Characterization of Knapsack Polytopes using Minimal Cover Inequalities

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Abstract

In this paper, we compare the strength of alternate formulations (polyhedra) of the binary knapsack set. We introduce a specific class of knapsack sets for which we prove that the polyhedra based on their minimal cover inequalities (together with the bounds on the variables) are strictly contained inside the polyhedra defined by their continuous knapsack relaxations. Furthermore, we answer an open question in the literature by establishing that for the knapsack sets belonging to this class, the formulations based on minimal covers provide their complete convex hull. Finally, we prove that the convex hull of a knapsack set violating some of the conditions required to define the above specific class can never be completely described just by its minimal cover inequalities.

Keywords: Knapsack, Polyhedron, Minimal cover inequality, Convex hull

MSC Classification: 90C10

1 Introduction

Consider a polyhedron P defined as:

$$P = \left\{ \mathbf{x} \in \mathbb{R}^{|N|} \mid \mathbf{0} \leq \mathbf{x} \leq \mathbf{1}, \sum_{i \in N} a_i x_i \leq b,\ a_i > 0\ \forall i \in N,\ b > 0 \right\}. \tag{1}$$

P is a formulation of the binary knapsack set X if $X = P \cap \{0,1\}^{|N|}$. A set $C \subseteq N$ is called a *cover* of X if $\sum_{i \in C} a_i > b$. Given a cover C, the following inequality, called the *cover inequality*, is valid for conv(X):

$$\sum_{i \in C} x_i \leq |C| - 1$$

A cover M is *minimal* if none of its proper subset is a cover. Let $\mathcal{M}$ be the collection of all minimal covers of X, and the polyhedron $P^{\mathcal{M}}$ defined as follows:

$$P^{\mathcal{M}} = \left\{ \mathbf{x} \in \mathbb{R}^{|N|} \mid \mathbf{0} \leq \mathbf{x} \leq \mathbf{1}, \sum_{i \in M} x_i \leq |M| - 1, \quad \forall M \in \mathcal{M} \right\}. \tag{2}$$

Then, $P^{\mathcal{M}}$ is a formulation of $X^{\mathcal{M}}$, where $X^{\mathcal{M}} = P^{\mathcal{M}} \cap \{0,1\}^{|N|}$.

It is well known that $X = X^{\mathcal{M}}$ [1]. Thus, P and $P^{\mathcal{M}}$ are alternate formulations of X. In general, it is unknown which of the following is true for a given X: (i) $P^{\mathcal{M}} \subset P$; (ii) $P^{\mathcal{M}} \supset P$; (iii) P and $P^{\mathcal{M}}$ are incomparable. If, however, one can show $P^{\mathcal{M}} \subset P$, then $P^{\mathcal{M}}$ is a better formulation (for linear programming relaxation based algorithms for solving an optimization problem over X). In this paper, we show that for a class of knapsack sets X, it is indeed the case that $P^{\mathcal{M}} \subset P$. Thus, we are able to prove that for such knapsack sets X, (2) is a better formulation than (1). We further provide an answer to the following open question [2]: is there a class of knapsack sets X for which formulation (2) defines the complete convex hull of X (referred to as the knapsack polytope), i.e., $P^{\mathcal{M}} = \text{conv}(X)$?

In Section 2, we identify a class of knapsack sets X for which we prove that $P^{\mathcal{M}} \subset P$. We further prove, in Section 3, that for this class of identified knapsack sets, $P^{\mathcal{M}} = \text{conv}(X)$. In Section 4, we prove that $P^{\mathcal{M}} \neq \text{conv}(X)$ for a given X if some of the conditions required to define the above specific class are not satisfied, i.e., those conditions are necessary.

2 P versus $P^{\mathcal{M}}$

Consider a set $X = P \cap \{0,1\}^{|N|}$, where P is given by (1). Further, assume, without loss of generality, $a_1 \geq a_2 \geq \cdots \geq a_{|N|} > 0$. We also assume that $b \geq a_1$ and $\sum_{i \in N} a_i > b$ to ensure X is full-dimensional [2]. For an $h \in \{1, \ldots, |N|-1\}$, define

$$H := \{1, \ldots, h\}, \quad L := \{h+1, \ldots, |N|\}, \quad S_{\bar{N}} := \sum_{i \in \bar{N}} a_i \; \forall \bar{N} \subseteq N, \quad \Delta := b - S_H.$$

Definition 1 A knapsack set X belongs to the class $\mathcal{K}_S$ if there exists an h for which the following conditions hold:

(1) $S_H \leq b$;

(2) $S_H + a_{h+1} > b$;

(3) $(S_H - a_h) + S_L \leq b$;

(4) For each $j \in L$, there exists exactly one $L_j \subseteq L$ such that: (a) $\sum_{i \in L_j} a_i > \Delta$ and $\sum_{i \in L_j \setminus \{j\}} a_i \leq \Delta$; or (b) $L_j = \emptyset$.

