NORMAL CONES AND SUBDIFFERENTIALS AT INFINITY FOR CONVEX ANALYSIS AND OPTIMIZATION

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Abstract. Motivated by recent developments, this paper further investigates normal cones and subdifferentials at infinity within the framework of convex analysis. We establish fundamental properties of these constructions and derive basic calculus rules. The obtained results extend and refine existing concepts in variational analysis and nonsmooth optimization, providing new insights into the asymptotic structure of functions and sets. Several applications to optimality conditions in convex optimization are also presented.

1. Introduction

Convex analysis plays a fundamental role in modern mathematics and its applications. As the mathematical foundation of convex optimization, it provides powerful tools for studying functions, sets, and mappings with convex structures. The theory offers elegant results such as separation theorems, subdifferential calculus, and duality principles, which form the backbone of many optimization algorithms. Beyond pure mathematics, convex analysis is widely used in economics, engineering, data science, and machine learning, where many problems can be formulated as convex optimization models [1, 2].

Sufficient conditions for the existence of solutions in nonsmooth optimization via asymptotic cones and generalized asymptotic functions were given in [4–6, 8–11, 14, 17, 18]. Recently, the concepts of normal cones at infinity for unbounded sets, together with limiting and singular subdifferentials at infinity for extended real-valued functions were used to investigate optimality conditions and the existence of error bounds in research articles [12, 15, 21, 22]. In [19], Clarke's tangent cones at infinity for unbounded sets, subgradients at infinity for extended real-valued functions, and necessary optimality conditions at infinity for optimization problems were studied. Anh and Hung [3] investigated properties of normal cones with respect to a set and developed calculus rules for subdifferentials relative to a set at infinity. Furthermore, the authors also obtained necessary optimality conditions at infinity, establish the compactness of the solution set, and verify the coercivity in optimization problems with unbounded feasible sets. Very recently, the authors in [13] introduced new concepts, namely directional normal cones at infinity for unbounded sets, along with limiting and singular subdifferentials at infinity in the direction for extended real-valued functions. Moreover, they

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applied them to obtain directional optimality conditions at infinity, analyzing the coercivity, proving the compactness of the global solution set, and examining properties such as weak sharp minima and error bounds at infinity.

However, if we consider the case where the set and the function are convex, what can be said about the normal cone at infinity of the set and the subdifferential at infinity of the function? To the best of our knowledge, the definitions of the normal cone of a convex set at infinity and the subdifferential of a convex function at infinity have not been studied previously.

In this paper, motivated by [1,2,13,15], we study the normal cone of a convex set at infinity and the subdifferential of a convex function at infinity and their applications. The rest of the paper is organized as follows. Section 1 and Section 2 present introduction, notations and preliminaries. In Section 3, we introduce and study the normal cones to convex sets at infinity. In Section 4, we study the subdifferentials of convex functions at infinity. In Section 5, we investigate the directional differentiability of convex functions at infinity. In Section 6, we obtain subgradients of supremum functions at infinity. Finally, Section 7 is devoted to applications in convex optimization problems.

2. Preliminaries

2.1. **Notation and definitions.** In this paper we deal with the Euclidean space $\mathbb{R}^n$ equipped with the usual scalar product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\|\cdot\|$. We denote by $\mathbb{B}_r(x)$ the closed ball centered at x with radius r; when x is the origin of $\mathbb{R}^n$ we write $\mathbb{B}_r$ instead of $\mathbb{B}_r(x)$, and when $r = 1$ we write $\mathbb{B}$ instead of $\mathbb{B}_1$. For any two different points x and x' in $\mathbb{R}^n$, the *open line segment* joining x and x' is the set

$$(x, x') \quad := \quad \{(1-t)x + tx' \mid 0 < t < 1\}.$$

We will adopt the convention that $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$, and that $\lambda_1 + \lambda_2 = +\infty$ if either λ_1 or λ_2 is $+\infty$ (even if the other is $-\infty$). It also is expedient to set

$$\begin{aligned}
(\pm\infty) + \lambda \quad &= \quad \lambda + (\pm\infty) \;=\; \pm\infty \quad \text{for any real } \lambda,\\
\lambda \cdot (\pm\infty) \quad &= \quad (\pm\infty) \cdot \lambda \;=\; \pm\infty \quad \text{for all } \lambda > 0,\\
\lambda \cdot (\pm\infty) \quad &= \quad (\pm\infty) \cdot \lambda \;=\; \mp\infty \quad \text{for all } \lambda < 0.
\end{aligned}$$

The order on $\mathbb{R} \cup \{\pm\infty\}$ is extended in the natural way to

$$-\infty < r < +\infty$$

for all $r \in \mathbb{R}$.

Let C be a nonempty subset of $\mathbb{R}^n$. The closure, interior, boundary, and convex hull of the set $C \subset \mathbb{R}^n$ will be written as $\mathrm{cl}C$, $\mathrm{int}C$, ∂C, and $\mathrm{co}C$, respectively.

Let $\mathbb{R}_+ := [0, +\infty)$ and $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$. For an extended-real-valued function $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, we denote its *effective domain* and *epigraph* by, respectively,

$$\begin{aligned} \text{dom} f &:= \{x \in \mathbb{R}^n \mid f(x) < +\infty\}, \\ \text{epi} f &:= \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \le r\}. \end{aligned}$$

We call f a *proper* function if $f(x) < \infty$ for at least one $x \in \mathbb{R}^n$, or in other words, if dom$f$ is a nonempty set. The function f is said to be lower semi-continuous if for each $x \in \mathbb{R}^n$ the inequality $\liminf_{x' \to x} f(x') \ge f(x)$ holds. The function f is upper semicontinuous if for every sequence $x \in \mathbb{R}^n$ the inequality $\limsup_{x' \to x} f(x') \le f(x)$ holds.

Definition 2.1. Given a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ (not necessarily convex), its Fenchel conjugate $f^* : \mathbb{R}^n \to [-\infty, \infty]$ is

$$f^*(v) := \sup\{\langle v, x\rangle - f(x) | x \in \mathbb{R}^n\}, v \in \mathbb{R}^n. \tag{1}$$

Note that the case where $f^*(v) = \infty$ is not excluded in (1), while we have $f^*(v) > -\infty$ for all $v \in \mathbb{R}^n$ if dom$f \neq \emptyset$.

Definition 2.2. The biconjugate of a proper function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ as the conjugate of f^*, i.e.,$f^{**}(x) := (f^*)^*(x)$.

It follows from the definition that $f^{**} : \mathbb{R}^n \to [-\infty, \infty]$ is given by

$$f^{**}(x) = \sup\{\langle x, v\rangle - f^*(v) | v \in \mathbb{R}^n, x \in \mathbb{R}^n\}.$$

Definition 2.3. Given a nonempty subset Ω of $\mathbb{R}^n$, the support function associated with Ω is defined by

$$\sigma_\Omega(v) := \sup\{\langle v, x\rangle \,| x \in \Omega\}, v \in \mathbb{R}^n.$$

2.2. **Normal cones to convex sets and subdifferentials of convex functions.**

Definition 2.4. (See [2]) A subset Ω of $\mathbb{R}^n$ is convex if $\lambda a + (1 - \lambda) b \in \Omega$ for all $a, b \in \Omega$ and $0 \le \lambda \le 1$.

Definition 2.5. (See [2]) Let $f : \Omega \to \overline{\mathbb{R}}$ be an extended-real-valued function defined on a convex set $\Omega \subset \mathbb{R}^n$. Then f is convex on Ω if

$$f(\lambda x + (1 - \lambda) y) \le \lambda f(x) + (1 - \lambda) f(y) \text{ for all } x, y \in \Omega \text{ and } 0 < \lambda < 1.$$

Definition 2.6. ([2]) Let Ω_1 and Ω_2 be nonempty subsets of $\mathbb{R}n$. It is said that Ω_1 and Ω_2 can be separated by a hyperplane if there exists a nonzero element $v \in \mathbb{R}^n$ such that we have

$$\sup\ \{\langle v, x\rangle | x \in \Omega_1\} \le\ \inf\ \{\langle v, x\rangle | x \in \Omega_2\}. \tag{2}$$

If it holds in addition that

$$\inf\ \{\langle v, x\rangle | x \in \Omega_1\} \le\ \sup\ \{\langle v, x\rangle | x \in \Omega_2\}. \tag{3}$$

which means that there exist $x_1 \in \Omega_1$ and $x_2 \in \Omega_2$ with $\langle v, x_1\rangle < \langle v, x_2\rangle$, then the sets Ω_1 and Ω_2 can be properly separated by the hyperplane.

