

# Separable QCQPs and Their Exact SDP Relaxations

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## Abstract

This paper studies exact semidefinite programming relaxations (SDPRs) for separable quadratically constrained quadratic programs (QCQPs). We consider the construction of a larger separable QCQP from multiple QCQPs with exact SDPRs. We show that exactness is preserved when such QCQPs are combined through a separable horizontal connection, where the coupling is induced through the right-hand-side parameters of the constraints. The proposed framework provides a simple sufficient condition for exactness of the resulting SDPR. We then identify notable classes of QCQPs for which this condition holds, including convex QCQPs, QCQPs defined by sign-pattern and graph-structural conditions, and separable homogeneous QCQPs with a limited number of constraints. Two examples illustrate the constructive nature of the proposed framework, showing how heterogeneous QCQPs can be combined to yield new instances with exact SDP relaxations.

**Key words.** Separable QCQP, semidefinite programming relaxation, exact relaxation, horizontal connection, new rank-based exactness results, structured bilevel optimization.

**AMS Classification.** 90C20, 90C22, 90C25, 90C26,

## 1 Introduction

Quadratically constrained quadratic programs (QCQPs) form a fundamental class of generally nonconvex and computationally challenging optimization problems (see, e.g., [7]). They arise in a wide range of applications, including signal processing [16, 9], control [19], finance [10], and combinatorial optimization [17, 22]. Among convex relaxation techniques for QCQPs, semidefinite programming relaxations (SDPRs) have been studied extensively as both a theoretical framework and a practical tool for computing lower bounds and approximate solutions. In general, the optimal value of an SDPR provides a lower bound for that of the original QCQP. Let  $\eta$  and  $\zeta$  denote the optimal values of the SDPR and the original QCQP, respectively. The SDPR is said to be exact if  $\eta = \zeta$ . When an optimal solution of

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the QCQP exists, the SDPR is exact if and only if it admits a rank-one optimal solution corresponding to an optimal solution of the original QCQP.

Over the past two decades, many structured classes of QCQPs have been identified for which the standard SDPR is exact. Representative examples include

- (a) QCQPs characterized by convexity [4],
- (b) QCQPs characterized by sign-pattern conditions and associated graph-structural conditions [3, 24],
- (c) Separable homogeneous QCQPs characterized by a limited number of constraints [9, 16],
- (d) QCQPs characterized by the rank-one-generated (ROG) property [1, 14],
- (e) QCQPs characterized by various non-intersecting quadratic constraint (NIQC) conditions [2, 11, 25].

These classes have largely been studied independently. Recently, [2] investigated the relationship between classes (d) and (e), showing that (e) can be regarded as a special case of (d).

In parallel with this line of research, Kojima, Kim, and Arima [15] proposed a unified framework for extending QCQPs that admit exact convex relaxations by adding quadratic inequality constraints under suitable NIQC assumptions, without changing the variable set. Starting from QCQPs in classes (a)–(e), their framework provides sufficient conditions under which the extended problem continues to admit an exact SDP, CPP, or DNN relaxation. In this sense, the framework enlarges the class of QCQPs with exact convex relaxations “vertically” by adding constraints while preserving exactness.

In contrast to the above line of work, which extends QCQPs with exact SDPRs by adding constraints, we pursue a different but complementary direction: combining QCQPs with exact SDPRs through a separable coupling structure. Specifically, we consider  $\hat{p}$  QCQPs, each parameterized by the right-hand-side vector of its constraints, and construct a larger QCQP by coupling these problems through shared constraint parameters. We interpret this construction as a horizontal connection of QCQPs with exact SDPRs.

More precisely, for each  $p = 1, \dots, \hat{p}$ , consider the QCQP

$$\zeta^p(\boldsymbol{\delta}^p) = \inf \{ f_0^p(\mathbf{u}^p) : f_k^p(\mathbf{u}^p) \triangleleft_k \delta_k^p \ (1 \leq k \leq m) \}. \quad (1)$$

Here,  $f_k^p$  denotes a real-valued quadratic function on the  $n^p$ -dimensional Euclidean space  $\mathbb{R}^{n^p}$ ,  $\boldsymbol{\delta}^p = (\delta_1^p, \dots, \delta_m^p) \in \mathbb{R}^m$ , and  $\triangleleft_k$  denotes either ‘ $\leq$ ’, ‘ $=$ ’, or ‘ $\geq$ ’. Without loss of generality, we assume that  $f_k^p(\mathbf{0}) = 0$  ( $0 \leq k \leq m, 1 \leq p \leq \hat{p}$ ). We also note that each QCQP may involve a redundant constraint such that  $f_k^p$  and  $\delta_k$  are identically 0 for some  $k \in \{1, \dots, m\}$ . Therefore, the number of nonredundant constraints may vary across the QCQPs. The resulting horizontally connected QCQP is

$$\zeta(\boldsymbol{\gamma}) = \inf \left\{ \sum_{p=1}^{\hat{p}} f_0^p(\mathbf{u}^p) : \sum_{p=1}^{\hat{p}} f_k^p(\mathbf{u}^p) \triangleleft_k \gamma_k \ (1 \leq k \leq m) \right\}, \quad (2)$$

where  $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_m) \in \mathbb{R}^m$ . We call each QCQP (1) a sub-QCQP of QCQP (2), and refer to this construction as *a horizontal connection of sub-QCQPs* (hereafter, the horizontal connection). This contrasts with vertical extensions, which enlarge QCQPs by adding constraints without changing the variable set.

A central feature of the horizontal connection is the separable structure of both the objective and the constraints. QCQPs with this structure arise, for example, in unicast downlink transmit beamforming [5, 9, 16]. In that setting, existing exactness results concern the homogeneous case, where all functions  $f_k^p$  ( $0 \leq k \leq m, 1 \leq p \leq \hat{p}$ ) are homogeneous quadratic functions, and show that the associated SDPR is exact under conditions including that the number  $m$  of constraints does not exceed  $\hat{p} + 1$ , i.e.,  $m \leq \hat{p} + 1$ . This is precisely the class of separable homogeneous QCQPs with a limited number of constraints described in (c) above. The following theorem extends this exactness result by allowing more general classes of sub-QCQPs within the horizontal connection framework. The theorem shows that, under mild assumptions, exactness of the SDPR is preserved under the horizontal connection of sub-QCQPs whose SDPRs are exact.

**Theorem 1.1.** *Assume that*

(I) *for each  $p = 1, \dots, \hat{p}$ , the SDPR (6) of QCQP (1) is exact whenever  $-\infty < \eta^p(\boldsymbol{\delta}^p) < \infty$ , and*

(II) *the SDPR (7) of QCQP (2) admits an optimal solution.*

*Then the SDPR (7) is exact.*

Assumption (I) is weaker than requiring exactness for every  $\boldsymbol{\delta}^p \in \mathbb{R}^m$ , since it only requires exactness when the optimal value of SDPR (6) is finite. This distinction is essential, as the SDPR may attain the value  $-\infty$  even when the original QCQP has a finite optimal value. Such a phenomenon can occur even in simple instances; see [12, Example 4.2].

Theorem 1.1 applies not only to class (c) but also to QCQPs in classes (a) and (b). Indeed, for these classes, the exactness of the SDPR depends only on the quadratic and linear terms and is independent of the constant terms in the quadratic inequality constraints. This property is consistent with the horizontal connection framework, where sub-QCQPs are coupled through their right-hand-side parameters. Thus, Theorem 1.1 provides a unified way to construct separable QCQPs with exact SDPR by combining heterogeneous sub-QCQPs from classes (a), (b), and (c). In contrast, classes (d) and (e) do not fit this framework directly, since their defining conditions depend essentially on the constant terms of the quadratic inequalities.

QCQP (2) also admits an interpretation as a bilevel optimization problem:

$$\left. \begin{array}{ll} \text{Minimize} & \zeta^1(\mathbf{v}^1) + \dots + \zeta^{\hat{p}}(\mathbf{v}^{\hat{p}}) \\ \text{subject to} & \mathbf{v}^1 + \dots + \mathbf{v}^{\hat{p}} = \boldsymbol{\gamma}, \\ & \zeta^p(\mathbf{v}^p) = \inf \{f_0^p(\mathbf{u}^p) : f_k^p(\mathbf{u}^p) \triangleleft_k v_k^p \ (1 \leq k \leq m)\} \ (1 \leq p \leq \hat{p}). \end{array} \right\} \quad (3)$$

Here, the upper-level problem allocates the parameters  $\mathbf{v}^1, \dots, \mathbf{v}^{\hat{p}}$  subject to  $\mathbf{v}^1 + \dots + \mathbf{v}^{\hat{p}} = \boldsymbol{\gamma}$ , while each lower-level problem computes the optimal value of a QCQP whose constraints are parameterized by the corresponding right-hand-side vector  $\mathbf{v}^p$ . Thus, QCQP (2) may be viewed as a special class of bilevel optimization problems. In contrast to standard resource allocation problems [6, 16], the variables  $\mathbf{v}^p$  are not restricted to be nonnegative. This difference simplifies the problem structure, as it can be reduced to the single-level separable QCQP (2). Theorem 1.1 ensures that, under the stated assumptions, the corresponding single-level formulation QCQP (2) admits an exact SDPR.

