

OPTIMAL TRANSPORT ON LIE GROUP ORBITS

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ABSTRACT. In its most general form, the optimal problem is an infinite-dimensional optimization problem, yet certain notable instances admit closed-form solutions. We identify the common source of this tractability as *symmetry* and formalize it using Lie group theory. Fixing a Lie group action and a reference distribution, we study optimal transport between measures in the same orbit. For such pairs of measures, restricting the Monge problem to transport maps induced by the group action yields the Lie group orbit transport problem, an optimization over the stabilizer of the reference law that provides an upper bound on the Monge problem. Under suitable regularity conditions, a feasible group-induced map satisfying a c -convex first-order certificate attains this bound, solves the original Monge problem, and induces an optimal Kantorovich plan. For the quadratic cost, we establish a structural criterion based on a global Cartan decomposition. Exploiting the nonuniqueness of orbit representations, the criterion constructs an optimal map through a path of group transformations with self-adjoint derivatives. Once verified for an action and reference law, it applies to every pair of orbit measures with finite second moments. For affine-induced actions, Cartan structure on the linear image suffices, and the means and covariance operators determine the optimal map and value explicitly. The framework recovers classical elliptical transport formulas and yields closed-form solutions on congruence orbits generated by absolutely continuous, spectrally invariant reference laws, including Wishart, inverse-Wishart, and matrix beta type II families with fixed shape parameters. It also recovers the common-copula optimal coupling for multivariate normal sinh-arcsinh laws with unit tailweight parameters sharing a Gaussian copula, and evaluates the optimal transport value in closed form.

1. INTRODUCTION

Originating in the seminal work of [Monge, 1781] and later given its analytic form by [Kantorovich, 1942], the optimal transport problem seeks the minimal total cost of transforming one probability measure into another with respect to a prescribed transportation cost function defined on the underlying space. When both probability measures are discrete and explicitly specified by enumerating their atoms and associated probabilities, the optimal transport problem reduces to a finite-dimensional linear program. Thus, classical polynomial-time algorithms (e.g., interior-point methods [Karmarkar, 1984]) provide efficient solutions whose complexity scales polynomially with the input size. In sharp contrast, when discrete measures have implicitly defined supports, such as when distributions factorize across multiple dimensions, the number of atoms can grow exponentially with the dimension, resulting in a problem that remains polynomially describable yet is provably $\#P$ -hard [Taşkesen et al., 2023b]. Likewise, the optimal transport problem between a generic (possibly continuous) measure and a discrete measure is also known to be $\#P$ -hard [Taşkesen et al., 2023a]. When both probability measures are *continuous*, the optimal transport problem manifests as an infinite-dimensional linear program. Following the previously observed complexity trend, one might naturally conjecture that optimal transport between two continuous, non-atomic measures, where mass is spread over a continuum rather than finitely many atoms,

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would be equally formidable. Curiously, however, certain instances of the optimal transport problem between continuous probability measures admit explicit closed-form solutions, rendering these problems uniquely tractable despite their inherently infinite-dimensional structure.

Early breakthroughs showed that infinite-dimensionality does not preclude analytic tractability. In one dimension, [Dall’Aglio, 1956] showed that for any power of absolute distance of order at least one, the optimal transport problem admits an explicit solution in terms of the marginal quantile functions. For multivariate Gaussian distributions, [Dowson and Landau, 1982] and [Olkin and Pukelsheim, 1982] derived an explicit solution for the optimal transport problem induced by the quadratic cost function in terms of the means and covariance matrices. [Gelbrich, 1990] then generalized this to elliptically contoured distributions sharing a common characteristic generator and having finite second moments, yielding a closed-form formula for the optimal transport problem and the associated optimal solution. For a comprehensive survey of closed-form solutions of optimal transport, see [Rachev and Rüschendorf, 1998, §3].

Taken together, these examples appear disparate: it is not evident a priori why an inherently infinite-dimensional problem should admit explicit closed-form solutions in these particular cases. We show that their common structural mechanism is symmetry, and we use Lie group actions to formalize and exploit this structure across families of distributions.

Specifically, we fix a Lie group action on the outcome space and a reference distribution, and study optimal transport between distributions lying on the same Lie group orbit. This orbit perspective induces a natural equivalence among distributions, allowing familiar families to be recognized as members of a single orbit. Restricting attention to transport maps induced by the group yields a structured upper bound on the Monge value, formulated as an optimization over the stabilizer subgroup of the reference law. Under suitable regularity conditions, we prove that this upper bound is exact for both the Monge and Kantorovich problems whenever an orbit-induced candidate map satisfies a c -convex first-order certificate. In that case, the optimal transport is realized by a suitable group element, and the Monge, Kantorovich, and orbit transport values coincide.

For the quadratic cost, we give a Cartan-based structural criterion that exploits the nonuniqueness of orbit representations to construct an orbit-induced map. By connecting this map to the identity through group transformations with self-adjoint, invertible derivatives, the criterion ensures that it satisfies the c -convex first-order certificate. Once verified for a group action and reference law, the criterion certifies optimality for every pair of distributions in the corresponding orbit with finite second moments.

When the action is induced by an affine representation on the ambient Hilbert space, the structural criterion can be imposed on the linear image of the acting group. In this setting, the resulting optimal map is explicitly determined by the means and covariance operators of the source and target distributions. Its transport value attains the lower bound expressed in terms of these first two moments in [Gelbrich, 1990, Theorem 2.1]. For elliptical distributions sharing a common characteristic generator, this recovers the classical closed-form transport formula and the corresponding equality case of that bound.

We then apply the affine criterion to congruence orbits generated by absolutely continuous, spectrally invariant reference laws on the cone of positive-definite matrices. This yields closed-form quadratic optimal transport maps and values between any two distributions in each such orbit. In particular, we obtain explicit formulas for Wishart, inverse-Wishart, and matrix beta type II distributions with fixed shape parameters and varying positive-definite scale matrices. Viewed as probability distributions on $\text{Sym}(d)$, these matrix-valued families are not elliptically contoured

and therefore furnish a natural non-elliptical equality class for the moment-based lower bound in [Gelbrich, 1990, Theorem 2.1]. To the best of our knowledge, the corresponding closed-form transport formulas for these three families are new. Finally, we apply the general Cartan criterion to an orbit of multivariate normal sinh-arcsinh distributions with unit tailweight parameters and a common Gaussian copula, recovering from the orbit structure the common-copula optimal coupling of [Alfonsi and Jourdain, 2014, Proposition 1.1] and evaluating the corresponding quadratic transport value in closed form.

Our main results are organized in three stages. First, Theorem 4.1 establishes the general orbit-reduction principle for arbitrary costs and gives a c -convex certificate under which the Monge, Kantorovich, and Lie group orbit transport values coincide. Second, for the quadratic cost, Theorem 4.2 provides a structural Cartan criterion that produces an orbit-induced optimal map. Along the resulting continuous path from the identity, self-adjointness and invertibility force the derivatives to remain positive definite, so the terminal map is the gradient of a strictly convex potential. Third, Proposition 1 specializes this criterion to actions induced by affine representations on the ambient Hilbert space. In this setting, the means and covariance operators of the source and target measures explicitly determine the optimal map, and the corresponding transport value coincides with the moment-based lower bound in [Gelbrich, 1990, Theorem 2.1]. Section 5 applies these results to elliptical distributions, spectral-congruence orbits of matrix-valued distributions on the positive-definite cone, and multivariate normal sinh-arcsinh distributions.

Notation. We write $\mathbb{N}_{>0} = \{1, 2, \dots\}$, $\mathbb{R}_{>0} = (0, \infty)$, and $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ for the positive integers, the positive real line, and the upper extended real line, respectively. For $n \in \mathbb{N}_{>0}$, $\mathbb{R}^n$ is equipped with its Borel σ -algebra $\mathcal{B}(\mathbb{R}^n)$, Lebesgue measure $\mathcal{L}^n$, Euclidean inner product $\langle \cdot, \cdot \rangle$, and induced norm $\|\cdot\|$. Unless stated otherwise, $\mathcal{X}$ denotes a nonempty open subset of $\mathbb{R}^n$, equipped with the Borel σ -algebra $\mathcal{B}(\mathcal{X})$ inherited from $\mathbb{R}^n$. For any set E , $\text{id}_E : E \rightarrow E$ denotes the identity map. All measures are understood to be Borel measures on the indicated spaces. For two measures μ and ν on the same measurable space, $\mu \ll \nu$ denotes absolute continuity, and $d\mu/d\nu$ denotes the corresponding Radon-Nikodym derivative. If ν is a measure on $\mathbb{R}^n$, then $\nu|_{\mathcal{X}}$ denotes its restriction to $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. We write $\mathcal{P}(\mathcal{E})$ for the set of Borel probability measures on a Borel space $\mathcal{E}$. For a Borel-measurable map $T : \mathcal{E} \rightarrow \mathcal{F}$ and $\mu \in \mathcal{P}(\mathcal{E})$, the push-forward $T_{\#}\mu \in \mathcal{P}(\mathcal{F})$ is defined by $T_{\#}\mu(\mathcal{A}) = \mu(T^{-1}(\mathcal{A}))$ for every $\mathcal{A} \in \mathcal{B}(\mathcal{F})$. The Dirac probability measure at $\mathbf{x} \in \mathcal{E}$ is denoted by $\delta_{\mathbf{x}}$. For a differentiable map F between open subsets of finite-dimensional Euclidean spaces, $D_{\mathbf{x}}F$ denotes its derivative at $\mathbf{x}$. When F depends on additional variables, the subscript $\mathbf{x}$ indicates differentiation with respect to the state variable. For a differentiable real-valued function f , $\nabla_{\mathbf{x}}f(\mathbf{x})$ denotes its Euclidean gradient. For a smooth map F between smooth manifolds, dF_p denotes its differential at p . For a Borel subset $\mathcal{X}$ of a finite-dimensional real Hilbert space $\mathcal{H}$, we write

$$\mathcal{P}_2(\mathcal{X}) = \left\{ \mu \in \mathcal{P}(\mathcal{X}) : \int_{\mathcal{X}} \|\mathbf{x}\|_{\mathcal{H}}^2 d\mu(\mathbf{x}) < \infty \right\}.$$

We write $\text{GL}(\mathcal{H})$ for the group of invertible linear operators on $\mathcal{H}$, A^* for the adjoint of A , and $\text{Tr}_{\mathcal{H}}(A)$ for its operator trace. If $\mathbf{X} \sim \mu \in \mathcal{P}_2(\mathcal{H})$, then $\mathbf{m}_{\mu} = \mathbb{E}[\mathbf{X}]$ denotes its mean and $\mathbf{C}_{\mu} : \mathcal{H} \rightarrow \mathcal{H}$ its covariance operator, defined by $\langle \mathbf{u}, \mathbf{C}_{\mu}\mathbf{v} \rangle_{\mathcal{H}} = \mathbb{E}[\langle \mathbf{u}, \mathbf{X} - \mathbf{m}_{\mu} \rangle_{\mathcal{H}} \langle \mathbf{v}, \mathbf{X} - \mathbf{m}_{\mu} \rangle_{\mathcal{H}}]$. We denote the Lie-group exponential by $\exp : \text{Lie}(G) \rightarrow G$; for a matrix Lie group, this agrees with the usual matrix exponential.

2. PRELIMINARIES

This section introduces mass transportation problems and develops the Lie group background needed for our main results.

2.1. Mass transportation problems. In his memoir, [Monge, 1781] formulated the problem of transporting one distribution of mass into another at minimal cost. Throughout the paper, we assume that the transportation cost $c : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ is Borel measurable and bounded below. Given two measures $\mu_0, \mu_1 \in \mathcal{P}(\mathcal{X})$, the *Monge problem* induced by c is defined as

$$(MP) \quad \mathbb{M}_c(\mu_0, \mu_1) = \inf_{T \in \mathcal{T}(\mu_0, \mu_1)} \int_{\mathcal{X}} c(\mathbf{x}, T(\mathbf{x})) d\mu_0(\mathbf{x}),$$

where $\mathcal{T}(\mu_0, \mu_1) = \{T : \mathcal{X} \rightarrow \mathcal{X} \text{ Borel-measurable} \mid T_{\#}\mu_0 = \mu_1\}$ is the set of admissible transport maps. If a map $T^* \in \mathcal{T}(\mu_0, \mu_1)$ attains the infimum in (MP), then it is called an *optimal Monge map* between μ_0 and μ_1 .

The admissible set $\mathcal{T}(\mu_0, \mu_1)$ can be empty. For instance, if μ_0 has an atom while μ_1 is atomless, no measurable (deterministic) map can push μ_0 to μ_1 because a map cannot split mass. Even when $\mathcal{T}(\mu_0, \mu_1)$ is nonempty, the constraint $T_{\#}\mu_0 = \mu_1$ of (MP) is non-convex. To see this, suppose that $\mu_0 = \mu_1 = \mathcal{U}([0, 1])$. Now, let $T_1(x) = x$ and $T_2(x) = 1 - x$. Then, a simple calculation shows that $T_1, T_2 \in \mathcal{T}(\mu_0, \mu_1)$. Their midpoint $\bar{T} = 0.5(T_1 + T_2)$ is the constant map sending every x to $1/2$; that is, it pushes μ_0 to a Dirac measure $\delta_{1/2}$, $\bar{T}_{\#}\mu_0 = \delta_{1/2} \neq \mu_1$ implying that $\bar{T} \notin \mathcal{T}(\mu_0, \mu_1)$. Thus the Monge problem (MP) is an infinite-dimensional optimization over a generally nonconvex feasible set.

In 1942, [Kantorovich, 1942] introduced a convex relaxation of (MP) by replacing maps with probability couplings. Given $\mu_0, \mu_1 \in \mathcal{P}(\mathcal{X})$, the *Kantorovich problem* induced by cost function c is defined as

$$(KTP) \quad \mathbb{K}_c(\mu_0, \mu_1) = \inf_{\Gamma \in \Pi(\mu_0, \mu_1)} \int_{\mathcal{X} \times \mathcal{X}} c(\mathbf{x}, \mathbf{y}) d\Gamma(\mathbf{x}, \mathbf{y}),$$

where $\Pi(\mu_0, \mu_1)$ denotes the set of all probability measures on $\mathcal{X} \times \mathcal{X}$ with first marginal μ_0 and second marginal μ_1 . If $\Gamma^* \in \Pi(\mu_0, \mu_1)$ solves (KTP), then it is called an *optimal transportation plan* between μ_0 and μ_1 . The optimization in (KTP) is commonly called the optimal transport problem; to avoid ambiguity we will refer to (KTP) as the *Kantorovich problem* and to (MP) as the *Monge problem*.

The deterministic nature of the Monge problem is appealing as it yields an explicit transport map that relocates mass without splitting it, in contrast to the probabilistic transportation plans of the Kantorovich formulation in the form of couplings. This brings interpretability and aligns with settings where splitting is not physically meaningful or desirable (e.g., moving a pile of soil or routing indivisible items). There is a tight connection between the Monge problem (MP) and the Kantorovich problem (KTP). Under suitable regularity assumptions, optimal Kantorovich plans concentrate on graphs of transport maps, and these maps solve the corresponding Monge problem. Existence of Monge solutions for the quadratic cost on Euclidean space was first developed in [Brenier, 1987, Rüschendorf and Rachev, 1990]; for non-quadratic costs, existence was investigated in [Rüschendorf, 1991, Smith and Knott, 1992, McCann, 1995]. One of the most general results linking Monge solutions to gradients of c -convex functions is due to [Villani, 2008]. For ease of reference, we now adapt [Villani, 2008, Theorem 10.28] to our notation and restate it here. Before doing so, we record the regularity hypotheses on cost function c and recall the definition of c -convexity required by that theorem.

Assumption 1. *The cost function $c : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ is continuous and bounded below. Additionally, for every $\mathbf{y} \in \mathcal{X}$, the map $\mathbf{x} \mapsto c(\mathbf{x}, \mathbf{y})$ belongs to $\mathcal{C}^1(\mathcal{X})$, and for every $\mathbf{x} \in \mathcal{X}$, the map $\mathcal{X} \ni \mathbf{y} \mapsto \nabla_{\mathbf{x}} c(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n$ is injective.*

Assumption 1 contains the twist condition (see [Villani, 2008, §10]): for each fixed $\mathbf{x}$, no two distinct targets $\mathbf{y} \neq \mathbf{y}'$ yield the same $\mathbf{x}$ -gradient of the cost. Geometrically, different destinations exert different first-order “forces” at $\mathbf{x}$.

Definition 1 (*c*-convexity and *c*-subdifferential). *Suppose $c : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$. A function $\psi : \mathcal{X} \rightarrow \bar{\mathbb{R}}$ is said to be *c*-convex if it is not identically $+\infty$ and there exists a function $\phi : \mathcal{X} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ such that $\psi(\mathbf{x}) = \sup_{\mathbf{y} \in \mathcal{X}} \{\phi(\mathbf{y}) - c(\mathbf{x}, \mathbf{y})\}$, $\forall \mathbf{x} \in \mathcal{X}$. For a *c*-convex function $\psi : \mathcal{X} \rightarrow \bar{\mathbb{R}}$, its *c*-subdifferential at a point $\mathbf{x} \in \mathcal{X}$ with $\psi(\mathbf{x}) < \infty$ is defined by $\partial^c \psi(\mathbf{x}) = \{\mathbf{y} \in \mathcal{X} : \psi(\mathbf{z}) \geq \psi(\mathbf{x}) + c(\mathbf{x}, \mathbf{y}) - c(\mathbf{z}, \mathbf{y}) \forall \mathbf{z} \in \mathcal{X}\}$. If $\psi(\mathbf{x}) = +\infty$, we set $\partial^c \psi(\mathbf{x}) = \emptyset$. We say that ψ is *c*-subdifferentiable at $\mathbf{x}$ if $\partial^c \psi(\mathbf{x}) \neq \emptyset$, and we call $\text{dom}(\partial^c \psi) = \{\mathbf{x} \in \mathcal{X} : \partial^c \psi(\mathbf{x}) \neq \emptyset\}$ the domain of *c*-subdifferentiability.*

The notion of *c*-convexity is a cost-dependent analogue of ordinary convexity. In classical convex analysis, convex functions arise as pointwise suprema of affine functions. Here, the affine family is replaced by the cost-generated family $\mathbf{x} \mapsto \phi(\mathbf{y}) - c(\mathbf{x}, \mathbf{y})$, $\mathbf{y} \in \mathcal{X}$, which leads to the definition above. In the same spirit, the notion of the *c*-subdifferential is the corresponding cost-dependent analogue of the ordinary subdifferential. In classical convex analysis, a vector $\mathbf{p} \in \mathbb{R}^n$ belongs to the subdifferential of a convex function ψ at $\mathbf{x}$ if the affine function $\mathbf{z} \mapsto \psi(\mathbf{x}) + \langle \mathbf{p}, \mathbf{z} - \mathbf{x} \rangle$ agrees with ψ at $\mathbf{x}$ and lies below ψ for all $\mathbf{z} \in \mathcal{X}$. Here, affine supporting functions are replaced by the cost-adjusted family $\mathbf{z} \mapsto \psi(\mathbf{x}) + c(\mathbf{x}, \mathbf{y}) - c(\mathbf{z}, \mathbf{y})$, $\mathbf{y} \in \mathcal{X}$, which leads to the definition above.

Assumption 2. *The cost function $c : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ and the measure $\mu_0 \in \mathcal{P}(\mathcal{X})$ are such that, for every *c*-convex function $\psi : \mathcal{X} \rightarrow \bar{\mathbb{R}}$, the function ψ is differentiable μ_0 -almost everywhere on $\text{dom}(\partial^c \psi)$.*

Assumption 2 allows the geometric condition $\mathbf{y} \in \partial^c \psi(\mathbf{x})$ to be converted into a first-order identity. Indeed, let ψ be *c*-convex, let $\mathbf{x} \in \text{dom}(\partial^c \psi)$ be a point at which ψ is differentiable, and choose $\mathbf{y} \in \partial^c \psi(\mathbf{x})$. By the definition of the *c*-subdifferential, $\psi(\mathbf{z}) \geq \psi(\mathbf{x}) + c(\mathbf{x}, \mathbf{y}) - c(\mathbf{z}, \mathbf{y})$ for all $\mathbf{z} \in \mathcal{X}$. Equivalently, $\psi(\mathbf{z}) + c(\mathbf{z}, \mathbf{y}) \geq \psi(\mathbf{x}) + c(\mathbf{x}, \mathbf{y})$ for all $\mathbf{z} \in \mathcal{X}$. Hence the function $\mathbf{z} \mapsto \psi(\mathbf{z}) + c(\mathbf{z}, \mathbf{y})$ attains a minimum at $\mathbf{z} = \mathbf{x}$. Since $\mathcal{X}$ is open and $\mathbf{z} \mapsto c(\mathbf{z}, \mathbf{y})$ is $\mathcal{C}^1$, the first-order optimality condition gives $\nabla_{\mathbf{x}} \psi(\mathbf{x}) + \nabla_{\mathbf{x}} c(\mathbf{x}, \mathbf{y}) = 0$. Together with the twist condition in Assumption 1, this identity identifies $\mathbf{y}$ uniquely.

Assumption 3. *The cost $c : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ satisfies condition (H_∞) in the sense of [Villani, 2008, Chapter 10]. More explicitly, for a set $\mathcal{S} \subseteq \mathcal{X}$ and a point $\mathbf{x} \in \bar{\mathcal{S}}$, write $\mathcal{T}(\mathcal{S}, \mathbf{x}) = \{\lim_{k \rightarrow \infty} \frac{\mathbf{x}_k - \mathbf{x}}{t_k} : \mathbf{x}_k \in \mathcal{S}, \mathbf{x}_k \rightarrow \mathbf{x}, t_k > 0, t_k \rightarrow 0\}$ for the tangent cone to $\mathcal{S}$ at $\mathbf{x}$. Then:*

- (i) *For every measurable set $\mathcal{S} \subseteq \mathcal{X}$ and every $\mathbf{x} \in \bar{\mathcal{S}} \cap \mathcal{X}$ whose tangent cone $\mathcal{T}(\mathcal{S}, \mathbf{x})$ is not contained in a half-space, there exist points $\mathbf{z}_1, \dots, \mathbf{z}_k \in \mathcal{S}$ and a ball $B \subseteq \mathcal{X}$ containing $\mathbf{x}$ such that, for all $\mathbf{y}$ outside a compact subset of $\mathcal{X}$, $\inf_{\mathbf{w} \in B} c(\mathbf{w}, \mathbf{y}) \geq \inf_{1 \leq j \leq k} c(\mathbf{z}_j, \mathbf{y})$.*
- (ii) *For every $\mathbf{x} \in \mathcal{X}$ and every neighborhood $\mathcal{U} \subseteq \mathcal{X}$ of $\mathbf{x}$, there exists a ball $B \subseteq \mathcal{X}$ containing $\mathbf{x}$ such that $\lim_{\mathbf{y} \rightarrow \infty} \sup_{\mathbf{w} \in B} \inf_{\mathbf{z} \in \mathcal{U}} \{c(\mathbf{z}, \mathbf{y}) - c(\mathbf{w}, \mathbf{y})\} = -\infty$.*

Here $\mathbf{y} \rightarrow \infty$ means that $\mathbf{y}$ eventually leaves every compact subset of $\mathcal{X}$.

Assumption 3 is a no-escape-at-infinity condition on the cost. It controls what happens when the second argument $\mathbf{y}$ leaves every compact subset of $\mathcal{X}$. The first part says that, near any point $\mathbf{x}$

and along any set $\mathcal{S}$ that has enough directions around $\mathbf{x}$, the cost of sending nearby points $\mathbf{w} \in B$ to a far-away target $\mathbf{y}$ is bounded from below by the cost of sending finitely many comparison points $\mathbf{z}_1, \dots, \mathbf{z}_k \in \mathcal{S}$ to the same target. Thus far-away targets cannot distinguish the point $\mathbf{x}$ from its surrounding directions in an uncontrolled way. The second part is a stronger separation condition. It says that, if $\mathbf{z}$ is allowed to range in any prescribed neighborhood $\mathcal{U}$ of $\mathbf{x}$, then for targets $\mathbf{y}$ going to infinity the cost difference $c(\mathbf{z}, \mathbf{y}) - c(\mathbf{w}, \mathbf{y})$ can be made uniformly very negative, with $\mathbf{w}$ ranging over a small ball B around $\mathbf{x}$. In other words, when the target moves far away, points near $\mathbf{x}$ cannot all behave as equally good first-order competitors. The assumption therefore rules out pathological behavior caused by target points escaping to infinity.

For ease of reference, we now recall the relevant consequences of [Villani, 2008, Theorem 10.28] in the notation of this paper.

Theorem 2.1. *Let $\mu_0, \mu_1 \in \mathcal{P}(\mathcal{X})$. Suppose that $c : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ satisfies Assumption 1, that the pair (c, μ_0) satisfies Assumption 2, and that $\mathbb{K}_c(\mu_0, \mu_1) < \infty$. Then:*

- (i) (KTP) admits a unique (in law) optimal transportation plan $\Gamma^* \in \Pi(\mu_0, \mu_1)$.
 - (ii) There exists a unique optimal Monge map $T^* : \mathcal{X} \rightarrow \mathcal{X}$, up to μ_0 -almost everywhere equality solving (MP).
 - (iii) There exists a c -convex function $\varphi : \mathcal{X} \rightarrow \bar{\mathbb{R}}$ such that
- (♡) $\nabla_{\mathbf{x}} \varphi(\mathbf{x}) + \nabla_{\mathbf{x}} c(\mathbf{x}, T^*(\mathbf{x})) = 0 \quad \mu_0\text{-almost surely.}$
- (iv) Γ^* is concentrated on the graph of T^* , that is, $\Gamma^* = (\text{id}_{\mathcal{X}}, T^*)_{\#} \mu_0$.
 - (v) If, in addition, c satisfies Assumption 3, then (♡) characterizes the optimal coupling in the following sense: if $\Gamma \in \Pi(\mu_0, \mu_1)$ is concentrated on pairs $(\mathbf{x}, \mathbf{y})$ satisfying $\nabla_{\mathbf{x}} \varphi(\mathbf{x}) + \nabla_{\mathbf{x}} c(\mathbf{x}, \mathbf{y}) = 0$ for some c -convex function φ , then Γ is the unique optimal Kantorovich plan. In particular, if a feasible map $T : \mathcal{X} \rightarrow \mathcal{X}$ satisfies (♡) for some c -convex function φ , then T is the unique optimal Monge map up to μ_0 -almost everywhere equality.

Under Assumptions 1, 2, and 3, Theorem 2.1(v) delineates conditions under which the Monge map is characterized via the gradient condition (♡).

2.2. Lie group theory. Originating with [Lie, 1891], Lie groups formalize continuous symmetry, that is, they are groups that are simultaneously smooth manifolds, with multiplication and inversion smooth. A canonical example is the circle (planar rotations), where composition and inversion vary smoothly with the angle. In what follows, we will review the minimal Lie-theoretic background used throughout; for comprehensive treatments see [Boumal, 2023, Duistermaat and Kolk, 2012, Hall, 2013].

Definition 2 (Lie group). *Let G be both a smooth manifold and a group. If the product map $\text{prod} : G \times G \rightarrow G : (g, h) \mapsto \text{prod}(g, h) = gh$ and the inverse map $\text{inv} : G \rightarrow G : g \mapsto \text{inv}(g) = g^{-1}$ are smooth, then G is a Lie group. Smoothness of prod is understood with respect to the product manifold structure on $G \times G$.*

We now illustrate Definition 2 with a standard example: the general linear group $\text{GL}(d)$. Note that determinant is a polynomial map, so $\text{GL}(d) = \det^{-1}(\mathbb{R} \setminus \{0\})$ is an open subset of $\mathbb{R}^{d^2}$. Hence, $\text{GL}(d)$ is a smooth manifold of dimension d^2 . The group operation is matrix multiplication and inversion. The multiplication map $\text{prod} : \text{GL}(d) \times \text{GL}(d) \rightarrow \text{GL}(d)$, with $(\mathbf{A}, \mathbf{B}) \mapsto \mathbf{AB}$ is smooth because each entry of $\mathbf{AB}$ is a polynomial in the entries of $\mathbf{A}$ and $\mathbf{B}$. The inverse map $\text{inv} : \text{GL}(d) \rightarrow \text{GL}(d)$, with $\mathbf{A} \mapsto \mathbf{A}^{-1}$ is smooth on $\text{GL}(d)$ since $\mathbf{A}^{-1} = \text{adj}(\mathbf{A}) / \det(\mathbf{A})$, where $\text{adj}(\mathbf{A})$ is polynomial in the entries of $\mathbf{A}$ and $\det(\mathbf{A}) \neq 0$ on $\text{GL}(d)$. Thus both prod and inv are

smooth, so $\text{GL}(d)$ is a Lie group. Many other familiar matrix groups are Lie groups including $\mathcal{O}(d)$ and $\text{SL}(d) = \{\mathbf{A} \in \mathbb{R}^{d \times d} \mid \det(\mathbf{A}) = 1\}$; see [Hall, 2013, Chapter 1].

Definition 3 (Left action and orbit). *Let G be a Lie group and let $\mathcal{M}$ be a set. A left action of G on $\mathcal{M}$ is a map $\phi : G \times \mathcal{M} \rightarrow \mathcal{M}$ such that:*

- (i) *for all $\rho \in \mathcal{M}$, $\phi(e, \rho) = \rho$,*
- (ii) *for all $g, h \in G$ and $\rho \in \mathcal{M}$, $\phi(gh, \rho) = \phi(g, \phi(h, \rho))$,*

where e is the identity element of G . The orbit of $\rho \in \mathcal{M}$ under the action ϕ of G is the set $G_\rho = \{\phi(g, \rho) \mid g \in G\}$.

Orbits induce an equivalence relation on $\mathcal{M}$: two points $\rho_1, \rho_2 \in \mathcal{M}$ are equivalent whenever one can be reached from the other by applying an element of G , that is, whenever there exists $g \in G$ such that $\rho_2 = \phi(g, \rho_1)$. Thus, the equivalence classes are the orbits of the action.

3. DISTRIBUTIONS IN LIE GROUP ORBITS

We denote the infinite-dimensional group of all smooth diffeomorphisms of $\mathcal{X}$ by $\text{Diff}(\mathcal{X})$, defined as $\text{Diff}(\mathcal{X}) = \{\Phi : \mathcal{X} \rightarrow \mathcal{X} \mid \Phi \text{ is a } \mathcal{C}^\infty \text{ bijection and } \Phi^{-1} \text{ is } \mathcal{C}^\infty\}$. The group operation on $\text{Diff}(\mathcal{X})$ is composition, the identity element is $\text{id}_{\mathcal{X}}$, and inversion is the usual map inverse.

In the remainder of the paper, let G be a Lie group acting smoothly on $\mathcal{X}$. Thus we are given a smooth map $G \times \mathcal{X} \rightarrow \mathcal{X}$, $(g, \mathbf{x}) \mapsto \alpha(g)(\mathbf{x})$, such that $\alpha(e) = \text{id}_{\mathcal{X}}$ and $\alpha(gh) = \alpha(g) \circ \alpha(h)$ for all $g, h \in G$. Equivalently, the action determines a group homomorphism $\alpha : G \rightarrow \text{Diff}(\mathcal{X})$, $g \mapsto \alpha(g)$, whose associated evaluation map $(g, \mathbf{x}) \mapsto \alpha(g)(\mathbf{x})$ is smooth. We refer to the image $\alpha(G) \subseteq \text{Diff}(\mathcal{X})$ as the associated acting transformation group. For example, let $G = \text{GL}(d)$ and $\mathcal{X} = \mathbb{R}^d$, and define $\alpha : G \rightarrow \text{Diff}(\mathbb{R}^d)$ by $\alpha(\mathbf{A})(\mathbf{x}) = \mathbf{A}\mathbf{x}$. Then α is a group homomorphism, and $\alpha(G)$ is the group of linear diffeomorphisms of $\mathbb{R}^d$.

