^0 \cup S^1)$. Suppose that \bar{S}^0 and \bar{S}^1 are extended formulations for S^0 and S^1 , respectively. Then $\bar{C} := \text{conv}(\bar{S}^0 \cup \bar{S}^1)$ is an extended formulation for \mathcal{C} .*

Proof. Suppose that the sets \mathcal{C}, S^0, S^1 lie in the space $x \in \mathbb{R}^n$. Since \bar{S}^0 and \bar{S}^1 are extended formulations for S^0 and S^1 , we can write $S_0 = \text{proj}_x \bar{S}_0$ and $S_1 = \text{proj}_x \bar{S}_1$, where proj_x denotes the projection onto the x space. We then have:

$$\begin{aligned} C &= \text{conv}((\text{proj}_x \bar{S}_0) \cup (\text{proj}_x \bar{S}_1)) \\ &= \text{conv}(\text{proj}_x(\bar{S}_0 \cup \bar{S}_1)) \\ &= \text{proj}_x \text{conv}(\bar{S}_0 \cup \bar{S}_1) \\ &= \text{proj}_x \bar{C}, \end{aligned}$$

where the second equality follows since projection commutes with union, and the third equality follows since projection commutes with taking the convex hull. \square

The next result implies that if the graph G has two plus loops, then the proposed SDP relaxation is an extended formulation for $\text{QP}(G)$.

Theorem 4. *Let $G = (V, E, L)$ be a complete hypergraph and $L = L^+ = \{\{i, i\}, \{j, j\}\}$ for some $i \neq j \in V$. Consider the convex set $\text{PP}(G)$ defined by (5). Then an extended formulation for $\text{PP}(G)$ with $|V| + |E| - 1$ additional variables is given by:*

$$\begin{bmatrix} \ell(J, V \setminus (J \cup \{i, j\})) & \ell(J \cup \{i\}, V \setminus (J \cup \{i, j\})) & \ell(J \cup \{j\}, V \setminus (J \cup \{i, j\})) \\ \ell(J \cup \{i\}, V \setminus (J \cup \{i, j\})) & \rho(i, J, V \setminus (J \cup \{i, j\})) & \ell(J \cup \{i, j\}, V \setminus (J \cup \{i, j\})) \\ \ell(J \cup \{j\}, V \setminus (J \cup \{i, j\})) & \ell(J \cup \{i, j\}, V \setminus (J \cup \{i, j\})) & \rho(j, J, V \setminus (J \cup \{i, j\})) \end{bmatrix} \succeq 0, \quad \forall J \subseteq V \setminus \{i, j\} \quad (44)$$

$$\begin{bmatrix} \ell(J, V \setminus (J \cup \{i\})) & \ell(J \cup \{i\}, V \setminus (J \cup \{i\})) \\ \ell(J \cup \{i\}, V \setminus (J \cup \{i\})) & \rho(i, J, V \setminus (J \cup \{i\})) \end{bmatrix} \succeq 0, \quad \forall J \subseteq V \setminus \{i\} \quad (45)$$

$$\begin{bmatrix} \ell(J, V \setminus (J \cup \{j\})) & \ell(J \cup \{j\}, V \setminus (J \cup \{j\})) \\ \ell(J \cup \{j\}, V \setminus (J \cup \{j\})) & \rho(j, J, V \setminus (J \cup \{j\})) \end{bmatrix} \succeq 0, \quad \forall J \subseteq V \setminus \{j\} \quad (46)$$

$$\ell(J, V \setminus J) \geq 0, \quad \forall J \subseteq V. \quad (47)$$

Proof. First, observe that system (44)- (47) is obtained from system (17) by defining $P := \{i, j\}$ and $M := V \setminus \{i, j\}$. Namely, LMI (44) is obtained by letting $R = P$, LMI (45) is obtained by letting $R = \{i\}$, LMI (46) is obtained by letting $R = \{j\}$, and the RLT inequalities (47) are obtained by letting $R = \emptyset$. Therefore, system (44)- (47) is a valid convex relaxation for $\text{PP}(G)$. We next prove that this system provides an extended formulation for $\text{PP}(G)$.

We prove by induction on the number of nodes in M . In the base case, we have $M = \emptyset$, implying that $V = P = \{i, j\}$. Using the identities $\ell(\emptyset, \emptyset) = 1, \ell(i, \emptyset) = z_i, \rho(i, \emptyset, \emptyset) = z_{ii}$, it can

be checked that, in this case system (44)- (47) simplifies to system (24) (after replacing $V = \{1, 2\}$ with $V = \{i, j\}$). By Lemma 5 and Observation 3, system (24) is an extended formulation for $\text{PP}(G)$.

Henceforth, let $|M| \geq 1$. Let $k \in M$. Let $\text{PP}^0(G)$ (resp. $\text{PP}^1(G)$) denote the face of $\text{PP}(G)$ defined by $z_k = 0$ (resp. $z_k = 1$). Since by part (i) of Lemma 1, $\text{PP}(G)$ is a closed set, by part (ii) of Lemma 1 at every extreme point of $\text{PP}(G)$ we have $z_k \in \{0, 1\}$, and by part (iii) of Lemma 1 at every extreme direction of $\text{PP}(G)$ we have $z_k = 0$, it follows that:

$$\text{PP}(G) = \text{conv}(\text{PP}^0(G) \cup \text{PP}^1(G)). \quad (48)$$

Let $\bar{G} = (V \setminus \{k\}, \bar{E}, L)$ be a complete hypergraph, and let $L = L^+ = \{\{i, i\}, \{j, j\}\}$. Since \bar{G} is a complete hypergraph with two plus loops and with one fewer node in M than the hypergraph G , by the induction hypothesis, an extended formulation for $\text{PP}(\bar{G})$ is given by:

$$\begin{aligned} & \begin{bmatrix} \ell(J, V \setminus (J \cup \{i, j, k\})) & \ell(J \cup \{i\}, V \setminus (J \cup \{i, j, k\})) & \ell(J \cup \{j\}, V \setminus (J \cup \{i, j, k\})) \\ \ell(J \cup \{i\}, V \setminus (J \cup \{i, j, k\})) & \rho(i, J, V \setminus (J \cup \{i, j, k\})) & \ell(J \cup \{i, j\}, V \setminus (J \cup \{i, j, k\})) \\ \ell(J \cup \{j\}, V \setminus (J \cup \{i, j, k\})) & \ell(J \cup \{i, j\}, V \setminus (J \cup \{i, j, k\})) & \rho(j, J, V \setminus (J \cup \{i, j, k\})) \end{bmatrix} \succeq 0, \\ & \forall J \subseteq V \setminus \{i, j, k\} \\ & \begin{bmatrix} \ell(J, V \setminus (J \cup \{i, k\})) & \ell(J \cup \{i\}, V \setminus (J \cup \{i, k\})) \\ \ell(J \cup \{i\}, V \setminus (J \cup \{i, k\})) & \rho(i, J, V \setminus (J \cup \{i, k\})) \end{bmatrix} \succeq 0, \quad \forall J \subseteq V \setminus \{i, k\} \\ & \begin{bmatrix} \ell(J, V \setminus (J \cup \{j, k\})) & \ell(J \cup \{j, k\}, V \setminus (J \cup \{j, k\})) \\ \ell(J \cup \{j, k\}, V \setminus (J \cup \{j, k\})) & \rho(j, J, V \setminus (J \cup \{j, k\})) \end{bmatrix} \succeq 0, \quad \forall J \subseteq V \setminus \{j, k\} \\ & \ell(J, V \setminus (J \cup \{k\})) \geq 0, \quad \forall J \subseteq V \setminus \{k\}. \end{aligned}$$