**Remark 1.2.** In Theorem 1.1, we have assumed neither the existence of a rank-one optimal solution for SDPR (6) in Assumption I nor that for SDPR (7) in Assumption II. Under our definition, exactness refers only to the equality of the optimal values of a QCQP and its SDPR, and does not guarantee the existence of a rank-one optimal solution of the SDPR corresponding to an optimal solution of the QCQP in general. See [12, Example 4.3].

The main contribution of this paper is to show that the exactness of the SDPR is preserved under a horizontal connection of QCQPs whose individual SDPRs are exact, thereby providing a unified framework for constructing separable QCQPs with exact SDP relaxations from heterogeneous building blocks. In addition, we establish a rank-based exactness result for a class of separable homogeneous QCQPs, extending existing results based on constraint bounds [16]. Although the present paper does not focus on specific applications, the proposed horizontal connection framework provides a systematic mechanism for constructing larger-scale QCQPs with guaranteed exact SDP relaxations from smaller building blocks. Such constructions are potentially useful in applications where complex systems are naturally decomposed into loosely coupled subsystems. To complement the theoretical development, we present two examples that demonstrate how the proposed framework can be used constructively. The first example examines a separable homogeneous QCQP, while the second illustrates how heterogeneous sub-QCQPs can be combined to form a larger QCQP with an exact SDP relaxation. These examples highlight the role of the framework as a tool for generating new problem instances with guaranteed exactness.

The remainder of the paper is organized as follows. Section 2 introduces the SDPR formulations and notation, Section 3 proves Theorem 1.1, Section 4 discusses representative classes of sub-QCQPs satisfying Assumption (I), Section 5 presents two examples illustrating the main results, and Section 6 concludes the paper.

## 2 SDPRs of QCQPs (1) and (2)

SDPR (6) is the standard Shor semidefinite programming relaxation of QCQP (1). If this relaxation admits a rank-one optimal solution, then it yields an optimal solution of the original QCQP. For QCQP (2), the separable structure induces a chordal sparsity structure in the Shor relaxation. Exploiting this structure, the lifted matrix variable can be decomposed into block matrices, leading to the sparse SDPR (7). In this formulation, a block-wise rank-one optimal solution corresponds to an optimal solution of QCQP (2). This construction can be viewed as a special case of the matrix completion framework of [8]; see also Remark 2.1.

To describe the SDPRs of QCQPs (1) and (2), we first introduce notation. Let  $\mathbb{S}^n$  denote the space of  $n \times n$  symmetric matrices equipped with the inner product

$$\langle \mathbf{A}, \mathbf{B} \rangle = \sum_{i=1}^n \sum_{j=1}^n A_{ij} B_{ij} \text{ for } \mathbf{A}, \mathbf{B} \in \mathbb{S}^n,$$

and let  $\mathbb{S}_+^n$  denote the cone of  $n \times n$  symmetric positive semidefinite matrices. We represent each quadratic function  $f_k^p : \mathbb{R}^{n^p} \rightarrow \mathbb{R}$  ( $1 \leq p \leq \hat{p}, 0 \leq k \leq m$ ) as

$$f_k^p(\mathbf{u}^p) = \begin{pmatrix} \mathbf{u}^p \\ 1 \end{pmatrix}^T \mathbf{B}_k^p \begin{pmatrix} \mathbf{u}^p \\ 1 \end{pmatrix} \text{ for every } \mathbf{u}^p \in \mathbb{R}^{n^p},$$

where  $\mathbf{B}_k^p \in \mathbb{S}^{n^p+1}$  and  $[B_k^p]_{n^p+1, n^p+1} = 0$ , reflecting the assumption  $f_k^p(0) = 0$ . Equivalently, this can be expressed in inner product form as

$$f_k^p(\mathbf{u}^p) = \left\langle \mathbf{B}_k^p, \begin{pmatrix} \mathbf{u}^p(\mathbf{u}^p)^T & \mathbf{u}^p \\ (\mathbf{u}^p)^T & 1 \end{pmatrix} \right\rangle \quad \text{for every } \mathbf{u}^p \in \mathbb{R}^{n^p}.$$

With this representation, QCQPs (1) and (2) admit the following equivalent vector and lifted-matrix formulations:

$$\begin{aligned} \zeta^p(\boldsymbol{\delta}^p) &= \inf \left\{ f_0^p(\mathbf{u}^p) : \mathbf{u}^p \in \mathbb{R}^{n^p}, f_k^p(\mathbf{u}^p) \leq_k \delta_k^p \ (1 \leq k \leq m) \right\} \\ &= \inf \left\{ \left\langle \mathbf{B}_0^p, \begin{pmatrix} \mathbf{u}^p(\mathbf{u}^p)^T & \mathbf{u}^p \\ (\mathbf{u}^p)^T & 1 \end{pmatrix} \right\rangle : \begin{array}{l} \mathbf{u}^p \in \mathbb{R}^{n^p}, \\ \left\langle \mathbf{B}_k^p, \begin{pmatrix} \mathbf{u}^p(\mathbf{u}^p)^T & \mathbf{u}^p \\ (\mathbf{u}^p)^T & 1 \end{pmatrix} \right\rangle \leq_k \delta_k^p \\ (1 \leq k \leq m) \end{array} \right\}, \quad (4) \end{aligned}$$

( $1 \leq p \leq \hat{p}$ ) and

$$\begin{aligned} \zeta(\boldsymbol{\gamma}) &= \inf \left\{ \sum_{p=1}^{\hat{p}} f_0^p(\mathbf{u}^p) : \begin{array}{l} \mathbf{u}^p \in \mathbb{R}^{n^p} \ (1 \leq p \leq \hat{p}), \\ \sum_{p=1}^{\hat{p}} f_k^p(\mathbf{u}^p) \leq_k \gamma_k \ (1 \leq k \leq m) \end{array} \right\} \\ &= \inf \left\{ \sum_{p=1}^{\hat{p}} \left\langle \mathbf{B}_0^p, \begin{pmatrix} \mathbf{u}^p(\mathbf{u}^p)^T & \mathbf{u}^p \\ (\mathbf{u}^p)^T & 1 \end{pmatrix} \right\rangle : \begin{array}{l} \mathbf{u}^p \in \mathbb{R}^{n^p} \ (1 \leq p \leq \hat{p}), \\ \sum_{p=1}^{\hat{p}} \left\langle \mathbf{B}_k^p, \begin{pmatrix} \mathbf{u}^p(\mathbf{u}^p)^T & \mathbf{u}^p \\ (\mathbf{u}^p)^T & 1 \end{pmatrix} \right\rangle \leq_k \gamma_k \\ (1 \leq k \leq m) \end{array} \right\}. \quad (5) \end{aligned}$$

By replacing the rank-one matrix  $\begin{pmatrix} \mathbf{u}^p(\mathbf{u}^p)^T & \mathbf{u}^p \\ (\mathbf{u}^p)^T & 1 \end{pmatrix}$  with a matrix variable  $\mathbf{X}^p \in \mathbb{S}_+^{n^p+1}$  satisfying  $X_{n^p+1, n^p+1}^p = 1$ , we obtain the following (Shor-type) SDPRs of QCQPs (1) and (2):

$$\eta^p(\boldsymbol{\delta}^p) = \inf \left\{ \left\langle \mathbf{B}_0^p, \mathbf{X}^p \right\rangle : \begin{array}{l} \mathbf{X}^p \in \mathbb{S}_+^{n^p+1}, X_{n^p+1, n^p+1}^p = 1, \\ \left\langle \mathbf{B}_k^p, \mathbf{X}^p \right\rangle \leq_k \delta_k^p \ (1 \leq k \leq m) \end{array} \right\} \quad (6)$$

( $1 \leq p \leq \hat{p}$ ), and

$$\eta(\boldsymbol{\gamma}) = \inf \left\{ \sum_{p=1}^{\hat{p}} \left\langle \mathbf{B}_0^p, \mathbf{X}^p \right\rangle : \begin{array}{l} \mathbf{X}^p \in \mathbb{S}_+^{n^p+1} \ (1 \leq p \leq \hat{p}), \\ X_{n^p+1, n^p+1}^p = 1 \ (1 \leq p \leq \hat{p}), \\ \sum_{p=1}^{\hat{p}} \left\langle \mathbf{B}_k^p, \mathbf{X}^p \right\rangle \leq_k \gamma_k \ (1 \leq k \leq m) \end{array} \right\}. \quad (7)$$

In this relaxation, the rank-one constraint on each matrix variable is dropped.

