Strong convergence, perturbation resilience and superiorization of Generalized Modular String-Averaging with infinitely many input operators

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September 15, 2025. Revised: December 23, 2025.

Abstract

We study the strong convergence and bounded perturbation resilience of iterative algorithms based on the Generalized Modular String-Averaging (GMSA) procedure for infinite sequences of input operators under a general admissible control. These methods address a variety of feasibility-seeking problems in real Hilbert spaces, including the common fixed point problem and the convex feasibility problem. In addition to the general case, involving certain strongly quasi-nonexpansive input operators, we consider a specific subclass of their corresponding relaxed firmly nonexpansive operators. This subclass proves useful for establishing bounded perturbation resilience. We further demonstrate the applicability of our strong convergence results, within the GMSA framework, to the Superiorization Methodology and to Dynamic String-Averaging, analyzing the behavior of a superiorized version of our main algorithm. The novelty and significance of this work is that it not only includes a variety of earlier algorithms as special cases but, more importantly, it allows the use of modular options of string-averaging that give rise to new, hitherto unavailable, algorithmic schemes with emphasis on infinitely many input operators. The strong convergence guarantees and the applications for superiorization and dynamic string-averaging are also important facets.

Keywords: Approximately shrinking operator, bounded perturbation resilience, bounded regularity, common fixed point problem, convex feasibility problem, dynamic string-averaging, Fejér monotonicity, nonexpansive operators, strongly quasi-nonexpansive operators, superiorization.

1 Introduction

The Modular String-Averaging (MSA) procedure for a finite number of input operators was developed by Reich and Zalas in [46], based on the original string-averaging scheme introduced by [19], and on the work in [22], to provide a flexible algorithmic framework for a large family of iterative methods. The MSA thus not only includes a variety of earlier algorithms as special cases but, more importantly, it allows the use of new modular options of string-averaging that give rise to new, hitherto not available, algorithmic schemes.

On the other hand, Bauschke and Combettes suggested in [9] the notion of coherence which turned out to be an important approach in proving weak and strong convergence of such iterative processes. The stronger version of coherence, was presented by Barshad, Reich and Zalas in [6], as a useful tool for establishing the coherence of a sequence of certain operators.

In 2023 Barshad, Gibali and Reich combined the ideas of both MSA and strong coherence, by proposing in [5] the Generalized Modular String-Averaging (GMSA) procedure for an infinite number of input operators and showing the strong coherence of its output operators, where the general admissible control was used. In that work, the weak convergence of GMSA methods was proved based on the theory of coherence, while the strong convergence necessitated the use of Haugazeau projections.

In many problems the computation of output operators can be inexact and produce some small errors which we call perturbations. Unfortunately, it is difficult and not convenient to consider perturbations by using Haugazeau projection methods when we are interested in the strong convergence of iterative algorithms in the infinite-dimensional Hilbert space. For this reason formulating alternative conditions on the input operators under which the strong convergence will hold in the infinite-dimensional space is desirable. Such conditions were provided in [46], where the properties of the MSA procedure for a finite family of input operators under an s-intermittent control were studied.

In this paper we develop extensions of these conditions to a much more general method, which involves relaxation parameters, based on the GMSA procedure for an infinite family of input operators under a more general admissible control, where we additionally show its bounded perturbation resilience. Of particular interest is the case of relaxed firmly nonexpansive operators, where we can guarantee the bounded perturbation resilience of our strongly convergent methods. This enables us to study the properties of the superiorized version of our GMSA algorithm.

The situation of an infinite family of input operators is mathematically interesting but also has its roots in modeling of real-world problems. Combettes [26, 1997] noted that finding a common point of a family of closed and convex sets in a Hilbert space, known as the hilbertian convex feasibility problem, captures problems in disciplines as diverse as approximation theory, integral equations, control theory, signal and image processing, biomedical engineering, communications, and geophysics. The "extrapolated method of parallel projections" (EMOPP), which allows the total number of sets to be countably infinite, was proposed in that paper. Kong, Pajoohesh and Herman [43] study string-averaging algorithms for convex feasibility with infinitely many sets and make a convincing case, backed by many references, for the importance of infinitely many sets. The latter translate to infinitely many input operators in our framework, see also [49] and [50].

Since its inception in 2007, the superiorization method (SM) has evolved and gained ground. Recent review papers on the subject are Herman's [38] and [39] and the review in [18]. The superiorization method was born when the terms and notions "superiorization" and "perturbation resilience", in the present context, first appeared in the 2009 paper [33] which followed its 2007 forerunner by Butnariu et al. [11]. The ideas have some of their roots in the 2006 and 2008 papers of Butnariu et al. [12, 13], where it was shown that if iterates of a nonexpansive operator converge for any initial point, then its inexact iterates with summable errors also converge.

Bounded perturbation resilience of a parallel projection method was observed as early as 2001 in [28, Theorem 2] (without using this term). All these culminated in Ran Davidi's 2010 PhD dissertation [32] and the many papers that appeared since then and are cited in [17] which is a Webpage dedicated to superiorization and perturbation resilience of algorithms that contains a continuously updated bibliography on the subject. This Webpage[1] is source for the wealth of work done in this field to date, including two special issues of journals [20] and [37] dedicated to research of the SM. Interestingly, [1] notices some structural similarities of the SM with incremental proximal gradient methods.

[1] http://math.haifa.ac.il/yair/bib-superiorization-censor.html#top, last updated on February 22, 2026, with 204 items

The 2001 paper of Combettes [29] also investigated the use summable perturbations on cutter-type methods. For example, Theorem 5.2 of [29] is a bounded perturbation resilience result for Algorithm 5.1 there, without using this term. Many more results in [29] are related directly to the SM, without using the language of the SM.

The "adaptive steepest descent projections onto convex sets" (ASD-POCS) algorithm described in [47] and in many subsequent works on it, has some similarities to the SM. However, it is not as general as the SM; see [40] for a comparison between it and the SM.

The paper is laid out as follows. We set up notations and present preliminaries in Section 2 with emphasis on various types of algorithmic operators in Subsection 2.1. General bounded regularity and approximate shrinking properties of operators are discussed in Subsection 2.2 and some additional notions and results are presented in Subsection 2.3. In Subsection 2.4 the general modular string-averaging procedure is recalled. In Section 3 we establish our results concerning the strong convergence properties of the GMSA procedure based methods. Namely, Subsection 3.1 describes results in the general case of strongly quasi nonexpansive operators, while in Subsection 3.2 we focus on the particular case of relaxed firmly nonexpansive operators. In Section 4 we investigate certain properties of the general superiorization algorithm. Finally, in the last Section 5 we present applications of our results to the Superiorization Methodology and Dynamic String Averaging.

2 Preliminaries

Throughout this paper, $\mathbb{Z}$ denotes the set of integer numbers, $\mathbb{N}$ denotes the set of natural numbers (including 0), and for any two integers m and n, with $m \leq n$, we denote by $\{m, m+1, \ldots, n\}$ the set of all integers between m and n. For a set A, we denote by $|A|$ the cardinality of A. For a real Hilbert space $\mathcal{H}$, we use the following notations:

- $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathcal{H}$.
- $\| \cdot \|$ denotes the norm on $\mathcal{H}$ induced by $\langle \cdot, \cdot \rangle$.
- Id denotes the identity operator on $\mathcal{H}$.
- $\mathrm{Fix}T$ denotes the set $\mathrm{Fix}T := \{x \in \mathcal{H} \mid T(x) = x\}$ of fixed points of an operator $T : \mathcal{H} \to \mathcal{H}$.
- For a nonempty and convex subset C of $\mathcal{H}$, we denote by P_C the (unique) metric projection onto C, the existence of which is guaranteed if C is, in addition, closed.
- The expression $x^k \to x$ denotes the strong convergence to x of a sequence $\{x^k\}_{k=0}^{\infty}$ in $(\mathcal{H}, \| \cdot \|)$ when $k \to \infty$.
- For a convex function $\phi : \mathcal{H} \to \mathbb{R}$, where $\mathbb{R}$ denotes the real line, and a point $x \in \mathcal{H}$, we denote by $\partial\phi(x)$ the subdifferential set of ϕ at x, that is,

$$\partial\phi(x) := \{g \in \mathcal{H} \mid \langle g, y - x \rangle \leq \phi(y) - \phi(x) \text{ for all } y \in \mathcal{H}\}.$$

- For a function $f : \mathcal{H} \to \mathbb{R}$ and a subset A of $\mathcal{H}$, we denote by $\underset{x \in A}{\mathrm{Argmin}} f(x)$ the set of minimizers of f on the set A.
- $B(x, r)$ denotes the open ball centered at $x \in \mathcal{H}$ of radius $r > 0$.
- For a nonempty subset C of $\mathcal{H}$ and $x \in \mathcal{H}$, we denote by $d(x, C)$ the distance from x to C, that is, $d(x, C) := \inf_{y \in C} \|x - y\|$.

We study the properties of the following algorithm, which serves as the algorithmic framework for our GMSA methods.

Algorithm 2.1 (The algorithmic framework)**.** *Given* $\varepsilon \in (0,1]$, $x^0 \in \mathcal{H}$ *and a sequence* $\{T_k\}_{k=0}^{\infty}$ *of operators,* $T_k : \mathcal{H} \to \mathcal{H}$ *for each* $k \in \mathbb{N}$*, the algorithm is defined by the recurrence*

$$x^{k+1} := x^k + \lambda_k \left(T_k\left(x^k\right) - x^k\right),$$

where $\lambda_k \in [\varepsilon, 2-\varepsilon]$ *for each* $k \in \mathbb{N}$*.*

2.1 Various types of algorithmic operators and their properties

We review the following classes of algorithmic operators together with key results regarding their properties. For additional details, see, for instance, [14].

Definition 2.2. Let $T : \mathcal{H} \to \mathcal{H}$ be an operator and let $\lambda \in [0,2]$. The operator $T_\lambda : \mathcal{H} \to \mathcal{H}$ defined by $T_\lambda := (1-\lambda)\,\mathrm{Id} + \lambda T$ is called a λ-*relaxation* of the operator T. The operator T_2 is called a *reflection* of the operator T.

Remark 2.3. Clearly, $\mathrm{Fix}T = \mathrm{Fix}T_\lambda$ for every operator $T : \mathcal{H} \to \mathcal{H}$ and every $\lambda \in (0,2]$.

Definition 2.4. An operator $T : \mathcal{H} \to \mathcal{H}$ is *nonexpansive* if

$$(\forall x, y \in \mathcal{H}) \;\; \|T(x) - T(y)\| \le \|x - y\|.$$

For every ordered pair $(x,y) \in \mathcal{H}^2$, we define the closed and convex set $H(x,y)$ by

$$H(x,y) := \{u \in \mathcal{H} \mid \langle u - y, x - y\rangle \le 0\}.$$

Definition 2.5. An operator $T : \mathcal{H} \to \mathcal{H}$ is called a *cutter* if it satisfies

$$\mathrm{Fix}T \subseteq H(x, T(x)), \; \forall x \in \mathcal{H},$$

or, equivalently, if $\langle z - T(x), x - T(x)\rangle \le 0$ for each $z \in \mathrm{Fix}T$ and $x \in \mathcal{H}$. For $\lambda \in [0,2]$, an operator $T : \mathcal{H} \to \mathcal{H}$ is a λ-*relaxed* cutter if T is a λ-relaxation of a cutter U, that is, $T = U_\lambda = (1-\lambda)\,Id + \lambda U$.

Proposition 2.6 ([9, Proposition 2.6])**.** *Let* T *be a cutter. Then* $\mathrm{Fix}T = \cap_{x\in\mathcal{H}} H(x, T(x))$ *and hence* $\mathrm{Fix}T$ *is a closed and convex subset of* $\mathcal{H}$*, as an intersection of half-spaces.*

Remark 2.7. The class of cutters was originally introduced by Bauschke and Combettes in [9] under a different terminology, where it was referred to as the "class $\mathfrak{T}$". The term "cutter" was later proposed in [15]. Other names are used in the literature for these operators, for instance, "firmly quasi-nonexpansive" (see, for example, Definition 4.1 and Proposition 4.2 in [10]), which also contains various properties and examples of these operators. It is important to alert the reader about some ambiguity in the literature regarding this term. Definition 9.2 of [15] provides the original definition of cutters in which they are defined without any condition on the non-emptiness of their fixed points sets. In this sense cutters are simply another name for the members of the original "class $\mathfrak{T}$" of Bauschke and Combettes in [9] which are also defined there without such a condition. However, in Definition 2.1.30 in [14] the definition was modified to require that the fixed points sets of the cutter operators be nonempty. This has led to some ambiguity, as some later publications either include or do not include this non-emptiness assumption.

We follow the original definition (Definition 9.2 of [15]) and explicitly assume the non-emptiness of the fixed point set of a cutter only when required.

Definition 2.8. We say that an operator $T : \mathcal{H} \to \mathcal{H}$ is:

(i) *Quasi-nonexpansive* if

$$(\forall x \in \mathcal{H})\,(z \in \mathrm{Fix}T)\,\|T(x) - z\| \leq \|x - z\|.$$

(ii) *ρ-strongly quasi-nonexpansive* for some $0 \leq \rho \in \mathbb{R}$ if

$$(\forall x \in \mathcal{H})\,(z \in \mathrm{Fix}T)\,\|T(x) - z\|^2 \leq \|x - z\|^2 - \rho\,\|T(x) - x\|^2. \tag{2.1}$$

If T satisfies (2.1) for some $\rho > 0$, then it is called strongly quasi-nonexpansive.

Theorem 2.9 ([14, Theorem 2.1.39])**.** *Let $T : \mathcal{H} \to \mathcal{H}$ be an operator and let $\lambda \in (0, 2]$. Then T is a cutter if and only if its relaxation T_λ is $\frac{2-\lambda}{\lambda}$-strongly quasi-nonexpansive, that is,*

$$\|T_\lambda x - z\|^2 \leq \|x - z\|^2 - \frac{2-\lambda}{\lambda}\|T_\lambda x - x\|^2$$

for all $x \in \mathcal{H}$ and for all $z \in \mathrm{Fix}T$.