Denote by \bar{z} the vector consisting of all variables z_v , $v \in V \setminus \{k\}$, z_e for all $e \in \bar{E}$, z_{ii}^J for all $J \subseteq V \setminus \{i, k\}$ and z_{jj}^J for all $J \subseteq V \setminus \{j, k\}$, where as before we define $z_{ii}^\emptyset := z_{ii}$ and $z_{jj}^\emptyset := z_{jj}$. It then follows that

$$\begin{aligned} \text{PP}^0(G) = & \left\{ z \in \mathbb{R}^{V \cup E \cup L} : \exists z_{ii}^J, J \subseteq V \setminus \{i\}, J \neq \emptyset, z_{jj}^J, J \subseteq V \setminus \{j\}, J \neq \emptyset \text{ such that } z_k = 0, z_e = 0, \right. \\ & \left. \forall e \in E : e \ni k, z_{ii}^J = 0, \forall J \subseteq V \setminus \{i\} : J \ni k, z_{jj}^J = 0, \forall J \subseteq V \setminus \{j\} : J \ni k, \bar{z} \in \text{PP}(\bar{G}) \right\}, \end{aligned}$$

$$\begin{aligned} \text{PP}^1(G) = & \left\{ z \in \mathbb{R}^{V \cup E \cup L} : \exists z_{ii}^J, J \subseteq V \setminus \{i\}, J \neq \emptyset, z_{jj}^J, J \subseteq V \setminus \{j\}, J \neq \emptyset \text{ such that } z_k = 1, z_e = z_{e \setminus \{k\}}, \right. \\ & \left. \forall e \in E : e \ni k, z_{ii}^J = z_{ii}^{J \setminus \{k\}}, \forall J \subseteq V \setminus \{i\} : J \ni k, z_{jj}^J = z_{jj}^{J \setminus \{k\}}, \forall J \subseteq V \setminus \{j\} : J \ni k, \right. \\ & \left. \bar{z} \in \text{PP}(\bar{G}) \right\}. \end{aligned}$$

From (48), Lemma 6, and the standard disjunctive programming technique [6, 29], it follows that an extended formulation for $\text{PP}(G)$ is given by:

$$\lambda_0 + \lambda_1 = 1, \quad \lambda_0 \geq 0, \quad \lambda_1 \geq 0 \quad (49)$$

$$z_v = z_v^0 + z_v^1, \quad \forall v \in V \quad (50)$$

$$z_e = z_e^0 + z_e^1, \quad \forall e \in E \quad (51)$$

$$z_{ii}^J = z_{ii}^{0;J} + z_{ii}^{1;J}, \quad \forall J \subseteq V \setminus \{i\} \quad (52)$$

$$z_{jj}^J = z_{jj}^{0,J} + z_{jj}^{1,J}, \quad \forall J \subseteq V \setminus \{j\} \quad (53)$$

$$z_k^0 = 0 \quad (54)$$

$$z_e^0 = 0, \quad \forall e \in E : e \ni k \quad (55)$$

$$z_{ii}^{0,J} = 0, \quad \forall J \subseteq V \setminus \{i\} : J \ni k \quad (56)$$

$$z_{jj}^{0,J} = 0, \quad \forall J \subseteq V \setminus \{j\} : J \ni k \quad (57)$$

$$\begin{bmatrix} \ell^0(J, V \setminus (J \cup \{i, j, k\})) & \ell^0(J \cup \{i\}, V \setminus (J \cup \{i, j, k\})) & \ell^0(J \cup \{j\}, V \setminus (J \cup \{i, j, k\})) \\ \ell^0(J \cup \{i\}, V \setminus (J \cup \{i, j, k\})) & \rho^0(i, J, V \setminus (J \cup \{i, j, k\})) & \ell^0(J \cup \{i, j\}, V \setminus (J \cup \{i, j, k\})) \\ \ell^0(J \cup \{j\}, V \setminus (J \cup \{i, j, k\})) & \ell^0(J \cup \{i, j\}, V \setminus (J \cup \{i, j, k\})) & \rho^0(j, J, V \setminus (J \cup \{i, j, k\})) \end{bmatrix} \succeq 0, \quad \forall J \subseteq V \setminus \{i, j, k\} \quad (58)$$

$$\begin{bmatrix} \ell^0(J, V \setminus (J \cup \{i, k\})) & \ell^0(J \cup \{i\}, V \setminus (J \cup \{i, k\})) \\ \ell^0(J \cup \{i\}, V \setminus (J \cup \{i, k\})) & \rho^0(i, J, V \setminus (J \cup \{i, k\})) \end{bmatrix} \succeq 0, \quad \forall J \subseteq V \setminus \{i, k\} \quad (59)$$

$$\begin{bmatrix} \ell^0(J, V \setminus (J \cup \{j, k\})) & \ell^0(J \cup \{j, k\}, V \setminus (J \cup \{j, k\})) \\ \ell^0(J \cup \{j, k\}, V \setminus (J \cup \{j, k\})) & \rho^0(j, J, V \setminus (J \cup \{j, k\})) \end{bmatrix} \succeq 0, \quad \forall J \subseteq V \setminus \{j, k\} \quad (60)$$

$$\ell^0(J, V \setminus (J \cup \{k\})) \geq 0, \quad \forall J \subseteq V \setminus \{k\} \quad (61)$$

$$z_k^1 = \lambda_1 \quad (62)$$

$$z_e^1 = z_{e \setminus \{k\}}^1, \quad \forall e \in E : e \ni k \quad (63)$$

$$z_{ii}^{1,J} = z_{ii}^{J \setminus \{k\}}, \quad \forall J \subseteq V \setminus \{i\} : J \ni k \quad (64)$$

$$z_{jj}^{1,J} = z_{jj}^{J \setminus \{k\}}, \quad \forall J \subseteq V \setminus \{j\} : J \ni k \quad (65)$$

$$\begin{bmatrix} \ell^1(J, V \setminus (J \cup \{i, j, k\})) & \ell^1(J \cup \{i\}, V \setminus (J \cup \{i, j, k\})) & \ell^1(J \cup \{j\}, V \setminus (J \cup \{i, j, k\})) \\ \ell^1(J \cup \{i\}, V \setminus (J \cup \{i, j, k\})) & \rho^1(i, J, V \setminus (J \cup \{i, j, k\})) & \ell^1(J \cup \{i, j\}, V \setminus (J \cup \{i, j, k\})) \\ \ell^1(J \cup \{j\}, V \setminus (J \cup \{i, j, k\})) & \ell^1(J \cup \{i, j\}, V \setminus (J \cup \{i, j, k\})) & \rho^1(j, J, V \setminus (J \cup \{i, j, k\})) \end{bmatrix} \succeq 0, \quad \forall J \subseteq V \setminus \{i, j, k\} \quad (66)$$