Definition 2.7. ([2]) Two nonempty sets Ω_1 and Ω_2 form an extremal system if for any $\varepsilon > 0$ there exists $a \in \mathbb{R}^n$ such that $\| a \| < \varepsilon$ and

$$(\Omega_1 - a) \cap \Omega_2 = \emptyset.$$

Lemma 2.8. ([2]) *Let Ω_1 and Ω_2 be two nonempty convex subsets of $\mathbb{R}^n$. Then Ω_1 and Ω_2 form an extremal system if and only if the sets Ω_1 and Ω_2 can be separated by a hyperplane .*

Definition 2.9. ([2]) Let $\Omega \subset \mathbb{R}^n$ be a nonempty convex set with $\bar{x} \in \Omega$. The normal cone to Ω at $\bar{x}$ is

$$N(\bar{x}; \Omega) := \{v \in \mathbb{R}^n | \langle v, x - \bar{x} \rangle \leq 0 \text{ for all } x \in \Omega\}.$$

By convention, we let $N(\bar{x}; \Omega) = \emptyset$ if $\bar{x} \notin \Omega$.

Lemma 2.10. ([2]) *Let Ω_1 and Ω_2 be convex subsets of $\mathbb{R}^n$ with $\bar{x} \in \Omega_1 \cap \Omega_2$. Suppose that the qualification condition is satisfied:*

$$N(\bar{x}; \Omega_1) \cap [-N(\bar{x}; \Omega_2)] = \{0\}.$$

Then

$$N(\bar{x}; \Omega_1 \cap \Omega_2) = N(\bar{x}; \Omega_1) + N(\bar{x}; \Omega_2).$$

Definition 2.11. ([2])Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a convex function, and let $\bar{x} \in \text{dom } f$.

(i) The subdifferential of the function f at $\bar{x}$ is defined by

$$\partial f(\bar{x}) := \{v \in \mathbb{R}^n | (v, -1) \in N((\bar{x}, f(\bar{x})); \text{ epi } f)\}.$$

(ii) The singular subdifferential of the function f at $\bar{x}$ is defined by

$$\partial^\infty f(\bar{x}) := \{v \in \mathbb{R}^n | (v, 0) \in N((\bar{x}, f(\bar{x})); \text{ epi } f)\}.$$

If $x \notin \text{dom} f$, we put $\partial f(x) := \emptyset$, and $\partial^\infty f(x) := \emptyset$.

Lemma 2.12 ([2],Fermat rule)**.** *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a convex function with $\bar{x} \in \text{dom } f$. Then f has alocal/global minimum at $\bar{x}$ if and only if $0 \in \partial f(\bar{x})$.*

Lemma 2.13. ([2]) *Let $f_i : \mathbb{R}^n \to \overline{\mathbb{R}}$ for $i = 1, 2$ be convex functions and let $\bar{x} \in \text{dom } f_1 \cap \text{dom } f_2$. Impose the singular subdifferential qualification condition*

$$\partial^\infty f_1(\bar{x}) \cap [-\partial^\infty f_2(\bar{x})] = \{0\}.$$

Then we have the subdifferential sum rule

(i), $\partial(f_1 + f_2)(\bar{x}) = \partial f_1(\bar{x}) + \partial f_2(\bar{x})$.
(ii), $\partial^\infty(f_1 + f_2)(\bar{x}) = \partial^\infty f_1(\bar{x}) + \partial^\infty f_2(\bar{x})$.

Remark 2.14. [2] Given a convex set $\Omega \subset \mathbb{R}^n$. The indicator function $\delta(\cdot; \Omega) \colon \mathbb{R}^n \to \overline{\mathbb{R}}$ of Ω is defined by

$$\delta(x; \Omega) := \begin{cases} 0 & \text{if } x \in \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

For any $x \in \Omega$, it holds that $\partial \delta(x; \Omega) = \partial^\infty \delta(x; \Omega) = N(x; \Omega)$.

The following Ekeland variational principle is an useful tool in establishing our main results.

Lemma 2.15 (see [7, Theorem 1]). *Let $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper lower semi-continuous function and bounded from below. Let $\epsilon > 0$ and $u \in \mathbb{R}^n$ be satisfied*

$$f(u) \quad \leq \quad \inf_{x\in\mathbb{R}^n} f(x) + \epsilon.$$

Then, for any $\lambda > 0$ there exists $v \in \mathbb{R}^n$ such that

(i) $f(v) \leq f(u)$,
(ii) $\|v - u\| \leq \lambda$, *and*
(iii) $f(v) \leq f(x) + \dfrac{\epsilon}{\lambda}\|x - v\|$ *for all* $x \in \mathbb{R}^n$.

3. Normal cones to convex sets at infinity

In this section, we introduce and study the notion of normal cones to convex sets at infinity and derive some fundamental rules of normal cone calculus.

Let Ω be a convex subset of $\mathbb{R}^n$. Suppose that Ω is unbounded. In what follows, the notation $x \xrightarrow{\Omega} \infty$ means that $x \in \Omega$ and $||x|| \to \infty$.

Definition 3.1. The normal cone to the set Ω at infinity is defined by

$$N(\infty;\Omega) := \{v \in \mathbb{R}^n | \langle v, x' - x\rangle \leq 0 \text{ such that } x \xrightarrow{\Omega} \infty \text{ for all } x' \in \Omega\}.$$

Remark 3.2. $N(\infty;\Omega)$ is nontrivial if and only if $\partial\Omega$ is unbounded. Furthermore, by definition, it is easy to see that

$$N(\infty;\Omega) = \bigcup_{i=1}^{n} N(\infty_{\{i\}};\Omega),$$

where

$$N(\infty_{\{i\}};\Omega) := \{v \in \mathbb{R}^n | \langle v, x' - x\rangle \leq 0 \text{ such that } x \xrightarrow{\Omega} \infty \text{ with } |x_i| \to \infty \text{ for all } x' \in \Omega\}$$

for $i = 1, \cdots, n$.

Next, we establish several basic properties of the normal cone at infinity.

Proposition 3.3. *The set $N(\infty;\Omega)$ is a closed convex cone that contains the origin.*

Proof. Fix $v_i \in N(\infty;\Omega)$ and $\lambda_i \geq 0$ for $i = 1, 2$. By definition, one has $0 \in N(\infty;\Omega)$ and $\langle v_i, x' - x\rangle \leq 0$ such that $x \xrightarrow{\Omega} \infty$ for all $x' \in \Omega$ and $i = 1, 2$. This implies that

$$\langle \lambda_1 v_1 + \lambda_2 v_2, x' - x\rangle \leq 0 \text{ such that } x \xrightarrow{\Omega} \infty \text{ whenever } x' \in \Omega.$$

Thus $\lambda_1 v_1 + \lambda_2 v_2 \in N(\infty;\Omega)$, and hence $N(\infty;\Omega)$ is a convex cone. To check its closedness fix a sequence $\{v_k\} \subset N(\infty;\Omega)$ that converges to v. Then passing to the limit in the inequalities

$$\langle v_k, x' - x\rangle \leq 0 \text{ such that } x \xrightarrow{\Omega} \infty \text{ for all } x' \in \Omega \text{ as } k \to \infty$$

tells us that $\langle v, x' - x\rangle \leq 0$ such that $x \xrightarrow{\Omega} \infty$ whenever $x \in \Omega$, and so $v \in N(\infty;\Omega)$. The proof is complete. □

Proposition 3.4. *Let Ω_1 and Ω_2 be nonempty, convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Suppose that Ω_1 and Ω_2 are unbounded. Then, we have*

$$N(\infty;\Omega_1)\times N(\infty;\Omega_2)=N(\infty;\Omega_1\times\Omega_2).$$

Proof. Obviously, we have

$$N(\infty;\Omega_1)\times N(\infty;\Omega_2)\subset N(\infty;\Omega_1\times\Omega_2).$$

We next prove that

$$N(\infty;\Omega_1\times\Omega_2)\subset N(\infty;\Omega_1)\times N(\infty;\Omega_2).$$

Fix $(v_1,v_2)\in N(\infty;\Omega_1\times\Omega_2)$ and get that

$$\begin{aligned}&\langle (v_1,v_2),(x_1',x_2')-(x_1,x_2)\rangle=\langle v_1,x_1'-x_1\rangle+\langle v_2,x_2'-x_2\rangle\le 0 \text{ such that } x_1\xrightarrow{\Omega_1}\infty, x_2\xrightarrow{\Omega_2}\infty\\ &\text{whenever } (x_1',x_2')\in\Omega_1\times\Omega_2.\end{aligned}\tag{4}$$