**Remark 2.1.** This remark clarifies that the separable structure assumed in QCQP (2) is not intrinsic, but depends on the representation of the problem. Consider a general QCQP

$$\zeta(\boldsymbol{\gamma}) = \inf \{ f_0(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n, f_k(\mathbf{x}) \leq \gamma_k \ (1 \leq k \leq m) \}, \quad (8)$$

where

$$f_k(\mathbf{x}) = \mathbf{x}^T \mathbf{A}_k \mathbf{x} + \mathbf{a}_k^T \mathbf{x} \quad \text{for every } x \in \mathbb{R}^n, \quad \mathbf{A}_k \in S^n, \quad \mathbf{a}_k \in \mathbb{R}^n \quad (0 \leq k \leq m).$$

It is straightforward to see that QCQP (8) is invariant under orthogonal linear transformations. Namely, for any  $n \times n$  orthogonal matrix  $\mathbf{P}$ , QCQP (8) is equivalent to

$$\zeta'(\boldsymbol{\gamma}) = \inf\{f_0(\mathbf{P}\mathbf{y}) : \mathbf{y} \in \mathbb{R}^n, f_k(\mathbf{P}\mathbf{y}) \leq \gamma_k \ (1 \leq k \leq m)\}. \quad (9)$$

Thus, a given QCQP may admit different equivalent representations. In particular, whether and how one can choose an  $n \times n$  orthogonal matrix  $\mathbf{P}$  so that QCQP (9) exhibits a separable structure is an interesting problem. This problem reduces to finding an  $n \times n$  orthogonal matrix  $\mathbf{P}$  such that  $\mathbf{P}^T \mathbf{A}_0 \mathbf{P}, \dots, \mathbf{P}^T \mathbf{A}_m \mathbf{P}$  share a common block-diagonal structure. Such a block-diagonal structure of the matrices directly induces a separable structure of QCQP (9), in which each block corresponds to a sub-QAP. See [18] and the references therein for more details.

### 3 Proof of Theorem 1.1

Let  $(\widetilde{\mathbf{X}}^1, \dots, \widetilde{\mathbf{X}}^{\hat{p}})$  be an optimal solution of SDPR (7), the existence of which is guaranteed by Assumption (II). For every  $p \in \{1, \dots, \hat{p}\}$ , let

$$\boldsymbol{\delta}^p = (\delta_1^p, \dots, \delta_m^p), \quad \delta_k^p = \langle \mathbf{B}_k^p, \widetilde{\mathbf{X}}^p \rangle \quad (1 \leq k \leq m).$$

By construction, each  $\widetilde{\mathbf{X}}^p$  satisfies the constraints of SDPR (6) with the right-hand side  $\boldsymbol{\delta}^p$ ; hence each  $\widetilde{\mathbf{X}}^p$  is a feasible solution of SDPR (6).

We claim that  $\widetilde{\mathbf{X}}^p$  is in fact optimal for SDPR (6) with the right-hand side  $\boldsymbol{\delta}^p$ . Suppose not. Then there exists a feasible solution  $\overline{\mathbf{X}}^p \in \mathbb{S}_+^{n_p+1}$  of SDPR (6) such that  $\langle \mathbf{B}_0^p, \overline{\mathbf{X}}^p \rangle < \langle \mathbf{B}_0^p, \widetilde{\mathbf{X}}^p \rangle$  and  $\langle \mathbf{B}_k^p, \overline{\mathbf{X}}^p \rangle \leq_k \delta_k^p \quad (1 \leq k \leq m)$ . Hence

$$\begin{aligned} \langle \mathbf{B}_0^p, \overline{\mathbf{X}}^p \rangle + \sum_{p' \neq p} \langle \mathbf{B}_0^{p'}, \widetilde{\mathbf{X}}^{p'} \rangle &< \sum_{p'=1}^{\hat{p}} \langle \mathbf{B}_0^{p'}, \widetilde{\mathbf{X}}^{p'} \rangle = \eta(\boldsymbol{\gamma}), \\ \langle \mathbf{B}_k^p, \overline{\mathbf{X}}^p \rangle + \sum_{p' \neq p} \langle \mathbf{B}_k^{p'}, \widetilde{\mathbf{X}}^{p'} \rangle &\leq_k \delta_k^p + \sum_{p' \neq p} \delta_k^{p'} \leq_k \gamma_k \quad (1 \leq k \leq m). \end{aligned}$$

Thus, replacing  $\widetilde{\mathbf{X}}^p$  in the optimal solution  $(\widetilde{\mathbf{X}}^1, \dots, \widetilde{\mathbf{X}}^{\hat{p}})$  of SDPR (7) by  $\overline{\mathbf{X}}^p$ , we obtain a feasible solution of SDPR (7) with a strictly smaller objective value than the optimal value  $\eta(\boldsymbol{\gamma})$ , a contradiction. Therefore we have shown that  $-\infty < \eta^p(\boldsymbol{\delta}^p) = \langle \mathbf{B}_0^p, \widetilde{\mathbf{X}}^p \rangle < \infty \quad (1 \leq p \leq \hat{p})$ .

By Assumption (I),  $\eta^p(\boldsymbol{\delta}^p) = \zeta^p(\boldsymbol{\delta}^p) \quad (1 \leq p \leq \hat{p})$ . Thus

$$\eta(\boldsymbol{\gamma}) = \sum_{p=1}^{\hat{p}} \langle \mathbf{B}_0^p, \widetilde{\mathbf{X}}^p \rangle = \sum_{p=1}^{\hat{p}} \eta^p(\boldsymbol{\delta}^p) = \sum_{p=1}^{\hat{p}} \zeta^p(\boldsymbol{\delta}^p).$$

Since any collection  $\{\mathbf{u}^p : 1 \leq p \leq \hat{p}\}$  of feasible solutions of QCQP (4) yields a feasible solution  $(\mathbf{u}^1, \dots, \mathbf{u}^{\hat{p}})$  of QCQP (5),  $\eta(\boldsymbol{\gamma}) = \sum_{p=1}^{\hat{p}} \zeta^p(\boldsymbol{\delta}^p) \geq \zeta(\boldsymbol{\gamma})$ . We also know that  $\eta(\boldsymbol{\gamma}) \leq \zeta(\boldsymbol{\gamma})$  in general. Therefore  $\eta(\boldsymbol{\gamma}) = \zeta(\boldsymbol{\gamma})$ , *i.e.*, SDPR (7) is exact.  $\square$

## 4 Eligible classes of sub-QCQPs with exact SDPRs

Section 1 describes three classes of QCQPs relevant to Theorem 1.1: (a) convex QCQPs, (b) QCQPs characterized by sign-pattern and graph-structural conditions, and (c) separable homogeneous QCQPs with a limited number of constraints. In this section, we show that these classes can serve as eligible sub-QCQPs in the framework of Theorem 1.1. Specifically, Sections 4.1, 4.2, and 4.3 discuss these three classes, respectively. Whenever the sub-QCQPs belong to any of these classes, Assumption (I) of Theorem 1.1 is satisfied; if Assumption (II) also holds, then SDPR (7) is exact, *i.e.*,  $\eta(\gamma) = \zeta(\gamma)$ .

For simplicity of notation, we drop the superscript  $p$  and write a sub-QCQP embedded in QCQP (2) as

$$\begin{aligned} \zeta(\boldsymbol{\delta}) &= \inf \{f_0(\mathbf{u}) : \mathbf{u} \in \mathbb{R}^n, f_k(\mathbf{u}) \preceq_k \delta_k (1 \leq k \leq m)\} \\ &= \inf \left\{ \left\langle \mathbf{B}_0, \begin{pmatrix} \mathbf{u}\mathbf{u}^T & \mathbf{u} \\ \mathbf{u}^T & 1 \end{pmatrix} \right\rangle : \begin{matrix} \mathbf{u} \in \mathbb{R}^n, \\ \left\langle \mathbf{B}_k, \begin{pmatrix} \mathbf{u}\mathbf{u}^T & \mathbf{u} \\ \mathbf{u}^T & 1 \end{pmatrix} \right\rangle \preceq_k \delta_k (1 \leq k \leq m) \end{matrix} \right\}, \end{aligned} \quad (10)$$

and its SDPR as

$$\eta(\boldsymbol{\delta}) = \inf \left\{ \left\langle \mathbf{B}_0, \mathbf{X} \right\rangle : \begin{matrix} \mathbf{X} \in \mathbb{S}_+^{n+1}, [\mathbf{X}]_{n+1, n+1} = 1, \\ \left\langle \mathbf{B}_k, \mathbf{X} \right\rangle \preceq_k \delta_k (1 \leq k \leq m) \end{matrix} \right\}. \quad (11)$$

Here

$$\begin{aligned} \mathbf{B}_k &\in \mathbb{S}^{n+1} (0 \leq k \leq m), \\ f_k(\mathbf{u}) &= \left\langle \mathbf{B}_k, \begin{pmatrix} \mathbf{u}\mathbf{u}^T & \mathbf{u} \\ \mathbf{u}^T & 1 \end{pmatrix} \right\rangle \text{ for every } \mathbf{u} \in \mathbb{R}^n (0 \leq k \leq m). \end{aligned}$$

We now present three classes of QCQP (10) for which SDPR (11) is exact.