Theorem 2.10 ([14, Theorems 2.1.48 and 2.1.50])**.** *Let m be a positive integer. For each $i = 1, 2, \ldots, m$, let $\rho_i > 0$ and assume that $U_i : \mathcal{H} \to \mathcal{H}$ is a ρ_i-strongly quasi-nonexpansive operator. Suppose that $\bigcap_{i=1}^m \mathrm{Fix}U_i \neq \emptyset$. Define $\rho := \min_{i \in \{1,2,\ldots,m\}} \rho_i$. Then the following two assertions hold:*

(i) *The convex combination $U := \sum_{i=1}^m \omega_i U_i$, where $\omega_i \in (0, 1]$ for each $i = 1, 2, \ldots, m$, and $\sum_{i=1}^m \omega_i = 1$, is ρ-strongly quasi-nonexpansive.*

(ii) *The composition $V := U_m \cdots U_2 U_1$ is ρm^{-1}-strongly quasi-nonexpansive.*

Remark 2.11. In a manner similar to Remark 2.7 regarding cutters, we note that in the literature (for instance, in Definitions 2.1.19 and 2.1.38 of [14]) quasi-nonexpansive and strongly quasi-nonexpansive operators are defined with the requirement that their fixed point sets be nonempty. Here we define these operators without imposing this condition, while we assume the non-emptiness of their fixed points sets only if we need it.

The following corollary is an immediate consequence of Theorem 2.9.

Corollary 2.12. *Let $U : \mathcal{H} \to \mathcal{H}$ be an operator and let $T := Id + \frac{1+\rho}{2}(U - Id)$ for some $\rho \geq 0$. Then U is ρ-strongly quasi-nonexpansive if and only if T is a cutter. In particular, U is quasi-nonexpansive if and only if $T := \frac{1}{2}(U + Id)$ is a cutter.*

Definition 2.13. We say that an operator $T : \mathcal{H} \to \mathcal{H}$ is:

(i) *Firmly nonexpansive* if

$$(\forall x, y \in \mathcal{H})\,\langle T(x) - T(y), x - y\rangle \geq \|T(x) - T(y)\|^2.$$

(ii) *ρ-firmly nonexpansive*, where $\rho \geq 0$ is a real number, if

$$(\forall x, y \in \mathcal{H})\,\|T(x) - T(y)\|^2 \leq \|x - y\|^2 - \rho\,\|(x - T(x)) - (y - T(y))\|^2.$$

Remark 2.14. Clearly, for each $\rho \geq 0$, a ρ-firmly nonexpansive operator is, in particular, nonexpansive and ρ-strongly quasi-nonexpansive.

Definition 2.15. For $\lambda \in [0, 2]$, an operator $T : \mathcal{H} \to \mathcal{H}$ is called λ*-relaxed firmly nonexpansive* if T is a λ-relaxation of a firmly nonexpansive operator U, that is, $T = U_\lambda = (1 - \lambda)\, Id + \lambda U$.

Theorem 2.16 ([14, Theorems 2.2.4 and 2.2.5])**.** *If $T : \mathcal{H} \to \mathcal{H}$ is firmly nonexpansive, then T is a nonexpansive cutter.*

Theorem 2.17 ([14, Corollary 2.2.15])**.** *For any $\lambda \in (0, 2]$, an operator $T : \mathcal{H} \to \mathcal{H}$ is firmly nonexpansive if and only if its relaxation T_λ is $(2 - \lambda)\,\lambda^{-1}$-firmly nonexpansive, that is, if and only if*

$$(\forall x, y \in \mathcal{H})\ \|T_\lambda(x) - T_\lambda(y)\|^2 \leq \|x - y\|^2 - (2 - \lambda)\,\lambda^{-1}\,\|(x - T_\lambda(x)) - (y - T_\lambda(y))\|^2 .$$

The following corollary is immediate from Theorem 2.17.

Corollary 2.18. *Let $U : \mathcal{H} \to \mathcal{H}$ be an operator and let $T := Id + 2^{-1}(1 + \rho)(U - Id)$ for some $\rho \geq 0$, that is, $T = U_{2^{-1}(1+\rho)}$. Then U is ρ-firmly nonexpansive if and only if T is firmly nonexpansive. In particular, U is nonexpansive if and only if $T := \frac{1}{2}(U + Id)$ is firmly nonexpansive.*

Theorem 2.19 ([14, Theorems 2.2.35 and 2.2.42])**.** *For a positive integer m, let $\{U_i\}_{i=1}^m$ be a finite family of λ_i-relaxed firmly nonexpansive operators, where $U_i : \mathcal{H} \to \mathcal{H}$ and $\lambda_i \in (0, 2]$ for each $i = 1, 2, \ldots, m$. Then:*

(i) *For each finite set of numbers $\{\omega_i\}_{i=1}^m \subset [0, 1]$ such that $\sum_{i=1}^m \omega_i = 1$, the convex combination $U := \sum_{i=1}^m \omega_i U_i$ is $\sum_{i=1}^m \omega_i \lambda_i$-relaxed firmly nonexpansive.*

(ii) *The composition $V := U_m \cdots U_2 U_1$ is λ-relaxed firmly nonexpansive, where*

$$(0, 2] \ni \lambda = \begin{cases} 2, & \text{if } \max_{i \in \{1,2,\ldots,m\}} = 2, \\ 2\left(\left(\sum_{i=1}^m \lambda_i (2 - \lambda_i)^{-1}\right)^{-1} + 1\right)^{-1}, & \text{otherwise.} \end{cases}$$

An immediate consequence of of Theorems 2.17 and 2.19 is presented in the following corollary.

Corollary 2.20. *For a positive integer m, let $\{U_i\}_{i=1}^m$ be a finite family of ρ_i-firmly nonexpansive operators, where $U_i : \mathcal{H} \to \mathcal{H}$ and $\rho_i \in [0, \infty)$ for each $i = 1, 2, \ldots, m$. Define $\rho := \min_{i \in \{1,2,\ldots,m\}} \rho_i$. Then:*

(i) *For each finite set of numbers $\{\omega_i\}_{i=1}^m \subset [0, 1]$ such that $\sum_{i=1}^m \omega_i = 1$, the convex combination $U := \sum_{i=1}^m \omega_i U_i$ is ρ- firmly nonexpansive.*

(ii) *The composition $V := U_m \cdots U_2 U_1$ is ρm^{-1}- firmly nonexpansive.*

Example 2.21. Given a nonempty, closed and convex subset C of $\mathcal{H}$, the metric projection P_C onto C is firmly nonexpansive (see, for instance, Theorem 2.2.21 in [14]) and hence, it is a nonexpansive cutter (by Theorem 2.16). Moreover, $\mathrm{Fix} P_C = C$.

Proposition 2.22 ([16, Propositions 4.5 and 4.6])**.** *Let m be a positive integer. For each $i = 1, 2, \ldots, m$, let $\rho_i > 0$ and assume that $U_i : \mathcal{H} \to \mathcal{H}$ is a ρ_i-strongly quasi nonexpansive operator. Suppose that $\bigcap_{i=1}^m \mathrm{Fix} U_i \neq \emptyset$. Let $U := \sum_{i=1}^m \omega_i U_i$, where $\omega_i \in (0, 1]$ for each $i = 1, 2, \ldots, m$ and $\sum_{i=1}^m \omega_i = 1$, be a convex combination of $\{U_i\}_{i=1}^m$ and let $V := U_m U_{m-1} \cdots U_1$ be a composition of $\{U_i\}_{i=1}^m$. Moreover, let $x \in \mathcal{H}$ and let $z \in \bigcap_{i=1}^m \mathrm{Fix} U_i$ be arbitrary. Then the following inequalities hold:*

$$\frac{1}{2R} \sum_{i=1}^m \omega_i \rho_i \|U_i x - x\|^2 \leq \|Ux - x\|$$

and

$$\frac{1}{2R}\sum_{i=1}^{m}\rho_i\left\|S_ix-S_{i-1}x\right\|^2\leq\left\|Vx-x\right\|,$$

for any positive $R\geq\|x-z\|$*, where* $S_i=\prod_{j=1}^{i}U_j=U_iU_{i-1}\cdots U_1$ *for each* $i=1,2,\ldots,m$ *(by definition, the composition* $S_0=Id$*).*

2.2 The general bounded regularity and approximate shrinking properties of operators

The notions of bounded regularity and approximate shrinking are needed for establishing the strong convergence of the methods which we discuss in this paper. The bounded regularity of a finite family of sets was examined in [8, Section 5] and [7]. This property was extended to an infinite family of sets in [3]. We recall it briefly below.

Definition 2.23. For a nonempty index set I (either finite or infinite), the family $\{C_i\}_{i\in I}$ of nonempty, closed and convex subsets of $\mathcal{H}$ with nonempty intersection C is *boundedly regular* if for any bounded sequence $\left\{x^k\right\}_{k=0}^{\infty}$ in $\mathcal{H}$, the following implication holds:

$$\lim_{k\to\infty}d\left(x^k,C_i\right)=0\text{ for each }i\in I\implies\lim_{k\to\infty}d\left(x^k,C\right)=0.$$

The next proposition establishes sufficient conditions for bounded regularity. Recall that a topological space X is locally compact if each $x\in X$ has a compact neighborhood with respect to the topology inherited from X.

Proposition 2.24 ([3, Proposition 3.2])**.** *Let* $\{C_i\}_{i\in I}$ *be a family of nonempty, closed and convex subsets of* $\mathcal{H}$ *with a nonempty intersection* C*. Then the following assertions hold:*

- ***(i)*** *If there is an* $i_0\in I$ *for which the set* C_{i_0} *is a locally compact topological space (with respect to the norm topology inherited from* $\mathcal{H}$*), then the family* $\{C_i\}_{i\in I}$ *is boundedly regular.*
- ***(ii)*** *If* $\mathcal{H}$ *is of finite dimension, then the family* $\{C_i\}_{i\in I}$ *is boundedly regular.*

We also use the concept of approximate shrinking, which has been the subject of extensive study in [16].

Definition 2.25. A quasi-nonexpansive operator $T:\mathcal{H}\to\mathcal{H}$ *is approximately shrinking* if for each bounded sequence $\left\{x^k\right\}_{k=0}^{\infty}$ in $\mathcal{H}$, the following implication holds:

$$\lim_{k\to\infty}\left\|T\left(x^k\right)-x^k\right\|=0\implies\lim_{k\to\infty}d\left(x^k,\mathrm{Fix}T\right)=0.$$

Example 2.26. Given a nonempty, closed and convex subset C of $\mathcal{H}$, the metric projection P_C onto C is approximately shrinking (see Example 3.5 in [16]).

2.3 Additional notions and results which we use in our work

Definition 2.27. For a nonempty, closed and convex subset C of $\mathcal{H}$, a sequence $\left\{x^k\right\}_{k=0}^{\infty}$ in $\mathcal{H}$ is

- **(i)** *Fejér monotone with respect to* C if for each $z\in C$ and each $k\in\mathbb{N}$,

$$\left\|x^{k+1}-z\right\|\leq\left\|x^k-z\right\|.$$

(ii) *Strictly Fejér monotone with respect to* C if for each $z \in C$ and each $k \in \mathbb{N}$,

$$\left\|x^{k+1} - z\right\| < \left\|x^k - z\right\|.$$

(iii) *Strongly Fejér monotone with respect to* C if there exists a constant $\alpha > 0$ such that

$$\left\|x^{k+1} - z\right\|^2 \leq \left\|x^k - z\right\|^2 - \alpha \left\|x^{k+1} - x^k\right\|^2.$$

for each $z \in C$ and each $k \in \mathbb{N}$.

For a discussion of properties of *Fejér* monotone sequences see, for example, Subsection 3.3 in [14].

Theorem 2.28 ([8, Theorem 2.16(v)])**.** *Let* C *be a nonempty, closed and convex subset of* $\mathcal{H}$*. If* $\left\{x^k\right\}_{k=0}^{\infty}$ *is Fejér monotone with respect to* C*, then it converges in the norm of* $\mathcal{H}$ *to some point in* C *if and only if*

$$\lim_{k\to\infty} d\left(x^k, C\right) = 0.$$

The next definition is fundamental in the analysis of the superiorization methodology. It defines bounded perturbations resilience of an iterative algorithm that is governed by an infinite sequence of algorithmic operators. See [12], where it was first shown that if the exact iterates of a nonexpansive operator converge, then the inexact iterates, subject to summable errors, also converge.

Definition 2.29 ([3, Definition 2.24])**.** Let $\Gamma \subseteq \mathcal{H}$ be a given nonempty subset of $\mathcal{H}$ and $\{T_k\}_{k=0}^{\infty}$ be a sequence of operators, $T_k : \mathcal{H} \to \mathcal{H}$ for each $k \in \mathbb{N}$. The algorithm $x^{k+1} := T_k(x^k)$, for all $k \in \mathbb{N}$, is said to be *bounded perturbations resilient* with respect to Γ if the following is true: If a sequence $\{x^k\}_{k=0}^{\infty}$, generated by the algorithm, converges in the norm of $\mathcal{H}$ to a point in Γ for all $x^0 \in \mathcal{H}$, then any sequence $\{y^k\}_{k=0}^{\infty}$ in $\mathcal{H}$ that is generated by the algorithm $y^{k+1} := T_k(y^k + \beta_k v^k)$, for all $k \in \mathbb{N}$, also converges in the norm of $\mathcal{H}$ to a point in Γ for all $y^0 \in \mathcal{H}$, provided that $\{\beta_k v^k\}_{k=0}^{\infty}$ are bounded perturbations, meaning that $\{\beta_k\}_{k=0}^{\infty}$ is a sequence of positive real numbers such that $\sum_{k=0}^{\infty} \beta_k < \infty$ and that the vector sequence $\{v^k\}_{k=0}^{\infty}$ is a bounded sequence in $\mathcal{H}$.

Remark 2.30. In this work we consider inner perturbations, where bounded perturbations are applied before the evaluation of the underlying operators. An alternative perturbation model consists of outer perturbations in which summable error terms are added to the operator outputs; such perturbations naturally model inexact operator evaluations and have been discussed in relation to the superiorization method in Section 8.1 in [21], see Proposition 5 therein. Further results on the the Krasnosel'skiĭ-Mann iterative method with perturbations can be found in [34]. Outer perturbations in projection methods for the split equality problem appeared in [35]. Although inner and outer perturbations are defined differently, under the nonexpansiveness assumptions imposed throughout our paper they are asymptotically equivalent in the sense that both lead to the same convergence behavior of the generated iterates. This shows that the present analysis is consistent and compatible with results established in the outer-perturbation framework.

