Integral Inverse Optimization Problems

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Abstract

Inverse optimization problems are bilevel optimization problems in which the leader modifies the follower's objective such that a prescribed feasible solution becomes an optimal solution of the follower. They capture hierarchical decision-making problems like parameter estimation tasks or situations where a planner wants to steer an agent's choice.

In this work, we study *integral* inverse optimization problems, in which the leader's cost modifications must be integral. We focus on the version where the modifications are minimized with respect to the ℓ_1-norm. Adding this integrality constraint for the cost modifications changes the optimal objective value for some inverse optimization problems like the inverse linear program, the inverse min weight perfect matching problem, the inverse knapsack problem and the inverse travelling salesman problem. In contrast, others like the inverse shortest path problem, the inverse assignment problem, the inverse min cut problem or the inverse min spanning tree problem have optimal integral solutions if all original weights are integer. Interestingly, the integrality constraint does not affect the complexity of the problem in many cases. For instance, we prove that (primal bound verification for) integral inverse mixed-integer linear optimization lies in co$\mathcal{NP}$, similar to (continuous) inverse mixed-integer linear optimization. Thus, for example, the integral inverse knapsack problem is co$\mathcal{NP}$-complete. In contrast to the continuous inverse problem, the optimal objective values of the optimistic and pessimistic version of the integral inverse problem can be arbitrarily far apart.

1 Introduction

An inverse optimization problem asks for minimal modifications of the weight coefficients in the objective of an optimization problem such that a given feasible solution is optimal. We study the case where modifications can only be integral, which has only received very little attention compared to (continuous) inverse optimization so far. For integral weight modifications, the scaling of weight factors in the objective of the optimization problem plays a crucial role. This is in contrast to the single-level optimization problem where the optimality is not affected by the scaling of the weight factors.

Formally, an integral inverse optimization problem is a bilevel problem of the form

$$\begin{aligned} \min_{w} \quad & \|w\| \\ \text{s.t.} \quad & y^0 \in \operatorname*{arg\,min}_{y \in Y} \, (d+w)^\top y \\ & w \in \mathbb{Z}^n, \end{aligned}$$

where a required lower-level solution y^0 shall become an optimal solution to the lower-level optimization problem $\min_{y \in Y} (d + w)^\top y$. We focus on *combinatorial* and *mixed-integer linear* optimization problems on the lower level. Note that the domain of the lower-level problem is not modified by the upper-level decisions w; only the objective of the lower-level problem changes. Unless stated explicitly otherwise, we consider the ℓ_1-norm of the modifications in the objective.

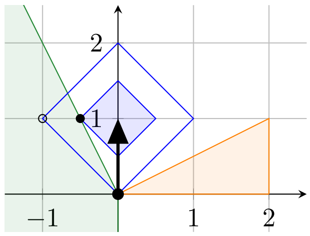
Integral inverse problems have only rarely been considered in the literature. The integral inverse knapsack problem has been studied both with respect to ℓ_1 and ℓ_∞ in [RFS13]. In particular, it is shown that the integral inverse knapsack problem with respect to ℓ_∞ is co$\mathcal{NP}$-complete and with respect to ℓ_1 is co$\mathcal{NP}$-hard. For some integral inverse problems, mainly with respect to ℓ_∞, rounding a continuous inverse problem solution provides an approximation [ABSS18].

Inverse discrete optimization problems have mainly been studied with continuous modifications. If the lower-level problem is solvable in polynomial time, the corresponding continuous inverse problem is still solvable in polynomial time [AO01]. Efficient solution methods for the continuous inverse problem have been studied for several polynomial-time solvable combinatorial optimization problems, including the min spanning tree (MST) problem [AO00], the shortest path problem [AO01], and network flow problems like the min cut problem [AO02]. The inverse MIP has been shown to be co$\mathcal{NP}$-complete [BR21] which can be translated in at least some cases to the integral version of it, see Section 3. There are polyhedral though exponential-sized formulations for the inverse IP and inverse MIP based on supperadditive duality [Sch09, LS15] that can be directly extended for the integral inverse IP respectively MIP. The inverse MIP can be solved by a cutting plane approach [Wan09] where a trust region can be used to improve performance [BCZ22].

