On vehicle routing problems with stochastic demands – Scenario-optimal recourse policies

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Two-Stage Vehicle Routing Problems with Stochastic Demands (VRPSDs) form a class of stochastic combinatorial optimization problems where routes are planned in advance, demands are revealed upon vehicle arrival, and recourse actions are triggered whenever capacity is exceeded. Following recent works, we consider VRPSDs where demands are given by an empirical probability distribution of scenarios. Existing approaches rely on integer L-shaped (ILS) cuts, whose coefficients are tailored for specific recourse policies. In contrast, we propose a framework that casts recourse policies as solutions of a higher-dimensional mixed-integer program, and we characterize its convex hull in the original lower-dimensional space via a new class of inequalities called scenario recourse inequalities (SRIs). We show that SRIs are valid for any recourse policy satisfying mild assumptions and are sufficient for formulating the VRPSD under a scenario-optimal recourse policy, where the recourse actions are chosen optimally for each scenario. Under this latter policy, we also show that SRIs dominate several ILS cuts. We conduct computational experiments on the VRPSD with scenarios under both the classical and the scenario-optimal recourse policies. By using the SRIs, our algorithm solves 329 more instances to optimality than the previous state-of-the-art ILS algorithm.

Keywords: integer programming, stochastic programming, vehicle routing problem.

1 Introduction

Two-Stage Vehicle Routing Problems with Stochastic Demands (VRPSDs) constitute a class of stochastic variants of the classic *Capacitated Vehicle Routing Problem* (CVRP) in which routing decisions are fixed in advance, customer demands are revealed upon vehicle arrival, and capacity violations trigger additional *recourse actions*. Since their introduction over 50 years ago (Tillman, 1969), VRPSDs have attracted considerable attention, with strong interest in recent years (Gendreau et al., 2016; Louveaux and Salazar-González, 2018; Hoogendoorn and Spliet, 2023; Florio et al., 2022, 2020; Ota and Fukasawa, 2024; Parada et al., 2024; Salavati-Khoshghalb et al., 2019b,a,c; Legault et al., 2025; Ota and Fukasawa, 2025; Hoogendoorn and Spliet, 2025).

A key limitation of most existing literature is its reliance on simplifying assumptions on the demand distributions; in particular, most studies assume independence (Gendreau et al., 2016; Laporte et al., 2002; Jabali et al., 2014; Salavati-Khoshghalb et al., 2019b,a,c; Hoogendoorn and Spliet, 2023; Parada et al., 2024; Legault et al., 2025). Motivated by scenario-based approaches in the general stochastic programming literature (Birge and Louveaux, 2011), Ota and Fukasawa (2025) recently addressed this limitation by using *demand scenarios*. The authors also

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generalized previous *integer L-shaped* (ILS) formulations for VRPSDs, allowing them to derive the first branch-and-cut algorithm for the VRPSD with scenarios under the *classical recourse policy*, where a vehicle traverses the route and, as soon as the capacity gets exceeded, it returns to the depot to unload.

For the most part, the framework of Ota and Fukasawa (2025) is independent of the scenario assumption and does not rely on a specific choice of recourse function. However, it relies on ILS cuts to capture recourse costs, and the coefficients of these cuts depend on the chosen recourse policy. As a result, ILS cuts derived for one policy cannot be directly applied to other policies. This policy-specific dependence is also present in all other ILS-based methods, whether they consider the classical (Laporte and Nobert, 1983; Jabali et al., 2014; Parada et al., 2024; Gendreau et al., 1995, 2016), *optimal* (Salavati-Khoshghalb et al., 2019b; Hoogendoorn and Spliet, 2023; Legault et al., 2025), or *rule-based* (Salavati-Khoshghalb et al., 2019c) recourse policies. It is not entirely clear how approaches developed for one policy can be adapted or extended to different policies.

In this work, we develop an approach tailored to scenario-based VRPSDs that yields cuts whose validity is not tied to specific recourse policies. To accomplish this, we propose a common framework that can accommodate all recourse policies satisfying a natural assumption. Within this framework, recourse policies are represented as feasible solutions of a higher-dimensional mixed-integer program (MIP). This naturally leads us to define *scenario-optimal recourse policies* as the optimal solutions of such a MIP, which correspond to policies that optimally select the recourse actions for each scenario.

Beyond unifying the modeling of recourse policies, this representation reveals a previously unexplored polyhedral structure, which we use to derive our main results. First, although recourse policies are defined in a higher-dimensional space, we characterize their convex hull using inequalities in the original space. This leads to the introduction of *scenario recourse inequalities* (SRIs), which are valid for any recourse policy satisfying the assumptions of our framework. Second, we show that SRIs enable a formulation for the VRPSD under a scenario-optimal recourse policy that includes decision variables for both the route edges and the recourse actions. Finally, we project this last formulation onto the space of ILS formulations. Using our convex hull characterization, we express recourse lower bounds for scenario-optimal recourse policies as optimal values of linear programs (LPs). This allows us to apply LP duality to show that several ILS cuts are dominated by SRIs.

In summary, our contributions are:

- We propose a common framework that accommodates several recourse policies and requires only mild assumptions (see Definition 1 and Assumptions 1, 2, 3 and 4);
- We describe the convex hull of recourse policies (Theorem 2);
- We define scenario-optimal recourse policies as optimal solutions within this framework, and we formulate the VRPSD under a scenario-optimal recourse policy using SRIs (Definition 2 and Theorem 3);
- We demonstrate that SRIs can be used to solve the VRPSD with scenarios under any recourse policy that satisfies the assumptions of our framework (Proposition 1);
- We show that, under a scenario-optimal recourse policy, several ILS inequalities are dominated by the SRIs (Theorem 5, Corollary 3 and Theorem 4);
- We present computational experiments that validate our theoretical results and show that SRIs can substantially enhance the performance of a state-of-the-art branch-and-cut approach (Section 6). In particular, by incorporating SRIs, we are able to solve 329 more instances to optimality compared to the approach in (Ota and Fukasawa, 2025).

The rest of this paper is organized as follows. Section 2 formally defines the class of problems we study and briefly reviews concepts introduced by Ota and Fukasawa (2025). Section 3 defines recourse policies and characterizes their convex hull. Section 4 introduces scenario-optimal recourse policies and a VRPSD formulation based on SRIs. Section 5 examines the projection of the polytope defined by SRIs onto the space of ILS formulations. Section 6 presents computational results, and Section 7 concludes the paper.

Notation. We let $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote the nonnegative and positive real numbers, respectively, with analogous notation for $\mathbb{Q}$ and $\mathbb{Z}$. For any real number a, we define $(a)^+ := \max\{0, a\}$. For any integer a, we define $[a] := \{1, \dots, a\}$ if $a > 0$, and $[a] := \emptyset$ otherwise. The symbol $\mathbb{I}(\cdot)$ represents the indicator function. For any vector f, we use f_i and $f(i)$ interchangeably. For any function (or vector) f and a subset S of its domain (or coordinates), $f(S) := \sum_{i \in S} f(i)$. We write $\mathbf{1}$ and $\mathbf{0}$ to refer to the all-ones and all-zeroes vectors, respectively.

For an undirected graph G, $V(G)$ and $E(G)$ refer to the sets of vertices and edges, while for a directed graph, $A(G)$ denotes its arcs. We sometimes write an edge $\{u, v\}$ (or an arc (u, v)) as uv. For every $S \subseteq V(G)$, $\delta_G(S)$ (respectively, $E_G(S)$) is the set of edges with exactly one endpoint (respectively, two endpoints) in S, omitting the subscript when the graph is clear. When S is a singleton $\{v\}$, we abbreviate S to v. For any two disjoint subsets $S, T \subseteq V(G)$, $E_G(S, T) := \{e \in E(G) : |e \cap S| = |e \cap T| = 1\}$.