Putting $x_2'=x_2$ in (4) gives us

$$\langle v_1,x_1'-x_1\rangle\le 0 \text{ such that } x_1\xrightarrow{\Omega_1}\infty \text{ for all } x_1'\in\Omega_1.$$

meaning that $v_1\in N(\infty;\Omega_1)$. Similarly, taking $x_1'=x_1$ in (4) gives us

$$\langle v_2,x_2'-x_2\rangle\le 0 \text{ such that } x_2\xrightarrow{\Omega_2}\infty \text{ for all } x_2'\in\Omega_2.$$

meaning that $v_2\in N(\infty;\Omega_2)$. Therefore,

$$N(\infty;\Omega_1\times\Omega_2)\subset N(\infty;\Omega_1)\times N(\infty;\Omega_2).$$

□

Proposition 3.5. *Let Ω_1 and Ω_2 be convex subsets of $\mathbb{R}^n$. Suppose that Ω_1 and Ω_2 are unbounded. Then, we have*

$$N(\infty;\Omega_1+\Omega_2)=N(\infty;\Omega_1)\cap N(\infty;\Omega_2).$$

Proof. Obviously, we have

$$N(\infty;\Omega_1)\cap N(\infty;\Omega_2)\subset N(\infty;\Omega_1+\Omega_2).$$

We will prove that

$$N(\infty;\Omega_1+\Omega_2)\subset N(\infty;\Omega_1)+N(\infty;\Omega_2).$$

Fix $v\in N(\infty;\Omega_1+\Omega_2)$ and get by definition that

$$\langle v,x_1'+x_2'-(x_1+x_2)\rangle\le 0 \text{ such that } x_1\xrightarrow{\Omega_1}\infty, x_2\xrightarrow{\Omega_2}\infty \text{ whenever } x_1'\in\Omega_1, x_2'\in\Omega_2.\tag{5}$$

Putting $x_1'=x_1$ in (5), one has

$$\langle v,x_2'-x_2\rangle\le 0 \text{ suc that } x_2\xrightarrow{\Omega_2}\infty \text{ for all } x_2'\in\Omega_2.$$

Therefore $v\in N(\infty;\Omega_2)$. Putting $x_2'=x_2$ in (5), one has

$$\langle v,x_1'-x_1\rangle\le 0 \text{ suc that } x_1\xrightarrow{\Omega_1}\infty \text{ for all } x_1'\in\Omega_1.$$

Thus $v\in N(\infty;\Omega_1)$. This implies that $v\in N(\infty;\Omega_1)\cap N(\infty;\Omega_2)$. □

Proposition 3.6. *Let Ω_1 and Ω_2 be convex sets in $\mathbb{R}^n$. Suppose that Ω_1 and Ω_2 are unbounded. Then the following assertions are equivalent:*

(i) $\{\Omega_1, \Omega_2\}$ *forms an extremal system.*

(ii) *The sets Ω_1 and Ω_2 can be separated by a hyperplane.*

(iii) $N(\infty; \Omega_1) \cap (-N(\infty; \Omega_2)) \neq \{0\}$.

Proof. The equivalence of (i) and (ii) follows from Lemma 2.8. Suppose that (ii) is satisfied. By definition, there exits $v \neq 0$ such that

$$\langle v, x' \rangle \leq \langle v, y \rangle \text{ whenever } x' \in \Omega_1, y \in \Omega_2.$$

Since Ω_2 is unbounded, there exists $x \in \Omega_2$ such that $x \to \infty$ and

$$\langle v, x' \rangle \leq \langle v, x \rangle,$$

and thus $\langle v, x' - x \rangle \leq 0$ for all $x' \in \Omega_1$. Since Ω_1 is unbounded, there exists $x \in \Omega_1$ such that $x \to \infty$ and $\langle v, x' - x \rangle \leq 0$ for all $x' \in \Omega_1$, i.e., $v \in N(\infty; \Omega_1)$. Similarly, we can verify the inclusion $-v \in N(\infty; \Omega_2)$ and hence arrive at (iii). Suppose further that (iii) is satisfied, and let $0 \neq v \in N(\infty; \Omega) \cap (-N(\infty; \Omega_2))$. Then (2) follows directly from the definition of the normal cone at infinity. Indeed, for any $x' \in \Omega_1$ and $y' \in \Omega_2$, we have

$$\langle v, x' - x \rangle \leq 0 \text{ and } \langle -v, y' - x \rangle \leq 0 \text{ such that } x \in \Omega_1 \cap \Omega_2, x \to \infty,$$

which imply that $\langle v, x' - x \rangle \leq 0 \leq \langle v, y' - x \rangle$, and hence $\langle v, x' \rangle \leq \langle v, y' \rangle$. Therefore, we obtain (ii) and complete the proof of the proposition. □

Proposition 3.7. *Let Ω_1 and Ω_2 be closed convex subsets of $\mathbb{R}^n$. Suppose that Ω_1 and Ω_2 are unbounded. Then for any $v \in N(\infty; \Omega_1 \cap \Omega_2)$ there exist $\lambda \geq 0$ and $v_i \in N(\infty; \Omega_i), i = 1, 2$, such that*

$$\lambda v = v_1 + v_2, (\lambda, v_1) \neq (0, 0). \tag{6}$$

Proof. Fixing $v \in N(\infty; \Omega_1 \cap \Omega_2)$, we get by the normal cone definition that

$$\langle v, x' - x \rangle \leq 0 \text{ such that } x \xrightarrow{\Omega_1 \cap \Omega_2} \infty \text{ for all } x' \in \Omega_1 \cap \Omega_2.$$

Consider the two convex sets $\Theta_1 = \Omega_1 \times [0, \infty)$ and $\Theta_2 = \{(x', \lambda) \in \mathbb{R}^n \times \mathbb{R} \mid x' \in \Omega_2, \lambda \leq \langle v, x' - x \rangle\}$. It is easy to see that Θ_1 and Θ_2 are unbounded. These convex sets form extremal system since we obviously have $(x, 0) \in \Theta_1 \cap \Theta_2$ and

$$\Theta_1 \cap (\Theta_2 - (0, a)) = \emptyset \text{ whenever } a > 0.$$

Then applying Lemma 2.8 to these convex sets in $\mathbb{R}^{n+1}$ gives us a nonzero pair $(w, \gamma) \in \mathbb{R}^n \times \mathbb{R}$ such that

$$\langle w, x' \rangle + \gamma \lambda_1 \leq \langle w, y' \rangle + \gamma \lambda_2 \text{ for all } (x', \lambda_1) \in \Theta_1, (y', \lambda_2) \in \Theta_2. \tag{7}$$

It is easy to see that $\gamma < 0$. Indeed, assuming $\gamma > 0$ and taking into account that $(x, k) \in \Theta_1$ when $k > 0$ and $(x, 0) \in \Theta_2$, we get

$$\langle w, x \rangle + k\gamma < \langle w, x \rangle,$$

which leads to a contradiction. There are two possible cases to consider.

Case 1: $\gamma = 0$. In this case we have $w \neq 0$ and

$$\langle w, x' \rangle \leq \langle w, y' \rangle \text{ for all } x' \in \Omega_1, y' \in \Omega_2.$$

From Proposition 3.6, one has $w \in N(\infty; \Omega_1) \cap (-N(\infty; \Omega_2))$, therefore (6) holds for $\lambda = 0, v_1 = w$, and $v_2 = -w$.

Case 2: $\gamma < 0$. In this case we let $\mu := -\gamma > 0$ and deduce from (7), by taking into account that $(x', 0) \in \Theta_1$ if $x' \in \Omega_1$ and that $(x, 0) \in \Theta_2$, the inequality

$$\langle w, x' \rangle \leq \langle w, x \rangle \text{ such that } x \xrightarrow{\Omega_1 \cap \Omega_2} \infty \text{ for all } x' \in \Omega_1.$$

This implies therefore that $w \in N(\infty; \Omega_1)$ and so $\dfrac{w}{\mu} \in N(\infty; \Omega_1)$. Moreover, we get from (7), due to $(x, 0) \in \Theta_1$ and $(y', \alpha) \in \Theta_2$ for all $y' \in \Omega_2$ with $\alpha := \langle v, y' - x \rangle$, that

$$\langle w, x \rangle \leq \langle w, y' \rangle + \gamma \langle v, y' - x \rangle \text{ whenever } y' \in \Omega_2.$$

Dividing both sides therein by γ, we arrive at the relationship

$$\langle \frac{w}{\gamma}, x \rangle \leq \langle \frac{w}{\gamma}, y' \rangle + \langle v, y' - x \rangle \text{ whenever } y' \in \Omega_2.$$