Theorem 2.31 ([13, Theorems 3.2 and 5.2])**.** *Let* $C \subset \mathcal{H}$ *be a nonempty and closed subset. Let* $\{T_k\}_{k=0}^{\infty}$*,* $T_k : \mathcal{H} \to \mathcal{H}$ *for each* $k \in \mathbb{N}$*, be a sequence of nonexpansive operators satisfying* $C \subset \cap_{k=0}^{\infty} \mathrm{Fix} T_k$*. Assume that for each* $y \in \mathcal{H}$ *and each* $q \in \mathbb{N}$*, the sequence* $\{T_{q+k} \cdots T_{q+1} T_q(y)\}_{k=0}^{\infty}$ *converges in the norm of* $\mathcal{H}$ *to an element of* C*. Let* $\{\gamma_k\}_{k=0}^{\infty} \subset [0, \infty)$ *be a real sequence such that* $\sum_{k=0}^{\infty} \gamma_k < \infty$ *and let* $\left\{y^k\right\}_{k=0}^{\infty} \subset \mathcal{H}$*. Further assume that for each* $k \in \mathbb{N}$*,*

$$\left\|y^{k+1} - T_k\left(y^k\right)\right\| \leq \gamma_k.$$

Then the sequence $\left\{y^k\right\}_{k=0}^{\infty}$ *converges in the norm of* $\mathcal{H}$ *to an element of* C*.*

The following well-known property of a convex and continuous function will be used in the sequel.

Theorem 2.32 ([10, Theorem 16.17(ii)]). *Let $\phi : \mathcal{H} \to \mathbb{R}$ be convex and continuous function at the point $x \in \mathcal{H}$. Then the subgradient set $\partial\phi(x)$ is nonempty.*

2.4 The general modular string-averaging procedure

The MSA procedure for a finite number of input operators was introduced by Reich and Zalas in [46], based on the original string-averaging scheme introduced by Censor, Elfving and Herman in [19]. It was later generalized by Barshad, Gibali and Reich to the GMSA procedure in [5], where an infinite number of input operators along with further properties of this procedure were studied. Below we recall some of these properties which we need for our results in the sequel.

Assume that $\{U_n\}_{n=0}^{\infty}$ is a sequence of input operators and $\varepsilon \in (0,1]$. Let $\{N_k\}_{k=0}^{\infty}$ be a sequence of positive integers which we call "the sequence of numbers of recursive steps". For every $n \in \mathbb{N}$, define $L_n := \{m \in \mathbb{Z} \mid m \le n\}$. For each $k \in \mathbb{N}$, we consider the following settings:

- Let $c_k : \mathbb{N}\setminus\{0\} \cap L_{N_k} \to \{0,1,2\}$ be a function.
- For each positive integer $n \in L_{N_k}$, let J_n^k be a nonempty finite subset of L_{n-1} satisfying the following assertion: if $n \in c_k^{-1}(\{0\})$, then $\left|J_n^k\right| = 1$ and $J_n^k \subset L_0$.
- For each $n \in c_k^{-1}(\{0\})$, let j_n^k be the unique element of J_n^k and define $P_n^k := 1$.
- For each $n \in c_k^{-1}(\{1\})$, let $\omega_n^k : J_n^k \to [\varepsilon, 1]$ be a function which we call "a weight function", satisfying $\sum_{j\in J_n^k} \omega_n^k(j) = 1$, and define $P_n^k := \left|J_n^k\right|$.
- For each $n \in c_k^{-1}(\{2\})$, let $o_n^k : \left\{1, 2, \ldots, P_n^k\right\} \to J_n^k$ be a function onto J_n^k, which we call "an order function", and $P_n^k \ge \left|J_n^k\right|$ is a positive integer.
- Let $\alpha_k : c_k^{-1}(\{0\}) \to [\varepsilon, 2-\varepsilon]$ be another function.
- For each $n \in L_{N_k}$, we define a set $I_n^k \subset \mathbb{N}$ (recursively) by

$$I_n^k := \begin{cases} \{-n\} & \text{if } n \le 0, \\ \cup_{j\in J_n^k} I_j^k & \text{if } n > 0. \end{cases} \tag{2.2}$$

Now we are ready to describe our GMSA procedure.

For each $k \in \mathbb{N}$, define a sequence $\left\{V_n^k\right\}_{n=-\infty}^{N_k}$ of intermediate modules V_n^k (recursively) with respect to N_k recursive steps by

$$V_n^k := \begin{cases} V_n^k := U_{-n} & \text{if } n \le 0, \\ \text{(Relaxation) } Id + \alpha_k(n)\left(V_{j_n^k}^k - Id\right), & \text{if } n \in c_k^{-1}(\{0\}), \\ \text{(Convex combination) } \sum_{j\in J_n^k} \omega_n^k(j) V_j^k, & \text{if } n \in c_k^{-1}(\{1\}), \\ \text{(Composition) } V_{o_n^k(P_n^k)}^k \cdots V_{o_n^k(2)}^k V_{o_n^k(1)}^k, & \text{if } n \in c_k^{-1}(\{2\}) \end{cases} \tag{2.3}$$

for each $n \in L_{N_k}$. Following the work described in [46], we say that the sequence $\{T_k\}_{k=0}^{\infty}$, defined by $T_k := V_{N_k}^k$ for each $k \in \mathbb{N}$, is "generated by the GMSA procedure".

Note that for each $k \in \mathbb{N}$ and each positive integer $n \in L_{N_k}$, the set J_n^k stores the indices of all the previous modules which were used in order to define the module V_n^k, while for each $n \in L_{N_k}$, the set I_n^k stores the indices of those input operators which were actually applied in the construction of V_n^k.

Lemma 2.33 ([5, Lemma 3.1])**.** *Assume that* $\{\ell_k\}_{k=0}^{\infty}$ *is a strictly increasing sequence of natural numbers. For each* $k \in \mathbb{N}$ *and* $n \in L_{N_{\ell_k}}$*, Define* $N'_k := N_{\ell_k}$*,* $J'^k_n := J^{\ell_k}_n$*,* $c'_k := c_{\ell_k}$*,* $\alpha'_k := \alpha_{\ell_k}$*,* $P'^k_n := P^{\ell_k}_n$*,* $j'^k_n := j^{\ell_k}_n$ *the unique element of* J'^k_n *if* $n \in c'^{-1}_k(\{0\})$*,* $\omega'^k_n := \omega^{\ell_k}_n$ *if* $n \in c'^{-1}_k(\{1\})$ *and* $o'^k_n := o^{\ell_k}_n$ *if* $n \in c'^{-1}_k(\{2\})$*. Then for each* $k \in \mathbb{N}$ *and each* $n \in L_{N'_k}$*, we have* $I'^k_n = I^{\ell_k}_n$ *and* $V'^k_n = V^{\ell_k}_n$*, where the family* $\left\{\{I'^k_n\}_{n \in L_{N'_k}}\right\}_{k=0}^{\infty}$ *is defined by* (2.2) *with respect to the sets* $\left\{J'^k_n\right\}_{k \in \mathbb{N},\, n \in L_{N'_k}}$ *and the family* $\left\{\{V'^k_n\}_{n=-\infty}^{N_k}\right\}_{k \in \mathbb{N}}$ *is defined by* (2.3) *with respect to the sequence* $\{N'_k\}_{k=0}^{\infty}$ *and the above parameters. Consequently, the sequence* $\{S_k\}_{k=0}^{\infty}$ *that is generated by the GMSA procedure with respect to the sequence* $\{N'_k\}_{k=0}^{\infty}$ *and the above parameters satisfies* $S_k = T_{\ell_k}$ *for each* $k \in \mathbb{N}$*.*

Lemma 2.34 ([5, Lemma 3.2])**.** *In the settings of the GMSA procedure above, for each* $k \in \mathbb{N}$ *and each non-positive integer* n *such that* U_{-n} *is a* 2^{-1}*-strongly quasi-nonexpansive operator, the intermediate module* V^k_n*, generated by the GMSA procedure, is* $2^{-1}\varepsilon$*-strongly quasi-nonexpansive and*

$$\mathrm{Fix} V^k_n = \cap_{i \in I^k_n} \mathrm{Fix} U_i.$$

Lemma 2.35 ([5, Lemma 3.4])**.** *Let* $\{U_n\}_{n=0}^{\infty}$ *be a sequence of* 2^{-1}*-strongly quasi-nonexpansive operators such that* $\cap_{i \in I^k_n} \mathrm{Fix} U_i \neq \emptyset$ *for each* $n \in \mathbb{N}$*, let* $k \in \mathbb{N}$ *and let* $n \in L_{N_k}$*. Then*

$$\mathrm{Fix} V^k_n = \cap_{i \in I^k_n} \mathrm{Fix} U_i \neq \emptyset.$$

Moreover, if n *is positive, then we have*

$$\cap_{j \in J^k_n} \mathrm{Fix} V^k_j = \cap_{i \in I^k_n} \mathrm{Fix} U_i \neq \emptyset$$

and the following assertions hold:

- ***(i)*** *If* $n \in c^{-1}_k(\{0\})$ *and, in addition,* $U_{-j^k_n}$ *is a cutter, or if* $\alpha_k(n) = 1$*, then the intermediate module* V^k_n*, generated by the GMSA procedure, is an* $\frac{\varepsilon}{2\Pi_{i=1}^n P^k_i}$*-strongly quasi-nonexpansive operator.*
- ***(ii)*** *If* $n \in c^{-1}_k(\{1\}) \cup c^{-1}_k(\{2\})$*, then the intermediate module* V^k_n*, generated by the GMSA procedure, is an* $\frac{\varepsilon}{2\Pi_{i=1}^n P^k_i}$*-strongly quasi-nonexpansive operator.*

We use the convention that for a non-positive integer n*, the empty product* $\Pi_{i=1}^n P^k_i = 1$*.*

Under the assumptions made in Subsection 2.4, we consider the following condition on the sequence $\left\{I^k_{N_K}\right\}_{k=0}^{\infty}$, which is known in the literature as an "admissible control" (see, for example, [26]):

For each $n \in \mathbb{N}$, there is an integer $M_n > 0$ such that

$$n \in \bigcup_{k=i}^{i+M_n-1} I^k_{N_k} \text{ for all } i \in \mathbb{N}. \tag{2.4}$$

In the case where for each $n \in \mathbb{N}$, M_n in the condition of (2.4) equals some $s > 0$, this condition is known in the literature as an "s-intermittent control" (see, for instance, [14, Definition 5.8.10], [6] and [46]).

The next example (see Example 4.3 in [5]) demonstrates a particular structure of the GMSA procedure, where (2.4) is satisfied. It should be noted that the GMSA studied here is a general modular algorithmic scheme which can cover algorithmic structures investigated earlier. In this respect, one can find examples of similar nature elsewhere, for instance, in [25].

Example 2.36. For each $i \in \mathbb{N}$, let $\max_i$ be a maximal natural number such that $i+1 = 2^{\max_i} p$, where $p \in \mathbb{N}$ is odd. Define a sequence $\{f_i\}_{i=0}^{\infty}$ of natural numbers by $f_i := \max_i$. For each $n \in \mathbb{N}$, Define $M_n := 2^{n+1}$. Then for each $n, i \in \mathbb{N}$, we have $n \in f(\{i, i+1, \ldots, i+M_n-1\})$ because $n = f_{2^n(2m+1)-1}$ for each $m \in \mathbb{N}$. The first 20 elements of $\{f_i\}_{i=0}^{\infty}$ are:

$$0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 4, 0, 1, 0, 2.$$

Now for each $k \in \mathbb{N}$, let $c_k : \{1\} \to \{0, 1, 2\}$ be a function. Define a sequence $\{N_k\}_{k=0}^{\infty}$ by $N_k := 1$ for each $k \in \mathbb{N}$. Clearly, $L_{N_k} \cap \mathbb{N} \setminus \{0\} = \{1\}$. Pick a bounded sequence $\left\{P_1^k\right\}_{k=0}^{\infty}$ of positive integers and a sequence $\left\{J_1^k\right\}_{k=0}^{\infty}$ of subsets of L_0 such that $-f_k \in J_1^k$, $\left|J_1^k\right| \leq P_1^k$ and $\left|J_1^k\right| = \begin{cases} 1, & \text{if } c_k(1) = 0, \\ P_1^k, & \text{if } c_k(1) = 1, \end{cases}$ for each $k \in \mathbb{N}$. Define the rest of parameters of GMSA procedure presented in Section 2.4. Clearly, then $I_{N_k}^k = -J_1^k$, where $-J_1^k = \left\{-j \middle| j \in J_1^k\right\}$, and hence for each $k \in \mathbb{N}$ and each $n \in \mathbb{N}$, Equation (2.4) is satisfied.

In the next example we recall the MSA procedure presented in [46], where the finite sequence of input operators $\{U_i'\}_{i=0}^{\mathcal{M}}$ was considered for some positive integer $\mathcal{M}$ and U_0' was set to be the identity Id. We show that it is a particular case of the GMSA procedure above. See also Example 4.4 in [5] in this connection.