2 Problem description and ILS formulations

As mentioned earlier, we consider in this work the *VRPSD with scenarios* (Ota and Fukasawa, 2025), a class of problems where customer demands are uncertain and modeled via a finite set of scenarios. We adopt the *two-stage* (or *a priori*) paradigm (Oyola et al., 2018; Gendreau et al., 2016), where the *first-stage* decisions specify a feasible *routing plan*, while during the *second-stage*, the planned routes are traversed, and the customer demands are revealed upon vehicle arrival. Consequently, a vehicle may have insufficient capacity to serve a customer, leading to a *route failure*. To handle such failures, a given *recourse policy* prescribes certain *recourse actions* that the vehicle should execute — typically, these are either *back-and-forth* trips between the failure location and the depot, or *preventive returns*, where the vehicle returns to the depot to unload and then proceeds directly to the next customer in the route.

In this sense, all the different recourse policies previously proposed share the key property that the vehicle never carries more load than its capacity, and they only differ in when the trips to the depot are performed to unload (Dror et al., 1989; Yee and Golden, 1980; Yang et al., 2000; Salavati-Khoshghalb et al., 2019b,a,c). The goal of the VRPSD with scenarios is to find a routing plan that minimizes the sum of the first-stage routing costs and the expected costs of executing the recourse actions.

In Section 2.1, we describe the input of the VRPSD with scenarios, and in Section 2.2, we cast it as an instance of the more general class of problems introduced in (Ota and Fukasawa, 2025). This allows us to represent several variants of the VRPSD with scenarios studied in this paper under a common notation. Sections 2.3 and 2.4 then briefly review key concepts from the ILS-based approach of Ota and Fukasawa (2025), which will serve as a baseline for our approach.

We highlight that while Section 2.4 presents in detail the ILS cuts from Ota and Fukasawa (2025), these cuts are not used in this paper until Section 5, where we compare certain ILS cuts with the SRIs.

2.1 Input data

Let $G = (V, E)$ be a complete undirected graph with edge weights $c \in \mathbb{Q}_+^E$. The vertex set V is partitioned as $\{0\}\dot{\cup}V_+$, where 0 represents the depot and V_+ denotes the set of customers. We denote by $D = (V, A)$ the digraph obtained from G by replacing each edge with two arcs in opposite directions. Each vehicle has a capacity of $C \in \mathbb{Q}_{++}$, and the demands are modeled by a random vector d, which follows a probability distribution $\mathbb{P}$ and whose components $d(v)$ correspond to the random demand of customer $v \in V_+$. As mentioned earlier, we assume that $\mathbb{P}$ *is given by scenarios*, and we describe the associated input data next.

Let N be a positive integer. We refer to any $\xi \in [N]$ as a *scenario*, and we associate with this scenario a *demand vector* $d^\xi \in \mathbb{Q}_+^{V_+}$ and a *realization probability* $p_\xi \in [0, 1] \cap \mathbb{Q}_+$. As their names suggest, these parameters satisfy $\sum_{\xi\in[N]} p_\xi = 1$ and $\mathbb{P}(d = d^\xi) = p_\xi$, for all $\xi \in [N]$. For convenience, we use $\bar{d}$ as an abbreviation for $\mathbb{E}[d]$. An instance of the VRPSD with scenarios is given by the tuple $\mathcal{I} = (G, c, C, d^1, \dots, d^N, p_1, \dots, p_N)$, which we assume is fixed throughout the entire paper.

As in (Ota and Fukasawa, 2025), we also assume that the customer demands in each scenario are never larger than the vehicle capacity:

Assumption 1. For every $\xi \in [N]$ and $v \in V_+$, we have that $d^\xi(v) \leq C$.

The reasoning for Assumption 1 is that, whenever scenario ξ is realized, the vehicle is guaranteed to execute $\lfloor d^\xi(v)/C \rfloor$ back-and-forth trips between the depot and customer v in the second stage. Therefore, if $\lfloor d^\xi(v)/C \rfloor \geq 1$, we can add $p_\xi(2c_{0v}) \cdot \lfloor d^\xi(v)/C \rfloor$ to the objective function and preprocess the demand accordingly.

2.2 Problem description

A *route* R is a simple undirected cycle in G that includes the depot, i.e., $V(R) = \{0, v_1, v_2, \dots, v_\ell\}$ and $E(R) = \{\{0, v_1\}, \{v_1, v_2\}, \dots, \{v_\ell, 0\}\}$, where $v_1, \dots, v_\ell \in V_+$ are all distinct. The set of customers in R is denoted $V_+(R) = \{v_1, \dots, v_\ell\}$ and we often represent R by the tuple $(v_1, \dots, v_\ell)$ (implicitly assuming $v_0 = v_{\ell+1} = 0$). A *subroute* of R is any route of the form $R' = (v_i, \dots, v_j)$ with $j \in [\ell]$ and $i \in [j]$, and we write $R' \subseteq R$ to indicate that R' is a subroute of R (even though R' is not necessarily a subgraph of R). The notation $c(R)$ is a shorthand for $c(E(R))$. A *routing plan* $\mathcal{R} = \{R_1, \dots, R_k\}$ is a collection of routes such that $\{V_+(R_i)\}_{i=1}^k$ forms a partition of V_+ (note that k is not necessarily fixed here).

When discussing specific recourse policies, it is often necessary to distinguish between the two orientations of a route. Accordingly, we associate with each route $R = (v_1, \dots, v_\ell)$ two *directed routes* $\vec{R}, \overleftarrow{R} \subseteq D$, which are digraphs with the same vertex set as R, but with arc sets $A(\vec{R}) = \{(0, v_1), \dots, (v_\ell, 0)\}$ and $A(\overleftarrow{R}) = \{(0, v_\ell), \dots, (v_1, 0)\}$. Similarly to undirected routes, we represent directed routes using tuples, so $\vec{R} = (v_1, \dots, v_\ell)$ and $\overleftarrow{R} = (v_\ell, \dots, v_1)$ (see Ota and Fukasawa (2025) for a clarification on a minor ambiguity with this notation).

It is well known (Toth and Vigo, 2002; Dantzig et al., 1954) that routing plans can be represented as integer vectors inside the polytope

$$\mathcal{X}_{\text{SUB}} = \left\{ x \in [0, 2]^E : \begin{array}{ll} x(\delta(v)) = 2, & \forall v \in V_+ \\ x(E(S)) \leq |S| - 1, & \forall \emptyset \subsetneq S \subseteq V_+ \end{array} \right\}. \tag{$\mathcal{X}_{\text{SUB}}$}$$

In this sense, we sometimes refer to a vector $\bar{x} \in \mathcal{X}_{\text{SUB}} \cap \mathbb{Z}^E$ as a routing plan, and we use $\mathcal{R}(\bar{x})$ to denote its corresponding collection of routes. As mentioned before, routing plans $\bar{x} \in \mathcal{X}_{\text{SUB}} \cap \mathbb{Z}^E$ may not contain a given fixed number k of routes.