This implies that

$$\left\langle \frac{w}{\gamma} + v, y' - x \right\rangle \leq 0 \text{ such that } x \xrightarrow{\Omega_1 \cap \Omega_2} \infty \text{ for all } y' \in \Omega_2,$$

and thus $\dfrac{w}{\gamma} + v = -\dfrac{w}{\mu} + v \in N(\infty; \Omega_2)$. Letting $v_1 := \dfrac{w}{\mu} \in N(\infty; \Omega_1)$ and $v_2 := -\dfrac{w}{\mu} + v \in N(\infty; \Omega_2)$ gives us $v = v_1 + v_2$, and thus (6) holds with $\lambda = 1$. □

Proposition 3.8. *Let Ω_1 and Ω_2 be closed convex subsets of $\mathbb{R}^n$ satisfying the qualification condition at infinity*

$$N(\infty; \Omega_1) \cap [-N(\infty; \Omega_2)] = \{0\}. \tag{8}$$

Then

$$N(\infty; \Omega_1 \cap \Omega_2) = N(\infty; \Omega_1) + N(\infty; \Omega_2).$$

Proof. We now prove that

$$N(\infty; \Omega_1 \cap \Omega_2) \subset N(\infty; \Omega_1) + N(\infty; \Omega_2).$$

For any $v \in N(\infty; \Omega_1 \cap \Omega_2)$, we find by Proposition 3.7 a number $\lambda \geq 0$ and element $v_i \in N(\infty; \Omega_i)$ as $i = 1, 2$ such that the conditions in (6) hold. Assuming that $\lambda = 0$ in (6) gives us that $0 \neq v_1 = -v_2 \in N(\infty; \Omega_1) \cap [-N(\infty; \Omega_2)]$, which contradicts (8). Thus $\lambda > 0$ and

$$v = \frac{v_1}{\lambda} + \frac{v_2}{\lambda} \in N(\infty; \Omega_1) + N(\infty; \Omega_2).$$

We next prove that

$$N(\infty; \Omega_1) + N(\infty; \Omega_2) \subset N(\infty; \Omega_1 \cap \Omega_2).$$

Indeed, fix any $v \in N(\infty;\Omega_1)+N(\infty;\Omega_2)$ with $v = v_1+v_2$, where $v_i \in N(\infty;\Omega_i)$ for $i = 1, 2$. For any $x' \in \Omega_1 \cap \Omega_2$, one has

$$\langle v, x'-x\rangle = \langle v_1+v_2, x'-x\rangle = \langle v_1, x'-x\rangle + \langle v_2, x'-x\rangle \le 0 \text{ such that } x \xrightarrow{\Omega_1\cap\Omega_2} \infty.$$

This implies that $x \in N(\infty;\Omega_1 \cap \Omega_2)$. □

4. Subdifferentials of convex functions at infinity

This section presents the notions of subdifferentials at infinity for extended-real-valued convex functions and then describes some of their fundamental properties.

Let $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ be an extended-real-valued convex functions. Consider the projection

$$\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n (x, y) \mapsto x.$$

By definition, we have

$$\pi(\text{epi} f) = \{x \in \mathbb{R}^n \mid f(x) < \infty\} = \text{dom} f.$$

To avoid triviality, we will assume that f is proper at infinity in the sense that the set $\text{dom} f$ is unbounded. In what follows, the notation $x \xrightarrow{\text{dom} f} \infty$ means that $x \in$ dom f and $x \to \infty$.

Definition 4.1. An element $v \in \mathbb{R}^n$ is called a subgradient of f at infinity if

$$\langle v, x'-x\rangle \le f(x') - f(x) \;\text{ such that } x \xrightarrow{\text{dom} f} \infty \text{ for all } x' \in \mathbb{R}^n.$$

The set of all the subgradients of f at infinity is called the subdifferential of the function at infinity and is denoted by $\partial f(\infty)$.

The following proposition provides a relationship between the subdifferential of f at infinity and the normal cone to the epigraph of f at infinity.

Proposition 4.2. *The following relation holds*

$$\partial f(\infty) = \{v \in \mathbb{R}^n \mid (v,-1) \in N((x, f(x)); \text{epi } f) \textit{ such that } x \xrightarrow{\text{dom} f} \infty\}.$$

Proof. Firstly, we will prove that

$$\partial f(\infty) \subset \{v \in \mathbb{R}^n \mid (v,-1) \in N((x, f(x)); \text{epi } f) \text{ such that } x \xrightarrow{\text{dom} f} \infty\}.$$

Fix $v \in \partial f(\infty)$ and $(x', \lambda) \in$ epi f. Since $\lambda \ge f(x')$, by Definition 4.1, we have

$$\begin{aligned} \langle (v,-1),(x',\lambda)-(x,f(x))\rangle &= \langle v, x'-x\rangle - (\lambda - f(x)) \\ &\le \langle v, x'-x\rangle - (f(x') - f(x)) \le 0, \end{aligned}$$

such that $x \xrightarrow{\text{dom} f} \infty$. This readily implies that $(v,-1) \in N((x, f(x)); \text{epi } f)$ with $x \xrightarrow{\text{dom} f} \infty$.

We next prove that

$$\{v \in \mathbb{R}^n \mid (v,-1) \in N((x, f(x)); \text{epi } f) \text{ such that } x \xrightarrow{\text{dom} f} \infty\} \subset \partial f(\infty).$$

Fix $v \in \mathbb{R}^n$ with $(v, -1) \in N((x, f(x)); \text{epi } f)$ such that $x \xrightarrow{\text{dom}f} \infty$. For any $x' \in \text{dom } f$, we have $(x', f(x')) \in \text{epi } f$. Thus

$$\langle v, x' - x\rangle - (f(x') - f(x)) = \langle (v, -1), (x', f(x')) - (x, f(x))\rangle \leq 0,$$

such that $x \xrightarrow{\text{dom}f} \infty$, which shows that $v \in \partial f(\infty)$. □

Definition 4.3. The singular subdifferential of f at infinity is defined

$$\partial^\infty f(\infty) := \{v \in \mathbb{R}^n \mid (v, 0) \in N((x, f(x)); \text{epi } f) \text{ such that } x \xrightarrow{\text{dom}f} \infty\}.$$

Proposition 4.4. *The following relation holds*

$$\partial^\infty f(\infty) = N(\infty; \text{dom } f).$$

Proof. Firstly, we will prove that

$$\partial^\infty f(\infty) \subset N(\infty; \text{dom } f).$$

Fix any $v \in \partial^\infty f(\infty)$ with $x' \in \text{dom } f$ and observe that $(x', f(x')) \in \text{epi } f$. Then using Definition 4.3 and Definition 3.1 gives us

$$\langle v, x' - x\rangle = \langle v, x' - x\rangle + 0.(f(x') - f(x)) \leq 0,$$

such that $x \xrightarrow{\text{dom}f} \infty$, which shows that $v \in N(\infty; \text{dom } f)$.

We next prove that

$$N(\infty; \text{dom } f) \subset \partial^\infty f(\infty).$$

Suppose that $v \in N(\infty; \text{dom } f)$ and fix any $(x', \lambda) \in \text{epi } f$. Then we have $f(x') \leq \lambda$, and so $x' \in \text{dom } f$. Thus

$$\langle v, x' - x\rangle + 0.(\lambda - f(x)) = \langle v, x' - x\rangle \leq 0,$$

such that $x \xrightarrow{\text{dom}f} \infty$, which implies that $(v, 0) \in N((x, f(x)); \text{epi } f)$ with $x \xrightarrow{\text{dom}f} \infty$. Therefore, $v \in \partial^\infty f(\infty)$. □

Remark 4.5. From Remark 2.14, one has $\partial\delta(\infty; \Omega) = \partial^\infty\delta(\infty; \Omega) = N(\infty; \Omega)$.

Proposition 4.6 (sum rule)**.** *Let $f_1, f_2 \colon \mathbb{R}^n \to \overline{\mathbb{R}}$ be convex functions satisfying the (singular subdifferential) qualification condition at infinity*

$$\partial^\infty f_1(\infty) \cap \big(-\partial^\infty f_2(\infty)\big) = \{0\}. \tag{9}$$

Then we have $\partial(f_1 + f_2)(\infty) = \partial f_1(\infty) + \partial f_2(\infty)$.