The goal of this paper is to identify when and explain why the optimal transport problem admits closed-form solutions. We show that many tractable instances occur when the source and target measures lie in the same orbit of a group acting on the outcome space. Accordingly, we begin by describing the push-forward action of subgroups of diffeomorphisms on probability measures, which lets us speak of orbits exactly as in classical group actions, but now at the level of distributions.

The action α induces a left action of G on $\mathcal{P}(\mathcal{X})$ by push-forward: $\phi : G \times \mathcal{P}(\mathcal{X}) \rightarrow \mathcal{P}(\mathcal{X})$, $\phi(g, \mu) = \alpha(g)_\# \mu$. Indeed, since $\alpha(e) = \text{id}_{\mathcal{X}}$ and $\alpha(gh) = \alpha(g) \circ \alpha(h)$, it follows that $\alpha(e)_\# \mu = \mu$, and $\alpha(gh)_\# \mu = \alpha(g)_\# (\alpha(h)_\# \mu)$ for all $g, h \in G$ and $\mu \in \mathcal{P}(\mathcal{X})$.

With this action in hand, we introduce the basic objects it generates.

Definition 4 (Orbit of a measure). *For $\rho \in \mathcal{P}(\mathcal{X})$, we define its G -orbit as $G_\# \rho = \{\alpha(g)_\# \rho \mid g \in G\} \subset \mathcal{P}(\mathcal{X})$.*

Thus two measures are equivalent if one is the push-forward of the other by some $g \in G$.

Lemma 1 (Orbit measures remain absolutely continuous). *If $\rho \ll \mathcal{L}^n|_{\mathcal{X}}$, then for every $g \in G$ we have $\alpha(g)_\# \rho \ll \mathcal{L}^n|_{\mathcal{X}}$ and, for $\mathcal{L}^n$ -almost every $\mathbf{y} \in \mathcal{X}$,*

$$\frac{d(\alpha(g)_\# \rho)}{d\mathcal{L}^n}(\mathbf{y}) = \frac{d\rho}{d\mathcal{L}^n}(\alpha(g)^{-1}(\mathbf{y})) \left| \det(D(\alpha(g)^{-1})(\mathbf{y})) \right|,$$

where $D(\alpha(g)^{-1})(\mathbf{y})$ denotes the Jacobian matrix of $\alpha(g)^{-1}$ evaluated at $\mathbf{y}$.

Proof. Since $\alpha(g) \in \text{Diff}(\mathcal{X})$, the map $\alpha(g)$ is a $\mathcal{C}^1$ -diffeomorphism of $\mathcal{X}$. The conclusion therefore follows from the change-of-variables formula. $\square$

Definition 5 (Stabilizer subgroup). *For a reference measure $\rho \in \mathcal{P}(\mathcal{X})$, the stabilizer of ρ in G is $\text{Stab}_G(\rho) = \{h \in G \mid \alpha(h)_\# \rho = \rho\}$.*

Thus $\text{Stab}_G(\rho)$ consists precisely of those elements of G whose induced push-forward action leaves the reference measure ρ invariant. $\text{Stab}_G(\rho)$ is nonempty because $\alpha(e) = \text{id}_\mathcal{X}$, and hence $\alpha(e)_\# \rho = \rho$.

With the action, orbits, and stabilizer in place, the subsequent section studies the optimal transport problem between measures lying on a common orbit.

4. TRANSPORT OF DISTRIBUTIONS WITHIN LIE GROUP ORBITS

For the remainder of this paper, we fix a reference probability measure $\rho \in \mathcal{P}(\mathcal{X})$. We study the Monge transportation problem (MP) induced by a Borel-measurable cost function $c : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$, between two measures in the orbit of ρ , that is, $\mu_0, \mu_1 \in G_\#(\rho)$. Equivalently, there exist $g_0, g_1 \in G$, such that $\mu_0 = \alpha(g_0)_\# \rho$ and $\mu_1 = \alpha(g_1)_\# \rho$. We fix one such pair (g_0, g_1) throughout this section. We assume $\rho \ll \mathcal{L}^n|_\mathcal{X}$, then by Lemma 1, μ_0 and μ_1 are absolutely continuous as well, and we write $r_i = d\mu_i/d\mathcal{L}^n$ for their densities.

A crucial observation underpinning our analysis is that, while the Monge problem is typically an infinite-dimensional and notoriously challenging optimization problem, a subtle algebraic structure emerges when the measures reside in a common G -orbit. First, the set of admissible Monge maps can be empty in general. When the source and target measures reside on a common orbit, the map $T = \alpha(g_1) \circ \alpha(g_0)^{-1} \in \alpha(G)$ satisfies $T_\# \mu_0 = \mu_1$, so $\mathcal{T}(\mu_0, \mu_1)$ is nonempty. Moreover, every admissible transport admits a canonical factorization through the reference law: it can be pulled back to the ρ -coordinates. The next lemma formalizes this observation and serves as our bridge from arbitrary transportation maps to ρ -preserving transformations.

Lemma 2. *If a measurable map $T : \mathcal{X} \rightarrow \mathcal{X}$ satisfies $T_\# \mu_0 = \mu_1$, then the composite $H = \alpha(g_1)^{-1} \circ T \circ \alpha(g_0)$ is such that $H_\# \rho = \rho$ and $T = \alpha(g_1) \circ H \circ \alpha(g_0)^{-1}$.*

Proof. By the functoriality of push-forward measures and the identities $\mu_i = \alpha(g_i)_\# \rho$, $i = 0, 1$, we have

$$\begin{aligned} H_\# \rho &= (\alpha(g_1)^{-1} \circ T \circ \alpha(g_0))_\# \rho = (\alpha(g_1)^{-1})_\# T_\# (\alpha(g_0)_\# \rho) = (\alpha(g_1)^{-1})_\# (T_\# \mu_0) \\ &= (\alpha(g_1)^{-1})_\# \mu_1 = (\alpha(g_1)^{-1})_\# (\alpha(g_1)_\# \rho) = \rho. \end{aligned}$$

The identity $T = \alpha(g_1) \circ H \circ \alpha(g_0)^{-1}$ follows directly from the definition of H . $\square$

Lemma 2 shows that any admissible map $T \in \mathcal{T}(\mu_0, \mu_1)$ can be pulled back to a ρ -preserving transformation H on the reference space, and then recovered by composing with the orbit representatives $\alpha(g_0)$ and $\alpha(g_1)$. Thus, the Monge problem can be viewed as a search over ρ -preserving maps on the reference space. If, in addition, H belongs to the acting transformation group $\alpha(G)$, say $H = \alpha(h)$ for some $h \in G$, then $h \in \text{Stab}_G(\rho)$. To systematically exploit this structure, we introduce the *Lie group orbit transport problem* between $\alpha(g_0)_\# \rho$ and $\alpha(g_1)_\# \rho$ induced by the cost function c as follows:

$$\text{(LGOP)} \quad \mathbb{J}_c(g_0, g_1) = \inf_{h \in \text{Stab}_G(\rho)} \int_{\mathcal{X}} c(\alpha(g_0)(\mathbf{x}), \alpha(g_1 h)(\mathbf{x})) d\rho(\mathbf{x}).$$

Lemma 3. *Let $s_0, s_1 \in \text{Stab}_G(\rho)$. Then $\mathbb{J}_c(g_0 s_0, g_1 s_1) = \mathbb{J}_c(g_0, g_1)$.*

Proof. By definition, $\mathbb{J}_c(g_0 s_0, g_1 s_1) = \inf_{h \in \text{Stab}_G(\rho)} \int_{\mathcal{X}} c(\alpha(g_0 s_0)(\mathbf{x}), \alpha(g_1 s_1 h)(\mathbf{x})) d\rho(\mathbf{x})$.

As $s_0 \in \text{Stab}_G(\rho)$, the change of variables $\mathbf{z} = \alpha(s_0)(\mathbf{x})$ preserves ρ . Therefore $\mathbb{J}_c(g_0 s_0, g_1 s_1)$ equals

$$\inf_{h \in \text{Stab}_G(\rho)} \int_{\mathcal{X}} c(\alpha(g_0)(\mathbf{z}), \alpha(g_1 s_1 h s_0^{-1})(\mathbf{z})) d\rho(\mathbf{z}).$$

Since $\text{Stab}_G(\rho)$ is a subgroup of G , the map $h \mapsto s_1 h s_0^{-1}$ is a bijection of $\text{Stab}_G(\rho)$ onto itself. Hence

$$\mathbb{J}_c(g_0 s_0, g_1 s_1) = \inf_{\tilde{h} \in \text{Stab}_G(\rho)} \int_{\mathcal{X}} c(\alpha(g_0)(\mathbf{z}), \alpha(g_1 \tilde{h})(\mathbf{z})) d\rho(\mathbf{z}) = \mathbb{J}_c(g_0, g_1).$$

□

By Lemma 3, one may equivalently regard $\mathbb{J}_c$ as a function of the quotient representatives $g_0 \text{Stab}_G(\rho)$ and $g_1 \text{Stab}_G(\rho)$, or equivalently as a function of the orbit measures μ_0, μ_1 .

Moreover, because every $h \in \text{Stab}_G(\rho)$ produces an admissible transport map $T_h = \alpha(g_1 h g_0^{-1})$, (LGOP) provides a structured upper bound on (MP) indexed by the stabilizer of the reference law. Since $\text{Stab}_G(\rho)$ is finite-dimensional, this reduces the search to a finite-dimensional problem; when the stabilizer is trivial, it reduces to evaluating a single candidate map.

The key issue is when this upper bound is tight. Under the hypotheses of Theorem 2.1, optimality of a candidate map is certified by the existence of a c -convex potential satisfying (♥). In general, finding such a potential is an analytic problem and, for smooth costs, leads to the partial differential equations of optimal transport [Villani, 2003, § 12], for which explicit solutions are rarely available.

The point of the orbit framework is that, for maps of the special form $T_h = \alpha(g_1 h g_0^{-1})$, this analytic certificate can sometimes be verified algebraically. When this happens, T_h is optimal for the Monge problem, the orbit upper bound is tight, and the same element h automatically solves (LGOP). The following theorem formalizes this certificate.

Theorem 4.1. *The following statements hold.*

- (i) $\mathbb{M}_c(\mu_0, \mu_1) \leq \mathbb{J}_c(g_0, g_1)$.
- (ii) Suppose that $\mathbb{K}_c(\mu_0, \mu_1) < \infty$, that c satisfies Assumptions 1 and 3, and that the pair (c, μ_0) satisfies Assumption 2. Suppose further that there exist $h \in \text{Stab}_G(\rho)$ and a c -convex function φ such that the map $T_h = \alpha(g_1 h g_0^{-1})$ satisfies (♥) μ_0 -almost surely. Then:
 - (a) T_h solves (MP), and it is the unique optimal Monge map up to μ_0 -almost everywhere equality,
 - (b) h solves (LGOP),
 - (c) $(\text{id}_{\mathcal{X}}, T_h)_{\#} \mu_0$ solves (KTP) and it is the unique optimal Kantorovich plan,
 - (d) $\mathbb{K}_c(\mu_0, \mu_1) = \mathbb{M}_c(\mu_0, \mu_1) = \mathbb{J}_c(g_0, g_1)$.

Proof. We first show that (MP) provides a lower bound on (LGOP), and this proves the first claim in the theorem statement. To this end, choose any h feasible in (LGOP), and define $T_h = \alpha(g_1 h g_0^{-1})$. Since $h \in \text{Stab}_G(\rho)$, we have

$$(T_h)_{\#} \mu_0 = \alpha(g_1 h g_0^{-1})_{\#} \alpha(g_0)_{\#} \rho = \alpha(g_1 h)_{\#} \rho = \alpha(g_1)_{\#} \alpha(h)_{\#} \rho = \alpha(g_1)_{\#} \rho = \mu_1.$$

Thus $T_h \in \mathcal{T}(\mu_0, \mu_1)$. We next verify that the objective value attained by h in (LGOP) coincides with the value attained by T_h in (MP):

$$\int_{\mathcal{X}} c(\alpha(g_0)(\mathbf{x}), \alpha(g_1 h)(\mathbf{x})) d\rho(\mathbf{x}) = \int_{\mathcal{X}} c(\alpha(g_0)(\mathbf{x}), (\alpha(g_1) \circ \alpha(h))(\mathbf{x})) d\rho(\mathbf{x})$$

$$\begin{aligned}
&= \int_{\mathcal{X}} c(\alpha(g_0)(\mathbf{x}), (T_h \circ \alpha(g_0))(\mathbf{x})) d\rho(\mathbf{x}) \\
&= \int_{\mathcal{X}} c(\mathbf{x}, T_h(\mathbf{x})) d\mu_0(\mathbf{x}).
\end{aligned}$$

Taking the infimum over h in both sides of the equality above implies that (MP) provides a lower bound on (LGOP). This observation proves assertion (i).

In the remainder of the proof, assume that c satisfies Assumptions 1 and 3, that the pair (c, μ_0) satisfies Assumption 2, and that there exist $h \in \text{Stab}_G(\rho)$ and a c -convex function φ such that T_h satisfies (♥) μ_0 -almost surely. Set $\Gamma_h = (\text{id}_{\mathcal{X}}, T_h)_{\#} \mu_0$. Since $T_h \in \mathcal{T}(\mu_0, \mu_1)$, we have $\Gamma_h \in \Pi(\mu_0, \mu_1)$. Moreover, the assumed identity in (♥) is equivalent to $\nabla_{\mathbf{x}} \varphi(\mathbf{x}) + \nabla_{\mathbf{x}} c(\mathbf{x}, \mathbf{y}) = 0$, Γ_h -almost surely. By Theorem 2.1(v), the coupling Γ_h is the unique optimal Kantorovich plan. This proves assertion (ii)(c). Since Γ_h is induced by the deterministic map T_h , the map T_h solves the Monge problem. The uniqueness of T_h up to μ_0 -almost everywhere equality follows from Theorem 2.1. This proves (ii)(a). Hence $\mathbb{M}_c(\mu_0, \mu_1) = \int_{\mathcal{X}} c(\mathbf{x}, T_h(\mathbf{x})) d\mu_0(\mathbf{x})$. As $T_h = \alpha(g_1 h g_0^{-1})$, we have

$$\mathbb{M}_c(\mu_0, \mu_1) = \int_{\mathcal{X}} c(\alpha(g_0)(\mathbf{z}), \alpha(g_1 h)(\mathbf{z})) d\rho(\mathbf{z}).$$

Since $h \in \text{Stab}_G(\rho)$, this h is feasible for (LGOP). Therefore

$$\mathbb{J}_c(g_0, g_1) \leq \int_{\mathcal{X}} c(\alpha(g_0)(\mathbf{z}), \alpha(g_1 h)(\mathbf{z})) d\rho(\mathbf{z}) = \mathbb{M}_c(\mu_0, \mu_1).$$

Together with assertion (i), the inequality above implies $\mathbb{M}_c(\mu_0, \mu_1) = \mathbb{J}_c(g_0, g_1)$, and that the feasible element h attains the infimum in (LGOP), proving (ii)(b).

Since $(\text{id}_{\mathcal{X}}, T_h)_{\#} \mu_0$ has the same cost as the optimal Monge map, we obtain $\mathbb{K}_c(\mu_0, \mu_1) = \mathbb{M}_c(\mu_0, \mu_1) = \mathbb{J}_c(g_0, g_1)$. This proves (ii)(d). $\square$

4.1. Quadratic cost. We now specialize to the quadratic cost. Let $\mathcal{H}$ be a finite-dimensional real Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and induced norm $\|\cdot\|_{\mathcal{H}}$, and let $\mathcal{X} \subseteq \mathcal{H}$ be a nonempty open convex set. Unless stated otherwise, the cost function is the quadratic cost $c(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_{\mathcal{H}}^2$, $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, and the reference measure $\rho \in \mathcal{P}(\mathcal{X})$ satisfies $\rho \ll \mathcal{L}_{\mathcal{H}}|_{\mathcal{B}(\mathcal{X})}$ and $\int_{\mathcal{X}} \|\mathbf{x}\|_{\mathcal{H}}^2 d\rho(\mathbf{x}) < \infty$, where $\mathcal{L}_{\mathcal{H}}$ denotes the Lebesgue measure on $\mathcal{H}$ induced by its Hilbert structure.

The following lemma records the basic convex-analytic identities for the quadratic cost that will be used repeatedly in the remainder of the paper.

Lemma 4 (Quadratic c -convexity and c -subdifferential). *For a function $\psi : \mathcal{X} \rightarrow \bar{\mathbb{R}}$, define $\bar{\psi}(\mathbf{x}) = \psi(\mathbf{x}) + \|\mathbf{x}\|_{\mathcal{H}}^2$. Then the following hold:*

- (i) *If ψ is c -convex, then $\bar{\psi}$ is convex.*
- (ii) *For $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, $\mathbf{y} \in \partial^c \psi(\mathbf{x})$ if and only if $2\mathbf{y} \in \partial \bar{\psi}(\mathbf{x})$, where $\partial \bar{\psi}(\mathbf{x})$ denotes the convex subdifferential of $\bar{\psi}$ at $\mathbf{x}$.*
- (iii) *If $\mu_0 \ll \mathcal{L}_{\mathcal{H}}$, then Assumption 2 holds.*
- (iv) *c satisfies Assumption 3 on $\mathcal{H}$.*

Proof. Recall that $\partial \bar{\psi}(\mathbf{x}) = \{\mathbf{p} \in \mathcal{H} : \bar{\psi}(\mathbf{z}) \geq \bar{\psi}(\mathbf{x}) + \langle \mathbf{p}, \mathbf{z} - \mathbf{x} \rangle_{\mathcal{H}} \forall \mathbf{z} \in \mathcal{X}\}$. For the quadratic cost, $c(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_{\mathcal{H}}^2 = \|\mathbf{x}\|_{\mathcal{H}}^2 - 2\langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{H}} + \|\mathbf{y}\|_{\mathcal{H}}^2$.

(i) Suppose that ψ is c -convex. Then there exists a function $\phi : \mathcal{X} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ such that $\psi(\mathbf{x}) = \sup_{\mathbf{y} \in \mathcal{X}} \{\phi(\mathbf{y}) - c(\mathbf{x}, \mathbf{y})\}$ for all $\mathbf{x} \in \mathcal{X}$. Then, we have $\bar{\psi}(\mathbf{x}) = \sup_{\mathbf{y} \in \mathcal{X}} \{\phi(\mathbf{y}) - \|\mathbf{y}\|_{\mathcal{H}}^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{H}}\}$. For each fixed $\mathbf{y} \in \mathcal{X}$, the map $\mathbf{x} \mapsto \phi(\mathbf{y}) - \|\mathbf{y}\|_{\mathcal{H}}^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{H}}$ is affine on $\mathcal{H}$. Hence $\bar{\psi}$ is the pointwise supremum of affine functions, and therefore is convex on $\mathcal{X}$. (ii) Fix $\mathbf{x}, \mathbf{y} \in \mathcal{X}$. By

definition, $\mathbf{y} \in \partial^c \psi(\mathbf{x})$ if and only if $\psi(\mathbf{z}) + \|\mathbf{z}\|_{\mathcal{H}}^2 \geq \psi(\mathbf{x}) + \|\mathbf{x}\|_{\mathcal{H}}^2 + 2\langle \mathbf{y}, \mathbf{z} - \mathbf{x} \rangle_{\mathcal{H}}$ for all $\mathbf{z} \in \mathcal{X}$, that is, $\bar{\psi}(\mathbf{z}) \geq \bar{\psi}(\mathbf{x}) + \langle 2\mathbf{y}, \mathbf{z} - \mathbf{x} \rangle_{\mathcal{H}}$ for all $\mathbf{z} \in \mathcal{X}$, which corresponds to the condition $2\mathbf{y} \in \partial \bar{\psi}(\mathbf{x})$. (iii) Let ψ be c -convex. By part (i), $\bar{\psi}$ is convex. Let $\text{dom}_{\text{eff}}(\bar{\psi}) = \{\mathbf{x} \in \mathcal{X} : \bar{\psi}(\mathbf{x}) < \infty\}$ be the effective domain of $\bar{\psi}$. Since $\bar{\psi}$ is convex, $\text{dom}_{\text{eff}}(\bar{\psi})$ is convex. By [Borwein and Vanderwerff, 2010, Theorem 2.1.12], $\bar{\psi}$ is locally Lipschitz on the interior of its effective domain. Therefore, by [Evans and Gariepy, 1991, Theorem 3.2], $\bar{\psi}$ is differentiable $\mathcal{L}_{\mathcal{H}}$ -almost everywhere on $\text{int}(\text{dom}_{\text{eff}}(\bar{\psi}))$. Since $\psi = \bar{\psi} - \|\cdot\|_{\mathcal{H}}^2$, and $\mathbf{x} \mapsto \|\mathbf{x}\|_{\mathcal{H}}^2$ is smooth, it follows that ψ is also differentiable $\mathcal{L}_{\mathcal{H}}$ -almost everywhere on $\text{int}(\text{dom}_{\text{eff}}(\bar{\psi}))$. Now let $\mathbf{x} \in \text{dom}(\partial^c \psi)$. Then $\partial^c \psi(\mathbf{x}) \neq \emptyset$. By part (ii), $\partial \bar{\psi}(\mathbf{x}) \neq \emptyset$, and therefore $\bar{\psi}(\mathbf{x}) < \infty$. Thus $\text{dom}(\partial^c \psi) \subseteq \text{dom}_{\text{eff}}(\bar{\psi})$. Since $\text{dom}_{\text{eff}}(\bar{\psi})$ is convex, its boundary has $\mathcal{L}_{\mathcal{H}}$ -measure zero. Hence ψ is $\mathcal{L}_{\mathcal{H}}$ -almost everywhere differentiable on $\text{dom}(\partial^c \psi)$. If $\mu_0 \ll \mathcal{L}_{\mathcal{H}}$, the same holds μ_0 -almost surely. Hence, Assumption 2 holds. (iv) $\mathcal{H}$ is a flat Riemannian manifold, its Riemannian distance is $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_{\mathcal{H}}$, and its sectional curvature is identically zero. Therefore, [Villani, 2008, Example 10.36] applied with $M = \mathcal{X} = \mathcal{Y} = \mathcal{H}$ and $c(\mathbf{x}, \mathbf{y}) = d(\mathbf{x}, \mathbf{y})^2$, implies that the quadratic cost $c(x, y) = \|x - y\|_{\mathcal{H}}^2$ satisfies Assumption 3. $\square$

The preceding lemma shows that, for the quadratic cost, the shifted potential $\bar{\psi} = \psi + \|\cdot\|_{\mathcal{H}}^2$ is convex, and the c -subdifferential relation becomes an ordinary subgradient relation for this shifted potential. Consequently, when the potential is differentiable, the first-order condition in Theorem 2.1 identifies the optimal map as the gradient of a convex function, in agreement with Brenier's convex-gradient characterization of quadratic optimal transport [Brenier, 1991, Theorem 1.1-1.2].

We now turn to the additional structure supplied by the group action. While the group describes the transformations relating the measures, optimal transport imposes a much more restrictive geometric requirement on the transformation between them. To identify when these two structures are compatible, it is useful to first understand the group transformations infinitesimally, through their behavior near the identity. For a Lie group, this local structure is encoded by its Lie algebra. We begin with the necessary Lie-algebraic background. Recall that the identity element of G is denoted by e . We write $\text{Lie}(G)$ for its Lie algebra, that is, the tangent space $T_e G$ endowed with its canonical Lie bracket. A *Lie group involution* is a smooth group automorphism $\theta : G \rightarrow G$ satisfying $\theta^2 = \text{id}_G$, where $\text{id}_G : G \rightarrow G$ satisfies $\text{id}_G(g) = g$ for every $g \in G$. Differentiating this identity at the identity element e gives $(d\theta_e)^2 = \text{id}_{\text{Lie}(G)}$. Thus $d\theta_e$ is a linear involution on the finite-dimensional real vector space $\text{Lie}(G)$. Define

$$\mathfrak{k} = \{X \in \text{Lie}(G) : d\theta_e(X) = X\}, \quad \mathfrak{p} = \{X \in \text{Lie}(G) : d\theta_e(X) = -X\}.$$

For every $X \in \text{Lie}(G)$, we have $X = \frac{1}{2}(X + d\theta_e(X)) + \frac{1}{2}(X - d\theta_e(X))$. Since $(d\theta_e)^2 = \text{id}_{\text{Lie}(G)}$, the first term belongs to $\mathfrak{k}$ and the second belongs to $\mathfrak{p}$. Moreover, $\mathfrak{k} \cap \mathfrak{p} = \{0\}$, since $d\theta_e(X) = X = -X$ for every $X \in \mathfrak{k} \cap \mathfrak{p}$. Therefore, $\text{Lie}(G) = \mathfrak{k} \oplus \mathfrak{p}$.

Definition 6 (Cartan decomposition). *Let $\theta : G \rightarrow G$ be a Lie group involution. Write*

$$\mathfrak{g} = \text{Lie}(G), \quad \mathfrak{k} = \{X \in \mathfrak{g} : d\theta_e(X) = X\}, \quad \mathfrak{p} = \{X \in \mathfrak{g} : d\theta_e(X) = -X\},$$

and let $G^\theta = \{\ell \in G : \theta(\ell) = \ell\}$. We say that (G, θ) admits a global Cartan decomposition if the map $G^\theta \times \mathfrak{p} \rightarrow G$, $(k, S) \mapsto k \exp(S)$, is a diffeomorphism.

Definition 6 imposes a global property on (G, θ) that is stronger than the infinitesimal splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. In particular, it implies that every $g \in G$ admits a unique factorization $g = k \exp(S)$ with $k \in G^\theta$ and $S \in \mathfrak{p}$, and that these factors depend smoothly on g . In the standard reductive setting,

this factorization separates a compact, rotation-like component from a noncompact, stretching-like component. More precisely, when G is a real reductive linear Lie group and θ is a Cartan involution, G^θ is the compact factor and the map $G^\theta \times \mathfrak{p} \rightarrow G$, $(k, S) \mapsto k \exp(S)$, gives the global Cartan decomposition; see, for example, [Knapp, 1996, § VII.2].

Lemma 5 (Cartan absorption). *Let (G, θ) admit a global Cartan decomposition in the sense of Definition 6. Then, for every $\ell_0, \ell_1 \in G$, there exists $h \in G^\theta$ such that $\ell_1 h \ell_0^{-1} \in \exp(\mathfrak{p})$.*

Proof. Define $\eta : G \rightarrow G$ such that $\eta(\ell) = \theta(\ell)^{-1} \ell$. We will first show that $\{\eta(\ell) : \ell \in G\} = \exp(\mathfrak{p})$. Let $\ell \in G$. By the Cartan decomposition, there exist $k \in G^\theta$ and $X \in \mathfrak{p}$ such that $\ell = k \exp(X)$. Since $\theta(k) = k$ and $d\theta_e(X) = -X$, we have

$$(1) \quad \theta(\exp(X)) = \exp(d\theta_e(X)) = \exp(-X).$$

Therefore

$$\begin{aligned} \eta(\ell) &= \theta(k \exp(X))^{-1} k \exp(X) = (\theta(k) \theta(\exp(X)))^{-1} k \exp(X) \\ &= (k \exp(-X))^{-1} k \exp(X) = \exp(2X) \in \exp(\mathfrak{p}), \end{aligned}$$

which implies $\{\eta(\ell) : \ell \in G\} \subseteq \exp(\mathfrak{p})$. Conversely, if $p = \exp(Y) \in \exp(\mathfrak{p})$ with $Y \in \mathfrak{p}$, then

$$\eta(\exp(Y/2)) = \theta(\exp(Y/2))^{-1} \exp(Y/2) = \exp(-Y/2)^{-1} \exp(Y/2) = \exp(Y) = p,$$

where the second equality follows by (1). Hence $\exp(\mathfrak{p}) \subseteq \{\eta(\ell) : \ell \in G\}$, and thus $\{\eta(\ell) : \ell \in G\} = \exp(\mathfrak{p})$.

Next, for each $a \in G$, define the map $\tau_a : \exp(\mathfrak{p}) \rightarrow G$ as $\tau_a(p) = \theta(a)^{-1} p a$. In what follows, we will show that $\{\tau_a(p) : p \in \exp(\mathfrak{p})\} \subseteq \exp(\mathfrak{p})$ for every $a \in G$. Indeed, if $p \in \exp(\mathfrak{p})$, then by $\{\eta(\ell) : \ell \in G\} = \exp(\mathfrak{p})$, there exists $x \in G$ such that $p = \eta(x) = \theta(x)^{-1} x$. Therefore

$$\tau_a(p) = \theta(a)^{-1} \theta(x)^{-1} x a = \theta(xa)^{-1} (xa) = \eta(xa) \in \exp(\mathfrak{p}).$$

Thus $\tau_a(\exp(\mathfrak{p})) \subseteq \exp(\mathfrak{p})$ for every $a \in G$. Now fix $\ell_0, \ell_1 \in G$. By the Cartan decomposition, we may write $\theta(\ell_0)^{-1} \ell_1 = kp$ for some $k \in G^\theta$, $p \in \exp(\mathfrak{p})$. Set $h = k^{-1} \in G^\theta$. Since $k \in G^\theta$, we have $\theta(k) = k$ by definition of G^θ . As group automorphisms preserve inverses $\theta(h) = \theta(k)^{-1} = k^{-1} = h$. Hence h indeed belongs to G^θ and we have $\theta(h)^{-1} = k$. Therefore,

$$kpk^{-1} = \theta(k^{-1})^{-1} pk^{-1} = \tau_{k^{-1}}(p) \in \exp(\mathfrak{p}).$$

Now, define $q = kpk^{-1} \in \exp(\mathfrak{p})$. Then

$$\ell_1 h \ell_0^{-1} = \ell_1 k^{-1} \ell_0^{-1} = \theta(\ell_0) kpk^{-1} \ell_0^{-1} = \theta(\ell_0) q \ell_0^{-1} = \tau_{\ell_0^{-1}}(q) \in \exp(\mathfrak{p}),$$

where the second equality follows because $\theta(\ell_0)^{-1} \ell_1 = kp$, and thus $\ell_1 = \theta(\ell_0) kp$, and the inclusion follows because $\tau_a(\exp(\mathfrak{p})) \subseteq \exp(\mathfrak{p})$ for every $a \in G$. This concludes our proof. $\square$

In words, Lemma 5 shows that, after right-composing by a suitable element of G^θ , the relative element between ℓ_0 and ℓ_1 can always be moved into $\exp(\mathfrak{p})$. Thus, the fixed-point factor in the Cartan decomposition (Definition 6) can be absorbed, leaving only the $\exp(\mathfrak{p})$ -factor.

Theorem 4.2 (Cartan criterion). *Let $\mu_i = \alpha(g_i)_{\#} \rho \in \mathcal{P}_2(\mathcal{X})$, $i = 0, 1$ and let $\theta : G \rightarrow G$ be a Lie group involution. Suppose that:*

- (i) (G, θ) admits a global Cartan decomposition in the sense of Definition 6,
- (ii) $G^\theta \subseteq \text{Stab}_G(\rho)$, and
- (iii) $D_{\mathbf{x}} \alpha(\exp(S)) = D_{\mathbf{x}} \alpha(\exp(S))^*$ for all $S \in \mathfrak{p}$ and $\mathbf{x} \in \mathcal{X}$.

Then, there exists $h \in G^\theta$, such that:

- (a) $T_h = \alpha(g_1 h g_0^{-1})$ solves (MP), and is the unique optimal Monge map up to μ_0 -almost everywhere equality,
- (b) $(\text{id}_{\mathcal{X}}, T_h)_{\#} \mu_0$ solves (KTP), and is the unique optimal Kantorovich plan,
- (c) h solves (LGOP),
- (d) $\mathbb{M}_c(\mu_0, \mu_1) = \mathbb{K}_c(\mu_0, \mu_1) = \mathbb{J}_c(g_0, g_1)$.