### 4.1 QCQPs characterized by convexity

We assume that ' $\preceq_k$ ' = ' $\leq$ ' ( $1 \leq k \leq m$ ), and that the objective and constraint functions are convex. Then each QCQP (10) is a convex quadratic optimization problem. For this class of QCQPs, SDPR (11) is exact for every right-hand-side vector  $\boldsymbol{\delta} \in \mathbb{R}^m$ , and hence Assumption (I) of Theorem 1.1 is automatically satisfied. The following result is well-known ([23],[4, Section 4.2]), and can be proved easily.

**Theorem 4.1.** *Assume that  $f_k : \mathbb{R}^n \rightarrow \mathbb{R}$  is convex ( $0 \leq k \leq m$ ).*

- (i) *Let  $\bar{\mathbf{X}} = \begin{pmatrix} \bar{\mathbf{U}} & \bar{\mathbf{u}} \\ \bar{\mathbf{u}}^T & 1 \end{pmatrix}$  be an optimal solution of SDPR (11). Then  $\bar{\mathbf{u}}$  is an optimal solution of QCQP (10).*
- (ii) *Let  $\bar{\mathbf{u}} \in \mathbb{R}^n$  be an optimal solution of QCQP (10). Then  $\bar{\mathbf{X}} = \begin{pmatrix} \bar{\mathbf{u}}\bar{\mathbf{u}}^T & \bar{\mathbf{u}} \\ \bar{\mathbf{u}}^T & 1 \end{pmatrix} \in \mathbb{S}_+^{n+1}$  is an optimal solution of SDPR (11).*

In particular, for this class, the exactness of the SDPR is independent of the right-hand-side vector  $\boldsymbol{\delta}$ .

## 4.2 QCQPs characterized by sign-pattern conditions and associated graph structures

We next consider a class of generally nonconvex QCQPs whose SDPR exactness is guaranteed by sign-pattern and graph-structural conditions on the coefficient matrices  $\mathbf{B}_k$  ( $0 \leq k \leq m$ ) and the associated sparsity graph. We assume that  $\preceq_k = \leq$  ( $1 \leq k \leq m$ ). As in the convex case, the exactness result for this class holds for every right-hand-side vector  $\boldsymbol{\delta}$ . Hence these QCQPs also satisfy Assumption (I) of Theorem 1.1.

Let  $\mathcal{V} = \{1, \dots, n\}$  and  $\mathcal{E} = \{(i, j) \in \mathcal{V} \times \mathcal{V} : i \neq j, [\mathbf{B}_k]_{ij} \neq 0 \text{ for some } k \in \{0, 1, \dots, m\}\}$ . We call  $G(\mathcal{V}, \mathcal{E})$  the *aggregated sparsity pattern graph* associated with  $\mathbf{B}_k$  ( $0 \leq k \leq m$ ). For every  $(i, j) \in \mathcal{E}$ , define

$$\sigma_{ij} = \begin{cases} +1 & \text{if } [\mathbf{B}_k]_{ij} \geq 0 \text{ for all } k \in \{0, 1, \dots, m\}, \\ -1 & \text{if } [\mathbf{B}_k]_{ij} \leq 0 \text{ for all } k \in \{0, 1, \dots, m\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let  $\{C_1, \dots, C_r\}$  denote a cycle basis for  $G(\mathcal{V}, \mathcal{E})$ .

**Theorem 4.2.** ([24, Theorem 2]) *Assume that*

- (i)  $\sigma_{ij} \in \{-1, 1\}$  for every  $(i, j) \in \mathcal{E}$ ,
- (ii)  $\prod_{(i,j) \in C_s} \sigma_{ij} = (-1)^{|C_s|}$  for every  $s = 1, \dots, r$ .

*Then  $\eta(\boldsymbol{\delta}) = \zeta(\boldsymbol{\delta})$  for every  $\boldsymbol{\delta} \in \mathbb{R}^m$ .*

As special cases, we have the following result.

**Corollary 4.3.** ([24, Corollary 1])  *$\eta(\boldsymbol{\delta}) = \zeta(\boldsymbol{\delta})$  for every  $\boldsymbol{\delta} \in \mathbb{R}^m$  if one of the following holds:*

- (i) *The graph  $G(\mathcal{V}, \mathcal{E})$  is arbitrary and  $\sigma_{ij} = -1$  for every  $(i, j) \in \mathcal{E}$  (or equivalently all off-diagonal elements of  $\mathbf{B}_k$  ( $0 \leq k \leq m$ ) are nonpositive).*
- (ii) *The graph  $G(\mathcal{V}, \mathcal{E})$  is forest and  $\sigma_{ij} \in \{-1, 1\}$  for every  $(i, j) \in \mathcal{E}$ .*
- (iii) *The graph  $G(\mathcal{V}, \mathcal{E})$  is bipartite and  $\sigma_{ij} = 1$  for every  $(i, j) \in \mathcal{E}$ .*

## 4.3 Separable homogeneous QCQPs characterized by a limited number of constraints

We now consider a homogeneous QCQP whose objective and constraint functions share a separable structure. Unlike the previous two classes, exactness of the SDPR for this class is established through a rank argument that depends on the number of constraints relative to the number of separable blocks. Let  $\boldsymbol{\delta} \in \mathbb{R}^m$ , and for each  $q = 1, \dots, \hat{q}$  and  $k = 0, \dots, m$ ,  $g_k^q : \mathbb{R}^{\bar{n}^q} \rightarrow \mathbb{R}$  be a homogeneous quadratic function (*i.e.*, without linear and constant terms) such that

$$g_k^q(\mathbf{v}^q) = \langle \mathbf{C}_k^q, \mathbf{v}^q(\mathbf{v}^q)^T \rangle \text{ for every } \mathbf{v}^q \in \mathbb{R}^{\bar{n}^q},$$

where  $C_k^q \in \mathbb{S}^{\bar{n}^q}$ . Then we consider the following QCQP and its SDPR:

$$\begin{aligned} \zeta(\boldsymbol{\delta}) &= \inf \left\{ \sum_{q=1}^{\hat{q}} g_0^q(\mathbf{v}^q) : \sum_{q=1}^{\hat{q}} g_k^q(\mathbf{v}^q) \preceq_k \delta_k \ (1 \leq k \leq m) \right\} \\ &= \inf \left\{ \sum_{q=1}^{\hat{q}} \langle \mathbf{C}_0^q, \mathbf{v}^q(\mathbf{v}^q)^T \rangle : \sum_{q=1}^{\hat{q}} \langle \mathbf{C}_k^q, \mathbf{v}^q(\mathbf{v}^q)^T \rangle \preceq_k \delta_k \ (1 \leq k \leq m) \right\}, \end{aligned} \quad (12)$$

and

$$\eta(\boldsymbol{\delta}) = \inf \left\{ \sum_{q=1}^{\hat{q}} \langle \mathbf{C}_0^q, \mathbf{V}^q \rangle : \sum_{q=1}^{\hat{q}} \langle \mathbf{C}_k^q, \mathbf{V}^q \rangle \preceq_k \delta_k \ (1 \leq k \leq m) \right\}, \quad (13)$$

respectively.