Proof. We will show that

$$\partial(f_1 + f_2)(\infty) \subset \partial f_1(\infty) + \partial f_2(\infty).$$

Take any $v \in \partial(f_1 + f_2)(\infty)$, and define two convex sets

$$\Omega_1 = \{(x', \lambda_1, \lambda_2) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} | \lambda_1 \geq f_1(x')\}.$$

$$\Omega_1 = \{(x', \lambda_1, \lambda_2) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} | \lambda_2 \geq f_1(x')\}.$$

Let us first verify the inclusion

$$(v,-1,-1)\in N((x,f_1(x),f_2(x));\Omega_1\cap\Omega_2) \text{ with } x \xrightarrow{\text{dom}f_1\cap\,\text{dom}f_2} \infty. \tag{10}$$

For any $(x',\lambda_1,\lambda_2)\in\Omega_1\cap\Omega_2$, we have $\lambda_1\ge f_1(x')$ and $\lambda_2\ge f_2(x')$. Then

$$\begin{aligned}&\langle v,x'-x\rangle+(-1)(\lambda_1-f_1(x))+(-1)(\lambda_2-f_2(x))\\ &\le\langle v,x'-x\rangle-(f_1(x')+f_2(x')-f_1(x)-f_2(x))\le 0,\end{aligned}$$

where the last estimate holds due to $v\in\partial(f_1+f_2)(\infty)$. Thus (10) is satisfied.

To proceed with employing the normal cone intersection rule in (10), let us first check that the assumed qualification condition (9) yields

$$N((x,f_1(x),f_2(x));\Omega_1)\cap(-N((x,f_1(x),f_2(x));\Omega_2))=\{0\}, \tag{11}$$

which is the qualification condition (8) needed for the application of Proposition 3.8 in the setting of (11). Indeed, we have from $\Omega_1=\text{epi}f_1\times\mathbb{R}$ that

$$N((x,f_1(x),f_2(x));\Omega_1)=N((x,f_1(x));\text{epi}f_1)\times\{0\}$$

and observe similarly the representation

$$N((x,f_1(x),f_2(x));\Omega_2)=\{(v,0,\gamma)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}|(v,\gamma)\in N((x,f_2(x));\text{epi}f_2)\}.$$

Futher, fix any element

$$(v,\gamma_1,\gamma_2)\in N((x,f_1(x),f_2(x));\Omega_1)\cap(-N((x,f_1(x),f_2(x));\Omega_2)).$$

Then $\gamma_1=\gamma_2=0$, and hence

$$(v,0)\in N((x,f_1(x));\text{epi}f_1) \text{ and } (-v,0)\in N((x,f_2(x));\text{epi}f_2).$$

It implies by the definition of the singular subdifferential that

$$v\in\partial^\infty f_1(\infty)\cap\big(-\partial^\infty f_2(\infty)\big)=\{0\}.$$

It deduces $(v,\gamma_1,\gamma_2)=(0,0,0)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}$. Applying Proposition 3.8 in (10) shows that

$$N((x,f_1(x),f_2(x));\Omega_1\cap\Omega_2)=N((x,f_1(x),f_2(x));\Omega_1)+N((x,f_1(x),f_2(x));\Omega_2)$$

which implies in turn that

$$(v,-1,-1)=(v_1,-\gamma_1,0)+(v_2,0,-\gamma_2),$$

where $(v_1,-\gamma_1)\in N((x,f_1(x));\ \text{epi}\ f_1)$ and $(v_2,-\gamma_2)\in N((x,f_2(x));\ \text{epi}\ f_2)$. Then we get $\gamma_1=\gamma_2=1$ and $v=v_1+v_2$, where $v_i\in\partial f_i(\infty)$ for $i=1,2$. This verifies the inclusion $\partial(f_1+f_2)(\infty)\subset\partial f_1(\infty)+\partial f_2(\infty)$.

We next show that

$$\partial f_1(\infty)+\partial f_2(\infty)\subset\partial(f_1+f_2)(\infty).$$

Indeed, any subgradient $v \in \partial f_1(\infty) + \partial f_2(\infty)$ can be represented as $v = v_1 + v_2$ where $v_i \in \partial f_i(\infty)$ for $i = 1, 2$. Then we have

$$\begin{aligned}\langle v, x' - x\rangle &= \langle v_1, x' - x\rangle + \langle v_2, x' - x\rangle \\ &\le f_1(x') - f_1(x) + f_2(x') - f_2(x) \\ &= (f_1 + f_1)(x') - (f_1 + f_2)(x)\end{aligned}$$

such that $x \xrightarrow{\text{dom}f} \infty$ for all $x' \in \mathbb{R}^n$ which readily implies that $v \in \partial(f_1 + f_2)(\infty)$ and therefore

$$\partial(f_1 + f_2)(\infty) = \partial f_1(\infty) + \partial f_2(\infty).$$

□

Definition 4.7. We say that f is Lipschitz at infinity if there exist $L > 0$ and $R > 0$ such that

$$|f(x) - f(y)| \le L\|x - y\| \ \ \forall x, y \in \mathbb{R}^n \setminus \mathbb{B}_R.$$

Remark 4.8. From Corollary 1.62 in [2], we can deduce that if $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is convex function, then f is Lipschitz at infinity.

Proposition 4.9. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be convex. Then $\partial f(\infty)$ is nonempty and compact.*

Proof. Assume that f is Lipschitz at infinity. Then there exist $L > 0$ and $R > 0$ such that

$$|f(x) - f(y)| \le L\|x - y\| \ \ \forall x, y \in \mathbb{R}^n \setminus \mathbb{B}_R.$$

Thus for any $v \in \partial f(\infty)$, one has

$$\langle v, h - x\rangle \le f(h) - f(x) \text{ such that } x \xrightarrow{\text{dom}f} \infty \text{ for all } h \in \mathbb{R}^n.$$

Combining this with Lipschitz property of f at infinity gives us

$$\langle v, h - x\rangle \le L\|h - x\| \text{ such that } x \xrightarrow{\text{dom}f} \infty \text{ for all } h \in \mathbb{R}^n \setminus \mathbb{B}_R,$$

which yields that

$$\langle v, u + x - x\rangle \le L\|u + x - x\| \text{ such that } x \xrightarrow{\text{dom}f} \infty \text{ for all } u \in \mathbb{R}^n \setminus \mathbb{B}_R,$$

or

$$\langle v, u\rangle \le L\|u\| \text{ for all } u \in \mathbb{R}^n \setminus \mathbb{B}_R,$$

Take $u := v$, we arrive at the conclusion $||v|| \le L$. This implies that $\partial f(\infty)$ is bounded. The closedness of $\partial f(\infty)$ follows from Definition 4.1. Therefore, one has $\partial f(\infty)$ is compact.

We next prove that $\partial f(\infty) \neq \emptyset$. We can deduce Corollary 2.54 in [2] that $N(\infty; \text{epi}f) \neq \{0\}$. Take such an element $(0, 0) \neq (v, -\alpha) \in N(\infty; \text{epi}f)$ and get by Definition 3.1 that

$$\langle v, x' - x\rangle - \alpha\,(\lambda - f(x)) \text{ such that } x \xrightarrow{\text{dom}f} \infty \text{ for all } (x, \lambda) \in \text{epi}f.$$

Letting $x' := x$ and $\lambda = f(x) + 1$ gives us $\alpha \ge 0$. If $\alpha = 0$, then $0 \neq v \in \partial^\infty f(\infty) = \{0\}$, which it contradicts Proposition 3.16 in [2]. Thus $\alpha > 0$ and $\left(\frac{v}{\alpha}, -1\right) \in N(\infty; \text{epi}f)$. Hence $\frac{v}{\alpha} \in \partial f(\infty)$, which implies that $\partial f(\infty) \neq \emptyset$. □

5. Directional differentiability of convex functions at infinity

This section presents the notions of directional derivative at infinity for extended-real-valued convex functions and then describes some of their fundamental properties. Let $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ be an extended-real-valued functions. Consider the projection $\pi : \mathbb{R}^n \times \mathbb{R}, (x, y) \mapsto x$. By definition, then $\pi(\text{epi} f) = \{x \in \mathbb{R}^n \mid f(x) < \infty\} = \text{dom} f$. To avoid triviality, we will assume that f is proper at infinity in the sense that the set $\text{dom} f$ is unbounded. In what follows, the notation $x \xrightarrow{\text{dom} f} \infty$ means that $x \in$ dom f and $x \to \infty$.

Definition 5.1. Let $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ be an extended-real-valued functions. The directional derivative of the function f at infinity with respect to $d \in \mathbb{R}^n$ is defined by

$$f'(\infty; d) := \lim_{x \xrightarrow{\text{dom} f} \infty, t \to 0^+} \frac{f(x + td) - f(x)}{t}.$$

Remark 5.2. Let $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ be a convex functions. From Corollary 4.25 in [2], we can follow that $f'(\infty; d)$ is a real number for all $d \in \mathbb{R}^n$.

Proposition 5.3. *Let $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ be a convex functions. The following assertions are equivalent:*

(i) $v \in \partial f(\infty)$.
(ii) $\langle v, d\rangle \leq f'(\infty; d)$ *for all* $d \in \mathbb{R}^n$.