In a partial inverse problem, only the values of some of the follower's variables are required, see e.g. the survey [Heu04]. The complexity in partial inverse problems with a polynomial-time solvable lower-level problem can change for several reasons. First, in contrast to the polynomial-time solvable (complete) inverse min cut problem the partial inverse min cut problem is already $\mathcal{NP}$-complete [Gas10]. Furthermore, restricting weight modifications to be non-negative can increase the complexity. For example, the partial inverse shortest path problem is solvable in polynomial time while the partial inverse shortest path problem with only weight increases is $\mathcal{NP}$-complete [LM25]. Even the 'size' of the partial required solution can influence the complexity. The partial inverse min spanning tree problem with only weight increases is solvable in polynomial time if only a single edge is required to be included while it turns $\mathcal{NP}$-complete once at least two edges are required to be part of an optimal solution [LZD18]. In contrast, adding an integrality constraint for the weight modifications to a partial inverse shortest path problem with integral original weights does not change the complexity. Whether the integrality constraint changes an inverse problem directly translates to corresponding partial inverse problems. Similarly, partial inverse knapsack problem can have only fractional optimal solutions. The partial inverse knapsack problem can be solved via a branch and cut method [LLMT25].

In this paper, we show that the integrality constraint on the weight modifications can change an inverse problem substantially. This includes both problems solvable in polynomial time like linear programs (LPs) or the min weight perfect matching problem on a non-bipartite graph and $\mathcal{NP}$-hard problems like the knapsack problem or the travelling salesman problem (TSP). The optimal value of an integral and the corresponding continuous inverse problem can lay arbitrarily far apart. Furthermore, rounding an optimal solution of the continuous inverse LP provides in general no constant-factor approximation for the corresponding integral inverse LP.

In contrast, we will see that for certain lower-level problems there is an integral solution to the inverse problem without loss of generality if the original weights d are integral. This is the case for LPs that can be formulated via a totally unimodular (TU) matrix and consequently combinatorial problems like the shortest path, assignment and min cut problem. Furthermore, also the min spanning tree (MST) problem has an integral optimal solution for integral original weights d.



● $y^0 = (0,0)^\top$
□ $Y(k) = \{(y_1,y_2) \in \mathbb{R}^2 \mid -y_1 + ky_2 \le 0, y_1 \le k, y_2 \ge 0\}$ for $k = 2$
□ $\mathcal{C}^o(y^0) = \{d + w \in \mathbb{R}^n : (d+w)^\top (y - y^0) \le 0 \; \forall y \in Y\}$
↑ $d = (0,1)^\top$
□ $\mathcal{K}(\gamma) = \{d + w \in \mathbb{R}^n : \|w\| \le \gamma\}$ for $\gamma = \frac{1}{2}$
□ $\mathcal{K}(1)$

Figure 1: Example of the inverse LP with worse solution for integral weight modifications: For $k \in \mathbb{R}_{>1}$, consider the LP $\max\{y_2 \mid (y_1,y_2) \in Y(k)\}$ with feasible region $Y(k)$. The required solution y^0 is an optimal solution if the modified cost vector $d+w$ is in the normal cone at the required solution $\mathcal{C}^o(y^0)$. An optimal solution of the continuous inverse LP is to subtract $\frac{1}{k}$ from the second coordinate (minimal γ such that $\mathcal{K}(\gamma)$ and $\mathcal{C}^o(y^0)$ have non-empty intersection). In contrast, the optimal integral inverse LP solution has value 1.

We show that the integral inverse mixed-integer program (MIP) and the integral inverse knapsack problem remain in co$\mathcal{NP}$ similarly to their continuous relaxations. This answers an open question from [RFS13], namely whether the integral inverse knapsack problem is co$\mathcal{NP}$-complete. Integral inverse problems with a lower-level problem that can be solved in polynomial time are both in $\mathcal{NP}$ and co$\mathcal{NP}$.

For most of this paper we study the optimistic version of integral inverse optimization problems. However, we also show that the objective value of an optimistic and a pessimistic optimal solution of an integral inverse problem can lay arbitrarily far apart. In contrast to the pessimistic continuous inverse problem, the pessimistic integral inverse problem attains its optimum if feasible.

This paper is structured as follows: In Section 2, we study whether the integrality constraint for the weight modifications changes the inverse problem for different lower-level problems. Afterwards, we study the complexity of some integral inverse problems in Section 3. In Section **??**, we consider the difference between the optimistic and pessimistic integral inverse problem.

2 Continuous vs. Integral

As we will see, adding an integrality constraint for the weight modifications changes the inverse linear program (LP), inverse integer program (IP) and inverse combinatorial problems in general. However, for certain classes of inverse LPs and some inverse combinatorial optimization problems, there are always integral optimal solutions for integral original weights.