Proof. By Lemma 5, there exists $h \in G^\theta$ such that $g_1 h g_0^{-1} \in \exp(\mathfrak{p})$. Hence there exists $S \in \mathfrak{p}$ satisfying $g_1 h g_0^{-1} = \exp(S)$. Moreover, assumption (ii) implies that $h \in \text{Stab}_G(\rho)$.

Set $T_h = \alpha(g_1 h g_0^{-1}) = \alpha(\exp(S))$. Since $h \in \text{Stab}_G(\rho)$,

$$(T_h)_{\#} \mu_0 = \alpha(g_1 h g_0^{-1})_{\#} (\alpha(g_0)_{\#} \rho) = \alpha(g_1)_{\#} (\alpha(h)_{\#} \rho) = \alpha(g_1)_{\#} \rho = \mu_1.$$

Hence $T_h \in \mathcal{T}(\mu_0, \mu_1)$.

For $t \in [0, 1]$, define $F_t = \alpha(\exp(tS))$ and $J_t(\mathbf{x}) = D_{\mathbf{x}} F_t$. Since $\mathfrak{p}$ is a vector space, $tS \in \mathfrak{p}$ for every $t \in [0, 1]$. Assumption (iii) therefore yields $J_t(\mathbf{x}) = J_t(\mathbf{x})^*$ for every $t \in [0, 1]$ and $\mathbf{x} \in \mathcal{X}$. For each t , the map F_t is a diffeomorphism of $\mathcal{X}$, with inverse $F_{-t} = \alpha(\exp(-tS))$. Differentiating $F_{-t} \circ F_t = \text{id}_{\mathcal{X}}$ at $\mathbf{x}$ gives $D_{F_t(\mathbf{x})} F_{-t} \circ J_t(\mathbf{x}) = \text{id}_{\mathcal{H}}$. Hence $J_t(\mathbf{x})$ is invertible for every $t \in [0, 1]$ and $\mathbf{x} \in \mathcal{X}$. Since the map $(t, \mathbf{x}) \mapsto F_t(\mathbf{x}) = \alpha(\exp(tS))(\mathbf{x})$ is smooth, the map $t \mapsto J_t(\mathbf{x})$ is continuous for every fixed $\mathbf{x} \in \mathcal{X}$. Moreover, $J_0(\mathbf{x}) = \text{id}_{\mathcal{H}}$. Thus $\{J_t(\mathbf{x})\}_{t \in [0, 1]}$ is a continuous path of invertible self-adjoint operators beginning at the identity. Since $J_t(\mathbf{x})$ is self-adjoint, all of its eigenvalues are real. Moreover, $J_t(\mathbf{x})$ is invertible for every $t \in [0, 1]$, so none of its eigenvalues can be zero. As $t \mapsto J_t(\mathbf{x})$ is continuous, its eigenvalues can be chosen continuously as functions of t . Since $J_0(\mathbf{x}) = \text{id}_{\mathcal{H}}$ has only positive eigenvalues, no eigenvalue can become negative without first passing through zero. Therefore, $J_t(\mathbf{x}) \succ 0$ for every $t \in [0, 1]$.

Fix $\mathbf{x}_o \in \mathcal{X}$ and define $\Phi(\mathbf{x}) = \int_0^1 \langle T_h(\mathbf{x}_o + s(\mathbf{x} - \mathbf{x}_o)), \mathbf{x} - \mathbf{x}_o \rangle_{\mathcal{H}} ds$ for $\mathbf{x} \in \mathcal{X}$. This is well defined because $\mathcal{X}$ is convex. Fix $\mathbf{x} \in \mathcal{X}$ and $\mathbf{v} \in \mathcal{H}$, and write $\mathbf{z}_s = \mathbf{x}_o + s(\mathbf{x} - \mathbf{x}_o)$. Differentiating under the integral sign gives $D\Phi(\mathbf{x})[\mathbf{v}] = \int_0^1 \langle D_{\mathbf{z}_s} T_h[s\mathbf{v}], \mathbf{x} - \mathbf{x}_o \rangle_{\mathcal{H}} ds + \int_0^1 \langle T_h(\mathbf{z}_s), \mathbf{v} \rangle_{\mathcal{H}} ds$. Since $D_{\mathbf{z}_s} T_h$ is self-adjoint, the first integrand may be rewritten as $\langle sD_{\mathbf{z}_s} T_h[\mathbf{x} - \mathbf{x}_o], \mathbf{v} \rangle_{\mathcal{H}}$. Consequently, $D\Phi(\mathbf{x})[\mathbf{v}] = \int_0^1 \langle sD_{\mathbf{z}_s} T_h[\mathbf{x} - \mathbf{x}_o] + T_h(\mathbf{z}_s), \mathbf{v} \rangle_{\mathcal{H}} ds$. Since $\frac{d}{ds}(sT_h(\mathbf{z}_s)) = T_h(\mathbf{z}_s) + sD_{\mathbf{z}_s} T_h[\mathbf{x} - \mathbf{x}_o]$, the fundamental theorem of calculus yields $D\Phi(\mathbf{x})[\mathbf{v}] = \langle sT_h(\mathbf{z}_s), \mathbf{v} \rangle_{\mathcal{H}}|_{s=0}^{s=1} = \langle T_h(\mathbf{x}), \mathbf{v} \rangle_{\mathcal{H}}$. Therefore $\nabla_{\mathbf{x}} \Phi(\mathbf{x}) = T_h(\mathbf{x})$ for every $\mathbf{x} \in \mathcal{X}$. Since $T_h = F_1$, we have $\nabla_{\mathbf{x}}^2 \Phi(\mathbf{x}) = D_{\mathbf{x}} T_h = J_1(\mathbf{x})$. By the previous step, $J_t(\mathbf{x}) \succ 0$ for every $t \in [0, 1]$ and thus $J_1(\mathbf{x}) \succ 0$ for every $\mathbf{x} \in \mathcal{X}$. Hence Φ is strictly convex on $\mathcal{X}$.

To obtain a convex function on the ambient Hilbert space, define $\bar{\Phi} : \mathcal{H} \rightarrow (-\infty, +\infty]$ by $\bar{\Phi}(\mathbf{y}) = \sup_{\mathbf{x} \in \mathcal{X}} \{\Phi(\mathbf{x}) + \langle T_h(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle_{\mathcal{H}}\}$. As the supremum of continuous affine functions, $\bar{\Phi}$ is lower semicontinuous and convex. For every $\mathbf{y} \in \mathcal{X}$, convexity and differentiability of Φ give $\Phi(\mathbf{y}) \geq \Phi(\mathbf{x}) + \langle T_h(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle_{\mathcal{H}}$ for every $\mathbf{x} \in \mathcal{X}$. Hence $\bar{\Phi}(\mathbf{y}) \leq \Phi(\mathbf{y})$. Taking $\mathbf{x} = \mathbf{y}$ in the definition of $\bar{\Phi}$ gives the reverse inequality, and therefore $\bar{\Phi}(\mathbf{y}) = \Phi(\mathbf{y})$ for every $\mathbf{y} \in \mathcal{X}$. Thus $\bar{\Phi}$ is proper and agrees with Φ on the open set $\mathcal{X}$. In particular, $\bar{\Phi}$ is differentiable on $\mathcal{X}$ and $\nabla_{\mathbf{x}} \bar{\Phi}(\mathbf{x}) = T_h(\mathbf{x})$ for every $\mathbf{x} \in \mathcal{X}$.

We now regard μ_0 and μ_1 as probability measures on $\mathcal{H}$ supported on $\mathcal{X}$. More precisely, if $\iota : \mathcal{X} \hookrightarrow \mathcal{H}$ denotes the inclusion map, we identify each $\mu_i \in \mathcal{P}(\mathcal{X})$ with $\iota_{\#} \mu_i \in \mathcal{P}(\mathcal{H})$. Define the ambient quadratic cost $\bar{c} : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ by $\bar{c}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_{\mathcal{H}}^2$.

Define $\varphi : \mathcal{H} \rightarrow \bar{\mathbb{R}}$ by $\varphi(\mathbf{x}) = 2\bar{\Phi}(\mathbf{x}) - \|\mathbf{x}\|_{\mathcal{H}}^2$. Let $\bar{\Phi}^\dagger$ denote the Fenchel conjugate of $\bar{\Phi}$ on $\mathcal{H}$, and define $\chi : \mathcal{H} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ by $\chi(\mathbf{y}) = \|\mathbf{y}\|_{\mathcal{H}}^2 - 2\bar{\Phi}^\dagger(\mathbf{y})$.

We first show that φ is $\bar{c}$ -convex on $\mathcal{H}$. For every $\mathbf{x} \in \mathcal{H}$, the definitions of χ and $\bar{c}$ give $\sup_{\mathbf{y} \in \mathcal{H}} \{\chi(\mathbf{y}) - \bar{c}(\mathbf{x}, \mathbf{y})\} = -\|\mathbf{x}\|_{\mathcal{H}}^2 + 2 \sup_{\mathbf{y} \in \mathcal{H}} \{\langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{H}} - \bar{\Phi}^\dagger(\mathbf{y})\} = -\|\mathbf{x}\|_{\mathcal{H}}^2 + 2\bar{\Phi}^{\dagger\dagger}(\mathbf{x})$. Since $\bar{\Phi}$ is proper, lower semicontinuous, and convex, the Fenchel-Moreau theorem gives $\bar{\Phi}^{\dagger\dagger} = \bar{\Phi}$. Consequently, $\sup_{\mathbf{y} \in \mathcal{H}} \{\chi(\mathbf{y}) - \bar{c}(\mathbf{x}, \mathbf{y})\} = 2\bar{\Phi}(\mathbf{x}) - \|\mathbf{x}\|_{\mathcal{H}}^2 = \varphi(\mathbf{x})$. Moreover, φ is not identically $+\infty$, because $\bar{\Phi}$ is finite on the nonempty set $\mathcal{X}$. Thus, φ is $\bar{c}$ -convex on $\mathcal{H}$.

For completeness, the maximizing point in the preceding representation can be identified on $\mathcal{X}$. Indeed, for every $\mathbf{x} \in \mathcal{X}$, the identity $\nabla \bar{\Phi}(\mathbf{x}) = T_h(\mathbf{x})$ implies that $T_h(\mathbf{x}) \in \partial \bar{\Phi}(\mathbf{x})$. Therefore, the equality condition in the Fenchel-Young inequality gives $\bar{\Phi}(\mathbf{x}) + \bar{\Phi}^\dagger(T_h(\mathbf{x})) = \langle \mathbf{x}, T_h(\mathbf{x}) \rangle_{\mathcal{H}}$. It follows that $\chi(T_h(\mathbf{x})) - \bar{c}(\mathbf{x}, T_h(\mathbf{x})) = \varphi(\mathbf{x})$. Thus, for every $\mathbf{x} \in \mathcal{X}$, the supremum in the $\bar{c}$ -convex representation of $\varphi(\mathbf{x})$ is attained at $\mathbf{y} = T_h(\mathbf{x})$.

Since $\nabla \bar{\Phi} = T_h$ on $\mathcal{X}$, we have $\nabla_{\mathbf{x}} \varphi(\mathbf{x}) = 2T_h(\mathbf{x}) - 2\mathbf{x}$ for every $\mathbf{x} \in \mathcal{X}$. Moreover, $\nabla_{\mathbf{x}} \bar{c}(\mathbf{x}, T_h(\mathbf{x})) = 2(\mathbf{x} - T_h(\mathbf{x}))$. Therefore, $\nabla_{\mathbf{x}} \varphi(\mathbf{x}) + \nabla_{\mathbf{x}} \bar{c}(\mathbf{x}, T_h(\mathbf{x})) = 0$ for every $\mathbf{x} \in \mathcal{X}$. Thus, with the ambient quadratic cost $\bar{c}$, the map T_h satisfies the optimality condition $(\heartsuit)$ at every $\mathbf{x} \in \mathcal{X}$, and hence μ_0 -almost surely.

We next verify the hypotheses of Theorem 2.1 on the ambient state space $\mathcal{H}$. Since $\rho \ll \mathcal{L}_{\mathcal{H}}|_{\mathcal{X}}$ and $\alpha(g_0)$ is a diffeomorphism of $\mathcal{X}$, Lemma 1 gives $\mu_0 \ll \mathcal{L}_{\mathcal{H}}|_{\mathcal{X}}$. Under the preceding ambient identification, this implies $\mu_0 \ll \mathcal{L}_{\mathcal{H}}$. Indeed, if $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ satisfies $\mathcal{L}_{\mathcal{H}}(\mathcal{A}) = 0$, then $\mu_0(\mathcal{A}) = \mu_0(\mathcal{A} \cap \mathcal{X}) = 0$.

Thus, Lemma 4(iii) implies that $(\bar{c}, \mu_0)$ satisfies Assumption 2, while Lemma 4(iv) implies that $\bar{c}$ satisfies Assumption 3 on $\mathcal{H}$. The ambient quadratic cost also satisfies Assumption 1. Moreover, since $\mu_0, \mu_1 \in \mathcal{P}_2(\mathcal{X})$, their ambient extensions belong to $\mathcal{P}_2(\mathcal{H})$. The independent coupling therefore has finite quadratic cost, because $\bar{c}(\mathbf{x}, \mathbf{y}) \leq 2\|\mathbf{x}\|_{\mathcal{H}}^2 + 2\|\mathbf{y}\|_{\mathcal{H}}^2$. Hence, the ambient Kantorovich value is finite.

Set $\Gamma_h = (\text{id}_{\mathcal{X}}, T_h)_{\#} \mu_0$ and regard Γ_h as a probability measure on $\mathcal{H} \times \mathcal{H}$. Since $(T_h)_{\#} \mu_0 = \mu_1$, we have $\Gamma_h \in \Pi(\mu_0, \mu_1)$. Furthermore, Γ_h is concentrated on pairs $(\mathbf{x}, \mathbf{y})$ satisfying $\nabla \varphi(\mathbf{x}) + \nabla_{\mathbf{x}} \bar{c}(\mathbf{x}, \mathbf{y}) = 0$. Therefore, Theorem 2.1(v), applied with state space $\mathcal{H}$, cost $\bar{c}$, and potential φ , implies that Γ_h is the unique optimal Kantorovich plan for the ambient problem.

Because both marginals are supported on $\mathcal{X}$, every coupling of μ_0 and μ_1 on $\mathcal{H}$ is concentrated on $\mathcal{X} \times \mathcal{X}$. Since $\bar{c}$ agrees with c on $\mathcal{X} \times \mathcal{X}$, the ambient and original Kantorovich problems have the same feasible couplings and the same optimal value. Consequently, $(\text{id}_{\mathcal{X}}, T_h)_{\#} \mu_0$ is the unique optimal Kantorovich plan for the original problem on $\mathcal{X}$.

Since $T_h \in \mathcal{T}(\mu_0, \mu_1)$, we have

$$\mathbb{K}_c(\mu_0, \mu_1) \leq \mathbb{M}_c(\mu_0, \mu_1) \leq \int_{\mathcal{X}} c(\mathbf{x}, T_h(\mathbf{x})) d\mu_0(\mathbf{x}) = \mathbb{K}_c(\mu_0, \mu_1).$$

Hence T_h solves (MP). If $\tilde{T} \in \mathcal{T}(\mu_0, \mu_1)$ is any other optimal Monge map, then $(\text{id}_{\mathcal{X}}, \tilde{T})_{\#} \mu_0$ is an optimal Kantorovich plan. By uniqueness of Γ_h , we have $(\text{id}_{\mathcal{X}}, \tilde{T})_{\#} \mu_0 = (\text{id}_{\mathcal{X}}, T_h)_{\#} \mu_0$, which implies $\tilde{T} = T_h$ μ_0 -almost surely. This proves assertions (a) and (b).

By Theorem 4.1(i), we have $\mathbb{M}_c(\mu_0, \mu_1) \leq \mathbb{J}_c(g_0, g_1)$. On the other hand, the element h constructed above belongs to $\text{Stab}_G(\rho)$ and is therefore feasible for (LGOP). Using $\mu_0 = \alpha(g_0)_{\#} \rho$, the left-action property, and the definition of T_h , we obtain

$$\mathbb{J}_c(g_0, g_1) \leq \int_{\mathcal{X}} c(\alpha(g_0)(z), \alpha(g_1 h)(z)) d\rho(z) = \int_{\mathcal{X}} c(\mathbf{x}, T_h(\mathbf{x})) d\mu_0(\mathbf{x}) = \mathbb{M}_c(\mu_0, \mu_1).$$

Combining the two inequalities gives $\mathbb{J}_c(g_0, g_1) = \mathbb{M}_c(\mu_0, \mu_1)$, and the constructed element h attains the infimum in (LGOP). This proves assertion (c).

Finally, the preceding argument already gives $\mathbb{K}_c(\mu_0, \mu_1) = \mathbb{M}_c(\mu_0, \mu_1)$. Together with $\mathbb{J}_c(g_0, g_1) = \mathbb{M}_c(\mu_0, \mu_1)$, this yields $\mathbb{K}_c(\mu_0, \mu_1) = \mathbb{M}_c(\mu_0, \mu_1) = \mathbb{J}_c(g_0, g_1)$, which proves assertion (d). $\square$

The central insight of Theorem 4.2 is that the symmetry-induced nonuniqueness of orbit representations is a resource for identifying the optimal transport map. Under assumption (ii), replacing g_1 by $g_1 h$ with $h \in G^\theta$ does not change the represented measure, since $\alpha(g_1 h)_\# \rho = \alpha(g_1)_\# \rho$. The Cartan absorption mechanism uses this freedom to choose h so that the relative transformation satisfies $g_1 h g_0^{-1} = \exp(S)$, $S \in \mathfrak{p}$. Thus, the component that can be absorbed without changing the target measure is removed, leaving a stretch-like transformation canonically connected to the identity through $F_t = \alpha(\exp(tS))$, $t \in [0, 1]$. Self-adjointness of $D_{\mathbf{x}} F_t$ plays two complementary roles. It rules out an infinitesimal rotational component, allowing the terminal map to be integrated to a scalar potential on the convex domain $\mathcal{X}$, and it ensures that the spectrum of $D_{\mathbf{x}} F_t$ remains real along the path. Since every F_t is a diffeomorphism, $D_{\mathbf{x}} F_t$ remains invertible; because the path begins at $D_{\mathbf{x}} F_0 = \text{id}_{\mathcal{H}}$, none of its eigenvalues can change from positive to negative without passing through zero. Hence the terminal map is the gradient of a strictly convex potential on $\mathcal{X}$. Convexity is therefore not imposed or verified separately for each candidate map, but inherited from the identity through the geometry of the action. The theorem consequently replaces the pair-by-pair search for a convex transport potential by structural conditions on the action and the symmetries of the reference measure. Once these conditions are verified at the level of $(G, \theta, \alpha, \rho)$, they certify, for every pair of measures in the resulting orbit family with finite second moments, the unique optimal Monge map, the unique optimal Kantorovich plan, and the equality $\mathbb{M}_c(\mu_0, \mu_1) = \mathbb{K}_c(\mu_0, \mu_1) = \mathbb{J}_c(g_0, g_1)$.

In what follows, we consider actions induced by affine representations on the ambient Hilbert space. In this setting, the full acting group is not required to admit a Cartan decomposition. Instead, it suffices to impose a Cartan structure on the linear image $L = \pi(G)$. The Cartan decomposition then selects an orbit-induced affine map with a positive-definite linear part. This map is determined explicitly by the means and covariance operators of the source and target measures, and the resulting optimal transport value attains the lower bound on $\mathbb{K}_c(\mu_0, \mu_1)$ provided in [Gelbrich, 1990, Theorem 2.1].

Assumption 4 (Affine-induced action). *There exist a smooth group homomorphism $\pi : G \rightarrow \text{GL}(\mathcal{H})$ and a smooth map $b : G \rightarrow \mathcal{H}$ satisfying $b(gh) = b(g) + \pi(g)b(h)$ for every $g, h \in G$, such that $\alpha(g)(\mathbf{x}) = b(g) + \pi(g)\mathbf{x}$ for every $g \in G$ and $\mathbf{x} \in \mathcal{X}$.*

Proposition 1 (Affine Cartan criterion). *Let $\mathcal{X} \subseteq \mathcal{H}$ be nonempty, open, and convex, and let G act smoothly on $\mathcal{X}$ through α . Suppose that $\rho \in \mathcal{P}(\mathcal{X})$ is absolutely continuous with respect to $\mathcal{L}_{\mathcal{H}}|_{\mathcal{X}}$ and satisfies $\int_{\mathcal{X}} \|\mathbf{x}\|_{\mathcal{H}}^2 d\rho(\mathbf{x}) < \infty$. Fix $g_0, g_1 \in G$, set $\mu_i = \alpha(g_i)_\# \rho$ for $i = 0, 1$, and let $c(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_{\mathcal{H}}^2$.*

Suppose that Assumption 4 holds, and set $L = \pi(G) \subseteq \text{GL}(\mathcal{H})$. Assume that L is a Lie subgroup of $\text{GL}(\mathcal{H})$, and let $\theta : L \rightarrow L$ be a Lie group involution. Write $\text{Lie}(L) = \mathfrak{k} \oplus \mathfrak{p}$ for the corresponding eigenspace decomposition. Suppose further that:

- (i) (L, θ) admits a global Cartan decomposition in the sense of Definition 6;
- (ii) $L^\theta \subseteq \pi(\text{Stab}_G(\rho))$;
- (iii) every $S \in \mathfrak{p}$ is self-adjoint on $\mathcal{H}$.

Let $\mathbf{m}_i \in \mathcal{H}$ and $\mathbf{C}_i : \mathcal{H} \rightarrow \mathcal{H}$ denote the mean and covariance operator of μ_i , respectively, for $i = 0, 1$. Then $\mathbf{C}_0$ and $\mathbf{C}_1$ are self-adjoint and positive definite. Moreover, there exists $h \in \text{Stab}_G(\rho)$ satisfying $\pi(h) \in L^\theta$ such that, upon defining $T_h := \alpha(g_1 h g_0^{-1})$ and $\mathbf{A}_h = \pi(g_1 h g_0^{-1}) \in \exp(\mathfrak{p}) \subseteq$

$L \subseteq \text{GL}(\mathcal{H})$, the operator A_h is self-adjoint and positive definite and

$$(2) \quad T_h(\mathbf{x}) = \mathbf{m}_1 + A_h(\mathbf{x} - \mathbf{m}_0), \quad \mathbf{x} \in \mathcal{X}, \quad A_h = C_0^{-1/2} \left(C_0^{1/2} C_1 C_0^{1/2} \right)^{1/2} C_0^{-1/2}.$$

Here the inverses and square roots in (2) are taken in the cone of self-adjoint positive-definite operators on $\mathcal{H}$.

For this choice of h , the following conclusions hold: (a) The map T_h solves (MP) and is the unique optimal Monge map from μ_0 to μ_1 , up to μ_0 -almost-everywhere equality. (b) The coupling $(\text{id}_{\mathcal{X}}, T_h)_{\#} \mu_0$ solves (KTP) and is the unique optimal Kantorovich plan. (c) The element h solves the Lie group orbit transport problem (LGOP). (d) $\mathbb{K}_c(\mu_0, \mu_1) = \mathbb{M}_c(\mu_0, \mu_1) = \mathbb{J}_c(g_0, g_1)$ and

$$(3) \quad \mathbb{K}_c(\mu_0, \mu_1) = \|\mathbf{m}_0 - \mathbf{m}_1\|_{\mathcal{H}}^2 + \text{Tr}_{\mathcal{H}} \left(C_0 + C_1 - 2 \left(C_0^{1/2} C_1 C_0^{1/2} \right)^{1/2} \right).$$

Proof. Set $\ell_i = \pi(g_i) \in L$ for $i = 0, 1$. By Lemma 5, there exists $k \in L^\theta$ such that $\ell_1 k \ell_0^{-1} \in \exp(\mathfrak{p})$. Assumption (ii) therefore yields $h \in \text{Stab}_G(\rho)$ satisfying $\pi(h) = k$. In particular, $\pi(h) \in L^\theta$.

Define $T_h := \alpha(g_1 h g_0^{-1})$ and $A_h := \pi(g_1 h g_0^{-1})$. Since π is a group homomorphism, $A_h = \ell_1 k \ell_0^{-1} \in \exp(\mathfrak{p})$. Thus there exists $S \in \mathfrak{p}$ such that $A_h = \exp(S)$. By assumption (iii), S is self-adjoint on $\mathcal{H}$. The spectral theorem then implies that A_h is self-adjoint and positive definite.

Set $\mathbf{a}_h = b(g_1 h g_0^{-1})$. By Assumption 4, we have $T_h(\mathbf{x}) = \mathbf{a}_h + A_h \mathbf{x}$ for every $\mathbf{x} \in \mathcal{X}$. Moreover, since $h \in \text{Stab}_G(\rho)$ and α is a left action, $(T_h)_{\#} \mu_0 = \alpha(g_1 h g_0^{-1})_{\#} (\alpha(g_0)_{\#} \rho) = \alpha(g_1)_{\#} (\alpha(h)_{\#} \rho) = \alpha(g_1)_{\#} \rho = \mu_1$. Hence $T_h \in \mathcal{T}(\mu_0, \mu_1)$.

Assumption 4 defines an affine action $\hat{\alpha}$ of G on $\mathcal{H}$ by $\hat{\alpha}(g)(\mathbf{x}) = b(g) + \pi(g)\mathbf{x}$. Indeed, the homomorphism property of π and the cocycle identity for b imply $\hat{\alpha}(gh) = \hat{\alpha}(g) \circ \hat{\alpha}(h)$, and $\hat{\alpha}(g^{-1})$ is the inverse of $\hat{\alpha}(g)$. Moreover, $\hat{\alpha}(g)|_{\mathcal{X}} = \alpha(g)$ for every $g \in G$. We regard ρ , μ_0 , and μ_1 as probability measures on $\mathcal{H}$ supported on $\mathcal{X}$.

Since $\rho \ll \mathcal{L}_{\mathcal{H}}|_{\mathcal{X}}$, its density extends by zero to a density on $\mathcal{H}$. Each $\hat{\alpha}(g_i)$ is an affine diffeomorphism of $\mathcal{H}$, and hence $\mu_i \ll \mathcal{L}_{\mathcal{H}}$ for $i = 0, 1$. Furthermore, the finite second moment of ρ and the bound $\|b(g_i) + \pi(g_i)\mathbf{x}\|_{\mathcal{H}}^2 \leq 2\|b(g_i)\|_{\mathcal{H}}^2 + 2\|\pi(g_i)\|_{\text{op}}^2 \|\mathbf{x}\|_{\mathcal{H}}^2$ imply that $\mu_i \in \mathcal{P}_2(\mathcal{H})$.

We next show that the covariance operators are positive definite. Fix $i \in \{0, 1\}$ and suppose that $\langle \mathbf{u}, C_i \mathbf{u} \rangle_{\mathcal{H}} = 0$ for some $\mathbf{u} \neq \mathbf{0}$. Then $\langle \mathbf{u}, \mathbf{X} - \mathbf{m}_i \rangle_{\mathcal{H}} = 0$ holds μ_i -almost surely, so μ_i is concentrated on the affine hyperplane $\{\mathbf{x} \in \mathcal{H} : \langle \mathbf{u}, \mathbf{x} - \mathbf{m}_i \rangle_{\mathcal{H}} = 0\}$. This hyperplane has $\mathcal{L}_{\mathcal{H}}$ -measure zero, contradicting $\mu_i \ll \mathcal{L}_{\mathcal{H}}$. Thus $C_0, C_1 \succ 0$.

Let $\hat{T}_h : \mathcal{H} \rightarrow \mathcal{H}$ be the affine extension $\hat{T}_h(\mathbf{x}) = \mathbf{a}_h + A_h \mathbf{x}$. Then $\hat{T}_h = \hat{\alpha}(g_1 h g_0^{-1})$ and $\hat{T}_h|_{\mathcal{X}} = T_h$. Define $\Phi_h : \mathcal{H} \rightarrow \mathbb{R}$ by $\Phi_h(\mathbf{x}) = \langle \mathbf{a}_h, \mathbf{x} \rangle_{\mathcal{H}} + \frac{1}{2} \langle \mathbf{x}, A_h \mathbf{x} \rangle_{\mathcal{H}}$. Since $A_h = A_h^* \succ 0$, the function Φ_h is strictly convex and satisfies $\nabla \Phi_h = \hat{T}_h$.

Let $\hat{c}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_{\mathcal{H}}^2$ on $\mathcal{H} \times \mathcal{H}$, define $\varphi_h(\mathbf{x}) = 2\Phi_h(\mathbf{x}) - \|\mathbf{x}\|_{\mathcal{H}}^2$, and let $\Phi_h^\dagger$ denote the Fenchel conjugate of Φ_h . If $\chi_h(\mathbf{y}) = \|\mathbf{y}\|_{\mathcal{H}}^2 - 2\Phi_h^\dagger(\mathbf{y})$, then the Fenchel-Moreau theorem gives $\varphi_h(\mathbf{x}) = \sup_{\mathbf{y} \in \mathcal{H}} \{\chi_h(\mathbf{y}) - \hat{c}(\mathbf{x}, \mathbf{y})\}$ for every $\mathbf{x} \in \mathcal{H}$. Hence φ_h is $\hat{c}$ -convex.

Moreover, $\nabla \varphi_h(\mathbf{x}) = 2\hat{T}_h(\mathbf{x}) - 2\mathbf{x}$ and $\nabla_{\mathbf{x}} \hat{c}(\mathbf{x}, \hat{T}_h(\mathbf{x})) = 2(\mathbf{x} - \hat{T}_h(\mathbf{x}))$. Therefore, $\nabla \varphi_h(\mathbf{x}) + \nabla_{\mathbf{x}} \hat{c}(\mathbf{x}, \hat{T}_h(\mathbf{x})) = 0$ for every $\mathbf{x} \in \mathcal{H}$.

Because $\hat{\alpha}(g)|_{\mathcal{X}} = \alpha(g)$ and ρ is supported on $\mathcal{X}$, the stabilizer of ρ is the same for the two actions. In particular, the element h selected above is feasible for the ambient Lie group orbit problem. We now apply Theorem 4.1(ii) with state space $\mathcal{H}$, action $\hat{\alpha}$, and cost $\hat{c}$.

The quadratic cost satisfies Assumption 1; by Lemma 4(iii), the absolute continuity of μ_0 implies Assumption 2; and by Lemma 4(iv), $\hat{c}$ satisfies Assumption 3 on $\mathcal{H}$. Finally, since $\mu_0, \mu_1 \in \mathcal{P}_2(\mathcal{H})$, the independent coupling has finite quadratic cost, and hence the ambient Kantorovich value

is finite. Thus Theorem 4.1(ii) implies that $\hat{T}_h$ is the unique optimal Monge map up to μ_0 -almost-everywhere equality, its graph coupling is the unique optimal Kantorovich plan, h solves the ambient Lie group orbit transport problem, and the three ambient optimal values coincide.

These ambient problems agree with the corresponding problems on $\mathcal{X}$. Indeed, ρ , μ_0 , and μ_1 are supported on $\mathcal{X}$, and $\hat{\alpha}(g)|_{\mathcal{X}} = \alpha(g)$ for every $g \in G$. Hence the ambient and original stabilizers and Lie group orbit objectives coincide. Every coupling of μ_0 and μ_1 on $\mathcal{H}$ is supported on $\mathcal{X} \times \mathcal{X}$, so the ambient and original Kantorovich problems coincide.

Finally, every Borel map from $\mathcal{X}$ to $\mathcal{X}$ can be extended to a Borel map on $\mathcal{H}$, whereas an ambient transport map can be modified on a μ_0 -null set and restricted to $\mathcal{X}$ to obtain a transport map from $\mathcal{X}$ to $\mathcal{X}$ with the same cost. Thus the ambient and original Monge problems also coincide. Since $\hat{T}_h|_{\mathcal{X}} = T_h$, conclusions (a)-(c) and the equality of the three values in (d) follow.