Most works on the VRPSD (see Table 1 of Hoogendoorn and Spliet (2025)) assume that feasible routing plans satisfy the classical *CVRP feasibility* conditions with respect to the expected demands. That is, $k \in \mathbb{Z}_{++}$ is now given as a fixed input parameter and routing plans x

belong to the set

$$\mathcal{X}_{\text{CVRP}} := \mathcal{X}_{\text{SUB}} \cap \left\{ x \in [0,2]^E : \begin{array}{l} x(\delta(0)) = 2k, \\ x(E(S)) \le |S| - \bar{k}(S), \quad \forall \emptyset \subsetneq S \subseteq V_+ \end{array} \right\}, \tag{$\mathcal{X}_{\text{CVRP}}$}$$

where $\bar{k}(S) := \lceil \bar{d}(S)/C \rceil$.

However, Hoogendoorn and Spliet (2025) question the use of $\mathcal{X}_{\text{CVRP}}$ as somewhat arbitrary, and show that, in some settings, using $\mathcal{X}_{\text{CVRP}}$ instead of $\mathcal{X}_{\text{SUB}}$ can significantly increase the solution cost. Therefore, we consider both sets in this work, and henceforth, whenever we write $\mathcal{X}$, we assume that $\mathcal{X} \in \{\mathcal{X}_{\text{SUB}}, \mathcal{X}_{\text{CVRP}}\}$ (if $\mathcal{X} = \mathcal{X}_{\text{CVRP}}$, we also assume that k is part of the input $\mathcal{I}$).

To model the expected recourse cost of a route, we let $\mathcal{Q}$ be a *recourse function* (Ota and Fukasawa, 2024), meaning that it takes the fixed input $\mathcal{I}$ as a parameter and maps each route R to a nonnegative rational number, i.e., $\mathcal{Q}(R; \mathcal{I}) \in \mathbb{Q}_+$ for every route R. Since $\mathcal{I}$ is fixed, we omit it from the notation. Also with respect to $\mathcal{I}$, we define the *VRPSD with scenarios* with respect to $\mathcal{X} \in \{\mathcal{X}_{\text{SUB}}, \mathcal{X}_{\text{CVRP}}\}$ and $\mathcal{Q}$ as

$$\min \left\{ \sum_{R \in \mathcal{R}(x)} [c(R) + \mathcal{Q}(R)] : x \in \mathcal{X} \cap \mathbb{Z}^E \right\}. \tag{VRPSD($\mathcal{X}, \mathcal{Q}$)}$$

We remark that the problem above has precisely the form of the *vehicle routing problems with recourse (VRPRs)* introduced by Ota and Fukasawa (2024). In this way, problem VRPSD($\mathcal{X}, \mathcal{Q}$) can be viewed as a subclass of the VRPRs where the input is fixed as in Section 2.1, and the recourse function $\mathcal{Q}$ takes as input the route R (together with the scenario demands and probabilities) and outputs the expected cost of the trips to the depot that ensures the vehicle never carries more load than its capacity. Section 3 will more formally define the conditions on $\mathcal{Q}$.

With this notation, we can conveniently refer to several variants of the VRPSD with scenarios. For instance, if $\mathcal{Q}_C$ corresponds to the *classical recourse policy* (see Example 1), then the problem considered in the computational experiments of Ota and Fukasawa (2025) is denoted as VRPSD($\mathcal{X}_{\text{CVRP}}, \mathcal{Q}_C$). Following the work of Hoogendoorn and Spliet (2025), we may also drop the assumption of CVRP feasibility, which yields the variant VRPSD($\mathcal{X}_{\text{SUB}}, \mathcal{Q}_C$). Furthermore, Section 4 introduces a new *scenario-optimal* recourse function $\mathcal{Q}^*$, which gives rise to problems VRPSD($\mathcal{X}_{\text{CVRP}}, \mathcal{Q}^*$) and VRPSD($\mathcal{X}_{\text{SUB}}, \mathcal{Q}^*$).

2.3 Recourse disaggregations and feasible regions

In this subsection, we review the *recourse disaggregation* model studied in Ota and Fukasawa (2025), which was motivated by the computational success of different algorithms that disaggregate the recourse function (Hoogendoorn and Spliet, 2023; Parada et al., 2024; Legault et al., 2025). We refer the reader to Ota and Fukasawa (2025) for a more detailed discussion. For our purposes, we take from that work the assumption that any recourse function $\mathcal{Q}$ satisfies the following:

Assumption 2. For each route R, we have access to values $\{\mathcal{Q}(R, v)\}_{v \in V_+}$ such that $\mathcal{Q}(R) = \sum_{v \in V_+(R)} \mathcal{Q}(R, v)$, and $\mathcal{Q}(R, v) = 0$, for every customer $v \notin V_+(R)$.

Note that Ota and Fukasawa (2025) has a discussion on why Assumption 2 is both natural and theoretically justified. The $\mathcal{Q}(R, v)$-values are referred to as the *disaggregation* of $\mathcal{Q}$.

For any recourse function $\mathcal{Q}$, there is always a choice of disaggregation that satisfies Assumption 2 (e.g., pick an arbitrary customer $v \in V_+(R)$ and assign $\mathcal{Q}(R, v) = \mathcal{Q}(R)$). More interestingly, the following example shows that, in some cases, the disaggregation can be chosen so that each term $\mathcal{Q}(R, v)$ corresponds to the expected cost of executing recourse actions *at customer* $v \in V_+(R)$ while traversing route R.

Example 1. Intuitively, the *classical recourse policy* simply determines that the vehicle executes back-and-forth trips whenever it observes a failure. Since we assume $\mathbb{P}$ is given by scenarios, we compute the recourse cost of the directed route $\vec{R} = (v_1, \ldots, v_\ell)$ under the classical recourse policy using the following formula (Dror et al., 1989):

$$\mathcal{Q}_C(\vec{R}) = \sum_{j \in [\ell]} 2\, c_{0v_j} \sum_{\xi \in [N]} p_\xi \sum_{t=1}^{\infty} \mathbb{I}\left(\sum_{i \in [j-1]} d^\xi(v_i) \le tC < \sum_{i \in [j]} d^\xi(v_i) \right). \tag{$\mathcal{Q}_C$}$$

Without loss of generality, assume $\mathcal{Q}_C(\vec{R}) \le \mathcal{Q}_C(\bar{R})$ and define $\mathcal{Q}_C(R) := \mathcal{Q}_C(\vec{R})$. The disaggregation used in Ota and Fukasawa (2025) is such that, for each $j \in [\ell]$,

$$\mathcal{Q}_C(R, v_j) = 2\, c_{0v_j} \sum_{\xi \in [N]} p_\xi \sum_{t=1}^{\infty} \mathbb{I}\left(\sum_{i \in [j-1]} d^\xi(v_i) \le tC < \sum_{i \in [j]} d^\xi(v_i) \right).$$

□

Having established Assumption 2, Ota and Fukasawa (2025) associate with a recourse function $\mathcal{Q}$ (and its disaggregation) the feasible region

$$\mathcal{F}(\mathcal{X}, \mathcal{Q}) := \left\{ (x, \theta) \in (\mathcal{X} \cap \mathbb{Z}^E) \times \mathbb{R}_+^{V_+} : \theta_v \ge \sum_{R \in \mathcal{R}(x)} \mathcal{Q}(R, v),\ \forall v \in V_+ \right\}, \tag{$\mathcal{F}(\mathcal{X}, \mathcal{Q})$}$$

and show that problem $\text{VRPSD}(\mathcal{X}, \mathcal{Q})$ is equivalent to $\min\{c^\intercal x + \mathbb{1}^\intercal \theta : (x, \theta) \in \mathcal{F}(\mathcal{X}, \mathcal{Q})\}$.