2.1 Inverse Problems with Fractional Optimal Solutions

Inverse optimization problems can in general have (unique) fractional optimal solutions such that adding an integrality constraint for the weight modifications changes the problem.

The following theorem follows from the examples given in Figure 1 and Figure 2 which have a strictly better fractional optimal solution than their integral versions. In particular, the examples also shows that solvability in polynomial time of the lower-level problem does not imply integrality of an optimal solution for integral original weights.

Theorem 2.1. *The inverse LP and the inverse matching problem with integral original weights do not have an integral optimal solution in general even if the original weights are integral.*

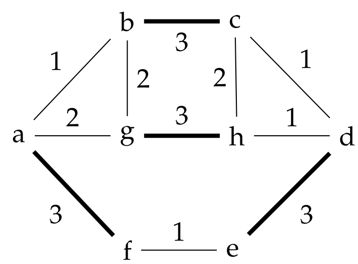


Figure 2: Example of the inverse minimum weight perfect matching problem on a non-bipartite graph with worse solution for integral weight modifications: The required solution $\{af, bc, gh, de\}$ has an original weight of 12. All three other perfect matchings $\{ab, cd, ef, gh\}$, $\{ag, dh, bc, ef\}$ and $\{af, de, bg, ch\}$ are initially better. A fractional optimal solution is to modify the weights in total by 6.5, e.g. reducing the weights of required edges bc by 1.5, af by 4.5 and gh by 0.5. An integer optimal solution modifies weights in total by 7, e.g. reducing the weight of bc by 2 and af by 5.

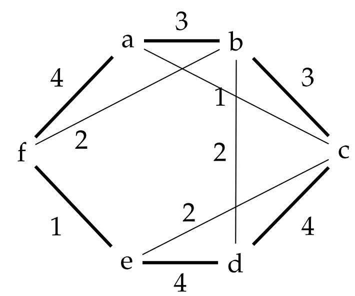


Figure 3: Example of the inverse TSP with worse solution for integral weight modifications: The required tour $abcdef$ has an original weight of 19 which is strictly larger than the weights of tours $abfedc$ (with original weight 15), $abdcef$ (14) and $acbdef$ (15). An optimal fractional solution to the inverse problem is to decrease the weight of edge bc by 1.5 and to increase the weights of edges ac by 2.5 and bd by 1.5, i.e. a total weight modification of 5.5.

The analogue of the previous theorem also holds for the inverse travelling salesman (TSP) problem, see Figure 3, and the inverse knapsack problem [Lyk25, Beispiel 4.1], also see the proof of the following Proposition 2.2. Combining copies of such an example, we can construct instances where the optimal values of the continuous and the integral versions lie arbitrarily far apart. The construction in the proof of Proposition 2.2 can also be applied to the examples in Figure 1 and Figure 2 to show the analogue for the inverse LP and the inverse matching problem on a non-bipartite graph, respectively.

Proposition 2.2. *The difference between the optimal values of the continuous and the integral version of an inverse knapsack problem and an inverse IP can be unbounded.*

Proof. The following example of an inverse knapsack problem instance combines copies of the example [Lyk25, Beispiel 4.1] with different weight scaling. Let $n \in \mathbb{N}$ and the budget be $\sum_{i=1}^{n} 3 \cdot 10^{i-1}$ to choose from the $4n$ items $\{a_i, b_i, c_i, d_i : i \in [n]\}$ with values and weights as specified in Table 1. Let the required solution consist of the set of items $\{a_i, b_i : i \in [n]\}$.

The optimal continuous solution is to increase the values of items $a_i, i \in [n]$ by 0.5 each while decreasing the values of $c_i, d_i, i \in [n]$ by 0.5 each with a total weight modification of $1.5n$. In contrast, the optimal integer solution is to increase the value of items a_i and $b_i, i \in [n]$ by one each with a total value of $2n$. □

Item, $i \in [n]$	a_i	b_i	c_i	d_i
Value d	1	3	2	2
Weight q	10^{i-1}	$2 \cdot 10^{i-1}$	10^{i-1}	10^{i-1}
Cont. optimal solution w^{cont}	0.5	0	-0.5	-0.5
Int. optimal solution w^{int}	1	1	0	0

Table 1: Item data for inverse knapsack instance with budget $\sum_{i=1}^{n} 3 \cdot 10^{i-1}$

Rounding the fractional solution of the inverse knapsack problem instance in the proof of Proposition 2.2 or the matching problem instance in Figure 2 does not result in an optimal integer solution. Moreover, rounding up also does not yield a constant-factor-approximation in general. Neither the inverse LP nor the inverse min weight perfect matching is half-integral. Note, however, that the construction for the following theorem does not work with a 0-1 problem on the lower level.