**Theorem 4.4.** *Assume that*

(A) *for every optimal solution  $(\mathbf{V}^1, \dots, \mathbf{V}^{\hat{q}})$  of SDPR (13), at least  $m - 1$  of*

$$\mathbf{V}^q \in \mathbb{S}^n \ (1 \leq q \leq \hat{q}) \quad \text{and} \quad \delta_k - \sum_{q=1}^{\hat{q}} \langle \mathbf{C}_k^q, \mathbf{V}^q \rangle \in \mathbb{R} \ (1 \leq k \leq m)$$

*are nonzero.*

(B) *SDPR (13) has an optimal solution,*

*Then SDPR (13) admits an optimal solution  $(\tilde{\mathbf{V}}^1, \dots, \tilde{\mathbf{V}}^{\hat{q}})$  such that  $\text{rank}(\tilde{\mathbf{V}}^q) \leq 1$  ( $1 \leq q \leq \hat{q}$ ). Consequently,  $\tilde{\mathbf{V}}^q = \tilde{\mathbf{v}}^q(\tilde{\mathbf{v}}^q)^T$  for some  $\tilde{\mathbf{v}}^q \in \mathbb{R}^{\bar{n}^q}$ , and  $(\tilde{\mathbf{v}}^1, \dots, \tilde{\mathbf{v}}^{\hat{q}})$  is an optimal solution of QCQP (12).*

*Proof.* Introduce slack variables  $s_k \in \mathbb{R}$  ( $1 \leq k \leq m$ ) and rewrite SDPR (13) as

$$\eta(\boldsymbol{\delta}) = \inf \left\{ \sum_{q=1}^{\hat{q}} \langle \mathbf{C}_0^q, \mathbf{V}^q \rangle : \sum_{q=1}^{\hat{q}} \langle \mathbf{C}_k^q, \mathbf{V}^q \rangle - s_k = \delta_k \ (1 \leq k \leq m), \ 0 \preceq_k s_k, \ \mathbf{V}^q \in \mathbb{S}_+^{\bar{n}^q} \ (1 \leq q \leq \hat{q}) \right\}. \quad (14)$$

By Assumption (B), this problem admits an optimal solution. Then we can apply Theorem 2.2 of [20] to SDPR (14), which provides a rank bound for feasible solutions of an SDP, for the existence of an optimal solution  $(\tilde{\mathbf{V}}^1, \dots, \tilde{\mathbf{V}}^{\hat{q}}, \bar{s}_1, \dots, \bar{s}_m)$  satisfying

$$\sum_{q \in Q} \frac{\text{rank}(\tilde{\mathbf{V}}^q)(\text{rank}(\tilde{\mathbf{V}}^q) + 1)}{2} + \sum_{k \in K} 1 \leq m, \quad (15)$$

where  $Q = \{q : \tilde{\mathbf{V}}^q \neq \mathbf{O}\}$  and  $K = \{k : \bar{s}_k \neq 0\}$ . Suppose, to the contrary, that  $\text{rank}(\tilde{\mathbf{V}}^{\bar{q}}) \geq 2$  for some  $\bar{q} \in Q$ . Then

$$\frac{\text{rank}(\tilde{\mathbf{V}}^{\bar{q}})(\text{rank}(\tilde{\mathbf{V}}^{\bar{q}}) + 1)}{2} \geq 3.$$

By Assumption (A), at least  $m - 1$  of the matrices  $\tilde{\mathbf{V}}^q$  ( $1 \leq q \leq \hat{q}$ ) and the residuals  $\delta_k - \sum_{q=1}^{\hat{q}} \langle \mathbf{C}_k^q, \tilde{\mathbf{V}}^q \rangle$  ( $1 \leq k \leq m$ ) are nonzero. Hence  $|Q \setminus \{\tilde{q}\}| + |K| \geq m - 2$ . Therefore,

$$\begin{aligned} m + 1 &= 3 + (m - 2) \\ &\leq \frac{\text{rank}(\tilde{\mathbf{V}}^{\tilde{q}})(\text{rank}(\tilde{\mathbf{V}}^{\tilde{q}}) + 1)}{2} + \sum_{q \in Q \setminus \{\tilde{q}\}} \frac{\text{rank}(\tilde{\mathbf{V}}^q)(\text{rank}(\tilde{\mathbf{V}}^q) + 1)}{2} + \sum_{k \in K} 1. \end{aligned}$$

This contradicts (15). Thus  $\text{rank}(\tilde{\mathbf{V}}^q) \leq 1$  for every  $q = 1, \dots, \hat{q}$ .

Since each  $\tilde{\mathbf{V}}^q \in \mathbb{S}_+^{\bar{n}^q}$  has rank at most one, there exists  $\bar{\mathbf{v}}^q \in \mathbb{R}^{\bar{n}^q}$  such that  $\tilde{\mathbf{V}}^q = \bar{\mathbf{v}}^q(\bar{\mathbf{v}}^q)^T$  ( $1 \leq q \leq \hat{q}$ ). Substituting these factorizations into SDPR (13) shows that  $(\bar{\mathbf{v}}^1, \dots, \bar{\mathbf{v}}^{\hat{q}})$  is feasible for QCQP (12) and attains the same objective value. Hence it is an optimal solution of QCQP (12).  $\square$

To better understand Assumption (A), consider the case where  $\ell$  of the relations  $\preceq_k$  ( $1 \leq k \leq m$ ) are equalities, and the remaining ones are inequalities. Without loss of generality, assume that  $\preceq_k = '='$  for  $1 \leq k \leq \ell$ , and  $\preceq_k = '\leq'$  for  $\ell + 1 \leq k \leq m$ . Then, for every optimal solution, the residual  $\delta_k - \sum_{q=1}^{\hat{q}} \langle \mathbf{C}_k^q, \mathbf{V}^q \rangle$  vanishes for  $1 \leq k \leq \ell$ . Hence, Assumption (A) requires that at least  $m - 1$  of the matrices  $\mathbf{V}^q$  ( $1 \leq q \leq \hat{q}$ ) and the remaining residuals corresponding to inequality constraints be nonzero. This implies that  $m - 1 \leq \hat{q} + (m - \ell)$ , *i.e.*,  $\ell - 1 \leq \hat{q}$ . In particular, when  $\ell = m$ , we must have  $m \leq \hat{q} + 1$ , whereas when  $\ell \leq 2$ , Assumption (A) can be satisfied for arbitrary values of  $m \geq \ell$  and  $\hat{q} \geq 1$ .

**Remark 4.5.** Theorem 4.4 can be compared with the result in [16], where the assumptions  $m \leq \hat{q} + 1$  and

(ii)'  $\mathbf{V}^q \neq \mathbf{O}$  ( $1 \leq q \leq \hat{q}$ ) for every optimal solution  $(\mathbf{V}^1, \dots, \mathbf{V}^{\hat{q}})$  of SDPR (13)

were imposed. Under these assumptions, (15) implies

$$m - 1 \leq \hat{q} \leq \sum_{q=1}^{\hat{q}} \frac{\text{rank}(\tilde{\mathbf{V}}^q)(\text{rank}(\tilde{\mathbf{V}}^q) + 1)}{2} \leq m,$$

and hence either  $\hat{q} = m - 1$  or  $\hat{q} = m$ ; equivalently, either  $m = \hat{q} + 1$  or  $m = \hat{q}$ . In contrast, Assumption (A) of Theorem 4.4 lightens this restriction considerably as discussed above.

Whether Assumption (A) holds depends not only on the data matrices  $\mathbf{C}_k^q$  ( $1 \leq q \leq \hat{q}$ ,  $0 \leq k \leq m$ ) but also on the right-hand-side vector  $\boldsymbol{\delta}$  and the relations ' $\preceq_k$ ' ( $1 \leq k \leq m$ ). Consequently, unlike the convex and sign-pattern classes, homogeneous QCQP (12) does not automatically qualify as an eligible sub-QCQP in the framework of Theorem 1.1. When  $m \leq 2$ , the conclusion of Theorem 4.4 holds without any additional assumption on  $\boldsymbol{\delta}$ , ' $\preceq_k$ ' and  $\mathbf{C}_k^q$  ( $1 \leq q \leq \hat{q}$ ,  $0 \leq k \leq m$ ) as shown below. (This result is known [2, 21, 26]). Therefore, homogeneous QCQP (12) with  $m \leq 2$  can be incorporated as an eligible sub-QCQP in separable QCQP (2) whose SDPR is guaranteed to be exact by Theorem 1.1.