Ota and Fukasawa (2025) also argue that previous approaches (Hoogendoorn and Spliet, 2023; Parada et al., 2024; Legault et al., 2025) all replace the nonlinear constraints on the θ-variables with certain (linear) ILS cuts that preserve the optimal value of the problem. Moreover, they show that these ILS cuts can be generalized to accommodate essentially any choice of $\mathcal{X} \subseteq \mathcal{X}_{\text{SUB}}$ and $\mathcal{Q}$. We briefly discuss these cuts in the next subsection.

2.4 ILS cuts

Roughly speaking, ILS cuts are valid inequalities for $\mathcal{F}(\mathcal{X}, \mathcal{Q})$ that have the following simple structure. Given a subset $\mathcal{X}' \subseteq \mathcal{X} \cap \mathbb{Z}^E$ and a lower bound $\mathcal{L} \in \mathbb{Q}_+$, an ILS cut is "active" whenever $x \in \mathcal{X}'$, in which case it enforces a lower bound of $\mathcal{L}$ on the sum of certain θ-variables; otherwise, the cut is "inactive" and does not impose any additional restrictions.

Formally, an affine function $W(x; \mathcal{X}')$ is called an *activation function* (with respect to $\mathcal{X}' \subseteq \mathcal{X} \cap \mathbb{Z}^E$) if, for every $x \in \mathcal{X} \cap \mathbb{Z}^E$, we have $W(x; \mathcal{X}') = 1$ whenever $x \in \mathcal{X}'$, and $W(x; \mathcal{X}') \le 0$ otherwise. Given a subset of customers $S \subseteq V_+$ and a recourse function $\mathcal{Q}$ that satisfies Assumption 2, we define an ILS cut as a valid inequality for $\mathcal{F}(\mathcal{X}, \mathcal{Q})$ of the form

$$\theta(S) \ge \mathcal{L} \cdot W(x; \mathcal{X}'),$$

where $\mathcal{L} \in \mathbb{Q}_+$ is called a *recourse lower bound* with respect to $\mathcal{X}' \subseteq \mathcal{X} \cap \mathbb{Z}^E$ and $S \subseteq V_+$. By the definition of $\mathcal{F}(\mathcal{X}, \mathcal{Q})$, for every $\bar{x} \in \mathcal{X}'$, we have that $\mathcal{L} \le \sum_{v \in S} \sum_{R \in \mathcal{R}(\bar{x})} \mathcal{Q}(R, v)$.

We now briefly describe two of the main classes of ILS cuts discussed in Ota and Fukasawa (2025).

Partial route cuts. Partial routes are generalizations of routes that were introduced by Hjorring and Holt (1999) and are commonly used in ILS-based algorithms for VRPSDs. A *partial route* is a tuple $H = (S_1, \ldots, S_\ell)$ of disjoint customer subsets such that there exists no index $i \in [\ell]$ for which both S_i and S_{i+1} are not singletons (for convenience, $S_0 = S_{\ell+1} = \{0\}$). The customers in H are represented by $V_+(H) := \cup_{i=1}^{\ell} S_i$.

Intuitively, partial route H compactly represents all routes that visit the customers in $V_+(H)$ in the same order as prescribed by H. In this sense, we say that a route $R = (v_1, \ldots, v_h)$ *adheres* to H if $|V_+(R)| = |V_+(H)|$ and, for each $i \in [\ell]$, we can label $S_i = \{v_1^i, \ldots, v_{t_i}^i\}$, so that $(v_1, \ldots, v_h) = (v_1^1, \ldots, v_{t_1}^1, \ldots, v_1^\ell, \ldots, v_{t_\ell}^\ell)$.

To refer to the set of routing plans containing routes (or subroutes) that adhere to H, define

$$\mathcal{X}_=(H) := \left\{x \in \mathcal{X} \cap \mathbb{Z}^E : \ \exists R \in \mathcal{R}(x) \text{ s.t. } R \text{ adheres to } H\right\}, \tag{$\mathcal{X}_=(H)$}$$

$$\mathcal{X}_\supseteq(H) := \left\{x \in \mathcal{X} \cap \mathbb{Z}^E : \ \exists R \in \mathcal{R}(x), R' \subseteq R \text{ s.t. } R' \text{ adheres to } H\right\}. \tag{$\mathcal{X}_\supseteq(H)$}$$

Ota and Fukasawa (2025) introduced an activation function $W_{OF}(x; \mathcal{X}_\supseteq(H))$ and showed that the activation function $W_{HS}(x; \mathcal{X}_=(H))$ of Hoogendoorn and Spliet (2023) can be obtained by adding nonpositive terms to $W_{OF}(x; \mathcal{X}_\supseteq(H))$.

In this paper, we define a *partial route cut* as an ILS cut (with respect to $\mathcal{F}(\mathcal{X}, \mathcal{Q})$) with one of the following two forms:

$$\theta(V_+(H)) \geq \mathcal{L}_=(H) \cdot W_{HS}(x; \mathcal{X}_=(H)), \tag{1}$$

$$\theta(V_+(H)) \geq \mathcal{L}_\supseteq(H) \cdot W_{OF}(x; \mathcal{X}_\supseteq(H)), \tag{2}$$

where $\mathcal{L}_=(H)$ and $\mathcal{L}_\supseteq(H)$ are recourse lower bounds. We refer the reader to Ota and Fukasawa (2025) for details on how such bounds are derived when $\mathcal{Q} = \mathcal{Q}_C$. Note that when H corresponds to a route R (i.e., every set in H is a singleton), Ota and Fukasawa (2025) set $\mathcal{L}_=(H) = \mathcal{Q}_C(R)$, which implies that inequalities (1) ensure that the θ-variables correctly capture the recourse cost of a solution.

Set cuts. Set cuts are ILS cuts that enforce a lower bound on the recourse cost incurred when visiting a set of customers using the minimum required number of vehicles. They were introduced by Parada et al. (2024) for the VRPSD under the classical recourse policy with specific demand distributions, and later extended to more general recourse functions by Legault et al. (2025) and Ota and Fukasawa (2025).

Let $\emptyset \subsetneq S \subseteq V_+$ and suppose that $k' \in \mathbb{Z}_{++}$ is a lower bound on the number of routes required to serve the customers in S, i.e., inequality $x(E(S)) \leq |S| - k'$ is valid for $\mathcal{X} \cap \mathbb{Z}^E$. Define the set

$$\mathcal{X}(S, k') := \left\{x \in \mathcal{X} \cap \mathbb{Z}^E : x(E(S)) = |S| - k'\right\}. \tag{$\mathcal{X}(S,k')$}$$

A *set cut* is an ILS cut of the form

$$\theta(S) \geq \mathcal{L}(S) \cdot W_{DL}(x; \mathcal{X}(S, k')), \tag{3}$$

where $W_{DL}(x; \mathcal{X}(S, k')) := 1 + (x(E(S)) - |S| + k')$ is the activation function employed by the DL-shaped method of Parada et al. (2024), and $\mathcal{L}(S)$ is a recourse lower bound. We present the specific recourse lower bound used by Ota and Fukasawa (2025) in Section 5, where we compare their set cuts with the SRIs.

3 Recourse policies

As mentioned in the beginning of Section 2, there are multiple different recourse functions that have been studied in the literature. A key difficulty in placing these functions within a common framework is that they are typically defined through sets of rules used to compute the recourse, as illustrated in Example 1. In this section, we take a different view and cast *recourse policies* and *recourse actions* as feasible solutions of a MIP based on a network-flow formulation.

Section 3.1 provides the formal definition of recourse policies under this framework. Section 3.2 then leverages polyhedra and network-flow theory to characterize the convex hull of the set of recourse policies. While these structural results do not seem directly connected with the solution of problem $\textsc{vrpsd}(\mathcal{X}, \mathcal{Q})$, subsequent sections will use these results for that purpose. In particular, Section 4 shows that, by adding an objective function to the MIP formulation studied here, we are able to derive valid inequalities for problem $\textsc{vrpsd}(\mathcal{X}, \mathcal{Q})$.