$$\begin{bmatrix} \ell^1(J, V \setminus (J \cup \{i, k\})) & \ell^1(J \cup \{i\}, V \setminus (J \cup \{i, k\})) \\ \ell^1(J \cup \{i\}, V \setminus (J \cup \{i, k\})) & \rho^1(i, J, V \setminus (J \cup \{i, k\})) \end{bmatrix} \succeq 0, \quad \forall J \subseteq V \setminus \{i, k\} \quad (67)$$

$$\begin{bmatrix} \ell^1(J, V \setminus (J \cup \{j, k\})) & \ell^1(J \cup \{j, k\}, V \setminus (J \cup \{j, k\})) \\ \ell^1(J \cup \{j, k\}, V \setminus (J \cup \{j, k\})) & \rho^1(j, J, V \setminus (J \cup \{j, k\})) \end{bmatrix} \succeq 0, \quad \forall J \subseteq V \setminus \{j, k\} \quad (68)$$

$$\ell^1(J, V \setminus (J \cup \{k\})) \geq 0, \quad \forall J \subseteq V \setminus \{k\}, \quad (69)$$

where we define

$$z_\emptyset^0 := \lambda_0$$

$$\ell^0(J, V \setminus (J \cup \{k\})) := \sum_{t: t \subseteq V \setminus (J \cup \{k\})} (-1)^{|t|} z_{J \cup t}^0, \quad \forall J \subseteq V \setminus \{k\}$$

$$\rho^0(i, J, V \setminus (J \cup \{i, k\})) = \sum_{t: t \subseteq V \setminus (J \cup \{i, k\})} (-1)^{|t|} z_{ii}^{0, J \cup t}, \quad \forall J \subseteq V \setminus \{i, k\}$$

$$z_\emptyset^1 := \lambda_1$$

$$\ell^1(J, V \setminus (J \cup \{k\})) := \sum_{t: t \subseteq V \setminus (J \cup \{k\})} (-1)^{|t|} z_{J \cup t}^1, \quad \forall J \subseteq V \setminus \{k\}$$

$$\rho^1(i, J, V \setminus (J \cup \{i, k\})) := \sum_{t: t \subseteq V \setminus (J \cup \{i, k\})} (-1)^{|t|} z_{ii}^{1, J \cup t}, \quad \forall J \subseteq V \setminus \{i, k\}.$$

That is, $\ell^0(\cdot, \cdot)$ (resp. $\ell^1(\cdot, \cdot)$) is obtained from $\ell(\cdot, \cdot)$ by replacing z_v with z_v^0 (resp. z_v^1) for all $v \in V$, z_e with z_e^0 (resp. z_e^1) for all $e \in E$ and z_\emptyset , with z_\emptyset^0 (resp. z_\emptyset^1). Similarly, $\rho^0(i, \cdot, \cdot)$ (resp. $\rho^1(i, \cdot, \cdot)$) is obtained from $\rho(i, \cdot, \cdot)$ by replacing z_{ii}^J with $z_{ii}^{0,J}$ (resp. $z_{ii}^{1,J}$) for all $J \subseteq V \setminus \{i\}$.

In the remainder of the proof, we project out variables $\lambda_0, \lambda_1, z^0, z^1$ from the above system and show that the resulting system coincides with system (44)- (47). First, from (49), (50) (54), and (62) it follows that

$$\lambda_0 = 1 - z_k, \quad \lambda_1 = z_k. \quad (70)$$

Next, from (51), (55), and (63), we get that:

$$z_e^1 = z_{e \setminus \{k\}}^1 = z_e \quad \forall e \in E : e \ni k \quad (71)$$

$$z_{e \setminus \{k\}}^0 = z_{e \setminus \{k\}} - z_e \quad \forall e \in E : e \ni k \quad (72)$$

Similarly, from (52), (53), (56), (57), (64), and (65) it follows that

$$z_{ii}^{1,J} = z_{ii}^{1,J \setminus \{k\}} = z_{ii}^J, \quad \forall J \subseteq V \setminus \{i\} : J \ni k \quad (73)$$

$$z_{ii}^{0,J \setminus \{k\}} = z_{ii}^{J \setminus \{k\}} - z_{ii}^J, \quad \forall J \subseteq V \setminus \{i\} : J \ni k \quad (74)$$

$$z_{jj}^{1,J} = z_{jj}^{1,J \setminus \{k\}} = z_{ii}^J, \quad \forall J \subseteq V \setminus \{j\} : J \ni k \quad (75)$$

$$z_{jj}^{0,J \setminus \{k\}} = z_{jj}^{J \setminus \{k\}} - z_{jj}^J, \quad \forall J \subseteq V \setminus \{j\} : J \ni k. \quad (76)$$

We use equation (70) to project out λ_0, λ_1 , equation (71) to project out z_v^1 for all $v \in V \setminus \{k\}$ and for all z_e^1 for all $e \in \bar{E}$, equation (72) to project out z_v^0 for all $v \in V \setminus \{k\}$ and z_e^0 for all $e \in \bar{E}$, equation (73) (resp. equation (75)) to project out $z_{ii}^{1,J}$ (resp. $z_{jj}^{1,J}$) for all $J \subseteq V \setminus \{i, k\}$ (resp. $J \subseteq V \setminus \{j, k\}$), and we use equation (74) (resp. (76)) to project out $z_{ii}^{0,J}$ (resp. $z_{jj}^{0,J}$) for all $J \subseteq V \setminus \{i, k\}$ (resp. $J \subseteq V \setminus \{j, k\}$). By making these substitutions, for any $J \subseteq V \setminus \{k\}$, we get:

$$\begin{aligned} \ell^0(J, V \setminus (J \cup \{k\})) &= \sum_{t \subseteq V \setminus (J \cup \{k\})} (-1)^t (z_{J \cup t} - z_{J \cup \{k\} \cup t}) \\ &= \ell(J, V \setminus (J \cup \{k\})) - \ell(J \cup \{k\}, V \setminus (J \cup \{k\})) \\ &= \ell(J, V \setminus J) \end{aligned} \quad (77)$$

$$\begin{aligned} \ell^1(J, V \setminus (J \cup \{k\})) &= \sum_{t \subseteq V \setminus (J \cup \{k\})} (-1)^t z_{J \cup t} \\ &= \ell(J \cup \{k\}, V \setminus (J \cup \{k\})). \end{aligned} \quad (78)$$