Example 2.37 (MSA methods with an admissible control)**.** For a finite sequence of operators $\{U_i'\}_{i=0}^{\mathcal{M}}$ (where $\mathcal{M} \geq 0$ is an integer and $U_i : \mathcal{H} \to \mathcal{H}$ for each $i \in \{1, 2, \ldots, \mathcal{M}\}$) and a sequence $\{N_k'\}_{k=0}^{\infty}$ of positive integers, the parameters of the MSA procedure are defined similarly, where the sets $\left\{\left\{J_n'^k\right\}_{n \in \mathbb{N} \setminus \{0\} \cap L_{N_k'}}\right\}_{k \in \mathbb{N}}$ are the subsets of $\{-\mathcal{M}, -\mathcal{M}+1, \ldots, n-1\}$ and the intermediate modules $\left\{\left\{V_n'^k\right\}_{n=-\mathcal{M}}^{N_k}\right\}_{k \in \mathbb{N}}$ are considered (instead of $\left\{\left\{V_n'^k\right\}_{n=-\infty}^{N_k}\right\}_{k \in \mathbb{N}}$) and, consequently, the finite sequence of sets of indices $\left\{\left\{I_n'^{N_k'}\right\}_{n=-\mathcal{M}}^{N_k'}\right\}_{k \in \mathbb{N}}$ are of interest (instead of $\left\{\left\{I_n'^{N_k'}\right\}_{n=-\infty}^{N_k'}\right\}_{k \in \mathbb{N}}$). Let $\{f_n\}_{n=0}^{\infty}$ be a sequence defined in Example 2.36. Define a sequence $\{U_n\}_{n=0}^{\infty}$ by

$$U_n := \begin{cases} U_n', & \text{if } n \in \{0, 1, \ldots, \mathcal{M}\}, \\ Id, & \text{if } n \notin \{0, 1, \ldots, \mathcal{M}\}, \end{cases}$$

for each $n \in \mathbb{N}$, and define a sequence $\{N_k\}_{k=0}^{\infty}$ by

$$N_k := \begin{cases} N_k', & \text{if } k \in f^{-1}(\{0, 1, \ldots, \mathcal{M}\}), \\ N_k' + 1, & \text{if } k \notin f^{-1}(\{0, 1, \ldots, \mathcal{M}\}), \end{cases}$$

for each $k \in \mathbb{N}$. Define

$$J_n^k := \begin{cases} J_n'^k, & \text{if } n \in L_{N_k'}, \\ \{N_k', -f_k\}, & \text{if } n = N_k' + 1, \end{cases}$$

and

$$P_n^k := \begin{cases} P_n'^k, & \text{if } n \in L_{N_k'}, \\ 2, & \text{if } n = N_k' + 1, \end{cases}$$

for each $k \in \mathbb{N}$ and each positive integer $n \in L_{N_k}$. For each $k \in \mathbb{N}$, define a function $c_k : \mathbb{N} \setminus \{0\} \cap L_{N_k} \to \{0, 1, 2\}$ by

$$c_k(n) := \begin{cases} c_k'(n), & \text{if } n \in L_{N_k'}, \\ 2, & \text{if } n = N_k' + 1, \end{cases}$$

for each positive integer $n \in L_{N_k}$, and for each $k \notin f^{-1}(\{0,1,\dots,\mathcal{M}\})$, define a function $o^k_{N_k} : \{1,2\} \to J^k_{N_k}$ by

$$o^k_{N_k}(m) := \begin{cases} N'_k, & \text{if } m = 1, \\ -f_k, & \text{if } m = 2, \end{cases}$$

for each $m \in \{1,2\}$. Now set the rest of parameters of the GMSA procedure with respect to the operators $\{U_n\}_{n=0}^{\infty}$ to be the same as the parameters of the MSA procedure above. By (2.2).

$$I^k_{N_k} = \begin{cases} I'^k_{N'_k}, & \text{if } k \in f^{-1}(\{0,1,\dots,\mathcal{M}\}), \\ I'^k_{N'_k} \cup \{f_k\}, & \text{if } k \notin f^{-1}(\{0,1,\dots,\mathcal{M}\}). \end{cases} \tag{2.5}$$

By (2.3), the sequence $\{T'_k\}_{k=0}^{\infty}$ of the operators generated by the MSA procedure above satisfies $T'_k = V'^k_{N'_k} = V^k_{N_k} = T_k$ for each $k \in \mathbb{N}$. If, in addition, the sequence $\left\{I'^k_{N_k}\right\}_{k=0}^{\infty}$ satisfies Equation (2.4) with respect to the set $\{0,1,\dots,\mathcal{M}\}$, that is, for each $n \in \{0,1,\dots,\mathcal{M}\}$, Equation (2.4) holds, then it is easily verified by (2.5) that the sequence $\left\{I^k_{N_k}\right\}_{k=0}^{\infty}$ satisfies Equation (2.4) with respect to $\mathbb{N}$.

Remark 2.38. Note that if for each $n \in \mathbb{N}$, the operator U_n has a fixed point and is ρ-strongly nonexpansive for some $\rho \geq 0$, then, by Theorem 2.9, Remark 2.3 and Remark 2.6, $\mathrm{Fix}U_n$ is closed and convex for each $n \in \mathbb{N}$ and, consequently, $\cap_{n\in\mathbb{N}}\mathrm{Fix}U_n$ is closed and convex.

Throughout the rest of the paper we consider the settings of the GMSA procedure presented in this section.

3 The strong convergence properties of the GMSA procedure based methods

In this section we investigate the strong convergence and the bounded perturbation resilience of Algorithm 2.1 with respect to the operators $\{T_k\}_{k=0}^{\infty}$ generated by the GMSA procedure described in Section 2.4, and deduce important corollaries for the case where the input operators are certain relaxations of firmly nonexpansive ones.

3.1 General results concerning strongly quasi-nonexpansive input operators

We begin by proving the following key auxiliary lemma.

Lemma 3.1. *Let $\varepsilon \in (0,1]$, let $\{U_n\}_{n=0}^{\infty}$ be a sequence of 2^{-1}-strongly quasi-nonexpansive and approximately shrinking operators and suppose that the sequence of positive integers $\{N_k\}_{k=0}^{\infty}$ is bounded. Let $\{T_k\}_{k=0}^{\infty}$ be a sequence of operators generated by the GMSA procedure as in (2.3). Assume also that for each $k \in \mathbb{N}$ and each $n \in c_k^{-1}(\{0\})$, the operator $U_{j^k_n}$ is a cutter, or that $\alpha_k(n) = 1$. Further assume that there exists $r > 0$ such that for each $k \in \mathbb{N}$ and each $n \in c_k^{-1}(\{1\}) \cup c_k^{-1}(\{2\})$, we have $\cap_{i\in I^k_n}\mathrm{Fix}U_i \cap B(0,r) \neq \emptyset$, that is, the finitely many members of the sequence $\{U_n\}_{n=0}^{\infty}$ with a common fixed point in the ball $B(0,r)$ are used in the construction of the intermediate modules V^k_n. Then for any bounded sequence $\left\{x^k\right\}_{k=0}^{\infty} \subset \mathcal{H}$, the following implication holds:*

$$T_k\left(x^k\right) - x^k \to 0 \implies \lim_{k\to\infty} \max_{i\in I^k_{N_k}} d\left(x^k, \mathrm{Fix}U_i\right) = 0.$$

Proof. Since the sequence $\{N_k\}_{k=0}^{\infty}$ is a bounded sequence of positive integers, it attains only a finite number of values. As a result, passing to subsequence if necessary and using Lemma 2.33, we may assume without any loss of generality that $N_k = N$ for all $k \in \mathbb{N}$, where N is some positive integer. Clearly, it is enough to show the following assertion:

For each positive integer $n \in L_N$, we have the implication (which holds, in particular, for $n = N$)

$$V_n^k\left(x^k\right) - x^k \to 0 \implies \lim_{k\to\infty} \max_{i\in I_n^k} d\left(x^k, \mathrm{Fix}U_i\right) = 0 \tag{3.1}$$

for each bounded sequence $\left\{x^k\right\}_{k=0}^{\infty} \subset \mathcal{H}$. Let n be a positive integer in L_N. We prove that n satisfies implication (3.1) by induction on the set $\mathbb{N}\setminus\{0\}\cap L_N$. Assume that for each positive integer $j < n$ in L_N, this implication holds for each bounded sequence $\left\{x^k\right\}_{k=0}^{\infty} \subset \mathcal{H}$. Let $\left\{x^k\right\}_{k=0}^{\infty} \subset \mathcal{H}$ be bounded. Suppose that

$$V_n^k\left(x^k\right) - x^k \to 0. \tag{3.2}$$

By the definition of the GMSA procedure and since for each $k \in \mathbb{N}$, the image of the function c_k is contained in the finite set $\{0, 1, 2\}$, we consider, passing to subsequence if necessary and by Lemma 2.33, the following three cases:

Case 1: $c_k(n) = 0$ for each $k \in \mathbb{N}$. In this case (by (2.3)) we have for each $k \in \mathbb{N}$,

$$V_n^k = Id + \alpha_k(n)\left(V_{j_n^k}^k - Id\right) = Id + \alpha_k(n)\left(U_{-j_n^k} - Id\right), \tag{3.3}$$

where $\alpha_k(n) \in [\varepsilon, 2-\varepsilon]$ and $j_n^k \in J_n^k \subset L_0$. Clearly, $J_n^k = \left\{j_n^k\right\}$. By (2.2),

$$I_n^k = \cup_{j\in J_n^k} I_j^k = I_{j_n^k}^k = \left\{-j_n^k\right\} \tag{3.4}$$

for each $k \in \mathbb{N}$. By (3.3), for each $k \in \mathbb{N}$, we have

$$\left\|V_n^k\left(x^k\right) - x^k\right\| = \alpha_k(n)\left\|U_{-j_n^k}\left(x^k\right) - x^k\right\| \geq \varepsilon\left\|U_{-j_n^k}\left(x^k\right) - x^k\right\|$$

and, hence, by (3.2), $\left\|U_{-j_n^k}\left(x^k\right) - x^k\right\| \to 0$. Since $U_{-j_n^k}$ is approximately shrinking, it follows from (3.4) that

$$\lim_{k\to\infty} \max_{i\in I_n^k} d\left(x^k, \mathrm{Fix}U_i\right) = \lim_{k\to\infty} d\left(x^k, \mathrm{Fix}U_{-j_n^k}\right) = 0,$$

which proves (3.1).

For the next two cases, assume that $c_k(n) \neq 0$ for each $k \in \mathbb{N}$ and let $\left\{z^k\right\}_{k=0}^{\infty}$ be a sequence such that $z^k \in \cap_{i\in I_n^k}\mathrm{Fix}U_i \cap B(0,r)$ for each $k \in \mathbb{N}$ and $R > 0$ so that $\left\|x^k - z^k\right\| < R$ for all $k \in \mathbb{N}$ is satisfied.

Case 2: $c_k(n) = 1$ for each $k \in \mathbb{N}$. In this case (by (2.3)) $V_n^k = \sum_{j\in J_n^k} \omega_n^k(j) V_j^k$, where $J_n^k \subset L_{n-1}$ and the weights $\left\{\omega_n^k(j)\right\}_{j\in J_n^k}$ are in the interval $[\varepsilon, 1]$ for each $k \in \mathbb{N}$. Moreover, $I_n^k = \cup_{j\in J_n^k} I_j^k$ for each $k \in \mathbb{N}$ (by (2.2)). By Lemmata 2.34 and 2.35*(ii)*, V_j^k is $\frac{\varepsilon}{2\Pi_{i=1}^{j} P_i^k}$-strongly-quasi nonexpansive and hence $\frac{\varepsilon}{2\Pi_{i=1}^{n} P_i^k}$-strongly-quasi nonexpansive, for each $k \in \mathbb{N}$ and each $j \in J_n^k$, and by Lemma 2.35,

$$\cap_{j\in J_n^k}\mathrm{Fix}V_j^k = \cap_{i\in I_n^k}\mathrm{Fix}U_i \neq \emptyset$$

for each $k \in \mathbb{N}$. By Proposition 2.22 and (3.2), for each $k \in \mathbb{N}$, we have

$$\left\|V_n^k\left(x^k\right) - x^k\right\| \geq \frac{\varepsilon^2}{4R\Pi_{i=0}^{n} P_i^k} \sum_{j\in J_n^k} \left\|V_j^k\left(x^k\right) - x^k\right\|^2 \to 0,$$

which implies $\left\|V_j^k\left(x^k\right)-x^k\right\|^2 \to 0$ for each $j \in J_n^k$. Now, by the induction hypothesis and (2.2) (since U_{-j} is approximately shrinking for a non-positive j), $\lim_{k\to\infty}\max_{i\in I_j^k} d\left(x^k, \mathrm{Fix}U_i\right)=0$ for each $j \in J_n^k$. It follows then from (2.2) that

$$\lim_{k\to\infty}\max_{i\in I_n^k} d\left(x^k, \mathrm{Fix}U_i\right)=0.$$

Case 3: $c_k(n)=2$ for each $k\in\mathbb{N}$. In this case (by (2.3)) for each $k\in\mathbb{N}$, we have $V_n^k = V^k_{o_n^k\left(P_n^k\right)}\cdots V^k_{o_n^k(2)}V^k_{o_n^k(1)}$, where $o_n^k:\left\{1,2,\ldots,P_n^k\right\}\to J_n^k$ is an order function and $J_n^k\subset L_{n-1}$. Moreover, $I_n^k=\cup_{j\in J_n^k} I_j^k$ for each $k\in\mathbb{N}$. By Lemmata 2.34 and 2.35*(ii)*, for each $k\in\mathbb{N}$ and each $q=1,2,\ldots,P_n^k$, the operator $V^k_{o_n^k(q)}$ is $\frac{\varepsilon}{2\Pi_{i=1}^{o_n^k(q)}P_i^k}$-strongly-quasi nonexpansive and hence $\frac{\varepsilon}{2\Pi_{i=1}^{n}P_i^k}$-strongly-quasi nonexpansive. By Lemma 2.35,

$$\cap_{q=1}^{P_n^k}\mathrm{Fix}V^k_{o_n^k(q)}=\cap_{j\in J_n^k}\mathrm{Fix}V_j^k=\cap_{i\in I_n^k}\mathrm{Fix}U_i\neq\emptyset. \tag{3.5}$$

By Proposition 2.22 and (3.2), for each $k\in\mathbb{N}$, we have

$$\left\|V_n^k\left(x^k\right)-x^k\right\|\geq\frac{\varepsilon}{4R\Pi_{i=0}^{n}P_i^k}\sum_{q=1}^{P_n^k}\left\|S_q\left(x^k\right)-S_{q-1}\left(x^k\right)\right\|\to 0, \tag{3.6}$$

where $S_q=\prod_{i=1}^{q}V^k_{o_n^k(q)}=V^k_{o_n^k(q)}\cdots V_{o_n^k(2)}V_{o_n^k(1)}$ for each $q=1,2,\ldots,P_n^k$. Let $j\in J_n^k$ be arbitrary. Since the mapping o_n^k is onto J_n^k, there is $q_0\in\left\{1,2,\ldots,P_n^k\right\}$ such that $j=o_n^k\left(q_0\right)$. This, combined with (3.6), implies that for each $k\in\mathbb{N}$,

$$\begin{aligned}\left\|S_{q_0}\left(x^k\right)-S_{q_0-1}\left(x^k\right)\right\| &= \left\|V^k_{o_n^k(q_0)}S_{q_0-1}\left(x^k\right)-S_{q_0-1}\left(x^k\right)\right\|\\ &= \left\|V_j^kS_{q_0-1}\left(x^k\right)-S_{q_0-1}\left(x^k\right)\right\|\to 0.\end{aligned} \tag{3.7}$$