Proof. We first prove that (i) ⇒ (ii). Taking any $v \in \partial f(\infty)$ and $t > 0$, we get

$$\langle v, td\rangle \leq f(x + td) - f(x) \text{ such that } x \xrightarrow{\text{dom} f} \infty \text{ whenever } d \in \mathbb{R}^n,$$

which implies that

$$\langle v, d\rangle \leq \frac{f(x + td) - f(x)}{t} \text{ such that } x \xrightarrow{\text{dom} f} \infty \text{ whenever } d \in \mathbb{R}^n.$$

Thus,

$$\langle v, d\rangle \leq \lim_{x \xrightarrow{\text{dom} f} \infty, t \to 0^+} \frac{f(x + td) - f(x)}{t} = f'(\infty; d).$$

We next prove that (ii) ⇒ (i). Assuming now that assertion (ii) holds,

$$\langle v, d\rangle \leq f'(\bar{x}; d) \text{ for all } d \in \mathbb{R}^n.$$

It follow directly from Lemma 4.26 in [2] that

$$\langle v, d\rangle \leq f'(\bar{x}; d) \leq f(x + d) - f(x) \text{ such that } x \xrightarrow{\text{dom} f} \infty \text{ for all } d \in \mathbb{R}^n.$$

This implies by Definition 4.1 that $v \in \partial f(\infty)$. □

Proposition 5.4. *Let $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ be a convex functions. Define the directional function $\psi(d) = f'(\infty; d)$ for all $d \in \mathbb{R}^n$. Then this directional function satisfies the following properties:*

(i) $\psi(0) = 0$.

(ii) $\psi(d_1+d_2) \le \psi(d_1)+\psi(d_2)$ *for all* $d_1, d_2 \in \mathbb{R}^n$ *provided that the right-hand side is well-defined.*

(iii) $\psi(\alpha d) = \alpha\psi(d)$ *whenever* $d \in \mathbb{R}^n$ *and* $\alpha > 0$.

(iv) ψ *is finite on* $\mathbb{R}^n$.

(v) $\partial f(\bar{x}) = \partial\psi(0)$.

Proof. (i) From Definition 5.1, one has

$$\psi(0) = f'(\infty; 0) = \lim_{x \xrightarrow{\text{dom} f} \infty, t \to 0^+} \frac{f(x+0)-f(x)}{t} = 0.$$

(ii) It is easyly to imply from Denifition 5.1 that

$$\begin{aligned}
\psi(d_1+d_2) = f'(\infty; d_1+d_2) &= \lim_{x \xrightarrow{\text{dom} f} \infty, t \to 0^+} \frac{f(x+t(d_1+d_2))-f(x)}{t} \\
&= \lim_{x \xrightarrow{\text{dom} f} \infty, t \to 0^+} \frac{f\left(\dfrac{x+2td_1+x+2td_2}{2}\right)-f(x)}{t} \\
&\le \lim_{x \xrightarrow{\text{dom} f} \infty, t \to 0^+} \frac{f(\bar{x}+2td_1)-f(\bar{x})}{2t} \\
&+ \lim_{x \xrightarrow{\text{dom} f} \infty, t \to 0^+} \frac{f(\bar{x}+2td_2)-f(\bar{x})}{2t} \\
&= \psi(d_1)+\psi(d_2).
\end{aligned}$$

(iii) For any $d \in \mathbb{R}^n$ and $\alpha > 0$, one has

$$\begin{aligned}
\psi(\alpha d) = f'(\infty; \alpha d) &= \lim_{x \xrightarrow{\text{dom} f} \infty, t \to 0^+} \frac{f(x+\alpha td)-f(x)}{t} \\
&= \lim_{x \xrightarrow{\text{dom} f} \infty, t \to 0^+} \alpha \cdot \left[\frac{f(x+\alpha td)-f(x)}{\alpha t}\right] \\
&= \alpha \cdot \lim_{x \xrightarrow{\text{dom} f} \infty, s \to 0^+} \frac{f(x+sd)-f(x)}{s} \\
&= \alpha f'(\infty; d) \\
&= \alpha\psi(\alpha d).
\end{aligned}$$

(iv) By Remark 5.2, ψ is finite on $\mathbb{R}^n$.

(v) It follows from Proposition 5.3 that $v \in \partial f(\bar{x})$ if and only if

$$\langle v, d-0\rangle = \langle v, d\rangle \le f'(\infty; d) = \psi(d) = \psi(d) - \psi(0) \text{ for all } d \in \mathbb{R}^n.$$

This is equivalent to $v \in \partial\psi(0)$. □

Proposition 5.5. *For any proper convex function* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *and* $x \xrightarrow{domf} \infty$, *we have that* $v \in \partial f(\infty)$ *if and only if* $f(x) + f^*(v) = \langle v, x\rangle$.

Proof. Taking any $v \in \partial f(\infty)$, one has

$$f(x) + \langle v, x'\rangle - f(x') \le \langle v, x\rangle \quad \text{such that } x \xrightarrow{\text{dom} f} \infty \text{ for all } x' \in \mathbb{R}^n.$$

This readily implies the inequality

$$f(x)+f^*(v)=f(x)+\sup\{\langle v,x'\rangle-f(x')|x'\in\mathbb{R}^n\}\le\langle v,x\rangle\,.$$

Since dom$f$ is unbounded, one gets

$$\langle v,x\rangle-f(x)\le\sup\{\langle v,x\rangle-f(x)|x\in\mathrm{dom}f\}=f^*(v).$$

This implies that $f(x)+f^*(v)\ge\langle v,x\rangle$. Therefore, one has $f(x)+f^*(v)=\langle v,x\rangle$.

Conversely, suppose that $f(x')+f^*(v)=\langle v,x\rangle$ with $x\xrightarrow{\mathrm{dom}f}\infty$ and get $f^*(v)=\langle v,x\rangle-f(x)$. Note that we have $\langle v,x'\rangle-f(x')\le f^*(v)$ for every $x'\in\mathbb{R}^n$. This yields

$$\langle v,x'\rangle-f(x')\le\langle v,x\rangle-f(x)\text{ with }x\xrightarrow{\mathrm{dom}f}\infty\text{ for every }\ x'\in\mathbb{R}^n,$$

which shows that $v\in\partial f(\infty)$. □

Proposition 5.6. *Given a convex function $f:\mathbb{R}^n\to\overline{\mathbb{R}}$, we have the relationship*

$$f'(\infty;d)=\max\{\langle v,d\rangle\,|v\in\partial f(\infty)\}\textit{ for every }d\in\mathbb{R}^n.$$

Proof. It follows from Proposition 4.8 in [2], with taking into account the properties of the function $\psi(d)=f'(\infty;d),d\in\mathbb{R}^n$, listed in Proposition 5.4 that

$$f'(\bar{x};d)=\psi(d)=\psi^{**}(d)\text{ whenever }d\in\mathbb{R}^n.$$

Note that ψ is a real-valued convex function in this setting. Employing now Lemma 4.29 in [2] tells us that $\psi^*(v)=\delta_\Omega(v)$, where $\Omega=\partial\psi(0)=\partial f(\infty)$. Hence we have by Proposition 4.12 in [2] that

$$\psi^{**}(d)=\sigma_\Omega(d)=\sigma_\Omega(d)=\sup\{\langle v,d\rangle\,|v\in\Omega\}.$$

Since the subgradient set $\Omega=\partial f(\infty)$ is compact by Proposition 4.9, we thus complete the proof of the theorem. □

6. Subgradients of supremum functions at infinity

For a function $f:\mathbb{R}^n\to\mathbb{R}^m$, we use the notation $f(\infty)$ for the set of values of f at infinity,

$$f(\infty):=\{\,y\in\mathbb{R}^m\cup\{\infty\}\mid\exists x^k\to\infty,\ f(x^k)\to y\,\}.$$

Note that the set $f(\infty)$ may contain the element ∞ (in $\mathbb{R}^m$). For example, if $f(x)=e^x$ for all $x\in\mathbb{R}$, then $f(\infty)=\{0,\infty\}$. If $f(x)=\sin x$ for all $x\in\mathbb{R}$, then $f(\infty)=[-1,1]$.