3.1 Definition via network flows

Henceforth, we reserve the notation y for a vector in $\mathbb{R}^{[N]\times V_+}$ with entries y_v^ξ, for each scenario $\xi \in [N]$ and customer $v \in V_+$. We often represent y with the tuple $(y^1, \dots, y^N)$, where, for each scenario $\xi \in [N]$, $y^\xi \in \mathbb{R}^{V_+}$ is the restriction of y to the entries $\{y_v^\xi\}_{v\in V_+}$.

Based on the intuitive description of the VRPSD given in the beginning of Section 2 (where a vehicle never carries more load than its capacity), we formalize the notions of *recourse actions* and *recourse policies* as follows (see Figure 1).

Definition 1. Fix a directed route $\vec{R} = (v_1, \dots, v_\ell)$ and let $\xi \in [N]$. (Recall that $v_0 = v_{\ell+1} = 0$.) A vector $y^\xi \in \mathbb{R}_+^{V_+}$ is a *recourse action for* $\vec{R}$ *and scenario* ξ if it is integer and there exist $f^\xi \in \mathbb{R}_+^A$ and $g^\xi \in \mathbb{R}_+^{V_+}$ such that

$$f^\xi_{(v_{i-1},v_i)} + d^\xi(v_i) = f^\xi_{(v_i,v_{i+1})} + g^\xi_{v_i}, \qquad \forall i \in [\ell], \tag{4a}$$

$$f^\xi_{(v_{i-1},v_i)} \le C, \qquad \forall i \in [\ell+1] \tag{4b}$$

$$g^\xi_{v_i} \le C \cdot y^\xi_{v_i}, \qquad \forall i \in [\ell]. \tag{4c}$$

The set of all recourse actions for $\vec{R}$ and ξ is denoted $\mathcal{Y}^\xi(\vec{R})$. A *recourse policy for* $\vec{R}$ is a vector $y = (y^1, \dots, y^N) \in \Pi(\vec{R})$, where $\Pi(\vec{R}) \coloneqq \mathcal{Y}^1(\vec{R}) \times \ldots \times \mathcal{Y}^N(\vec{R})$ is the set of all recourse policies for $\vec{R}$.

In a sense, Definition 1 is natural: for each scenario $\xi \in [N]$, the flow variables f^ξ determine the load of the vehicle along the route, the recourse action y^ξ specifies at which customers and how many times we unload at the depot, and the flow variables g^ξ determine the corresponding amount that we unload. The following example shows that previously proposed recourse policies are indeed covered by Definition 1.

Example 2. Fix a directed route $\vec{R} = (v_1, \dots, v_\ell)$. Suppose that a given recourse policy specifies, for each customer v_i, a threshold value $\tau_i \in [0, C] \cap \mathbb{Q}_+$ indicating that the vehicle should execute a recourse action whenever the accumulated load upon arrival at v_i (including the demand of v_i) exceeds τ_i. As observed in Salavati-Khoshghalb et al. (2019a), several recourse policies fall within this setup: the classical recourse policy (Gendreau et al., 1995; Laporte et al., 2002; Jabali et al., 2014; Christiansen and Lysgaard, 2007; Gauvin et al., 2014) sets all the threshold values to C; preventive recourse policies (Yee and Golden, 1980; Hoogendoorn and Spliet, 2023; Yang et al., 2000; Salavati-Khoshghalb et al., 2019b; Florio et al., 2020) sets the threshold values according to a dynamic-programming algorithm; finally, rule-based recourse policies assign thresholds according to other route measures such as volume, risk and distances (see Salavati-Khoshghalb et al. (2019c,a) for more details).

Let d^ξ be any realization of the random vector d and let $\bar{y}^\xi \in \mathbb{Z}_+^{V_+}$ be such that each entry $\bar{y}^\xi_{v_i}$, with $i \in [\ell]$, counts how many times the given recourse policy executes a recourse action at customer v_i (either through back-and-forth or preventive returns) under this realization. One can check that $\bar{y}^\xi$ is feasible for Formulation (4). In fact, $\bar{y}^\xi$ is feasible for a more restricted variant of Formulation (4) where we add the constraints $f^\xi_{(v_{i-1},v_i)} + d^\xi(v_i) \le \tau_i + (C + d^\xi(v_i) - \tau_i)\, y^\xi_{v_i}$, for all $i \in [\ell]$. $\square$

Before continuing, we need to address a technicality concerning the directions of the routes. The next lemma shows that, whenever we speak of recourse policies (in the sense of Definition 1), we may ignore route directions. In other words, for every scenario $\xi \in [N]$ and route R, we may write $\Pi(R) := \Pi(\vec{R}) = \Pi(\overleftarrow{R})$ and $\mathcal{Y}^\xi(R) := \mathcal{Y}^\xi(\vec{R}) = \mathcal{Y}^\xi(\overleftarrow{R})$. The proof is given in Appendix A and was inspired by the work of Hernández-Pérez and Salazar-González (2003).

Lemma 1. *Let R be a route and $\xi \in [N]$. For any vector $y^\xi \in \mathbb{Z}^{V_+}$, we have that $y^\xi \in \mathcal{Y}^\xi(\vec{R})$ if and only if there exist $f^\xi \in \mathbb{R}^A_+$ and $g^\xi \in \mathbb{R}^{V_+}_+$ such that*

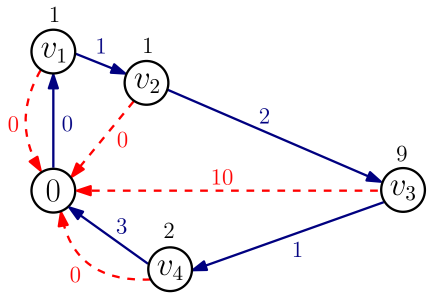
$$f^\xi_{(v_{i-1},v_i)} + f^\xi_{(v_{i+1},v_i)} + d^\xi(v_i) = f^\xi_{(v_i,v_{i+1})} + f^\xi_{(v_i,v_{i-1})} + g^\xi_{v_i}, \qquad \forall i \in [\ell], \tag{5a}$$

$$f^\xi_{(v_{i-1},v_i)} \leq \frac{C}{2}, \qquad \forall i \in [\ell+1], \tag{5b}$$

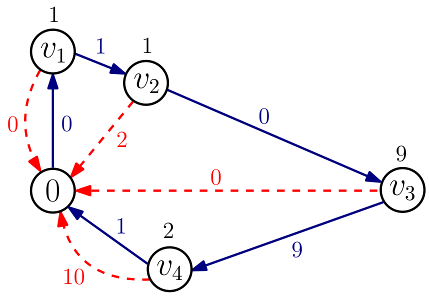
$$f^\xi_{(v_i,v_{i-1})} \leq \frac{C}{2}, \qquad \forall i \in [\ell+1], \tag{5c}$$

$$g^\xi_{v_i} \leq C \cdot y^\xi_{v_i}, \qquad \forall i \in [\ell]. \tag{5d}$$

In particular, this implies that $\mathcal{Y}^\xi(\vec{R}) = \mathcal{Y}^\xi(\overleftarrow{R})$.



(a) Recourse action $y^\xi_{v_1} = y^\xi_{v_2} = y^\xi_{v_4} = 0$ and $y^\xi_{v_3} = 1$ corresponding to the classical recourse policy.