Theorem 2.3. *Rounding up an optimal solution of the continuous inverse LP is no constant factor approximation of the corresponding integral inverse LP.*

Proof. Let $n \in \mathbb{Z}$ and $2 \leq n \leq k$. The LP

$$\max \{y_1 \mid ky_1 - y_i \leq 0, y_i \leq k, 2 \leq i \leq n, y_1 \geq 0\}$$

with required solution in the origin $0 \in \mathbb{R}^n$ has as continuous optimal solution to subtract $\frac{1}{k}$ from the objective coefficient of each y_i, $2 \leq i \leq n$. It has an optimal value of $\frac{n-1}{k} < 1$. The rounded integral solution has a value of $n-1$. In contrast, the optimal integral solution value is to subtract one from the objective coefficient of y_1, with constant value 1. Thus, the approximation ratio of rounding up the continuous solution is $n-1$ for this example. $\square$

2.2 Inverse Problems That Always Have Integral Optimal Solutions

For certain LPs and several combinatorial problems, the corresponding inverse problems have integral optimal solutions for integral original weights. First, we give a criterion for inverse LPs to have an integral solution based on a *totally unimodular (TU)* matrix to describe the feasible area. Based on this, the inverse shortest path, inverse assignment problem and inverse min cut problem have integral solutions for integral original weights. Afterwards, we show that also the inverse min spanning tree (MST) problem has integral solutions for integral original weights based on a combinatorial solution method for it. Note that a total dual integral (TDI) description of the feasible area does not suffice as there exists one for every integral polyhedron, see e.g. [KV18, Theorem 5.18], i.e. also for the example given in Figure 1.

Theorem 2.4. *Let $A \in \{0, \pm 1\}^{m \times n}$ be a TU matrix, $b \in \mathbb{Z}^m$, y^0 be a vertex of $P = \{y : Ay \geq b\}$ and $d \in \mathbb{Z}^n$. Then there is an optimal solution to the inverse LP for the LP $\min_y \{d^\top y : y \in P\}$ that is integral.*

Proof. Using the approach of [AO01], we write down the inverse LP problem w.r.t. ℓ_1 as an LP: The dual of the LP $\min_y \{d^\top y : Ay \geq b\}$ is given by $\max_\lambda \{b^\top \lambda : \lambda^\top A = d, \lambda \geq 0\}$. Let the index set of inequalities $Ay \geq b$ that are fulfilled with equality for y^0 be given by $B(y^0)$, i.e. let $B(y^0) = \left\{i \in [m] : \sum_{j \in [n]} a_{ij} y_j^0 = b_i\right\}$. Then, by complementary slackness, we need $\lambda_i = 0$ for $i \notin B(y^0)$ for y^0 to be an optimal solution of the LP.

Thus, the inverse LP problem w.r.t. ℓ_1 is given by

$$\begin{array}{rrcll}
\min\limits_{\lambda, w^+, w^-} & \sum\limits_{j \in [n]} w_j^+ + w_j^- & & & \\
\text{s.t.} & \sum\limits_{i \in B(y^0)} a_{ij}\lambda_i & = & d_j + w_j^+ - w_j^- & \forall j \in [n] \\
& \lambda_i & \geq & 0 & \forall i \in B(y^0) \\
& w_j^+, w_j^- & \geq & 0 & \forall j \in [n].
\end{array}$$

Due to A being TU and d being integral, there is an integral optimal solution to this LP. □

The following statement follows from the previous theorem using that the incidence matrix of directed graphs and bipartite graphs are TU, see e.g. [KV18, Theorems 5.26/5.27].

Corollary 2.5. *For integral original costs d, there exists an integral optimal solution to the inverse shortest path problem, the inverse assignment problem and the inverse min cut problem.*

Alternatively, one can obtain these results based on combinatorial solution methods. For example, for the inverse min cut problem first evaluate a maximum flow and then reduce the capacities of arcs on the cut such that they are saturated by the flow [AO02]. If all capacities are integral, there is an integral max flow and the difference of two integers is again integral.