Similarly, for each $J \subseteq V \setminus \{i, k\}$, we get:

$$\begin{aligned} \rho^0(i, J, V \setminus (J \cup \{i, k\})) &= \sum_{t \subseteq V \setminus (J \cup \{i, k\})} (-1)^t (z_{ii}^{J \cup t} - z_{ii}^{J \cup \{k\} \cup t}) \\ &= \rho(i, J, V \setminus (J \cup \{i, k\})) - \rho(i, J \cup \{k\}, V \setminus (J \cup \{i, k\})) \\ &= \rho(i, J, V \setminus (J \cup \{i\})) \end{aligned} \quad (79)$$

$$\begin{aligned} \rho^1(i, J, V \setminus (J \cup \{i, k\})) &= \sum_{t \subseteq V \setminus (J \cup \{i, k\})} (-1)^t z_{ii}^{J \cup \{k\} \cup t} \\ &= \rho(i, J \cup \{k\}, V \setminus (J \cup \{i, k\})). \end{aligned} \quad (80)$$

First, substituting (77) and (78) into (61) and (69) we get RLT inequalities (47). Next, substituting (77) and (79) into LMI (58), we get

$$\begin{bmatrix} \ell(J, V \setminus (J \cup \{i, j\})) & \ell(J \cup \{i\}, V \setminus (J \cup \{i, j\})) & \ell(J \cup \{j\}, V \setminus (J \cup \{i, j\})) \\ \ell(J \cup \{i\}, V \setminus (J \cup \{i, j\})) & \rho(i, J, V \setminus (J \cup \{i, j\})) & \ell(J \cup \{i, j\}, V \setminus (J \cup \{i, j\})) \\ \ell(J \cup \{j\}, V \setminus (J \cup \{i, j\})) & \ell(J \cup \{i, j\}, V \setminus (J \cup \{i, j\})) & \rho(j, J, V \setminus (J \cup \{i, j\})) \end{bmatrix} \succeq 0, \forall J \subseteq V \setminus \{i, j, k\}, \quad (81)$$

Similarly, substituting (78) and (80) into LMI (66), we get

$$\begin{aligned} & \begin{bmatrix} \ell(J \cup \{k\}, V \setminus (J \cup \{i, j, k\})) & \ell(J \cup \{i, k\}, V \setminus (J \cup \{i, j, k\})) & \ell(J \cup \{j, k\}, V \setminus (J \cup \{i, j, k\})) \\ \ell(J \cup \{i, k\}, V \setminus (J \cup \{i, j, k\})) & \rho(i, J \cup \{k\}, V \setminus (J \cup \{i, j, k\})) & \ell(J \cup \{i, j, k\}, V \setminus (J \cup \{i, j, k\})) \\ \ell(J \cup \{j, k\}, V \setminus (J \cup \{i, j, k\})) & \ell(J \cup \{i, j, k\}, V \setminus (J \cup \{i, j, k\})) & \rho(j, J \cup \{k\}, V \setminus (J \cup \{i, j, k\})) \end{bmatrix} \succeq 0, \\ & \forall J \subseteq V \setminus \{i, j, k\} \end{aligned} \quad (82)$$

The two LMI systems (81) and (82) can be equivalently written as one LMI system given by (44). Similarly, substituting (77) and (79) into LMI (58), we get

$$\begin{bmatrix} \ell(J, V \setminus (J \cup \{i\})) & \ell(J \cup \{i\}, V \setminus (J \cup \{i\})) \\ \ell(J \cup \{i\}, V \setminus (J \cup \{i\})) & \rho(i, J, V \setminus (J \cup \{i\})) \end{bmatrix} \succeq 0, \quad \forall J \subseteq V \setminus \{i, k\}, \quad (83)$$

while substituting (78) and (80) into LMI (67), we get

$$\begin{bmatrix} \ell(J \cup \{k\}, V \setminus (J \cup \{i, k\})) & \ell(J \cup \{i, k\}, V \setminus (J \cup \{i, k\})) \\ \ell(J \cup \{i, k\}, V \setminus (J \cup \{i, k\})) & \rho(i, J \cup \{k\}, V \setminus (J \cup \{i, k\})) \end{bmatrix} \succeq 0, \quad \forall J \subseteq V \setminus \{i, k\} \quad (84)$$

The two LMI systems (83) and (84) can be equivalently written as one LMI system given by (45). By symmetry, following an identical line of argument, we deduce that by projecting out the extra variables from LMIs (60) and (68), we obtain LMIs (46). Therefore, system (44)-(47) is an extended formulation for $\text{PP}(G)$. Finally, observe that the extra variables in this extended formulation are z_{ii}^J for all nonempty $J \subseteq V \setminus \{i\}$ and z_{jj}^J for all nonempty $J \subseteq V \setminus \{j\}$. Since G is a complete hypergraph, $V \cup E$ consists of all nonempty subsets of V . Therefore, we have $|V| + |E| - 1$ extra variables. \square

By Theorem 4 and Lemma 2, the following result is immediate:

Corollary 1. *Let $G = (V, E, L)$ be a complete graph with $L = L^+ = \{\{i, i\}, \{j, j\}\}$. Then an extended formulation for $\text{QP}(G)$ is given by inequalities (44)-(47).*

We next show that if the graph G is sparse, then one can obtain a significant generalization of Corollary 1. Consider a graph $G = (V, E, L)$ and as before denote by V_c , $c \in \mathcal{C}$ the connected components of G_{V^+} , where G_{V^+} is the subgraph of G induced by V^+ defined by (7). Henceforth, if $|V_c| \leq 2$ for all $c \in \mathcal{C}$, we say that G does not have any *connected-plus-triplet*. Thanks to Proposition 1 and Theorem 4, we next present a sufficient condition under which $\text{QP}(G)$ is SDP-representable.

Theorem 5. *Consider a graph $G = (V, E, L)$. If G does not have any connected-plus-triplets, then $\text{QP}(G)$ admits a SDP-representable formulation with $O(2^{|V|})$ variables and inequalities.*

Proof. Since G does not have any connected-plus-triplets, we have $|V_c| \leq 2$ for all $c \in \mathcal{C}$. Therefore, by Proposition 1, an extended formulation for $\text{QP}(G)$ is given by putting together formulations for $\text{PP}(G'_c)$, $c \in \mathcal{C} \cup \{0\}$, where for each $c \in \mathcal{C}$, the hypergraph G'_c has either one or two plus loops and the hypergraph G'_0 has no plus loops. Consider some $c \in \mathcal{C}$; if G'_c has one plus loop, then from Lemma 2, Theorem 2 and Observation 2 it follows that $\text{PP}(G'_c)$ is SDP-representable. If G'_c has two plus loops, then from Lemma 2 and Theorem 4 it follows that $\text{PP}(G'_c)$ is SDP-representable. In either case, the extended formulation contains $O(2^{|V \setminus V^+|})$ variables and inequalities. Finally, consider G'_0 ; since G'_0 has no plus loops, $\text{PP}(G'_0)$ coincides with the multilinear polytope $\text{MP}(G'_0)$ and therefore an extended formulation for it is given by Theorem 1 which has $O(2^{|V \setminus V^+|})$ variables and inequalities. We then conclude that $\text{QP}(G)$ is SDP-representable. Moreover, using the fact that $(|V^+| + 1)2^{|V \setminus V^+|} \leq 2^{|V|}$, we infer that this extended formulation has $O(2^{|V|})$ variables and inequalities. \square

It is important to note that the proof of Theorem 5 is constructive. However, the resulting extended formulation may not be of polynomial size. In the next section, we obtain a sufficient condition under which $QP(G)$ admits a polynomial-size SDP-representable formulation that can be constructed in polynomial time.

