Bregman Regularized Proximal Point Methods for Computing Projected Solutions of Quasi-equilibrium Problems

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Abstract

In this paper, we propose two Bregman regularized proximal point methods that provide flexibility to compute projected solutions for quasi-equilibrium problems. Each method has one Bregman projection onto the feasible set and the regularized equilibrium problem. Under standard assumptions, we prove that the methods are well-defined and that the sequences they generate converge to a projected solution of the quasi-equilibrium problem. Additionally, we prove that both methods attain an R-linear rate of convergence under the relatively strong monotonicity assumption. Furthermore, we perform numerical experiments on some test problems to illustrate the effectiveness of the proposed methods. The results obtained in this paper can be considered as the generalization and improvement of some existing works in the field of equilibrium and quasi-equilibrium problems.

Keywords: Quasi-equilibrium problem, Bregman distance, Proximal point method, Projected solution
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1. Introduction

Throughout the paper, unless otherwise specified, $\mathbb{R}^n$ denotes the n dimensional Euclidean space whose scalar product and induced norm are respectively denoted by $\langle x, y\rangle = x^\top y$ and $\|x\| = \sqrt{\langle x, x\rangle}$, for any $x, y \in \mathbb{R}^n$. Let $\mathcal{C}$ be a nonempty, closed, and convex subset of $\mathbb{R}^n$ and $\mathcal{G} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a real-valued bifunction with $\mathcal{G}(x, x) = 0$, for all $x \in \mathcal{C}$.

The equilibrium problem(EP) in the sense of Blum and Oettli [1] consists of finding $\widetilde{x} \in \mathcal{C}$ such that

$$\mathcal{G}(\widetilde{x}, y) \geq 0, \text{ for all } y \in \mathcal{C}. \tag{1.1}$$

We denote the problem (1.1) and its solution set by $EP(\mathcal{G}, \mathcal{C})$ and $S_{EP}(\mathcal{G}, \mathcal{C})$, respectively. A problem associated to $EP(\mathcal{G}, \mathcal{C})$, is that of finding $\widetilde{y} \in \mathcal{C}$ such that

$$\mathcal{G}(x, \widetilde{y}) \leq 0, \text{ for all } x \in \mathcal{C},$$

which is called the dual of $EP(\mathcal{G}, \mathcal{C})$, and its solution set is denoted by $S^d_{EP}(\mathcal{G}, \mathcal{C})$. It is worth pointing out that important problems, such as scalar and vector optimization, saddle-point (minimax), Nash equilibrium, variational inequalities, complementarity, fixed point, and many problems arising in applied nonlinear analysis can be formulated as special cases of the equilibrium problem (1.1); see, for instance, [1, 8, 9] and references cited therein. If $\mathcal{G}(x, y) = \langle \mathcal{J}(x), y - x\rangle$ for some operator $\mathcal{J}$:

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$\mathbb{R}^n \to \mathbb{R}^n$, the problem (1.1) becomes the classical variational inequality problem (VIP) introduced by Stampacchia [12].

On the other hand, one of the most widely used techniques for solving EP is the so-called regularization method. In particular, Moudafi [5] and Iusem and Sosa [25], proposed iterative schemes that solve the regularized equilibrium problem $EP(\mathcal{G}_k, \mathcal{C})$ at each iteration, where regularized bifunction $\mathcal{G}_k$ is given by

$$\mathcal{G}_k(x, y) = \mathcal{G}(x, y) + \gamma_k \langle x - x^k, y - x \rangle, \tag{1.2}$$

for a positive bounded sequence of regularization parameters $\{\gamma_k\}$.

In 2012, Burachik and Kassy [26] introduced a method that incorporates a Bregman function into the regularization term. Specifically, they define

$$\mathcal{G}_k(x, y) = \mathcal{G}(x, y) + \gamma_k \langle \nabla\psi(x) - \nabla\psi(x^k), y - x \rangle, \tag{1.3}$$

where ψ is the Bregman function. This approach can be viewed as a generalization of (1.2). Further details can be found in [26].

In 2022, Bento et al. [35] introduced a new regularization scheme for solving equilibrium problems on Hadamard manifolds. In particular, in the Euclidean setting, their method solves at each iteration the regularized equilibrium problem $EP(\mathcal{G}_k, \mathcal{C})$, where $\mathcal{G}_k$ is defined as follows:

$$\mathcal{G}_k(x, y) = \mathcal{G}(x, y) + \gamma_k \left(\|y - x^k\|^2 - \|x - x^k\|^2 \right). \tag{1.4}$$

It is easy to see that (1.2) is equivalent to

$$\mathcal{G}_k(x, y) = \mathcal{G}(x, y) + \frac{\gamma_k}{2} \left(\|y - x^k\|^2 - \|x - x^k\|^2 - \|y - x\|^2 \right). \tag{1.5}$$

Thus, the regularization in (1.4) eliminates the term $\|y - x\|^2$ present in (1.5), thereby becoming separable in the variables x and y.

The broader formulation than the classical equilibrium problem (EP) is called the quasi-equilibrium problem (QEP):

$$\text{find } \widetilde{x} \in \mathcal{Q}(\widetilde{x}) \text{ such that } \mathcal{G}(\widetilde{x}, y) \geq 0, \text{ for all } y \in \mathcal{Q}(\widetilde{x}), \tag{1.6}$$

where $\mathcal{Q} : \mathcal{C} \rightrightarrows \mathbb{R}^n$ is a constraint map for a given set $\mathcal{C} \subset \mathbb{R}^n$. The problem (1.6) and its solution set will be denoted by $QEP(\mathcal{G}, \mathcal{Q})$ and $S_{QEP}(\mathcal{G}, \mathcal{Q})$, respectively. The problem (1.6) is very general in the sense that it includes many relevant mathematical problems such as quasi-optimization problems, quasi-variational inequalities, mixed quasi-variational inequalities, generalized Nash equilibrium problems, as well as all special cases of the equilibrium problem (1.1); see, for instance, [32, 41].

Clearly, when $\mathcal{Q}(x) = \mathcal{C}$, for all $x \in \mathcal{C}$, the quasi-equilibrium problem (1.6) becomes the equilibrium problem (1.1). Moreover, if $\mathcal{G}(x, y) = \langle \mathcal{J}(x), y - x \rangle$ for some operator $\mathcal{J} : \mathbb{R}^n \to \mathbb{R}^n$, then the QEP becomes a quasi-variational inequality problem introduced in [13] of finding $\widetilde{x} \in \mathcal{Q}(\widetilde{x})$ such that

$$\langle \mathcal{J}(\widetilde{x}), y - \widetilde{x} \rangle \geq 0, \text{ for all } y \in \mathcal{Q}(\widetilde{x}). \tag{1.7}$$

Existence results for quasi-equilibrium and quasi-variational inequality problems have been established in the literature under the assumption that the constraint set-valued mapping $\mathcal{Q}$ is a self-map, that is, $\mathcal{Q}(\mathcal{C}) \subseteq \mathcal{C}$; see, for example, [32, 33] and references therein. However, such a requirement is quite restrictive and is not always satisfied in applications.

For instance, in the electricity market model described by Aussel et al. [18] the set-valued constraint map may not be a self-map, that is, $\mathcal{Q}(\mathcal{C}) \not\subseteq \mathcal{C}$. Consequently, the corresponding quasi-variational inequality problem may fail to admit a solution. To address this limitation, Aussel et al. [18] studied quasi-variational inequalities in the more general setting where the constraint set-valued mapping $\mathcal{Q}$ is not a self-map, that is, the case where the constraint map may take values outside the feasible set. In this case, the main challenge is that the existence of a point $x \in \mathcal{Q}(x)$ satisfying (1.7) cannot, in general, be guaranteed. For this reason, the authors introduced the concept of projected solutions for quasi-variational inequalities. Later, Cotrina and Zúñiga [19] adapted the notion of projected solution for quasi-equilibrium problems. It is worth pointing out that, in recent years, considerable progress has been made in establishing existence results for projected solutions both in finite dimensions [19, 42] and in Banach spaces [45] settings.

In the case where the constraint set-valued mapping is a self-map, that is, $\mathcal{Q}(\mathcal{C}) \subseteq \mathcal{C}$, numerous numerical algorithms have been developed for solving $QEP(\mathcal{G}, \mathcal{Q})$; see, for instance, [2, 6, 7, 16]. In particular, Santos and Souza [2] proposed a proximal point method that extends the iterative methods for equilibrium problems introduced in [5, 11] to the quasi-equilibrium problem.

More recently, Aussel et al. [36] investigated the more general case where the constraint set-valued mapping is not necessarily a self-map and proposed a proximal point method for computing projected solutions of $QEP(\mathcal{G}, \mathcal{Q})$. Their method can be viewed as a generalization of the proximal point approach proposed in [2].

One other strategy to potentially enhance numerical algorithms is the incorporation of Bregman distances. These distances induce a geometry-adaptive measure of proximity that can be substantially more appropriate than the Euclidean squared norm, particularly when the feasible set or underlying model exhibits intrinsic non-Euclidean structure, such as positivity constraints or entropic geometry. In the equilibrium framework of Blum and Oettli [1], proximal regularization is classically built with quadratic terms and leads to Rockafellar-type proximal point iterations; this Euclidean viewpoint has been widely developed for EP and QEP in Hilbert or Banach settings [1, 2, 5, 10, 11, 25]. Replacing the quadratic stabilization by a Bregman distance d_ψ (generated by a Legendre or strictly convex kernel ψ) yields non-Euclidean proximal steps and projections, recovering the Euclidean case when $\psi = \frac{1}{2}\|\cdot\|^2$, and providing a natural descent in d_ψ [20, 23, 38]. This perspective is particularly attractive in settings where Bregman proximal mappings admit simple closed forms and improved numerical behavior [28, 29]. Recent works have consequently developed Bregman regularized proximal point schemes for equilibrium and quasi-equilibrium problems, including separable regularizations, and reported advantages over Euclidean counterparts in geometry-sensitive regimes; see, for instance [26, 27, 31, 34, 35].

Recently, Dias et al. [27] proposed a Bregman regularized version of the proximal point method in [2] for solving quasi-equilibrium problems. The results obtained in [27] may be viewed as a twofold generalization: first, it extends the equilibrium problem framework of [26] to quasi-equilibrium problems; second, it replaces the standard squared-norm regularization used in [2] with a Bregman distance.

Inspired by the regularization technique (1.4), Soubeyran et al. [34] studied the convergence of the separable proximal point method proposed in [35] for a more general context of quasi-equilibrium problem by replacing the Euclidean distance in (1.4) with a Bregman distance as follows: Given x^k, compute $x^{k+1} \in S_{EP}(\mathcal{G}_k, \mathcal{Q}_k)$, where

$$\mathcal{G}_k(x, y) = \mathcal{G}(x, y) + \gamma_k \left(d_\psi(y, x^k) - d_\psi(x, x^k) \right), \ \mathcal{Q}_k := \mathcal{Q}\left(x^k\right)$$

and $\{\gamma_k\} \subset (0, \infty)$ is a bounded sequence of regularization parameters.

Motivated by the above literature, in particular the works in [27, 34, 36], in this paper, we consider quasi-equilibrium problems in which the constraint map $\mathcal{Q}$ need not be a self-map, that is, $\mathcal{Q}(\mathcal{C}) \nsubseteq \mathcal{C}$, and propose two Bregman-regularized proximal point methods for computing projected solutions of such problems. The resulting schemes extend proximal point method for computing projected solutions of quasi-equilibrium problems proposed in [36] and generalize several existing methods in the literature, including those in [2, 27, 34]. A distinctive feature of the present work is the systematic replacement of classical Euclidean projections with Bregman projections, which both broadens applicability to non-self constraint geometries and constitutes a central novelty of the paper.

The paper is organized as follows. Section 2 presents the basic definitions, concepts, and preliminary results that are needed for later sections. Section 3 is devoted to describing the proposed algorithms and establishing its convergence properties. In Section 4, we provide numerical experiments that illustrate the performance of the proposed algorithms in comparison with the other existing methods and conclude in Section 5.

2. Preliminaries

In this section, we introduce several important definitions, concepts, and lemmas that constitute the theoretical foundation of the proposed algorithms. These include the Bregman distance, the Bregman projection, and related notions of equilibrium problems.

Definition 2.1. [8] *A bifunction $\mathcal{G} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is said to be*

(i) μ-strongly monotone on $\mathcal{C}$ if there exists a constant $\mu > 0$ such that

$$\mathcal{G}(x, y) + \mathcal{G}(y, x) \leq -\mu \|x - y\|^2, \text{ for all } x, y \in \mathcal{C};$$

(ii) monotone on $\mathcal{C}$ if

$$\mathcal{G}(x, y) + \mathcal{G}(y, x) \leq 0, \text{ for all } x, y \in \mathcal{C};$$

(iii) strictly monotone on $\mathcal{C}$ if

$$\mathcal{G}(x, y) + \mathcal{G}(y, x) < 0, \text{ for all } x, y \in \mathcal{C} \text{ with } x \neq y;$$

(iv) pseudo-monotone on $\mathcal{C}$ if

$$\mathcal{G}(x, y) \geq 0 \text{ implies } \mathcal{G}(y, x) \leq 0, \text{ for all } x, y \in \mathcal{C}.$$

The following implications are true between these concepts:

$$\text{(i)} \implies \text{(iii)} \implies \text{(ii)} \implies \text{(iv)},$$

however, the converse implications do not hold in general [8].