Proposition 1 *For any* $X \in \mathcal{K}_S$, $H \subset M \; \forall M \in \mathcal{M}$.

Proof Let M be a minimal cover. Further, let $k \in H$ and $k \notin M$. Then, $\sum_{i \in M} a_i \leq (S_H - a_k) + S_L \leq b$ (using condition (3) in Definition 1). Hence, M cannot be a cover. This proves $H \subseteq M$. Again, by Condition (1) in Definition 1, $S_H \leq b$. Hence, $H \subset M$. ■

Remark 1 As a result of Proposition 1, every minimal cover of X is of the form $H \cup L'$ for some nonempty $L' \subseteq L$.

Proposition 2 *For any* $X \in \mathcal{K}_S, j \in L \cap M \implies j \notin M' \; \forall M, M' \in \mathcal{M}$.

Proof By Proposition 1, every minimal cover has the form $H \cup L'$ for some nonempty $L' \subseteq L$ such that $\sum_{i \in L'} a_i > \Delta$. Condition (4) in Definition 1 states that, for each $j \in L$, there exists at most one such minimal set containing j. Therefore, $\nexists \; j \in L$ which belongs to two distinct minimal covers. ■

We now demonstrate that for the special class $\mathcal{K}_S$, its formulation (2) is strictly tighter than formulation (1) by proving that for $X \in \mathcal{K}_S$: (i)$P_{\mathcal{M}} \subseteq P$; and (ii) the fractional extreme points of P does not belong to $P_{\mathcal{M}}$.

Theorem 1 $P_{\mathcal{M}} \subseteq P$ *for any* $X \in \mathcal{K}_S$.

Proof To prove Theorem 1, we show that there exists a weight u_r for each minimal cover inequality $\sum_{i \in M_r} x_i \leq |M_r| - 1$ for $r \in \{1, \ldots, |\mathcal{M}|\}$, and v_i for each trivial upper bound $x_i \leq 1$ such that the following sets of inequalities hold [1]:

(1) $u_r \geq 0$ for all $r \in \{1, \ldots, |\mathcal{M}|\}$; $v_i \geq 0$ for all $i \in N$.

(2) $\sum_{r: i \in M_r} u_r + v_i \geq a_i$ for all $i \in N$.

(3) $\sum_{r=1}^{|\mathcal{M}|} (|M_r| - 1) u_r + \sum_{i \in N} v_i \leq b$.

By Condition (4) of class $\mathcal{K}_S$, each item $j \in L$ belongs to at most one minimal cover. Therefore, the minimal covers in $\mathcal{M}$ take the form $M_r = H \cup L^r$ for $r \in \{1, \ldots, |\mathcal{M}|\}$, where $L^1, \ldots, L^{|\mathcal{M}|}$ are strictly disjoint subsets of L. For this, let us consider $u_r \in \mathbb{R}$ for $r \in \{1, \ldots, |\mathcal{M}|\}$ satisfying the following conditions:

(i) $u_r \geq 0$,

(ii) $u_r \leq \min_{j \in L^r} \{a_j\}$,

(iii) $\sum_{r=1}^{|\mathcal{M}|} u_r = S_N - b$.

Using these chosen values for u_r, we fix v_i as follows:

(iv) For $i \in H$: $v_i = a_i - (S_N - b)$,

(v) For $j \in L^r$: $v_j = a_j - u_r$,

(vi) For $j \in L \setminus \bigcup_{r=1}^{|\mathcal{M}|} L^r$: $v_j = a_j$.

We now show any combination of $(\mathbf{u}, \mathbf{v})$ satisfying the conditions (i)-(vi) constitutes a feasible solution to (1)-(3) with the help of Claims 1.1 to 1.4.

Claim 1.1 *The system (i)-(iii), which defines* $\mathbf{u}$*, admits a feasible solution.*

Proof Feasibility requires $0 \le S_N - b \le \sum_{r=1}^{|\mathcal{M}|} \min_{j \in L^r} a_j$. The lower bound holds trivially since $S_N > b$ by assumption.

Let $j_r = \arg\min_{j \in L^r} a_j$ and define $L' = \{j_1, \ldots, j_{|\mathcal{M}|}\}$. We must show $\sum_{j \in L'} a_j \ge S_N - b = S_L - \Delta$. Hence, $\sum_{j \in L \setminus L'} a_j \le \Delta$.

Suppose for contradiction that $\sum_{j \in L \setminus L'} a_j > \Delta$. This implies that $H \cup (L \setminus L')$ is a cover for X.

By Condition (4), $\exists\, r^* \in \{1, \ldots, |\mathcal{M}|\}$ such that $L^{r^*} \subseteq L \setminus L'$ and $H \cup L^{r^*}$ is a minimal cover of X.