5 Polynomial-size SDP-representable formulations

In this section, we obtain a sufficient condition under which the extended formulation of Theorem 5 is of polynomial size. Consider a graph $G = (V, E, L)$. Recall that if $L = \emptyset$, then $QP(G)$ coincides with the Boolean quadric polytope $BQP(G)$, a well-known polytope that has been thoroughly studied by the integer programming community [27, 19]. As we discussed in Section 1, a bounded treewidth for G is a necessary and sufficient condition for the existence of polynomial-size linear extended formulations for $BQP(G)$. This suggests that assuming

$$\text{tw}(G) \in O(\log |V|), \quad (85)$$

is reasonable for obtaining a polynomial-size SDP-representable formulation for $QP(G)$ as well. Now consider the hypergraphs G'_c for $c \in \mathcal{C}$, as defined in the proof of Theorem 5. Clearly, Theorem 2 and Theorem 4 provide polynomial-size extended formulations for $PP(G'_c)$, $c \in \mathcal{C}$, only if

$$|V_c| + |N(V_c)| \in O(\log |V|), \quad \forall c \in \mathcal{C}, \quad (86)$$

where $N(V_c)$ is defined by (8). We next prove that conditions (85) and (86) are, in fact, sufficient to guarantee that the extended formulation of Theorem 5 is of polynomial size.

Theorem 6. *Consider a graph $G = (V, E, L)$. Suppose that the following conditions are satisfied:*

- (i) *G does not have any connected-plus-triplets.*
- (ii) *The treewidth of G is bounded; i.e., $\text{tw}(G) \in O(\log |V|)$.*
- (iii) *The degree of each node with a plus loop is bounded; i.e., $\deg(v) \in O(\log |V|)$ for all $v \in V^+$, where V^+ is defined by (7).*

Then $QP(G)$ admits a polynomial-size SDP-representable formulation that can be constructed in polynomial time.

Notice that, given a graph $G = (V, E, L)$, one can check in polynomial time whether it satisfies conditions (i)-(iii) of Theorem 6.

To prove Theorem 6, we need to introduce some tools and terminology. Given a graph $G = (V, E)$, a *tree decomposition* of G is a pair (\mathcal{X}, T) , where $\mathcal{X} = \{X_1, \dots, X_m\}$ is a family of subsets of V , called *bags*, and T is a tree with m nodes, where each node of T corresponds to a bag such that:

1. $V = \bigcup_{i \in [m]} X_i$.
2. For every edge $\{v_j, v_k\} \in E$, there is a bag X_i for some $i \in [m]$ such that $X_i \ni v_j, v_k$.
3. For each node $v \in V$, the set of all bags containing v induces a connected subtree of T .

The *width* $\omega(\mathcal{X})$ of a tree decomposition (\mathcal{X}, T) is the size of its largest bag X_i minus one. The *treewidth* $\text{tw}(G)$ of a graph G is the minimum width among all possible tree decompositions of G . Given a hypergraph $G = (V, E)$, the *intersection graph* of G is the graph with node set V , where two nodes $v, v' \in V$ are adjacent if $v, v' \in e$ for some $e \in E$. We define the tree decomposition and treewidth of a hypergraph $G = (V, E)$, as the tree decomposition and treewidth

of its intersection graph. For a graph (resp. hypergraph) with loops, i.e., $G = (V, E, L)$, we define its tree decomposition and treewidth as the tree decomposition and treewidth of the corresponding loopless graph (resp. hypergraph); i.e., (V, E) .

In [35, 24, 8], the authors proved that if a hypergraph G has a bounded treewidth, then the multilinear polytope $MP(G)$ admits a polynomial-size extended formulation.

Proposition 5. *Let $G = (V, E)$ be a hypergraph with $\text{tw}(G) = \kappa$. Then $MP(G)$ has a linear extended formulation with $O(2^\kappa |V|)$ variables and inequalities. Moreover, if $\kappa \in O(\log \text{poly}(|V|, |E|))$, then $MP(G)$ has a polynomial-size extended formulation that can be constructed in polynomial time.*

To prove Theorem 6, we need to show that the treewidth of the intersection graph of the hypergraph G'_0 defined in the proof of Theorem 5 is not “too large”. The next two lemmata establish this fact. In the following, given a graph $G = (V, E)$ and a node $v \in V$, we say that the graph $G - v := (V', E')$ is obtained from G by *removing* v if $V' = V \setminus \{v\}$ and $E' = \{\{i, j\} \in E : i \neq v, j \neq v\}$.

Lemma 7. *Let $G = (V, E)$ be a graph with $\text{tw}(G) = \kappa$. Let $C \subseteq V$ be such that the induced subgraph G_C of G is a connected graph. Denote by H a graph obtained from G by (i) adding all edges between nodes of $N(C)$, where $N(C)$ is defined by (8), and (ii) removing all nodes in C . Then the treewidth of H is upper bounded by*

$$\text{tw}(H) \leq \max(\kappa, |N(C)| - 1). \quad (87)$$

Proof. If $|N(C)| \leq 1$, then the statement holds trivially, because H is a subgraph of G . Henceforth, let $|N(C)| \geq 2$. Consider a tree-decomposition (\mathcal{X}, T) of G with width $\omega(\mathcal{X}) = k$. For each $\bar{v} \in V$, denote by $W_{\bar{v}}$ the set of nodes in T whose corresponding bags contain \bar{v} . Note that by Property 3 of a tree decomposition, $W_{\bar{v}}$ induces a connected subtree of T , which we denote by $T_{\bar{v}}$. Define $W_C := \cup_{v \in C} W_v$. Since for each $v \in C$, T_v is a connected subtree and G_C is a connected graph, from property 2 of a tree decomposition, we deduce that W_C induces a connected subtree of T , which we denote by T_C . Let $t_0 \in W_C$. Denote by U the node set of the tree T . Define a new tree \bar{T} obtained from T by adding a new leaf \bar{t} to the parent node t_0 . Define $\bar{\mathcal{X}} := \{\bar{X}_t, t \in U\} \cup \{\bar{X}_{\bar{t}}\}$, where:

$$\bar{X}_t := \begin{cases} X_t \setminus C, & t \in U, \\ N(C), & t = \bar{t}. \end{cases} \quad (88)$$

We claim that $(\bar{T}, \bar{\mathcal{X}})$ is a valid tree decomposition for H . To see this, we show that $(\bar{T}, \bar{\mathcal{X}})$ satisfies the three properties of a tree decomposition:

1. For any $v \in V \setminus C$, we have $v \in X_t$ for some $t \in U$, and by (88), we have $v \in \bar{X}_t$. Hence, all nodes of H are present in some bag of $\bar{\mathcal{X}}$.
2. Consider an edge $\{u, v\}$ of H . There are two cases. In the first case, we have $\{u, v\} \in E$ implying that $u, v \notin C$. In this case, we have $\{u, v\} \in X_t$ for some $t \in U$. From (88), it follows that $\{u, v\} \in \bar{X}_t$. In the second case, we have $\{u, v\} \notin E$ implying that $u, v \in N(C)$, in which case by (88) we have $\{u, v\} \in \bar{X}_{\bar{t}}$. Therefore, all edges of H are in some bag in $\bar{\mathcal{X}}$.
3. Consider some $v \in V \setminus C$. Recall that W_v denotes the set of nodes of T whose bags contain v . By property 3 of a tree decomposition, the subtree of T induced by W_v is a connected subtree. Two cases arise:
 - if $v \notin N(C)$, then the set of nodes of \bar{T} whose bags contain v coincides with W_v . Therefore, property 3 holds.