Let $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a function. We denote by $\text{dom}(\psi)$, the domain of ψ, that is $\text{dom}(\psi) := \{x \in \mathbb{R}^n : \psi(x) < +\infty\}$ and also denoted by $\text{int}(\text{dom}(\psi))$, the interior of the domain of ψ.

Definition 2.2. [14] *The function $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be:*

(i) *proper if* $\mathrm{dom}(\psi) \neq \varnothing$;

(ii) *lower semi-continuous if the set* $\{(x, \tau) \in \mathbb{R}^n \times \mathbb{R} : \psi(x) \leq \tau\}$ *is closed in* $\mathbb{R}^n \times \mathbb{R}$;

(iii) *convex if*

$$\psi(\beta x + (1-\beta)y) \leq \beta\psi(x) + (1-\beta)\psi(y), \quad \textit{for ll } x, y \in \mathrm{dom}(\psi) \textit{ and } \beta \in [0,1];$$

(iv) *strictly convex if*

$$\psi(\beta x + (1-\beta)y) < \beta\psi(x) + (1-\beta)\psi(y), \quad \textit{for all } x, y \in \mathrm{dom}(\psi) \textit{ with } x \neq y \textit{ and } \beta \in (0,1).$$

2.1. Bregman functions, distances and projection

Let $\Omega \subseteq \mathbb{R}^n$ be a closed and convex set with $\mathrm{int}\,\Omega \neq \varnothing$, where $\mathrm{int}\,\Omega$ denotes the interior of Ω. Consider $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ a strictly convex, proper and lower semi-continuous function with the closed domain $\mathcal{D} := \mathrm{dom}(\psi)$ and continuously differentiable on $\mathrm{int}\,\Omega$.

Definition 2.3. (see; [20, 21]) *The Bregman distance (or Bregman divergence) associated to* ψ *with zone* Ω *is the bifunction function* $d_\psi : \Omega \times \mathrm{int}\,\Omega \to [0, +\infty)$ *defined by*

$$d_\psi(x, y) = \psi(x) - \psi(y) - \langle \nabla\psi(y),\, x - y \rangle. \tag{2.1}$$

The following are the examples of Bregman functions with their respective Bregman distances, which are well documented in the literature; see, for instance, [20, 24, 28, 39]:

Let $x = (x_1, x_2, \ldots, x_n)^\top$ and $y = (y_1, y_2, \ldots, y_n)^\top$ be two points in $\mathbb{R}^n$.

Example 2.1. *The Squared Euclidean distance (SED) given by*

$$d_\psi(x, y) = \frac{1}{2}\|x - y\|^2$$

generated from the squared norm function $\psi(x) = \frac{1}{2}\|x\|^2$ *with* $\Omega = \mathbb{R}^n$ *and gradient* $\nabla\psi(x) = x$.

Example 2.2. *The Kullback–Leibler distance (KLD) generated by the Boltzmann–Shannon entropy function*

$$\psi(x) = \sum_{i=1}^{n} x_i \log(x_i), \quad \textit{with } \Omega = \mathbb{R}^n_+,$$

is given by

$$d_\psi(x, y) = \sum_{i=1}^{n} \left(x_i \log\left(\frac{x_i}{y_i}\right) + y_i - x_i \right)$$

and the gradient of ψ *is* $\nabla\psi(x) = (1 + \log x_1,\, 1 + \log x_2,\, \ldots,\, 1 + \log x_n)^\top$.

Example 2.3. *The Itakura–Saito distance (ISD) generated by the Burg entropy function*

$$\psi(x) = -\sum_{i \in I(x)} \log(x_i), \quad \textit{with } \Omega = \mathbb{R}^n_+,$$

is given by

$$d_\psi(x, y) = \sum_{i \in I(x)} \left(\frac{x_i}{y_i} - \log\left(\frac{x_i}{y_i}\right) - 1 \right),$$

where $I(x) = \{ i = 1, \dots, n : x_i > 0 \}$ *and the gradient of* ψ *is* $\nabla\psi(x) = -\left(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}\right)^\top$.

The following properties of the Bregman distance follow directly from (2.1):

Lemma 2.1. [38, Proposition 2.3] *For any* $x \in \mathcal{D}$, *and* $y, w \in \operatorname{int}\mathcal{D}$, *the following hold:*

(i) (The two–point identity) $d_\psi(w, y) + d_\psi(y, w) = \langle \nabla\psi(w) - \nabla\psi(y), w - y\rangle$;

(ii) (The three–point identity) $d_\psi(x, y) - d_\psi(x, w) - d_\psi(w, y) = \langle \nabla\psi(y) - \nabla\psi(w), w - x\rangle$.

The following definition introduces the notion of relative strong monotonicity for the bifunction $\mathcal{G}$:

Definition 2.4. [44] *An operator* $\mathcal{J} : \mathbb{R}^n \to \mathbb{R}^n$ *is said to be* μ*-strongly monotone relative to* ψ *if,*

$$\langle \mathcal{J}x - \mathcal{J}y, x - y\rangle \geq \mu\langle \nabla\psi(x) - \nabla\psi(y), x - y\rangle, \quad \text{for all } x, y \in \operatorname{dom}(\mathcal{J}) \cap \operatorname{int}\mathcal{D}, \tag{2.2}$$

In particular, when $\mathcal{G}(x, y) = \langle \mathcal{J}(x), y - x\rangle$ the left hand side of (2.2) becomes

$$\langle \mathcal{J}x - \mathcal{J}y, x - y\rangle = -\left(\mathcal{G}(x, y) + \mathcal{G}(y, x)\right).$$

This relationship provides the basis for defining μ-strong monotonicity of the bifunction $\mathcal{G}$ relative to ψ, as follows:

Definition 2.5. *A bifunction* $\mathcal{G} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *is said to be* μ*-strongly monotone relative to* ψ *if, for any* $x, y \in \operatorname{dom}(\mathcal{G}) \cap \operatorname{int}\mathcal{D}$,

$$\mathcal{G}(x, y) + \mathcal{G}(y, x) \leq -\mu\langle \nabla\psi(x) - \nabla\psi(y), x - y\rangle = -\mu\left(d_\psi(x, y) + d_\psi(y, x)\right).$$

In particular, when $\psi = \frac{1}{2}\|\cdot\|^2$, *the bifunction* $\mathcal{G}$ *is* μ*-strongly monotone in the usual sense.*

Definition 2.6. [20] *Let* $\mathcal{C} \subset \operatorname{int}\mathcal{D}$ *be a nonempty, closed, and convex set. The Bregman projection of* $x \in \operatorname{int}\mathcal{D}$ *onto* $\mathcal{C}$ *is the unique point* $P^\psi_{\mathcal{C}}(x) \in \mathcal{C}$ *that satisfies*

$$d_\psi(P^\psi_{\mathcal{C}}(x), x) = \inf\{d_\psi(y, x) : y \in \mathcal{C}\}. \tag{2.3}$$

Remark 2.1. *The operator* $P^\psi_{\mathcal{C}}$ *is continuous (see Bauschke et al. [24, Theorem 4.3]). In addition,* $P^\psi_{\mathcal{C}}$ *satisfies the following properties (see [21, Corollary 4.4]): for each* $x \in \operatorname{int}\mathcal{D}$,

(i) $z = P^\psi_{\mathcal{C}}(x)$ *if and only if* $\langle \nabla\psi(x) - \nabla\psi(z), y - z\rangle \leq 0, \quad$ *for all* $y \in \mathcal{C}$;

(ii) $d_\psi(y, z) + d_\psi(z, x) \leq d_\psi(y, x), \quad$ *for all* $y \in \mathcal{C}$.

Following the work of Burachik and Scheimbeber [22], in this paper we adopt the following assumptions on ψ:

(A1) The right level sets of $d_\psi(y, \cdot)$:

$$L_2(y, \tau) := \{x \in \operatorname{int}\mathcal{D} : d_\psi(y, x) \leq \tau\}$$

are bounded for all $\tau \geq 0$ and for all $y \in \mathcal{D}$.

(A2) If $\{x^k\}$ and $\{y^k\} \subset \operatorname{int}\mathcal{D}$ with $\lim_{k\to+\infty} x^k = \widetilde{x}$, $\lim_{k\to+\infty} y^k = \widetilde{x}$, and $\lim_{k\to+\infty} d_\psi(x^k, y^k) = 0$, then

$$\lim_{k\to+\infty}\left(d_\psi(\widetilde{x}, x^k) - d_\psi(\widetilde{x}, y^k)\right) = 0.$$

(A3) If $\{x^k\} \subset \mathcal{D}$ and $\{y^k\} \subset \operatorname{int} \mathcal{D}$ are sequences such that $\{x^k\}$ is bounded, $\lim_{k\to+\infty} y^k = \widetilde{y}$ and $\lim_{k\to+\infty} d_\psi(x^k, y^k) = 0$, then $\lim_{k\to+\infty} x^k = \widetilde{y}$.

(A4) For every $y \in \Omega$, there exists $x \in \operatorname{int} \mathcal{D}$ such that $\nabla\psi(x) = y$.

(A5) If $\{y^k\} \subset \operatorname{int} \mathcal{D}$ and $\lim_{k\to+\infty} \|y^k\| = +\infty$, then $\lim_{k\to+\infty} d_\psi(y^k, v) = +\infty$, for every $v \in \operatorname{int} \mathcal{D}$.

Remark 2.2. *The Bregman distances presented in Examples 2.1–2.3 satisfy assumptions* (A1)–(A5)*; see [30, 31].*

Definition 2.7. [38] *A sequence $\{x^k\} \subset \operatorname{int} \mathcal{D}$ is said to be d_ψ-Fejér convergent to a nonempty set $U \subset \operatorname{int} \mathcal{D}$ if, for all $k \in \mathbb{N}$,*

$$d_\psi(x, x^{k+1}) \leq d_\psi(x, x^k), \quad \textit{for all } x \in U.$$

Lemma 2.2. [38] *Let $\{x^k\}$ be the d_ψ-Fejér convergent sequence to a set $U \subset \operatorname{int} \mathcal{D}$. Then, the following hold:*

(i) *for all $x \in U$, $\{d_\psi(x, x^k)\}$ is decreasing;*

(ii) *$\{x^k\}$ is bounded if* (A1) *is satisfied.*

2.2. Basic concepts and some important results

Given a nonempty set $\mathcal{B} \subseteq \mathbb{R}^n$, we denote by $\widetilde{co}(\mathcal{B})$ the closure of the convex hull of $\mathcal{B}$. For the constraint map $\mathcal{Q} : \mathcal{C} \rightrightarrows \mathbb{R}^n$ in $QEP(\mathcal{G}, \mathcal{Q})$ (1.6), we define $\mathcal{M} := \widetilde{co}(\mathcal{Q}(\mathcal{C}))$.

Next, we introduce the notion of a Bregman projected solution for quasi-equilibrium problems with possibly non-self constraint maps. This definition extends the notion of projected solution introduced by Cotrina and Zúñiga [19].

Definition 2.8. *Let $\mathcal{C}, \mathcal{M} \subset \operatorname{int} \mathcal{D}$. Let $\mathcal{Q} : \mathcal{C} \rightrightarrows \mathbb{R}^n$ be a set-valued mapping and $\mathcal{G} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ be a bifunction. A point $\widetilde{x} \in \mathcal{C}$ is called a Bregman projected solution of $QEP(\mathcal{G}, \mathcal{Q})$ if there exists $\widetilde{y} \in \mathcal{Q}(\widetilde{x})$ such that*

(i) *$\widetilde{x} = P^\psi_{\mathcal{C}}(\widetilde{y})$;*

(ii) *$\mathcal{G}(\widetilde{y}, u) \geq 0$, for all $u \in \mathcal{Q}(\widetilde{x})$.*

The set of Bregman projected solutions is denoted by $S^{Bp}(\mathcal{G}, \mathcal{Q})$. Clearly, if $\widetilde{y} \in \mathcal{Q}(\widetilde{x}) \cap \mathcal{C}$, then no Bregman projection is required. Moreover, if such a point $\widetilde{y}$ satisfies condition (ii) of Definition 2.8, then, it is a classical solution of the quasi-equilibrium problem. Hence, $\mathcal{Q}(\mathcal{C}) \cap \mathcal{C} \neq \varnothing$ is the necessary condition for the existence of a classical solution of $QEP(\mathcal{G}, \mathcal{Q})$.

Remark 2.3. *In the particular case $\psi = \frac{1}{2}\|\cdot\|^2$, the operators $P^\psi_{\mathcal{C}}$ and $P_{\mathcal{C}}$ coincide, where $P_{\mathcal{C}}$ denotes the classical Euclidean projection onto $\mathcal{C}$. That is, for any $x \in \mathbb{R}^n$, the projection $P_{\mathcal{C}}(x)$ is the unique point in $\mathcal{C}$ satisfying*

$$\|x - P_{\mathcal{C}}(x)\| = \inf\{\|x - y\| : y \in \mathcal{C}\}.$$

In light of this remark, Definition 2.8 reduces to the well-known definition of a projected solution introduced in [19]*, as follows:*

Definition 2.9. *Let $\mathcal{Q} : \mathcal{C} \rightrightarrows \mathbb{R}^n$ be a set-valued mapping and $\mathcal{G} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ be a bifunction. A point $\widetilde{x} \in \mathcal{C}$ is said to be a projected solution of $QEP(\mathcal{G}, \mathcal{Q})$, if there exists $\widetilde{y} \in \mathcal{Q}(\widetilde{x})$ such that*

(i) $\widetilde{x} = P_{\mathcal{C}}(\widetilde{y})$;

(ii) $\mathcal{G}(\widetilde{y}, u) \geq 0$, *for all* $u \in \mathcal{Q}(\widetilde{x})$.