Let $\mathbf{X} \sim \mu_0$ and set $\mathbf{Y} = T_h(\mathbf{X}) \sim \mu_1$. Taking expectations in $\mathbf{Y} = \mathbf{a}_h + \mathbf{A}_h \mathbf{X}$ gives $\mathbf{m}_1 = \mathbf{a}_h + \mathbf{A}_h \mathbf{m}_0$. Therefore, $\mathbf{a}_h = \mathbf{m}_1 - \mathbf{A}_h \mathbf{m}_0$, and hence $T_h(\mathbf{x}) = \mathbf{m}_1 + \mathbf{A}_h(\mathbf{x} - \mathbf{m}_0)$ for every $\mathbf{x} \in \mathcal{X}$.

Since $\mathbf{Y} - \mathbf{m}_1 = \mathbf{A}_h(\mathbf{X} - \mathbf{m}_0)$, the covariance operators satisfy $\mathbf{C}_1 = \mathbf{A}_h \mathbf{C}_0 \mathbf{A}_h^* = \mathbf{A}_h \mathbf{C}_0 \mathbf{A}_h$. To see this directly, for $\mathbf{u}, \mathbf{v} \in \mathcal{H}$ the covariance bilinear form gives $\langle \mathbf{u}, \mathbf{C}_1 \mathbf{v} \rangle_{\mathcal{H}} = \langle \mathbf{A}_h^* \mathbf{u}, \mathbf{C}_0 \mathbf{A}_h^* \mathbf{v} \rangle_{\mathcal{H}} = \langle \mathbf{u}, \mathbf{A}_h \mathbf{C}_0 \mathbf{A}_h^* \mathbf{v} \rangle_{\mathcal{H}}$.

Set $\mathbf{R}_h = \mathbf{C}_0^{1/2} \mathbf{A}_h \mathbf{C}_0^{1/2}$. Since $\mathbf{C}_0 \succ 0$ and $\mathbf{A}_h \succ 0$, the operator $\mathbf{R}_h$ is self-adjoint and positive definite. Moreover, $\mathbf{R}_h^2 = \mathbf{C}_0^{1/2} \mathbf{A}_h \mathbf{C}_0 \mathbf{A}_h \mathbf{C}_0^{1/2} = \mathbf{C}_0^{1/2} \mathbf{C}_1 \mathbf{C}_0^{1/2}$. By uniqueness of the positive-definite square root, $\mathbf{R}_h = (\mathbf{C}_0^{1/2} \mathbf{C}_1 \mathbf{C}_0^{1/2})^{1/2}$. Multiplying on the left and right by $\mathbf{C}_0^{-1/2}$ proves (2).

Let $\mathbf{Z} = \mathbf{X} - \mathbf{m}_0$. Then $\mathbb{E}[\mathbf{Z}] = \mathbf{0}$ and $\mathbf{X} - T_h(\mathbf{X}) = \mathbf{m}_0 - \mathbf{m}_1 + (\text{id}_{\mathcal{H}} - \mathbf{A}_h) \mathbf{Z}$. The cross term vanishes after taking expectations.

Using the identity $\mathbb{E}[\|\mathbf{BZ}\|_{\mathcal{H}}^2] = \text{Tr}_{\mathcal{H}}(\mathbf{B} \mathbf{C}_0 \mathbf{B}^*)$ for a linear operator $\mathbf{B}$ on $\mathcal{H}$, together with $\mathbf{A}_h = \mathbf{A}_h^*$ and $\mathbf{A}_h \mathbf{C}_0 \mathbf{A}_h = \mathbf{C}_1$, we obtain $\mathbb{E}[\|\mathbf{X} - T_h(\mathbf{X})\|_{\mathcal{H}}^2] = \|\mathbf{m}_0 - \mathbf{m}_1\|_{\mathcal{H}}^2 + \text{Tr}_{\mathcal{H}}(\mathbf{C}_0 + \mathbf{C}_1 - 2\mathbf{A}_h \mathbf{C}_0)$.

Finally, cyclicity of the operator trace and the identity for $\mathbf{R}_h$ give

$$\text{Tr}_{\mathcal{H}}(\mathbf{A}_h \mathbf{C}_0) = \text{Tr}_{\mathcal{H}}(\mathbf{C}_0^{1/2} \mathbf{A}_h \mathbf{C}_0^{1/2}) = \text{Tr}_{\mathcal{H}}((\mathbf{C}_0^{1/2} \mathbf{C}_1 \mathbf{C}_0^{1/2})^{1/2}).$$

Substitution proves (3) and completes the proof. $\square$

The expression on the right-hand side of (3) is the lower bound on the quadratic Kantorovich value established in [Gelbrich, 1990, Theorem 2.1]. Proposition 1 provides a sufficient structural criterion for this lower bound to be attained. Indeed, the global Cartan decomposition of $L = \pi(G)$, together with the inclusion $L^\theta \subseteq \pi(\text{Stab}_G(\rho))$, produces $h \in \text{Stab}_G(\rho)$ such that $\pi(g_1 h g_0^{-1}) \in \exp(\mathfrak{p})$; the self-adjointness of $\mathfrak{p}$ then implies that this operator is positive definite. The resulting orbit-induced affine map transports μ_0 to μ_1 and attains the lower bound.

5. EXAMPLES

This section applies the Cartan criteria in Theorem 4.2 and proposition 1 to derive closed-form quadratic optimal transport maps and values for several distribution families. We begin with elliptical distributions, for which the affine Cartan criterion recovers the classical optimal transport formula for laws sharing a common characteristic generator, equivalently the corresponding equality case of Gelbrich's covariance lower bound [Gelbrich, 1990, Theorem 2.1]. We then turn to spectral-congruence orbits of matrix-valued distributions and conclude with the sinh-arcsinh family generated by a nonlinear group action.

5.1. Elliptical distributions. A random vector $\mathbf{x} \in \mathbb{R}^d$ has an elliptical distribution with characteristic generator $\varsigma : \mathbb{R}_{\geq} \rightarrow \mathbb{R}$ if its characteristic function evaluated at $\mathbf{t} \in \mathbb{R}^d$ is $\exp(it^\top \mathbf{m}) \varsigma(\mathbf{t}^\top \mathbf{S} \mathbf{t})$, for some location $\mathbf{m} \in \mathbb{R}^d$ and dispersion $\mathbf{S} \in \mathbb{S}_{\succ}^d$; and we write $\mathbf{x} \sim \mathcal{E}_{\varsigma}(\mathbf{m}, \mathbf{S})$. If $\varsigma'(0)$ exists

and is finite, then the location parameter is the mean, $\mathbb{E}_{\mathbf{x} \sim \mathcal{E}_\varsigma(\mathbf{m}, \mathbf{S})}[\mathbf{x}] = \mathbf{m}$ and the covariance matrix of $\mathbf{x} \sim \mathcal{E}_\varsigma(\mathbf{m}, \mathbf{S})$, satisfies $\text{Cov}(\mathbf{x}) = (-2\varsigma'(0))\mathbf{S}$. Reparameterizing the generator ς as $\tilde{\varsigma}(s) = \varsigma(s/(-2\varsigma'(0)))$ yields $\text{Cov}(\mathbf{x}) = \mathbf{S}$ with $\mathbf{x} \sim \mathcal{E}_\varsigma(\mathbf{m}, \mathbf{S})$.

The elliptical distribution class is broad and includes heavy-tailed examples such as Cauchy and multivariate stable laws. In this subsection, however, we restrict to absolutely continuous elliptical laws with finite second moment, so that the quadratic cost is finite. This finite-moment subclass includes the multivariate Gaussian, finite-variance Student- t , and multivariate Laplace families. We henceforth write ς for the normalized generator and let $\rho = \mathcal{E}_\varsigma(\mathbf{0}_d, \mathbf{I}_d)$, which is absolutely continuous and has finite second moment. Hence, ρ has mean $\mathbf{0}_d$ and covariance matrix $\mathbf{I}_d$.

Let $\mathcal{X} = \mathcal{H} = \mathbb{R}^d$, equipped with the standard inner product $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{H}} = \mathbf{x}^\top \mathbf{y}$. Let $G_{\text{aff}} = \mathbb{R}^d \rtimes \text{GL}(d)$ be the affine group with multiplication $(\mathbf{m}, \mathbf{A})(\mathbf{n}, \mathbf{B}) = (\mathbf{m} + \mathbf{A}\mathbf{n}, \mathbf{A}\mathbf{B})$. Its identity element is $(\mathbf{0}_d, \mathbf{I}_d)$, and $(\mathbf{m}, \mathbf{A})^{-1} = (-\mathbf{A}^{-1}\mathbf{m}, \mathbf{A}^{-1})$. The group G_{aff} acts smoothly on $\mathbb{R}^d$ through $\alpha((\mathbf{m}, \mathbf{A}))(\mathbf{x}) = \mathbf{m} + \mathbf{A}\mathbf{x}$.

Define $b : G_{\text{aff}} \rightarrow \mathcal{H}$ by $b((\mathbf{m}, \mathbf{A})) = \mathbf{m}$ and $\pi : G_{\text{aff}} \rightarrow \text{GL}(\mathcal{H})$ by $\pi((\mathbf{m}, \mathbf{A})) = \mathbf{A}$, and set $L = \pi(G_{\text{aff}})$.

Corollary 1 (Closed-form quadratic transport between elliptical distributions). *Let $\mu_i = \mathcal{E}_\varsigma(\mathbf{m}_i, \Sigma_i)$, where $\mathbf{m}_i \in \mathbb{R}^d$ and $\Sigma_i \in \mathbb{S}_{>}^d$ for $i = 0, 1$, and set $g_i = (\mathbf{m}_i, \Sigma_i^{1/2}) \in G_{\text{aff}}$. Define $\mathbf{A}^* = \Sigma_0^{-1/2}(\Sigma_0^{1/2}\Sigma_1\Sigma_0^{1/2})^{1/2}\Sigma_0^{-1/2}$, $\mathbf{Q}^* = \Sigma_1^{-1/2}\mathbf{A}^*\Sigma_0^{1/2}$, and $h^* = (\mathbf{0}_d, \mathbf{Q}^*)$. Then $\mathbf{Q}^* \in \mathcal{O}(d)$ and $h^* \in \text{Stab}_{G_{\text{aff}}}(\rho)$. Moreover, all the conclusions of Proposition 1 hold with $h = h^*$ and $\mathbf{C}_i = \Sigma_i$ for $i = 0, 1$.*

The affine orbit representation and the Cartan structure play complementary roles in this example. The former identifies the elliptical family and its stabilizer, while the latter uses $L^\theta = \pi(\text{Stab}_{G_{\text{aff}}}(\rho)) = \mathcal{O}(d)$ and $\mathfrak{p} = \text{Sym}(d)$ to select the stabilizing element h^* for which the relative linear transformation is self-adjoint and positive definite. Consequently, Proposition 1 identifies the induced map T_{h^*} as the optimal affine transport and yields the common Monge, Kantorovich, and Lie group orbit transport value. When $\varsigma(s) = \exp(-s/2)$, the reference law ρ is the standard Gaussian distribution, and Corollary 1 recovers the classical Gaussian formulas of [Dowson and Landau, 1982, Olkin and Pukelsheim, 1982]. For a general common characteristic generator, it recovers [Gelbrich, 1990, Theorem 2.1].

5.2. Spectrally invariant distributions. We next consider probability distributions on $\mathbb{S}_{>}^d$ generated by congruence transformations of an absolutely continuous, spectrally invariant reference law ρ , whose density depends on $\mathbf{X}$ only through its unordered eigenvalues. We first introduce the congruence action and verify the Cartan structure required by Proposition 1. We then identify the class of reference distributions for which the fixed-point subgroup of this Cartan decomposition is contained in the stabilizer. The resulting orbit contains, in particular, Wishart, inverse-Wishart, and matrix beta type II families with general positive-definite scale matrices; although the reference law is spectrally invariant, a general member of its congruence orbit need not be.

Let the state space be $\mathcal{X} = \mathbb{S}_{>}^d$, the cone of $d \times d$ symmetric positive-definite matrices, and let $G = \text{GL}(d)$ act on $\mathcal{X}$ by congruence: $\alpha(\mathbf{A})(\mathbf{X}) = \mathbf{A}\mathbf{X}\mathbf{A}^\top$ for $\mathbf{A} \in \text{GL}(d)$ and $\mathbf{X} \in \mathbb{S}_{>}^d$. The ambient Hilbert space is $\mathcal{H} = \text{Sym}(d)$, equipped with the Frobenius inner product $\langle \mathbf{X}, \mathbf{Y} \rangle_{\mathcal{H}} = \text{Tr}(\mathbf{X}\mathbf{Y})$. Let $\mathcal{L}_{\text{Sym}(d)}$ denote the Euclidean volume measure on the finite-dimensional Hilbert space $\text{Sym}(d)$ induced by the Frobenius inner product. Equivalently, it is the $d(d+1)/2$ -dimensional Hausdorff measure induced by the Frobenius norm. All densities on $\mathbb{S}_{>}^d$ in this subsection are taken with respect to the restriction $\mathcal{L}_{\text{Sym}(d)}|_{\mathbb{S}_{>}^d}$.

Definition 7 (Spectrally invariant distribution). *A probability measure $\rho \in \mathcal{P}(\mathbb{S}_{>}^d)$ is spectrally invariant if $\alpha(\mathbf{Q})_{\#}\rho = \rho$ for every $\mathbf{Q} \in \mathcal{O}(d)$.*

Orthogonal congruence preserves the unordered eigenvalues of a symmetric matrix and acts orthogonally on $\text{Sym}(d)$ with respect to the Frobenius inner product. Consequently, any density on $\mathbb{S}_{>}^d$ that depends on $\mathbf{X}$ only through its unordered eigenvalues defines a spectrally invariant distribution. For $\Sigma \in \mathbb{S}_{>}^d$, write $\mu_{\Sigma} = \alpha(\Sigma^{1/2})_{\#}\rho$.

Define $b : G \rightarrow \mathcal{H}$ by $b \equiv 0$ and $\pi : G \rightarrow \text{GL}(\mathcal{H})$ by $\pi(\mathbf{A})(\mathbf{X}) = \mathbf{A}\mathbf{X}\mathbf{A}^{\top}$. For $\mathbf{H} \in \mathfrak{gl}(d)$, define $\mathcal{D}_{\mathbf{H}}$ by $\mathcal{D}_{\mathbf{H}}(\mathbf{X}) = \mathbf{H}\mathbf{X} + \mathbf{X}\mathbf{H}^{\top}$, and set $L = \pi(G)$.

Lemma 6. *Suppose that $\rho \in \mathcal{P}_2(\mathbb{S}_{>}^d)$ is spectrally invariant. Then the following statements hold.*

- (i) *The map π is a smooth group homomorphism, and $\alpha(\mathbf{A})(\mathbf{X}) = b(\mathbf{A}) + \pi(\mathbf{A})(\mathbf{X})$ for every $\mathbf{A} \in G$ and $\mathbf{X} \in \mathbb{S}_{>}^d$. Hence Assumption 4 holds.*
- (ii) *The subgroup L is a closed Lie subgroup of $\text{GL}(\text{Sym}(d))$. The map $\theta : L \rightarrow L$ defined by $\theta(\ell) = (\ell^*)^{-1}$ is a Lie group involution. Moreover, $\text{Lie}(L) = \{\mathcal{D}_{\mathbf{H}} : \mathbf{H} \in \mathfrak{gl}(d)\}$, $L^{\theta} = \pi(\mathcal{O}(d))$, $\mathfrak{k} = \{\mathcal{D}_{\mathbf{K}} : \mathbf{K}^{\top} = -\mathbf{K}\}$, and $\mathfrak{p} = \{\mathcal{D}_{\mathbf{S}} : \mathbf{S} \in \text{Sym}(d)\}$. For every $\mathbf{S} \in \text{Sym}(d)$, $\exp(\mathcal{D}_{\mathbf{S}}) = \pi(\exp(\mathbf{S}))$, and (L, θ) admits a global Cartan decomposition in the sense of Definition 6.*
- (iii) *Every element of $\mathfrak{p}$ is self-adjoint on $\text{Sym}(d)$ with respect to the Frobenius inner product, and $L^{\theta} = \pi(\text{Stab}_G(\rho))$.*

Consequently, Assumption 4 and all three structural hypotheses of Proposition 1 are satisfied.

Lemma 7 (Moment structure of a spectrally invariant law). *Let $d \geq 2$, and let $\rho \in \mathcal{P}(\mathbb{S}_{>}^d)$ be spectrally invariant with finite second moment. Define $\tau_{\rho} = d^{-1}\mathbb{E}_{\mathbf{X} \sim \rho}[\text{Tr}(\mathbf{X})]$. Then $\mathbb{E}_{\mathbf{X} \sim \rho}[\mathbf{X}] = \tau_{\rho}\mathbf{I}_d$ and $\tau_{\rho} > 0$. Moreover, there exist unique constants $a_{\rho}, b_{\rho} \in \mathbb{R}$ such that*

$$(4) \quad \mathbb{E}_{\mathbf{X} \sim \rho}[X_{ij}X_{kl}] = a_{\rho}\delta_{ij}\delta_{kl} + b_{\rho}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \quad \text{for all } i, j, k, l \in \{1, \dots, d\}.$$

Consequently, for every $\mathbf{U}, \mathbf{V} \in \mathbb{R}^{d \times d}$,

$$(5) \quad \mathbb{E}_{\mathbf{X} \sim \rho}[\text{Tr}(\mathbf{U}\mathbf{X}\mathbf{V}\mathbf{X})] = a_{\rho}\text{Tr}(\mathbf{U}\mathbf{V}) + b_{\rho}(\text{Tr}(\mathbf{U}\mathbf{V}^{\top}) + \text{Tr}(\mathbf{U})\text{Tr}(\mathbf{V})).$$

The constants are determined by the two spectral moments through

$$(6) \quad b_{\rho} = \frac{\mathbb{E}_{\mathbf{X} \sim \rho}[\text{Tr}(\mathbf{X}^2)] - d^{-1}\mathbb{E}_{\mathbf{X} \sim \rho}[(\text{Tr} \mathbf{X})^2]}{d^2 + d - 2}, \quad a_{\rho} = \frac{\mathbb{E}_{\mathbf{X} \sim \rho}[(\text{Tr} \mathbf{X})^2] - 2db_{\rho}}{d^2}.$$

Finally, for $\Sigma \in \mathbb{S}_{>}^d$, the mean $\bar{\mathbf{M}}_{\Sigma}$ and covariance operator $\mathbf{C}_{\Sigma}$ of $\mu_{\Sigma} = \alpha(\Sigma^{1/2})_{\#}\rho$ satisfy

$$(7) \quad \bar{\mathbf{M}}_{\Sigma} = \tau_{\rho}\Sigma, \quad \mathbf{C}_{\Sigma}(\mathbf{H}) = 2b_{\rho}\Sigma\mathbf{H}\Sigma + (a_{\rho} - \tau_{\rho}^2)\text{Tr}(\Sigma\mathbf{H})\Sigma, \quad \mathbf{H} \in \text{Sym}(d).$$

Theorem 5.1 (Closed-form transport on spectral congruence orbits). *Let $d \geq 2$, and let $\rho \in \mathcal{P}(\mathbb{S}_{>}^d)$ be absolutely continuous with respect to $\mathcal{L}_{\text{Sym}(d)}|_{\mathbb{S}_{>}^d}$, spectrally invariant, and have a finite second moment with respect to the Frobenius norm. Let $c(\mathbf{X}, \mathbf{Y}) = \|\mathbf{X} - \mathbf{Y}\|_{\text{F}}^2$. For $\Sigma_i \in \mathbb{S}_{>}^d$, set $g_i = \Sigma_i^{1/2}$, $\mu_i = \alpha(g_i)_{\#}\rho$, $i = 0, 1$. Let a_{ρ} and b_{ρ} be given by (6), and define $\mathbf{C} = (\Sigma_0^{1/2}\Sigma_1\Sigma_0^{1/2})^{1/2}$, $\mathbf{A}^* = \Sigma_0^{-1/2}\mathbf{C}\Sigma_0^{-1/2}$, $\mathbf{Q}^* = \Sigma_1^{-1/2}\mathbf{A}^*\Sigma_0^{1/2}$, $h^* = \mathbf{Q}^*$. Then $\mathbf{Q}^* \in \mathcal{O}(d)$ and $h^* \in \text{Stab}_{\text{GL}(d)}(\rho)$. Moreover, all the conclusions of Proposition 1 hold with $h = h^*$. The corresponding optimal Monge map is*

$$(8) \quad T^*(\mathbf{X}) = \mathbf{A}^*\mathbf{X}\mathbf{A}^*, \quad \mathbf{X} \in \mathbb{S}_{>}^d,$$

and

$$(9) \quad \mathbb{K}_c(\mu_0, \mu_1) = (a_{\rho} + b_{\rho})\|\Sigma_0 - \Sigma_1\|_{\text{F}}^2 + b_{\rho}((\text{Tr} \Sigma_0)^2 + (\text{Tr} \Sigma_1)^2 - 2(\text{Tr} \mathbf{C})^2).$$

We conclude by presenting three families of spectrally invariant reference measures, each yielding a closed-form optimal transport value through (9). Table 1 summarizes these families and the corresponding coefficients a_ρ and b_ρ . We relegate the derivations to Section C.2.

TABLE 1. Congruence families generated by spectrally invariant reference laws on $\mathbb{S}_>^d$, $d \geq 2$, their densities, reference laws, and second-moment constants entering the transport value (9).

Family	Density ($\propto$)	ρ	a_ρ	b_ρ
Wishart ($\Sigma \in \mathbb{S}_>^d$, $p > d - 1$)				
$\mathcal{W}_d(\Sigma, p)$	$\det(\mathbf{X})^{(p-d-1)/2} \exp(-\frac{1}{2} \text{Tr}(\Sigma^{-1} \mathbf{X}))$	$\mathcal{W}_d(\mathbf{I}_d, p)$	p^2	p
Inverse-Wishart ($\Psi \in \mathbb{S}_>^d$, $p > d+3$, $m = p-d$)				
$\mathcal{IW}_d(\Psi, p)$	$\det(\mathbf{X})^{-(p+d+1)/2} \exp(-\frac{1}{2} \text{Tr}(\Psi \mathbf{X}^{-1}))$	$\mathcal{IW}_d(\mathbf{I}_d, p)$	$\frac{m-2}{m(m-1)(m-3)}$	$\frac{1}{m(m-1)(m-3)}$
Matrix beta type II ($\Sigma \in \mathbb{S}_>^d$, $q_1 > (d-1)/2$, $2q_2 > d+3$, $q = 2q_2-d$)				
$\mathcal{MB}_d^{\text{II}}(q_1, q_2, \Sigma)$	$\det(\mathbf{X})^{q_1-(d+1)/2} \det(\Sigma + \mathbf{X})^{-(q_1+q_2)}$	$\mathcal{MB}_d^{\text{II}}(q_1, q_2, \mathbf{I}_d)$	$\frac{4q_1(q_1(q-2)+1)}{q(q-1)(q-3)}$	$\frac{2q_1(2q_1+q-1)}{q(q-1)(q-3)}$

Remark 1 (Tightness of [Gelbrich, 1990, Eq. (3)] beyond elliptical families). *The expression in (3) coincides with the lower bound for quadratic optimal transport established in [Gelbrich, 1990, Theorem 2.1]. A classical equality class for this bound is given by elliptically contoured distributions sharing a common characteristic generator. Theorem 5.1 shows that the same bound is attained on congruence orbits of absolutely continuous, spectrally invariant reference laws with finite second moment. The Wishart, inverse-Wishart, and matrix beta type II families in Table 1 are not elliptically contoured when viewed as probability distributions on $\text{Sym}(d)$. Indeed, the Wishart, inverse-Wishart, and matrix beta type II densities are strictly positive throughout $\mathbb{S}_>^d$. Consequently, when these laws are viewed as probability measures on $\text{Sym}(d)$, their topological support is $\mathbb{S}_>^d$. This support is not centrally symmetric about any $\mathbf{M} \in \text{Sym}(d)$: for all sufficiently large $t > 0$, $t\mathbf{I}_d \in \mathbb{S}_>^d$, whereas $2\mathbf{M} - t\mathbf{I}_d \notin \mathbb{S}_>^d$. By contrast, an elliptically contoured distribution is centrally symmetric about its location and therefore has centrally symmetric support. Hence these matrix-valued distributions are not elliptically contoured when viewed as probability measures on $\text{Sym}(d)$. Thus, Theorem 5.1 identifies a matrix-valued, non-elliptical equality class for the lower bound established in [Gelbrich, 1990, Theorem 2.1].*

5.3. Multivariate normal sinh-arcsinh distributions. [Jones and Pewsey, 2009, Section 7] introduced multivariate normal sinh-arcsinh distributions by transforming the marginals of a standardized, correlated multivariate normal vector. To describe this family, for $\epsilon \in \mathbb{R}$ and $\delta > 0$, define $S_{\epsilon, \delta}(x) = \sinh(\epsilon + \delta \text{arsinh}(x))$. Here ϵ is a skewness parameter and δ is a tailweight parameter. A random variable X has the canonical univariate normal sinh-arcsinh distribution with parameters (ϵ, δ) if $S_{\epsilon, \delta}(X)$ is standard normal. Equivalently, if $Z \sim \mathcal{N}(0, 1)$, then $S_{\epsilon, \delta}^{-1}(Z) = \sinh((\text{arsinh}(Z) - \epsilon)/\delta)$ has the canonical normal sinh-arcsinh distribution with parameters (ϵ, δ) . For the multivariate construction, let $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_d) \in \mathbb{R}^d$, $\boldsymbol{\delta} = (\delta_1, \dots, \delta_d) \in \mathbb{R}_{>0}^d$, and let $\boldsymbol{\Omega} \in \mathbb{S}_>^d$ be a correlation matrix. If $\mathbf{Z} = (Z_1, \dots, Z_d) \sim \mathcal{N}(\mathbf{0}_d, \boldsymbol{\Omega})$, define $\mathbf{X} = (X_1, \dots, X_d)$ by $X_j = S_{\epsilon_j, \delta_j}^{-1}(Z_j)$ for $j = 1, \dots, d$. We denote the law of $\mathbf{X}$ by $\text{NSAS}_d(\boldsymbol{\epsilon}, \boldsymbol{\delta}; \boldsymbol{\Omega})$. Its j th marginal

is the canonical univariate normal sinh-arcsinh law with parameters (ϵ_j, δ_j) , while its dependence is inherited from the Gaussian copula with correlation matrix Ω .

In this subsection, we fix the tailweight vector at $\delta = \mathbf{1}_d$, fix a positive-definite correlation matrix Ω , and allow the skewness vector ϵ to vary. Let $\rho = \mathcal{N}(\mathbf{0}_d, \Omega)$. For $\epsilon \in \mathbb{R}^d$, define $T_\epsilon : \mathbb{R}^d \rightarrow \mathbb{R}^d$ coordinatewise by $T_\epsilon(\mathbf{z})_j = \sinh(\operatorname{arsinh}(z_j) - \epsilon_j)$, and set $\mu_\epsilon = (T_\epsilon)_\# \rho$. Thus $\mu_\epsilon = \mathcal{NSAS}_d(\epsilon, \mathbf{1}_d; \Omega)$. For convenience, define $\mathbf{u}_\epsilon : \mathbb{R}^d \rightarrow \mathbb{R}^d$ by $u_{\epsilon,j}(\mathbf{x}) = \sinh(\epsilon_j + \operatorname{arsinh}(x_j))$. Then $\mathbf{u}_\epsilon = T_\epsilon^{-1}$, and μ_ϵ has density

$$r_{\Omega, \epsilon}(\mathbf{x}) = \frac{1}{(2\pi)^{d/2} \det(\Omega)^{1/2}} \exp\left(-\frac{1}{2} \mathbf{u}_\epsilon(\mathbf{x})^\top \Omega^{-1} \mathbf{u}_\epsilon(\mathbf{x})\right) \times \prod_{j=1}^d \frac{\cosh(\epsilon_j + \operatorname{arsinh}(x_j))}{\sqrt{1+x_j^2}}, \quad \mathbf{x} \in \mathbb{R}^d.$$

Equation (10) is the multivariate normal sinh-arcsinh density of [Jones and Pewsey, 2009, Section 7], specialized to unit tailweight parameters. Since each marginal transformation T_{ϵ_j} is a strictly increasing diffeomorphism, all measures μ_ϵ have the Gaussian copula with correlation matrix Ω .

Let $\mathfrak{B}_d \subseteq \mathcal{O}(d)$ be the hyperoctahedral group of signed permutation matrices, and define $\mathfrak{B}_\Omega = \{\mathbf{Q} \in \mathfrak{B}_d : \mathbf{Q}\Omega\mathbf{Q}^\top = \Omega\}$. Consider the semidirect product $G_\Omega = \mathbb{R}^d \rtimes \mathfrak{B}_\Omega$ with multiplication $(\epsilon, \mathbf{Q})(\eta, \mathbf{P}) = (\epsilon + \mathbf{Q}\eta, \mathbf{Q}\mathbf{P})$. Its identity is $e = (\mathbf{0}_d, \mathbf{I}_d)$, and $(\epsilon, \mathbf{Q})^{-1} = (-\mathbf{Q}^{-1}\epsilon, \mathbf{Q}^{-1})$. Let $\mathcal{X} = \mathcal{H} = \mathbb{R}^d$, equipped with the standard inner product, and let $c(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|^2$. Define $\alpha((\epsilon, \mathbf{Q}))(\mathbf{x}) = \sinh(\mathbf{Q} \operatorname{arsinh}(\mathbf{x}) - \epsilon)$, $(\epsilon, \mathbf{Q}) \in G_\Omega$, $\mathbf{x} \in \mathbb{R}^d$, where $\sinh$ and arsinh are applied componentwise.

Define the involution $\theta : G_\Omega \rightarrow G_\Omega$ by $\theta(\epsilon, \mathbf{Q}) = (-\epsilon, \mathbf{Q})$. Under the identification $\operatorname{Lie}(G_\Omega) \simeq \mathbb{R}^d$, the exponential satisfies $\exp(\mathbf{S}) = (\mathbf{S}, \mathbf{I}_d)$; the Cartan structure is verified in Lemma 19.

Corollary 2 (Closed-form transport between multivariate normal sinh-arcsinh laws). *Let $g_i = (\epsilon_i, \mathbf{Q}_i) \in G_\Omega$ and $\mu_i = \alpha(g_i)_\# \rho$ for $i = 0, 1$. Set $\Delta = \epsilon_1 - \epsilon_0$ and $h^* = (\mathbf{0}_d, \mathbf{Q}_1^{-1} \mathbf{Q}_0)$. Then $h^* \in G_\Omega^\theta = \operatorname{Stab}_{G_\Omega}(\rho)$ and $g_1 h^* g_0^{-1} = (\Delta, \mathbf{I}_d)$. Moreover, all the conclusions of Theorem 4.2 hold with $h = h^*$. The corresponding unique optimal Monge map from μ_0 to μ_1 , up to μ_0 -almost-everywhere equality, is $T^*(\mathbf{x}) = \sinh(\operatorname{arsinh}(\mathbf{x}) - \Delta)$. For $j = 1, \dots, d$, set $\Delta C_j = \cosh(\epsilon_{1,j}) - \cosh(\epsilon_{0,j})$ and $\Delta S_j = \sinh(\epsilon_{1,j}) - \sinh(\epsilon_{0,j})$. Then $\mathbb{K}_c(\mu_0, \mu_1) = \sum_{j=1}^d ((\Delta C_j)^2 + 2(\Delta S_j)^2)$.*

For every $\epsilon \in \mathbb{R}$, the map $t_\epsilon(z) = \sinh(\operatorname{arsinh}(z) - \epsilon)$ is a strictly increasing diffeomorphism of $\mathbb{R}$. Consequently, each measure in the orbit is obtained from $\mathcal{N}(\mathbf{0}_d, \Omega)$ by coordinatewise strictly increasing transformations, and all the orbit measures share the Gaussian copula with correlation matrix Ω . Let $F_{i,j}$ denote the distribution function of the j th marginal of μ_i , and let Φ denote the standard normal distribution function. Since the j th marginal of μ_i is the push-forward of a standard normal random variable by $t_{\epsilon_{i,j}}$, its quantile function is $F_{i,j}^{-1} = t_{\epsilon_{i,j}} \circ \Phi^{-1}$. It follows that $F_{1,j}^{-1} \circ F_{0,j} = t_{\epsilon_{1,j} - \epsilon_{0,j}}$. Hence the coordinatewise quantile map associated with the common copula is precisely the optimal map identified in Corollary 2. Since the squared Euclidean cost is the sum of the coordinatewise quadratic costs, the optimality of the corresponding common-copula coupling also follows from [Alfonsi and Jourdain, 2014, Proposition 1.1]. Thus, Corollary 2 recovers the common-copula optimal transport for this family through the nonlinear Cartan criterion.