- if $v \in N(C)$, then the set of nodes of \bar{T} whose bags contain v is $W_v \cup \{\bar{t}\}$. Since $\{v, v'\} \in E$ for some $v' \in C$, there exists $t' \in U$ such that $v, v' \in X_{t'}$. Therefore $t' \in W_v \cap W_{v'} \subseteq W_v \cap W_C$. Because W_C induces a connected subtree containing t_0 , the unique path in T from t' to t_0 lies in the subtree induced by W_C . Attaching \bar{t} to t_0 , yields a path from t' to \bar{t} in \bar{T} , implying that $W_v \cup \{\bar{t}\}$ induces a connected subtree of \bar{T} . Therefore, property 3 holds.

Therefore, $(\bar{T}, \bar{\mathcal{X}})$ is a valid tree decomposition for H . By (88), we have $|\bar{X}_t| \leq |X_t|$ for all $t \in U$ and $|\bar{X}_{\bar{t}}| = |N(C)|$, implying the validity of (87). \square

Lemma 8. *Consider a graph $G = (V, E)$ and let $S \subseteq V$ be nonempty. Denote by G_S the subgraph of G induced by S . Denote by V_1, \dots, V_p the connected components of G_S , for some $p \geq 1$. Define $G_0 := G$. For each $i \in [p]$, let G_i be the graph obtained from G_{i-1} by removing the nodes in V_i and adding edges between the pairs of nodes in $G_{i-1} - V_i$ that are both adjacent to some node in V_i in G_{i-1} . Define*

$$N(V_i) := \{u \in V \setminus V_i : \exists v \in V_i \text{ with } \{u, v\} \in E\}, \quad \forall i \in [p],$$

and

$$d_{\max} := \max_{i \in [p]} |N(V_i)|.$$

If $\text{tw}(G) = \kappa$, then

$$\text{tw}(G_p) \leq \max(\kappa, d_{\max} - 1),$$

where G_p is the graph obtained from G after p component removal and clique addition operations.

Proof. We prove by induction on the number p of component removal and clique addition operations. In the base case, we have $p = 1$, and the proof follows immediately from Lemma 7. Next, by the induction hypothesis, suppose that after $p = k$ component removal and clique addition operations, we obtain a graph G_k with $\text{tw}(G_k) \leq \max(\kappa, d' - 1)$, where $d' = \max_{i \in [k]} |N(V_i)|$. Let $G_k = (V', E')$. Consider the connected component V_{k+1} . Define

$$N'(V_{k+1}) := \{u \in V' \setminus V_{k+1} : \exists v \in V_{k+1} \text{ with } \{u, v\} \in E'\}.$$

Since $V_{k+1} \cap N(V_i) = \emptyset$ for all $i \in [k]$ and no two nodes in V_i and V_j are adjacent for any $i \neq j \in [k+1]$, we deduce that

$$N'(V_{k+1}) = N(V_{k+1}).$$

Therefore, letting $C = V_{k+1}$ in Lemma 7, we deduce that

$$\text{tw}(G_{k+1}) \leq \max(\text{tw}(G_k), |N(V_{k+1})|) = \max(\kappa, d'' - 1),$$

where $d'' = \max_{i \in [k+1]} |N(V_i)|$, and this completes the proof. \square

We are now ready to prove Theorem 6.

Proof of Theorem 6. Since by property (i) the graph G does not have any connected-plus-triplets, we have $|V_c| \leq 2$ for all $c \in \mathcal{C}$. Therefore, by Proposition 1, an extended formulation for $\text{QP}(G)$ is given by putting together formulations for $\text{PP}(G'_c)$, $c \in \mathcal{C} \cup \{0\}$, where for each $c \in \mathcal{C}$, the hypergraph G'_c has one or two plus loops and the hypergraph G'_0 has no plus loops. Consider some $c \in \mathcal{C}$. From the proof of Proposition 1 it follows that the hypergraph G'_c has $|V_c| + |N(V_c)|$ nodes, where $N(V_c)$ is defined by (8). Moreover, $|N(V_c)| \leq \sum_{v \in V_c} \deg(v)$, where $\deg(v)$ denotes the degree of node v . Since $|V_c| \leq 2$ for all $c \in \mathcal{C}$, from Theorem 2 and Theorem 4 it follows that an extended formulation for $\text{PP}(G'_c)$ for any $c \in \mathcal{C}$ contains $O(2^{\deg(v_1)})$ variables and inequalities if $V_c = \{v_1\}$

and $O(2^{\deg(v_1)+\deg(v_2)})$ variables and inequalities if $V_c = \{v_1, v_2\}$. Since by property (iii), $\deg(v) \in O(\log |V|)$ for any $v \in V^+$, it follows that $\text{PP}(G'_c)$ admits a polynomial-size SDP-representable formulation for any $c \in \mathcal{C}$. Next, consider the hypergraph G_0 and denote by H the intersection graph of G_0 . Moreover, denote by G' the intersection graph of the original hypergraph G . It then follows that H can be obtained from G' by performing $|\mathcal{C}|$ component removals and clique additions, where these operations are defined in the statement of Lemma 7. Therefore, from Lemma 8 and properties (i)-(iii) it follows that $\text{tw}(G_0) = \text{tw}(H) \in O(\log |V|)$. Therefore, by Proposition 5, the polytope $\text{PP}(G_0)$ admits a polynomial-size linear extended formulation. Finally, note that properties (i)-(iii) can be checked in polynomial time and from the proof of Theorem 5 it follows that the above extended formulation can be constructed in polynomial time as well. Therefore, $\text{QP}(G)$ has a polynomial-size SDP-representable formulation that can be constructed in polynomial time. \square

In [18], the authors present a sufficient condition under which $\text{QP}(G)$ admits a polynomial-size SOC-representable formulation using a different proof technique. Their sufficient condition states that the graph G admits a tree decomposition (\mathcal{X}, T) , with $\mathcal{X} = \{X_1, \dots, X_m\}$ satisfying the following properties:

- (I) for each bag $X_j \in \mathcal{X}$, there exists at most one node $v \in X_j$ such that $v \in V^+$.
- (II) the width of the tree decomposition is bounded; i.e., $\omega(\mathcal{X}) \in O(\log |V|)$,
- (III) for each node $v \in V^+$, the *spread* of node v is bounded; i.e.,

$$s_v(\mathcal{X}) := \sum_{i \in [m]: X_i \ni v} (|X_i| - 1) \in O(\log |V|).$$

By property 2 of a tree decomposition, condition (I) implies that V^+ is a stable set of G . Again, from property 2 of a tree decomposition it follows that for any tree decomposition (\mathcal{X}, T) of G , we have:

$$s_v(\mathcal{X}) \geq \deg(v), \quad \forall v \in V.$$

Moreover, the authors of [18] leave open the question of whether a tree decomposition of G satisfying conditions (I)-(III) can be constructed in polynomial time. Thanks to Theorem 6, we obtain a stronger result on the SOC-representability of $\text{QP}(G)$:

Corollary 2. *Consider a graph $G = (V, E, L)$. Suppose that the following conditions are satisfied:*

- (i) V^+ is a stable set of G .
- (ii) The treewidth of G is bounded; i.e., $\text{tw}(G) \in O(\log |V|)$.
- (iii) The degree of each node with a plus loop is bounded; i.e., $\deg(v) \in O(\log |V|)$ for all $v \in V^+$, where V^+ is defined by (7).