**Corollary 4.6.** *Assume that  $m \leq 2$  and that SDPR (13) has an optimal solution. Then SDPR (13) has an optimal solution  $(\tilde{\mathbf{V}}^1, \dots, \tilde{\mathbf{V}}^{\hat{q}})$  such that  $\text{rank}(\tilde{\mathbf{V}}^q) \leq 1$  ( $1 \leq q \leq \hat{q}$ ).*

*Proof.* If Assumption (A) of Theorem 4.4 holds with  $m = 2$ , then the conclusion follows immediately from Theorem 4.4. Otherwise, Assumption (A) fails. Since  $m - 1 = 1$ , there exists an optimal solution  $(\tilde{\mathbf{V}}^1, \dots, \tilde{\mathbf{V}}^{\hat{q}})$  of SDPR (13) such that all matrices  $\tilde{\mathbf{V}}^q$  ( $1 \leq q \leq \hat{q}$ ) and all residuals  $\delta_k - \sum_{q=1}^{\hat{q}} \langle \mathbf{C}_k^q, \tilde{\mathbf{V}}^q \rangle$  ( $k = 1, 2$ ) are zero. In particular,  $\tilde{\mathbf{V}}^q = \mathbf{O}$  for every  $q$ , and hence  $\text{rank}(\tilde{\mathbf{V}}^q) = 0 \leq 1$  for every  $q$ . This proves the result.  $\square$

## 5 Examples

This section presents two examples illustrating the main results. The first example is a separable homogeneous QCQP and shows when Assumption (A) of Theorem 4.4 holds. The second example illustrates how Theorem 1.1 can be used to construct a separable QCQP with an exact SDPR by combining heterogeneous sub-QCQPs from different classes, including convex QCQPs, QCQPs characterized by sign-pattern conditions, and separable homogeneous QCQPs. It demonstrates a systematic integration within a unified framework and shows how the exactness of the overall SDPR can be derived from the exactness of the individual subproblems.

**Example 5.1.** (A slight modification of [12, Example 4.1]) This example illustrates the role of Assumption (A) in Theorem 4.4. In general, whether Assumption (A) holds depends on the problem data, including the objective and constraint functions. To make this dependence explicit, we introduce a parameter  $\alpha \in \mathbb{R}$  into the constraints and consider the following parametric separable homogeneous QCQP. Let

$$\begin{aligned} g_0^1(\mathbf{v}) &= (v_1)^2, \quad g_1^1(\mathbf{v}) = (v_2)^2, \quad g_2^1(\mathbf{v}) = (v_1 - \alpha v_2)(v_1 - 4v_2), \\ g_3^1(\mathbf{v}) &= -(v_1 - 2v_2)(v_1 - 3v_2) \text{ for every } \mathbf{v}^1 \in \mathbb{R}^2, \\ g_0^2(w) &= -w^2, \quad g_1^2(w) = g_2^2(w) \equiv 0, \quad g_3^2(w) = w^2, \text{ for every } w \in \mathbb{R}, \quad \boldsymbol{\delta} = (1, 0, 0), \end{aligned}$$

where  $\alpha \in [0, 4]$  is a parameter, which will be specified later. We consider the QCQP

$$\begin{aligned} \zeta(\boldsymbol{\delta}) &= \inf \left\{ g_0^1(\mathbf{v}) + g_0^2(w) : \begin{array}{l} \mathbf{v} \in \mathbb{R}^2, w \in \mathbb{R}^1, g_1^1(\mathbf{v}) + g_1^2(w) = \delta_1, \\ g_2^1(\mathbf{v}) + g_2^2(w) \leq \delta_2, g_3^1(\mathbf{v}) + g_3^2(w) \leq \delta_3 \end{array} \right\} \\ &= \inf \left\{ v_1^2 - w^2 : \begin{array}{l} \mathbf{v} \in \mathbb{R}^2, w \in \mathbb{R}^1, v_2^2 = 1, \\ (v_1 - \alpha v_2)(v_1 - 4v_2) \leq 0, \\ -(v_1 - 2v_2)(v_1 - 3v_2) + w^2 \leq 0 \end{array} \right\} \\ &= \inf \left\{ v_1^2 - w^2 : \begin{array}{l} \mathbf{v} \in \mathbb{R}^2, w \in \mathbb{R}^1, \alpha \leq v_1 \leq 4, \\ v_1 \leq \frac{5 - \sqrt{1 + 4w^2}}{2} \text{ or } \frac{5 + \sqrt{1 + 4w^2}}{2} \leq v_1 \end{array} \right\} \end{aligned} \quad (16)$$

Here  $v_2^2 = 1$  in the constraint of QCQP (16) implies either  $v_2 = 1$  or  $v_2 = -1$ . We have assumed  $v_2 \geq 0$  without loss of generality since  $g_k^1(\mathbf{v}) = g_k^1(-\mathbf{v})$  ( $0 \leq k \leq 3$ ); hence  $v_2 = 1$ . This QCQP (16) can be formulated as a special case of the separable homogeneous QCQP (12) with

$$\begin{aligned} \hat{q} &= 2, \quad m = 3, \\ \mathbf{C}_0^1 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{C}_1^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{C}_2^1 = \begin{pmatrix} 1 & -\frac{4+\alpha}{2} \\ -\frac{4+\alpha}{2} & 4\alpha \end{pmatrix}, \quad \mathbf{C}_3^1 = \begin{pmatrix} -1 & \frac{5}{2} \\ \frac{5}{2} & -6 \end{pmatrix}, \\ \mathbf{C}_0^2 &= -1, \quad \mathbf{C}_1^2 = \mathbf{C}_2^2 = 0, \quad \mathbf{C}_3^2 = 1, \quad \text{'}\preceq_1\text{' = '}\text{=}\text{'}, \quad \text{'}\preceq_2\text{' = '}\leq\text{'}, \quad \text{'}\preceq_3\text{' = '}\leq\text{'}. \end{aligned}$$

We denote each feasible solution  $(\mathbf{V}^1, \mathbf{V}^2) \in \mathbb{S}_+^2 \times \mathbb{S}_+^1$  of SDPR of QCQP (12) as  $(\mathbf{V}, \mathbf{W})$ . The optimal solutions  $(\tilde{\mathbf{v}}, \tilde{\mathbf{w}})$  of QCQP (16) and  $(\tilde{\mathbf{V}}, \tilde{\mathbf{W}})$  of its SDPR are summarized in Table 1. We note that  $\tilde{\mathbf{W}} = \tilde{\mathbf{w}}^2$  holds. In particular, we compare the optimal solution  $(\tilde{\mathbf{V}}, \tilde{\mathbf{W}})$  and optimal value of the SDPR for different values of  $\alpha$ .

	QCQP (16)		SDPR of QCQP (16)		
	Opt.sol. $(\tilde{\mathbf{v}}, \tilde{\mathbf{w}})$	Opt.val.	Opt.val.	Opt.sol. $(\tilde{\mathbf{V}}, \tilde{\mathbf{W}})$	rank $(\tilde{\mathbf{V}})$
$0 \leq \alpha < 2$	$\tilde{\mathbf{v}} = (\alpha, 1), \tilde{\mathbf{w}} \neq 0$	$\alpha^2 - \tilde{\mathbf{w}}^2$	$\alpha^2 - \tilde{\mathbf{w}}^2$	$\tilde{\mathbf{V}} = \begin{pmatrix} \alpha^2 & \alpha \\ \alpha & 1 \end{pmatrix}, \tilde{\mathbf{W}} \neq \mathbf{O}$	1
$\alpha = 2$	$\tilde{\mathbf{v}} = (2, 1), \tilde{\mathbf{w}} = 0$	4	4	$\tilde{\mathbf{V}} = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}, \tilde{\mathbf{W}} = \mathbf{O}$	1
$2 < \alpha < 3$	$\tilde{\mathbf{v}} = (3, 1), \tilde{\mathbf{w}} = 0$	9	$\frac{14\alpha-24}{\alpha-1}$	$\tilde{\mathbf{V}} = \begin{pmatrix} \frac{14\alpha-24}{\alpha-1} & \frac{4\alpha-6}{\alpha-1} \\ \frac{4\alpha-6}{\alpha-1} & 1 \end{pmatrix}, \tilde{\mathbf{W}} = \mathbf{O}$	2
$\alpha = 3$	$\tilde{\mathbf{v}} = (3, 1), \tilde{\mathbf{w}} = 0$	9	9	$\tilde{\mathbf{V}} = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}, \tilde{\mathbf{W}} = \mathbf{O}$	1
$3 < \alpha \leq 4$	$\tilde{\mathbf{v}} = (\alpha, 1), \tilde{\mathbf{w}} \neq 0$	$\alpha^2 - \tilde{\mathbf{w}}^2$	$\alpha^2$	$\tilde{\mathbf{V}} = \begin{pmatrix} \alpha^2 & \alpha \\ \alpha & 1 \end{pmatrix}, \tilde{\mathbf{W}} \neq \mathbf{O}$	1

Table 1: Optimal solutions of QCQP (16) and its SDPR. Here  $(\tilde{\mathbf{v}}, \tilde{\mathbf{w}})$  denotes an optimal solution of QCQP (16), and  $(\tilde{\mathbf{V}}, \tilde{\mathbf{W}})$  an optimal solution of its SDPR.