Let T be a nonempty subset of $\mathbb{R}^p$, and let $g:T\times\mathbb{R}^n\to\overline{\mathbb{R}}$ be a extended-real-valued function. For convenience, we also use the notation $g_t(x)=g(t,x)$ for $(t,x)\in T\times\mathbb{R}^n$. The supremum function $f:\mathbb{R}^n\to\overline{\mathbb{R}}$ for g_t over T is

$$f(x)=\sup_{t\in T}g(t,x)=\sup_{t\in T}g_t(x),x\in\mathbb{R}^n.\tag{12}$$

If the supremum in (12) is attained (this happens, in particular, when the index set T is compact and $g(.,x)$ is continuous, then (12) reduces to the maximum function, which canbe written in form

$$f(x) = \max_{i=1,\ldots,m} g_i(x), x \in \mathbb{R}^n,$$

when T is a finite set. The main goal of this section is to calculate the subdifferential of (12) when the functions g_t are convex. Note that in this case the supremum function (12) is also convex. In what follows, we assume without mentioning it that the functions $g_t : \mathbb{R}^n \to \overline{\mathbb{R}}$ are convex for all $t \in T$. We define the active index set at infinity

$$\mathcal{S}(\infty) = \{t \in T | 0 \in (g_t - f)(\infty)\}.$$

Proposition 6.1. *Consider the supremum function (12) over a nonempty index set T. Then, we have the inclusion*

$$\text{clco}\left\{ \cup\, \partial g_t(\infty) \mid t \in \mathcal{S}(\infty) \right\} \subset \partial f(\infty). \tag{13}$$

Proof. This inclusion obviously holds if $\mathcal{S}(\infty) = \emptyset$. If $\mathcal{S}(\infty) \neq \emptyset$, pick $t \in \mathcal{S}(\infty)$ and $v \in \partial g_t(\infty)$. Then we get $0 \in (g_t - f)(\infty)$ and hence

$$\langle v, x' - x\rangle \le g_t(x') - g_t(x) = g_t(x') - f(x) \le f(x') - f(\bar{x}) \text{ such that } x \xrightarrow{\text{dom} f} \infty \text{ for all } x' \in \mathbb{R}^n,$$

which shows that $v \in \partial f(\infty)$. Since the subgradient set $\partial f(\infty)$ is a closed and convex, we arrive at (13) and thus complete the proof. □

Proposition 6.2. *Let T be a nonempty compact subset of $\mathbb{R}^p$, and let $g(\cdot, x)$ be upper semicontinuous on T for every $x \in \mathbb{R}^n$. Then the function f defined in (12) is a convex function and obtain real value at infinity.*

Proof. We need to check that $f(x) < \infty$ for every $x \in \mathbb{R}^n$. Assume on the contrary that $f(x) = \infty$ for some $x \in \mathbb{R}^n$. Then there exists a sequence $\{t_k\} \subset T$ such that $g(t_k, x) \to \infty$. Since T is compact, suppose without loss of generality that $t_k \to \bar{t} \in T$. It follows from the upper semicontinuity of $g(., \bar{x})$ on T that

$$\infty = \lim_{k\to\infty} \sup g(t_k, x) \le g(\bar{t}, x) < \infty,$$

which is a contradiction that completes the proof of the proposition. □

Proposition 6.3. *Let the index set $\emptyset \neq T \subset \mathbb{R}^p$ be compact, and let $g(\cdot, x)$ be upper semicontinuous on T for every $x \in \mathbb{R}^n$. Then the active index set at infinity $\mathcal{S}(\infty) \subset T$ is nonempty and compact. Furthermore, we have the compactness in $\mathbb{R}^n$ of the convex hull*

$$C = \text{co}\left\{ \cup\, \partial g_t(\infty) \mid t \in \mathcal{S}(\infty) \right\}.$$

Proof. Fix $x \in \mathbb{R}^n$ and get from Proposition 6.2 that $f(x) \in \mathbb{R}$ for the supremum function (12). Consider an arbitrary sequence $\{t_k\} \subset T$ with $g(t_k, x) \to f(x)$ as $k \to \infty$. By the compactness of T, we find $\bar{t} \in T$ such that $t_k \to \bar{t}$ along some subsequence. It follows from the imposed upper semicontinuity of the functions $g(.,x)$ on T that

$$f(x) = \lim_{k\to\infty} \sup g(t_k, x) \le g(\bar{t}, x) \le f(x),$$

which ensures that $t \in \mathcal{S}(\infty)$, and thus $\mathcal{S}(\infty) \neq \emptyset$. Since $g(t,x) \le f(x)$, for any $t \in T$ we get the representation

$$\mathcal{S}(x) = \{t \in T | g(t,x) \le f(x)\}$$

implying the closedness of $\mathcal{S}(\infty)$ (and hence its compactness since $\mathcal{S}(\infty) \in T$), which is an immediate consequence of the upper semicontinuity of $g(.,x)$.

It remains to show that the subgradient set

$$Q = \bigcup_{t\in \mathcal{S}(\infty)} \partial g_t(\infty)$$

is compact in $\mathbb{R}^n$. This yields the claimed compactness of $\mathrm{C} = \mathrm{co}\, Q$, since the convex hull of a compact set is compact as follows from the classical Carathéodory theorem (Corollary 5.7 in [2]). To proceed with verifying the compactness of Q, we recall first that $Q \subset \partial f(\infty)$ by Proposition 6.1. This implies that the set Q is bounded in $\mathbb{R}^n$, since the subgradient set of a real-valued convex function is compact by Proposition 3.3 in [2].

To check the closedness of Q, take a sequence $\{v_k\} \subset Q$ converging to some $\bar{v}$ and find $t_k \in \mathcal{S}(\infty)$ such that $v_k \in \partial g_{t_k}(\infty), k \in \mathbb{N}$. Since $\mathcal{S}(\infty)$ is compact, we may assume that $t_k \to \bar{t} \in \mathcal{S}(\infty)$. Then $g(t_k, x) = g(\bar{t}, x) = f(x)$ and

$$\langle v_k, x' - x\rangle \le g(t_k, x') - g(t_k, x) = g(t_k, x') - g(\bar{t}, \bar{x}) \text{ with } x \xrightarrow{\text{dom}g} \infty \text{ for all } x' \in \mathbb{R}^n.$$

This gives us the relationships

$$\begin{aligned}\langle \bar{v}, x' - x\rangle &= \lim_{k\to\infty} \sup \langle v_k, x' - x\rangle \\ &\le \lim_{k\to\infty} \sup g(t_k, x') - g(\bar{t}, x) \\ &\le g(\bar{t}, x') - g(\bar{t}, x) = g_{\bar{t}}(x') - g_{\bar{t}}(x) \;\text{ with } x \xrightarrow{\text{dom}g} \infty \text{ or all } x' \in \mathbb{R}^n,\end{aligned}$$

which verify that $\bar{v} \in \partial g_{\bar{t}}(\infty) \subset Q$ and thus complete the proof. □

Definition 6.4. Given a nonempty subset Ω of $\mathbb{R}^n$, the support function associated with Ω is defined by

$$\sigma_\Omega(v) := \sup\{\langle v, x\rangle \,| x \in \Omega\}, v \in \mathbb{R}^n.$$

Proposition 6.5. *Let the index set $\emptyset \neq T \subset \mathbb{R}^p$ be compact, and let $g(\cdot, x)$ be upper semi continuous on T at infinity, we have*

$$f'(\infty; v) \le \sigma_C(v) \textit{ for all } v \in \mathbb{R}^n$$

via the support function of the set C defined therein.

Proof. It follows from Propositions 4.24 and 4.25 in [2] that

$$-\infty < f'(\infty; v) = \inf_{\lambda>0} \varphi(\lambda) < \infty,$$

where the function φ is defined in Lemma 4.23 in [2] and is proved there to be nondecreasing on $(0, \infty)$. Thus there exists a strictly decreasing sequence $\lambda_k \downarrow 0$ as $k \to \infty$ along which we have

$$f'(\infty; v) = \lim_{x \xrightarrow{\text{dom} f} \infty, k\to\infty} \frac{f(x + \lambda_k v) - f(x)}{\lambda_k}, k \in \mathbb{N}.$$

For every k, we select $t_k \in T$ such that $f(x + \lambda_k v) = g(t_k, x + \lambda_k v)$. The compactness of T allows us to find $\bar{t} \in T$ such that $t_k \to \bar{t} \in T$ along some subsequence. Let us show that $\bar{t} \in \mathcal{S}(\infty)$. Since $g(t_k, .)$ is convex and $\lambda_k < 1$ for large k, we get the relationships

$$\begin{aligned} g(t_k, x + \lambda_k v) &= g(t_k, \lambda_k(x + v) + (1 - \lambda_k)x) \\ &\leq \lambda_k g(t_k, x + v) + (1 - \lambda_k) g(t_k, x), \end{aligned} \tag{14}$$

which imply by the continuity of f(since any real-valued convex function is continuous) and the upper semicontinuity of $g(.; x)$ for $x \in \mathbb{R}^n$ that

$$f(x) = \lim_{k\to\infty} f(x + \lambda_k v) = \lim_{k\to\infty} g(t_k, x + \lambda_k v) \leq g(\bar{t}, x).$$