Next, we show the integrality of the inverse MST for integral original costs based on its dual problem. First, the inverse MST can be formulated based on the optimality condition for an MST. The weight of any non-tree edge $e = \{u, v\}$ must not be smaller than the weight of a tree-edge on the path that connects u and v within the tree. Furthermore, weights of used elements are not increased and weights of unused elements are not decreased in an optimal solution of an inverse combinatorial problem. The dual problem of this formulation of the inverse MST is an assignment problem with cost coefficients being differences of two original edge costs.

Lemma 2.6 ([AO00]). *The dual problem of the inverse min spanning tree problem on a graph $G = (V, E)$ with $E = \{e_1, e_2, ..., e_m\}$ and a required spanning tree $T^0 = \{e_1, ..., e_{n-1}\}$ is the assignment problem*

$$\begin{array}{rrcll}
\min\limits_{x} & \sum\limits_{\{i,j\} \in E'} (d_j - d_i)x_{ij} & & & \\
\text{s.t.} & \sum\limits_{j:\{i,j\} \in E'} x_{ij} & \leq & 1 & \forall i \in N_1 \\
& \sum\limits_{i:\{i,j\} \in E'} x_{ij} & \leq & 1 & \forall j \in N_2 \\
& x & \geq & 0 &
\end{array}$$

on the bipartite path graph $G' = (N_1 \cup N_2, E')$ with nodes $N_1 = \{1, 2, ..., n-1\}$, $N_2 = \{n, n+1, ..., m\}$ and edges $E' = \{\{i, j\} : e_i \in P[e_j], i \in N_1, j \in N_2\}$ where $P[e_j]$ is the unique path from the end vertices of e_j within T^0.

Theorem 2.7. *For integral original costs d, there exists an optimal solution to the inverse min spanning tree problem that is integral.*

Proof. The inverse min spanning tree problem can be formulated as the dual problem of the assignment problem in Lemma 2.6. First, the constraint matrix of the assignment problem is the incidence matrix of a bipartite graph that is totally unimodular (TU), see e.g. [KV18, Theorem 5.26]. Note that the constraint matrix remains TU when formulating the dual problem. Furthermore, the cost coefficients of the assignment problem in Lemma 2.6 are integral for integral original costs d. Thus, both right-hand-side and the cost coefficients are integral in the inverse MST formulation via the TU matrix which implies the existence of an integral optimal solution. □

3 Complexity

3.1 Integral Inverse MIP

The primal bound verification problem for the integral inverse mixed-integer program (MIP) lies in co$\mathcal{NP}$, similarly to the continuous version of it. We obtain this result by only slightly modifying the proof for the continuous inverse MIP by Bulut and Ralphs [BR21]. Their central result is to give an equivalency statement corresponding to a NO-instance. They show that an empty intersection between two particular sets corresponding to upper- and lower-level feasibility can equivalently be shown via the existence of an element in the intersection between two other sets. In the integral inverse MIP, there is the additional integrality constraint on the upper level. Thus, the norm ball used for the continuous version's upper-level feasibility needs to be replaced by the subset of integer points in the norm ball. We then also replace the dual cone of the norm cone by the dual cone of the according integer set. Note that we use a slightly different problem formulation where given weights are modified instead of searching a new weight vector close to the original one. While [BR21] provides the containment in co$\mathcal{NP}$ for general p-norms in the objective, we show the last step only for the ℓ_1- and ℓ_∞-norms.

Definition 3.1 (Primal bound verification problem)**.** For a given integral inverse MIP problem instance and a bound $\gamma \in \mathbb{Q}$, the primal bound verification problem is whether there is a solution for the problem instance with $\|w\| \leq \gamma$.

Definition 3.2. We define the sets (similarly to [BR21])

$$\begin{aligned}
\mathcal{C}^o(y^0) &:= \left\{d + w \in \mathbb{R}^n : (d+w)^\top (y - y^0) \leq 0 \;\forall y \in Y\right\} \\
\mathcal{K}(\gamma) &:= \{d + w \in \mathbb{R}^n : \|w\| \leq \gamma\} \\
\mathcal{K}^*(\gamma) &:= \left\{y \in \mathbb{R}^n : (d+w)^\top (y^0 - y) \leq 0 \;\forall (d+w) \in \mathcal{K}(\gamma)\right\} \\
\mathcal{K}_{\mathbb{Z}}(\gamma) &:= \{d + w : \|w\| \leq \gamma, w \in \mathbb{Z}^n\} \\
\mathcal{K}^*_{\mathbb{Z}}(\gamma) &:= \left\{y \in \mathbb{R}^n : (d+w)^\top (y^0 - y) \leq 0 \;\forall (d+w) \in \mathcal{K}_{\mathbb{Z}}(\gamma)\right\}.
\end{aligned}$$