Definition 2.10. [17] *The multivalued mapping* $\mathcal{Q} : \mathcal{C} \rightrightarrows \mathbb{R}^n$ *is said to be M-continuous on* $\mathcal{C}$ *if:*

(i) For any two sequences $\{x^k\} \subset \mathcal{C}, \{y^k\} \subset \mathcal{M}$ *with* $x^k \to x$, $y^k \in \mathcal{Q}(x^k)$ *and* $y^k \to y$ *implies that* $y \in \mathcal{Q}(x)$, *which means that the graph of* $\mathcal{Q}$ *is sequentially weakly closed.*

(ii) For any sequence $\{x^k\} \subset \mathcal{C}$ *with* $x^k \to x$ *and for each* $y \in \mathcal{Q}(x)$ *there exists a sequence* $\{y^k\} \subset \mathcal{M}$ *with* $y^k \in \mathcal{Q}(x^k)$ *such that* $y^k \to y$.

The following lemma is a Bregman distance analogue of the celebrated Opial's lemma and plays a key role in establishing the convergence of sequences:

Lemma 2.3. [40, Lemma 2.8] *Let* $\mathcal{U} \subset \mathbb{R}^n$ *be a nonempty, closed and convex set, and let* $\{x^k\}$ *be a sequence in* $\mathbb{R}^n$ *such that the two conditions hold:*

(i) *for every* $x \in \mathcal{U}$, $\lim_{k\to\infty} d_\psi(x, x^k)$ *exists;*

(ii) *all cluster point of* $\{x^k\}$ *belongs to* $\mathcal{U}$.

Then $\{x^k\}$ *converges to a point in* $\mathcal{U}$.

3. Projected Solutions of $QEP(\mathcal{G}, \mathcal{Q})$

In this section, we present our proposed methods for finding projected solutions of $QEP(\mathcal{G}, \mathcal{Q})$ and provide their convergence analysis. We begin with the following assumptions:

Let $\mathcal{G} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ be a bifunction. We consider the following assumptions:

(P1) $\mathcal{G}(x, x) = 0$, for all $x \in \mathcal{M}$;

(P2) $y \mapsto \mathcal{G}(x, y)$ is convex, for every $x \in \mathcal{M}$;

(P3) $\mathcal{G}(.,.) : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ is jointly weakly continuous on $\mathcal{M} \times \mathcal{M}$ (or, the graph of $\mathcal{G}$ is sequentially weakly closed) in the sense that, if $x, y \in \mathcal{M}, \{x^k\}$ and $\{y^k\}$ are sequences in $\mathcal{M}$ weakly converging to x and y, respectively, then $\mathcal{G}(x^k, y^k)$ converges to $\mathcal{G}(x, y)$;

(P4) $\mathcal{G}$ is monotone;

(P5) For any sequence $\{z^k\} \subset \mathcal{M}$ such that $\lim_{k\to\infty} \|z^k\| = +\infty$, there exists $v \in \mathcal{M}$ and $k_0 \in \mathbb{N}$ such that $\mathcal{G}(z^k, v) \leq 0$, for all $k \geq k_0$.

Throughout this paper, we consider the quasi-equilibrium problem:

$$\text{find } \widetilde{x} \in \mathcal{Q}(\widetilde{x}) \text{ such that } \mathcal{G}(\widetilde{x}, y) \geq 0, \text{ for all } y \in \mathcal{Q}(\widetilde{x}), \tag{3.1}$$

where $\mathcal{G} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ and $\mathcal{Q} : \mathcal{C} \rightrightarrows \mathbb{R}^n$ is the constraint map for a given set $\mathcal{C} \subseteq \mathbb{R}^n$.

We assume that $\mathcal{M}$, $\mathcal{C} \subset \operatorname{int} \mathcal{D}$ and that d_ψ is the the Bregman distance with zone $\mathcal{M}$ generated by a strictly convex function ψ satisfying assumptions (A1)–(A5), and $P^\psi_{\mathcal{C}}$ denotes the associated Bregman projection. Moreover, the bifunction $\mathcal{G}\colon \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ is assumed to satisfy assumptions (P1)–(P5). In addition, let $\mathcal{Q}\colon \mathcal{C} \rightrightarrows \mathbb{R}^n$ be an M-continuous set-valued mapping such that, for every $x \in \mathcal{C}$, the set $\mathcal{Q}(x)$ is nonempty, closed, and convex subset of $\mathcal{M}$.

The convergence analysis of our Algorithms can be achieved through the following special type of solution set:

$$S^* = \left\{ \widetilde{y} \in \bigcap_{z\in\mathcal{C}} \mathcal{Q}(z) : \mathcal{G}(\widetilde{y}, y) \geq 0, \ \forall y \in \bigcup_{z\in\mathcal{C}} \mathcal{Q}(z) \right\}. \tag{3.2}$$

We define

$$\widetilde{S}_{QEP}(\mathcal{G}, \mathcal{Q}) = \left\{ \widetilde{y} \in \mathcal{M} : P_{\mathcal{C}}^{\psi}(\widetilde{y}) \in S^{Bp}(\mathcal{G}, \mathcal{Q}) \right\}.$$

Observe that $S^* \subset \widetilde{S}_{QEP}(\mathcal{G}, \mathcal{Q})$. We assume that the set S^*, defined in (3.2), is nonempty. Furthermore, we define

$$S^*_{Bp} := P_{\mathcal{C}}^{\psi}(S^*) = \bigcup_{\widetilde{y}\in S^*} P_{\mathcal{C}}^{\psi}(\widetilde{y}).$$

These sets will play a vital role in our convergence analysis and have been considered in the convergence analysis of several algorithms for quasi-equilibrium problems; see, for instance, [3, 4, 6, 7, 36] and the references cited therein.

The first Bregman regularized proximal point method is proposed as follows::

Algorithm 3.1 Bregman Proximal Point Method (B-PPM) to find projected solution of (3.1)

Step 0: Take a bounded sequence of regularization parameters $\{\gamma_k\} \subset (0, \infty)$, tolerance $\varepsilon > 0$ and choose $y^0 \in \mathcal{M}$. Set $k := 0$;

Step 1: (Outer loop) Set $x^0 = P_{\mathcal{C}}^{\psi}(y^0)$;

Step 2: (Inner loop) Given x^k and y^k, compute $y^{k+1} \in S_{EP}(\mathcal{G}_k, \mathcal{Q}_k)$, where

$$\mathcal{G}_k(x, y) = \mathcal{G}(x, y) + \gamma_k \langle \nabla\psi(x) - \nabla\psi(y^k), \, y - x\rangle \tag{3.3}$$

and $\mathcal{Q}_k := \mathcal{Q}(x^k)$.

Step 3: If $\|y^{k+1} - y^k\| < \varepsilon$, then STOP. Otherwise, set $k := k + 1$, and go to **Step 1** by setting $x^{k+1} = P_{\mathcal{C}}^{\psi}(y^{k+1})$.

Remark 3.1.

1. *At each iteration, Algorithm 3.1 solves a Bregman regularized equilibrium problem. Thus, the proposed method is well defined whenever the solution to* (3.3) *exists. It is shown in [26, Corollary 3.2] that* (3.3) *admits a solution if the following assumption is satisfied: for a fixed* $\widetilde{y} \in \mathcal{M}$*, every sequence* $\{y^k\} \subset \mathcal{M}$ *satisfies* $\lim_{k\to\infty} \|y^k\| = \infty$*, we have*

$$\liminf_{k\to\infty} \left(\mathcal{G}(\widetilde{y}, y^k) + \gamma\langle \nabla\psi(\widetilde{y}) - \nabla\psi(y^k), \widetilde{y} - y^k\rangle \right) > 0. \tag{3.4}$$

Moreove, if ψ *is strictly convex, then* (3.3) *admits a unique solution. On the other hand, Censor et al. [37] pointed out that, if* ψ *is a Bregman function with zone* A *and* $\widetilde{A} \subset A$ *is closed and convex, then* ψ *can also be considered as a Bregman function with zone* $\widetilde{A}$*. This fact applies to Algorithm 3.1, since* $\mathcal{Q}_k \subset \mathcal{M}$ *is convex and closed for all* $k \in \mathbb{N}$*, together with the assumptions imposed the bifunction* $\mathcal{G}$ *and on the Bregman distance* d_ψ *with zone* $\mathcal{M}$*. Thus, by assuming that* (3.4) *holds,* ***Step 2*** *of Algorithm 3.1 is well defined; see also [27, Remark 3.3].*

2. *The following are some special cases of our algorithm:*
   - *In the special case* $\psi = \frac{1}{2}\|\cdot\|^2$, *Algorithm 3.1 reduces to one proposed in [36, Algorithm 3].*
   - *If* $\mathcal{Q}(\mathcal{C}) \subseteq \mathcal{C}$, *then [27, Algorithm 3.2] is recovered from Algorithm 3.1.*

3.1. Convergence Analysis

In this subsection, we establish the convergence properties of Algorithm 3.1 under a standard set of assumptions.

The following proposition shows that, whenever $y^{k+1} = y^k$, Algorithm 3.1 produces the exact solution. This supports that our stopping rule $\|y^{k+1} - y^k\| < \varepsilon$ is practical.

Proposition 3.1. *Let* $\{y^k\}$ *be the sequence generated by Algorithm 3.1 such that* $y^{k+1} = y^k$, *for some* $k \geq 0$, *then* $P^{\psi}_{\mathcal{C}}(y^k)$ *is a Bregman projected solution of* (3.1).

Proof. From Algorithm 3.1, we have that $y^{k+1} \in S_{EP}(\mathcal{G}_k, \mathcal{Q}_k)$ implies that $y^{k+1} \in \mathcal{Q}(x^k)$ and

$$\mathcal{G}(y^{k+1}, y) + \gamma_k \left\langle \nabla\psi(y^{k+1}) - \nabla\psi(y^k), y - y^{k+1} \right\rangle \geq 0, \quad \forall y \in \mathcal{Q}(x^k).$$

Since, $y^{k+1} \in \mathcal{Q}(x^k)$ with $y^{k+1} = y^k$, we have $\mathcal{G}(y^{k+1}, y) \geq 0$, for all $y \in \mathcal{Q}(x^k)$. Furthermore, $P^{\psi}_{\mathcal{C}}(y^{k+1}) = P^{\psi}_{\mathcal{C}}(y^k)$ implies that $x^k = P^{\psi}_{\mathcal{C}}(y^{k+1})$. Thus, x^k is the Bregman projected solution of (3.1). □

Next, we recall the following result from [27] which establishes the relationship between the solutions of the dual equilibrium problem and the regularized equilibrium problem in Bregman setting:

For some fixed $\gamma > 0$ and $\widetilde{y} \in \mathbb{R}^n$, a bifunction $\widetilde{\mathcal{G}} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ given by

$$\widetilde{\mathcal{G}}(x, y) = \mathcal{G}(x, y) + \gamma \left\langle \nabla\psi(x) - \nabla\psi(\widetilde{y}), y - x \right\rangle, \tag{3.5}$$

is known as the regularization of $\mathcal{G}$ and the equilibrium problem $EP(\widetilde{\mathcal{G}}, \mathcal{Y})$ for some $\mathcal{Y} \subseteq \mathcal{M}$ is called the regularized equilibrium problem.

Lemma 3.2. [27, Proposition 3.7] *Suppose that the bifunction* $\mathcal{G} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ *satisfies* (P1)–(P3) *and* $\mathcal{Y}$ *be a nonempty, closed and convex subset of* $\mathcal{M}$. *Let* $\widetilde{y} \in \mathcal{M}$ *be an arbitrary point,* $\hat{y}, y^* \in \mathcal{M}$ *such that* $\hat{y} \in S_{EP}(\widetilde{\mathcal{G}}, \mathcal{Y})$ *and* $y^* \in S^d_{EP}(\mathcal{G}, \mathcal{Y})$, *where* $\widetilde{\mathcal{G}}$ *is given by* (3.5). *Then*

$$d_\psi(y^*, \hat{y}) + d_\psi(\hat{y}, \widetilde{y}) \leq d_\psi(y^*, \widetilde{y}).$$

Proposition 3.3. *Let* $\{y^k\}$ *be the sequence generated by Algorithm 3.1. Then, the following assertions hold:*

(i) $\{y^k\}$ *is* d_ψ*-Fejér convergent to* S^**;*

(ii) $\{y^k\}$ *is bounded;*

(iii) $\lim\limits_{k\to\infty} d_\psi(y^{k+1}, y^k) = 0.$

Proof. (i) Let $y^* \in S^*$ be an arbitrary point. Since $S^* \subset S_{EP}(\mathcal{G}, \mathcal{Q}(z))$, for all $z \in \mathcal{C}$, it follows that $y^* \in S_{EP}(\mathcal{G}, \mathcal{Q}(z))$, and hence $\mathcal{G}(y^*, y) \geq 0$, for all $y \in \mathcal{Q}(z)$. By the monotonicity of $\mathcal{G}$, we obtain $\mathcal{G}(y, y^*) \leq 0$, for all $y \in \mathcal{Q}(z)$. Therefore, $y^* \in S^d_{EP}(\mathcal{G}, \mathcal{Q}(z))$, for all $y \in \mathcal{Q}(z)$. In particular, let us take $z = x^k$. By Algorithm 3.1, we have $y^{k+1} \in S_{EP}(\mathcal{G}_k, \mathcal{Q}(x^k))$. Applying Lemma 3.2 with $\widetilde{\mathcal{G}} = \mathcal{G}_k$ in (3.3), $\hat{y} = y^{k+1}$, and $\widetilde{y} = y^k$, we have

$$d_\psi(y^*, y^{k+1}) + d_\psi(y^{k+1}, y^k) \leq d_\psi(y^*, y^k), \quad \text{for all } k \in \mathbb{N}. \tag{3.6}$$

Since $d_\psi(y^{k+1}, y^k) \geq 0$, we have

$$d_\psi(y^*, y^{k+1}) \leq d_\psi(y^*, y^k), \quad \text{for all } k \in \mathbb{N},$$

and y^* was arbitrary taken from S^*, $\{y^k\}$ is d_ψ-Fejér convergent to S^*.