Since for each $k\in\mathbb{N}$, the operator S_{q_0} is quasi-nonexpansive (by (3.5), Lemma 2.34, Lemma 2.35*(ii)* and Theorem 2.10*(ii)*), we see, from (3.5), that

$$\left\|S_{q_0-1}\left(x^k\right)-z^k\right\|\leq\left\|x^k-z^k\right\|<R$$

for each $k\in\mathbb{N}$, which implies that the sequence $\left\{S_{q_0-1}\left(x^k\right)\right\}_{k=0}^{\infty}$ is bounded because the sequence $\left\{z^k\right\}_{k=0}^{\infty}$ is bounded. By (3.7), the induction hypothesis and (2.2) (since U_{-j} is approximately shrinking if j is non-positive), we obtain

$$\lim_{k\to\infty}\max_{i\in I_j^k} d\left(S_{q_0-1}\left(x^k\right),\mathrm{Fix}U_i\right)=0. \tag{3.8}$$

Now for each $k\in\mathbb{N}$, since $\cap_{i\in I_n^k}\mathrm{Fix}U_i\neq\emptyset$, by Remark 2.38, the properties of the metric projection, the triangle inequality, (3.6) and (3.8), we have

$$\begin{aligned}\max_{i\in I_j^k} d\left(x^k, \mathrm{Fix}U_i\right) &= \max_{i\in I_j^k}\left\|x^k - P_{\mathrm{Fix}U_i}\left(x^k\right)\right\| \le \max_{i\in I_j^k}\left\|x^k - P_{\mathrm{Fix}U_i}S_{q_0-1}\left(x^k\right)\right\| \\ &\le \max_{i\in I_j^k}\left\|S_{q_0-1}\left(x^k\right) - P_{\mathrm{Fix}U_i}S_{q_0-1}\left(x^k\right)\right\| + \left\|S_{q_0-1}\left(x^k\right) - x^k\right\| \\ &\le \max_{i\in I_j^k}\left\|S_{q_0-1}\left(x^k\right) - P_{\mathrm{Fix}U_i}S_{q_0-1}\left(x^k\right)\right\| + \sum_{q=1}^{q_0-1}\left\|S_q\left(x^k\right) - S_{q-1}\left(x^k\right)\right\| \\ &= \max_{i\in I_j^k} d\left(S_{q_0-1}\left(x^k\right), \mathrm{Fix}U_i\right) + \sum_{q=1}^{q_0-1}\left\|S_q\left(x^k\right) - S_{q-1}\left(x^k\right)\right\| \to 0.\end{aligned}$$

As a result, by (2.2) , $\lim_{k\to\infty}\max_{i\in I_n^k} d\left(x^k, \mathrm{Fix}U_i\right) = 0$. Lemma 3.1 is now proved. ■

The next theorem specifies conditions under which the sequence $\left\{x^k\right\}_{k=0}^{\infty}$, generated by Algorithm 2.1, is Fejér monotone, converges in the norm of $\mathcal{H}$ to a point $x \in \cap_{n=0}^{\infty}\mathrm{Fix}U_n$, and is bounded perturbations resilient with respect to $\cap_{n=0}^{\infty}\mathrm{Fix}U_n$.

Theorem 3.2. *Let $\{U_n\}_{n=0}^{\infty}$ be a sequence of 2^{-1}-strongly quasi-nonexpansive operators such that $\cap_{n=0}^{\infty}\mathrm{Fix}U_n \neq \emptyset$ and let $\rho \in [0,1]$. Assume that the sequence $\{N_k\}_{k=0}^{\infty}$ is bounded. Suppose further that for each $k \in \mathbb{N}$ and each $n \in c_k^{-1}(\{0\})$, in addition, $U_{j_n^k}$ is a cutter, or that $\alpha_k(n) = 1$. Let $\{T_k\}_{k=0}^{\infty}$ be a sequence of operators generated by the GMSA procedure as in (2.3), and assume that T_k is ρ-strongly quasi-nonexpansive operator for each $k \in \mathbb{N}$. Let $\{\lambda_k\}_{k=0}^{\infty} \subset [\varepsilon, 1+\rho-\varepsilon]$ (where $\varepsilon \in (0,1]$) be a sequence. Then the following statements hold:*

(i) *For each $x^0 \in \mathcal{H}$, the sequence $\left\{x^k\right\}_{k=0}^{\infty}$ generated by Algorithm 2.1 with respect to the sequences $\{T_k\}_{k=0}^{\infty}$ and $\{\lambda_k\}_{k=0}^{\infty}$ is strongly Fejér monotone with respect to $\cap_{n=0}^{\infty}\mathrm{Fix}U_n$. Namely,*

$$\left\|x^{k+1} - z\right\|^2 \le \left\|x^k - z\right\|^2 - \varepsilon\left(1+\rho-\varepsilon\right)^{-1}\left\|\left(x^{k+1} - x^k\right)\right\|^2$$

for each $k \in \mathbb{N}$, and as a result, $\left\|x^{k+1} - x^k\right\| \to 0$.

(ii) *If, in addition, the sequence $\left\{I_{N_k}^k\right\}_{k=0}^{\infty}$ satisfies Equation (2.4), the family $\{\mathrm{Fix}U_n\}_{n=0}^{\infty}$ is boundedly regular and the operator U_n is approximately shrinking for each $n \in \mathbb{N}$, then for each $x^0 \in \mathcal{H}$, the sequence $\left\{x^k\right\}_{k=0}^{\infty}$ generated by Algorithm 2.1 with respect to the sequences $\{T_k\}_{k=0}^{\infty}$ and $\{\lambda_k\}_{k=0}^{\infty}$ converges in the norm of $\mathcal{H}$ to a point $x \in \cap_{n=0}^{\infty}\mathrm{Fix}U_n$.*

(iii) *Under the assumptions of (ii), if for each $k \in \mathbb{N}$, the λ_k-relaxation of the operator T_k is nonexpansive, then Algorithm 2.1 with respect to the sequences $\{T_k\}_{k=0}^{\infty}$ and $\{\lambda_k\}_{k=0}^{\infty}$ is bounded perturbations resilient with respect to $\cap_{n=0}^{\infty}\mathrm{Fix}U_n$.*

Proof. For each $k \in \mathbb{N}$, denote by S_k the λ_k-relaxation of T_k.
(i) Let $x^0 \in \mathcal{H}$ and let $\left\{x^k\right\}_{k=0}^{\infty}$ be the sequence generated by Algorithm 2.1 with respect to the sequences $\{T_k\}_{k=0}^{\infty}$ and $\{\lambda_k\}_{k=0}^{\infty}$. Assume that $k \in \mathbb{N}$ and that $z \in \cap_{n=0}^{\infty}\mathrm{Fix}U_n$. By Corollary 2.12, the $2^{-1}(1+\rho)$-relaxation of T_k is a cutter, that is, is 1-strongly quasi-nonexpansive. Therefore,

its $2\lambda_k (1+\rho)^{-1}$-relaxation, which is the λ_k-relaxation of T_k, is $(1+\rho-\lambda_k)\lambda_k^{-1}$-strongly quasi-nonexpansive by Theorem 2.9. It follows that

$$\begin{aligned}\left\|x^{k+1}-z\right\|^2 &= \left\|S_k x^k - z\right\|^2 \leq \left\|x^k - z\right\|^2 - (1+\rho-\lambda_k)\lambda_k^{-1}\left\|S_k x^k - x^k\right\|^2 \\ &= \left\|x^k - z\right\|^2 - (1+\rho-\lambda_k)\lambda_k^{-1}\left\|x^{k+1} - x^k\right\|^2 \\ &\leq \left\|x^k - z\right\|^2 - \varepsilon(1+\rho-\varepsilon)^{-1}\left\|x^{k+1} - x^k\right\|^2. \end{aligned} \tag{3.9}$$

Since $\varepsilon(1+\rho-\varepsilon)^{-1} > 0$, we see from (3.9) that $\left\{x^k\right\}_{k=0}^{\infty}$ is strongly Fejér monotone with respect to $\cap_{n=0}^{\infty}\mathrm{Fix}U_n$. Since the real sequence $\left\{\left\|x^k - z\right\|^2\right\}_{k=0}^{\infty}$ is monotonically decreasing and bounded from below by 0, it converges and, by (3.9), $\left\|x^{k+1} - x^k\right\| \to 0$.

(ii) Let $x^0 \in \mathcal{H}$ and let $\left\{x^k\right\}_{k=0}^{\infty}$ be the sequence generated by Algorithm 2.1 with respect to the sequences $\{T_k\}_{k=0}^{\infty}$ and $\{\lambda_k\}_{k=0}^{\infty}$. Assume that the sequence $\left\{I_{N_K}^k\right\}_{k=0}^{\infty}$ satisfies Equation (2.4), that the family $\{\mathrm{Fix}U_n\}_{n=0}^{\infty}$ is boundedly regular and that the operator U_n is approximately shrinking for each $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ be a natural number. Clearly, there is a sequence $\{\ell_k\}_{k=0}^{\infty} \subset \{0,1,\ldots,M_n-1\}$ such that $n \in I_{N_{k+\ell_k}}^{k+\ell_k}$ for each $k \in \mathbb{N}$. We have

$$\lambda_{k+\ell_k}\left(T_{k+\ell_k}\left(x^{k+\ell_k}\right) - x^{k+\ell_k}\right) = S_{k+\ell_k}\left(x^{k+\ell_k}\right) - x^{k+\ell_k} = x^{k+\ell_k+1} - x^{k+\ell_k} \tag{3.10}$$

for each $k \in \mathbb{N}$. Since $\{\lambda_{k+\ell_k}\}_{k=0}^{\infty} \subset [\varepsilon, 1+\rho-\varepsilon]$, we obtain, from (3.10) and *(i)*, that

$$\lim_{k\to\infty}\left\|T_{k+\ell_k}\left(x^{k+\ell_k}\right) - x^{k+\ell_k}\right\| = 0.$$

The sequence $\left\{x^{k+\ell_k}\right\}_{k=0}^{\infty}$ is bounded because it is Fejér monotone with respect to $\cap_{n=0}^{\infty}\mathrm{Fix}U_n$ by *(i)*. Hence, by Lemmata 2.33 and 3.1,

$$\lim_{k\to\infty}\max_{i\in I_{N_{k+\ell_k}}^{k+\ell_k}} d\left(x^{k+\ell_k}, \mathrm{Fix}U_i\right) = 0. \tag{3.11}$$

Now for each $k \in \mathbb{N}$,

$$\left\|x^{k+\ell_k} - x^k\right\| = \left\|\sum_{i=0}^{\ell_k-1}\left(x^{k+i+1} - x^{k+i}\right)\right\| \leq \sum_{i=0}^{\ell_k-1}\left\|\left(x^{k+i+1} - x^{k+i}\right)\right\|. \tag{3.12}$$

Due to the finite number of summands in (3.12) we deduce from *(i)* that

$$\lim_{k\to\infty}\left\|x^{k+\ell_k} - x^k\right\| = 0. \tag{3.13}$$

Since $n \in I_{N_{k+\ell_k}}^{k+\ell_k}$ for each $k \in \mathbb{N}$ and $\mathrm{Fix}U_n \neq \emptyset$, we see, by (3.13), Remark 2.38, the definition of the metric projection and the triangle inequality, that for each $k \in \mathbb{N}$,

$$\begin{aligned} d\left(x^k, \mathrm{Fix}U_n\right) &= \left\|x^k - P_{\mathrm{Fix}U_n}\left(x^k\right)\right\| \leq \left\|x^k - P_{\mathrm{Fix}U_n}\left(x^{k+\ell_k}\right)\right\| \\ &\leq \left\|x^{k+\ell_k} - x^k\right\| + \left\|x^{k+\ell_k} - P_{\mathrm{Fix}U_n}\left(x^{k+\ell_k}\right)\right\| \\ &= \left\|x^{k+\ell_k} - x^k\right\| + d\left(x^{k+\ell_k}, \mathrm{Fix}U_n\right) \\ &\leq \left\|x^{k+\ell_k} - x^k\right\| + \max_{i\in I_{N_{k+\ell_k}}^{k+\ell_k}} d\left(x^{k+\ell_k}, \mathrm{Fix}U_i\right). \end{aligned} \tag{3.14}$$

Combining (3.14) with (3.11) and (3.13), we obtain that $\lim_{k\to\infty} d\left(x^k, \mathrm{Fix}U_n\right) = 0$. Since $n \in \mathbb{N}$ is arbitrary, the bounded regularity of the family $\{\mathrm{Fix}U_n\}_{n=0}^{\infty}$ implies that

$$\lim_{k\to\infty} d\left(x^k, \cap_{n=0}^{\infty}\mathrm{Fix}U_n\right) = 0. \tag{3.15}$$

It now follows, from (3.15), Remark 2.38 and Theorem 2.28, that the sequence $\left\{x^k\right\}_{k=0}^{\infty}$ converges in the norm of $\mathcal{H}$ to a point $x \in \cap_{n=0}^{\infty}\mathrm{Fix}U_n$.