As shown in Table 1, the behavior of the SDPR depends strongly on  $\alpha$ . If  $\alpha \in [0, 2) \cup (3, 4]$ , then Assumption (A) of Theorem 4.4 is satisfied, and hence the SDPR admits a rank-one optimal solution. In this case, the SDPR is exact. On the other hand, if  $\alpha \in [2, 3]$ , Assumption (A) is violated. More precisely, the total number of nonzeros among

$$\tilde{\mathbf{V}}, \tilde{\mathbf{W}}, \delta_k - \langle \mathbf{C}_k^1, \tilde{\mathbf{V}} \rangle - \langle \mathbf{C}_k^2, \tilde{\mathbf{W}} \rangle \quad (k = 1, 2, 3);$$

is less than  $m - 1 = 2$ ; only  $\tilde{\mathbf{V}}$  is nonzero among these quantities. Particularly, if  $\alpha \in (2, 3)$ ,  $\tilde{\mathbf{V}}$  has rank two, and hence the SDPR is not exact. In the case where  $\alpha = 2$  or  $3$ , we have  $\text{rank}(\tilde{\mathbf{V}}) = 1$  and  $\text{rank}(\tilde{\mathbf{W}}) = 0$ . Although Assumption (A) is violated, the SDPR remains exact. This shows that Assumption (A) is a sufficient but not necessary for exactness.

The loss of exactness for  $\alpha \in (2, 3)$  can be understood from the different behaviors of the feasible regions of QCQP (16) and its SDPR. At  $\alpha = 2$ , the feasible region of QCQP (16) undergoes a discontinuous structural change. In contrast, the feasible region of the SDPR varies continuously with respect to  $\alpha$ . As a result, in the range  $\alpha \in (2, 3)$ , the SDP relaxation no longer captures the geometry of the original problem exactly, leading to a rank-two optimal solution. This example demonstrates that Assumption (A) is not merely technical, but is essential for guaranteeing exactness. It also shows that exactness may fail even for a small perturbation of constraints.

**Example 5.2.** We construct a separable QCQP with exact SDPR by combining three heterogeneous sub-QCQPs: a convex QCQP for  $p = 1$ , a QCQP characterized by sign-pattern conditions for  $p = 2$ , and a separable homogeneous QCQP for  $p = 3$ . More precisely, we connect two sub-QCQPs of the form (4) (for  $p = 1, 2$ ) and one separable homogeneous

sub-QCQP of the form (12) with  $\hat{q} = 3$  for  $p = 3$ , and obtain the QCQP

$$\begin{aligned} \zeta(\boldsymbol{\gamma}) &= \inf \left\{ \sum_{p=1}^3 f_0^p(\mathbf{u}^p) : \mathbf{u}^p \in \mathbb{R}^{n^p+1}, \sum_{p=1}^3 f_k^p(\mathbf{u}^p) \triangleleft_k \gamma_k \ (p = 1, 2, 3, 1 \leq k \leq 7) \right\} \\ &= \inf \left\{ \sum_{p=1}^2 f_0^p(\mathbf{u}^p) + \sum_{q=1}^3 \langle \mathbf{C}_0^q, \mathbf{v}^q(\mathbf{v}^q)^T \rangle : \sum_{p=1}^2 f_k^p(\mathbf{u}^p) + \sum_{q=1}^3 \langle \mathbf{C}_k^q, \mathbf{v}^q(\mathbf{v}^q)^T \rangle \triangleleft_k \gamma_k \right. \\ &\quad \left. \begin{array}{l} \mathbf{u}^p \in \mathbb{R}^{n^p}, \mathbf{v}^q \in \mathbb{R}^{\bar{n}^q} \ (p = 1, 2, q = 1, 2, 3), \\ (1 \leq k \leq 7) \end{array} \right\}, \quad (17) \end{aligned}$$

and its SDPR

$$\eta(\boldsymbol{\gamma}) = \inf \left\{ \sum_{p=1}^2 \langle \mathbf{B}_0^p, \mathbf{X}^p \rangle + \sum_{q=1}^3 \langle \mathbf{C}_0^q, \mathbf{V}^q \rangle : \sum_{p=1}^2 \langle \mathbf{B}_k^p, \mathbf{X}^p \rangle + \sum_{q=1}^3 \langle \mathbf{C}_k^q, \mathbf{V}^q \rangle \triangleleft_k \gamma_k \right. \\ \left. \begin{array}{l} \mathbf{X}^p \in \mathbb{S}_+^{n^p+1}, \mathbf{V}^q \in \mathbb{S}_+^{\bar{n}^q}, \\ X_{n^p+1n^p+1}^p = 1 \ (p = 1, 2, q = 1, 2, 3), \\ (1 \leq k \leq 7) \end{array} \right\}, \quad (18)$$

where

$$f_k^p(\mathbf{u}^p) = \langle \mathbf{B}_k^p, \begin{pmatrix} \mathbf{u}^p \\ 1 \end{pmatrix} \begin{pmatrix} \mathbf{u}^p \\ 1 \end{pmatrix}^T \rangle \text{ for every } \mathbf{u}^p \in \mathbb{R}^{n^p} \ (p = 1, 2, 0 \leq k \leq 7),$$

$$f_k^3(\mathbf{u}^3) = \sum_{q=1}^3 \langle \mathbf{C}_k^q, \mathbf{v}^q(\mathbf{v}^q)^T \rangle \text{ for every } \mathbf{u}^3 = (\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3) \in \mathbb{R}^{n^3} \ (q = 1, 2, 3, 0 \leq k \leq 7),$$

$$\mathbf{B}_k^p \in \mathbb{S}^{n^p+1} \ (p = 1, 2), \ \mathbf{C}_k^q \in \mathbb{S}^{\bar{n}^q} \ (q = 1, 2, 3), \ n^3 = \bar{n}^1 + \bar{n}^2 + \bar{n}^3 \ (0 \leq k \leq 7).$$

We assume that

$$f_k^1 \text{ is convex } (0 \leq k \leq 7), \ f_1^1 \equiv f_2^1 \equiv 0 \text{ (equivalently, } \mathbf{B}_1^1 = \mathbf{B}_2^1 = \mathbf{O}), \quad (19)$$

$$\text{all off-diagonal elements of } \mathbf{B}_k^2 \ (0 \leq k \leq 7) \text{ are nonpositive, } \mathbf{B}_1^2 = \mathbf{B}_2^2 = \mathbf{O}, \quad (20)$$

$$\mathbf{C}_1^2 = \mathbf{O}, \ \mathbf{C}_1^3 = \mathbf{O}, \ \triangleleft_1 = \text{'='}, \ \gamma_1 \neq 0, \quad (21)$$

$$\mathbf{C}_2^1, \mathbf{C}_2^3 \text{ are negative semidefinite, } \triangleleft_2 = \text{'\geq'}, \ \gamma_2 > 0, \quad (22)$$

$$\mathbf{B}_3^1, \mathbf{B}_3^2, \mathbf{C}_3^1, \mathbf{C}_3^2 \text{ are positive semidefinite, } \triangleleft_3 = \text{'\leq'}, \ \gamma_3 < 0, \quad (23)$$

$$\alpha > 0, \ \mathbf{C}_5^q = \alpha \mathbf{C}_4^q \ (q = 1, 2, 3), \quad (24)$$

$$\mathbf{C}_k^q = \mathbf{O} \ (1 \leq q \leq 3, k = 6, 7), \quad (25)$$

$$\triangleleft_k = \text{'\leq'} \ (3 \leq k \leq 7). \quad (26)$$

The conditions imposed on the data matrices and constraints are summarized in Table 2, where the structural properties of each sub-QCQP are included. Table 2 shows how the three different classes of QCQPs are combined within a unified separable structure.

To apply Theorem 1.1, suppose that SDPR (18) admits an optimal solution, as required by Assumption (II), and denote it by  $(\widetilde{\mathbf{X}}^1, \widetilde{\mathbf{X}}^2, \widetilde{\mathbf{V}}^1, \widetilde{\mathbf{V}}^2, \widetilde{\mathbf{V}}^3)$ . Let

$$\begin{aligned}\delta_k^1 &= \gamma_k - \langle \mathbf{B}_k^2, \widetilde{\mathbf{X}}^2 \rangle - \sum_{q=1}^3 \langle \mathbf{C}_k^q, \widetilde{\mathbf{V}}^q \rangle, \quad \delta_k^2 = \gamma_k - \langle \mathbf{B}_{1_k}, \widetilde{\mathbf{X}}^1 \rangle - \sum_{q=1}^3 \langle \mathbf{C}_k^q, \widetilde{\mathbf{V}}^q \rangle, \\ \delta_k^3 &= \gamma_k - \sum_{p=1}^2 \langle \mathbf{B}_k^p, \widetilde{\mathbf{X}}^p \rangle \quad (1 \leq k \leq 7).\end{aligned}$$

As shown in the proof of Theorem 1.1,  $\widetilde{\mathbf{X}}^1$  is optimal for the SDPR of the first sub-QCQP with right-hand side  $\boldsymbol{\delta}^1$ ,  $\widetilde{\mathbf{X}}^2$  is optimal for the SDPR of the second sub-QCQP with right-hand side  $\boldsymbol{\delta}^2$ , and  $(\widetilde{\mathbf{V}}^1, \widetilde{\mathbf{V}}^2, \widetilde{\mathbf{V}}^3)$  is optimal for SDPR (13) with right-hand side  $\boldsymbol{\delta}^3$ .