(b) Recourse action with $y^\xi_{v_1} = y^\xi_{v_3} = 0$ and $y^\xi_{v_2} = y^\xi_{v_4} = 1$.

Figure 1: Illustration for a route with 4 customers and $C = 10$. We show two recourse actions for a given scenario $\xi \in [N]$ and a directed route $\vec{R} = (v_1, v_2, v_3, v_4)$. The black numbers refer to the demands in scenario ξ, the blue numbers (solid arcs) indicate the values of f^ξ and the red numbers (dashed arcs) indicate the values of g^ξ.

3.2 The convex hull of recourse policies

The following extension of Hoffman's circulation theorem (Hoffman, 1958) (which is a classical application of the max-flow/min-cut theorem, see Section 3.3 of Cook et al. (2011)) will be key for our discussion.

Theorem 1 (Theorem 3.18 of Cook et al. (2011))**.** *Let $D' = (V', A')$ be a directed graph. Let $h \in \mathbb{R}^{A'}_+$ be a vector of arc capacities and let $d' \in \mathbb{R}^{V'}$ be a vector of node demands such that $d'(V') = 0$. There exists a vector $f \in \mathbb{R}^{A'}_+$ satisfying*

$$f(\delta^-(v)) - f(\delta^+(v)) = d'_v, \qquad \forall v \in V',$$
$$0 \leq f_a \leq h_a, \qquad \forall a \in A',$$

if and only if $h(\delta^-(S)) \geq d'(S)$, for all $\emptyset \subsetneq S \subsetneq V'$ (or equivalently, $h(\delta^+(S)) \geq d'(V' \setminus S)$, for all $\emptyset \subsetneq S \subsetneq V'$).

Let $R = (v_1, \ldots, v_\ell)$ be a route. From now on, we abbreviate $y^\xi(V_+(R))$ as $y^\xi(R)$. Moreover, for every scenario $\xi \in [N]$ and set $S \subseteq V_+$, we define $k_\xi(S) := \lceil d^\xi(S)/C \rceil$, and we use $k_\xi(R)$ as a shorthand for $k_\xi(V_+(R))$. The next lemma applies Hoffman's circulation theorem to project out the flow variables in Definition 1.

Lemma 2. *Let $\bar{y} \in \mathbb{R}_+^{[N]\times V_+}$ and $\xi \in [N]$. There exist vectors f^ξ and g^ξ feasible to Formulation 4 with respect to $\bar{y}^\xi$ if and only if*

$$\bar{y}^\xi(R') \geq \frac{d^\xi(R')}{C} - 1, \qquad \forall R' \subseteq R.$$

Proof. Let D' be a digraph with vertex set $V(R)$ and arc set given by the union of the sets $A_1 = A(R)$ and $A_2 = \{(v_i, 0)\}_{i\in[\ell]}$. Set $d'_0 = d^\xi(R)$ and $d'_{v_i} = -d^\xi(v_i)$, for each $i \in [\ell]$. Additionally, set arc arc capacities as $h_a = C$ if $a \in A_1$, and $h_a = C\,\bar{y}^\xi_{v_i}$ if $a = (v_i, 0) \in A_2$.

Theorem 1 implies that Formulation (4) is feasible if and only if $h(\delta^+(S)) \geq d'(V(R) \setminus S)$, for every $S \subseteq V_+(R)$. Fix one such set $S \subseteq V_+(R)$. Write $S = \{V_+(R_i)\}_{i\in[p]}$, where $R_1, \ldots, R_p$ are subroutes of R with $V_+(R_i) \cap V_+(R_j) = \emptyset$, for all $i, j \in [p]$, $i \neq j$. Since $h(\delta^+(S)) = C \cdot \sum_{i=1}^p (\bar{y}^\xi(R_i) + 1)$ and $d'(V(R) \setminus S) = d^\xi(S) = \sum_{i=1}^p d^\xi(R_i)$, it follows that $h(\delta^+(S)) \geq d'(V(R) \setminus S)$ is equivalent to $C \cdot \sum_{i=1}^p (\bar{y}^\xi(R_i) + 1) \geq \sum_{i=1}^p d^\xi(R_i)$, as desired (the necessity of the condition in the statement follows from the case $p = 1$). $\square$

Lemma 2 allows us to derive a *perfect formulation* (see Chapter 4 of Conforti et al. (2014)) for the set of recourse policies. The key observation that enables this result is that the matrix associated with the constraints in Lemma 2 has the *consecutive ones* property, meaning that it is *totally unimodular.*

Theorem 2. *For every route R,*

$$\mathrm{conv}(\Pi(R)) = \left\{ y \in \mathbb{R}_+^{[N]\times V_+} : \ y^\xi(R') \geq k_\xi(R') - 1, \ \ \forall R' \subseteq R, \ \xi \in [N] \right\}.$$

Proof. Fix a route R and let $\tilde{\Pi}$ be the set in the RHS of the statement. We first show that $\Pi(R) = \tilde{\Pi} \cap \mathbb{Z}^{[N]\times V_+}$. If $\bar{y} \in \tilde{\Pi} \cap \mathbb{Z}^{[N]\times V_+}$, then it follows from Lemma 2 that $\bar{y} \in \Pi(R)$. Conversely, if $y' \in \Pi(R)$, then y' satisfies the inequalities in Lemma 2, and as y' is integer (by Definition 1), we can round the RHS of these inequalities to learn that $y' \in \tilde{\Pi} \cap \mathbb{Z}^{V_+}$.

Let M be the matrix given by the inequalities defining $\tilde{\Pi}$. Note that we may rearrange the columns of M so that its rows have the consecutive ones property, meaning that M is totally unimodular (Corollary 2.10 of Nemhauser and Wolsey (1988) or Exercise 4.8 of Conforti et al. (2014)). Hence, Theorem 4.4 of Conforti et al. (2014) implies that $\tilde{\Pi}$ is integral, and by the definition of an integral convex set, $\tilde{\Pi} = \mathrm{conv}(\tilde{\Pi} \cap \mathbb{Z}^{[N]\times V_+}) = \mathrm{conv}(\Pi(R))$. $\square$

Using Theorem 4.5 of Conforti et al. (2014), one can also extend the previous characterization to the case in which upper bounds are imposed on the recourse action vectors.

Corollary 1. *For every route R and $b \in \mathbb{Z}_+^{V_+}$,* $\mathrm{conv}(\Pi(R) \cap [\mathbf{0}, b]^N) = \mathrm{conv}(\Pi(R)) \cap [\mathbf{0}, b]^N$.

4 Scenario-optimal recourse policies and scenario-recourse inequalities

As discussed in Sections 2.3 and 2.4, Ota and Fukasawa (2025) show that existing approaches for the VRPSD (Gendreau et al., 1995; Hoogendoorn and Spliet, 2023; Parada et al., 2024; Legault et al., 2025) replace the nonlinear constraints $\theta_v \geq \sum_{R\in\mathcal{R}(x)} \mathcal{Q}(R, v)$ in $\mathcal{F}(\mathcal{X}, \mathcal{Q})$ with families of valid linear inequalities known as *ILS cuts*. Importantly, these cuts rely on *recourse lower bounds* derived under very specific forms of the recourse function, which makes it difficult to extend their validity to other variants of the problem.