(ii) By Lemma 2.2 and (i) above, we conclude that $\{d_\psi(y^*, y^k)\}$ is decreasing, and by non-negativity, it converges. In particular, it is bounded. Thus, it follows from condition (A1) that $\{y^k\}$ is bounded.

(iii) Now, from (3.6), we have

$$0 \leq d_\psi(y^k, y^{k+1}) \leq d_\psi(y^*, y^k) - d_\psi(y^*, y^{k+1}), \quad \text{for all } k \in \mathbb{N}. \tag{3.7}$$

Summing up (3.7) from $k = 0$ to $k = n$, we obtain

$$\sum_{k=0}^{n} d_\psi(y^{k+1}, y^k) \leq d_\psi(y^*, y^k) - d_\psi(y^*, y^{k+1}) \leq d_\psi(y^*, y^0).$$

Letting k goes to infinity, we conclude that $\sum_{k=0}^{\infty} d_\psi(y^{k+1}, y^k) \leq \infty$ and, in particular, we have $\lim_{k\to\infty} d_\psi(y^{k+1}, y^k) = 0$. This completes the proof.

□

Theorem 3.4. *Every cluster point of the sequence $\{y^k\}$ generated by Algorithm 3.1 yields the Bregman projected solution of* (3.1).

Proof. By virtue of Proposition 3.3 (ii), $\{y^k\}$ is bounded. Let $\{y^{k_j}\}$ be a subsequence of $\{y^k\}$ converging to $\widetilde{y}$. Now, by Algorithm 3.1, we have that $y^{k_j+1} \in \mathcal{Q}(x^{k_j})$, for $x^{k_j} = P^\psi_\mathcal{C}(y^{k_j})$ such that $\mathcal{G}_{k_j}(y^{k_j+1}, y) \geq 0$, for all $y \in \mathcal{Q}(x^{k_j})$. Using Proposition 3.3 (iii) , we obtain

$$\lim_{k\to\infty} d_\psi(y^{k_j+1}, y^{k_j}) = 0,$$

and hence, condition (A3) guarantees that $\lim_{j\to\infty} y^{k_j+1} = \widetilde{y}$. By continuity of $P^\psi_\mathcal{C}$ (see Remark 2.1), it follows that

$$x^{k_j} = P^\psi_\mathcal{C}(y^{k_j}) \longrightarrow P^\psi_\mathcal{C}(\widetilde{y}) = \widetilde{x} \quad \text{as } j \to \infty.$$

Now, we show that $\widetilde{x}$ is a Bregman projected solution. It is suffice to show that $\widetilde{y} \in \mathcal{Q}(\widetilde{x})$ such that $\mathcal{G}(\widetilde{y}, y) \geq 0$, for all $y \in \mathcal{Q}(\widetilde{x})$.

Since $\mathcal{Q}$ is M−continuous, we obtain that $\widetilde{y} \in \mathcal{Q}(\widetilde{x})$, and, for an arbitrary $t \in \mathcal{Q}(\widetilde{x})$, there exists a sequence $\{t^{k_j}\}$ such that $t^{k_j} \to t$ and $t^{k_j} \in \mathcal{Q}(x^{k_j})$. Now, since $y^{k_j+1} \in S_{EP}(\mathcal{G}_{k_j}, \mathcal{Q}_{k_j})$, we have

$$\mathcal{G}_{k_j}(y^{k_j+1}, u) \geq 0, \quad \text{for all } u \in \mathcal{Q}(x^{k_j}),$$

which means in particular for $u = t^{k_j} \in \mathcal{Q}(x^{k_j})$ that

$$0 \leq \mathcal{G}(y^{k_j+1}, t^{k_j}) + \gamma_{k_j} \left\langle \nabla\psi(y^{k_j+1}) - \nabla\psi(y^{k_j}), t^{k_j} - y^{k_j+1} \right\rangle, \tag{3.8}$$

Since $\{\gamma_{k_j}\}$, $\{y^{k_j}\}$ and $\{t^{k_j}\}$ are bounded sequences, ψ is continuous differentiable and $\mathcal{G}$ fulfils (P3). Taking the limit $j \to +\infty$ in (3.8), we obtain

$$\mathcal{G}(\widetilde{y}, t) \geq 0.$$

Since $t \in \mathcal{Q}(\widetilde{x})$ was chosen arbitrary, it follows that $\mathcal{G}(\widetilde{y}, t) \geq 0$, for all $t \in \mathcal{Q}(\widetilde{x})$. Hence $\widetilde{x}$ is a Bregman projected solution, that is , $\widetilde{x} \in S^{Bp}(\mathcal{G}, \mathcal{Q})$. This completes the proof. □

Since $S^* \subset \widetilde{S}_{QEP}(\mathcal{G}, \mathcal{Q})$, it is true that Theorem 3.4 does not guarantee that the cluster point of $\{y^k\}$ is in S^*, so we cannot apply Lemma 2.3 to obtain the convergence of the whole sequence to an element of $\widetilde{S}_{QEP}(\mathcal{G}, \mathcal{Q})$. To overcome this, we shall assume that $\mathcal{G}$ is strictly monotone. Next result provides a sufficient condition for the convergence of the sequence $\{y^k\}$ to yield a Bregman projected solution of (3.1).

Corollary 3.5. *If $\mathcal{G}$ is strictly monotone, then, the sequence $\{y^k\}$ generated by Algorithm 3.1 converges to $\widetilde{y}$ such that $P^{\psi}_{\mathcal{C}}(\widetilde{y})$ is the Bregman projected solution to the $QEP(\mathcal{G}, \mathcal{Q})$ (3.1).*

Proof. First we prove that $S^* = \widetilde{S}_{QEP}(\mathcal{G}, \mathcal{Q}) = \{\widetilde{y}\}$, where $\widetilde{y}$ is the cluster point of $\{y^k\}$. By the proof of Theorem 3.4, we have that $\widetilde{x} = P^{\psi}_{\mathcal{C}}(\widetilde{y}) \in S^{Bp}(\mathcal{G}, \mathcal{Q})$. Then, $\widetilde{y} \in \mathcal{Q}(\widetilde{x})$ verifying

$$\mathcal{G}(\widetilde{y}, y) \geq 0, \qquad \text{for all } y \in \mathcal{Q}(\widetilde{x}). \tag{3.9}$$

On contrary, let $y^* \in S^*$ such that $\widetilde{y} \neq y^*$. Then, $y^* \in \bigcap_{x \in \mathcal{C}} \mathcal{Q}(x)$ satisfying

$$\mathcal{G}(y^*, y) \geq 0, \qquad \text{for all } y \in \bigcup_{x \in \mathcal{C}} \mathcal{Q}(x). \tag{3.10}$$

Suppose that $y = y^*$ in (3.9), then we have that $\mathcal{G}(\widetilde{y}, y^*) \geq 0$. On the other hand, by taking $y = \widetilde{y}$ in (3.10), we obtain $\mathcal{G}(y^*, \widetilde{y}) \geq 0$. By monotonicity of $\mathcal{G}$, we have $\mathcal{G}(y^*, \widetilde{y}) + \mathcal{G}(\widetilde{y}, y^*) = 0$ and this contradicts the fact that $\mathcal{G}$ is strictly monotone and $y^* \neq \widetilde{y}$. Hence, our assumption that there exists $y^* \in S^*$ such that $y^* \neq \widetilde{y}$ is false. Therefore, $S^* = \widetilde{S}_{QEP}(\mathcal{G}, \mathcal{Q}) = \{\widetilde{y}\}$.

Since $S^* = \widetilde{S}_{QEP}(\mathcal{G}, \mathcal{Q}) = \{\widetilde{y}\}$, it follows from Lemma 2.3 and Proposition 3.3(i) that $\{y^k\}$ converges to $\widetilde{y}$ such that $P^{\psi}_{\mathcal{C}}(\widetilde{y})$ is the Bregman projected solution of $QEP(\mathcal{G}, \mathcal{Q})$ (3.1). This completes the proof. □

3.1.1. Linear convergence of B-PPM

In this subsection, we establish R-linear convergence of the sequence generated by Algorithm 3.1. To this end, we make some additional assumptions as follows:

Assumption 3.1. *Suppose that $\mathcal{G}$ is μ-strongly monotone relative to ψ, and $0 < \underline{\gamma} \leq \gamma_k \leq \overline{\gamma} < \mu + \underline{\gamma}$, for all $k \in \mathbb{N}$.*

Before proceeding, we recall the notions of linear convergence for sequences in $\mathbb{R}^n$. A sequence $\{y^k\} \subset \mathbb{R}^n$ is said to converge

- Q-linearly to $\widetilde{y} \in \mathbb{R}^n$ if there is $\eta \in (0, 1)$ such that for all k sufficiently large $\|y^{k+1} - \widetilde{y}\| \leq \eta \|y^k - \widetilde{y}\|$.
- R-linearly to $\widetilde{y} \in \mathbb{R}^n$, if there exists a sequence of positive scalars $\{\rho_k\}$ such that $\{\rho_k\}$ converges Q-linearly to 0, and $\|y^k - \widetilde{y}\| \leq \rho_k$, for sufficiently large k.

Theorem 3.6. *Suppose that Assumptions 3.1, (P1)–(P5), and (A1)–(A5) hold. Let $\{y^k\}$ be the sequence generated by Algorithm 3.1. Then, for any $y^* \in S^*$, the sequence $\left\{d_\psi(y^*, y^k)\right\}$ converges R-linearly to 0.*

Proof. Let $y^* \in S^* \subset \widetilde{S}_{QEP}(\mathcal{G}, \mathcal{Q})$. By Algorithm 3.1, we have

$$y^{k+1} \in \mathcal{Q}(x^k) \text{ and } \mathcal{G}(y^{k+1}, y) + \gamma_k \left\langle \nabla\psi(y^{k+1}) - \nabla\psi(y^k),\, y - y^{k+1} \right\rangle \geq 0, \quad \text{for all } y \in \mathcal{Q}(x^k). \tag{3.11}$$

By choosing $y = y^*$ in (3.11), yields

$$\gamma_k \left\langle \nabla\psi(y^k) - \nabla\psi(y^{k+1}),\, y^* - y^{k+1} \right\rangle \leq \mathcal{G}(y^{k+1}, y^*) \tag{3.12}$$

Since $y^* \in S^* \subset \widetilde{S}_{QEP}(\mathcal{G}, \mathcal{Q})$, we have $\mathcal{G}(y^*, y^{k+1}) \geq 0$. Thus, by strong monotonicity of $\mathcal{G}$ relative to ψ (see Definition 2.5) , we obtain

$$\mathcal{G}(y^{k+1}, y^*) \leq -\mu \left(d_\psi(y^{k+1}, y^*) + d_\psi(y^*, y^{k+1}) \right). \tag{3.13}$$

Substituting (3.13) into (3.12), we obtain

$$\gamma_k \left\langle \nabla\psi(y^k) - \nabla\psi(y^{k+1}),\, y^* - y^{k+1} \right\rangle \leq -\mu\Big(d_\psi(y^{k+1}, y^*) + d_\psi(y^*, y^{k+1}) \Big). \tag{3.14}$$

Applying Lemma 2.1 (ii), we have

$$\left\langle \nabla\psi(y^k) - \nabla\psi(y^{k+1}),\, y^* - y^{k+1} \right\rangle = d_\psi(y^*, y^{k+1}) - d_\psi(y^*, y^k) + d_\psi(y^{k+1}, y^k). \tag{3.15}$$

Combining (3.14) and (3.15) yields

$$d_\psi(y^*, y^{k+1}) - d_\psi(y^*, y^k) + d_\psi(y^{k+1}, y^k) \leq -\frac{\mu}{\gamma_k}\Big(d_\psi(y^{k+1}, y^*) + d_\psi(y^*, y^{k+1}) \Big),$$

which is equivalent to

$$\left(1 + \frac{\mu}{\gamma_k}\right) d_\psi(y^*, y^{k+1}) + d_\psi(y^{k+1}, y^k) \leq d_\psi(y^*, y^k) - \frac{\mu}{\gamma_k} d_\psi(y^{k+1}, y^*). \tag{3.16}$$

Because $d_\psi(y^{k+1}, y^k)$ and $d_\psi(y^{k+1}, y^*)$ are non-negative, inequality (3.16) becomes

$$d_\psi(y^*, y^{k+1}) \leq \frac{1}{1 + \dfrac{\mu}{\gamma_k}}\, d_\psi(y^*, y^k). \tag{3.17}$$