However, by construction, $j_{r^*} \in L^{r^*} \cap L'$, meaning $L^{r^*} \not\subseteq L \setminus L'$. This is a direct contradiction. Thus, $\sum_{j \in L \setminus L'} a_j \le \Delta$ must hold, guaranteeing a feasible solution for $\mathbf{u}$. ■

Claim 1.2 $\mathbf{u} \ge 0$, $\mathbf{v} \ge 0$.

Proof The cover multipliers $u_r \ge 0$ by (i) in construction (i)-(iii).

For $j \in L^r$, $v_j \ge 0$ because $u_r \le \min_{j \in L^r} a_j \le a_j$. For $j \in L \setminus \bigcup_{r=1}^{|\mathcal{M}|} L^r$, $v_j = a_j \ge 0$. For $i \in H$, Condition (3) states that $(S_H - a_h) + S_L \le b$. Because $a_i \ge a_h \ \forall\, i \in H$, it follows that $S_N - b \le a_i$. Therefore, $v_i = a_i - (S_N - b) \ge 0$. All variables are strictly non-negative. ■

Claim 1.3 $\sum_{r: i \in M_r} u_r + v_i = a_i$ *for all* $i \in N$.

Proof We evaluate this over the three partitions of items in N:

Case 1.1 If $i \in H$: The item is present in every minimal cover. Thus, $\sum_{r: i \in M_r} u_r = \sum_{r=1}^{|\mathcal{M}|} u_r = S_N - b$. Hence, $\sum_{r: i \in M_r} u_r + v_i = (S_N - b) + [a_i - (S_N - b)] = a_i$.

Case 1.2 If $j \in L^r$: The item appears in exactly one cover, M_r. Hence, $\sum_{r: j \in M_r} u_r + v_j = u_r + (a_j - u_r) = a_j$.

Case 1.3 If $j \in L \setminus \bigcup_{r=1}^{|\mathcal{M}|} L^r$: The item appears in none of the minimal covers. So, $\sum_{r: j \in M_r} u_r + v_j = 0 + a_j = a_j$.

■

Claim 1.4 $\sum_{r=1}^{|\mathcal{M}|}(|M_r|-1)u_r + \sum_{i\in N} v_i = b$.

Proof

$$\begin{aligned}\sum_{i\in N} v_i &= \sum_{i\in H}[a_i - (S_N - b)] + \sum_{r=1}^{|\mathcal{M}|}\sum_{j\in L^r}(a_j - u_r) + \sum_{j\in L\setminus\bigcup L^r} a_j \\ &= (S_H - |H|(S_N - b)) + \left(S_L - \sum_{r=1}^{|\mathcal{M}|}|L^r|u_r\right) \\ &= S_N - |H|(S_N - b) - \sum_{r=1}^{|\mathcal{M}|}|L^r|u_r\end{aligned} \tag{3}$$

As, $M_r = H \cup L^r$, the size of each cover is $|M_r| = |H| + |L^r|$. Hence,

$$\begin{aligned}\sum_{r=1}^{|\mathcal{M}|}(|M_r|-1)u_r &= \sum_{r=1}^{|\mathcal{M}|}(|H|+|L^r|-1)u_r \\ &= (|H|-1)\sum_{r=1}^{|\mathcal{M}|}u_r + \sum_{r=1}^{|\mathcal{M}|}|L^r|u_r \\ &= (|H|-1)(S_N - b) + \sum_{r=1}^{|\mathcal{M}|}|L^r|u_r\end{aligned} \tag{4}$$

Adding (3) and (4), we get:

$$\sum_{r=1}^{|\mathcal{M}|}(|M_r|-1)u_r + \sum_{i\in N} v_i = (|H|-1)(S_N - b) - |H|(S_N - b) + S_N = b$$

■

Claims 1.1 to 1.4 together prove that there exists a feasible solution to 1-3. This proves $P_{\mathcal{M}} \subseteq P$ if $X \in \mathcal{K}_S$. ■

So far, we have established that $P_{\mathcal{M}} \subseteq P$. To prove the strict containment, we must also show the other direction: $P \not\subseteq P_{\mathcal{M}}$. We will show this by proving that a strictly fractional extreme point of P violates at least one minimal cover inequality.

Theorem 2 *If $X \in \mathcal{K}_S$ and $\bar{\mathbf{x}}$ be any strictly fractional extreme point of P. Then $\bar{\mathbf{x}} \notin P_{\mathcal{M}}$.*

Proof Any fractional extreme point of P contains exactly one strictly fractional variable. Let, $\bar{x}$ be such a fractional extreme point and $\tilde{k} \in N$ be this unique item such that $\bar{x}_{\tilde{k}} \in (0,1)$, while $\bar{x}_i \in \{0,1\}$ for all $i \in N \setminus \{\tilde{k}\}$.