Then $\text{QP}(G)$ admits a polynomial-size SOC-representable formulation that can be constructed in polynomial time.

Notice that, given a graph $G = (V, E, L)$, one can check in polynomial time whether it satisfies conditions (i)-(iii) of Corollary 2.

Recall that a graph is called *series-parallel* if and only if it does not contain a K_4 minor, where K_4 denotes a complete graph with four nodes. Series-parallel graphs subsume several families of graphs such as trees and forests, cycles and cactus graphs, among others. The treewidth of a series-parallel graph is at most two. Thanks to Theorem 6, we obtain the following result for $\text{QP}(G)$, where G is a series-parallel graph.

Corollary 3. *Let $G = (V, E, L)$ be a series-parallel graph. Suppose that G does not have any connected-plus-triplets and $\deg(v) \in O(\log |V|)$ for all $v \in V^+$, where V^+ is defined by (7). Then $QP(G)$ admits a polynomial-size SDP-representable formulation that can be constructed in polynomial time.*

We conclude this section by discussing the significance of the assumptions in Theorem 6. As noted above, assumption (i) is likely necessary. In contrast, assumptions (ii) and (iii) appear to be artifacts of our proof techniques. Relaxing assumption (ii) would require extending Theorem 4 from the case of a complete hypergraph with two plus loops to a complete hypergraph with a larger number of plus loops. Assumption (iii) is currently needed to control the size of extended formulations for both $PP(G'_c)$, $c \in \mathcal{C}$ and $PP(G'_0)$. Removing this assumption would require new tools and different proof techniques. We believe that the most natural direction to strengthen Theorem 6 is to relax assumption (ii); specifically, replacing condition $|V_c| \leq 2$ for all $c \in \mathcal{C}$ by the weaker requirement $|V_c| \leq k$ for some fixed k for all $c \in \mathcal{C}$. Finally, performing an extensive computational study with the proposed SDP relaxations for sparse box-constrained quadratic programs is a topic of future research.

6 Connections with SDP hierarchies

The moment/Sums of Squares (SOS) hierarchy provides a sequence of SDP relaxations for polynomial optimization problems, based on the duality between truncated moment sequences and SOS-based positivity certificates [22, 28]. For polynomial optimization over a compact basic semialgebraic set satisfying an Archimedean condition, the sequence of SOS relaxations converges *asymptotically* to the global optimum [21]. In particular, when minimizing a polynomial over the unit hypercube, the Archimedean property holds automatically, ensuring asymptotic convergence of the hierarchy [21]. Finite convergence at some fixed relaxation level is not guaranteed in general, but a result of Nie [26] shows that it holds under certain conditions at the global minimizers, and therefore finite convergence is achieved *generically* (that is, finite convergence holds in the entire space of input data except on a set of Lebesgue measure zero).

In the case of binary polynomial optimization, it is well-known that each level of RLT is implied by the same level of Lasserre relaxation [23]. This implication relies crucially on the identity $x_i^2 = x_i$ for $x_i \in \{0, 1\}$, which collapses higher-degree monomials and ensures that moment constraints encode all RLT inequalities. In contrast, for polynomial optimization over the unit hypercube, a given level of Lasserre hierarchy does not necessarily dominate the same level of RLT hierarchy. Since our proposed SDP relaxation (17) with $r := |P| + |M|$ implies level- r RLT, we deduce that this relaxation is not implied by level- r Lasserre relaxation either. However, as we describe next, our SDP relaxations can be obtained as a special case of a more general hierarchy, often referred to as the Schmüdgen hierarchy [30] or the Lasserre hierarchy with preordering in the literature.

We start by introducing some notation and terminology. We write $\mathbb{R}[x]$ for the ring of real polynomials in the x variables. A polynomial $\sigma \in \mathbb{R}[x]$ is called a SOS polynomial if it can be written as

$$\sigma(x) = \sum_{j=1}^m q_j(x)^2$$

for some polynomials $q_j \in \mathbb{R}[x]$. We denote by $\Sigma[x]$ the cone of all SOS polynomials in $\mathbb{R}[x]$. The

preordering with *square-free factors* associated with the unit hypercube is defined as:

$$\mathcal{T} := \left\{ \sum_{\substack{J_1, J_2 \subseteq [n]: \\ J_1 \cap J_2 = \emptyset}} \sigma_{J_1, J_2}(x) f(J_1, J_2) : \sigma_{J_1, J_2} \in \Sigma[x], \forall J_1, J_2 \subseteq [n] : J_1 \cap J_2 = \emptyset \right\},$$

where $f(J_1, J_2)$ is the polynomial factor as defined by (9). Since the unit hypercube is a compact set, Schmüdgen's Positivstellensatz [30] implies that every polynomial that is strictly positive on this set belongs to \mathcal{T} . For a polynomial $p \in \mathbb{R}[x]$, let $\deg(p)$ denote its degree. Fix an integer $d \geq 1$. The *truncated preordering* at level d is defined as

$$\mathcal{T}_d = \left\{ \sum_{\substack{J_1, J_2 \subseteq [n]: \\ J_1 \cap J_2 = \emptyset}} \sigma_{J_1, J_2}(x) f(J_1, J_2) : \sigma_{J_1, J_2} \in \Sigma[x], \deg(\sigma_{J_1, J_2} f(J_1, J_2)) \leq d \right\}.$$

If each multiplier σ_{J_1, J_2} is restricted to be a nonnegative constant, then \mathcal{T}_d yields exactly all RLT inequalities up to level- d . Now, consider the problem of minimizing a polynomial function $g(x)$ over the unit hypercube. The associated Schmüdgen-type lower bound is given by:

$$\max\{\gamma \in \mathbb{R} : g(x) - \gamma \in \mathcal{T}_d\}.$$

The moment/SOS duality framework introduced by Lasserre [22] shows that the above problem can be equivalently expressed as an SDP in terms of truncated moment sequences. Because the unit hypercube is compact, the bounds produced by this hierarchy converge monotonically to the global minimum of $g(x)$ over $[0, 1]^n$ as $d \rightarrow \infty$ [22].