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Appendix

This document contains supplementary material for the paper *Optimal Transport on Lie Group Orbits*.

APPENDIX A. BEYOND LIE GROUP ORBITS: ONE-DIMENSIONAL DISTRIBUTIONS

The preceding examples fall within the finite-dimensional Lie group framework developed in the main paper. We now record the classical one-dimensional case as a complementary example. This example still has an orbit and stabilizer structure, but the acting transformation group is infinite-dimensional and is not a finite-dimensional Lie group. The results in this section should therefore be read as an orbit-based reformulation of the classical monotone rearrangement formula [Cuesta-Albertos et al., 1993], rather than as an application of the finite-dimensional theory of the main paper.

For an absolutely continuous probability measure $\mu \ll \mathcal{L}^1$ on $\mathbb{R}$, let F_μ and r_μ denote its cumulative distribution function and density, respectively. We consider the class

$$\mathcal{P}^+ = \{\mu \in \mathcal{P}(\mathbb{R}) : \mu \ll \mathcal{L}^1, F_\mu \in \mathcal{C}^1, F'_\mu(x) > 0 \forall x \in \mathbb{R}\}$$

Fix a reference probability measure ρ with a smooth, strictly positive density $r \in \mathcal{C}^\infty(\mathbb{R})$; a concrete example is the logistic density $r(x) = \frac{1}{4}\text{sech}^2(x/2)$, and its cumulative distribution function $F_\rho(x) = \int_{-\infty}^x r(s)ds$ satisfies $F'_\rho(x) = r(x) > 0$ for all x , so $F_\rho : \mathbb{R} \rightarrow (0, 1)$ is a $\mathcal{C}^\infty$, strictly increasing bijection. By the inverse-function theorem, F_ρ is in fact a $\mathcal{C}^\infty$ diffeomorphism with smooth inverse $F_\rho^{-1} : (0, 1) \rightarrow \mathbb{R}$.

Let $\mathcal{G}^+$ be the group of orientation-preserving $\mathcal{C}^1$ diffeomorphisms of $\mathbb{R}$: $\mathcal{G}^+ = \{g \in \mathcal{C}^1(\mathbb{R}; \mathbb{R}) : g'(x) > 0 \forall x \in \mathbb{R}, g \text{ is bijective, } g^{-1} \in \mathcal{C}^1(\mathbb{R}; \mathbb{R})\}$, equipped with composition.

We regard $\mathcal{G}^+$ only as a transformation group. In particular, this one-dimensional example is not an application of the finite-dimensional Lie-group framework developed in the main paper. Nevertheless, it has the same orbit-stabilizer structure at the level of transformations. Namely, if $\mu_i = (g_i)_\# \rho$, $g_i \in \mathcal{G}^+$, $i = 0, 1$, then every map of the form $g_1 \circ h \circ g_0^{-1}$, $h \in \text{Stab}_{\mathcal{G}^+}(\rho)$, transports μ_0 to μ_1 . Thus we introduce the one-dimensional analogue of the orbit-reduced problem (LGOP). For this subsection only, we define

$$(11) \quad \mathbb{J}_c^1(g_0, g_1) = \inf_{h \in \text{Stab}_{\mathcal{G}^+}(\rho)} \int_{\mathbb{R}} c(g_0(x), (g_1 \circ h)(x)) d\rho(x),$$

where $\text{Stab}_{\mathcal{G}^+}(\rho) = \{h \in \mathcal{G}^+ : h_\# \rho = \rho\}$.

Lemma 8. *For any $\mu \in \mathcal{P}^+$, $(F_\mu^{-1} \circ F_\rho)_\# \rho = \mu$ and $\mathcal{G}_\#^+ \rho = \mathcal{P}^+$.*

Proof. First, we will show that $\mathcal{G}_\#^+ \rho \subseteq \mathcal{P}^+$. Take $g \in \mathcal{G}^+$ and set $\mu = g_\# \rho$. Since g is $\mathcal{C}^1$, strictly increasing and surjective, its inverse is likewise $\mathcal{C}^1$ and strictly increasing. Additionally, by the inverse function theorem $(g^{-1})'(t) = 1/g'(g^{-1}(t)) > 0$. A direct computation shows that the cumulative distribution function of μ is $F_\mu(x) = \int_{-\infty}^x r(g^{-1}(t))(g^{-1})'(t)dt$. By applying the change of variables in the form of $s = g^{-1}(t)$ (hence $dt = g'(s)ds$), we have $F_\mu(x) = \int_{-\infty}^{g^{-1}(x)} r(s)ds =$

$F_\rho(g^{-1}(x))$. Since $F_\rho : \mathbb{R} \rightarrow (0, 1)$ and $g^{-1} : \mathbb{R} \rightarrow \mathbb{R}$ are both $\mathcal{C}^1$ and strictly increasing, their composition $F_\rho \circ g^{-1}$ is $\mathcal{C}^1$ and strictly increasing on $\mathbb{R}$. Moreover,

$$F'_\mu(x) = r(g^{-1}(x))(g^{-1})'(x) = \frac{r(g^{-1}(x))}{g'(g^{-1}(x))} > 0.$$

Thus F_μ satisfies the defining properties of $\mathcal{P}^+$, so $\mu \in \mathcal{P}^+$.

Next, we will show that $\mathcal{P}^+ \subseteq \mathcal{G}_\#^+ \rho$. Take an arbitrary $\mu \in \mathcal{P}^+$ and write F_μ for its cumulative distribution function and r_μ for its density. Because F_μ is strictly increasing, the inverse $F_\mu^{-1} : (0, 1) \rightarrow \mathbb{R}$ is well defined and belongs to $\mathcal{C}^1((0, 1))$ and strictly increasing. Define $g = F_\mu^{-1} \circ F_\rho : \mathbb{R} \rightarrow \mathbb{R}$, which also belongs to $\mathcal{C}^1(\mathbb{R})$. Hence, $g \in \mathcal{G}^+$. For any $\mathcal{A} \in \mathcal{B}(\mathbb{R})$,

$$\begin{aligned} g_\# \rho(\mathcal{A}) &= \rho(g^{-1}(\mathcal{A})) \\ &= \rho(\{x \in \mathbb{R} : F_\mu^{-1}(F_\rho(x)) \in \mathcal{A}\}) \\ &= \rho(\{x \in \mathbb{R} : F_\rho(x) \in F_\mu(\mathcal{A})\}) \\ &= (F_\rho)_\# \rho(F_\mu(\mathcal{A})) = \lambda(F_\mu(\mathcal{A})) = \mu(\mathcal{A}), \end{aligned}$$

where λ is the Lebesgue measure on the unit interval $(0, 1)$. The first equality follows by the definition of the push-forward operator, the second equality by construction of g , the third equality follows because F_μ^{-1} is the inverse of the strictly increasing function F_μ . The fourth equality follows by the definition of the push-forward operator, the fifth equality follows because by the probability-integral transform $(F_\rho)_\# \rho = \lambda$. The last equality follows because $(F_\mu)_\# \mu = \lambda$, which equivalently means that for every $\mathcal{B}' \in \mathcal{B}((0, 1))$, we have $\lambda(\mathcal{B}') = \mu(F_\mu^{-1}(\mathcal{B}'))$. Taking $\mathcal{B}' = F_\mu(\mathcal{A})$ gives $\lambda(F_\mu(\mathcal{A})) = \mu(\mathcal{A})$. Hence, we may conclude that $g_\# \rho = \mu$, which implies that $\mathcal{P}^+ \subseteq \mathcal{G}_\#^+ \rho$. This observation completes our proof. $\square$

Lemma 9. *The stabilizer of ρ under $\mathcal{G}^+$ is $\text{Stab}_{\mathcal{G}^+}(\rho) = \{\text{id}_{\mathbb{R}}\}$.*

Proof. Fix $g \in \mathcal{G}^+$ with $g_\# \rho = \rho$. For every $x \in \mathbb{R}$ we have

$$F_\rho(x) = \rho((-\infty, x]) = \rho(g^{-1}((-\infty, x])) = \rho((-\infty, g^{-1}(x)]) = F_\rho(g^{-1}(x)).$$

Thus $F_\rho(x) = F_\rho(g^{-1}(x))$ for all $x \in \mathbb{R}$. Since $r > 0$, the function F_ρ is strictly increasing on $\mathbb{R}$. Hence F_ρ is injective, and the identity above implies $g^{-1}(x) = x$ for every $x \in \mathbb{R}$; equivalently, $g(x) = x$. Therefore $g = \text{id}_{\mathbb{R}}$, completing the proof. $\square$

Lemma 10. *Suppose that $g_i = F_{\mu_i}^{-1} \circ F_\rho$, $i = 0, 1$. Then, (11) induced by the cost function $c(x, y) = (x - y)^2$ is solved by $h^* = \text{id}_{\mathbb{R}}$ and admits the following closed form expression*

$$(12) \quad \mathbb{J}_c^1(g_0, g_1) = \int_0^1 (F_{\mu_1}^{-1}(t) - F_{\mu_0}^{-1}(t))^2 dt.$$

Proof. By Lemma 9, the stabilizer group is a singleton, and thus $h^* = \text{id}_{\mathbb{R}}$ solves (11). Next, we evaluate $\mathbb{J}_c^1$:

$$\begin{aligned} \mathbb{J}_c^1(g_0, g_1) &= \int_{\mathbb{R}} (g_0(x) - g_1(x))^2 d\rho(x) \\ &= \int_{\mathbb{R}} (F_{\mu_1}^{-1}(F_\rho(x)) - F_{\mu_0}^{-1}(F_\rho(x)))^2 r(x) dx \\ &= \int_0^1 (F_{\mu_1}^{-1}(t) - F_{\mu_0}^{-1}(t))^2 dt, \end{aligned}$$

where the last equality follows by the change-of-variables formula $t \leftarrow F_\rho(x)$, and thus $dt = r(x)dx$. This observation completes our proof. $\square$

Proposition 2. *Suppose that $\mu_0, \mu_1 \in \mathcal{P}^+$ have finite second moments and $c(x, y) = (x - y)^2$. Then, we have*

- (i) $\mathbb{M}_c(\mu_0, \mu_1) = \mathbb{J}_c^1(g_0, g_1)$, and $T^* = F_{\mu_1}^{-1} \circ F_{\mu_0}$ solves (MP),
- (ii) $\mathbb{K}_c(\mu_0, \mu_1) = \mathbb{J}_c^1(g_0, g_1)$, and $(\text{id}_{\mathbb{R}}, T^*)_{\#} \mu_0$ solves (KTP),

where $g_i = F_{\mu_i}^{-1} \circ F_\rho \in \mathcal{G}^+$ for $i = 0, 1$.

Proof. By Lemma 8, we have $(g_i)_{\#} \rho = \mu_i$ for $i = 0, 1$; hence $\mu_0, \mu_1 \in \mathcal{G}_{\#}^+ \rho$. Define $T^* = g_1 \circ h^* \circ g_0^{-1}$, where h^* is the map defined in Lemma 10. Consequently, we have $T^*(x) = F_{\mu_1}^{-1}(F_{\mu_0}(x))$.

In what follows, we define $\varphi(x) = 2 \int_{x_0}^x T^*(s) ds - x^2$ for some $x_0 \in \mathbb{R}$, and we compute its first and second order derivatives

$$\varphi'(x) = 2T^*(x) - 2x \text{ and } \varphi''(x) = 2(T^*)'(x) - 2.$$

As T^* is increasing, $(T^*)'(x) \geq 0$ for all $x \in \mathbb{R}$, implying $\varphi''(x) \geq -2$. Consequently, by [Villani, 2008, Example 13.6], φ is c -convex. Next, we evaluate ($\heartsuit$):

$$2T^*(x) - 2x + 2(x - T^*(x)) = 0.$$

The quadratic cost satisfies Assumption 1 directly. Since $\mu_0 \ll \mathcal{L}^1$, Lemma 4(iii) gives Assumption 2, while Lemma 4(iv) gives Assumption 3. The finite-second-moment assumption implies $\mathbb{K}_c(\mu_0, \mu_1) < \infty$. Therefore, by Theorem 2.1(v) T^* solves (MP) and the optimal Kantorovich plan is concentrated on the graph of T^* .

If $X \sim \mu_0$, then $U = F_{\mu_0}(X) \sim \mathcal{U}([0, 1])$. Hence

$$\int_{\mathbb{R}} |x - T^*(x)|^2 d\mu_0(x) = \int_0^1 (F_{\mu_0}^{-1}(t) - F_{\mu_1}^{-1}(t))^2 dt.$$

The equality with $\mathbb{J}_c^1(g_0, g_1)$ follows from Lemma 10. $\square$

Proposition 2 coincides with [Cuesta-Albertos et al., 1993, Corollary 2.7] when restricted to the quadratic cost.

APPENDIX B. ADDITIONAL EXAMPLE FOR SECTION 5

B.1. The diagonal scaling mechanism on $\mathbb{R}_{>0}^d$. Let the state space be $\mathcal{X} = \mathbb{R}_{>0}^d$, and let $G_{\text{diag}} = (\mathbb{R}_{>0}^d, \odot)$ be the componentwise multiplicative group. Its identity element is $\mathbf{1}_d$, and the inverse of $\mathbf{a} \in G_{\text{diag}}$ is $\mathbf{a}^{-1} = (a_1^{-1}, \dots, a_d^{-1})$. The componentwise logarithm is a Lie group isomorphism from G_{diag} onto $(\mathbb{R}^d, +)$, so G_{diag} is a d -dimensional Lie group.

The group acts smoothly on $\mathcal{X}$ by coordinatewise scaling, $\alpha(\mathbf{a})(\mathbf{x}) = \mathbf{a} \odot \mathbf{x}$ for $\mathbf{a}, \mathbf{x} \in \mathbb{R}_{>0}^d$. Equivalently, writing $D(\mathbf{a}) = \text{diag}(a_1, \dots, a_d)$, we have $\alpha(\mathbf{a})(\mathbf{x}) = D(\mathbf{a})\mathbf{x}$. Thus the action is induced by the linear representation on $\mathcal{H} = \mathbb{R}^d$, equipped with the standard inner product, given by $b \equiv 0$ and $\pi(\mathbf{a}) = D(\mathbf{a})$. Indeed, $\pi(\mathbf{a} \odot \mathbf{b}) = D(\mathbf{a} \odot \mathbf{b}) = D(\mathbf{a})D(\mathbf{b}) = \pi(\mathbf{a})\pi(\mathbf{b})$, and $\alpha(\mathbf{a})(\mathbf{x}) = b(\mathbf{a}) + \pi(\mathbf{a})\mathbf{x}$. Hence Assumption 4 holds.

Cartan structure. Set $L = \pi(G_{\text{diag}}) = \{D(\mathbf{a}) : \mathbf{a} \in \mathbb{R}_{>0}^d\}$. The map $\mathbb{R}^d \ni \mathbf{s} \mapsto \text{diag}(e^{s_1}, \dots, e^{s_d}) \in L$ is a Lie group isomorphism, and hence L is a Lie subgroup of $\text{GL}(\mathbb{R}^d)$. Define $\theta : L \rightarrow L$ by $\theta(D(\mathbf{a})) = D(\mathbf{a})^{-\top} = D(\mathbf{a}^{-1})$. Then θ is a Lie group involution. Its fixed-point subgroup is $L^\theta = \{\mathbf{I}_d\}$, because $a_i = a_i^{-1}$ and $a_i > 0$ imply $a_i = 1$ for every i .

The Lie algebra of L is the space of diagonal matrices, $\mathfrak{l} = \{\text{diag}(s_1, \dots, s_d) : s_i \in \mathbb{R}\}$. Since $d\theta_{\mathbf{I}_d}(\mathbf{S}) = -\mathbf{S}$ for every $\mathbf{S} \in \mathfrak{l}$, the corresponding eigenspaces are $\mathfrak{k} = \{0\}$ and $\mathfrak{p} = \mathfrak{l}$. The

exponential map restricts to a diffeomorphism from $\mathfrak{p}$ onto L , with $\text{diag}(s_1, \dots, s_d)$ mapped to $\text{diag}(e^{s_1}, \dots, e^{s_d})$ and inverse given by the componentwise logarithm. Consequently, the Cartan map $L^\theta \times \mathfrak{p} \rightarrow L$, $(\mathbf{I}_d, \mathbf{S}) \mapsto \exp(\mathbf{S})$, is a diffeomorphism. Thus (L, θ) admits a global Cartan decomposition in the sense of Definition 6. Moreover, every $\mathbf{S} \in \mathfrak{p}$ is diagonal and hence self-adjoint on $\mathbb{R}^d$.

Reference distribution. Let $\rho = \bigotimes_{i=1}^d \text{Exp}(1)$, where $\text{Exp}(\lambda)$ denotes the exponential distribution with rate parameter $\lambda > 0$, so that ρ has density $r(\mathbf{x}) = \prod_{i=1}^d e^{-x_i} \mathbb{1}_{\{x_i > 0\}}$ with respect to Lebesgue measure on $\mathbb{R}^d$. The state space $\mathbb{R}_{>0}^d$ is open and convex, while ρ is absolutely continuous and satisfies $\mathbb{E}_{\mathbf{Z} \sim \rho}[\|\mathbf{Z}\|^2] = 2d < \infty$. Furthermore, since $L^\theta = \{\mathbf{I}_d\}$ and the identity element $\mathbf{1}_d$ belongs to $\text{Stab}_{G_{\text{diag}}}(\rho)$, we have $L^\theta \subseteq \pi(\text{Stab}_{G_{\text{diag}}}(\rho))$. Therefore, all the standing assumptions and structural hypotheses of Proposition 1 are satisfied.

Lemma 11 (Product exponential orbit and stabilizer). *Let $\rho = \bigotimes_{i=1}^d \text{Exp}(1)$. Then:*

- (i) *for every $\beta \in \mathbb{R}_{>0}^d$, $\alpha(\beta^{-1})\#\rho = \bigotimes_{i=1}^d \text{Exp}(\beta_i)$, and hence $G_{\text{diag}}\#\rho = \{\bigotimes_{i=1}^d \text{Exp}(\beta_i) : \beta \in \mathbb{R}_{>0}^d\}$;*
- (ii) $\text{Stab}_{G_{\text{diag}}}(\rho) = \{\mathbf{1}_d\}$.

Proof. Let $\mathbf{Z} \sim \rho$ and set $\mathbf{X} = \mathbf{a} \odot \mathbf{Z}$. For each coordinate, $X_i = a_i Z_i$. Since $Z_i \sim \text{Exp}(1)$, the density of X_i is $x_i \mapsto a_i^{-1} e^{-x_i/a_i} \mathbb{1}_{\{x_i > 0\}}$. Taking $\mathbf{a} = \beta^{-1}$ gives $X_i \sim \text{Exp}(\beta_i)$ for every i , and independence is preserved under coordinatewise transformations. This proves (i).

For (ii), suppose that $\mathbf{a} \in \text{Stab}_{G_{\text{diag}}}(\rho)$. Then, by definition, $(\alpha(\mathbf{a}))\#\rho = \rho$. Hence, if $\mathbf{Z} \sim \rho$, then $\mathbf{a} \odot \mathbf{Z}$ also has distribution ρ . In particular, $a_i Z_i \stackrel{d}{=} Z_i$ for every coordinate i . Taking expectations and using $\mathbb{E}[Z_i] = 1$ gives $a_i = 1$ for every i . Hence $\mathbf{a} = \mathbf{1}_d$, which proves the claim. $\square$

Corollary 3 (Optimal transport between products of exponential laws). *For $j = 0, 1$, let $\mu_j = \bigotimes_{i=1}^d \text{Exp}(\beta_{j,i})$, where $\beta_j = (\beta_{j,1}, \dots, \beta_{j,d}) \in \mathbb{R}_{>0}^d$, and let $c(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|^2$. Set $g_j = \beta_j^{-1}$ and $h^* = \mathbf{1}_d$. Then h^* solves the Lie group orbit transport problem, and all the conclusions of Proposition 1 hold with $h = h^*$. In particular:*

- (a) *the unique optimal Monge map from μ_0 to μ_1 , up to μ_0 -almost-everywhere equality, is $T^*(\mathbf{x}) = (\frac{\beta_{0,1}}{\beta_{1,1}}x_1, \dots, \frac{\beta_{0,d}}{\beta_{1,d}}x_d)$;*
- (b) *$(\text{id}_{\mathbb{R}_{>0}^d}, T^*)\#\mu_0$ is the unique optimal Kantorovich plan;*
- (c) *the optimal Kantorovich value is $\mathbb{K}_c(\mu_0, \mu_1) = 2 \sum_{i=1}^d (\beta_{0,i}^{-1} - \beta_{1,i}^{-1})^2$;*
- (d) $\mathbb{M}_c(\mu_0, \mu_1) = \mathbb{K}_c(\mu_0, \mu_1) = \mathbb{J}_c(g_0, g_1)$.

Proof. By Lemma 11(i), $\mu_j = \alpha(g_j)\#\rho$ with $g_j = \beta_j^{-1} \in G_{\text{diag}}$ for $j = 0, 1$. The preceding verification shows that all the hypotheses of Proposition 1 are satisfied. By Lemma 11(ii), the stabilizer is trivial, so the stabilizing element furnished by that proposition must be $h^* = \mathbf{1}_d$.

The corresponding orbit-induced map is $T_{h^*} = \alpha(g_1 h^* g_0^{-1}) = \alpha(\beta_1^{-1} \odot \beta_0)$, and therefore $T_{h^*}(\mathbf{x}) = (\frac{\beta_{0,1}}{\beta_{1,1}}x_1, \dots, \frac{\beta_{0,d}}{\beta_{1,d}}x_d) = T^*(\mathbf{x})$. Thus the optimality and uniqueness assertions, as well as the equality of the Monge, Kantorovich, and Lie group orbit transport values, follow from Proposition 1.

It remains to evaluate the common value. Since $h^* = \mathbf{1}_d$ and $\mathbf{Z} \sim \rho$, we have $\mathbb{J}_c(g_0, g_1) = \mathbb{E}[\|g_0 \odot \mathbf{Z} - g_1 \odot \mathbf{Z}\|^2] = \sum_{i=1}^d (\beta_{0,i}^{-1} - \beta_{1,i}^{-1})^2 \mathbb{E}[Z_i^2]$. Because $Z_i \sim \text{Exp}(1)$, $\mathbb{E}[Z_i^2] = 2$. Hence $\mathbb{J}_c(g_0, g_1) = 2 \sum_{i=1}^d (\beta_{0,i}^{-1} - \beta_{1,i}^{-1})^2$, which proves the stated formula. $\square$

Remark 2. *In this example the Cartan decomposition is degenerate: the fixed-point subgroup L^θ is trivial. Thus there is no nontrivial stabilizer optimization or rotational component to absorb. The*

example nevertheless fits the same structural template as the affine and congruence mechanisms: the Cartan decomposition places the relative linear transformation in $\exp(\mathfrak{p})$, where it is self-adjoint and positive definite, and this algebraic structure certifies optimality.

Many other product-form scale families admit the same analysis, provided the shape parameters are fixed and the relevant second moments are finite. In such cases, the orbit is generated by coordinatewise positive scalings, the stabilizer is trivial, and the optimal map is again diagonal. The closed-form value reduces to a sum of one-dimensional second-moment calculations, exactly as in Corollary 3. Examples include products of Weibull, Rayleigh, Gamma, inverse-Gamma, Pareto, lognormal, and generalized-Gamma laws, under the corresponding parameter restrictions ensuring finite quadratic cost.

More generally, whenever the cost is additively separable across coordinates and measures are product measures, the Kantorovich problem decouples into one-dimensional problems: a product of one-dimensional optimal plans is optimal, and the Kantorovich value is the sum of the one-dimensional values. The one-dimensional case is treated in Section A.

APPENDIX C. PROOFS AND AUXILIARY RESULTS FOR SECTION 5

This appendix contains the technical proofs and auxiliary calculations used in Section 5, organized according to the corresponding distribution families.

C.1. Elliptical distributions.

Lemma 12. (i) $\{\alpha(g)_{\#}\rho : g \in G_{\text{aff}}\} = \{\mathcal{E}_{\zeta}(\mathbf{m}, \mathbf{S}) : \mathbf{m} \in \mathbb{R}^d, \mathbf{S} \in \mathbb{S}_{\succ}^d\}$, (ii) $\text{Stab}_{G_{\text{aff}}}(\rho) = \{(\mathbf{0}_d, \mathbf{Q}) : \mathbf{Q} \in \mathcal{O}(d)\}$.

Proof. (i) Follows by [Muirhead, 2009, § 1.5]. Indeed, if $\mathbf{X} \sim \rho$ and $\mathbf{Y} = \mathbf{m} + \mathbf{A}\mathbf{X}$, then the characteristic function of $\mathbf{Y}$ evaluated at $\mathbf{t} \in \mathbb{R}^d$ is $\exp(i\mathbf{t}^{\top}\mathbf{m})\zeta(\mathbf{t}^{\top}\mathbf{A}\mathbf{A}^{\top}\mathbf{t})$, so $\mathbf{Y} \sim \mathcal{E}_{\zeta}(\mathbf{m}, \mathbf{A}\mathbf{A}^{\top})$. Conversely, every $\mathbf{S} \in \mathbb{S}_{\succ}^d$ can be written as $\mathbf{S} = \mathbf{A}\mathbf{A}^{\top}$ with $\mathbf{A} = \mathbf{S}^{1/2}$. (ii) Let $(\mathbf{u}, \mathbf{A}) \in \text{Stab}_{G_{\text{aff}}}(\rho)$ and let $\mathbf{x} \sim \rho$. Then $\mathbf{u} + \mathbf{A}\mathbf{x} \sim \rho$. Since ρ has mean $\mathbf{0}_d$, taking expectations gives $\mathbf{u} + \mathbf{A}\mathbb{E}_{\mathbf{x} \sim \rho}[\mathbf{x}] = \mathbf{0}_d$, and hence $\mathbf{u} = \mathbf{0}_d$. Since ρ has covariance $\mathbf{I}_d$, taking covariances gives $\mathbf{A}\text{Cov}(\mathbf{x})\mathbf{A}^{\top} = \text{Cov}(\mathbf{x})$, that is, $\mathbf{A}\mathbf{A}^{\top} = \mathbf{I}_d$. Thus $\mathbf{A} \in \mathcal{O}(d)$, so $\text{Stab}_{G_{\text{aff}}}(\rho) \subseteq \{(\mathbf{0}_d, \mathbf{Q}) : \mathbf{Q} \in \mathcal{O}(d)\}$. Conversely, let $\mathbf{Q} \in \mathcal{O}(d)$. The characteristic function of ρ is $\phi_{\rho}(\mathbf{t}) = \zeta(\|\mathbf{t}\|^2)$. The characteristic function of $\mathbf{Q}\mathbf{x}$ is $\mathbf{t} \mapsto \phi_{\rho}(\mathbf{Q}^{\top}\mathbf{t}) = \zeta(\|\mathbf{Q}^{\top}\mathbf{t}\|^2) = \zeta(\|\mathbf{t}\|^2) = \phi_{\rho}(\mathbf{t})$. Hence $\mathbf{Q}\mathbf{x} \stackrel{d}{=} \mathbf{x}$, and therefore $(\mathbf{0}_d, \mathbf{Q}) \in \text{Stab}_{G_{\text{aff}}}(\rho)$. $\square$

Lemma 13. Let $\rho = \mathcal{E}_{\zeta}(\mathbf{0}_d, \mathbf{I}_d)$ be the standardized elliptical reference law introduced above. Then the following statements hold.

- (i) The map π is a smooth group homomorphism, b is a smooth π -cocycle, and $\alpha(g)(\mathbf{x}) = b(g) + \pi(g)\mathbf{x}$ for every $g \in G_{\text{aff}}$ and $\mathbf{x} \in \mathbb{R}^d$. Hence Assumption 4 holds.
- (ii) We have $L = \text{GL}(d)$, and L is a Lie subgroup of $\text{GL}(\mathcal{H})$. The map $\theta : L \rightarrow L$ defined by $\theta(\mathbf{A}) = \mathbf{A}^{-\top}$ is a Lie group involution. Moreover, $\text{Lie}(L) = \mathfrak{gl}(d)$, $L^{\theta} = \mathcal{O}(d)$, $\mathfrak{k} = \{\mathbf{K} \in \mathbb{R}^{d \times d} : \mathbf{K}^{\top} = -\mathbf{K}\}$, and $\mathfrak{p} = \text{Sym}(d)$. The Cartan map $L^{\theta} \times \mathfrak{p} \rightarrow L$, $(\mathbf{Q}, \mathbf{S}) \mapsto \mathbf{Q}\exp(\mathbf{S})$, is a diffeomorphism. Thus (L, θ) admits a global Cartan decomposition in the sense of Definition 6.
- (iii) Every element of $\mathfrak{p}$ is self-adjoint on $\mathcal{H}$, and $L^{\theta} = \pi(\text{Stab}_{G_{\text{aff}}}(\rho))$.

Consequently, Assumption 4 and all three structural hypotheses of Proposition 1 are satisfied.

Proof of Lemma 13. (i) For $g = (\mathbf{m}, \mathbf{A})$ and $\tilde{g} = (\mathbf{n}, \mathbf{B})$ in G_{aff} , the group law gives $g\tilde{g} = (\mathbf{m} + \mathbf{A}\mathbf{n}, \mathbf{A}\mathbf{B})$. Therefore, $\pi(g\tilde{g}) = \mathbf{A}\mathbf{B} = \pi(g)\pi(\tilde{g})$, so $\pi : G_{\text{aff}} \rightarrow \text{GL}(\mathbb{R}^d)$ is a group homomorphism.

Moreover, $b(g\tilde{g}) = \mathbf{m} + \mathbf{A}\mathbf{n} = b(g) + \pi(g)b(\tilde{g})$, and $\alpha(g)(\mathbf{x}) = \mathbf{m} + \mathbf{A}\mathbf{x} = b(g) + \pi(g)\mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^d$. The maps b and π are smooth because they are the coordinate projections of the smooth manifold $\mathbb{R}^d \times \mathrm{GL}(d)$. Hence (b, π) is an affine representation inducing α , and Assumption 4 holds. (ii) By part (i), π is a group homomorphism, so $L = \pi(G_{\mathrm{aff}})$ is a subgroup of $\mathrm{GL}(\mathbb{R}^d)$. Moreover, for every $\mathbf{A} \in \mathrm{GL}(d)$, we have $\pi((\mathbf{0}_d, \mathbf{A})) = \mathbf{A}$, and thus $L = \mathrm{GL}(d) = \mathrm{GL}(\mathcal{H})$. This implies L is a subgroup of $\mathrm{GL}(\mathcal{H})$.

Define $\theta : L \rightarrow L$ by $\theta(\mathbf{A}) = \mathbf{A}^{-\top}$. For every $\mathbf{A}, \mathbf{B} \in L$, we have $\theta(\mathbf{AB}) = (\mathbf{AB})^{-\top} = \mathbf{A}^{-\top}\mathbf{B}^{-\top} = \theta(\mathbf{A})\theta(\mathbf{B})$, and $\theta^2(\mathbf{A}) = \mathbf{A}$. Moreover, θ is smooth because matrix inversion is smooth on $\mathrm{GL}(d)$ and transposition is linear. Hence θ is a Lie group involution.