Since $\bar{\mathbf{x}}$ is a fractional extreme point, it must satisfy the knapsack capacity constraint at equality, i.e., $\sum_{i\in N} a_i\bar{x}_i = b$. We partition the integer variables into $N_1 = \{i \in N \setminus \{\tilde{k}\} \mid$

$\bar{x}_i = 1\}$ and $N_0 = \{i \in N \setminus \{\tilde{k}\} \mid \bar{x}_i = 0\}$. We analyze the configuration of $\bar{\mathbf{x}}$ based on the fractional item $\tilde{k}$ in three mutually exhaustive Cases 2.1 to 2.3.

Case 2.1 $\tilde{k} \in H$.
Since $\bar{\mathbf{x}}$ is an extreme point of P, the knapsack constraint must be tight, meaning $\sum_{i \in N_1} a_i + a_{\tilde{k}} \bar{x}_{\tilde{k}} = b$. Since $\bar{x}_{\tilde{k}} > 0$, it follows that $M' = N_1 \cup \{\tilde{k}\}$ is a minimal cover of X.

Now, for M':

$$\sum_{i \in M} \bar{x}_i = \sum_{i \in M \setminus \{\tilde{k}\}} \bar{x}_i + \bar{x}_{\tilde{k}} = (|M| - 1) + \bar{x}_{\tilde{k}} > |M| - 1.$$

Hence, $\bar{\mathbf{x}}$ violates the minimal cover inequality associated with M', proving that $\bar{\mathbf{x}} \notin P_{\mathcal{M}}$.

Case 2.2 $\tilde{k} \in L$ and belongs to some minimal cover $M \in \mathcal{M}$.
Let M be a minimal cover such that $\tilde{k} \in M$. Consider the fractional solution $\bar{\mathbf{x}}$ where we fully pack all items in M except $\tilde{k}$. We set $\bar{x}_i = 1$ for all $i \in M \setminus \{\tilde{k}\}$, $\bar{x}_j = 0$ for all $j \notin M$, and assign the remaining capacity to $\tilde{k}$, giving $\bar{x}_{\tilde{k}} = (b - \sum_{i \in M \setminus \{\tilde{k}\}} a_i)/a_{\tilde{k}}$. Because M is a *minimal* cover, removing $\tilde{k}$ ensures $\sum_{i \in M \setminus \{\tilde{k}\}} a_i \leq b$, implying $\bar{x}_{\tilde{k}} \geq 0$. Because M is a cover, $\sum_{i \in M} a_i > b$, implying $\bar{x}_{\tilde{k}} < 1$. Thus, $\bar{\mathbf{x}}$ is a valid fractional point in P. Evaluating the minimal cover inequality associated to M gives:

$$\sum_{i \in M} \bar{x}_i = \sum_{i \in M \setminus \{\tilde{k}\}} (1) + \bar{x}_{\tilde{k}} = |M| - 1 + \bar{x}_{\tilde{k}}.$$

Again, because $\bar{x}_{\tilde{k}} > 0$, the inequality is strictly violated. Thus, $\bar{\mathbf{x}} \notin P_{\mathcal{M}}$.

Case 2.3 $\tilde{k} \in L$ and $L_{\tilde{k}} = \emptyset$.
We show this case is trivial because such an extreme point cannot exist. Recall that for any fractional extreme point $\bar{\mathbf{x}}$, the capacity constraint is perfectly tight: $\sum_{i \in N_1} a_i + a_{\tilde{k}} \bar{x}_{\tilde{k}} = b$. Because $\bar{x}_{\tilde{k}} \in (0, 1)$, it strictly follows that:

$$\sum_{i \in N_1} a_i < b \quad \text{and} \quad \sum_{i \in N_1 \cup \{\tilde{k}\}} a_i > b.$$

This implies that the subset $N_1 \cup \{\tilde{k}\}$ is a cover. Consequently, it must contain at least one minimal cover $M \subseteq N_1 \cup \{\tilde{k}\}$. If $\tilde{k} \notin M$, then $M \subseteq N_1$. This would imply $\sum_{i \in M} a_i \leq \sum_{i \in N_1} a_i < b$, which contradicts the definition of a cover. Therefore, any minimal cover contained in $N_1 \cup \{\tilde{k}\}$ *must* contain $\tilde{k}$. Thus, any fractional extreme point of P always features a fractional variable that belongs to at least one minimal cover, implying that Case 2.3 is impossible.

■

Theorems 1 and 2 together imply that $P^{\mathcal{M}} \subset P$ for any $X \in \mathcal{K}_S$.