(iii) Assume that the assumptions of *(ii)* hold and that the λ_k-relaxation of the operator T_k is nonexpansive. Define $C := \cap_{n=0}^{\infty}\mathrm{Fix}U_n$. Let $\{\beta_k\}_{k=0}^{\infty}$ be a sequence of positive real numbers such that $\sum_{k=0}^{\infty}\beta_k < \infty$ and let $\{v^k\}_{k=0}^{\infty}$ be a bounded sequence in $\mathcal{H}$. Suppose that $y^0 \in \mathcal{H}$ and consider the sequence $\left\{y^k\right\}_{k=0}^{\infty}$ generated by the iterative process $y^{k+1} := S_k(y^k + \beta_k v^k)$. Let $q \in \mathbb{N}$ and $y \in \mathcal{H}$ be arbitrary. For each $k \in \mathbb{N}$, Define $\gamma_k := \beta_k \left\|v^k\right\| \in [0, \infty)$ and $\ell_k := q + k$. Define $x^0 := y$. Clearly, $\sum_{k=0}^{\infty}\gamma_k < \infty$ and the sequence $\left\{I_{N_{\ell_k}}^{\ell_k}\right\}_{k=0}^{\infty}$ satisfies Equation (2.4). By 2.33 and *(ii)*, the sequence $\left\{x^k\right\}_{k=0}^{\infty}$ generated by Algorithm 2.1 with respect to the sequences $\{T_{\ell_k}\}_{k=0}^{\infty}$ and $\{\lambda_{\ell_k}\}_{k=0}^{\infty}$ converges in the norm of $\mathcal{H}$ to a point in C. Then, for each $k \in \mathbb{N}$, we have

$$x^{k+1} = S_{\ell_k}\left(x^k\right) = S_{\ell_k}\cdots S_{\ell_1}S_{\ell_0}\left(y\right) = S_{q+k}\cdots S_{q+1}S_q\left(y\right)$$

and hence the sequence $\{S_{q+k}\cdots S_{q+1}S_q\left(y\right)\}_{k=0}^{\infty}$ converges in the norm of $\mathcal{H}$ to an element of C for an arbitrary $y \in \mathcal{H}$. Since for each $k \in \mathbb{N}$, the operator S_{q+k} is nonexpansive, we obtain

$$\left\|y^{k+1} - S_k\left(y^k\right)\right\| = \left\|S_k(y^k + \beta_k v^k) - S_k\left(y^k\right)\right\| \le \beta_k\left\|v^k\right\| = \gamma_k.$$

By Lemma 2.35, Equation (2.4) and Remark 2.38, $C = \cap_{k=0}^{\infty}\mathrm{Fix}T_k = \cap_{k=0}^{\infty}\mathrm{Fix}S_k$. We now deduce from Theorem 2.31 along with Remark 2.38 that the sequence $\left\{y^k\right\}_{k=0}^{\infty}$ converges in the norm of $\mathcal{H}$ to an element of C as well, proving that the sequence $\left\{x^k\right\}_{k=0}^{\infty}$ is bounded perturbations resilient with respect to C. ■

Corollary 3.3. *Let $\{U_n\}_{n=0}^{\infty}$ be a sequence of 2^{-1}-strongly quasi-nonexpansive operators such that $\cap_{n=0}^{\infty}\mathrm{Fix}U_n \neq \emptyset$. Suppose further that for each $k \in \mathbb{N}$ and each $n \in c_k^{-1}\left(\{0\}\right)$, $U_{j_n^k}$ is a cutter, or that $\alpha_k\left(n\right) = 1$. Assume that the sequences $\{N_k\}_{k=0}^{\infty}$ and $\left\{\{P_n^k\}_{n\in\mathbb{N}\setminus\{0\}\cap L_{N_k}}\right\}_{k=0}^{\infty}$ are bounded. Define $\rho := \frac{\varepsilon}{2M^K}$, where $K := \max_{k\in\mathbb{N}} N_k$ and $M := \max_{k\in\mathbb{N}}\left\{P_n^k\right\}_{n\in\mathbb{N}\setminus\{0\}\cap L_{N_k}}$. Let $\{T_k\}_{k=0}^{\infty}$ be a sequence generated by the GMSA procedure as in (2.3), and let $\{\lambda_k\}_{k=0}^{\infty} \subset [1 + \rho - \varepsilon]$ (where $\varepsilon \in (0, 1]$) be a sequence. Then, given $x^0 \in \mathcal{H}$, the sequence $\left\{x^k\right\}_{k=0}^{\infty}$ generated by Algorithm 2.1 with respect to the sequences $\{T_k\}_{k=0}^{\infty}$ and $\{\lambda_k\}_{k=0}^{\infty}$ satisfies all the statements of Theorem 3.2.*

Proof. Clearly, $\rho \in [0, 1]$. By Lemma 2.35, the operator T_k is $\frac{\varepsilon}{2\Pi_{i=1}^{N_k}P_i^k}$-strongly quasi-nonexpansive and hence ρ- strongly nonexpansive for each $k \in \mathbb{N}$.

The result now follows from Theorem 3.2. ■

3.2 The method based on relaxations of firmly nonexpansive operators

In this subsection we consider the particular case of 2^{-1}-strongly quasi-nonexpansive input operators, wherein they are 2^{-1}-firmly quasi-nonexpansive. In this case we establish the nonexpansiveness of the λ_k-relaxations of the operators $\{T_k\}_{k=0}^{\infty}$ from Theorem 3.2*(iii)*, as the following lemma shows.

Lemma 3.4. *Let $\{T_k\}_{k=0}^{\infty}$ be a sequence of operators generated by the GMSA procedure as in (2.3), let $\rho \in [0,1]$ and assume that T_k is a ρ-firmly nonexpansive operator for each $k \in \mathbb{N}$. Let $\varepsilon \in (0,1]$ and let $\{\lambda_k\}_{k=0}^{\infty} \subset [\varepsilon, 1+\rho-\varepsilon]$ be a sequence. Then for each $k \in \mathbb{N}$, the λ_k-relaxation of T_k, $T_{k\lambda_k}$ is nonexpansive.*

Proof. Let $k \in \mathbb{N}$. By Corollary 2.18, the $2^{-1}(1+\rho)$-relaxation of T_k is firmly nonexpansive, i.e., it is 1-firmly nonexpansive. Therefore, its $2\lambda_k (1+\rho)^{-1}$-relaxation, which is $T_{k\lambda_k}$, is $(1+\rho-\lambda_k)\lambda_k^{-1}$-firmly nonexpansive, by Theorem 2.17. So, by Remark 2.14, it is nonexpansive. ■

If we assume that for each $n \in \mathbb{N}$, the input operator U_n of the GMSA procedure is 2^{-1}-firmly nonexpansive (or, equivalently, $4 \cdot 3^{-1}$-relaxed firmly nonexpansive by Theorem 2.17), then we can guarantee in Theorem 3.2*(iii)* that the output operators $\{T_k\}_{k=0}^{\infty}$ are ρ-firmly nonexpansive and hence ρ-strongly quasi-nonexpansive for some $\rho \in [0,1]$. To this end we need the following auxiliary lemma.

Lemma 3.5. *Let $\varepsilon \in (0,1]$, let $\{U_n\}_{n=0}^{\infty}$ be a sequence of 2^{-1}-firmly nonexpansive operators, let $k \in \mathbb{N}$ and let $n \in L_{N_k}$. Then the following assertions hold:*

(i) *If $n \in c_k^{-1}(\{0\})$ and $U_{-j_n^k}$ is, in addition, a firmly nonexpansive operator, or if $\alpha_k(n) = 1$, then the intermediate module V_n^k generated by the GMSA procedure is $\frac{\varepsilon}{2\Pi_{i=1}^n P_i^k}$-firmly nonexpansive and hence nonexpansive.*

(ii) *If $n \in c_k^{-1}(\{1\}) \cup c_k^{-1}(\{2\})$, then the intermediate module V_n^k generated by the GMSA procedure is $\frac{\varepsilon}{2\Pi_{i=1}^n P_i^k}$-firmly nonexpansive and hence nonexpansive.*

We use the convention that for a non-positive integer n, the empty product $\Pi_{i=1}^n P_i^k = 1$.

Proof. If n is non-positive then, by (2.3), $V_n^k = U_{-n}$ and hence V_n^k is 2^{-1}-firmly nonexpansive, which implies that it is $\frac{\varepsilon}{2\Pi_{i=1}^n P_i^k}$-firmly nonexpansive. The proof proceeds by induction on the set $\mathbb{N} \setminus \{0\} \cap L_{N_k}$. Let n be a positive integer in L_{N_k} and assume that the statement of the lemma holds for each positive integer $j < n$ in $\mathbb{N} \setminus \{0\} \cap L_{N_k}$. We show that it also holds for n is this case. We have the following three cases:

Case 1: $c_k(n) = 0$. By (2.3),

$$V_n^k = Id + \alpha_k(n)\left(V_{j_n^k}^k - Id\right) = Id + \alpha_k(n)\left(U_{-j_n^k} - Id\right),$$

where $\alpha_k(n) \in [\varepsilon, 2-\varepsilon]$ and $j_n^k \in J_n^k \subset L_0$. Clearly, $J_n^k = \left\{j_n^k\right\}$ and, by (2.2),

$$I_n^k = \cup_{j \in J_n^k} I_j^k = I_{j_n^k}^k = \left\{-j_n^k\right\}.$$

Now, if $U_{-j_n^k}$ is, additionally, firmly nonexpansive then, by Theorem 2.17, the operator V_n^k is $\frac{2-\alpha_k(n)}{\alpha_k(n)}$-firmly nonexpansive. The inequality

$$\frac{2-\alpha_k(n)}{\alpha_k(n)} \geq \frac{\varepsilon}{2-\varepsilon} > \frac{\varepsilon}{2} \geq \frac{\varepsilon}{2\Pi_{i=1}^n P_i^k}$$

yields that V_n^k is $\frac{\varepsilon}{2\Pi_{i=1}^n P_i^k}$-firmly nonexpansive. If $\alpha_k(n) = 1$, then $V_n^k = U_{-j_n^k}$ is 2^{-1}-firmly nonexpansive, which implies that it is $\frac{\varepsilon}{2\Pi_{i=1}^n P_i^k}$-firmly nonexpansive and hence nonexpansive.

Case 2: $c_k(n) = 1$. In this case (by (2.3)) $V_n^k = \sum_{j \in J_n^k} \omega_n^k(j) V_j^k$, where $J_n^k \subset L_{n-1}$ and the weights $\left\{\omega_n^k(j)\right\}_{j \in J_n^k}$ are in the interval $[\varepsilon, 1]$. By the the induction hypothesis and since U_n is 2^{-1}-firmly nonexpansive for each $n \in \mathbb{N}$, the operator V_j^k is $\frac{\varepsilon}{2\Pi_{i=1}^{j} P_i^k}$-firmly nonexpansive for each $j \in J_n^k$. By Corollary 2.20*(i)*, V_n^k is $\min_{j \in J_n^k}\left\{\frac{\varepsilon}{2\Pi_{i=1}^{j} P_i^k}\right\}$-firmly nonexpansive. Since $\min_{j \in J_n^k}\left\{\frac{\varepsilon}{2\Pi_{i=1}^{j} P_i^k}\right\} \geq \frac{\varepsilon}{2\Pi_{i=0}^{n} P_i^k}$, it follows that V_n^k is $\frac{\varepsilon}{2\Pi_{i=1}^{n} P_i^k}$-firmly nonexpansive and hence nonexpansive also in this case.

Case 3: $c_k(n) = 2$. In this case (by (2.3)) $V_n^k = V_{o_n^k(P_n^k)}^k \cdots V_{o_n^k(1)}^k$, where o_n^k is an order function having its values in J_n^k. Similarly to **Case 2**, by Corollary 2.20*(ii)*, V_n^k is $\frac{\min_{j \in J_n^k}\left\{\frac{\varepsilon}{2\Pi_{i=1}^{j} P_i^k}\right\}}{P_n^k}$-firmly nonexpansive, which implies that it is $\frac{\varepsilon}{2\Pi_{i=1}^{n} P_i^k}$-firmly nonexpansive and hence nonexpansive. ■

Corollary 3.6. *Let $\{U_n\}_{n=0}^{\infty}$ be a sequence of approximately shrinking and γ_n-relaxed firmly nonexpansive operators, where $\gamma_n \in \left(0, 4 \cdot 3^{-1}\right]$ for each $n \in \mathbb{N}$, such that the family $\{\mathrm{Fix} U_n\}_{n=0}^{\infty}$ is boundedly regular and $\cap_{n=0}^{\infty} \mathrm{Fix} U_n \neq \emptyset$. Suppose that for each $k \in \mathbb{N}$ and each $n \in c_k^{-1}(\{0\})$, in addition, $U_{j_n^k}$ is firmly nonexpansive, or that $\alpha_k(n) = 1$. Suppose that the sequence $\{N_k\}_{k=0}^{\infty}$ is bounded. Let $\{T_k\}_{k=0}^{\infty}$ be a sequence generated by the GMSA procedure as in (2.3). Assume that the sequence $\left\{I_{N_k}^k\right\}_{k=0}^{\infty}$ satisfies Equation (2.4). Let $x^0 \in \mathcal{H}$. Then:*

- ***(i)*** *If there exists $\rho \in [0,1]$ such that for each $k \in \mathbb{N}$, the operator T_k is ρ-firmly nonexpansive, then the sequence $\left\{x^k\right\}_{k=0}^{\infty}$ generated by Algorithm 2.1 with respect to the sequences $\{T_k\}_{k=0}^{\infty}$ and $\{\lambda_k\}_{k=0}^{\infty} \subset [1+\rho-\varepsilon]$ is strongly Fejér monotone with respect to $\cap_{n=0}^{\infty} \mathrm{Fix} U_n$, converges in the norm of $\mathcal{H}$ to a point $x \in \cap_{n=0}^{\infty} \mathrm{Fix} U_n$ and is bounded perturbations resilient with respect to $\cap_{n=0}^{\infty} \mathrm{Fix} U_n$.*

- ***(ii)*** *If the sequence $\left\{\left\{P_n^k\right\}_{n \in \mathbb{N} \setminus \{0\} \cap L_{N_k}}\right\}_{k=0}^{\infty}$ is also bounded, then there exists $\rho \in [0,1]$ such that the operator T_k is ρ-firmly nonexpansive for each $k \in \mathbb{N}$. Consequently, by (i) above, the sequence $\left\{x^k\right\}_{k=0}^{\infty}$ generated by Algorithm 2.1 with respect to the sequences $\{T_k\}_{k=0}^{\infty}$ and $\{\lambda_k\}_{k=0}^{\infty} \subset [1+\rho-\varepsilon]$ is strongly Fejér monotone with respect to $\cap_{n=0}^{\infty} \mathrm{Fix} U_n$, converges in the norm of $\mathcal{H}$ to a point $x \in \cap_{n=0}^{\infty} \mathrm{Fix} U_n$ and is bounded perturbations resilient with respect to $\cap_{n=0}^{\infty} \mathrm{Fix} U_n$.*

Proof. *(i)* By Theorem 2.17, for each $n \in \mathbb{N}$, the operator U_n is $(2-\gamma_n)\gamma_n$-firmly nonexpansive and hence 2^{-1}-firmly nonexpansive. By Remark 2.14, the operator U_n is 2^{-1}-strongly quasi-nonexpansive for each $n \in \mathbb{N}$ and the operator T_k is ρ-strongly quasi nonexpansive for each $k \in \mathbb{N}$. Moreover, by Theorem 2.16, for each $n \in c_k^{-1}(\{0\})$, $U_{j_n^k}$ is, additionally, a nonexpansive cutter, or $\alpha_k(n) = 1$. The result now follows from Theorem 3.2 along with Lemma 3.4.