For  $p = 1$ , the first two constraints are variable-free because  $\mathbf{B}_1^1 = \mathbf{B}_2^1 = \mathbf{O}$ . After removing these constraints, the remaining sub-QCQP is convex by (19) and (26), and hence its SDPR is exact by Theorem 4.1. For  $p = 2$ , the first two constraints are again variable-free because  $\mathbf{B}_1^2 = \mathbf{B}_2^2 = \mathbf{O}$ . After removing them, the remaining sub-QCQP satisfies the sign-pattern condition in Corollary 4.3 by (20) and (26), and therefore its SDPR is exact.

For  $p = 3$ , sub-QCQP (12) is a separable homogeneous QCQP. We verify that its SDPR satisfies the assumptions of Theorem 4.4. By (25), the 6th and 7th constraints (the inequality constraints corresponding to  $k = 6, 7$ ) can be removed without affecting the feasible region of sub-SDPR (13). By (24), the 4th and 5th inequality constraints,

$$\sum_{q=1}^3 \langle \mathbf{C}_4^q, \mathbf{V}^q \rangle \leq \delta_4^3 \quad \text{and} \quad \sum_{q=1}^3 \langle \mathbf{C}_5^q, \mathbf{V}^q \rangle \leq \delta_5^3$$

have proportional coefficient matrices with  $\mathbf{C}_5^q = \alpha \mathbf{C}_4^q$  and  $\alpha > 0$ . Therefore one of them is redundant: the 4th constraint may be removed if  $\delta_4^3 \leq \delta_5^3/\alpha$ , and the 5th otherwise. Thus the sub-SDPR (13) reduces to a problem with at most four essential constraints.

Moreover, every optimal solution satisfies  $\mathbf{V}^1 \neq \mathbf{O}$ ,  $\mathbf{V}^2 \neq \mathbf{O}$ , and  $\mathbf{V}^3 \neq \mathbf{O}$ . Indeed, (21) forces  $\mathbf{V}^1 \neq \mathbf{O}$ , since  $\gamma_1 \neq 0$  while  $\mathbf{C}_1^2 = \mathbf{C}_1^3 = \mathbf{O}$ ; (22) forces  $\mathbf{V}^2 \neq \mathbf{O}$ , since  $\gamma_2 > 0$  while  $\mathbf{C}_2^1$  and  $\mathbf{C}_2^3$  are negative semidefinite; and (23) forces  $\mathbf{V}^3 \neq \mathbf{O}$ , since  $\gamma_3 < 0$  while  $\mathbf{C}_3^1$  and  $\mathbf{C}_3^2$  are positive semidefinite. Hence, after the above preprocessing, Assumption (A) of Theorem 4.4 is satisfied.

However, the right-hand-side values  $\delta_4^3$  and  $\delta_5^3$  depend on the optimal solution of the full SDPR (18), in particular on  $\widetilde{\mathbf{X}}^1$  and  $\widetilde{\mathbf{X}}^2$ . Therefore, it is not known a priori which of the 4th or 5th constraints can be removed in the analysis of the sub-SDPR (13). Moreover, these two constraints correspond to different coupling constraints in SDPR (18), so both are essential in SDPR (18). In particular, removing either of them would change the feasible region of SDPR (18). The elimination of one of these constraints is carried out solely within the analysis of the sub-SDPR (13), where it serves as a technical simplification, and should not be interpreted as indicating redundancy in the original problem.

Therefore, each of the three sub-QCQPs satisfies Assumption (I) of Theorem 1.1. We have shown that the SDPR of each sub-QCQP of QCQP (17) is exact: the case  $p = 1$  follows from convexity, the case  $p = 2$  from Corollary 4.3, and the case  $p = 3$  from Theorem 4.4 after the above preprocessing. Therefore, Assumption (I) of Theorem 1.1 is fulfilled for

QCQP (17) and its SDPR (18). By applying Theorem 1.1, we conclude that SDPR (18) is an exact relaxation of QCQP (17), *i.e.*, both problems attain the same optimal value, whenever SDPR (18) has an optimal solution (Assumption (II)).

$k$	$p = 1$ : Convex Sub-QCQP (4) See Section 4.1	$p = 2$ : Sign Pattern Sub-QCQP (4) See Section 4.2	$p = 3$ : Homogeneous Separable Sub-QCQP (12) See Section 4.3			$\trianglelefteq_k$	$\gamma_k$
0	$f_0^1$ :convex	$B_0^2$ :Off-diag $\ominus$	$\forall C_0^1$	$\forall C_0^2$	$\forall C_0^3$		
1	$B_1^1 = O$	$B_1^2 = O$	$\forall C_1^1$	$C_1^2 = O$	$C_1^3 = O$	=	$\neq 0$
2	$B_2^1 = O$	$B_2^2 = O$	$C_2^1$ : $\ominus$	$\forall C_2^2$	$C_2^3$ : $\ominus$	$\geq$	+
3	$f_3^1$ :convex $B_3^1$ : $\oplus$	$B_3^2$ :Off-diag $\ominus$ $B_3^2$ : $\oplus$	$C_3^1$ : $\oplus$	$C_3^2$ : $\oplus$	$\forall C_3^3$	$\leq$	-
4	$f_4^1$ :convex	$B_4^2$ :Off-diag $\ominus$	$\forall C_4^1$	$\forall C_4^2$	$\forall C_4^3$	$\leq$	$\forall$
5	$f_5^1$ :convex	$B_5^2$ :Off-diag $\ominus$	$C_5^1 = \alpha C_4^1$	$C_5^2 = \alpha C_4^2$	$C_5^3 = \alpha C_4^3$	$\leq$	$\forall$
6	$f_6^1$ :convex	$B_6^2$ :Off-diag $\ominus$	$C_6^1 = O$	$C_6^2 = O$	$C_6^3 = O$	$\leq$	$\forall$
7	$f_7^1$ :convex	$B_7^2$ :Off-diag $\ominus$	$C_7^1 = O$	$C_7^2 = O$	$C_7^3 = O$	$\leq$	$\forall$

Table 2: Structural conditions for the three sub-QCQPs. The table summarizes the assumptions (19)–(26), indicating how each constraint contributes to the convexity, sign-pattern condition, or separable homogeneous structure.  $\oplus$  : Positive semidefinite.  $\ominus$  : Negative semidefinite. Off-diag $\ominus$  : Off-diagonal nonpositive.  $\alpha \geq 0$ .

## 6 Concluding remarks

In this paper, we have shown that exactness of the SDPR for a class of separable QCQPs is preserved under a horizontal connection of sub-QCQPs whose SDPRs are exact. Theorem 1.1 provides a simple framework for constructing separable QCQPs by coupling sub-QCQPs through their right-hand-side constraint parameters. Rather than introducing a fundamentally new structural class, the framework clarifies how exactness results for existing QCQP classes can be systematically combined through separable structure. In addition, Theorem 4.4 presents a rank-based exactness condition for separable homogeneous QCQPs, which may be viewed as a refinement of earlier results based on the bound  $m \leq \hat{q} + 1$ . These results illustrate how exactness can be ensured in structured separable QCQPs.

Section 4 presents three classes of QCQPs satisfying Assumption (I) of Theorem 1.1, based on convexity, sign-pattern and graph-structural conditions, and separability with a limited number of constraints. Although the mechanisms guaranteeing exactness differ across these classes, they can all serve as sub-QCQPs in the horizontal connection of Theorem 1.1, leading to new separable QCQPs with exact SDPRs.

This horizontal connection is complementary to the vertical extension framework of Kojima, Kim, and Arima [15], which is particularly relevant to QCQPs arising from classes (d) and (e). Once the exactness of an SDPR is established for a given QCQP, the vertical extension framework can be applied to further enlarge the class of QCQPs by adding quadratic inequality constraints, while preserving exactness under suitable NIQC assump-

tions. Together, these horizontal and vertical extensions suggest a systematic way to construct structured QCQPs whose SDPR exactness is guaranteed by construction.

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