We next show that for a quadratic objective function $g(x)$, our proposed SDP relaxations can be considered as a special case of Schmüdgen hierarchy. Of course, the significance of our relaxations is that under certain assumptions they are tight (see Theorem 5), while no such a guarantee is available for the existing SDP hierarchies. Let $G = (V, E, L)$ be a graph with V^+ as defined by (7). By Proposition 1, without loss of generality we assume that the subgraph G_{V^+} of G induced by V^+ is connected and that every node in $V \setminus V^+$ is connected to some node in V^+ . For a polynomial $p \in \mathbb{R}[x]$, let $\text{supp}(p) \subseteq V$ denote the index set of variables on which p depends nontrivially. We then consider the following restricted version of the Schmüdgen hierarchy:

- Bounded SOS degree: each multiplier $\sigma_{J_1, J_2} \in \Sigma[x]$ is a sum of squares of affine polynomials; i.e.,

$$\deg(\sigma_{J_1, J_2}) \leq 2. \quad (89)$$

The intuition behind this restrictive assumption is that the objective function $g(x)$ is quadratic.

- Disjoint support: for each $J_1, J_2 \subseteq V$ with $J_1 \cap J_2 = \emptyset$, the variables on which σ_{J_1, J_2} depends correspond to some nodes of G with plus loops and are disjoint from the variables in the product factors; i.e.,

$$\text{supp}(\sigma_{J_1, J_2}) \subseteq V^+ \setminus (J_1 \cup J_2). \quad (90)$$

This assumption in particular implies that if $J_1 \cup J_2 \supseteq V^+$, then σ_{J_1, J_2} is a nonnegative constant.

Using assumptions (89) and (90), for each $d \in \{1, \dots, |V|\}$, we define the cone:

$$\mathcal{T}_d^{\text{disj}} = \left\{ \sum_{\substack{J_1, J_2 \subseteq V \\ J_1 \cap J_2 = \emptyset}} \sigma_{J_1, J_2}(x) f(J_1, J_2) : \sigma_{J_1, J_2} \in \Sigma[x], \deg(\sigma_{J_1, J_2}) \leq 2, \text{supp}(\sigma_{J_1, J_2}) \subseteq V^+ \setminus (J_1 \cup J_2), \right\}$$

$$\deg(\sigma_{J_1, J_2} f(J_1, J_2)) \leq d \Bigg\}. \quad (91)$$

We then obtain the following lower bound on the minimum value of a quadratic function $g(x)$ over the unit hypercube:

$$\max\{\gamma \in \mathbb{R} : g(x) - \gamma \in \mathcal{T}_d^{\text{disj}}\}. \quad (92)$$

We now obtain the dual of Problem (92) and relate it to our SDP relaxations. For notational simplicity, define $T_{J_1, J_2} := \text{supp}(\sigma_{J_1, J_2})$. Let $v_{T_{J_1, J_2}}(x) = (1, x_i : i \in T_{J_1, J_2})^\top$. Let $y = (y_\alpha)_{|\alpha| \leq d}$ be a truncated moment sequence of order d with $y_0 = 1$. For any polynomial p with monomial expansion $p(x) = \sum_{|\alpha| \leq d} p_\alpha x^\alpha$, define the associated linear functional

$$L_y(p) := \sum_{|\alpha| \leq d} p_\alpha y_\alpha.$$

Since $\deg(\sigma_{J_1, J_2}) \leq 2$, each multiplier admits a Gram representation:

$$\sigma_{J_1, J_2}(x) = v_{T_{J_1, J_2}}(x)^\top Q_{J_1, J_2, T_{J_1, J_2}} v_{T_{J_1, J_2}}(x),$$

where $Q_{J_1, J_2, T_{J_1, J_2}} \succeq 0$ and is of size $(1 + |T_{J_1, J_2}|) \times (1 + |T_{J_1, J_2}|)$. Applying L_y to the constraint set of Problem (92) yields

$$L_y(g) - \gamma = \sum_{J_1, J_2} \text{tr} \left(Q_{J_1, J_2, T_{J_1, J_2}} L_y \left(f(J_1, J_2) v_{T_{J_1, J_2}} v_{T_{J_1, J_2}}^\top \right) \right).$$

In the following, we refer to any triple (J_1, J_2, T_{J_1, J_2}) satisfying $J_1, J_2 \subseteq V$, $J_1 \cap J_2 = \emptyset$, $T_{J_1, J_2} \subseteq V^+ \setminus (J_1 \cup J_2)$, and $|J_1| + |J_2| + |T_{J_1, J_2}| \leq d$, as an *admissible* triple. For any admissible (J_1, J_2, T_{J_1, J_2}) , define the *localizing moment matrix* as

$$M_{T_{J_1, J_2}}^{J_1, J_2}(y) := L_y(f(J_1, J_2) v_{T_{J_1, J_2}} v_{T_{J_1, J_2}}^\top).$$

From the definitions (11) and (16), it follows that:

$$\ell(J_1, J_2) = L_y(f(J_1, J_2)), \quad \rho(i, J_1, J_2) = L_y(x_i^2 f(J_1, J_2)).$$

Therefore, letting $T_{J_1, J_2} = \{i_1, \dots, i_p\}$, we deduce that the localizing moment matrix $M_{T_{J_1, J_2}}^{J_1, J_2}(y)$ can be equivalently written as:

$$M_{T_{J_1, J_2}}^{J_1, J_2}(y) = \begin{pmatrix} \ell(J_1, J_2) & \ell(J_1 \cup \{i_1\}, J_2) & \dots & \ell(J_1 \cup \{i_p\}, J_2) \\ \ell(J_1 \cup \{i_1\}, J_2) & \rho(i_1, J_1, J_2) & \dots & \ell(J_1 \cup \{i_1, i_p\}, J_2) \\ \vdots & \vdots & \ddots & \vdots \\ \ell(J_1 \cup \{i_p\}, J_2) & \ell(J_1 \cup \{i_1, i_p\}, J_2) & \dots & \rho(i_p, J_1, J_2) \end{pmatrix}.$$

By self duality of the positive semidefinite cone, dualizing the constraints $Q_{J_1, J_2, T_{J_1, J_2}} \succeq 0$ we obtain $M_{T_{J_1, J_2}}^{J_1, J_2}(y) \succeq 0$ for all admissible (J_1, J_2, T_{J_1, J_2}) . It then follows that the dual of Problem (92) is the following SDP:

$$\begin{aligned} & \min_{y=(y_\alpha)_{|\alpha| \leq d}} L_y(g) \\ & \text{s.t.} \quad M_{T_{J_1, J_2}}^{J_1, J_2}(y) \succeq 0 \quad \text{for all admissible } (J_1, J_2, T_{J_1, J_2}). \end{aligned} \quad (93)$$

Finally, letting $R := T_{J_1, J_2}$, $J = J_1$, and $M_R = J_1 \cup J_2$, and invoking Proposition 2, we deduce the nonredundant LMIs defining the feasible set of Problem (93) coincide with the system of LMIs defined by (17). From Theorem 4, it then follows that for a quadratic objective function g with $|V^+| = 2$, if we let $d = |V|$, then the lower bound given by Problem (93) is sharp. We leave it as an open question whether a similar result can be obtained for $|V^+| > 2$.

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