(ii) Define $\rho := \frac{\varepsilon}{2M^K}$, where $K := \max_{k \in \mathbb{N}} N_k$ and $M := \max_{k \in \mathbb{N}}\left\{P_n^k\right\}_{n \in \mathbb{N} \setminus \{0\} \cap L_{N_k}}$. By Lemma 3.5, the operator T_k is $\frac{\varepsilon}{2\Pi_{i=1}^{n} P_i^k}$-firmly nonexpansive and hence ρ-firmly nonexpansive for each $k \in \mathbb{N}$. ■

Example 3.7. For each $n \in \mathbb{N}$, let C_n be an arbitrary finite-dimensional vector subspace of $\mathcal{H}$. Then by Proposition 2.24, the family $\{C_n\}_{n \in \mathbb{N}}$ is a boundedly regular family of nonempty, closed and convex subsets of $\mathcal{H}$ with the nonempty intersection which contains $\{0\}$, since each C_n is a finite-dimensional normed linear space and hence is locally compact. For each $n \in \mathbb{N}$, define $U_n := P_{C_n}$. By Examples (2.26) and (2.21), the operator U_n is approximately shrinking and firmly nonexpansive (that is, 1-firmly nonexpansive) for each $n \in \mathbb{N}$. We see that the family $\{\mathrm{Fix} U_n\}_{n=0}^{\infty}$

is boundedly regular and $\cap_{n=0}^{\infty}\text{Fix}U_n \neq \emptyset$. Choose the sequences $\{N_k\}_{k=0}^{\infty}$, $\left\{\{P_n^k\}_{n\in\mathbb{N}\setminus\{0\}\cap L_{N_k}}\right\}_{k=0}^{\infty}$ and $\left\{I_{N_k}^k\right\}_{k=0}^{\infty}$ and the rest of parameters of GMSA procedure as in Example 2.36. Clearly, the sequences $\{N_k\}_{k=0}^{\infty}$ and $\left\{\{P_n^k\}_{n\in\mathbb{N}\setminus\{0\}\cap L_{N_k}}\right\}_{k=0}^{\infty}$ are bounded and the sequence $\left\{I_{N_k}^k\right\}_{k=0}^{\infty}$ satisfies Equation (2.4).

These settings fit the framework of Corollary 3.6*(ii)* above and provide an example of a strongly convergent and perturbation resilient method, particularly, in case where $\mathcal{H}$ is an infinite-dimensional space.

4 The superiorization methodology

In many scientific or real-world problems that are modeled as constrained minimization problems, aiming for the (constrained) optimal solution may require high prices for the investment of time, energy, and resources.

The Superiorization Methodology (SM) addresses this by introducing low-cost perturbations into a feasibility-seeking algorithm. The resulting algorithm retains the convergence to a feasible point, while steering the iterates to a "superior" feasible point. In this context, a superior feasible point has a reduced (but not necessarily optimal) value of the associated objective function.

Additional information and references on the superiorization methodology are available in the papers listed in the bibliographic collection on the dedicated Webpage [17]. For recent works containing introductory material on the SM see, for example, [18], [36], [41] and [48].

We consider in the sequel the following algorithm which provides a general framework for superiorization methods with negative (sub)gradient perturbation.

Algorithm 4.1. *Given $y^0 \in \mathcal{H}$, $\varepsilon \in (0,1]$, $\phi : \mathcal{H} \to \mathbb{R}$ a convex and continuous function, a sequence $\{\mathcal{T}_k\}_{k=0}^{\infty}$ of operators, a sequence $\{M_k\}_{k=0}^{\infty}$ of positive integers and a family of positive real sequences $\left\{\{\beta_{k,n}\}_{n=1}^{M_k}\right\}_{k=0}^{\infty}$ such that $\sum_{k=0}^{\infty}\sum_{n=1}^{M_k}\beta_{k,n} < \infty$, the algorithm is defined by the recurrences*

$$y^{k+1} := \mathcal{T}_k\left(y^k + \sum_{n=1}^{M_k}\beta_{k,n}v^{k,n}\right)$$

wherein

$$v^{k,n+1} := \begin{cases} -\left\|s^{k,n}\right\|^{-1}s^{k,n}, & \text{if } 0 \notin \partial\phi\left(y^k + \sum_{i=1}^{n}\beta_{k,i}v^{k,i}\right), \\ 0, & \text{if } 0 \in \partial\phi\left(y^k + \sum_{i=1}^{n}\beta_{k,i}v^{k,i}\right), \end{cases} \tag{4.1}$$

for each $k \in \mathbb{N}$ and each $n = 0, 1, \ldots M_k - 1$, where $s^{k,n}$ is a selection of the subgradient $\partial\phi\left(y^k + \sum_{i=1}^{n}\beta_{k,i}v^{k,i}\right)$ (which exists by Theorem 2.32) for each $k \in \mathbb{N}$ and each $n = 0, 1, \ldots M_k - 1$ (recalling that, by definition, $\sum_{i=1}^{0}\beta_{k,i}v^{k,i} = 0$).

This formulation of the algorithm generalizes, in more than one way, [24, Algorithm 4.1] as will be clarified below.

We need the following auxiliary lemma to further investigate the behavior of Algorithm 4.1.

Lemma 4.2 ([3, Lemma 5.3])**.** *Let $\phi : \mathcal{H} \to \mathbb{R}$ be a convex and continuous real-valued objective function. For an arbitrary nonempty subset C of $\mathcal{H}$ and $y, z \in C$ such that $z \in \text{Argmin}_{x\in C}\phi(x)$ and $y \notin \text{Argmin}_{x\in C}\phi(x)$, there exist real numbers $r_1 > 0$ and $r_2 > 0$ so that for each $\overline{y} \in B(y, r_1)$ and $v \in \partial\phi(\overline{y})$, the following assertions are satisfied:*

(i) $0 \notin \partial\phi(\overline{y})$ *and for each* $\overline{z} \in B(z, r_2)$

$$\left\langle \|v\|^{-1} v, \overline{z} - \overline{y} \right\rangle < 0.$$

(ii) *We have*

$$\left\langle \|v\|^{-1} v, z - \overline{y} \right\rangle < -2^{-1} r_2.$$

(iii) *Let p be a nonnegative integer. Assume that $\{\alpha_n\}_{n=1}^{p}$ is a sequence of positive real numbers such that $\sum_{n=1}^{p} a_n < 2^{-1} r_1$ and $\{v^n\}_{n=1}^{p} \subset \mathcal{H} \backslash \{0\}$ is a sequence such that $v^n \in \partial\phi\left(\overline{y} - \sum_{i=1}^{n-1} \alpha_i \left\|v^i\right\|^{-1} v^i\right)$ for each $n = 1, 2, \ldots, p$. If, in addition, $\overline{y} \in B\left(y, 2^{-1} r_1\right)$, then*

$$\left\| \overline{y} - \sum_{n=1}^{p} \alpha_n \|v^n\|^{-1} v^n - z \right\|^2 \leq \|\overline{y} - z\|^2 - \sum_{n=1}^{p} (r_2 - \alpha_n)\, \alpha_n$$

(by definition $\sum_{n=1}^{0} \alpha_n \|v^n\|^{-1} v^n := \sum_{n=1}^{0} (r_2 - \alpha_n)\, \alpha_n := 0$).

The next theorem is "a theorem of alternatives" for the possible outputs of Algorithm 4.1. It generalizes Theorem 4.1 in [24] and Theorem 5.4 in [3] to the setting of Algorithm 4.1 which we apply in the next section to the "superiorized version of Algorithm 2.1".

Theorem 4.3. *Let $\phi : \mathcal{H} \to \mathbb{R}$ be a convex and continuous real valued objective function. Assume that $\{\mathcal{T}_k\}_{k=0}^{\infty}$ is a sequence of nonexpansive operators having a common fixed point, where $\varepsilon \in (0, 1]$. Let $y^0 \in \mathcal{H}$ and suppose that the sequence $\left\{y^k\right\}_{k=0}^{\infty}$, generated by Algorithm 4.1, converges in the norm of $\mathcal{H}$ to a point $y \in \cap_{k=0}^{\infty} \mathrm{Fix}\mathcal{T}_k$. Then exactly one of the following two alternatives holds:*

(i) $y \in \mathrm{Argmin}_{x \in \cap_{k=0}^{\infty} \mathrm{Fix}\mathcal{T}_k} \phi(x)$.

or

(ii) *$y \notin \mathrm{Argmin}_{x \in \cap_{k=0}^{\infty} \mathrm{Fix}\mathcal{T}_k} \phi(x)$ and there exists $k_0 \in \mathbb{N}$ such that $\left\{y^k\right\}_{k=k_0}^{\infty}$ is strictly Fejér monotone with respect to $\mathrm{Argmin}_{x \in \cap_{k=0}^{\infty} \mathrm{Fix}\mathcal{T}_k} \phi(x)$, that is, $\left\|y^{k+1} - z\right\|^2 < \left\|y^k - z\right\|^2$ for every $z \in \mathrm{Argmin}_{x \in \cap_{n=0}^{\infty} \mathrm{Fix}U_n} \phi(x)$ and for all natural $k \geq k_0$.*

Proof. Assume that $\left\{y^k\right\}_{k=0}^{\infty}$ converges in the norm of $\mathcal{H}$ to a point $y \notin \mathrm{Argmin}_{x \in \cap_{k=0}^{\infty} \mathrm{Fix}\mathcal{T}_k} \phi(x)$ and that $z \in \mathrm{Argmin}_{x \in \cap_{k=0}^{\infty} \mathrm{Fix}\mathcal{T}_k} \phi(x)$. By Lemma 4.2, since $y \in \cap_{k=0}^{\infty} \mathrm{Fix}\mathcal{T}_k$, there exist real numbers $r_1 > 0$ and $r_2 > 0$ such that each $\overline{y} \in B(y, r_1)$ and $v \in \partial\phi(\overline{y})$ satisfy its assertions. By using the strong convergence of $\left\{y^k\right\}_{k=0}^{\infty}$ to y and the convergence of the series $\sum_{k=0}^{\infty} \sum_{n=1}^{M_k} \beta_{k,n}$, choose $k_0 \in \mathbb{N}$ such that

$$y^k \in B\left(y, 2^{-1} r_1\right) \tag{4.2}$$

and

$$\sum_{n=1}^{M_k} \beta_{k,n} < \min\left\{2^{-1} r_1, r_2\right\} \tag{4.3}$$

for each integer $k \geq k_0$. This yields, for each $k \geq k_0$,

$$y^k + \sum_{i=1}^{n-1} \beta_{k,i} v^{k,i} \in B(y, r_1)$$

for each $n = 1, 2, \ldots, M_k$, and, consequently, by Lemma 4.2*(i)*,

$$0 \notin \partial\phi\left(y^k + \sum_{i=1}^{n-1} \beta_{k,i} v^{k,i}\right) \tag{4.4}$$

for each $n = 1, 2, \ldots, M_k$. Let $k \geq k_0$ be an integer. By (4.1) and (4.4),

$$v^{k,n} = -\left\|s^{k,n-1}\right\|^{-1} s^{k,n-1}, \tag{4.5}$$

where

$$s^{k,n-1} \in \partial\phi\left(y^k + \sum_{i=1}^{n-1} \beta_{k,i} v^{k,i}\right), \tag{4.6}$$

for each $n = 1, 2, \ldots, M_k$. Define $p := M_k$ and $\overline{y} := y^k$. For each $n = 1, 2, \ldots, p$, define $\alpha_n := \beta_{k,n} > 0$ and $v^n := s^{k,n-1}$. Then, by (4.2), (4.3), (4.4), (4.5) and (4.6), $\overline{y} \in B\left(y, 2^{-1}r_1\right)$, $\sum_{n=1}^{p} \alpha_n < 2^{-1}r_1$, $\{v^n\}_{n=1}^{p} \subset \mathcal{H} \backslash \{0\}$ and $v^n \in \partial\phi\left(\overline{y} - \sum_{i=1}^{n-1} \alpha_i \left\|v^i\right\|^{-1} v^i\right)$ for each $n = 1, 2, \ldots, p$.

Since the operator $\mathcal{T}_k$ is nonexpansive, we obtain from Lemma 4.2*(iii),* that

$$\begin{aligned}
\left\|y^{k+1} - z\right\|^2 &= \left\|\mathcal{T}_k\left(y^k + \sum_{n=1}^{M_k} \beta_{k,n} v^{k,n}\right) - z\right\|^2 \leq \left\|y^k + \sum_{n=1}^{M_k} \beta_{k,n} v^{k,n} - z\right\|^2 \\
&= \left\|\overline{y} - \sum_{n=1}^{p} \alpha_n \left\|v_n\right\|^{-1} v_n - z\right\|^2 \leq \left\|\overline{y} - z\right\|^2 - \sum_{n=1}^{p} \left(r_2 - \alpha_n\right) \alpha_n \\
&= \left\|y^k - z\right\|^2 - \sum_{n=1}^{M_k} \left(r_2 - \beta_{k,n}\right) \beta_{k,n}.
\end{aligned}$$

Since by (4.3), $\sum_{n=1}^{M_k} \beta_{k,n} < r_2$, we see that $\left\|y^{k+1} - z\right\| < \left\|y^{k+1} - z\right\|$ and the proof of the theorem is complete. ■

Remark 4.4. The fundamental Algorithm 4.1 is a "superiorization algorithm" meaning that it is not designed for reaching a constrained optimal point. Its intentional aim is "to do less than constrained optimization", namely, to enable to "improve" the feasibility-seeking algorithm by steering it to an asymptotic feasible point whose objective function value is smaller or equal to that of an asymptotic feasible point that would have been reached by the same feasibility-seeking algorithm under exact equal conditions without the perturbations. The theorem-of-alternatives (Theorem 4.3) is the best result that we can establish at this time. It does not rule out the possibility that Algorithm 4.3 will reach a constrained optimal point (see Remark 5.2 in the next section in this connection) but tells that if this does not happen then the generated sequence must be strictly Fejér monotone. It is probably possible to characterize when the superiorization algorithm can solve a constrained optimization problem but this will come at an unavoidable cost of imposing additional conditions. Such results may be derived from, or be developed further from, the work done in [27] and [30]. The leading motivation for splitting methods, as described in [30], is to break a complex task to more manageable sub-tasks. In this sense the superiorization can be viewed also as a splitting method. Its ties to the broad literature on splitting methods have not yet been explored.