Example 1 Consider the following binary knapsack set:

$$X = \left\{ \mathbf{x} \in \{0,1\}^6 \mid 10x_1 + 9x_2 + 8x_3 + 7x_4 + 4x_5 + 3x_6 \leq 33 \right\}.$$

This instance belongs to the class $\mathcal{K}_S$. By setting the split point at $h = 3$, we partition the items into $H = \{1, 2, 3\}$ and $L = \{4, 5, 6\}$. The residual capacity is $\Delta = 33 - (10 + 9 + 8) = 6$. The four structural conditions for $\mathcal{K}_S$ are satisfied:

1. $S_H = 27 \leq 33$.
2. $S_H + a_4 = 27 + 7 = 34 > 33$.
3. $(S_H - a_3) + S_L = (27 - 8) + 14 = 33 \leq 33$.
4. $L_4 = \{4\}$ (since $a_4 = 7 > \Delta$); $L_5 = L_6 = \{5, 6\}$ (since $a_5 + a_6 = 4 + 3 = 7 > \Delta$).

Using Remark 1, the only minimal covers of X are $H \cup L' = \{1, 2, 3, 4\}$ (for $L' = \{4\}$) and $\{1, 2, 3, 5, 6\}$ (for $L' = \{5, 6\}$). Hence, $P_{\mathcal{M}} = \{\mathbf{x} \in \mathbb{R}^6 : 0 \leq \mathbf{x} \leq 1, (5), (6)\}$, where (5) and (6) are given as:

$$x_1 + x_2 + x_3 + x_4 \leq 3 \tag{5}$$

$$x_1 + x_2 + x_3 + x_5 + x_6 \leq 4 \tag{6}$$

Using the weights $\mathbf{u} = (6, 2)$ for the minimal cover inequalities (5) and (6), respectively, and $\mathbf{v} = (2, 1, 0, 1, 2, 1)$ for the upper bound inequalities $x_i \leq 1$, the original knapsack inequality can be expressed as a conic combination of the inequalities in $P_{\mathcal{M}}$, as follows:

$$\begin{aligned}
&6(x_1 + x_2 + x_3 + x_4) + 2(x_1 + x_2 + x_3 + x_5 + x_6) \\
&\quad + 2x_1 + 1x_2 + 0x_3 + 1x_4 + 2x_5 + 1x_6 \\
&\leq 6(3) + 2(4) + 2(1) + 1(1) + 0(1) + 1(1) + 2(1) + 1(1),
\end{aligned}$$

which simplifies to:

$$10x_1 + 9x_2 + 8x_3 + 7x_4 + 4x_5 + 3x_6 \leq 33.$$

This proves $P_{\mathcal{M}} \subseteq P$.

Furthermore, consider $\bar{\mathbf{x}} = (1, 1, 0.875, 0, 1, 1)$. Clearly, $\bar{\mathbf{x}} \in P$ (since $10(1) + 9(1) + 8(0.875) + 4(1) + 3(1) = 33$). However, $\bar{\mathbf{x}} \notin P_{\mathcal{M}}$ as it violates (6). Hence, $P_{\mathcal{M}} \subset P$.

Example 1 (Contd.) Consider the optimization problem defined on the knapsack set X from Example 1:

$$z^* = \max\{15x_1 + 12x_2 + 10x_3 + 8x_4 + 5x_5 + 4x_6 \mid \mathbf{x} \in X\}$$

For this optimization problem, we have $z = \max\{15x_1 + 12x_2 + 10x_3 + 8x_4 + 5x_5 + 4x_6 \mid \mathbf{x} \in P\} = 44.75$, while $z' = \max\{15x_1 + 12x_2 + 10x_3 + 8x_4 + 5x_5 + 4x_6 \mid \mathbf{x} \in P_{\mathcal{M}}\} = 44 = z^*$.

3 Complete Linear Description of the Convex Hull

Having shown, in Section 2, that $P^{\mathcal{M}} \subset P$ for any $X \in \mathcal{K}_S$, the following question naturally arises: how tight is $P_{\mathcal{M}}$? The next theorem answers this question.

Theorem 3 *For any $X \in \mathcal{K}_S$, $P_{\mathcal{M}} = \mathrm{conv}(X)$.*

Proof Recall that $P_{\mathcal{M}} = \left\{\mathbf{x} \in \mathbb{R}^{|N|} \mid \mathbf{0} \leq \mathbf{x} \leq \mathbf{1},\ \sum_{i \in M} x_i \leq |M| - 1 \quad \forall M \in \mathcal{M}\right\}$.