A solution to the primal bound verification problem needs to fulfil properties that roughly correspond to upper- and lower-level feasibility. First, a solution to the primal bound verification problems satisfies the bound on the upper-level objective, i.e. the norm of the weight modification w has to be at most γ. This is captured in the norm ball $\mathcal{K}(\gamma)$. For the integral inverse MIP, we need additionally that the weight modifications are integral. We replace $\mathcal{K}(\gamma)$ by only the integer points in this set, defined in the following as the set $\mathcal{K}_{\mathbb{Z}}(\gamma)$. Furthermore, the required solution y^0 must be an optimal solution to the lower-level problem with weight modifications w. The required solution y^0 is optimal for cost vectors in the normal cone at y^0, defined as $\mathcal{C}^o(y^0)$. Thus, the primal bound verification problem asks whether the intersection of the norm ball and the normal cone is non-empty.

To show the containment in co$\mathcal{NP}$, we need a certificate for NO-instances that can be verified in polynomial time. The intersection of the normal cone $\mathcal{C}^o(y^0)$ and $\mathcal{K}_{\mathbb{Z}}(\gamma)$ is empty if and only if the convex hull of the lower-level feasible area shares a point with the interior of $\mathcal{K}^*_{\mathbb{Z}}(\gamma)$. This follows nearly directly from the according result without the integrality constraint [BR21, Lemma 13]. Similarly to Bulut and Ralphs, we here assume for simplicity that the lower-level domain Y and thus also the polar cone $\mathcal{C}^o(y^0)$ are full-dimensional.

Lemma 3.3. *For $\gamma \in \mathbb{Q}$ such that $0 \leq \gamma < \|d\|$, we have (for general p-norms, $p \in [1, \infty]$)*

$$\mathcal{K}_{\mathbb{Z}}(\gamma) \cap \mathcal{C}^o(y^0) = \emptyset \Leftrightarrow \mathrm{conv}(Y \cup \{y^0\}) \cap int(\mathcal{K}^*_{\mathbb{Z}}(\gamma)) \neq \emptyset.$$

Proof. By definition $\mathcal{K}_{\mathbb{Z}}(\gamma) \subseteq \mathcal{K}(\gamma)$. Thus, $\mathcal{K}^*_{\mathbb{Z}}(\gamma) \supseteq \mathcal{K}^*(\gamma)$. Hence, $int(\mathcal{K}^*_{\mathbb{Z}}(\gamma)) \neq \emptyset$ for $0 \leq \gamma < \|c\|$ follows from $int(\mathcal{K}^*(\gamma)) \neq \emptyset$. The remaining of the proof of [BR21, Lemma 13] also applies here. □

Note that $\mathcal{K}^*_{\mathbb{Z}}(\gamma)$ is the dual norm cone to $\mathcal{K}_{\mathbb{Z}}(\gamma)$. Thus, checking whether an element belongs to $\mathcal{K}^*_{\mathbb{Z}}(\gamma)$ cannot be done directly via the dual norm in contrast to doing this for $\mathcal{K}^*(\gamma)$. Next, we show how to verify containment of a point in $\mathcal{K}^*_{\mathbb{Z}}(\gamma)$ for ℓ_1 and ℓ_∞ still in polynomial time.

Lemma 3.4. *For an element $x \in \mathbb{R}^n$, one can test in polynomial time whether $x \in K^*_{\mathbb{Z}}(\gamma)$ w.r.t. ℓ_1.*

Proof. We reformulate the set $K^*_{\mathbb{Z}}$ plugging in the definition of $K_{\mathbb{Z}}(\gamma)$ and solving the resulting optimization problem. We get

$$\begin{aligned}
\mathcal{K}^*_{\mathbb{Z}}(\gamma) &= \left\{y \in \mathbb{R}^n : (d+w)^\top (y^0 - y) \leq 0 \; \forall (d+w) \in \mathcal{K}_{\mathbb{Z}}(\gamma)\right\} \\
&= \left\{y \in \mathbb{R}^n : \max_{d+w \in \mathcal{K}_{\mathbb{Z}}(\gamma)} \left\{(d+w)^\top (y^0 - y)\right\} \leq 0\right\} \\
&= \left\{y \in \mathbb{R}^n : \max_{w \in \mathbb{Z}^n : \|w\|_1 \leq \gamma} \left\{w^\top (y^0 - y)\right\} \leq -d^\top (y^0 - y)\right\} \\
&= \left\{y \in \mathbb{R}^n : \lfloor \gamma \rfloor |y^0_j - y_j| \leq -d^\top (y^0 - y), j \in \arg\max_{e \in [n]} \left\{|y_e - y^0_e|\right\}\right\}
\end{aligned}$$