This tells us that $f(x) = g(\bar{t}, x)$, and therefore $\bar{t} \in \mathcal{S}(\infty)$. Moreover, for any $\epsilon > 0$ and large k, we get the estimate

$$g(t_k, x + v) \leq \gamma + \epsilon \text{ with } \gamma = g(\bar{t}, x + v)$$

by the upper semicontinuity of $g(., x + v)$. It follows from (14) that

$$g(t_k, x) \geq \frac{g(t_k, x + \lambda_k v) - \lambda_k(\gamma + \epsilon)}{1 - \lambda_k} \to f(x),$$

which ensures that $\lim_{k\to\infty} g(t_k, x) = f(x)$ due to

$$f(x) \leq \liminf_{k\to\infty} g(t_k, x) \leq \limsup_{k\to\infty} g(t_k, x) \leq g(\bar{t}, x) = f(x).$$

Fix further any $\lambda > 0$ and see that $\lambda_k < \lambda$ for all k sufficiently large. Thus we can deducefrom Lemma 4.23 in [2] that

$$\begin{aligned} \frac{f(x + \lambda_k v) - f(x)}{\lambda_k} &= \frac{g(t_k, x + \lambda_k v) - g(\bar{t}, x)}{\lambda_k} \\ &\leq \frac{g(t_k, x + \lambda_k v) - g(t_k, x)}{\lambda_k} \\ &\leq \frac{g(t_k, x + \lambda v) - g(t_k, x)}{\lambda_k} \text{ for large } k. \end{aligned}$$

This implies in turn the conditions

$$\begin{aligned} \limsup_{k\to\infty} \frac{f(x + \lambda_k v) - f(x)}{\lambda_k} &= \limsup_{k\to\infty} \frac{g(t_k, x + \lambda v) - g(\bar{t}, x)}{\lambda} \\ &\leq \frac{g(\bar{t}, x + \lambda v) - g(\bar{t}, x)}{\lambda} \\ &= \frac{g_{\bar{t}}(x + \lambda v) - g_{\bar{t}}(x)}{\lambda}. \end{aligned}$$

which clearly yield the relationships

$$f'(\infty; v) \leq \lim_{\lambda > 0} \frac{g_{\bar{t}}(x + \lambda v) - g_{\bar{t}}(x)}{\lambda} = g'_{\bar{t}}(\infty; v).$$

Using Proposition 5.6 and the inclusion $\partial g_{\bar{t}}(\infty) \subset C$ obtained in Proposition 6.1 shows that

$$f'(\infty; v) \leq g'_{\bar{t}}(\infty; v) = \sup_{w \in \partial g_{\bar{t}}(\infty)} \langle w, v \rangle \leq \sup_{w \in C} \langle w, v \rangle = \sigma_C(v).$$

□

Proposition 6.6. *Let f be defined in (12), where the index set $\emptyset \neq T \subset \mathbb{R}^p$ is compact,and where the function $g(t, \cdot)$ is convex on $\mathbb{R}^n$ for every $t \in T$. Suppose in addition that the function $g(\cdot, x)$ is upper semicontinuous on T for every $\mathbb{R}^n$. Then we have the subdifferential formula*

$$\partial f(\infty) = \text{co}\left\{ \cup \partial g_t(\infty) \mid t \in \mathcal{S}(\infty) \right\}.$$

Proof. Taking any $w \in \partial f(\infty)$, we deduce from Proposition 5.6 and Proposition 6.5 that

$$\langle w, v \rangle \leq f'(\infty, v) \leq \sigma_C(v) \text{ for all } v \in \mathbb{R}^n,$$

where C is defined in Proposition 6.3. Thus $w \in \partial \sigma_C(0) = C$ for the set $\Omega = C$, which is a nonempty closed convex subset of $\mathbb{R}^n$. The reverse inclusion follows from Proposition 6.1. □

7. Applications to Optimization Problems

We consider the optimization problem (P):

$$\inf\{f(x) \mid x \in \Omega\},$$

where Ω be a convex subset of $\mathbb{R}^n$, $\text{dom} f \cap \Omega$ is unbounded, and $f \colon \mathbb{R}^n \to \overline{\mathbb{R}}$ be a convex and lower semi-continuous function bounded from below on Ω, i.e., $f_* := \inf_{x \in \Omega} f(x)$ is finite. We denote the optimal value of (P) by inf (P).

Theorem 7.1. *Consider the problem* (P)*. Assume that the following statements hold*

(i) There exists a sequence $x_k \xrightarrow{\Omega} \infty$ such that $f(x_k) \to \inf(\mathrm{P})$ as $k \to +\infty$.
(ii) For any sequence $\{x_k\}$ satisfying (i) and for all k sufficiently large, we have

$$\partial^\infty f_1(x_k) \cap \big(-\partial^\infty f_2(x_k) \big) = \{0\}.$$

Then, one has $0 \in \partial f(\infty)$.

Proof. We first consider the case that $\Omega = \mathbb{R}^n$. Then $N(\infty; \Omega) = \{0\}$. For each $k \in \mathbb{N}$, we have

$$f_* \leq f(x_k) \leq f_* + \left(f(x_k) - f_* + \frac{1}{k} \right).$$

Clearly, $\epsilon_k := f(x_k) - f_* + \frac{1}{k} > 0$ and $\epsilon_k \to 0$ as $k \to \infty$. Put $\lambda_k := \sqrt{\epsilon_k}$, then by the Ekeland variational principle (Lemma 2.15), there exists $z_k \in \mathbb{R}^n$ for $k > 0$ such that

$$\|x_k - z_k\| \leq \lambda_k,$$

$$f(z_k) \le f(x) + \lambda_k \|x - z_k\| \quad \text{for all} \quad x \in \mathbb{R}^n.$$

The first inequality and the fact that $x_k \xrightarrow{\Omega} \infty$ imply that $z_k \xrightarrow{\Omega} \infty$. While the second inequality says that z_k is a global minimizer of $\varphi(\cdot) := f(\cdot) + \lambda_k \|\cdot - z_k\|$ on $\mathbb{R}^n$. By the Fermat rule (Lemma 2.12), we obtain

$$0 \in \partial\left(f(\cdot) + \lambda_k \|\cdot - z_k\|\right)(z_k).$$

By the Lipschitzness of the function $\|\cdot - z_k\|$ and the sum rule (Lemma 2.13), we have

$$0 \in \partial f(z_k) + \lambda_k \partial(\|\cdot - z_k\|)(z_k) = \partial f(z_k) + \lambda_k \mathbb{B}$$

due to the fact that $\partial(\|\cdot - z_k\|)(z_k) = \mathbb{B}$. Since $\lambda_k \to 0$ and $z_k \to \infty$, by letting $k \to \infty$, we obtain $0 \in \partial f(\infty)$.

We now consider the case where Ω is an arbitrary subset of $\mathbb{R}^n$. We have

$$f_* = \inf_{x \in \mathbb{R}^n} \left(f + \delta(\infty; \Omega)\right)(x) = \inf_{x \in \Omega} f(x) \ > \ -\infty,$$

where $\delta(\cdot; \Omega)\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ stands for the indicator function of the set Ω. Clearly $f(x_k) + \delta(x_k; \Omega) \to f_*$, where x_k is a minimizer of $f(x) + \delta(x; \Omega) + \frac{1}{k}||x - x_k||$ on $\mathbb{R}^n$. Notably ,x_k is also a minimizer of $f(x) + \frac{1}{k}||x - x_k||$ on Ω. Therefore, $0 \in \partial(f + \frac{1}{k}|| \cdot - x_k||)(x_k)$ (by the argument employed in the first case). This, together with Proposition 4.6 and $\partial^\infty f_1(x_k) \cap \big(-\partial^\infty f_2(x_k)\big) = \{0\}$, yields

$$0 \in \partial f(x_k) + \frac{1}{k}\mathbb{B},$$

due to the fact that $\partial(\|\cdot - x_k\|)(x_k) = \mathbb{B}$. Since $\lambda_k \to 0$ and $x_k \to \infty$, by letting $k \to \infty$, we obtain $0 \in \partial f(\infty)$. The proof is complete. □

Corollary 7.2. *Consider the problem* (P). *Assume that for every sequence $x_k \xrightarrow{\Omega} \infty$ as $k \to +\infty$, the condition*

$$\partial^\infty f_1(x_k) \cap \big(-\partial^\infty f_2(x_k)\big) = \{0\}.$$

holds for all k sufficiently large. If f does not attain its infimum on Ω, then $0 \in \partial f(\infty)$.

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