We define $A \in \{0, 1\}^{|\mathcal{M}| \times |N|}$ as the coefficient matrix corresponding to the minimal cover inequalities. The rows of A are indexed by the minimal covers $M \in \mathcal{M}$, and the columns are indexed by the items $j \in N$. Using Remark 1, every minimal cover $M \in \mathcal{M}$ takes the form $H \cup L'$ for some $L' \subseteq L$. Thus, partitioning the columns of A into H and L, we obtain the matrix structure:

$$A = \left(\mathbf{1}_{|\mathcal{M}|\times|H|} \;\; E_L\right)$$

where $\mathbf{1}_{|\mathcal{M}|\times|H|}$ is a matrix of all ones, and $E_L \in \{0,1\}^{|\mathcal{M}|\times|L|}$ is the incidence matrix of the subsets $L' \subseteq L$.

Lemma 3.1 *For any $X \in \mathcal{K}_S$, the constraint matrix A associated with the minimal cover inequalities is totally unimodular.*

Proof We apply the Ghouila-Houri criterion, which states that a matrix A is totally unimodular if and only if for every subset of rows $R \subseteq \{1, \dots, |\mathcal{M}|\}$, there exists a partition $R = R_1 \cup R_2$ with $R_1 \cap R_2 = \emptyset$ such that for every column $j \in N$:

$$\left| \sum_{k \in R_1} A_{k,j} - \sum_{k \in R_2} A_{k,j} \right| \leq 1.$$

Let R be an arbitrary subset of rows. We construct the partition (R_1, R_2) such that their cardinalities differ by at most one, i.e., $\big||R_1| - |R_2|\big| \leq 1$. We verify the criterion for an arbitrary column $j \in N$ through two mutually exclusive and exhaustive Cases 3.1 and 3.2:

Case 3.1 $j \in H$.
Since every minimal cover contains H, $A_{k,j} = 1$ for all rows $k \in \mathcal{M}$. The alternating sum over the subset R is exactly the difference in partition sizes:

$$\left| \sum_{k \in R_1} 1 - \sum_{k \in R_2} 1 \right| = \big||R_1| - |R_2|\big| \leq 1.$$

Case 3.2 $j \in L$.
By Proposition 2, each element of L appears in at most one minimal cover. Therefore, the column vector corresponding to j in matrix E_L (and consequently A) contains at most a single 1. Summing over any subset R, we have $\sum_{k \in R} A_{k,j} \in \{0,1\}$. Thus, the absolute difference between the sums over R_1 and R_2 is trivially bounded by 1.

As the criterion holds for all columns $j \in N$ under any arbitrary row subset R, the matrix A is totally unimodular. ■

The polyhedron $P_{\mathcal{M}}$ is defined by the linear system:

$$\begin{pmatrix} A \\ I \\ -I \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} \mathbf{d} \\ \mathbf{1} \\ \mathbf{0} \end{pmatrix},$$

where $\mathbf{d}$ is the right-hand side vector with components $d_M = |M| - 1$ for all $M \in \mathcal{M}$, and I represents the $|N| \times |N|$ identity matrix governing the bounds $\mathbf{0} \leq \mathbf{x} \leq \mathbf{1}$.

By Lemma 3.1, A is totally unimodular. Appending the identity matrix and its negation preserves total unimodularity. Since the right-hand side vector $(\mathbf{d}, \mathbf{1}, \mathbf{0})^\top$ is strictly integral, the polyhedron $P_{\mathcal{M}}$ is integral.

Since every extreme point of $P_{\mathcal{M}}$ is integral, and $X = X^{\mathcal{M}}$, it follows that $P_{\mathcal{M}} = \text{conv}(X)$. ■

Example 1 (Contd.) For Example 1 the 14 inequalities which describe the complete convex hull of the polyhedron (obtained from PORTA [3]) is given in Table 1.

Table 1 Facets of Example 1 generated from PORTA

Sl. No.	Facets of Example 1	Type
1	$x_1 \geqslant 0$	Trivial
2	$x_2 \geqslant 0$	Trivial
3	$x_3 \geqslant 0$	Trivial
4	$x_4 \geqslant 0$	Trivial
5	$x_5 \geqslant 0$	Trivial
6	$x_6 \geqslant 0$	Trivial
7	$x_1 \leqslant 1$	Trivial
8	$x_2 \leqslant 1$	Trivial
9	$x_3 \leqslant 1$	Trivial
10	$x_4 \leqslant 1$	Trivial
11	$x_5 \leqslant 1$	Trivial
12	$x_6 \leqslant 1$	Trivial
13	$x_1 + x_2 + x_3 + x_4 \leqslant 3$	Minimal cover inequality
14	$x_1 + x_2 + x_3 + x_5 + x_6 \leqslant 4$	Minimal cover inequality

Theorem 3 demonstrates that the minimal cover inequalities, along with the bounds on the variables are sufficient to completely describe the convex hull for $X \in \mathcal{K}_S$. A natural related question is the following: can this complete linear description of $\text{Conv}(X)$ using minimal cover inequalities be extended to a broader class of knapsack polytopes? We investigate this question in the following section.