Thus, verifying whether an element is in $K^*_{\mathbb{Z}}(\gamma)$ w.r.t. ℓ_1 can be done in polynomial time. □

Lemma 3.5. *For an element $x \in \mathbb{R}^n$, one can test in polynomial time whether $x \in K^*_{\mathbb{Z}}(\gamma)$ w.r.t. ℓ_∞.*

Proof. Similarly to the previous case, we reformulate $K^*_{\mathbb{Z}}$ and get

$$\begin{aligned}
\mathcal{K}^*_{\mathbb{Z}}(\gamma) &= \left\{y \in \mathbb{R}^n : (d+w)^\top (y^0 - y) \leq 0 \; \forall (d+w) \in \mathcal{K}_{\mathbb{Z}}(\gamma)\right\} \\
&= \left\{y \in \mathbb{R}^n : \max_{d+w \in \mathcal{K}_{\mathbb{Z}}(\gamma)} \left\{(d+w)^\top (y^0 - y)\right\} \leq 0\right\} \\
&= \left\{y \in \mathbb{R}^n : \max_{w \in \mathbb{Z}^n : \|w\|_\infty \leq \gamma} \left\{w^\top (y^0 - y)\right\} \leq -d^\top (y^0 - y)\right\} \\
&= \left\{y \in \mathbb{R}^n : \lfloor \gamma \rfloor \sum_{e \in [n]} |y^0_e - y_e| \leq -d^\top (y^0 - y)\right\}.
\end{aligned}$$

The inequality in the last description can be verified in polynomial time for a given element $y \in \mathbb{R}^n$. □

Combining the previous lemmas, we obtain the main result.

Theorem 3.6. *The integral inverse mixed-integer problem w.r.t. ℓ_1 and ℓ_∞ lies in co$\mathcal{NP}$.*

Proof. Similarly as in the proof of [BR21, Theorem 15], we can provide a certificate for NO-instances based on Lemma 3.3 by a point in $\mathcal{K}^*_{\mathbb{Z}}(\gamma)$ and up to $n+1$ points in the feasible area of the lower-level problem. In contrast to the continuous case, verifying belonging to $\mathcal{K}^*_{\mathbb{Z}}(\gamma)$ is not possible via the dual norm anymore. We showed how it is still possible in polynomial time for ℓ_1 and ℓ_∞ in Lemma 3.4 and Lemma 3.5, respectively. □

Whether the integral inverse mixed-integer problem w.r.t. p-norms with $p \in (1, \infty)$ also lies in co$\mathcal{NP}$ remains open. The difference is that it is not clear whether the optimization problem in the second-to-last line in the proof of Lemmas 3.4/3.5 can be solved efficiently or has an according optimality certificate in these cases.

3.2 Integral Inverse Knapsack

We answer the open question in [RFS13] whether the integral inverse knapsack problem w.r.t. ℓ_1 is co$\mathcal{NP}$-complete. For the integral inverse knapsack problem with respect to the ℓ_1-norm, we can apply Theorem 3.6 to show that it is in co$\mathcal{NP}$. First, the required assumption of full-dimensionality is fulfilled when considering the standard MIP formulation for the knapsack problem. Furthermore, note that by construction all included variables are naturally included in the objective. In particular, no auxiliary variables are required. In addition, the size of the constraint matrix for the MIP is polynomial in the knapsack problem's data. Note that due to the need to check these points, Theorem 3.6 does not directly imply membership in co$\mathcal{NP}$ for any $\mathcal{NP}$-complete problem on the lower level. Together with the known co$\mathcal{NP}$-hardness of the integral inverse knapsack, see [RFS13, Theorem 4.3], we obtain the co$\mathcal{NP}$-completeness.

Corollary 3.7. *The integral inverse knapsack problem w.r.t. ℓ_1-norm is co$\mathcal{NP}$-complete.*

3.3 Integral Inverse Problems for P-Problems

For lower-level problems that are solvable in polynomial time, we get that the corresponding decision problem lies also in $\mathcal{NP}$.