Since $0 < \underline{\gamma} \leq \gamma_k \leq \overline{\gamma} < \mu + \underline{\gamma}$, foll all $k \in \mathbb{N}$, we obtain $\dfrac{1}{1 + \dfrac{\mu}{\gamma_k}} = \dfrac{\gamma_k}{\mu + \gamma_k} \leq \dfrac{\overline{\gamma}}{\mu + \underline{\gamma}}$. It then follows from (3.17) that

$$d_\psi(y^*, y^{k+1}) \;\leq \widetilde{\delta}\, d_\psi(y^*, y^k),$$

where $\widetilde{\delta} := \dfrac{\overline{\gamma}}{\mu + \underline{\gamma}}$. Thus, we have

$$d_\psi(y^*, y^k) \;\leq \widetilde{\delta}^k\, d_\psi(y^*, y^0). \tag{3.18}$$

Notice that $\widetilde{\delta} \in (0, 1)$. Hence we conclude that $\left\{ d_\psi(y^*, y^k) \right\}$ converges R-linearly to 0. This completes the proof. $\square$

Recall that, when $\psi = \frac{1}{2}\|\cdot\|^2$, Algorithm 3.1 recovers the classical proximal point method in [36, Algorithm 3] for computing projected solutions of $QEP(\mathcal{G}, \mathcal{Q})$, and the relatively strongly monotonicity coincides with strongly monotonicity assumption on bifunction $\mathcal{G}$. Then, the following result is obtained:

Corollary 3.7. *Let $\{y^k\}$ be the sequence generated by Algorithm 3.1 with $\psi = \frac{1}{2}\|\cdot\|^2$. Then, for any $y^* \in S^*$, the sequence $\{y^k\}$ converges R-linearly to y^*.*

Proof. Observe that $d_\psi(x, y) = \frac{1}{2}\|x - y\|^2$ when $\psi(x) = \frac{1}{2}\|x\|^2$. Thus, from (3.18), we obtain that

$$\frac{1}{2}\|y^k - y^*\|^2 \le \frac{1}{2}\widetilde{\delta}^k \, \|y^0 - y^*\|^2,$$

which implies that

$$\|y^k - y^*\| \le \left(\sqrt{\widetilde{\delta}}\,\right)^k \|y^0 - y^*\|, \text{ where } \widetilde{\delta} := \frac{\overline{\gamma}}{\mu + \underline{\gamma}} \in (0, 1).$$

Thus, the conclusion follows directly from the last inequality. □

Next, we present the separable Bregman regularized proximal point method as follows:

Algorithm 3.2 Separable Bregman Proximal Point Method (SB-PPM) to find projected solution of (3.1)

Step 0: Take a bounded sequence of regularization parameters $\{\gamma_k\} \subset (0, \infty)$, tolerance $\varepsilon > 0$ and choose $y^0 \in \mathcal{M}$. Set $k := 0$;

Step 1: (Outer loop) Set $x^0 = P^{\psi}_{\mathcal{C}}(y^0)$;

Step 2: (Inner loop) Given x^k and y^k, compute $y^{k+1} \in S_{EP}(\mathcal{G}_k, \mathcal{Q}_k)$, where

$$\mathcal{G}_k(x, y) = \mathcal{G}(x, y) + \gamma_k\big(d_\psi(y, y^k) - d_\psi(x, y^k)\big) \tag{3.19}$$

and $\mathcal{Q}_k := \mathcal{Q}(x^k)$.

Step 3: If $\|y^{k+1} - y^k\| < \varepsilon$, then STOP. Otherwise, set $k := k + 1$, and go to **Step 1** by setting $x^{k+1} = P^{\psi}_{\mathcal{C}}(y^{k+1})$.

Remark 3.2. *At each iteration, Algorithm 3.2 requires the solution of a Bregman regularized equilibrium problem. As a result, the method is well defined provided that* (3.19) *admits a solution. This fact is proved in [34, Proposition 3.1].*

Proposition 3.8. *Let $\{y^k\}$ be a sequence generated by Algorithm 3.2. Then, the following hold:*

(i) $\mathcal{G}(y^{k+1}, y) + \gamma_k\big(d_\psi(y, y^k) - d_\psi(y, y^{k+1})\big) \ge 0,$ *for all $y \in \mathcal{Q}_k$;*

(ii) $d_\psi(y^*, y^{k+1}) \le d_\psi(y^*, y^k), \quad (\forall y^* \in S^*)\, (\forall k \in \mathbb{N}).$

Proof. See [34, Proposition 3.2]. □

Proposition 3.9. *Let $\{y^k\}$ be the sequence generated by Algorithm 3.2.Then, the following assertions hold:*

(i) *$\{y^k\}$ is bounded;*

(ii) $\lim_{k\to\infty} d_\psi(y^{k+1}, y^k) = 0$.

Proof. The results come directly from [34, Corollary 3.1 and Corollary 3.2]. □

Next, we prove our main theorem.

Theorem 3.10. *Every cluster point of the sequence $\{y^k\}$ generated by Algorithm 3.2 yields the Bregman projected solution of* (3.1).

Proof. Since $\{y^k\}$ is bounded, let $\{y^{k_j}\}$ be a subsequence of $\{y^k\}$ converging to $\widetilde{y}$. Now, by Algorithm 3.2, we have that $y^{k_j+1} \in \mathcal{Q}(x^{k_j})$, for $x^{k_j} = P^\psi_\mathcal{C}(y^{k_j})$ such that $\mathcal{G}_{k_j}(y^{k_j+1}, y) \geq 0$, for all $y \in \mathcal{Q}(x^{k_j})$. Using Proposition 3.9 (iii), we have

$$\lim_{k\to\infty} d_\psi(y^{k_j+1}, y^{k_j}) = 0,$$

and hence, from (A3) we can guarantee that $\lim_{j\to\infty} y^{k_j+1} = \widetilde{y}$. By continuity of $P^\psi_\mathcal{C}$ (see Remark 2.1), it follows that

$$x^{k_j} = P^\psi_\mathcal{C}(y^{k_j}) \longrightarrow P^\psi_\mathcal{C}(\widetilde{y}) = \widetilde{x} \text{ as } j \to \infty.$$

Now, we show that $\widetilde{x}$ is a Bregman projected solution. It is sufficient to show that $\widetilde{y} \in \mathcal{Q}(\widetilde{x})$ such that $\mathcal{G}(\widetilde{y}, z) \geq 0$, for all $z \in \mathcal{Q}(\widetilde{x})$.

Since $\mathcal{Q}$ is M−continuous, we obtain that $\widetilde{y} \in \mathcal{Q}(\widetilde{x})$, and, for an arbitrary $u \in \mathcal{Q}(\widetilde{x})$, there exists a sequence $\{u^{k_j}\}$ such that $u^{k_j} \to u$ and $u^{k_j} \in \mathcal{Q}(x^{k_j})$. Now, from Proposition 3.8 with $y = u^{k_j} \in \mathcal{Q}(x^{k_j})$, we obtain

$$0 \leq \mathcal{G}(y^{k_j+1}, u^{k_j}) + \gamma_{k_j}\big(d_\psi(u^{k_j}, y^{k_j}) - d_\psi(u^{k_j}, y^{k_j+1})\big). \tag{3.20}$$

On the other hand, applying Lemma 2.1 (ii), we have

$$d_\psi(u^{k_j}, y^{k_j}) - d_\psi(u^{k_j}, y^{k_j+1}) = \big\langle \nabla\psi(y^{k_j+1}) - \nabla\psi(y^{k_j}), u^{k_j} - y^{k_j+1}\big\rangle + d_\psi(y^{k_j+1}, y^{k_j}). \tag{3.21}$$

Thus, substituting (3.21) into (3.20), we get

$$0 \leq \mathcal{G}(y^{k_j+1}, u^{k_j}) + \gamma_{k_j}\left(\big\langle \nabla\psi(y^{k_j+1}) - \nabla\psi(y^{k_j}), u^{k_j} - y^{k_j+1}\big\rangle + d_\psi(y^{k_j+1}, y^{k_j})\right). \tag{3.22}$$

Since $\{\gamma_{k_j}\}$ is bounded sequences and ψ is continuous differentiable, hence taking the limit $j \to +\infty$ in (3.22), we obtain

$$\mathcal{G}(\widetilde{y}, u) \geq 0.$$

Since we consider an arbitrary $u \in \mathcal{Q}(\widetilde{x})$ this means that $\mathcal{G}(\widetilde{y}, u) \geq 0$, for all $u \in \mathcal{Q}(\widetilde{x})$, and hence $\widetilde{x}$ is a Bregman projected solution, that is, $\widetilde{x} \in S^{Bp}(\mathcal{G}, \mathcal{Q})$. This completes the proof. □

3.1.2. Linear convergence of SB-PPM

In this subsection, we prove R-linear convergence of the sequence generated by Algorithm 3.2. To this end, we make some additional assumptions as follows:

Assumption 3.2. *Suppose that $\mathcal{G}$ is μ-strongly monotone relative to ψ, and $0 < \underline{\gamma} \leq \gamma_k \leq \overline{\gamma} < \mu$, for all $k \in \mathbb{N}$.*

Theorem 3.11. *Suppose that Assumptions 3.2,* (P1)–(P5), *and* (A1)–(A5) *hold. Let $\{y^k\}$ be the sequence generated by Algorithm 3.2. Then, for any $y^* \in S^*$, the sequence $\big\{d_\psi(y^*, y^k)\big\}$ converges R-linearly to* 0.

Proof. Let $y^* \in S^* \subset \widetilde{S}_{QEP}(\mathcal{G}, \mathcal{Q})$. By Algorithm 3.2, we have

$$y^{k+1} \in \mathcal{Q}(x^k) \text{ and } \mathcal{G}(y^{k+1}, y) + \gamma_k \Big(d_\psi(y, y^k) - d_\psi(y^{k+1}, y^k) \Big) \geq 0, \quad \text{for all } y \in \mathcal{Q}(x^k). \tag{3.23}$$

By taking $y = y^*$ in (3.23), we have

$$\gamma_k \Big(d_\psi(y^{k+1}, y^k) - d_\psi(y^*, y^k) \Big) \leq \ \mathcal{G}(y^{k+1}, y^*) \tag{3.24}$$

Since $y^* \in S^* \subset \widetilde{S}_{QEP}(\mathcal{G}, \mathcal{Q})$, we have $\mathcal{G}(y^*, y^{k+1}) \geq 0$. Thus, using the fact that $\mathcal{G}$ is strongly monotone relative to ψ, we have

$$\mathcal{G}(y^{k+1}, y^*) \leq -\mu \Big(d_\psi(y^{k+1}, y^*) + d_\psi(y^*, y^{k+1}) \Big). \tag{3.25}$$

Combining (3.25) and (3.24), we obtain

$$\gamma_k \Big(d_\psi(y^{k+1}, y^k) - d_\psi(y^*, y^k) \Big) \leq -\mu \Big(d_\psi(y^{k+1}, y^*) + d_\psi(y^*, y^{k+1}) \Big),$$

which implies that

$$\mu d_\psi(y^*, y^{k+1}) + \gamma_k d_\psi(y^{k+1}, y^k) \leq \gamma_k d_\psi(y^*, y^k) - \mu d_\psi(y^{k+1}, y^*). \tag{3.26}$$

Since $d_\psi(y^{k+1}, y^k)$ and $d_\psi(y^{k+1}, y^*)$ are non-negative, it follows from (3.26) that

$$d_\psi(y^*, y^{k+1}) \leq \frac{\gamma_k}{\mu} d_\psi(y^*, y^k) \leq \frac{\overline{\gamma}}{\mu} d_\psi(y^*, y^k),$$

where the second inequality is due to $0 < \gamma_k \leq \overline{\gamma}$. Thus, we get

$$d_\psi(y^*, y^k) \leq \left(\frac{\overline{\gamma}}{\mu} \right)^k d_\psi(y^*, y^0).$$

Observe that $\frac{\overline{\gamma}}{\mu} \in (0, 1)$, thus, $\big\{ d_\psi(y^*, y^k) \big\}$ converges R-linearly to 0. Hence, the proof is complete. □

4. Numerical Experiments

In this section, we consider some examples to illustrate the convergence behavior of the proposed algorithms and to compare their performance with that of an existing algorithm. For each example, we employ various choices of the Bregman functions and their associated Bregman distances. All codes were written in Python 3 and performed on a HP personal computer with Intel(R) Core(TM) i5-10500 CPU at 3.10GHz, RAM of 16 GB. To solve the subproblems (3.3) and (3.19) arising in the inner loops of the proposed algorithms, we employ the approaches of Muu and Quoc [15] and the Bregman regularization method of Flam and Antipin [43]. In their work, the authors propose the following iterative scheme for solving an equilibrium problem: given an initial point $y^0 \in \mathcal{M}$ and a sequence $\{\gamma_k\}$ with $\gamma_k > 0$, and for a given $y^k \in \mathcal{M}$, define

$$y^{k+1} = \arg\min_{y \in \mathcal{M}} \left\{ \gamma_k \, \mathcal{G}(y^k, y) + \frac{1}{2} \|y - y^k\|^2 \right\} \tag{4.1}$$

and

$$y^{k+1} = \underset{y\in\mathcal{M}}{\arg\min}\left\{\gamma_k\, \mathcal{G}(y^k, y) + d_\psi(y, y^k)\right\}, \tag{4.2}$$

respectively. In each test example, the regularization parameter γ_k is specified, taking either a constant value or a k-dependent sequence.