4 Beyond $\mathcal{K}_S$

In this section, we try to determine which of the conditions in Definition 1 can be relaxed such that $P_{\mathcal{M}} = \text{conv}(X)$ still holds. Note that for any X, there always exists some $h \in \{1, \ldots, |N|\}$ for which Conditions (1) and (2) hold. Hence, we only need to check if Conditions (3) and (4) in Definition 1 can be relaxed. We answer this question in the following Theorem 4 and Example 3.

Theorem 4 *If Condition (3) is violated, then* $P_{\mathcal{M}} \neq \text{conv}(X)$.

Proof Assume Condition (3) is violated. By definition, there exists at least one item $h^* \in H$ such that removing it still leaves the total weight greater than the capacity:

$$\sum_{i \in H \setminus \{h^*\}} a_i + \sum_{j \in L} a_j > b$$

This implies there exists a minimal cover $M^* \subseteq (H \setminus \{h^*\}) \cup L$. Let $L^* = M^* \cap L$. M^* must contain at least one $j \in L$.

By Condition (4), every item $j \in L$ belongs to exactly one minimal cover. This implies that the set of all minimal covers partitions the items in L into strictly disjoint subsets. Let $\{L^1, L^2, \dots, L^p\}$ be those specific disjoint subsets of L which intersects with L^*. Also, $M_r = H \cup L^r$ are minimal covers for $r \in \{1, \dots, p\}$.

Clearly, $p \geq 2$. Otherwise, if $p = 1$, then $L^* \subseteq L^1$, implying $M^* \subseteq (H \setminus \{h^*\}) \cup L^1$. Since $H \cup L^1$ is a minimal cover, removing h^* from it makes its weight less than b. Therefore, M^* could not be a cover if it were restricted to a single L^1. Thus, L^* must span across at least two disjoint subsets.

We now construct a specific fractional extreme point $\bar{\mathbf{x}} \in \mathbb{R}^{|N|}$. From each intersection $L^r \cap L^*$, we select exactly one item from L, denoting it as j_r. Let $J = \{j_1, j_2, \dots, j_p\}$ be the set of these p chosen items. We assign the coordinates of $\bar{\mathbf{x}}$ as follows:

$$\bar{x}_i = 1 \quad \forall i \in H \setminus \{h^*\} \tag{7}$$

$$\bar{x}_i = 1 \quad \forall i \in (L^* \setminus J) \cup \bigcup_{r=1}^{p} (L^r \setminus L^*) \tag{8}$$

$$\bar{x}_i = 0 \quad \text{for all other items not in } H, L^*, \text{ or } \bigcup L^r \tag{9}$$

$$\bar{x}_{h^*} = \frac{1}{p} \tag{10}$$

$$\bar{x}_j = \frac{p-1}{p} \quad \forall j \in J \tag{11}$$

Because $p \geq 2$, the coordinates for h^* and the items in J are strictly fractional, residing in $(0, 1)$. To prove $\bar{\mathbf{x}}$ is an extreme point of $P_{\mathcal{M}}$, we must show it is the unique solution to $|N|$ linearly independent active constraints.

First, we verify $\bar{\mathbf{x}}$ tightly satisfies the minimal cover inequalities for $M_1, \dots, M_p$ and M^*. For each $M_r = H \cup L^r$: Inside L^r, exactly one item (j_r) is fractional. All other items in L^r are set to 1. Hence,

$$\sum_{i \in M_r} \bar{x}_i = (|H| - 1) + \bar{x}_{h^*} + (|L^r| - 1) + \bar{x}_{j_r}$$

$$= |H| + |L^r| - 2 + \frac{1}{p} + \frac{p-1}{p} = |H| + |L^r| - 1 = |M_r| - 1$$

For $M^* = (H \setminus \{h^*\}) \cup L^*$: Inside L^*, exactly p items (the set J) are fractional. All other items in L^* are set to 1.

$$\sum_{i \in M^*} \bar{x}_i = (|H| - 1) + (|L^*| - p) + \sum_{j \in J} \frac{p-1}{p}$$

$$= |H| + |L^*| - p - 1 + p\left(\frac{p-1}{p}\right) = |H| + |L^*| - p - 1 + p - 1 = |M^*| - 1$$

All other variables are either 0 or 1, accounting for $|N| - (p+1)$ active constraints. The remaining variables are the $p+1$ fractional coordinates: x_{h^*} and $x_{j_1}, \dots, x_{j_p}$. The minimal cover inequalities yield the following algebraic subsystem for these variables:

$$x_{h^*} + x_{j_r} = 1 \quad \text{for } r = 1, \dots, p \quad (\text{from } M_r)$$

$$\sum_{r=1}^{p} x_{j_r} = p - 1 \quad (\text{from } M^*)$$

By summing the first p rows, we get $p \cdot x_{h^*} + \sum x_{j_r} = p$. Subtracting the final row gives $p \cdot x_{h^*} = 1 \implies x_{h^*} = 1/p$. Because this matrix has exactly one unique solution, it has full rank.