5 Applications to superiorization and Dynamic String Averaging

This section presents some consequences of the strong convergence properties of our GMSA procedure for iterative methods based on the GMSA procedure, and for the superiorization methodology, as developed in the preceding sections.

We start by observing that under the assumptions of Theorem 3.2*(iii)*, where, in particular, $\{\lambda_k\}_{k=0}^{\infty} \subset [\varepsilon, 1+\rho-\varepsilon]$ for $\varepsilon \in (0,1]$ and $\rho \in (0,1]$, the sequence $\left\{y^k\right\}_{k=0}^{\infty}$, generated by Algorithm 4.1 with respect to the sequences $\{\lambda_k\}_{k=0}^{\infty}$ and $\{T_{k\lambda_k}\}_{k=0}^{\infty}$ (the λ_k-relaxations of operators $\{T_k\}_{k=0}^{\infty}$ generated by the GMSA procedure obtained from the sequence of given input 2^{-1}-strongly quasi-nonexpansive operators $\{U_n\}_{n=0}^{\infty}$ with nonempty intersection) converges in the norm of $\mathcal{H}$ to a point $y \in \cap_{n=0}^{\infty}\mathrm{Fix}U_n$.

Indeed, define a sequence $\{\beta_k\}_{k=0}^{\infty}$ of positive real numbers by $0 < \beta_k = \sum_{n=1}^{M_k} \beta_{k,n}$ for each $k \in \mathbb{N}$. Then $\sum_{k=0}^{\infty} \beta_k < \infty$ and we have, by defining the operators involved in Algorithm 4.1 as $\mathcal{T}_k := T_{k\lambda_k}$ for each $k \in \mathbb{N}$,

$$y^{k+1} = T_{k\lambda_k}\left(y^k + \sum_{n=1}^{M_k} \beta_{k,n} v^{k,n}\right) = T_{k\lambda_k}\left(y^k + \beta_k \sum_{n=1}^{M_k} \beta_{k,n}\beta_k^{-1} v^{k,n}\right), \tag{5.1}$$

for each $k \in \mathbb{N}$. It follows, by the triangle inequality and by (4.1), that

$$\left\|\sum_{n=1}^{M_k} \beta_{k,n}\beta_k^{-1} v^{k,n}\right\| \le \sum_{n=1}^{M_k} \beta_{k,n}\beta_k^{-1} = 1,$$

i.e., the real sequence $\left\{\sum_{n=1}^{M_k} \beta_{k,n}\beta_k^{-1} v^{k,n}\right\}_{k=0}^{\infty}$ is bounded by 1 in $\mathcal{H}$. By Remark 2.3 and Lemma 2.35, we obtain in this case that

$$\cap_{k=0}^{\infty}\mathrm{Fix}T_{k\lambda_k} = \cap_{k=0}^{\infty}\mathrm{Fix}T_k = \cap_{k=0}^{\infty} \cap_{n\in I_{N_k}^k} U_n = \cap_{n=0}^{\infty} U_n. \tag{5.2}$$

By using (5.1), Corollary 3.6, Lemma 3.4, Theorem 4.3 and (5.2), we obtain the following corollary.

Corollary 5.1. *Let $\phi : \mathcal{H} \to \mathbb{R}$ be a convex and continuous real-valued objective function and let $\{U_n\}_{n=0}^{\infty}$ be a sequence of approximately shrinking and γ_n-relaxed firmly nonexpansive operators, where $\gamma_n \in \left(0, 4\cdot 3^{-1}\right]$ for each $n \in \mathbb{N}$, such that the family $\{\mathrm{Fix}U_n\}_{n=0}^{\infty}$ is boundedly regular and $\cap_{n=0}^{\infty}\mathrm{Fix}U_n \neq \emptyset$. Suppose that the sequence $\{N_k\}_{k=0}^{\infty}$ is bounded and that for each $k \in \mathbb{N}$ and each $n \in c_k^{-1}(\{0\})$, in addition, the operator $U_{j_n^k}$ is firmly nonexpansive, or that $\alpha_k(n) = 1$. Let $\{T_k\}_{k=0}^{\infty}$ be a sequence of operators generated by the GMSA procedure as in (2.3), and assume that there exists $\rho \in [0,1]$ such that the operators T_k are ρ-firmly nonexpansive for all $k \in \mathbb{N}$ (by Corollary 3.6(ii), this holds in particular when the sequence $\left\{\{P_n^k\}_{n\in\mathbb{N}\setminus\{0\}\cap L_{N_k}}\right\}_{k=0}^{\infty}$ is also bounded). Let $\varepsilon \in (0,1]$ and let $\{\lambda_k\}_{k=0}^{\infty} \subset [1+\rho-\varepsilon]$ be a sequence. Assume that the sequence $\left\{I_{N_k}^k\right\}_{k=0}^{\infty}$ satisfies Equation (2.4). Then the sequence $\left\{y^k\right\}_{k=0}^{\infty}$, generated by Algorithm 4.1 with respect to the sequence $\{\lambda_k\}_{k=0}^{\infty}$ and $\{T_{k\lambda_k}\}_{k=0}^{\infty}$ (the λ_k-relaxations of operators $\{T_k\}_{k=0}^{\infty}$) converges in the norm of $\mathcal{H}$ to a point $y \in \cap_{n=0}^{\infty}\mathrm{Fix}U_n$ and exactly one of the following two alternatives holds:*

(i) $y \in \mathrm{Argmin}_{x\in\cap_{n=0}^{\infty}\mathrm{Fix}U_n}\phi(x)$

or

(ii) *$y \notin \operatorname{Argmin}_{x\in\cap_{n=0}^{\infty}\mathrm{Fix}U_n}\phi(x)$ and there exists $k_0 \in \mathbb{N}$ such that $\{y^k\}_{k=k_0}^{\infty}$ is strictly Fejér monotone with respect to $\operatorname{Argmin}_{x\in\cap_{n=0}^{\infty}\mathrm{Fix}U_n}\phi(x)$, that is, $\left\|y^{k+1}-z\right\|^2 < \left\|y^k-z\right\|^2$ for every $z \in \operatorname{Argmin}_{x\in\cap_{n=0}^{\infty}\mathrm{Fix}U_n}\phi(x)$ and for all natural $k \geq k_0$.*

Remark 5.2. Alternative 2 in Corollary 5.1 is not merely theoretical. Using the framework of Example 2.37, we can embed concrete examples given, for instance, in [4] into the setting of this alternative.

A strongly convergent Dynamic String-Averaging Projection (DSAP) algorithm, based on convex combinations and compositions of operators, was introduced, along with its bounded perturbation resilience, by Censor and Zaslavski in [23] for a family of nonempty, closed and convex sets $\{C_i\}_{i=1}^m$, where the considered operators were given by the metric projections $\{P_{C_i}\}_{i=1}^m$ in the consistent case, that is when $\cap_{i=1}^m C_i \neq \emptyset$. A superiorized version of this algorithm appears in [24], where its properties were studied.

String-averaging projection (SAP) methods form a general algorithmic framework introduced in [19]. Subsequently, they were developed in a variety of situations such as for convex feasibility with infinitely many sets [42], for incremental stochastic subgradient algorithms [31] and for proton computed tomography image reconstruction [45], to name but a few. See also [2], where perturbation resilience of such methods was further studied.

In the recent paper by Barshad and Censor [3] the General Dynamic String-Averaging (GDSA) iterative scheme was presented covering the case of empty intersection of fixed points sets of the given operators (that is, the inconsistent case), wherein a finite family of relaxed firmly nonexpansive operators was considered. Since the emptiness of this intersection was assumed there, the conditions which guarantee the strong convergence of the GDSA algorithm had to be assumed on the output operators produced by it.

Using the above GMSA and SM machinery, we expand the GDSA method in the consistent case to cover the case of infinitely many input operators in the next example, where the aforesaid conditions are assumed on these input operators.

Example 5.3 (The General Dynamic String-Averaging method in the consistent case)**.** We consider a sequence $\{U_n\}_{n=0}^{\infty}$ of approximately shrinking and γ_n-relaxed firmly nonexpansive operators having a common fixed point, where $U_n : \mathcal{H} \to \mathcal{H}$ and $\gamma_n \in \left(0, 4\cdot 3^{-1}\right]$ for each $n \in \mathbb{N}$, such that the family $\{\mathrm{Fix}U_n\}_{n=0}^{\infty}$ is boundedly regular. Let $\{q_k\}_{k=0}^{\infty}$ be a bounded sequence of positive integers, let $\{J_k\}_{k=0}^{\infty}$ be a sequence of finite subsets of $\mathbb{N}$, whose cardinalities are bounded by a positive integer p, and let $\{\Omega_k\}_{k=0}^{\infty}$ be a sequence of nonempty sets such that $\Omega_k \subset J_k^{\{1,2,\ldots,q_k\}}$ (that is, Ω_k is a finite subset of the set of functions from $\{1,2,\ldots,q_k\}$ to J_k) for each $k \in \mathbb{N}$. For each $k \in \mathbb{N}$ and each $t \in \Omega_k$, define $V_k[t] := U_{t(q_k)} \cdots U_{t(2)} U_{t(1)}$ and let $\omega_k : \Omega_k \to [\varepsilon, 1]$ (where ε is a positive number) be a function such that $\sum_{t\in\Omega_k}\omega_k(t) = 1$. For each $k \in \mathbb{N}$, define $T_{(\Omega_k,\omega_k)} := \sum_{t\in\Omega_k}\omega_k(t) V_k[t]$.

Define $q := \max\{q_k\}_{k=0}^{\infty}$ and

$$\rho := \min\left\{q^{-1}\inf_{n\in\mathbb{N}}(2-\gamma_n)\gamma_n, 1\right\} \leq 1.$$

For each $k \in \mathbb{N}$, we say that the set Ω_k is called "fit" if the sequence of sets $\{I_k\}_{k=0}^{\infty}$, defined by $I_k := \cup_{t\in\Omega_k}\operatorname{Im} t$, for each $k \in \mathbb{N}$, where $\operatorname{Im} t$, which denotes the image of the mapping t, satisfies Equation (2.4) for all $n \in \mathbb{N}$. For example, if $f(k) \in \cup_{t\in\Omega_k}\operatorname{Im} t$ for each $k \in \mathbb{N}$, where $\{f_k\}_{k=0}^{\infty}$ is the sequence defined in Example 2.36, then the set Ω_k is a fit for each $k \in \mathbb{N}$.

The operators $T_{(\Omega_k,\omega_k)}$ defined above are "string-averaging operators" as introduced in [19] and further studied in various forms and settings, see, for instance, [3], Example 5.21 in [10], [42] and

[44], to name but a few. In those and other papers, the index vector t is called a "string", the composite operator $V_k[t]$ is called "a string operator" and ω_k are called "weight functions".

To properly embed this in the framework of Section (3) we do as follows. Define a sequence $\{N_k\}_{k=0}^{\infty}$ of positive integers by $N_k := |\Omega_k| + 1$ for each $k \in \mathbb{N}$. Clearly, $\{N_k\}_{k=0}^{\infty}$ is bounded, since $N_k \leq p^q + 1$ for each $k \in \mathbb{N}$. Let $t_k : \{1, 2, \ldots, |\Omega_k|\} \to \Omega_k$ be a bijection for each $k \in \mathbb{N}$. Now, for each $k \in \mathbb{N}$ and each positive integer $n \in L_{N_k - 1}$, define $J_n^k := -\operatorname{Im} t_k(n)$, where $\operatorname{Im} t_k(n)$ is the image of the mapping $t_k(n) \in \Omega_k$, $c_k(n) := 2$, $P_n^k := q_k$ and $o_n^k := -t_k(n)$. Define also $J_{N_k}^k := \{1, 2, \ldots, |\Omega_k|\}$, $c_k(N_k) := 1$, and $P_{N_k}^k := |\Omega_k|$ for each $k \in \mathbb{N}$. Define $\omega_n^k := \omega_k(t_k(n))$ for each positive integer $n \in L_{N_k - 1}$.

By (2.2),

$$I_n^k = \cup_{j \in J_n^k} I_j^k = -J_n^k = \operatorname{Im} t_k(n) \tag{5.3}$$

for each $k \in \mathbb{N}$ and each positive integer $n \in L_{N_k - 1} = L_{|\Omega_k|}$. By (5.3) and (2.2), since t_k is a bijection for each $k \in \mathbb{N}$, we have

$$I_{N_k}^k = \cup_{j \in J_{N_k}^k} I_j^k = \cup_{j \in \{1,2,\ldots,|\Omega_k|\}} I_j^k = \cup_{j \in \{1,2,\ldots,|\Omega_k|\}} \operatorname{Im} t_k(j) = \cup_{t \in \Omega_k} \operatorname{Im} t$$

for each $k \in \mathbb{N}$. Applying now the GMSA procedure with respect to the sequence $\{U_n\}_{n=0}^{\infty}$ we have, for each $k \in \mathbb{N}$ and each positive $n \in L_{N_k - 1}$,

$$V_n^k = V_k[t_k(n)] \text{ and } V_{N_k}^k = T_{(\Omega_k, \omega_k)}. \tag{5.4}$$

By Corollary 2.20*(ii)*, for each $k \in \mathbb{N}$, the operator $V_k(t)$ is ρ-firmly nonexpansive for each $t \in \Omega_k$. Therefore, by 2.20, $T_{(\Omega_k, \omega_k)}$ is ρ-firmly nonexpansive. Thus, assuming that the set Ω_k is a fit for each $k \in \mathbb{N}$, we obtain the GDSA iterative scheme which fits the framework of Corollaries 3.6 and 5.1.

We use Example 2.37 to embed in this scheme the GDSA method in the consistent case for a given finite family $\{U'_n\}_{n=1}^{m}$ of input operators. By Examples 2.21 and 2.26, we can choose, in particular, the input operators to be metric projections and obtain the DSAP method with infinitely many input operators, which was described for a finite number of these operators in [23].

Acknowledgments. We gratefully acknowledge the enlightening and constructive comments of the three referees which helped us improve the paper. This work is supported by U.S. National Institutes of Health (NIH) Grant Number R01CA266467 and by the Cooperation Program in Cancer Research of the German Cancer Research Center (DKFZ) and Israel's Ministry of Innovation, Science and Technology (MOST).

Data availability

No data was used for the research described in the article.

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