An element $\mathbf{A} \in L$ belongs to L^θ if and only if $\mathbf{A}^{-\top} = \mathbf{A}$, which is equivalent to $\mathbf{A}^\top \mathbf{A} = \mathbf{I}_d$. Therefore, $L^\theta = \mathcal{O}(d)$.

We next identify the eigenspaces of $d\theta_{\mathbf{I}_d}$. For $\mathbf{X} \in \mathfrak{gl}(d)$, the first-order expansion $(\mathbf{I}_d + t\mathbf{X})^{-\top} = \mathbf{I}_d - t\mathbf{X}^\top + o(t)$ shows that $d\theta_{\mathbf{I}_d}(\mathbf{X}) = -\mathbf{X}^\top$. It follows that the $+1$ eigenspace is $\mathfrak{k} = \{\mathbf{K} \in \mathbb{R}^{d \times d} : \mathbf{K}^\top = -\mathbf{K}\}$ and the -1 eigenspace is $\mathfrak{p} = \{\mathbf{S} \in \mathbb{R}^{d \times d} : \mathbf{S}^\top = \mathbf{S}\} = \mathrm{Sym}(d)$.

It remains to establish the global Cartan decomposition. Consider the Cartan map $\Phi : L^\theta \times \mathfrak{p} \rightarrow L$ given by $\Phi(\mathbf{Q}, \mathbf{S}) = \mathbf{Q} \exp(\mathbf{S})$. Let $\mathbf{A} \in \mathrm{GL}(d)$ and define $\mathbf{P} = (\mathbf{A}^\top \mathbf{A})^{1/2}$, $\mathbf{Q} = \mathbf{A}\mathbf{P}^{-1}$, and $\mathbf{S} = \log(\mathbf{P})$. Since $\mathbf{P} \in \mathbb{S}_>^d$, the principal logarithm $\mathbf{S}$ is well defined and belongs to $\mathrm{Sym}(d)$. Furthermore, $\mathbf{Q}^\top \mathbf{Q} = \mathbf{P}^{-1} \mathbf{A}^\top \mathbf{A} \mathbf{P}^{-1} = \mathbf{I}_d$, so $\mathbf{Q} \in \mathcal{O}(d)$, and $\mathbf{A} = \mathbf{Q}\mathbf{P} = \mathbf{Q} \exp(\mathbf{S})$. Thus Φ is surjective.

To prove uniqueness, suppose that $\mathbf{A} = \mathbf{Q} \exp(\mathbf{S})$ for some $\mathbf{Q} \in \mathcal{O}(d)$ and $\mathbf{S} \in \mathrm{Sym}(d)$. Since $\exp(\mathbf{S})$ is symmetric positive definite, we have $\mathbf{A}^\top \mathbf{A} = \exp(2\mathbf{S})$. Hence $(\mathbf{A}^\top \mathbf{A})^{1/2} = \exp(\mathbf{S})$ by uniqueness of the positive definite square root. Consequently, $\mathbf{S} = \log((\mathbf{A}^\top \mathbf{A})^{1/2})$ and $\mathbf{Q} = \mathbf{A}(\mathbf{A}^\top \mathbf{A})^{-1/2}$. Therefore, the factorization is unique.

The inverse of Φ is consequently given by $\mathbf{A} \mapsto (\mathbf{A}(\mathbf{A}^\top \mathbf{A})^{-1/2}, \log((\mathbf{A}^\top \mathbf{A})^{1/2}))$. The map Φ is smooth, and its inverse is smooth because matrix multiplication and transposition are smooth and the principal square root, inverse, and principal logarithm are smooth on $\mathbb{S}_>^d$. Hence Φ is a diffeomorphism. Therefore, (L, θ) admits a global Cartan decomposition in the sense of Definition 6.

(iii) Let $\mathbf{S} \in \mathfrak{p} = \mathrm{Sym}(d)$. For every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, we have $\langle \mathbf{S}\mathbf{x}, \mathbf{y} \rangle_{\mathcal{H}} = \mathbf{x}^\top \mathbf{S}^\top \mathbf{y} = \mathbf{x}^\top \mathbf{S} \mathbf{y} = \langle \mathbf{x}, \mathbf{S}\mathbf{y} \rangle_{\mathcal{H}}$. Thus $\mathbf{S}$ is self-adjoint on $\mathbb{R}^d$.

Finally, Lemma 12(ii) gives $\mathrm{Stab}_{G_{\mathrm{aff}}}(\rho) = \{(\mathbf{0}_d, \mathbf{Q}) : \mathbf{Q} \in \mathcal{O}(d)\}$. Therefore,

$$\pi(\mathrm{Stab}_{G_{\mathrm{aff}}}(\rho)) = \{\mathbf{Q} : \mathbf{Q} \in \mathcal{O}(d)\} = \mathcal{O}(d) = L^\theta.$$

Part (i) verifies Assumption 4, while parts (ii) and (iii) verify the three structural hypotheses of Proposition 1. This proves the claim. $\square$

Proof of Corollary 1. By the definition of $\mathbf{A}^*$, $\mathbf{A}^* \Sigma_0 \mathbf{A}^* = \Sigma_0^{-1/2} (\Sigma_0^{1/2} \Sigma_1 \Sigma_0^{1/2}) \Sigma_0^{-1/2} = \Sigma_1$. Since $\mathbf{A}^*$ is symmetric, it follows that $\mathbf{Q}^*(\mathbf{Q}^*)^\top = \Sigma_1^{-1/2} \mathbf{A}^* \Sigma_0 \mathbf{A}^* \Sigma_1^{-1/2} = \mathbf{I}_d$. Hence $\mathbf{Q}^* \in \mathcal{O}(d)$, and therefore $h^* = (\mathbf{0}_d, \mathbf{Q}^*) \in \mathrm{Stab}_{G_{\mathrm{aff}}}(\rho)$ by Lemma 12.

Moreover, $g_0^{-1} = (-\Sigma_0^{-1/2} \mathbf{m}_0, \Sigma_0^{-1/2})$, and the affine group law gives $g_1 h^* g_0^{-1} = (\mathbf{m}_1 - \mathbf{A}^* \mathbf{m}_0, \mathbf{A}^*)$, because $\Sigma_1^{1/2} \mathbf{Q}^* \Sigma_0^{-1/2} = \mathbf{A}^*$. Recall that $T_{h^*} = \alpha(g_1 h^* g_0^{-1})$ is the corresponding orbit-induced transport map. Thus, $T_{h^*}(\mathbf{x}) = \mathbf{m}_1 + \mathbf{A}^*(\mathbf{x} - \mathbf{m}_0)$. The reference law has mean $\mathbf{0}_d$ and covariance $\mathbf{I}_d$, so the mean and covariance operators of μ_i are $\mathbf{m}_i$ and Σ_i , respectively. All conclusions now follow from Proposition 1. $\square$

C.2. Spectrally invariant distributions.

Lemma 14 (Congruence orbit and stabilizer of a spectrally invariant law). *Let $\rho \in \mathcal{P}_2(\mathbb{S}_{>}^d)$ be spectrally invariant. Then the following statements hold.*

- (i) *For every $\mathbf{A} \in \text{GL}(d)$, $\alpha(\mathbf{A})_{\#}\rho = \mu_{\mathbf{A}\mathbf{A}^\top}$. Consequently, $G_{\#}\rho = \{\mu_{\Sigma} : \Sigma \in \mathbb{S}_{>}^d\}$.*
- (ii) *The stabilizer of ρ under the congruence action is $\text{Stab}_{\text{GL}(d)}(\rho) = \mathcal{O}(d)$.*

Proof. (i) Fix $\mathbf{A} \in \text{GL}(d)$. Its left polar decomposition is $\mathbf{A} = \mathbf{P}\mathbf{Q}$, where $\mathbf{P} = (\mathbf{A}\mathbf{A}^\top)^{1/2} \in \mathbb{S}_{>}^d$ and $\mathbf{Q} = \mathbf{P}^{-1}\mathbf{A} \in \mathcal{O}(d)$. Indeed, $\mathbf{Q}\mathbf{Q}^\top = \mathbf{P}^{-1}\mathbf{A}\mathbf{A}^\top\mathbf{P}^{-1} = \mathbf{I}_d$.

Since α is a left action, $\alpha(\mathbf{A}) = \alpha(\mathbf{P}) \circ \alpha(\mathbf{Q})$. Spectral invariance gives $\alpha(\mathbf{Q})_{\#}\rho = \rho$, and therefore $\alpha(\mathbf{A})_{\#}\rho = \alpha(\mathbf{P})_{\#}(\alpha(\mathbf{Q})_{\#}\rho) = \alpha(\mathbf{P})_{\#}\rho$. Since $\mathbf{P}^2 = \mathbf{A}\mathbf{A}^\top$, the definition of μ_{Σ} yields $\alpha(\mathbf{A})_{\#}\rho = \mu_{\mathbf{A}\mathbf{A}^\top}$.

It follows that every element of $G_{\#}\rho$ belongs to $\{\mu_{\Sigma} : \Sigma \in \mathbb{S}_{>}^d\}$. Conversely, for every $\Sigma \in \mathbb{S}_{>}^d$, we have $\mu_{\Sigma} = \alpha(\Sigma^{1/2})_{\#}\rho$, so $\mu_{\Sigma} \in G_{\#}\rho$. This proves (i).

(ii) Spectral invariance implies $\mathcal{O}(d) \subseteq \text{Stab}_{\text{GL}(d)}(\rho)$. It remains to prove the reverse inclusion: Let $\mathbf{X} \sim \rho$ and set $\mathbf{M} = \mathbb{E}[\mathbf{X}]$. The expectation is finite because $\rho \in \mathcal{P}_2(\mathbb{S}_{>}^d)$. For every $\mathbf{Q} \in \mathcal{O}(d)$, spectral invariance gives $\mathbf{Q}\mathbf{X}\mathbf{Q}^\top \sim \rho$, and hence $\mathbf{Q}\mathbf{M}\mathbf{Q}^\top = \mathbf{M}$. We first show that $\mathbf{M}$ is a scalar multiple of the identity. For each $r \in \{1, \dots, d\}$, let $\mathbf{D}_r$ be the diagonal orthogonal matrix whose r th diagonal entry is -1 and whose remaining diagonal entries are 1 . Since $\mathbf{D}_r\mathbf{M}\mathbf{D}_r = \mathbf{M}$, comparison of the (r, j) entry for $j \neq r$ gives $-M_{rj} = M_{rj}$. Thus $M_{rj} = 0$ whenever $r \neq j$, and therefore $\mathbf{M}$ is diagonal. Next, invariance under permutation matrices implies that all diagonal entries of $\mathbf{M}$ are equal. Consequently, $\mathbf{M} = \tau_{\rho}\mathbf{I}_d$, where $\tau_{\rho} = d^{-1}\mathbb{E}[\text{Tr}(\mathbf{X})]$.

Because $\mathbf{X} \in \mathbb{S}_{>}^d$ almost surely, $\text{Tr}(\mathbf{X}) > 0$ almost surely. The random variable $\text{Tr}(\mathbf{X})$ is integrable, and hence $\tau_{\rho} > 0$.

Now let $\mathbf{A} \in \text{Stab}_{\text{GL}(d)}(\rho)$. Then $\mathbf{A}\mathbf{X}\mathbf{A}^\top \stackrel{d}{=} \mathbf{X}$, so taking expectations gives $\mathbf{A}\mathbf{M}\mathbf{A}^\top = \mathbf{M}$. Substituting $\mathbf{M} = \tau_{\rho}\mathbf{I}_d$ and using $\tau_{\rho} > 0$ yields $\mathbf{A}\mathbf{A}^\top = \mathbf{I}_d$. Thus $\mathbf{A} \in \mathcal{O}(d)$.

We have proved $\text{Stab}_{\text{GL}(d)}(\rho) \subseteq \mathcal{O}(d)$. Together with the reverse inclusion obtained from spectral invariance, this gives $\text{Stab}_{\text{GL}(d)}(\rho) = \mathcal{O}(d)$. $\square$

Proof of Lemma 6. (i) For every $\mathbf{A} \in G = \text{GL}(d)$ and $\mathbf{X} \in \mathcal{H} = \text{Sym}(d)$, the matrix $\mathbf{A}\mathbf{X}\mathbf{A}^\top$ is symmetric. Moreover, $\pi(\mathbf{A}^{-1})$ is the inverse of $\pi(\mathbf{A})$, so $\pi(\mathbf{A}) \in \text{GL}(\mathcal{H})$.

For $\mathbf{A}, \mathbf{B} \in G$ and $\mathbf{X} \in \mathcal{H}$, we have $\pi(\mathbf{A}\mathbf{B})(\mathbf{X}) = \mathbf{A}\mathbf{B}\mathbf{X}\mathbf{B}^\top\mathbf{A}^\top = \pi(\mathbf{A})(\pi(\mathbf{B})(\mathbf{X}))$. Thus $\pi(\mathbf{A}\mathbf{B}) = \pi(\mathbf{A})\pi(\mathbf{B})$. The map $\pi : G \rightarrow \text{GL}(\mathcal{H})$ is smooth because its matrix coefficients, in any fixed basis of $\text{Sym}(d)$, are polynomial functions of the entries of $\mathbf{A}$.

Since $b \equiv 0$, for every $\mathbf{A}, \mathbf{B} \in G$ we have $b(\mathbf{A}\mathbf{B}) = \mathbf{0}_{\mathcal{H}} = b(\mathbf{A}) + \pi(\mathbf{A})b(\mathbf{B})$. Furthermore, for every $\mathbf{X} \in \mathcal{X} = \mathbb{S}_{>}^d$, $\alpha(\mathbf{A})(\mathbf{X}) = \mathbf{A}\mathbf{X}\mathbf{A}^\top = b(\mathbf{A}) + \pi(\mathbf{A})(\mathbf{X})$. Hence the congruence action is induced by the affine representation (b, π) , and Assumption 4 holds.

(ii) We first show that $L = \pi(G)$ is closed in $\text{GL}(\mathcal{H})$. Let $\|\cdot\|_{\text{op}}$ denote the operator norm on $\text{GL}(\mathcal{H})$ induced by the Frobenius norm on $\mathcal{H}$. For every $\mathbf{A} \in G$, $\|\pi(\mathbf{A})\|_{\text{op}} = \|\mathbf{A}\|_{\text{op}}^2$. Indeed, $\|\mathbf{A}\mathbf{X}\mathbf{A}^\top\|_{\text{F}} \leq \|\mathbf{A}\|_{\text{op}}^2\|\mathbf{X}\|_{\text{F}}$, which gives the upper bound. For the reverse inequality, let $\mathbf{v}$ be a unit right-singular vector of $\mathbf{A}$ corresponding to its largest singular value, and set $\mathbf{X} = \mathbf{v}\mathbf{v}^\top$. Then $\|\mathbf{X}\|_{\text{F}} = 1$ and $\|\pi(\mathbf{A})(\mathbf{X})\|_{\text{F}} = \|(\mathbf{A}\mathbf{v})(\mathbf{A}\mathbf{v})^\top\|_{\text{F}} = \|\mathbf{A}\mathbf{v}\|^2 = \|\mathbf{A}\|_{\text{op}}^2$. Applying the same identity to $\mathbf{A}^{-1}$ gives $\|\pi(\mathbf{A})^{-1}\|_{\text{op}} = \|\mathbf{A}^{-1}\|_{\text{op}}^2$.

Now suppose that $\pi(\mathbf{A}_n) \rightarrow \ell$ in $\text{GL}(\mathcal{H})$. Since inversion is continuous on $\text{GL}(\mathcal{H})$, we also have $\pi(\mathbf{A}_n)^{-1} \rightarrow \ell^{-1}$. The preceding norm identities imply that both $\{\mathbf{A}_n\}$ and $\{\mathbf{A}_n^{-1}\}$ are bounded. Passing to a subsequence, we may assume that $\mathbf{A}_n \rightarrow \mathbf{A}$ for some $\mathbf{A} \in \mathbb{R}^{d \times d}$.

There exists $M < \infty$ such that $\|\mathbf{A}_n^{-1}\|_{\text{op}} \leq M$ for every n . Hence $\|\mathbf{A}_n \mathbf{x}\| \geq M^{-1} \|\mathbf{x}\|$ for every $\mathbf{x} \in \mathbb{R}^d$. Passing to the limit gives $\|\mathbf{A} \mathbf{x}\| \geq M^{-1} \|\mathbf{x}\|$, so $\mathbf{A}$ is injective and therefore belongs to $\text{GL}(d)$. By continuity of π , $\ell = \pi(\mathbf{A}) \in L$. Thus L is closed in $\text{GL}(\mathcal{H})$ and hence is an embedded Lie subgroup by Cartan's closed subgroup theorem [Duistermaat and Kolk, 2012, Corollary 1.10.7].

For $\mathbf{A} \in G$ and $\mathbf{X}, \mathbf{Y} \in \mathcal{H}$, cyclicity of the trace gives $\langle \pi(\mathbf{A})(\mathbf{X}), \mathbf{Y} \rangle_{\text{F}} = \text{Tr}(\mathbf{A} \mathbf{X} \mathbf{A}^{\top} \mathbf{Y}) = \text{Tr}(\mathbf{X} \mathbf{A}^{\top} \mathbf{Y} \mathbf{A}) = \langle \mathbf{X}, \pi(\mathbf{A}^{\top})(\mathbf{Y}) \rangle_{\text{F}}$. Consequently, $\pi(\mathbf{A})^* = \pi(\mathbf{A}^{\top})$.

It follows that the map $\theta : L \rightarrow L$ defined by $\theta(\ell) = (\ell^*)^{-1}$ satisfies $\theta(\pi(\mathbf{A})) = (\pi(\mathbf{A})^*)^{-1} = \pi(\mathbf{A}^{\top})^{-1} = \pi(\mathbf{A}^{-\top})$. In particular, θ maps L into itself.

For $\ell_1, \ell_2 \in L$, the identity $(\ell_1 \ell_2)^* = \ell_2^* \ell_1^*$ gives $\theta(\ell_1 \ell_2) = ((\ell_1 \ell_2)^*)^{-1} = (\ell_1^*)^{-1} (\ell_2^*)^{-1} = \theta(\ell_1) \theta(\ell_2)$. Moreover, $\theta^2(\ell) = \ell$, because $((\ell^*)^{-1})^* = \ell^{-1}$. The map θ is smooth since the adjoint operation is linear and inversion is smooth on $\text{GL}(\mathcal{H})$. Hence θ is a Lie group involution.

We now identify its fixed-point subgroup. If $\mathbf{Q} \in \mathcal{O}(d)$, then $\mathbf{Q}^{-\top} = \mathbf{Q}$, and therefore $\theta(\pi(\mathbf{Q})) = \pi(\mathbf{Q})$. Thus $\pi(\mathcal{O}(d)) \subseteq L^{\theta}$.

Conversely, suppose that $\pi(\mathbf{A}) \in L^{\theta}$. Then $\pi(\mathbf{A}) = \pi(\mathbf{A}^{-\top})$. Evaluating both operators at $\mathbf{I}_d$ gives $\mathbf{A} \mathbf{A}^{\top} = \mathbf{A}^{-\top} \mathbf{A}^{-1} = (\mathbf{A} \mathbf{A}^{\top})^{-1}$. Since $\mathbf{A} \mathbf{A}^{\top}$ is symmetric positive definite, this implies $(\mathbf{A} \mathbf{A}^{\top})^2 = \mathbf{I}_d$. Every eigenvalue of $\mathbf{A} \mathbf{A}^{\top}$ is positive and has square equal to one, so $\mathbf{A} \mathbf{A}^{\top} = \mathbf{I}_d$. Hence $\mathbf{A} \in \mathcal{O}(d)$, and therefore $L^{\theta} = \pi(\mathcal{O}(d))$.

We next identify the Lie algebra of L . Since $\pi : G \rightarrow L$ is a surjective smooth Lie-group homomorphism, it is a submersion, and thus its differential at the identity, $d\pi_{\mathbf{I}_d} : \mathfrak{gl}(d) \rightarrow \text{Lie}(L)$, is surjective. For $\mathbf{H} \in \mathfrak{gl}(d)$ and $\mathbf{X} \in \mathcal{H}$, differentiating $\pi(\mathbf{I}_d + t\mathbf{H})(\mathbf{X}) = (\mathbf{I}_d + t\mathbf{H})\mathbf{X}(\mathbf{I}_d + t\mathbf{H})^{\top}$ at $t = 0$ gives $d\pi_{\mathbf{I}_d}(\mathbf{H})(\mathbf{X}) = \mathbf{H}\mathbf{X} + \mathbf{X}\mathbf{H}^{\top} = \mathcal{D}_{\mathbf{H}}(\mathbf{X})$. Consequently, $\text{Lie}(L) = \{\mathcal{D}_{\mathbf{H}} : \mathbf{H} \in \mathfrak{gl}(d)\}$.

We record that the map $\mathbf{H} \mapsto \mathcal{D}_{\mathbf{H}}$ is injective. Indeed, suppose that $\mathcal{D}_{\mathbf{H}} = 0$. Evaluating at $\mathbf{I}_d$ gives $\mathbf{H} + \mathbf{H}^{\top} = 0$, so $\mathbf{H}$ is skew-symmetric. It then follows that $\mathbf{H}\mathbf{X} - \mathbf{X}\mathbf{H} = 0$ for every $\mathbf{X} \in \text{Sym}(d)$. Choosing $\mathbf{X}$ to be diagonal with pairwise distinct diagonal entries shows that every off-diagonal entry of $\mathbf{H}$ vanishes. Since a skew-symmetric matrix has zero diagonal, we obtain $\mathbf{H} = 0$.

For $\mathbf{H} \in \mathfrak{gl}(d)$, we have $\theta(\pi(\exp(t\mathbf{H}))) = \pi(\exp(-t\mathbf{H}^{\top}))$. Differentiating at $t = 0$ yields $d\theta_e(\mathcal{D}_{\mathbf{H}}) = \mathcal{D}_{-\mathbf{H}^{\top}}$, where e denotes the identity of L . By injectivity of $\mathbf{H} \mapsto \mathcal{D}_{\mathbf{H}}$, $d\theta_e(\mathcal{D}_{\mathbf{H}}) = \mathcal{D}_{\mathbf{H}}$ if and only if $\mathbf{H}^{\top} = -\mathbf{H}$, whereas $d\theta_e(\mathcal{D}_{\mathbf{H}}) = -\mathcal{D}_{\mathbf{H}}$ if and only if $\mathbf{H}^{\top} = \mathbf{H}$. Therefore, $\mathfrak{k} = \{\mathcal{D}_{\mathbf{K}} : \mathbf{K}^{\top} = -\mathbf{K}\}$ and $\mathfrak{p} = \{\mathcal{D}_{\mathbf{S}} : \mathbf{S} \in \text{Sym}(d)\}$.

Compatibility of Lie-group homomorphisms with exponential maps gives $\exp(\mathcal{D}_{\mathbf{H}}) = \pi(\exp(\mathbf{H}))$ for every $\mathbf{H} \in \mathfrak{gl}(d)$, and in particular $\exp(\mathcal{D}_{\mathbf{S}}) = \pi(\exp(\mathbf{S}))$ for every $\mathbf{S} \in \text{Sym}(d)$.

We now prove the global Cartan decomposition. Let $\Phi : L^{\theta} \times \mathfrak{p} \rightarrow L$ denote the Cartan map $\Phi(k, Z) = k \exp(Z)$.

Fix $\ell = \pi(\mathbf{A}) \in L$ and let $\mathbf{A} = \mathbf{Q}\mathbf{P}$ be the right polar decomposition of $\mathbf{A}$, where $\mathbf{P} = (\mathbf{A}^{\top} \mathbf{A})^{1/2} \in \mathbb{S}_{>}^d$ and $\mathbf{Q} = \mathbf{A}\mathbf{P}^{-1} \in \mathcal{O}(d)$. Set $\mathbf{S} = \log(\mathbf{P}) \in \text{Sym}(d)$. Then $\pi(\mathbf{Q}) \in L^{\theta}$, $\mathcal{D}_{\mathbf{S}} \in \mathfrak{p}$, and $\exp(\mathcal{D}_{\mathbf{S}}) = \pi(\exp(\mathbf{S})) = \pi(\mathbf{P})$. Consequently, $\ell = \pi(\mathbf{Q}) \exp(\mathcal{D}_{\mathbf{S}})$, and hence Φ is surjective. Suppose that $\ell = k_i \exp(Z_i)$, where $k_i \in L^{\theta}$ and $Z_i \in \mathfrak{p}$ for $i = 1, 2$. By the description of $\mathfrak{p}$, there exist $\mathbf{S}_i \in \text{Sym}(d)$ such that $Z_i = \mathcal{D}_{\mathbf{S}_i}$. Set $\mathbf{P}_i = \exp(\mathbf{S}_i) \in \mathbb{S}_{>}^d$. Then $\exp(Z_i) = \pi(\mathbf{P}_i)$.

Every element of $L^{\theta} = \pi(\mathcal{O}(d))$ is an orthogonal operator on $\text{Sym}(d)$. Moreover, $\pi(\mathbf{P}_i)$ is self-adjoint because $\pi(\mathbf{P}_i)^* = \pi(\mathbf{P}_i^{\top}) = \pi(\mathbf{P}_i)$, and it is positive definite because, for every nonzero $\mathbf{X} \in \text{Sym}(d)$, $\langle \pi(\mathbf{P}_i)(\mathbf{X}), \mathbf{X} \rangle_{\text{F}} = \|\mathbf{P}_i^{1/2} \mathbf{X} \mathbf{P}_i^{1/2}\|_{\text{F}}^2 > 0$. Thus $\ell = k_i \pi(\mathbf{P}_i)$ is the polar decomposition of ℓ as an invertible operator on $\text{Sym}(d)$.

Uniqueness of the operator polar decomposition gives $k_1 = k_2$ and $\pi(\mathbf{P}_1) = \pi(\mathbf{P}_2)$. Evaluating the latter identity at $\mathbf{I}_d$ yields $\mathbf{P}_1^2 = \mathbf{P}_2^2$. Since both matrices are positive definite, uniqueness of

the positive-definite square root implies $\mathbf{P}_1 = \mathbf{P}_2$. Taking principal logarithms gives $\mathbf{S}_1 = \mathbf{S}_2$, and therefore $Z_1 = Z_2$. Hence Φ is bijective.

It remains to verify smoothness of the inverse. For $\ell \in L$, set $p_\ell = (\ell^* \ell)^{1/2}$, $k_\ell = \ell p_\ell^{-1}$, $\mathbf{S}_\ell = \frac{1}{2} \log(p_\ell(\mathbf{I}_d))$, and $Z_\ell = \mathcal{D}_{\mathbf{S}_\ell}$. To see that these formulas recover the preceding factorization, write $\ell = \pi(\mathbf{A})$ and let $\mathbf{A} = \mathbf{Q}\mathbf{P}$ be its right polar decomposition. Then $\ell^* \ell = \pi(\mathbf{A}^\top \mathbf{A}) = \pi(\mathbf{P}^2) = \pi(\mathbf{P})^2$. As shown above, $\pi(\mathbf{P})$ is self-adjoint and positive definite, and hence $p_\ell = \pi(\mathbf{P})$. It follows that $k_\ell = \pi(\mathbf{Q})$ and $p_\ell(\mathbf{I}_d) = \mathbf{P}^2$. Therefore, $\mathbf{S}_\ell = \log(\mathbf{P})$ and $Z_\ell = \mathcal{D}_{\log(\mathbf{P})}$. Thus $\Phi^{-1}(\ell) = (k_\ell, Z_\ell)$.

The map Φ is smooth. Its inverse is smooth because the adjoint, operator multiplication, positive-definite operator square root, operator inversion, evaluation at $\mathbf{I}_d$, and the principal matrix logarithm on $\mathbb{S}_{>}^d$ are smooth. Hence Φ is a diffeomorphism, and (L, θ) admits a global Cartan decomposition in the sense of Definition 6.

(iii) Let $Z \in \mathfrak{p}$. By the description of $\mathfrak{p}$ in part (ii), there exists $\mathbf{S} \in \text{Sym}(d)$ such that $Z = \mathcal{D}_{\mathbf{S}}$. For every $\mathbf{X}, \mathbf{Y} \in \text{Sym}(d)$, cyclicity of the trace and $\mathbf{S}^\top = \mathbf{S}$ give $\langle Z(\mathbf{X}), \mathbf{Y} \rangle_{\mathbb{F}} = \text{Tr}((\mathbf{S}\mathbf{X} + \mathbf{X}\mathbf{S})\mathbf{Y}) = \text{Tr}(\mathbf{X}(\mathbf{Y}\mathbf{S} + \mathbf{S}\mathbf{Y})) = \langle \mathbf{X}, Z(\mathbf{Y}) \rangle_{\mathbb{F}}$. Thus Z is self-adjoint on $\text{Sym}(d)$. Since $Z \in \mathfrak{p}$ was arbitrary, every element of $\mathfrak{p}$ is self-adjoint.

Finally, Lemma 14(ii) gives $\text{Stab}_G(\rho) = \mathcal{O}(d)$. Consequently, $L^\theta = \pi(\mathcal{O}(d)) = \pi(\text{Stab}_G(\rho))$.

Part (i) verifies Assumption 4. Part (ii) verifies that L is a Lie subgroup, that θ is a Lie group involution, and that (L, θ) admits a global Cartan decomposition. Part (iii) verifies the self-adjointness and stabilizer hypotheses. Hence all the structural hypotheses of Proposition 1 are satisfied. $\square$

Proof of Lemma 7. Let $\mathbf{M} = \mathbb{E}[\mathbf{X}]$. Spectral invariance gives $\mathbf{Q}\mathbf{M}\mathbf{Q}^\top = \mathbf{M}$ for every $\mathbf{Q} \in \mathcal{O}(d)$. Invariance under diagonal sign matrices implies that $\mathbf{M}$ is diagonal, while invariance under permutation matrices implies that all of its diagonal entries are equal. Thus $\mathbf{M} = \tau_\rho \mathbf{I}_d$, where $\tau_\rho = d^{-1} \mathbb{E}[\text{Tr}(\mathbf{X})]$. Since $\mathbf{X} \in \mathbb{S}_{>}^d$ almost surely and $\text{Tr}(\mathbf{X})$ is integrable, $\tau_\rho > 0$.

Let $\mathbf{X} \sim \rho$ and define the fourth-order tensor $T_{ijkl} = \mathbb{E}_{\mathbf{X} \sim \rho}[X_{ij}X_{kl}]$, $i, j, k, l \in \{1, \dots, d\}$. Since ρ is invariant under orthogonal congruence, we have $\mathbf{Q}\mathbf{X}\mathbf{Q}^\top \sim \rho$ for every $\mathbf{Q} \in \mathcal{O}(d)$. Therefore, for every $i, j, k, l \in \{1, \dots, d\}$,

$$T_{ijkl} = \mathbb{E}_{\mathbf{X} \sim \rho}[(\mathbf{Q}\mathbf{X}\mathbf{Q}^\top)_{ij}(\mathbf{Q}\mathbf{X}\mathbf{Q}^\top)_{kl}].$$

Using $(\mathbf{Q}\mathbf{X}\mathbf{Q}^\top)_{ij} = \sum_{i', j'=1}^d Q_{ii'} X_{i'j'} Q_{jj'}$, we obtain $T_{ijkl} = \sum_{i', j', k', l'=1}^d Q_{ii'} Q_{jj'} Q_{kk'} Q_{ll'} T_{i'j'k'l'}$.