Thus, $\bar{\mathbf{x}}$ is a fractional extreme point of $P_{\mathcal{M}}$. Because conv(X) is the integer hull, all of its extreme points must be integral. Since $\bar{\mathbf{x}} \in P_{\mathcal{M}}$ contains strictly fractional coordinates, $P_{\mathcal{M}} \neq \text{conv}(X)$. ■

Example 2 To illustrate the construction in Theorem 4, consider the knapsack set:

$$X = \left\{ \mathbf{x} \in \{0,1\}^4 \mid 4x_1 + 2x_2 + 2x_3 + 2x_4 \leq 5 \right\}.$$

Let $H = \{1\}$ and $L = \{2,3,4\}$. Condition (1) is satisfied since $4 \leq 5$. However, Condition (3) fails because $S_N - a_h = S_L = 2 + 2 + 2 = 6 > 5$.

Because Condition (3) fails, the item 1 is not forced into every minimal cover. The complete set of minimal covers $\mathcal{M}$ is:

$$M_1 = \{1,2\}, \quad M_2 = \{1,3\}, \quad M_3 = \{1,4\}, \quad M^* = \{2,3,4\}.$$

The linear relaxation $P_{\mathcal{M}}$ is defined by the corresponding cover inequalities: $x_1 + x_2 \leq 1$, $x_1 + x_3 \leq 1$, $x_1 + x_4 \leq 1$, and $x_2 + x_3 + x_4 \leq 2$, along with the bounds $0 \leq x_i \leq 1 \; \forall i = \{1, \cdots, 6\}$.

Following the construction in Theorem 4, M^* is the cover missing $h^* = 1$, with $L^* = \{2,3,4\}$. The residual capacity is $\Delta = 5 - 4 = 1$. The minimal covers for Δ within L form strictly disjoint subsets: $L^1 = \{2\}$, $L^2 = \{3\}$, and $L^3 = \{4\}$.

Because all of them intersect with L^*, we have $p = 3$ disjoint subsets. Selecting one item from each intersection yields the set $J = \{2,3,4\}$. Using the construction (10) and (11), we construct the fractional point $\bar{\mathbf{x}}$: $\bar{x}_1 = \frac{1}{p} = \frac{1}{3}$ and $\bar{x}_j = \frac{p-1}{p} = \frac{2}{3} \; \forall j \in J$. This yields the strictly fractional point $\bar{\mathbf{x}} = (1/3, 2/3, 2/3, 2/3)$.

Now, $\bar{\mathbf{x}} \in P_{\mathcal{M}}$, since $\bar{\mathbf{x}}$ satisfies all minimal cover inequalities tightly. Hence, $\bar{\mathbf{x}}$ is an extreme fractional point (vertex) of $P_{\mathcal{M}}$, which proves that $P_{\mathcal{M}} \neq \text{conv}(X)$.

In Theorem 4, we formally prove that Condition (3) in Definition 1 is necessary to prove $P_{\mathcal{M}} = \text{conv}(X)$. If this condition is relaxed, then $P_{\mathcal{M}}$ has at least one fractional extreme point. If we can show that Condition (4) is also necessary, then $\nexists X \notin \mathcal{K}_S$ such that $\text{conv}(X) = P_{\mathcal{M}}$. However, the following example shows that Condition (4) in Definition 1 is only sufficient but not necessary.

Example 3 Consider the knapsack set defined by:

$$X = \left\{ \mathbf{x} \in \{0,1\}^6 \mid 8x_1 + 8x_2 + 4x_3 + 2x_4 + x_5 + x_6 \leq 18 \right\}.$$

For $h = 2$, we have $H = \{1,2\}$, $L = \{3,4,5,6\}$, and $\Delta = 18 - 16 = 2$. For $j = 4$, Condition (4) of Definition 1 violated since the corresponding subset of L is not unique. This implies that $X \notin \mathcal{K}_{HS}$. However, the minimal cover inequalities for X, namely $x_1 + x_2 + x_3 \leq 2$, $x_1 + x_2 + x_4 + x_5 \leq 3$, and $x_1 + x_2 + x_4 + x_6 \leq 3$ (together with the bounds on the variables) are sufficient to completely characterize conv(X).

5 Conclusions

In this paper, we defined a specific class of binary knapsack sets for which its formulation based on its minimal cover inequalities is tighter than the one based on the the knapsack inequality. Furthermore, for any knapsack set in this specific class, the formulation based on minimal cover inequalities provides its complete convex hull. We also proved that all but one of the conditions in the definition of this special class of binary knapsack sets are necessary to establish the above result. To that extent, finding all the necessary conditions for the above result is a natural extension of our work.

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