Given our study in Section 3 of recourse policies as solutions to a network-flow model (Formulation (4)), we now assign weights to the variables y^ξ_v in order to obtain lower bounds on the terms $\mathcal{Q}(R, v)$. This allows us to derive stronger non-ILS cuts that remain valid for $\mathcal{F}(\mathcal{X}, \mathcal{Q})$ across different recourse functions $\mathcal{Q}$ associated with VRPSD recourse policies. Moreover, we show that, for a *scenario-optimal* recourse policy $\mathcal{Q}^*$, these new cuts can replace the constraints $\theta_v \geq \sum_{R\in\mathcal{R}(x)} \mathcal{Q}^*(R, v)$ in $\mathcal{F}(\mathcal{X}, \mathcal{Q}^*)$.

4.1 Assigning weights to recourse policies

To link a generic recourse function $\mathcal{Q}$ with a recourse policy, we assume that the recourse cost along a route can be lower bounded by a linear function of the recourse actions vector:

Assumption 3. The given recourse function $\mathcal{Q}$ satisfies Assumption 2. Moreover, we have vectors $w \in \mathbb{Q}_+^{V_+}$ and $b \in \mathbb{Z}_+^{V_+}$ such that, for every route R, there exists $\bar{y}_R \in \Pi(R) \cap [\mathbf{0}, b]^N$ satisfying

$$\mathcal{Q}(R, v) \geq \sum_{\xi \in [N]} p_\xi w_v (\bar{y}_R)_v^\xi, \qquad \forall v \in V_+(R).$$

For example, for the classical recourse policy considered in several works Gendreau et al. (1995); Laporte et al. (2002); Jabali et al. (2014); Hoogendoorn and Spliet (2023); Parada et al. (2024) and extended to scenarios in Ota and Fukasawa (2025), we can set $w_v = 2c_{0v}$. In fact, as we argue next (recall also Example 2), Assumption 3 captures a structural property shared by several recourse functions used in VRPSDs.

The strength of the inequalities derived in the next sections depends critically on the lower bounding vector $w \in \mathbb{R}_+^{V_+}$. For example, one may always set $w = \mathbf{0}$ and then Assumption 3 is trivially satisfied, but this would lead to weak inequalities. Intuitively, one should try to set w_v as a "tight" lower bound on the cost of executing recourse actions at customer $v \in V_+(R)$. For example, for recourse policies that allow preventive return trips between the customers and the depot (Yee and Golden, 1980; Salavati-Khoshghalb et al., 2019b,c; Laporte et al., 2002; Hoogendoorn and Spliet, 2023; Jabali et al., 2014; Gauvin et al., 2014; Florio et al., 2020; Christiansen and Lysgaard, 2007), we may use $w_v = (\min_{u \in V_+ \setminus \{v\}} \{2\, c_{0v}, c_{0u} + c_{0v} - c_{uv}\})^+$, for all $v \in V_+$ (see Remark 1).

Similarly, one should seek for the lowest possible upper bound vector b satisfying Assumption 3. In fact, for any recourse policy (in the sense of Definition 1), we may always assume that $b \leq 2 \cdot \mathbf{1}$. To see this, fix a directed route $\vec{R} = (v_1, \ldots, v_\ell)$ and let $(\bar{f}, \bar{g}, \bar{y})$ be feasible for (4). By Assumption 1, for any $i \in [\ell]$ and $\xi \in [N]$, we know that $\bar{f}^\xi_{(v_{i-1}, v_i)} + d^\xi(v_i) \leq 2C$, which implies that at most two unload trips are needed at v_i (i.e., we may assume that $\bar{y}^\xi_{v_i} \leq 2$). Actually, since the classical recourse policy never fails at the same customer more than once, we may set $b = \mathbf{1}$ whenever $\mathcal{Q} = \mathcal{Q}_C$.

Having said this, we now fix $\mathcal{Q}$ as a recourse function that satisfies Assumption 3 with parameters w and b. In view of the definition of recourse actions/policies from the previous section (Definition 1), we introduce a *scenario-optimal recourse policy* that, given a route, optimally selects the recourse actions for each scenario.

Definition 2. Let $w \in \mathbb{Q}_+^{V_+}$ and $b \in \mathbb{Z}_+^{V_+}$ be such that, for every route R, the set $\Pi(R) \cap [\mathbf{0}, b]^N$ is nonempty. The *scenario-optimal recourse cost* for a route R is given by the optimization problem:

$$\mathcal{Q}^*(R) := \min \left\{ \sum_{\xi \in [N]} \sum_{v \in V_+} p_\xi w_v y_v^\xi : \ y \in \Pi(R) \cap [\mathbf{0}, b]^N \right\}. \qquad (\mathcal{Q}^*)$$

A *scenario-optimal recourse policy* is a vector $y^* \in \Pi(R)$ that attains the above minimum.

Since we fixed w and b, we omit these parameters from the notation for $\mathcal{Q}^*$, and we sometimes refer to $\mathcal{Q}^*$ as the *scenario-optimal recourse function.* It will also be useful to refer to the scenario-optimal recourse cost of a route R in a specific scenario. Thus, for each $\xi \in [N]$, we define $\mathcal{Q}^*_\xi(R) := \min \left\{ \sum_{v \in V_+} w_v y_v^\xi : y^\xi \in \mathcal{Y}^\xi(R) \cap [\mathbf{0}, b] \right\}$, meaning that $\mathcal{Q}^*(R) = \sum_{\xi \in [N]} p_\xi\, \mathcal{Q}^*_\xi(R)$.

In the remainder of this section, we leverage the polyhedral results from Section 3 to derive a formulation for $\text{VRPSD}(\mathcal{X}, \mathcal{Q}^*)$ that does not rely on flow variables. This formulation captures the scenario-optimal recourse function through a new class of (non-ILS) cuts, which we call *scenario-recourse inequalities (SRIs)*. These inequalities not only model $\mathcal{Q}^*$ (Theorem 3), but they are also valid for any recourse function satisfying Assumption 3 (Proposition 1).

Remark 1. It was recently observed by Legault et al. (2025) that, in some cases, the lower bound $w_v = (\min_{u \in V_+ \setminus \{v\}} \{2\, c_{0v}, c_{0u} + c_{0v} - c_{uv}\})^+$ can be quite weak. This situation occurs, for example, when there exist two customers u and v which are located somewhat diametrically opposite to each other (with respect to the depot), so that the quantity $c_{0u} + c_{0v} - c_{uv}$ is close to zero. One might attempt to address such cases by modifying Definition 1 so that the recourse action vector is indexed by edges rather than vertices. However, this breaks the symmetry in Lemma 1, which is later used in Lemma 4. Thus, we propose instead the following weakening of Assumption 3:

Assumption 4. We are given $w \in \mathbb{Q}_+^{V_+ \times V_+}$ and $b \in \mathbb{Z}_+^{V_+}$ such that, for every route $R = (v_1, \ldots, v_\ell)$, there exists $\bar{y}_R \in \Pi(R) \cap [\mathbf{0}, b]^N$ satisfying

$$\mathcal{Q}(R, v_i) \geq \min\{w_{v_i,u} : u \in \{v_i, v_{i-1}, v_{i+1}\} \cap V_+\} \cdot \sum_{\xi \in [N]} p_\xi (\bar{y}_R)_{v_i}^\xi, \qquad \forall i \in [\ell].$$

Under Assumption 4, for every $u, v \in V_+$, we may set $w_{u,v} = c_{0u} + c_{0v} - c_{uv}$ ($c_{uv} = 0$, if $u = v$). In Appendix M, we extend part of the forthcoming results to the case where $\mathcal{Q}$ satisfies Assumption 4. □

4.2 Formulations under a scenario-optimal recourse policy

Recall that polytope $\mathcal{X}$ is either $\mathcal{X}_{\text{SUB}}$ or $\mathcal{X}_{\text{CVRP}}$. We start by extending the definition of recourse policies from routes to solutions (routing plans) in $\mathcal{X} \cap \mathbb{Z}^E$. Thus, for each $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$, we define $\Pi(\bar{x}) := \bigcap_{R \in \mathcal{R}(\bar{x})} \Pi(R)$. The next lemma shows that we can easily map recourse policies defined over solutions to those defined over the associated routes. The proof is given in Appendix B.