In other words, the tensor $T_{ijkl} = \mathbb{E}_{\mathbf{X} \sim \rho}[X_{ij}X_{kl}]$ is invariant under the simultaneous congruence $T \mapsto \mathbf{Q}^{\otimes 4} T$ for every $\mathbf{Q} \in \mathcal{O}(d)$. By the First Fundamental Theorem for the orthogonal group, every $\mathcal{O}(d)$ -invariant rank-four tensor is a linear combination of the three pairings $\delta_{ij}\delta_{kl}$, $\delta_{ik}\delta_{jl}$, $\delta_{il}\delta_{jk}$. Hence, for some scalars A, B, C , $T_{ijkl} = A\delta_{ij}\delta_{kl} + B\delta_{ik}\delta_{jl} + C\delta_{il}\delta_{jk}$. Since $\mathbf{X}$ is symmetric, we have $T_{ijkl} = T_{jikl} = T_{ijlk}$, which forces $B = C$. Writing $a_\rho = A$ and $b_\rho = B$ yields $T_{ijkl} = a_\rho\delta_{ij}\delta_{kl} + b_\rho(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. For the identity (5), note that $\text{Tr}(\mathbf{U}\mathbf{X}\mathbf{V}\mathbf{X}) = \sum_{i,j,k,l} U_{ij} V_{kl} X_{jk} X_{li}$. Substituting (4) for $\mathbb{E}_{\mathbf{X} \sim \rho}[X_{jk} X_{li}]$: the term $a_\rho\delta_{jk}\delta_{li}$ contributes $a_\rho \sum_{i,j} U_{ij} V_{ji} = a_\rho \text{Tr}(\mathbf{U}\mathbf{V})$; the term $b_\rho\delta_{jl}\delta_{ki}$ contributes $b_\rho \sum_{j,k} U_{kj} V_{kj} = b_\rho \text{Tr}(\mathbf{U}\mathbf{V}^\top)$; and the term $b_\rho\delta_{ji}\delta_{kl}$ contributes $b_\rho \sum_{i=1}^d U_{ii} \sum_{k=1}^d V_{kk} = b_\rho \text{Tr}(\mathbf{U}) \text{Tr}(\mathbf{V})$.

Next, evaluating (4) with $i = j, k = l$ gives $\mathbb{E}_{\mathbf{X} \sim \rho}[(\text{Tr}(\mathbf{X}))^2] = a_\rho d^2 + 2b_\rho d$, and with $i = k, j = l$ gives $\mathbb{E}_{\mathbf{X} \sim \rho}[\text{Tr}(\mathbf{X}^2)] = a_\rho d + b_\rho d(d+1)$. Solving this 2×2 system yields (6).

Finally, fix $\Sigma \in \mathbb{S}_{>}^d$, set $\mathbf{R} = \Sigma^{1/2}$, and let $\mathbf{Y} = \mathbf{R}\mathbf{X}\mathbf{R} \sim \mu_\Sigma$. Since $\mathbf{R}$ is symmetric and $\mathbf{R}^2 = \Sigma$, we have $\mathbb{E}[\mathbf{Y}] = \mathbf{R}\mathbb{E}[\mathbf{X}]\mathbf{R} = \mathbf{R}(\tau_\rho \mathbf{I}_d)\mathbf{R} = \tau_\rho \Sigma$. This proves the first identity in (7).

We next identify the covariance operator. Fix $\mathbf{H}, \mathbf{K} \in \text{Sym}(d)$ and define $\mathbf{U} = \mathbf{RHR}$ and $\mathbf{V} = \mathbf{RKR}$. Both $\mathbf{U}$ and $\mathbf{V}$ belong to $\text{Sym}(d)$. Moreover, cyclicity of the trace gives $\langle \mathbf{H}, \mathbf{Y} \rangle_{\text{F}} = \text{Tr}(\mathbf{HXR}) = \text{Tr}(\mathbf{RHRX}) = \text{Tr}(\mathbf{UX})$ and, similarly, $\langle \mathbf{K}, \mathbf{Y} \rangle_{\text{F}} = \text{Tr}(\mathbf{VX})$. All the expectations below are finite because $|\text{Tr}(\mathbf{UX})| \leq \|\mathbf{U}\|_{\text{F}} \|\mathbf{X}\|_{\text{F}}$ and ρ has finite second moment.

We first calculate the uncentered second moment. Since $\mathbf{X}$ is symmetric, $\text{Tr}(\mathbf{UX}) = \sum_{i,j=1}^d U_{ij} X_{ij}$ and $\text{Tr}(\mathbf{VX}) = \sum_{k,l=1}^d V_{kl} X_{kl}$. Consequently,

$$\mathbb{E}[\text{Tr}(\mathbf{UX}) \text{Tr}(\mathbf{VX})] = \sum_{i,j,k,l=1}^d U_{ij} V_{kl} \mathbb{E}[X_{ij} X_{kl}].$$

Substituting (4), the term containing $a_{\rho} \delta_{ij} \delta_{kl}$ contributes $a_{\rho} \sum_{i,k=1}^d U_{ii} V_{kk} = a_{\rho} \text{Tr}(\mathbf{U}) \text{Tr}(\mathbf{V})$. The term containing $b_{\rho} \delta_{ik} \delta_{jl}$ contributes $b_{\rho} \sum_{i,j=1}^d U_{ij} V_{ij} = b_{\rho} \text{Tr}(\mathbf{UV}^{\top}) = b_{\rho} \text{Tr}(\mathbf{UV})$, where the last equality uses $\mathbf{V}^{\top} = \mathbf{V}$. The term containing $b_{\rho} \delta_{il} \delta_{jk}$ contributes $b_{\rho} \sum_{i,j=1}^d U_{ij} V_{ji} = b_{\rho} \text{Tr}(\mathbf{UV})$. Therefore, $\mathbb{E}[\text{Tr}(\mathbf{UX}) \text{Tr}(\mathbf{VX})] = a_{\rho} \text{Tr}(\mathbf{U}) \text{Tr}(\mathbf{V}) + 2b_{\rho} \text{Tr}(\mathbf{UV})$.

On the other hand, $\mathbb{E}[\text{Tr}(\mathbf{UX})] = \text{Tr}(\mathbf{U} \mathbb{E}[\mathbf{X}]) = \tau_{\rho} \text{Tr}(\mathbf{U})$, and likewise $\mathbb{E}[\text{Tr}(\mathbf{VX})] = \tau_{\rho} \text{Tr}(\mathbf{V})$. Hence, by the definition of the covariance operator, $\langle \mathbf{H}, \mathbf{C}_{\Sigma}(\mathbf{K}) \rangle_{\text{F}} = 2b_{\rho} \text{Tr}(\mathbf{UV}) + (a_{\rho} - \tau_{\rho}^2) \text{Tr}(\mathbf{U}) \text{Tr}(\mathbf{V})$.

It remains to express this identity in terms of $\mathbf{H}$, $\mathbf{K}$, and Σ . Cyclicity of the trace and $\mathbf{R}^2 = \Sigma$ give $\text{Tr}(\mathbf{U}) = \text{Tr}(\mathbf{RHR}) = \text{Tr}(\Sigma \mathbf{H})$, $\text{Tr}(\mathbf{V}) = \text{Tr}(\Sigma \mathbf{K})$, and $\text{Tr}(\mathbf{UV}) = \text{Tr}(\mathbf{RHR}^2 \mathbf{KR}) = \text{Tr}(\mathbf{H} \Sigma \mathbf{K} \Sigma)$. It follows that $\langle \mathbf{H}, \mathbf{C}_{\Sigma}(\mathbf{K}) \rangle_{\text{F}} = 2b_{\rho} \text{Tr}(\mathbf{H} \Sigma \mathbf{K} \Sigma) + (a_{\rho} - \tau_{\rho}^2) \text{Tr}(\Sigma \mathbf{H}) \text{Tr}(\Sigma \mathbf{K})$.

Now define $\tilde{\mathbf{C}}_{\Sigma} : \text{Sym}(d) \rightarrow \text{Sym}(d)$ by $\tilde{\mathbf{C}}_{\Sigma}(\mathbf{K}) = 2b_{\rho} \Sigma \mathbf{K} \Sigma + (a_{\rho} - \tau_{\rho}^2) \text{Tr}(\Sigma \mathbf{K}) \Sigma$. For every $\mathbf{H} \in \text{Sym}(d)$, we have $\langle \mathbf{H}, \tilde{\mathbf{C}}_{\Sigma}(\mathbf{K}) \rangle_{\text{F}} = 2b_{\rho} \text{Tr}(\mathbf{H} \Sigma \mathbf{K} \Sigma) + (a_{\rho} - \tau_{\rho}^2) \text{Tr}(\Sigma \mathbf{H}) \text{Tr}(\Sigma \mathbf{K})$. Thus $\langle \mathbf{H}, \mathbf{C}_{\Sigma}(\mathbf{K}) \rangle_{\text{F}} = \langle \mathbf{H}, \tilde{\mathbf{C}}_{\Sigma}(\mathbf{K}) \rangle_{\text{F}}$ for every $\mathbf{H} \in \text{Sym}(d)$. By nondegeneracy of the Frobenius inner product, $\mathbf{C}_{\Sigma}(\mathbf{K}) = \tilde{\mathbf{C}}_{\Sigma}(\mathbf{K})$. Since $\mathbf{K} \in \text{Sym}(d)$ was arbitrary, this proves the second identity in (7) and completes the proof. $\square$

Proof of Theorem 5.1. The cone $\mathcal{X} = \mathbb{S}_{>}^d$ is an open convex subset of the finite-dimensional Hilbert space $\mathcal{H} = \text{Sym}(d)$. The absolute-continuity and finite-second-moment assumptions on ρ are precisely those required in Proposition 1, while Lemma 6 verifies Assumption 4 and all three structural hypotheses of that proposition. Hence Proposition 1 applies.

Let $\hat{h} \in \text{Stab}_G(\rho)$ be the element supplied by Proposition 1, and set $\mathbf{A}_{\hat{h}} = \pi(g_1 \hat{h} g_0^{-1})$. Then $\pi(\hat{h}) \in L^{\theta}$ and $\mathbf{A}_{\hat{h}} \in \exp(\mathfrak{p})$. By Lemma 6(ii), $\mathfrak{p} = \{\mathcal{D}_{\mathbf{S}} : \mathbf{S} \in \text{Sym}(d)\}$. Thus there exists $\mathbf{S} \in \text{Sym}(d)$ such that $\mathbf{A}_{\hat{h}} = \exp(\mathcal{D}_{\mathbf{S}})$. Set $\mathbf{P} = \exp(\mathbf{S})$. Since $\mathbf{S}$ is symmetric, $\mathbf{P} \in \mathbb{S}_{>}^d$, and the compatibility of the congruence representation with exponential maps gives $\mathbf{A}_{\hat{h}} = \pi(\mathbf{P})$. Because $b \equiv 0$, the associated orbit-induced map is therefore $T_{\hat{h}}(\mathbf{X}) = \mathbf{PXP}$ for every $\mathbf{X} \in \mathbb{S}_{>}^d$.

We next identify $\mathbf{P}$. By Lemma 7, the mean of μ_i is $\tau_{\rho} \Sigma_i$, where $\tau_{\rho} > 0$. If $\mathbf{X} \sim \mu_0$, then $T_{\hat{h}}(\mathbf{X}) = \mathbf{PXP} \sim \mu_1$. Taking expectations consequently yields $\tau_{\rho} \Sigma_1 = \mathbf{P}(\tau_{\rho} \Sigma_0) \mathbf{P}$. Since $\tau_{\rho} > 0$, it follows that $\mathbf{P} \Sigma_0 \mathbf{P} = \Sigma_1$. Set $\mathbf{R} = \Sigma_0^{1/2} \mathbf{P} \Sigma_0^{1/2}$. Then $\mathbf{R} \in \mathbb{S}_{>}^d$, and

$$\mathbf{R}^2 = \Sigma_0^{1/2} \mathbf{P} \Sigma_0 \mathbf{P} \Sigma_0^{1/2} = \Sigma_0^{1/2} \Sigma_1 \Sigma_0^{1/2} = \mathbf{C}^2.$$

Both $\mathbf{R}$ and $\mathbf{C}$ are positive definite. By uniqueness of the positive-definite square root, $\mathbf{R} = \mathbf{C}$, and hence $\mathbf{P} = \Sigma_0^{-1/2} \mathbf{C} \Sigma_0^{-1/2} = \mathbf{A}^*$. Consequently, $\mathbf{A}_{\hat{h}} = \pi(\mathbf{A}^*)$ and $T_{\hat{h}}(\mathbf{X}) = \mathbf{A}^* \mathbf{X} \mathbf{A}^* = T^*(\mathbf{X})$.

It remains to show that the explicitly defined element $h^* = \mathbf{Q}^*$ can be used in place of $\hat{h}$. The preceding argument gives $\mathbf{A}^* \Sigma_0 \mathbf{A}^* = \Sigma_1$. Since $\mathbf{A}^*$ is symmetric, the definition of $\mathbf{Q}^*$ gives

$$\begin{aligned} \mathbf{Q}^*(\mathbf{Q}^*)^{\top} &= \Sigma_1^{-1/2} \mathbf{A}^* \Sigma_0 \mathbf{A}^* \Sigma_1^{-1/2} = \mathbf{I}_d, \\ g_1 h^* g_0^{-1} &= \Sigma_1^{1/2} \mathbf{Q}^* \Sigma_0^{-1/2} = \mathbf{A}^*. \end{aligned}$$

Thus $\mathbf{Q}^* \in \mathcal{O}(d)$. Spectral invariance of ρ therefore implies $h^* = \mathbf{Q}^* \in \text{Stab}_G(\rho)$. Moreover, $\pi(h^*) \in \pi(\mathcal{O}(d)) = L^\theta$, and $\mathbf{A}_{h^*} = \pi(g_1 h^* g_0^{-1}) = \pi(\mathbf{A}^*) = \mathbf{A}_{\hat{h}}$. In particular, $T_{h^*} = T_{\hat{h}} = T^*$.

The conclusions of Proposition 1 concerning the linear operator, the optimal Monge map, the optimal Kantorovich plan, and the equality of the three optimal values therefore hold with $h = h^*$. To verify the remaining conclusion concerning the orbit problem, observe that for every $\tilde{h} \in \text{Stab}_G(\rho)$, the change of variables $\mathbf{X} = \alpha(g_0)(\mathbf{Z})$ gives

$$\int_{\mathbb{S}_{\geq}^d} c(\alpha(g_0)(\mathbf{Z}), \alpha(g_1 \tilde{h})(\mathbf{Z})) d\rho(\mathbf{Z}) = \int_{\mathbb{S}_{\geq}^d} c(\mathbf{X}, T_{\tilde{h}}(\mathbf{X})) d\mu_0(\mathbf{X}).$$

Since $T_{h^*} = T_{\hat{h}}$, the objective in (LGOP) has the same value at h^* and $\hat{h}$. The element $\hat{h}$ attains the infimum by Proposition 1; therefore h^* also solves (LGOP). This proves that all the conclusions of Proposition 1 hold with $h = h^*$.

It remains to evaluate the common optimal value. Let $\mathbf{Z} \sim \rho$, set $\mathbf{X}_0 = \Sigma_0^{1/2} \mathbf{Z} \Sigma_0^{1/2}$, and let $\mathbf{X}_1 = T^*(\mathbf{X}_0)$. Then $\mathbf{X}_i \sim \mu_i$ for $i = 0, 1$. For each i , cyclicity of the trace and (5), applied with $\mathbf{U} = \mathbf{V} = \Sigma_i$, give $\mathbb{E}[\text{Tr}(\mathbf{X}_i^2)] = (a_\rho + b_\rho) \text{Tr}(\Sigma_i^2) + b_\rho (\text{Tr} \Sigma_i)^2$.

By the definitions of $\mathbf{A}^*$ and $\mathbf{C}$, we have $\mathbf{C} = \Sigma_0^{1/2} \mathbf{A}^* \Sigma_0^{1/2}$. Consequently, cyclicity of the trace gives $\text{Tr}(\mathbf{X}_0 \mathbf{X}_1) = \text{Tr}(\mathbf{Z} \mathbf{C} \mathbf{Z} \mathbf{C}) = \text{Tr}(\mathbf{C} \mathbf{Z} \mathbf{C} \mathbf{Z})$. Since $\mathbf{C}$ is symmetric, another application of (5), now with $\mathbf{U} = \mathbf{V} = \mathbf{C}$, yields $\mathbb{E}[\text{Tr}(\mathbf{X}_0 \mathbf{X}_1)] = (a_\rho + b_\rho) \text{Tr}(\mathbf{C}^2) + b_\rho (\text{Tr} \mathbf{C})^2$. Furthermore, $\mathbf{C}^2 = \Sigma_0^{1/2} \Sigma_1 \Sigma_0^{1/2}$, and hence $\text{Tr}(\mathbf{C}^2) = \text{Tr}(\Sigma_0 \Sigma_1)$.

Since T^* is optimal, its expected transportation cost equals $\mathbb{K}_c(\mu_0, \mu_1)$. Combining the preceding identities, we obtain

$$\begin{aligned} \mathbb{K}_c(\mu_0, \mu_1) &= \mathbb{E}[\|\mathbf{X}_0 - \mathbf{X}_1\|_{\mathbb{F}}^2] \\ &= (a_\rho + b_\rho) (\text{Tr}(\Sigma_0^2) + \text{Tr}(\Sigma_1^2) - 2 \text{Tr}(\Sigma_0 \Sigma_1)) + b_\rho ((\text{Tr} \Sigma_0)^2 + (\text{Tr} \Sigma_1)^2 - 2(\text{Tr} \mathbf{C})^2) \\ &= (a_\rho + b_\rho) \|\Sigma_0 - \Sigma_1\|_{\mathbb{F}}^2 + b_\rho ((\text{Tr} \Sigma_0)^2 + (\text{Tr} \Sigma_1)^2 - 2(\text{Tr} \mathbf{C})^2). \end{aligned}$$

This is precisely (9), and the proof is complete. $\square$

In what follows, we collect the orbit, stabilizer, and second-moment computations for the distributional families summarized in Table 1. Each subsection verifies the hypotheses of Theorem 5.1 and derives the constants a_ρ and b_ρ that enter the optimal transport cost formula (9).

Wishart distributions. The Wishart distribution $\mathcal{W}_d(\Sigma, p)$ with scale matrix $\Sigma \in \mathbb{S}_{\geq}^d$ and degrees of freedom $p > d - 1$ is a probability law on $\mathbb{S}_{\geq}^d$ with density

$$(13) \quad \mathbf{X} \mapsto \frac{\det(\mathbf{X})^{(p-d-1)/2} \exp(-\frac{1}{2} \text{Tr}(\Sigma^{-1} \mathbf{X}))}{2^{pd/2} \kappa_d \det(\Sigma)^{p/2} \Gamma_d(p/2)},$$

with respect to $\mathcal{L}_{\text{Sym}(d)}|_{\mathbb{S}_{\geq}^d}$, where Γ_d denotes the multivariate gamma function. Here, $\kappa_d = 2^{d(d-1)/4}$. When $d = 1$, this reduces to a scaled χ^2 distribution with p degrees of freedom and scale parameter $\Sigma > 0$.

Lemma 15 (Wishart orbit, stabilizer, and second moments). *Let $\rho = \mathcal{W}_d(\mathbf{I}_d, p)$ with $p > d - 1$. Then:*

- (i) $\alpha(\mathbf{A})_{\#} \rho = \mathcal{W}_d(\mathbf{A} \mathbf{A}^\top, p)$ for every $\mathbf{A} \in \text{GL}(d)$,
- (ii) $\text{Stab}_{\text{GL}(d)}(\rho) = \mathcal{O}(d)$,
- (iii) for $d \geq 2$, the constants in Lemma 7 are $a_\rho = p^2$, $b_\rho = p$. For $d = 1$, the identifiable combination is $a_\rho + 2b_\rho = p(p + 2)$.

Proof. (i) This is the standard transformation property of the Wishart distribution; see, for example, [Eaton, 1983, Proposition 8.1] or [Muirhead, 2009, Theorem 3.2.5]. (ii) We next identify the stabilizer. If $\mathbf{Q} \in \mathcal{O}(d)$, then by (i), we have $\alpha(\mathbf{Q})_{\#}\rho = \mathcal{W}_d(\mathbf{I}_d, p) = \rho$. Thus $\mathcal{O}(d) \subseteq \text{Stab}_{\text{GL}(d)}(\rho)$. Conversely, suppose $\mathbf{A} \in \text{Stab}_{\text{GL}(d)}(\rho)$. Then $\mathcal{W}_d(\mathbf{A}\mathbf{A}^\top, p) = \alpha(\mathbf{A})_{\#}\rho = \rho = \mathcal{W}_d(\mathbf{I}_d, p)$. Then, the equality of the two densities implies $\det(\mathbf{A}\mathbf{A}^\top)^{-p/2} \exp(-\frac{1}{2} \text{Tr}((\mathbf{A}\mathbf{A}^\top)^{-1} - \mathbf{I}_d)\mathbf{Y}) = 1$ for almost every $\mathbf{Y} \in \mathbb{S}_>^d$. Since both sides are continuous in $\mathbf{Y}$, the identity holds for all $\mathbf{Y} \in \mathbb{S}_>^d$. The exponent is a linear function of $\mathbf{Y}$. Since the displayed identity holds for every $\mathbf{Y} \in \mathbb{S}_>^d$, this linear function is constant on the open cone $\mathbb{S}_>^d$. A linear function that is constant on a nonempty open set must vanish identically; so $(\mathbf{A}\mathbf{A}^\top)^{-1} - \mathbf{I}_d = 0$. Thus $\mathbf{A}\mathbf{A}^\top = \mathbf{I}_d$, and hence $\mathbf{A} \in \mathcal{O}(d)$. This proves $\text{Stab}_{\text{GL}(d)}(\rho) = \mathcal{O}(d)$.

(iii) It remains to compute the constants a_ρ and b_ρ . We compute them from the two scalar moments $M_1 = \mathbb{E}_{\mathbf{X} \sim \rho}[(\text{Tr}(\mathbf{X}))^2]$ and $M_2 = \mathbb{E}_{\mathbf{X} \sim \rho}[\text{Tr}(\mathbf{X}^2)]$.

Let $Y_{\mathbf{H}} = \text{Tr}(\mathbf{H}\mathbf{X})$, $\mathbf{H} \in \text{Sym}(d)$, where $\mathbf{X} \sim \rho = \mathcal{W}_d(\mathbf{I}_d, p)$. For t in a neighborhood of 0, the Laplace transform of $Y_{\mathbf{H}}$ is $L_{\mathbf{H}}(t) = \mathbb{E}_{\mathbf{X} \sim \rho}[\exp(-tY_{\mathbf{H}})] = \mathbb{E}_{\mathbf{X} \sim \rho}[\exp(-t \text{Tr}(\mathbf{H}\mathbf{X}))] = \det(\mathbf{I}_d + 2t\mathbf{H})^{-p/2}$, where the last equality follows from the characteristic function formula for the Wishart distribution in [Muirhead, 2009, Theorem 3.2.3] by evaluating it at the imaginary argument $it\mathbf{H}$. Since $L_{\mathbf{H}}(t) = \mathbb{E}_{\mathbf{X} \sim \rho}[\exp(-tY_{\mathbf{H}})]$, differentiation at $t = 0$ gives $L'_{\mathbf{H}}(0) = -\mathbb{E}_{\mathbf{X} \sim \rho}[Y_{\mathbf{H}}]$ and $L''_{\mathbf{H}}(0) = \mathbb{E}_{\mathbf{X} \sim \rho}[Y_{\mathbf{H}}^2]$. Equivalently, $L'_{\mathbf{H}}(0) = -\mathbb{E}_{\mathbf{X} \sim \rho}[\text{Tr}(\mathbf{H}\mathbf{X})]$ and $L''_{\mathbf{H}}(0) = \mathbb{E}_{\mathbf{X} \sim \rho}[(\text{Tr}(\mathbf{H}\mathbf{X}))^2]$.

To compute these quantities, we differentiate $\log(L_{\mathbf{H}}(t)) = -\frac{p}{2} \text{Tr}(\log(\mathbf{I}_d + 2t\mathbf{H}))$. Hence $(\log(L_{\mathbf{H}}))'(t) = -p \text{Tr}((\mathbf{I}_d + 2t\mathbf{H})^{-1}\mathbf{H})$, and

$$(\log(L_{\mathbf{H}}))''(t) = 2p \text{Tr}((\mathbf{I}_d + 2t\mathbf{H})^{-1}\mathbf{H}(\mathbf{I}_d + 2t\mathbf{H})^{-1}\mathbf{H}).$$

Evaluating the first and second derivatives of $\log(L_{\mathbf{H}})$ at $t = 0$ yields $(\log(L_{\mathbf{H}}))'(0) = -p \text{Tr}(\mathbf{H})$ and $(\log(L_{\mathbf{H}}))''(0) = 2p \text{Tr}(\mathbf{H}^2)$.

Now we use the identity: $(\log(L_{\mathbf{H}}))''(0) = L''_{\mathbf{H}}(0)/L_{\mathbf{H}}(0) - (L'_{\mathbf{H}}(0)/L_{\mathbf{H}}(0))^2$. Since $L_{\mathbf{H}}(0) = 1$, we have $L''_{\mathbf{H}}(0) = (\log L_{\mathbf{H}})''(0) + ((\log L_{\mathbf{H}})'(0))^2$. Therefore

$$(14) \quad \mathbb{E}_{\mathbf{X} \sim \rho}[(\text{Tr}(\mathbf{H}\mathbf{X}))^2] = 2p \text{Tr}(\mathbf{H}^2) + p^2(\text{Tr}(\mathbf{H}))^2.$$

Taking $\mathbf{H} = \mathbf{I}_d$ gives $M_1 = \mathbb{E}_{\mathbf{X} \sim \rho}[(\text{Tr}(\mathbf{X}))^2] = 2p \text{Tr}(\mathbf{I}_d^2) + p^2(\text{Tr}(\mathbf{I}_d))^2 = 2pd + p^2d^2$.

To compute M_2 , let

$$\mathcal{ON} = \{\mathbf{E}_{aa} : 1 \leq a \leq d\} \cup \left\{ \frac{\mathbf{E}_{ab} + \mathbf{E}_{ba}}{\sqrt{2}} : 1 \leq a < b \leq d \right\}$$

be the standard Frobenius-orthonormal basis of $\text{Sym}(d)$. Since $\text{Tr}(\mathbf{X}^2) = \|\mathbf{X}\|_{\text{F}}^2 = \sum_{\mathbf{H} \in \mathcal{ON}} (\text{Tr}(\mathbf{H}\mathbf{X}))^2$, where the sum is over this orthonormal basis, and thus we have

$$\begin{aligned} M_2 &= \mathbb{E}_{\mathbf{X} \sim \rho}[\text{Tr}(\mathbf{X}^2)] \\ &= \mathbb{E}_{\mathbf{X} \sim \rho} \left[\sum_{a=1}^d (\text{Tr}(\mathbf{E}_{aa}\mathbf{X}))^2 + \sum_{1 \leq a < b \leq d} \left(\text{Tr} \left(\frac{\mathbf{E}_{ab} + \mathbf{E}_{ba}}{\sqrt{2}} \mathbf{X} \right) \right)^2 \right] \\ &= \sum_{a=1}^d (p^2(\text{Tr}(\mathbf{E}_{aa}))^2 + 2p \text{Tr}(\mathbf{E}_{aa}^2)) \\ &\quad + \sum_{1 \leq a < b \leq d} \left(p^2 \left(\text{Tr} \left(\frac{\mathbf{E}_{ab} + \mathbf{E}_{ba}}{\sqrt{2}} \right) \right)^2 + 2p \text{Tr} \left(\left(\frac{\mathbf{E}_{ab} + \mathbf{E}_{ba}}{\sqrt{2}} \right)^2 \right) \right) \\ &= d(p^2 + 2p) + \frac{d(d-1)}{2} 2p = p^2d + pd(d+1). \end{aligned}$$

where the third equality follows by applying (14) to each element of the Frobenius orthonormal basis above.

For $d \geq 2$, Lemma 7 gives $b_\rho = \frac{M_2 - \frac{1}{d}M_1}{d^2 + d - 2}$. Substituting the values of M_1 and M_2 ,

$$b_\rho = \frac{p^2d + pd(d+1) - \frac{1}{d}(p^2d^2 + 2pd)}{d^2 + d - 2} = \frac{p(d^2 + d - 2)}{d^2 + d - 2} = p.$$

Then $a_\rho = \frac{M_1 - 2b_\rho d}{d^2} = \frac{p^2d^2 + 2pd - 2pd}{d^2} = p^2$. For $d = 1$, we can only identify the combination $a_\rho + 2b_\rho = \mathbb{E}_\rho[X^2] = p^2 + 2p = p(p+2)$, as claimed. $\square$

Corollary 4 (Optimal transport between Wishart distributions). *For $d \geq 2$, let $\mu_i = \mathcal{W}_d(\Sigma_i, p)$ with $\Sigma_i \in \mathbb{S}_\prec^d$, $p > d - 1$, $i = 0, 1$, and let $c(\mathbf{X}, \mathbf{Y}) = \|\mathbf{X} - \mathbf{Y}\|_F^2$. Set $\mathbf{C} = (\Sigma_0^{1/2}\Sigma_1\Sigma_0^{1/2})^{1/2}$. Then:*

- (a) *the unique optimal Monge map from μ_0 to μ_1 , up to μ_0 -almost-everywhere equality, is T^* , where T^* is as defined in (8),*
- (b) *$(\text{id}_{\mathbb{S}_\prec^d}, T^*)_{\#}\mu_0$ is the unique optimal Kantorovich plan,*
- (c) $\mathbb{K}_c(\mu_0, \mu_1) = p(p+1)\|\Sigma_0 - \Sigma_1\|_F^2 + p((\text{Tr}(\Sigma_0))^2 + (\text{Tr}(\Sigma_1))^2 - 2(\text{Tr}(\mathbf{C}))^2)$,
- (d) $\mathbb{M}_c(\mu_0, \mu_1) = \mathbb{K}_c(\mu_0, \mu_1)$.

Proof. By Lemma 15, $\rho = \mathcal{W}_d(\mathbf{I}_d, p)$ satisfies the hypotheses of Theorem 5.1 for $p > d - 1$. Substituting the corresponding values of a_ρ and b_ρ from Lemma 15 into (9) yields the stated value of $\mathbb{K}_c$. The remaining claims follow directly from Theorem 5.1. $\square$

Inverse-Wishart distributions. The inverse-Wishart distribution $\mathcal{IW}_d(\Psi, p)$ with scale matrix $\Psi \in \mathbb{S}_\prec^d$ and degrees of freedom $p > d - 1$ is a probability law on $\mathbb{S}_\prec^d$ with density

$$(15) \quad \mathbf{X} \mapsto \frac{\det(\Psi)^{p/2}}{2^{pd/2}\kappa_d\Gamma_d(p/2)} \det(\mathbf{X})^{-(p+d+1)/2} \exp\left(-\frac{1}{2}\text{Tr}(\Psi\mathbf{X}^{-1})\right),$$

with respect to $\mathcal{L}_{\text{Sym}(d)}|_{\mathbb{S}_\prec^d}$, where $\kappa_d = 2^{d(d-1)/4}$.

Its mean exists for $p > d + 1$, and the second moments of its matrix entries exist for $p > d + 3$.

Lemma 16 (Inverse-Wishart orbit, stabilizer, and second moments). *Let $\rho = \mathcal{IW}_d(\mathbf{I}_d, p)$ with $p > d + 3$, and set $m = p - d$. Then:*

- (i) *For every $\mathbf{A} \in \text{GL}(d)$, $\alpha(\mathbf{A})_{\#}\rho = \mathcal{IW}_d(\mathbf{A}\mathbf{A}^\top, p)$.*
- (ii) $\text{Stab}_{\text{GL}(d)}(\rho) = \mathcal{O}(d)$.
- (iii) *For $d \geq 2$, the constants in Lemma 7 are $a_\rho = (m-2)/(m(m-1)(m-3))$ and $b_\rho = (m(m-1)(m-3))^{-1}$. If $d = 1$, only the combination $a_\rho + 2b_\rho$ is identifiable, and in this case $a_\rho + 2b_\rho = ((p-2)(p-4))^{-1}$.*

Proof. Parts (i) and (ii) are proved by the same arguments as in the proof of Lemma 15. We therefore omit the proofs of those assertions.