Next, we present two examples of quasi-equilibrium problems and report the numerical performance of the proposed methods in comparison with an existing one

Example 4.1. *Consider the quasi-equilibrium problem with the bifunction $\mathcal{G} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by*

$$\mathcal{G}(x, y) = \sum_{i=1}^{2}(y_i^2 - x_i^2) - 6\sum_{i=1}^{2}(y_i - x_i).$$

We chose $\mathcal{C} = [1, 2] \times [1, 2]$ and the set-valued map $\mathcal{Q} : \mathcal{C} \rightrightarrows \mathbb{R}^2$ defined by

$$\mathcal{Q}(x) = \prod_{i=1}^{2}\left[\frac{x_i}{2},\, 1 + 2x_i\right],$$

It follows by direct computation that

$$\bigcap_{z\in\mathcal{C}} \mathcal{Q}(z) = [1, 3] \times [1, 3] \quad \text{and} \quad \bigcup_{z\in\mathcal{C}} \mathcal{Q}(z) = \left[\frac{1}{2}, 5\right] \times \left[\frac{1}{2}, 5\right],$$

hence, $\mathcal{Q}(\mathcal{C}) \nsubseteq \mathcal{C}$. The bifunction $\mathcal{G}$ in this example fulfills assumptions (P1)–(P5). *Moreover, the point $\widetilde{y} = (3, 3)$ belongs to the set $\bigcap_{z\in\mathcal{C}} \mathcal{Q}(z)$, and satisfies $\mathcal{G}(\widetilde{y}, y) \geq 0$, for all $y \in \mathcal{Q}(\mathcal{C}) = \left[\frac{1}{2}, 5\right] \times \left[\frac{1}{2}, 5\right]$. Therefore, $\widetilde{y} \in S^*$. Furthermore, $S^*_{Bp} = P^\psi_{\mathcal{C}}(S^*) = \{(2, 2)\}$ which is the subset of Bregman projected solution.*

To evaluate the performance of the proposed algorithms on this example, we set $\gamma_k = 1$, for all $k \in \mathbb{N}$ and use $\|y^{k+1} - y^k\| < 10^{-10}$ as the stopping criterion for all three algorithms with three distances, namely, SED, KLD, and ISD.

Figs. 1 and 2 show the decrease of the residual with respect to iterations and CPU time, respectively. In both figures, each faint line represents an individual run and the bold line indicates the mean behavior over 150 random starting points in $[0.5, 5]^2$. From Fig. 1, we observe that [36, Algorithm 3] requires more iterations than both Algorithm 3.1 and Algorithm 3.2, while the latter two exhibit very similar iteration performance. In contrast, Fig. 2 shows that Algorithm 3.2 achieves the lowest CPU time to reach the stopping criterion, outperforming both Algorithm 3.1 and [36, Algorithm 3]. On the other hand, Fig. 3 summarizes the CPU-time distributions with three distances settings. It shows that [36, Algorithm 3] is consistently slower and exhibits greater variability than the proposed Algorithms 3.1 and 3.2. The latter two perform comparably, with Algorithm 3.2 showing a slight advantage by attaining the lowest and most concentrated CPU times.

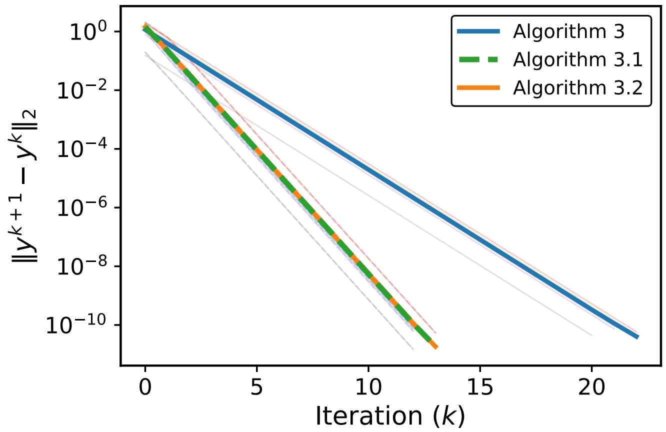
In Tables 1 and 2, we can see that, for different starting points (y^0), Algorithms 3.1 and 3.2 require a similar number of iterations and fewer iterations to reach the stopping criterion than [36, Algorithm 3]. On the other hand, Algorithm 3.2 outperforms both Algorithm 3.1 and [36, Algorithm 3] in terms of CPU time.

Table 1. Results for Example 4.1 with three starting points (y^0) in the box $[0.5, 5]^2$ using SED and KLD.

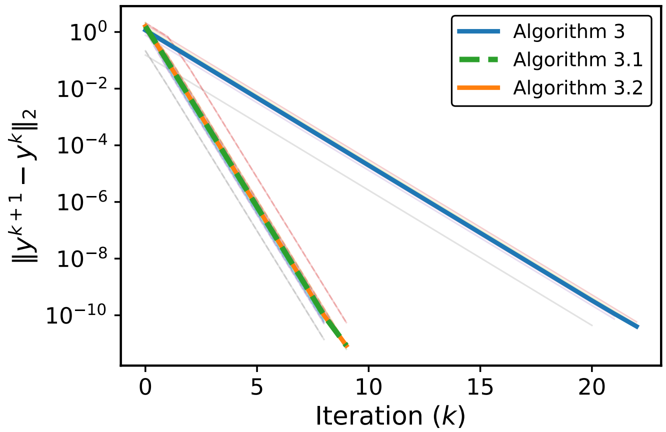
y^0	Algorithm	Solution ($\widetilde{y}$)	Projected ($\widetilde{x}$)	Iter. (k)	CPU (s)
(0.7, 2.6)	Algorithm 3.1	(3, 3)	(2, 2)	14	0.001483
	Algorithm 3.2			14	0.001240
	Algorithm 3 [36]			23	0.001222
(1.0, 3.0)	Algorithm 3.1	(3, 3)	(2, 2)	14	0.000532
	Algorithm 3.2			14	0.000416
	Algorithm 3 [36]			23	0.000626
(2.0, 4.7)	Algorithm 3.1	(3, 3)	(2, 2)	14	0.000541
	Algorithm 3.2			14	0.000480
	Algorithm 3 [36]			23	0.000590

Table 2. Results for Example 4.1 with three starting points (y^0) in the box $[0.5, 5]^2$ using SED and ISD.

y^0	Algorithm	Solution ($\widetilde{y}$)	Projected ($\widetilde{x}$)	Iter. (k)	CPU (s)
(0.7, 2.6)	Algorithm 3.1	(3, 3)	(2, 2)	10	0.000468
	Algorithm 3.2			10	0.000421
	Algorithm 3 [36]			23	0.000951
(1.0, 3.0)	Algorithm 3.1	(3, 3)	(2, 2)	10	0.000429
	Algorithm 3.2			10	0.000392
	Algorithm 3 [36]			23	0.001482
(2.0, 4.7)	Algorithm 3.1	(3, 3)	(2, 2)	10	0.000241
	Algorithm 3.2			10	0.000184
	Algorithm 3 [36]			23	0.000468

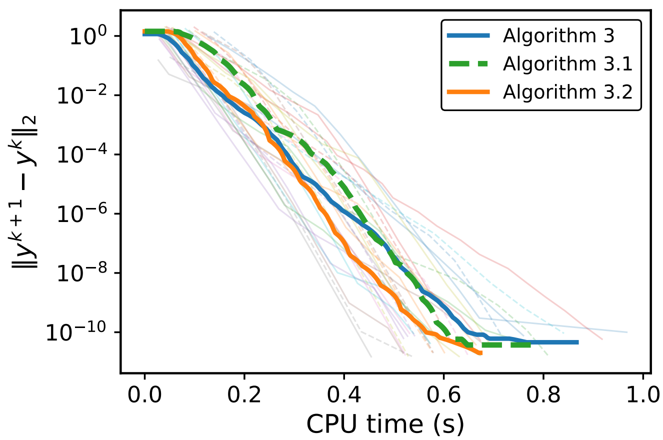


(a) Residual over iterations for KLD

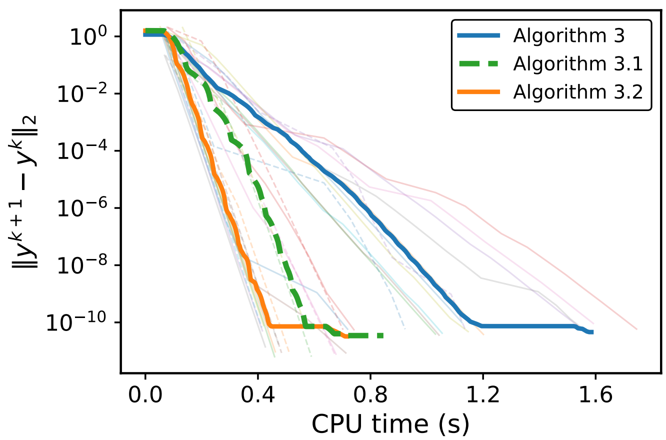


(b) Residual over iterations for ISD

Fig. 1. Numerical behavior of all three algorithms using SED, KLD and ISD.

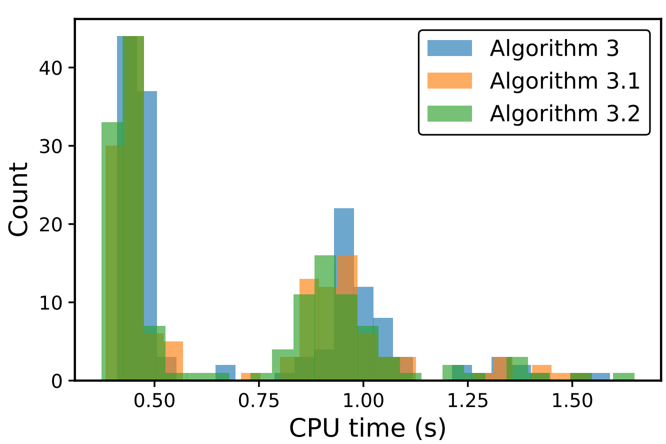


(a) Residual over CPU time for SED and KLD.

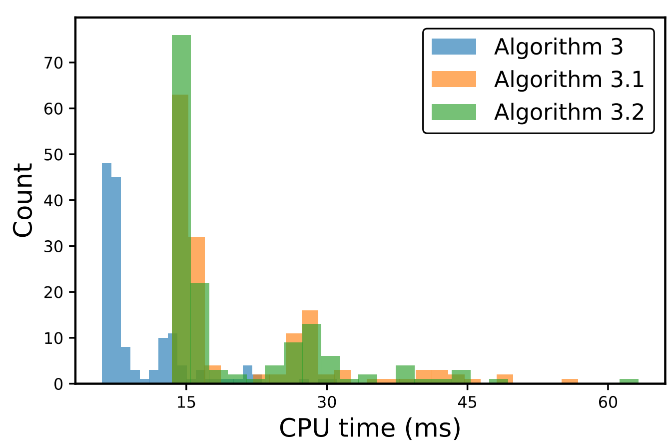


(b) Residual over CPU time for SED and ISD.

Fig. 2. Numerical behavior of all three algorithms using SED, KLD and ISD.



(a) Iteration counts over time for SED and KLD



(b) Iteration counts over time for SED and ISD

Fig. 3. Distribution of CPU times across trials, indicating iteration counts and computational efficiency of the algorithms using SED, KLD, and ISD.

Example 4.2. [36, Example 4] *Consider the quasi-equilibrium problem with* $\mathcal{C} = \left\{ x \in \mathbb{R}^n_+ : \sum_{i=1}^{n} x_i \geq n \right\}$ *and a multivalued mapping* $\mathcal{Q} : \mathcal{C} \rightrightarrows \mathbb{R}^n$ *given by*

$$\mathcal{Q}(x) = \left\{ y \in \mathbb{R}^n : \sum_{i=1}^{n} y_i \leq 1 + \frac{\|x\|}{1 + \|x\|}, \quad y_i \geq -\frac{1}{\|x\|} \right\}.$$

The bifunction $\mathcal{G} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *is defined by*

$$\mathcal{G}(x, y) = \left(\sum_{i=1}^{n} y_i \right)^2 - \left(\sum_{i=1}^{n} x_i \right)^2 - \exp\left(\sum_{i=1}^{n} x_i - 0.5 \right) + \exp\left(\sum_{i=1}^{n} y_i - 0.5 \right) + 2 \sum_{i=1}^{n} (x_i - y_i).$$

Clearly, the bifunction $\mathcal{G}$ *in this example satisfies assumptions* (P1)–(P5).

When $n = 1$, $\widetilde{y} = 0.5$ *is such that* $\widetilde{y} \in \bigcap_{z \in \mathcal{C}} \mathcal{Q}(z) = [0, 1.5]$, *and satisfies* $\mathcal{G}(\widetilde{y}, y) \geq 0$, *for all* $y \in \bigcup_{z \in \mathcal{C}} \mathcal{Q}(z) = [-1, 2]$. *Therefore,* $S^* \neq \emptyset$ *for* $n = 1$ *and hence* $S^*_p \neq \emptyset$.