Theorem 3.8. *The integral inverse problems where the lower-level problem is solvable in polynomial time lie in $\mathcal{NP}$.*

Proof. Verifying feasibility of given weight modifications can be done in polynomial time by solving the lower-level problem on the instance with modified weights and comparing it to the modified weights of the required solution. □

The results in Section 2.2 imply that several integral inverse problems can be solved in polynomial time for integral original weights by dropping the integrality constraint:

Theorem 3.9. *The integral inverse shortest path problem, the integral inverse assignment problem, the integral inverse min cut problem, the integral inverse min spanning tree problem and general integral inverse LPs given via a TU matrix can be solved in polynomial time if the original weights d are integral.*

The integral inverse LP and the integral inverse min weight perfect matching problem both lie in $\mathcal{NP}$ and co$\mathcal{NP}$ by Theorems 3.6 and 3.8. However, it remains open whether they are solvable in polynomial time.

4 Pessimistic Integral Inverse Problems

For the pessimistic integral inverse problem, an optimum is always attained, though there can be an arbitrarily large gap between the optimal objective value of the optimistic and pessimistic versions, as shown in the following. This is in contrast to the continuous inverse problem where a pessimistic optimal solution is not attained in general but for ε-optimality lies only ε far apart from an optimistic optimum, see [LM25, Theorem 2.12].

Proposition 4.1. *For a pessimistic integral inverse problem an optimum is always attained if the problem is feasible.*

Proof. By assumption, the feasible area of the pessimistic integral inverse problem is non-empty. Thus, we can choose any feasible modification $w' \in \mathbb{Z}^n$. If we add the constraint $\|w\| \leq \|w'\|$ to the pessimistic integral inverse problem, the optimum does not change. Due to the integrality constraint, only finitely many feasible weight modifications remain. Thus, the minimum is attained. □

Theorem 4.2. *The gap between the optimal objective value of the optimistic integral inverse knapsack problem and the pessimistic integral inverse knapsack problem can be arbitrarily large.*

Proof. For $n \in \mathbb{N}$, let the knapsack problem have $2n$ items, all with unit costs and unit weights. Furthermore, let the budget be n. Then, the knapsack problem is given by

$$\max_y \left\{ \sum_{i=1}^{2n} 1 \cdot y_i : \sum_{i=1}^{2n} y_i \leq n, y \in \{0,1\}^{2n} \right\}.$$

Fix an arbitrary set of n items to be the required optimal solution y^0.

For the optimistic integral inverse knapsack problem, no modifications are necessary, i.e. the optimal value is 0. In contrast, for the pessimistic integral inverse knapsack problem, at least costs of either all required or all forbidden elements have to be modified. By integrality, all modifications have a value of at least one. Thus, modifications of in total at least n are necessary. The optimal value of this pessimistic integral inverse knapsack problem instance is n. □

Similarly to the result for continuous inverse problems, the constructed example also shows the respective result for the integral inverse problems where weights can only be either increased or decreased by choosing the items for which costs are modified accordingly.

5 Conclusion

We studied inverse optimization problems in which an integrality constraint is imposed on the weight modifications. In general, this restriction changes the problem as illustrated by our examples for the integral inverse linear program, integral inverse min weight perfect matching and integral inverse knapsack problem. By contrast, for some inverse problems integrality can be assumed without loss of generality when the original weights are integral. This is the case, for example, for the minimum spanning tree problem and problems that admit a formulation with a totally unimodular matrix, such as the shortest path problem. We also investigated the complexity of integral inverse problems. We showed that, compared with its continuous counterpart, the complexity of integral inverse mixed-integer programming does not change. With respect to the ℓ_1- and ℓ_∞-norms, the problem remains in co$\mathcal{NP}$. In fact, we are not aware of any case in which the integral and continuous versions of an inverse discrete optimization problem belong to different complexity classes (assuming their hierarchy does not collapse) though it would be surprising if no such example existed. Candidates are the integral inverse MIP with respect to general p-norms with

$p \in (1, \infty)$, for which membership in co$\mathcal{NP}$ remains open, and integral inverse LPs, which we only know to be in both co$\mathcal{NP}$ and $\mathcal{NP}$. Finally, our observations about pessimistic integral inverse optimization problems suggest that their study might be more interesting than for the continuous inverse case.

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