(iii) The condition $p > d + 3$ is equivalent to $m = p - d > 3$, which is the condition under which the second moments of the inverse-Wishart entries are finite. Then, we have $\mathbb{E}_{\mathbf{X} \sim \rho}[\mathbf{X}] = (m-1)^{-1}\mathbf{I}_d$, so $\mathbb{E}_{\mathbf{X} \sim \rho}[X_{ab}] = \delta_{ab}/(m-1)$. Moreover, the entrywise covariance is

$$\text{Cov}(X_{ab}, X_{cd}) = \frac{2\delta_{ab}\delta_{cd} + (m-1)(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})}{m(m-1)^2(m-3)};$$

see, for example, [Mardia et al., 1979, §3.8]. Using $\mathbb{E}_{\mathbf{X} \sim \rho}[X_{ab}X_{cd}] = \mathbb{E}_{\mathbf{X} \sim \rho}[X_{ab}]\mathbb{E}_{\mathbf{X} \sim \rho}[X_{cd}] + \text{Cov}(X_{ab}, X_{cd})$, we get

$$\mathbb{E}_{\mathbf{X} \sim \rho}[X_{ab}X_{cd}] = \frac{\delta_{ab}\delta_{cd}}{(m-1)^2} + \frac{2\delta_{ab}\delta_{cd} + (m-1)(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})}{m(m-1)^2(m-3)}$$

$$= \left(\frac{1}{(m-1)^2} + \frac{2}{m(m-1)^2(m-3)} \right) \delta_{ab}\delta_{cd} + \frac{1}{m(m-1)(m-3)} (\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}).$$

A routine calculation shows that the coefficient of $\delta_{ab}\delta_{cd}$ simplifies as $\frac{m-2}{m(m-1)(m-3)}$. Therefore

$$\mathbb{E}[X_{ab}X_{cd}] = \frac{m-2}{m(m-1)(m-3)} \delta_{ab}\delta_{cd} + \frac{1}{m(m-1)(m-3)} (\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}).$$

Comparing this with the tensor form in Lemma 7 yields, for $d \geq 2$,

$$a_\rho = \frac{m-2}{m(m-1)(m-3)}, \quad b_\rho = \frac{1}{m(m-1)(m-3)}.$$

For $d = 1$, the coefficients a_ρ and b_ρ are not separately identifiable. Only the combination $a_\rho + 2b_\rho$ is meaningful. Substituting $d = 1$, so that $m = p - 1$, gives

$$a_\rho + 2b_\rho = \frac{m-2+2}{m(m-1)(m-3)} = \frac{1}{(m-1)(m-3)} = \frac{1}{(p-2)(p-4)}.$$

This completes the proof. $\square$

Corollary 5 (Optimal transport between inverse-Wishart distributions). *For $d \geq 2$ and $i = 0, 1$, let $\mu_i = \mathcal{IW}_d(\Sigma_i, p)$ with $\Sigma_i \in \mathbb{S}_{\succ}^d$, $p > d + 3$, and let $c(\mathbf{X}, \mathbf{Y}) = \|\mathbf{X} - \mathbf{Y}\|_{\mathbb{F}}^2$. Set $m = p - d$ and $\mathbf{C} = (\Sigma_0^{1/2} \Sigma_1 \Sigma_0^{1/2})^{1/2}$. Then,*

- (a) *the unique optimal Monge map from μ_0 to μ_1 , up to μ_0 -almost-everywhere equality, is T^* , where T^* is as defined in (8),*
- (b) *$(\text{id}_{\mathbb{S}_{\succ}^d}, T^*)_{\#} \mu_0$ is the unique optimal Kantorovich plan,*
- (c) *$\mathbb{K}_c(\mu_0, \mu_1) = (m(m-3))^{-1} \|\Sigma_0 - \Sigma_1\|_{\mathbb{F}}^2 + \frac{1}{m(m-1)(m-3)} ((\text{Tr } \Sigma_0)^2 + (\text{Tr } \Sigma_1)^2 - 2(\text{Tr } \mathbf{C})^2),$*
- (d) *$\mathbb{M}_c(\mu_0, \mu_1) = \mathbb{K}_c(\mu_0, \mu_1).$*

Proof. By Lemma 16, $\rho = \mathcal{IW}_d(\mathbf{I}_d, p)$ satisfies the hypotheses of Theorem 5.1 for $p > d + 3$. Substituting the corresponding values of a_ρ and b_ρ from Lemma 16 into (9) yields the stated value of $\mathbb{K}_c$. The remaining claims follow directly from Theorem 5.1. $\square$

Matrix beta type II distributions. The matrix beta type II distribution $\mathcal{MB}_d^{\text{II}}(q_1, q_2, \Sigma)$ with scale matrix $\Sigma \in \mathbb{S}^* \succ^d$ and shape parameters $q_1 > (d-1)/2$ and $q_2 > (d-1)/2$ is a probability law on $\mathbb{S}_{\succ}^d$ with density

$$(16) \quad \mathbf{X} \mapsto \frac{\det(\Sigma)^{q_2}}{\kappa_d \mathbf{B}_d(q_1, q_2)} \det(\mathbf{X})^{q_1 - (d+1)/2} \det(\Sigma + \mathbf{X})^{-(q_1 + q_2)},$$

with respect to $\mathcal{L}_{\text{Sym}(d)}|_{\mathbb{S}_{\succ}^d}$, where $\kappa_d = 2^{d(d-1)/4}$ and $\mathbf{B}_d$ denotes the multivariate beta function. When $q_1 > (d-1)/2$ and $2q_2 > d + 3$, the entry-wise second moments are finite; see [Gupta and Nagar, 2018, Theorem 5.3.20].

Lemma 17 (Matrix beta II orbit, stabilizer, and second moments). *Let $\rho = \mathcal{MB}_d^{\text{II}}(q_1, q_2, \mathbf{I}_d)$ with $q_1 > (d-1)/2$ and $2q_2 > d + 3$, and set $q = 2q_2 - d$. Then:*

- (i) $\alpha(\mathbf{A})_{\#} \rho = \mathcal{MB}_d^{\text{II}}(q_1, q_2, \mathbf{A}\mathbf{A}^\top)$ for every $\mathbf{A} \in \text{GL}(d)$;
- (ii) $\text{Stab}_{\text{GL}(d)}(\rho) = \mathcal{O}(d)$;
- (iii) for $d \geq 2$, the constants in Lemma 7 are

$$a_\rho = \frac{4q_1(q_1(q-2)+1)}{q(q-1)(q-3)} \text{ and } b_\rho = \frac{2q_1(2q_1+q-1)}{q(q-1)(q-3)}.$$

Proof. (i) follows by [Gupta and Nagar, 2018, Theorem 5.2.2].

(ii) If $\alpha(\mathbf{A})_{\#}\rho = \rho$, then by part (i), $\mathcal{MB}_d^{\text{II}}(q_1, q_2, \mathbf{A}\mathbf{A}^\top) = \mathcal{MB}_d^{\text{II}}(q_1, q_2, \mathbf{I}_d)$. Since the scale parameter is uniquely determined by the density in (16), it follows that $\mathbf{A}\mathbf{A}^\top = \mathbf{I}_d$. Conversely, if $\mathbf{Q} \in \mathcal{O}(d)$, then part (i) gives

$$\alpha(\mathbf{Q})_{\#}\rho = \mathcal{MB}_d^{\text{II}}(q_1, q_2, \mathbf{Q}\mathbf{Q}^\top) = \mathcal{MB}_d^{\text{II}}(q_1, q_2, \mathbf{I}_d) = \rho.$$

(iii) By [Gupta and Nagar, 2018, Theorem 5.2.5], if $\mathbf{W}_1 \sim \mathcal{W}_d(\mathbf{I}_d, 2q_1)$ and $\mathbf{W}_2 \sim \mathcal{W}_d(\mathbf{I}_d, 2q_2)$ are independent, then $\mathbf{X} = \mathbf{W}_2^{-1/2}\mathbf{W}_1\mathbf{W}_2^{-1/2}$ satisfies $\mathbf{X} \sim \mathcal{MB}_d^{\text{II}}(q_1, q_2, \mathbf{I}_d)$.

To compute the second moments of $\mathbf{X}$, we condition on $\mathbf{W}_2$. Given $\mathbf{W}_2$, the matrix $\mathbf{W}_2^{-1/2}$ is deterministic and $\mathbf{W}_1$ remains distributed as $\mathcal{W}_d(\mathbf{I}_d, 2q_1)$, because $\mathbf{W}_1$ and $\mathbf{W}_2$ are independent. Therefore, by the congruence transformation rule for the Wishart distribution,

$$\mathbf{X} \mid \mathbf{W}_2 = \mathbf{W}_2^{-1/2}\mathbf{W}_1\mathbf{W}_2^{-1/2} \sim \mathcal{W}_d(\mathbf{W}_2^{-1}, 2q_1).$$

Here the scale matrix $\mathbf{W}_2^{-1}$ is understood conditionally on $\mathbf{W}_2$.

Now we apply the Wishart second-moment formula to $\mathbf{X} \mid \mathbf{W}_2$. If $\mathbf{Y} \sim \mathcal{W}_d(\boldsymbol{\Sigma}, p)$, then by Lemma 15(iii)

$$\mathbb{E}_{\mathbf{Y} \sim \mathcal{W}_d(\boldsymbol{\Sigma}, p)}[Y_{ab}Y_{cd}] = p^2\Sigma_{ab}\Sigma_{cd} + p(\Sigma_{ac}\Sigma_{bd} + \Sigma_{ad}\Sigma_{bc}).$$

Taking here $p = 2q_1$ and $\boldsymbol{\Sigma} = \mathbf{W}_2^{-1}$ yields

$$(17) \quad \mathbb{E}_{\mathbf{X} \sim \mathcal{MB}_d^{\text{II}}(q_1, q_2, \mathbf{I}_d)}[X_{ij}X_{kl} \mid \mathbf{W}_2] = (2q_1)^2(W_2^{-1})_{ij}(W_2^{-1})_{kl} + 2q_1((W_2^{-1})_{ik}(W_2^{-1})_{jl} + (W_2^{-1})_{il}(W_2^{-1})_{jk}).$$

Next, since $\mathbf{W}_2^{-1} \sim \mathcal{IW}_d(\mathbf{I}_d, 2q_2)$, we may use the inverse-Wishart second-moment tensor from Lemma 16. Writing

$$\mathbb{E}_{\mathbf{W}_2^{-1} \sim \mathcal{IW}_d(\mathbf{I}_d, 2q_2)}[(W_2^{-1})_{ij}(W_2^{-1})_{kl}] = \alpha_{\text{IW}}\delta_{ij}\delta_{kl} + \beta_{\text{IW}}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

with

$$\alpha_{\text{IW}} = \frac{q-2}{q(q-1)(q-3)}, \quad \beta_{\text{IW}} = \frac{1}{q(q-1)(q-3)}, \quad q = 2q_2 - d,$$

we take expectations in (17).

For the first term,

$$\mathbb{E}_{\mathbf{W}_2^{-1} \sim \mathcal{IW}_d(\mathbf{I}_d, 2q_2)}[(W_2^{-1})_{ij}(W_2^{-1})_{kl}] = \alpha_{\text{IW}}\delta_{ij}\delta_{kl} + \beta_{\text{IW}}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

For the second term, apply the same formula twice:

$$\mathbb{E}_{\mathbf{W}_2^{-1} \sim \mathcal{IW}_d(\mathbf{I}_d, 2q_2)}[(W_2^{-1})_{ik}(W_2^{-1})_{jl}] = \alpha_{\text{IW}}\delta_{ik}\delta_{jl} + \beta_{\text{IW}}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}),$$

$$\mathbb{E}_{\mathbf{W}_2^{-1} \sim \mathcal{IW}_d(\mathbf{I}_d, 2q_2)}[(W_2^{-1})_{il}(W_2^{-1})_{jk}] = \alpha_{\text{IW}}\delta_{il}\delta_{jk} + \beta_{\text{IW}}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}).$$

Summing these two identities gives

$$\begin{aligned} & \mathbb{E}_{\mathbf{W}_2^{-1} \sim \mathcal{IW}_d(\mathbf{I}_d, 2q_2)}[(W_2^{-1})_{ik}(W_2^{-1})_{jl}] + \mathbb{E}_{\mathbf{W}_2^{-1} \sim \mathcal{IW}_d(\mathbf{I}_d, 2q_2)}[(W_2^{-1})_{il}(W_2^{-1})_{jk}] \\ &= 2\beta_{\text{IW}}\delta_{ij}\delta_{kl} + (\alpha_{\text{IW}} + \beta_{\text{IW}})(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \end{aligned}$$

Substituting back into (17), we obtain $\mathbb{E}[X_{ij}X_{kl}] = a_\rho\delta_{ij}\delta_{kl} + b_\rho(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, where

$$a_\rho = (2q_1)^2\alpha_{\text{IW}} + 2q_1 \cdot 2\beta_{\text{IW}}, \quad b_\rho = (2q_1)^2\beta_{\text{IW}} + 2q_1(\alpha_{\text{IW}} + \beta_{\text{IW}}).$$

Finally, inserting the values of α_{IW} and β_{IW} yields

$$a_\rho = \frac{4q_1^2(q-2) + 4q_1}{q(q-1)(q-3)} = \frac{4q_1(q_1(q-2) + 1)}{q(q-1)(q-3)},$$

$$b_\rho = \frac{4q_1^2 + 2q_1(q-1)}{q(q-1)(q-3)} = \frac{2q_1(2q_1 + q - 1)}{q(q-1)(q-3)}.$$

□

Corollary 6 (Optimal transport between matrix beta II distributions). *For $d \geq 2$ and $i = 0, 1$ let $\mu_i = \mathcal{MB}_d^{\text{II}}(q_1, q_2, \Sigma_i)$ with $\Sigma_i \in \mathbb{S}_>^d$, $q_1 > (d-1)/2$, $2q_2 > d+3$, and let $c(\mathbf{X}, \mathbf{Y}) = \|\mathbf{X} - \mathbf{Y}\|_{\text{F}}^2$. Set $q = 2q_2 - d$ and $\mathbf{C} = (\Sigma_0^{1/2} \Sigma_1 \Sigma_0^{1/2})^{1/2}$. Then,*

- (a) *the unique optimal Monge map from μ_0 to μ_1 , up to μ_0 -almost-everywhere equality, is T^* , where T^* is as defined in (8),*
- (b) *$(\text{id}_{\mathbb{S}_>^d}, T^*)_{\#} \mu_0$ is the unique optimal Kantorovich plan,*
- (c)

$$\begin{aligned} \mathbb{K}_c(\mu_0, \mu_1) &= \frac{2q_1(2q_1q - 2q_1 + q + 1)}{q(q-1)(q-3)} \|\Sigma_0 - \Sigma_1\|_{\text{F}}^2 + \\ &\quad \frac{2q_1(2q_1 + q - 1)}{q(q-1)(q-3)} ((\text{Tr } \Sigma_0)^2 + (\text{Tr } \Sigma_1)^2 - 2(\text{Tr } \mathbf{C})^2), \end{aligned}$$

- (d) $\mathbb{M}_c(\mu_0, \mu_1) = \mathbb{K}_c(\mu_0, \mu_1)$.

Proof. By Lemma 17, $\rho = \mathcal{MB}_d^{\text{II}}(q_1, q_2, \mathbf{I}_d)$ satisfies the hypotheses of Theorem 5.1 for $q_1 > (d-1)/2$ and $2q_2 > d+3$. Substituting the corresponding values of a_ρ and b_ρ from Lemma 17 into (9) yields the stated value of $\mathbb{K}_c$. The remaining claims follow directly from Theorem 5.1. □

C.3. Multivariate normal sinh–arcsinh distributions.

Lemma 18. *The following statements hold.*

- (i) *The group G_Ω is a d -dimensional Lie group, and α defines a smooth left action of G_Ω on $\mathbb{R}^d$. Moreover, $(G_\Omega)_{\#} \rho = \{\mu_\epsilon : \epsilon \in \mathbb{R}^d\}$.*
- (ii) *$\text{Stab}_{G_\Omega}(\rho) = \{(\mathbf{0}_d, \mathbf{Q}) : \mathbf{Q} \in \mathfrak{B}_\Omega\}$.*
- (iii) *Every orbit law μ_ϵ has density (10) and belongs to $\mathcal{P}_2(\mathbb{R}^d)$.*

Proof of Lemma 18. (i) The set $\mathfrak{B}_\Omega$ is a subgroup of the finite group $\mathfrak{B}_d$ and is therefore a zero-dimensional Lie group with the discrete topology. Consequently, $G_\Omega = \mathbb{R}^d \rtimes \mathfrak{B}_\Omega$ is the finite disjoint union of the components $\mathbb{R}^d \times \{\mathbf{Q}\}$, with $\mathbf{Q} \in \mathfrak{B}_\Omega$, each of which is diffeomorphic to $\mathbb{R}^d$. Multiplication and inversion are smooth on every component, so G_Ω is a Lie group of dimension d .

Every signed permutation matrix commutes with a componentwise odd function. Hence, for every $\mathbf{Q} \in \mathfrak{B}_d$ and $\mathbf{x} \in \mathbb{R}^d$, $\text{arsinh}(\mathbf{Q}\mathbf{x}) = \mathbf{Q} \text{arsinh}(\mathbf{x})$ and $\sinh(\mathbf{Q}\mathbf{x}) = \mathbf{Q} \sinh(\mathbf{x})$. Let $g = (\epsilon, \mathbf{Q})$ and $\tilde{g} = (\eta, \mathbf{P})$ belong to G_Ω . Since $\text{arsinh}(\sinh(\mathbf{z})) = \mathbf{z}$, we have $\alpha(g)(\alpha(\tilde{g})(\mathbf{x})) = \sinh(\mathbf{Q}\mathbf{P} \text{arsinh}(\mathbf{x}) - \mathbf{Q}\eta - \epsilon) = \alpha(g\tilde{g})(\mathbf{x})$, while $\alpha(e) = \text{id}_{\mathbb{R}^d}$. Thus α defines a left action. For each fixed $\mathbf{Q}$, the evaluation map is a composition of smooth maps in $(\epsilon, \mathbf{x})$; since $\mathfrak{B}_\Omega$ is discrete, the action is smooth on $G_\Omega \times \mathbb{R}^d$.

Using $\text{arsinh}(\mathbf{Q}\mathbf{x}) = \mathbf{Q} \text{arsinh}(\mathbf{x})$ for $\mathbf{Q} \in \mathfrak{B}_\Omega$, we obtain

$$\alpha((\epsilon, \mathbf{Q}))(\mathbf{x}) = \sinh(\mathbf{Q} \text{arsinh}(\mathbf{x}) - \epsilon) = \sinh(\text{arsinh}(\mathbf{Q}\mathbf{x}) - \epsilon) = T_\epsilon(\mathbf{Q}\mathbf{x}).$$

If $\mathbf{Q} \in \mathfrak{B}_\Omega$ and $\mathbf{Z} \sim \mathcal{N}(\mathbf{0}_d, \Omega)$, then $\mathbf{Q}\mathbf{Z} \sim \mathcal{N}(\mathbf{0}_d, \mathbf{Q}\Omega\mathbf{Q}^\top) = \rho$. Therefore, $\alpha((\epsilon, \mathbf{Q}))_{\#} \rho = (T_\epsilon)_{\#}((\mathbf{Q})_{\#} \rho) = (T_\epsilon)_{\#} \rho = \mu_\epsilon$. Conversely, $\mu_\epsilon = \alpha((\epsilon, \mathbf{I}_d))_{\#} \rho$. This proves (i).

(ii) If $\mathbf{Q} \in \mathfrak{B}_\Omega$, then $\alpha((\mathbf{0}_d, \mathbf{Q}))(\mathbf{x}) = \mathbf{Q}\mathbf{x}$, and the preceding Gaussian invariance gives $(\mathbf{0}_d, \mathbf{Q}) \in \text{Stab}_{G_\Omega}(\rho)$.

Conversely, suppose that $(\epsilon, Q) \in \text{Stab}_{G_\Omega}(\rho)$, let $Z \sim \rho$, and set $Y = \alpha((\epsilon, Q))(Z)$. Then $Y \sim \rho$ and $\text{arsinh}(Y) = Q \text{arsinh}(Z) - \epsilon$. The vector $\text{arsinh}(Z)$ is integrable because $|\text{arsinh}(z)| \leq |z|$. Each marginal of Z is centered and symmetric, so oddness of arsinh gives $\mathbb{E}[\text{arsinh}(Z)] = \mathbf{0}_d$. The same identity holds for Y , because Y and Z have the same law. Taking expectations in the preceding relation gives $\epsilon = \mathbf{0}_d$. Since the second component of every element of G_Ω belongs to $\mathfrak{B}_\Omega$, this proves $\text{Stab}_{G_\Omega}(\rho) = \{(\mathbf{0}_d, Q) : Q \in \mathfrak{B}_\Omega\}$.

(iii) The map T_ϵ is a smooth diffeomorphism with inverse $u_\epsilon = T_{-\epsilon}$. The Jacobian matrix of u_ϵ at x is diagonal, and its j th diagonal entry is $\cosh(\epsilon_j + \text{arsinh}(x_j)) / \sqrt{1 + x_j^2}$. Applying the change-of-variables formula to the density of $\mathcal{N}(\mathbf{0}_d, \Omega)$ gives (10).

The hyperbolic subtraction formula gives $T_\epsilon(x)_j = x_j \cosh(\epsilon_j) - \sqrt{1 + x_j^2} \sinh(\epsilon_j)$. Since $\sqrt{1 + x_j^2} \leq 1 + |x_j|$, there exists a finite constant C_ϵ such that $\|T_\epsilon(x)\|^2 \leq C_\epsilon(1 + \|x\|^2)$ for every $x \in \mathbb{R}^d$. Because the Gaussian reference law belongs to $\mathcal{P}_2(\mathbb{R}^d)$, it follows that $\mu_\epsilon \in \mathcal{P}_2(\mathbb{R}^d)$. $\square$

Lemma 19 (Cartan verification for the multivariate sinh-arcsinh action). *Define $\theta : G_\Omega \rightarrow G_\Omega$ by $\theta(\epsilon, Q) = (-\epsilon, Q)$. Then the following statements hold.*

- (i) *The map θ is a Lie group involution, and $G_\Omega^\theta = \{(\mathbf{0}_d, Q) : Q \in \mathfrak{B}_\Omega\}$ and $G_\Omega^\theta = \text{Stab}_{G_\Omega}(\rho)$.*
- (ii) *Under the identification $\text{Lie}(G_\Omega) \simeq \mathbb{R}^d$, the eigenspaces of $d\theta_e$ are $\mathfrak{k} = \{\mathbf{0}_d\}$ and $\mathfrak{p} = \mathbb{R}^d$. Moreover, $\exp(S) = (S, I_d)$ for every $S \in \mathfrak{p}$, and (G_Ω, θ) admits a global Cartan decomposition in the sense of Definition 6.*
- (iii) *For every $S \in \mathfrak{p}$ and $x \in \mathbb{R}^d$, $D_x \alpha(\exp(S))$ is self-adjoint on $\mathbb{R}^d$.*

Consequently, together with Lemma 18, all the hypotheses of Theorem 4.2 are satisfied.

Proof of Lemma 19. (i) Let $g = (\epsilon, Q)$ and $\tilde{g} = (\eta, P)$. The group law gives $\theta(g\tilde{g}) = (-\epsilon - Q\eta, QP) = \theta(g)\theta(\tilde{g})$. Moreover, $\theta^2 = \text{id}_{G_\Omega}$, and θ is smooth on each connected component. Hence θ is a Lie group involution. Its fixed-point subgroup is $G_\Omega^\theta = \{(\mathbf{0}_d, Q) : Q \in \mathfrak{B}_\Omega\}$, which equals $\text{Stab}_{G_\Omega}(\rho)$ by Lemma 18(ii).

(ii) A neighborhood of $e = (\mathbf{0}_d, I_d)$ is contained in the identity component $\mathbb{R}^d \times \{I_d\}$. Its tangent space at e is identified with $\mathbb{R}^d$, and under this identification $d\theta_e(S) = -S$. Therefore, $\mathfrak{k} = \{\mathbf{0}_d\}$ and $\mathfrak{p} = \mathbb{R}^d$. For $S \in \mathfrak{p}$, the curve $t \mapsto (tS, I_d)$ is the one-parameter subgroup with initial velocity S , so $\exp(S) = (S, I_d)$.

Let $\Psi : G_\Omega^\theta \times \mathfrak{p} \rightarrow G_\Omega$ be the Cartan map. The group law gives $\Psi((\mathbf{0}_d, Q), S) = (QS, Q)$. This map is bijective, with inverse $(\epsilon, Q) \mapsto ((\mathbf{0}_d, Q), Q^{-1}\epsilon)$. Both maps are smooth on every connected component. Hence Ψ is a diffeomorphism and (G_Ω, θ) admits a global Cartan decomposition.

(iii) Fix $S \in \mathfrak{p}$ and $x \in \mathbb{R}^d$. By Lemma 19(ii), $\exp(S) = (S, I_d)$. Hence, by the definition of α , $\alpha(\exp(S))(x)_j = \sinh(\text{arsinh}(x_j) - S_j)$ for $j = 1, \dots, d$. Using the hyperbolic subtraction identity $\sinh(a - b) = \sinh(a) \cosh(b) - \cosh(a) \sinh(b)$, together with $\sinh(\text{arsinh}(x_j)) = x_j$ and $\cosh(\text{arsinh}(x_j)) = \sqrt{1 + x_j^2}$, we obtain $\alpha(\exp(S))(x)_j = x_j \cosh(S_j) - \sqrt{1 + x_j^2} \sinh(S_j)$. Differentiating coordinatewise shows that $D_x \alpha(\exp(S))$ is diagonal, with j th diagonal entry $\cosh(S_j) - \frac{x_j}{\sqrt{1 + x_j^2}} \sinh(S_j)$. Hence $D_x \alpha(\exp(S))$ is self-adjoint.

By Lemma 18(i) and (iii), the action α is smooth and the reference and orbit laws satisfy the required absolute-continuity and second-moment conditions. Parts (i) and (ii) verify the stabilizer condition and global Cartan decomposition, while part (iii) verifies the differential-symmetry condition. Thus all the hypotheses of Theorem 4.2 are satisfied. $\square$

Proof of Corollary 2. By Lemmas 18 and 19, all the hypotheses of Theorem 4.2 are satisfied. Moreover, Lemma 18(i) gives $\mu_i = \mu_{\epsilon_i}$ for $i = 0, 1$.

Since $\mathfrak{B}_\Omega$ is a subgroup and $\mathbf{Q}_0, \mathbf{Q}_1 \in \mathfrak{B}_\Omega$, we have $\mathbf{Q}_1^{-1}\mathbf{Q}_0 \in \mathfrak{B}_\Omega$. Hence, Lemma 19(i) implies that $h^* = (\mathbf{0}_d, \mathbf{Q}_1^{-1}\mathbf{Q}_0)$ belongs to $G_\Omega^\theta = \text{Stab}_{G_\Omega}(\rho)$. Using $g_0^{-1} = (-\mathbf{Q}_0^{-1}\epsilon_0, \mathbf{Q}_0^{-1})$ and the group law, we obtain $g_1 h^* = (\epsilon_1, \mathbf{Q}_0)$ and $g_1 h^* g_0^{-1} = (\epsilon_1 - \epsilon_0, \mathbf{I}_d) = (\Delta, \mathbf{I}_d)$. By Lemma 19(ii), $\exp(\Delta) = (\Delta, \mathbf{I}_d)$, and therefore $g_1 h^* g_0^{-1} = \exp(\Delta)$. Thus h^* realizes the Cartan absorption used in the proof of Theorem 4.2, and the conclusions of that theorem hold with $h = h^*$.

By the definition of T_h in Theorem 4.2, $T_{h^*}(\mathbf{x}) = \alpha(g_1 h^* g_0^{-1})(\mathbf{x}) = \alpha((\Delta, \mathbf{I}_d))(\mathbf{x}) = \sinh(\text{arsinh}(\mathbf{x}) - \Delta)$. Thus $T_{h^*} = T^*$.

It remains to evaluate the common optimal value. Let $\mathbf{Z} \sim \rho = \mathcal{N}(\mathbf{0}_d, \Omega)$, and recall that $T_\epsilon(\mathbf{x}) = \sinh(\text{arsinh}(\mathbf{x}) - \epsilon)$. Because every $\mathbf{Q} \in \mathfrak{B}_\Omega$ is a signed permutation matrix and arsinh is odd and applied componentwise, we have $\text{arsinh}(\mathbf{Q}\mathbf{x}) = \mathbf{Q} \text{arsinh}(\mathbf{x})$. Consequently, $\alpha((\epsilon, \mathbf{Q}))(\mathbf{x}) = T_\epsilon(\mathbf{Q}\mathbf{x})$.

Since h^* solves (LGOP), its optimal value is $\int_{\mathbb{R}^d} c(\alpha(g_0)(z), \alpha(g_1 h^*)(z)) d\rho(z)$. Let $\mathbf{Z} \sim \rho$. Since $g_0 = (\epsilon_0, \mathbf{Q}_0)$ and $g_1 h^* = (\epsilon_1, \mathbf{Q}_0)$, and since $\alpha((\epsilon, \mathbf{Q}))(\mathbf{x}) = T_\epsilon(\mathbf{Q}\mathbf{x})$, the preceding optimal value is $\mathbb{E}[\|T_{\epsilon_0}(\mathbf{Q}_0 \mathbf{Z}) - T_{\epsilon_1}(\mathbf{Q}_0 \mathbf{Z})\|^2]$. Moreover, $\mathbf{Q}_0 \in \mathfrak{B}_\Omega$ implies $\mathbf{Q}_0 \Omega \mathbf{Q}_0^\top = \Omega$, and therefore $\mathbf{Q}_0 \mathbf{Z} \sim \mathcal{N}(\mathbf{0}_d, \mathbf{Q}_0 \Omega \mathbf{Q}_0^\top) = \mathcal{N}(\mathbf{0}_d, \Omega) = \rho$. Hence the optimal value of (LGOP) is $\mathbb{E}[\|T_{\epsilon_0}(\mathbf{Z}) - T_{\epsilon_1}(\mathbf{Z})\|^2]$. Finally, by Theorem 4.2, the Kantorovich, Monge, and Lie group orbit transport values coincide. Consequently, $\mathbb{K}_c(\mu_0, \mu_1) = \mathbb{E}[\|T_{\epsilon_1}(\mathbf{Z}) - T_{\epsilon_0}(\mathbf{Z})\|^2]$.

For each $j = 1, \dots, d$, the hyperbolic subtraction identity gives $T_{\epsilon_i}(\mathbf{Z})_j = Z_j \cosh(\epsilon_{i,j}) - \sqrt{1 + Z_j^2} \sinh(\epsilon_{i,j})$. Therefore, $T_{\epsilon_1}(\mathbf{Z})_j - T_{\epsilon_0}(\mathbf{Z})_j = \Delta C_j Z_j - \Delta S_j \sqrt{1 + Z_j^2}$. Since Ω is a correlation matrix, $Z_j \sim \mathcal{N}(0, 1)$ for every j . Thus $\mathbb{E}[Z_j^2] = 1$, $\mathbb{E}[1 + Z_j^2] = 2$, and $\mathbb{E}[Z_j \sqrt{1 + Z_j^2}] = 0$, where the last identity follows because the integrand is odd and the standard normal distribution is symmetric. It follows that $\mathbb{E}[(T_{\epsilon_1}(\mathbf{Z})_j - T_{\epsilon_0}(\mathbf{Z})_j)^2] = (\Delta C_j)^2 + 2(\Delta S_j)^2$. Summing the preceding identity over $j = 1, \dots, d$ yields $\mathbb{K}_c(\mu_0, \mu_1) = \sum_{j=1}^d ((\Delta C_j)^2 + 2(\Delta S_j)^2)$, which proves the stated formula. $\square$