For $n = 2$,

$$\bigcap_{z \in \mathcal{C}} \mathcal{Q}(z) = \left\{ y \in \mathbb{R}^2 : 0 < y_1 + y_2 \leq 3 - \sqrt{2} \right\}$$

and

$$\bigcup_{z \in \mathcal{C}} \mathcal{Q}(z) = \left\{ y \in \mathbb{R}^2 : -\sqrt{2} \leq y_1 + y_2 < 2 \right\}.$$

Also, $\widetilde{y} = (0.3675, 0.3675) \in \bigcap_{z \in \mathcal{C}} \mathcal{Q}(z)$ *and* $\mathcal{G}(\widetilde{y}, y) \geq 0$, *for all* $y \in \bigcup_{z \in \mathcal{C}} \mathcal{Q}(z)$.

To perform numerical experiments on this example, we set the values of the regularization parameter γ_k as indicated in Table 3 and use $\|y^{k+1} - y^k\| < 10^{-12}$ as the stopping criterion for all three algorithms with two distances, namely, SED and shifted Bregman distance defined below. To satisfy the domain constraint of the Bregman distance (Itakura–Saito distance) in this example, we employ a shifted Burg distance. Recall that the Burg entropy

$$\psi(x) = -\sum_{i=1}^{n} \log(x_i)$$

is well defined only for $x \in \mathbb{R}^n_{++}$. However, in Example 4.2 the feasible sets $\mathcal{Q}(x)$ may contain vectors whose components are not strictly positive. To ensure that the arguments of the Bregman generator remain strictly positive and the distance is well defined, we introduce a shift parameter $\nu > 0$ (typically very small) and consider the shifted generator

$$\psi_\nu(x) = -\sum_{i=1}^{n} \log(x_i + \nu), \quad x_i + \nu > 0.$$

The associated shifted Itakura–Saito distance (SISD) is

$$D_{\psi_\nu}(y, x) = \sum_{i=1}^{n} \left(\frac{y_i + \nu}{x_i + \nu} - \log\left(\frac{y_i + \nu}{x_i + \nu} \right) - 1 \right),$$

which is well defined for all $x, y \in \mathbb{R}^n$ such that $x_i + \nu > 0$ and $y_i + \nu > 0$ for all i. The shift preserves convexity and separability of the Burg generator while extending its domain. All numerical experiments in Example 4.2 were carried out with SISD with the shift parameter $\nu = \frac{1}{\sqrt{n}} + 10^{-20}$.

Validating the choice of the shift parameter. For $x \in \mathcal{C}$, we have $\|x\| \geq \sqrt{n}$. Hence, from the definition of $\mathcal{Q}(x)$ it follows that $y_i \geq -\frac{1}{\|x\|} \geq -\frac{1}{\sqrt{n}}, \quad$ for $i = 1, \ldots, n$. Therefore, to guarantee $y_i + \nu > 0$ for all admissible y, it is sufficient to choose $\nu > \frac{1}{\sqrt{n}}$.

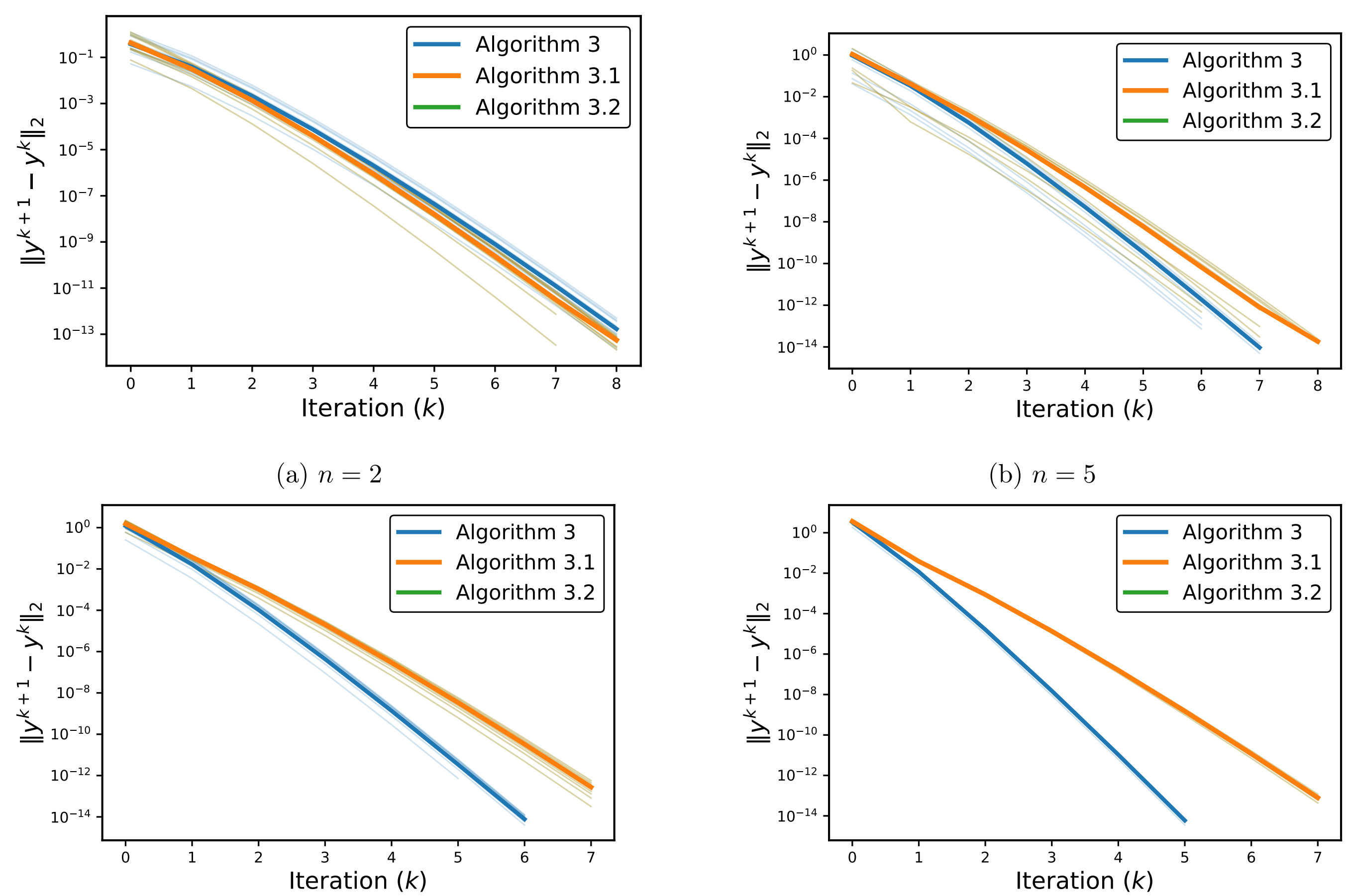
Across all tested dimensions in Fig. 6, [36, Algorithm 3] consistently achieves the lowest and most concentrated CPU times, indicating superior runtime performance. Algorithms 3.1 and 3.2 are slower and more variable, with Algorithm 3.2 showing a modest advantage over Algorithm 3.1, particularly as the dimension increases.

Figs. 4 and 5 show the decrease of the residual with respect to iterations and CPU time, respectively, across different dimensions. In both figures, each faint line corresponds to an individual instance arising from a single random realization of the starting point, while the bold line represents the overall trend. From Fig. 4, it is observed that across all dimensions the residual decreases approximately linearly with the number of iterations, indicating linear convergence of all three algorithms. For $n = 2$, Algorithms 3.1 and 3.2 slightly outperform [36, Algorithm 3] by attaining lower residuals in fewer iterations. As the dimension increases, however, [36, Algorithm 3] exhibits a faster decrease of the residual and becomes the superior method, whereas Algorithms 3.1 and 3.2 maintain nearly identical numerical behavior. In Fig. 5, we can see that in all given dimensions, [36, Algorithm 3] consistently exhibits the steepest decrease and reaches very small residuals in the shortest time compared with Algorithms 3.1 and 3.2, which again display almost identical runtime behavior, with only minor differences attributable to the particular random realization.

In Table 3, we observe that, for the given starting points (y^0) and γ_k, Algorithms 3.1 and 3.2 require a similar number of iterations to reach the stopping criterion. In contrast, [36, Algorithm 3] outperforms both Algorithm 3.1 and Algorithm 3.2 in terms of number of iterations. All three methods, however, yield solutions of similar accuracy.

Table 3. Results for Example 4.2 with different dimensions, starting points (y^0), and γ_k.

n	y^0	γ_k	Algorithm	Solution ($\widetilde{y}$)	Projected ($\widetilde{x}$)	Iter.(k)
			Algorithm 3.1			10
1	2	$\frac{1}{k}$	Algorithm 3.2	0.5	1	10
			Algorithm 3[36]			12
			Algorithm 3.1			9
2	(0.5, 0.5)	$\frac{1}{2k}$	Algorithm 3.2	(0.3675, 0.3675)	(1, 1)	9
			Algorithm 3[36]			9
			Algorithm 3.1			9
5	(1, ..., 1)	$\frac{1}{3k}$	Algorithm 3.2	(0.1952, ..., 0.1952)	(1, ..., 1)	9
			Algorithm 3[36]			8
			Algorithm 3.1			9
10	(2, ..., 2)	$\frac{1}{4k}$	Algorithm 3.2	(0.1079, ..., 0.1079)	(1, ..., 1)	9
			Algorithm 3[36]			7
			Algorithm 3.1			8
30	(1.5, ..., 1.5)	$\frac{1}{6k}$	Algorithm 3.2	(0.0385, ..., 0.0385)	(1, ..., 1)	8
			Algorithm 3[36]			6



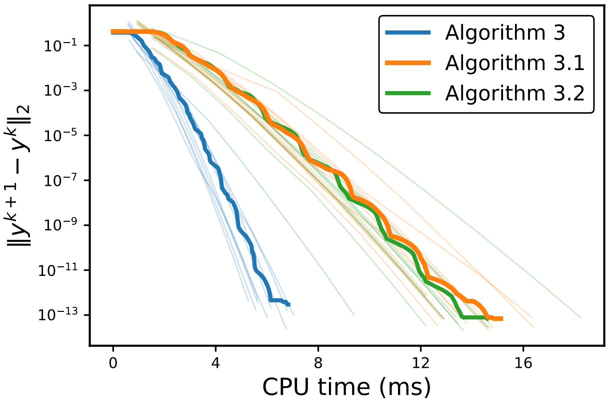
(a) $n = 2$

(b) $n = 5$

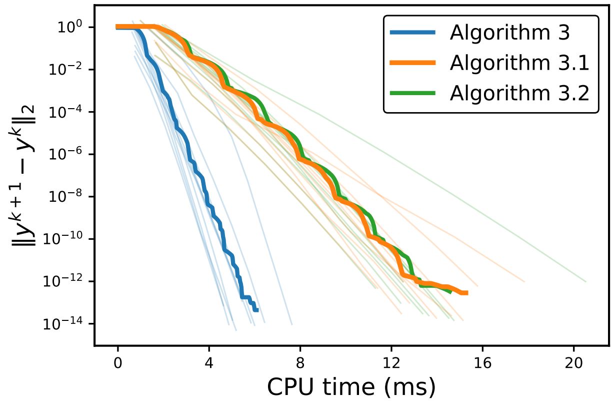
(c) $n = 10$

(d) $n = 30$

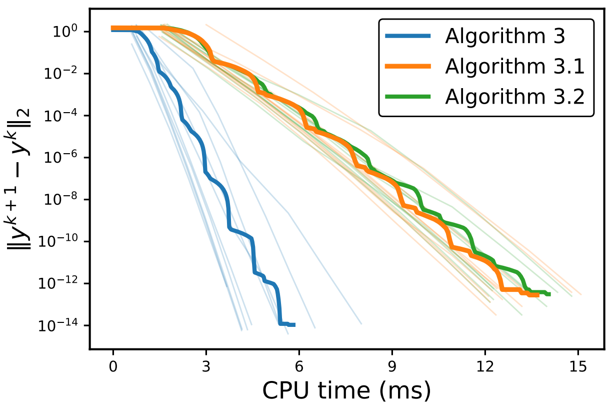
Fig. 4. Residual over iterations for all three algorithms at different dimensions in Example 4.2 using SED and SISD.



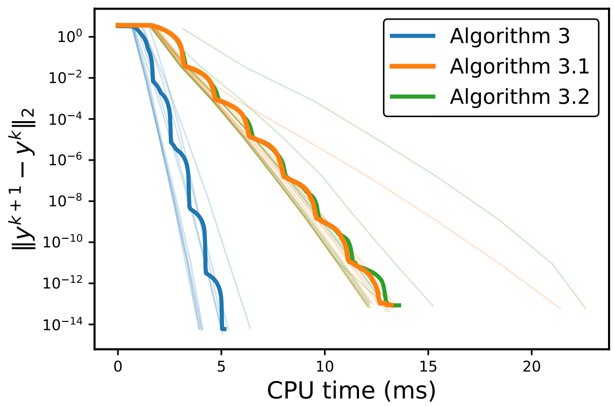
(a) $n = 2$



(b) $n = 5$

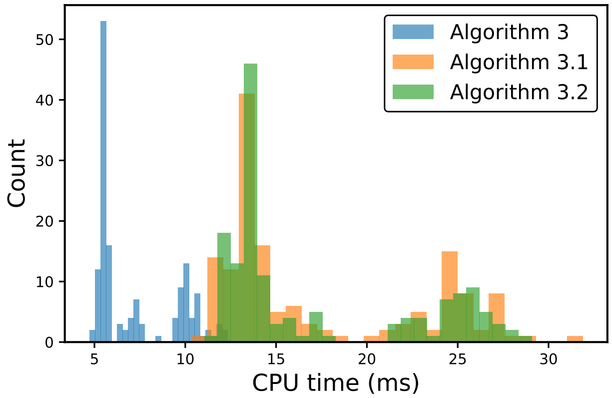


(c) $n = 10$

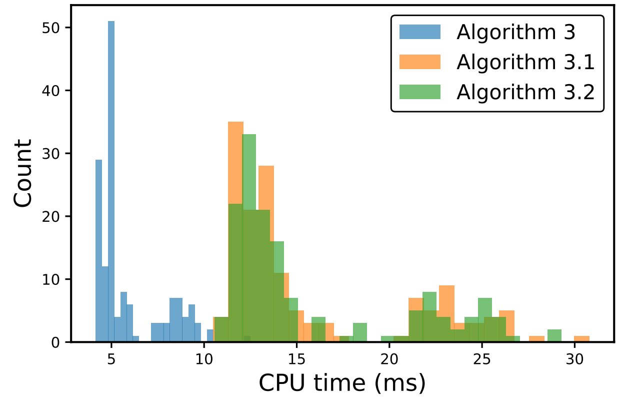


(d) $n = 30$

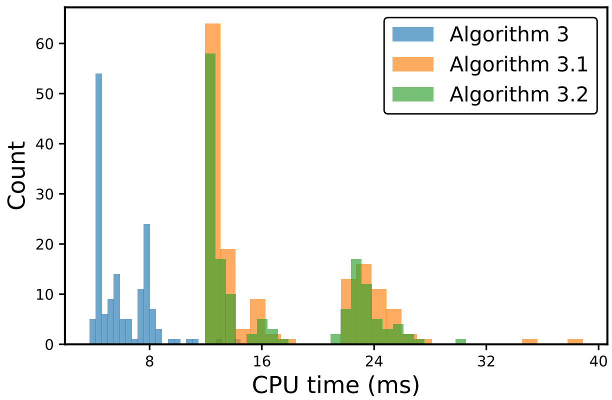
Fig. 5. Residual over CPU time for all three algorithms at different dimensions in Example 4.2 using SED and SISD.



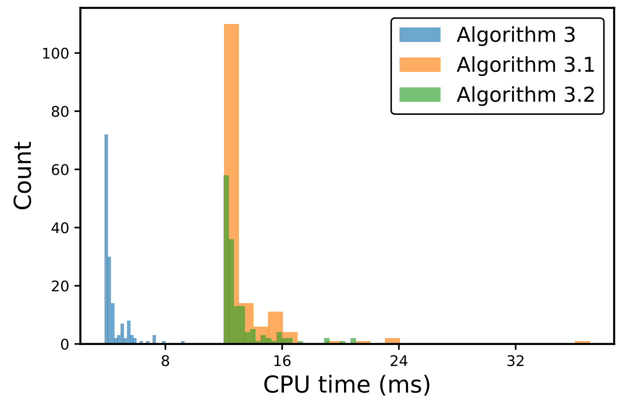
(a) $n = 2$



(b) $n = 5$



(c) $n = 10$



(d) $n = 30$

Fig. 6. Distribution of CPU time for the total number of iterations of the algorithms at different dimensions in Example 4.2 using SED and SISD.

5. Conclusion

In this work, we proposed two Bregman regularized proximal point methods for computing projected solutions of quasi-equilibrium problems. Under standard assumptions, we showed that the methods are well defined and that the generated sequences converge to a projected solution of the quasi-equilibrium problem. Moreover, under a relatively strong monotonicity condition, both methods achieved an R-linear rate of convergence. We complemented the theoretical analysis with numerical experiments on representative test problems. The results indicate that Bregman regularization can yield improved convergence behavior compared with the Euclidean case in certain settings. However, the relative performance depends on the structure of the underlying problem: while the Bregman-based methods outperform the Euclidean case in some instances, the Euclidean variant is more efficient in others. Overall, these findings highlight the practical flexibility of the proposed framework and suggest that an appropriate choice of Bregman geometry can enhance performance for specific classes of quasi-equilibrium problems.

The results obtained in this paper naturally open several directions for future research. It is important to note that Hadamard manifolds generally lack a linear structure. Consequently, many properties, techniques, and algorithms developed in linear spaces are not directly applicable in the Hadamard manifold setting. This motivates the study of algorithms for computing projected solutions of quasi-equilibrium problems on Hadamard manifolds. As a future direction, it would be interesting to see whether the proposed Bregman regularized algorithms and their convergence properties can be extended in the context of Hadamard manifolds.

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Data Availability No datasets were generated or analyzed during the course of this study.

Code Availability Python code used in this paper is available from the corresponding author upon reasonable request.

Conflict of Interest The authors declare that they have no conflict of interest in this paper.

References

[1] E. Blum and W. Oettli. From optimization and variational inequalities to equilibrium problems. *Math. Student*, 63:123–145, 1994.

[2] P. J. S. Santos and J. C. de O. Souza. A proximal point method for quasi-equilibrium problems in Hilbert spaces. *Optimization*, 71(1):55–70, 2022. doi:10.1080/02331934.2020.1810686.

[3] M. Mohammadi and G. Zamani Eskandani. Approximation solutions of quasi-equilibrium problems in Banach spaces. *Miskolc Math. Notes*, 21(1):261–272, 2020. doi:10.18514/MMN.2020.3077.

[4] J. Zhang, B. Qu, and N. Xiu. Some projection-like methods for the generalized Nash equilibria. *Comput. Optim. Appl.*, 45(1):89–109, 2010. doi:10.1007/s10589-008-9173-x.

[5] A. Moudafi. Proximal point algorithm extended to equilibrium problems. *J. Nat. Geometry*, 15(1–2):91–100, 1999.

[6] J. J. Strodiot, T. T. V. Nguyen, and V. H. Nguyen. A new class of hybrid extragradient algorithms for solving quasi-equilibrium problems. *J. Global Optim.*, 56(2):373–397, 2013. doi:10.1007/s10898-011-9814-y.

[7] N. T. T. Van, J. J. Strodiot, V. H. Nguyen, and P. T. Vuong. An extragradient-type method for solving nonmonotone quasi-equilibrium problems. *Optimization*, 67(5):651–664, 2018. doi:10.1080/02331934.2017.1416610.

[8] G. Bigi, M. Castellani, M. Pappalardo, and M. Passacantando. *Nonlinear Programming Techniques for Equilibria*. Springer, Switzerland, 2019. doi:10.1007/978-3-030-00205-3.

[9] G. Kassay and V. Rădulescu. *Equilibrium Problems and Applications*. Academic Press, London, 2018.

[10] R. T. Rockafellar. Monotone operators and the proximal point algorithm. *SIAM J. Control Optim.*, 14(5):877–898, 1976.

[11] A. N. Iusem and W. Sosa. Iterative algorithms for equilibrium problems. *Optimization*, 52(3):301–316, 2003. doi:10.1080/0233193031000120039.

[12] G. Stampacchia. Formes bilineaires coercitives sur les ensembles convexes. *Comptes Rendus Acad. Sci. Paris*, 258:4413, 1964.

[13] A. Bensoussan and J. L. Lions. Nouvelle formulation de problèmes de contrôle impulsionnel et applications. *C. R. Acad. Sci. Paris*, Ser. A, 276:1279–1284, 1973.

[14] H. H. Bauschke and P. L. Combettes. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*. 2nd ed., Springer, Cham, 2017. doi:10.1007/978-3-319-48311-5.

[15] L. D. Muu and T. D. Quoc. Regularization algorithms for solving monotone Ky Fan inequalities with application to a Nash-Cournot equilibrium model. *J. Optim. Theory Appl.*, 142(1):185–204, 2009. doi:10.1007/s10957-009-9529-0.

[16] G. Bigi and M. Passacantando. Gap functions for quasi-equilibria. *J. Global Optim.*, 66(4):791–810, 2016. doi:10.1007/s10898-016-0458-9.

[17] A. A. Khan, C. Tammer, and C. Zălinescu. Regularization of quasi-variational inequalities. *Optimization*, 64(8):1703–1724, 2015. doi:10.1080/02331934.2015.1028935.

[18] D. Aussel, A. Sultana, and V. Vetrivel. On the existence of projected solutions of quasi-variational inequalities and generalized Nash equilibrium problems. *J. Optim. Theory Appl.*, 170(3):818–837, 2016. doi:10.1007/s10957-016-0951-9.

[19] J. Cotrina and J. Zúñiga. Quasi-equilibrium problems with non-self constraint map. *J. Global Optim.*, 75(1):177–197, 2019.

[20] L. M. Bregman. The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming. *USSR Comput. Math. Math. Phys.*, 7(3):200–217, 1967.

[21] D. Butnariu and E. Resmerita. Bregman distances, totally convex functions, and a method for solving operator equations in Banach spaces. *Abstract Appl. Anal.*, 2006.

[22] R. S. Burachik and S. Scheimberg. A proximal point method for the variational inequality problem in Banach spaces. *SIAM J. Control Optim.*, 39(5):1633–1649, 2000.

[23] H. H. Bauschke and J. M. Borwein. Legendre functions and the method of random Bregman projections. *J. Convex Anal.*, 4(1):27–67, 1997.

[24] H. H. Bauschke, X. Wang, J. Ye, and X. Yuan. Bregman distances and Chebyshev sets. *J. Approx. Theory*, 159(1):3–25, 2009.

[25] A. N. Iusem and W. Sosa. On the proximal point method for equilibrium problems in Hilbert spaces. *Optimization*, 59(8):1259–1274, 2010.

[26] R. Burachik and G. Kassay. On a generalized proximal point method for solving equilibrium problems in Banach spaces. *Nonlinear Anal.*, 75(18):6456–6464, 2012.

[27] E. L. Dias Junior, P. J. Santos, and J. C. Souza. A Bregman regularized proximal point method for quasi-equilibrium problems. *J. Pure Appl. Funct. Anal.*, 8:817–828, 2023.

[28] M. Teboulle. Entropic proximal mappings with applications to nonlinear programming. *Math. Oper. Res.*, 17(3):670–690, 1992.

[29] G. Chen and M. Teboulle. Convergence analysis of a proximal-like minimization algorithm using Bregman functions. *SIAM J. Optim.*, 3(3):538–543, 1993.

[30] R. S. Burachik and A. N. Iusem. A generalized proximal point algorithm for the variational inequality problem in a Hilbert space. *SIAM J. Optim.*, 8(1):197–216, 1998.

[31] J. X. Cruz Neto, P. S. M. Santos, R. C. M. Silva, and J. C. O. Souza. On a Bregman regularized proximal point method for solving equilibrium problems. *Optim. Lett.*, 13(5):1143–1155, 2019.

[32] J. Cotrina and J. Zúñiga. A note on quasi-equilibrium problems. *Oper. Res. Lett.*, 46(1):138–140, 2018.

[33] D. Aussel and A. Sultana. Quasi-variational inequality problems with non-compact valued constraint maps. *J. Math. Anal. Appl.*, 456(2):1482–1494, 2017.

[34] A. Soubeyran and J. C. de O. Souza. A variational rationality perspective on the quasi-equilibrium problem: climbing the goal ladder. *J. Nonlinear Var. Anal.*, 9(3):461–478, 2025.

[35] G. C. Bento, J. X. Cruz Neto, P. A. Soares Jr., and A. Soubeyran. A new regularization of equilibrium problems on Hadamard manifolds: applications to theories of desires. *Ann. Oper. Res.*, 316(2):1301–1318, 2022.

[36] D. Aussel, A. Sultana, S. Valecha, et al. Iterative methods for the projected solutions of quasi-equilibrium problems. *arXiv:2506.22080*, 2025.

[37] Y. Censor, A. N. Iusem, and S. A. Zenios. An interior point method with Bregman functions for the variational inequality problem with paramonotone operators. *Math. Program.*, 81(3):373–400, 1998.

[38] H. H. Bauschke, J. M. Borwein, and P. L. Combettes. Bregman monotone optimization algorithms. *SIAM J. Control Optim.*, 42(2):596–636, 2003.

[39] Y. Censor and S. A. Zenios. *Parallel Optimization: Theory, Algorithms, and Applications.* Oxford University Press, 1997.

[40] A. Gibali. A new Bregman projection method for solving variational inequalities in Hilbert spaces. *Pure Appl. Funct. Anal.*, 3(3):403–415, 2018.

[41] D. Aussel, J. Cotrina, and A. Iusem. *Existence Results for Quasi-Equilibrium Problems.* IMPA, 2014.

[42] O. Bueno and J. Cotrina. Existence of projected solutions for generalized Nash equilibrium problems. *J. Optim. Theory Appl.*, 191(1):344–362, 2021.

[43] S. D. Flåm and A. S. Antipin. Equilibrium programming using proximal-like algorithms. *Math. Program.*, 78(1):29–41, 1996.

[44] K. Guo and C. Zhu. On the linear convergence of a Bregman proximal point algorithm. *J. Nonlinear Var. Anal.*, 6(2):5–14, 2022.

[45] M. Castellani, M. Giuli, M. Milasi, and D. Scopelliti. Projected solutions of generalized quasi-variational problems in Banach spaces. *Nonlinear Anal. Real World Appl.*, 76:104021, 2024.