Lemma 3. *Let $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$ and $\mathcal{R}(\bar{x}) = \{R_1, \ldots, R_k\}$. Let $\bar{y} \in \mathbb{R}^{[N] \times V_+}$ and $\bar{y}^1, \ldots, \bar{y}^k \in \mathbb{R}^{[N] \times V_+}$ be such that*

$$(\bar{y}^i)_v^\xi = \mathbb{I}(v \in V_+(R_i)) \cdot \bar{y}_v^\xi, \qquad \forall i \in [k], \xi \in [N], v \in V_+.$$

Then $\bar{y} \in \Pi(\bar{x})$ if and only if $(\bar{y}^1, \ldots, \bar{y}^k) \in \Pi(R_1) \times \ldots \times \Pi(R_k)$.

Combining Definition 2 and Lemma 3 yields the following nonlinear formulation for $\text{VRPSD}(\mathcal{X}, \mathcal{Q}^*)$:

$$\begin{aligned}
\min \quad & c^\intercal x + \sum_{v \in V_+} \theta_v, \\
\text{s.t.} \quad & \theta_v \geq \sum_{\xi \in [N]} p_\xi w_v y_v^\xi, && \forall v \in V_+, && \text{(7a)} \\
& x \in \mathcal{X} \cap \mathbb{Z}^E, && && \text{(7b)} \\
& y \in \Pi(x) \cap [\mathbf{0}, b]^N. && && \text{(7c)}
\end{aligned}$$

It follows from Assumption 3 that the projection of the feasible region of Formulation (7) onto the (x, θ)-space yields a relaxation of $\mathcal{F}(\mathcal{X}, \mathcal{Q})$. To present this result, for any set $\mathcal{H} \subseteq \mathbb{R}^{n_1+n_2}$ containing vectors of the form (h, g) with $h \in \mathbb{R}^{n_1}$ and $g \in \mathbb{R}^{n_2}$, we define $\text{proj}_h(\mathcal{H}) := \{h \in \mathbb{R}^{n_1} : \text{exists } g \in \mathbb{R}^{n_2} \text{ such that } (h, g) \in \mathcal{H}\}$.

Proposition 1. *For any recourse function $\mathcal{Q}$ that satisfies Assumption 3,*

$$\mathcal{F}(\mathcal{X}, \mathcal{Q}) \subseteq \text{PROJ}_{(x,\theta)}\left(\{(x, \theta, y) : (x, \theta, y) \text{ is feasible for Formulation (7)}\}\right).$$

Proof. Fix a recourse function $\mathcal{Q}$ as in the statement and let $(\bar{x}, \bar{\theta}) \in \mathcal{F}(\mathcal{X}, \mathcal{Q})$. We need to show that there exists $y \in \Pi(\bar{x}) \cap [\mathbf{0}, b]^N$ such that $(\bar{\theta}, y)$ satisfies (7a). Let $\mathcal{R}(\bar{x}) = \{R_1, \dots, R_k\}$ and, for each $i \in [k]$, choose $\bar{y}^i \in \Pi(R_i) \cap [\mathbf{0}, b]^N$ according to Assumption 3. Apply Lemma 3 to get a vector $\bar{y} \in \Pi(\bar{x}) \cap [\mathbf{0}, b]^N$. For every $i \in [k]$ and $v \in V_+(R_i)$,

$$\bar{\theta}_v \geq \sum_{R \in \mathcal{R}(\bar{x})} \mathcal{Q}(R, v) \overset{\text{Assumption 2}}{=} \mathcal{Q}(R_i, v) \overset{\text{Assumption 3}}{\geq} \sum_{\xi \in [N]} p_\xi w_v (\bar{y}^i)^\xi_v \overset{\text{Lemma 3}}{=} \sum_{\xi \in [N]} p_\xi w_v \bar{y}^\xi_v,$$

as desired. ☐

Proposition 1 motivates our study of valid inequalities for Formulation (7), since these inequalities can be applied to the recourse function $\mathcal{Q}$ that we fixed in the beginning of this section. As a first step in this direction, we build on Lemma 1 to reformulate the family of sets $\Pi(x)_{x \in \mathcal{X} \cap \mathbb{Z}^E}$ as a single MILP, parameterized by a vector $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$. We defer the proof to Appendix C.

Lemma 4. *Fix $b \in \mathbb{Z}_+^{V_+}$. For every $\bar{x} \in \mathcal{X}$, define*

$$\textsc{flow}(\bar{x}) := \left\{ y \in [\mathbf{0}, b]^N : \begin{array}{ll} f^\xi(\delta^-(v)) + d^\xi(v) = f^\xi(\delta^+(v)) + g^\xi_v, & \forall v \in V_+, \xi \in [N] \\ f^\xi_{(u,v)} \leq \dfrac{C}{2} \bar{x}_{\{u,v\}}, & \forall uv \in A, \xi \in [N] \\ g^\xi_v \leq C \cdot y^\xi_v, & \forall v \in V_+, \xi \in [N] \\ f \in \mathbb{R}_+^{[N] \times A}, g \in \mathbb{R}_+^{[N] \times V_+} & \end{array} \right\}.$$

Then $\textsc{flow}(\bar{x}) \cap \mathbb{Z}^{[N] \times V_+} = \Pi(\bar{x}) \cap [\mathbf{0}, b]^N$, for every $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$.

Lemma 4 yields an MILP model for Formulation (7), since we can replace constraint (7c) with the constraints and variables in the (parametric) MILP formulation for $\textsc{flow}(\bar{x}) \cap \mathbb{Z}^{[N] \times V_+}$. However, this approach may be inefficient, since it leads to a formulation for $\textsc{vrpsd}(\mathcal{X}, \mathcal{Q}^*)$ that has integrality on the y-variables and uses $N \cdot (|A| + |V_+|)$ additional flow variables.

Similarly to Lemma 2, we address this issue by using Hoffman's circulation theorem (Theorem 1) to project out the flow variables. The proof is left to Appendix D.

Proposition 2. *Fix $b \in \mathbb{Z}_+^{V_+}$. For every $\bar{x} \in \mathcal{X}$,*

$$\textsc{flow}(\bar{x}) = \left\{ y \in [\mathbf{0}, b]^N : y^\xi(S) \geq \frac{d^\xi(S)}{C} + \bar{x}(E(S)) - |S|, \ \forall \emptyset \subsetneq S \subseteq V_+, \ \xi \in [N] \right\}.$$

Recall that $k_\xi(S) = \lceil d^\xi(S)/C \rceil$. Since Formulation (7) enforces integrality on the (x, y)-variables, we may round the RHS of the inequalities in Proposition 2 to obtain

$$y^\xi(S) \geq k_\xi(S) + x(E(S)) - |S|, \tag{8}$$

which we call a *scenario-recourse inequality (SRI)*. (We invite the reader to compare the SRIs with the inequalities in Theorem 2.)

The following example illustrates that SRIs may indeed yield a stronger relaxation than the set $\textsc{flow}(\bar{x})$ in Lemma 4.

Example 3. Consider the setting in Figure 1b. Suppose that $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$ is such that $\mathcal{R}(\bar{x})$ only contains the route $R = (v_1, v_2, v_3, v_4)$ and let $(\bar{y}, \bar{f}, \bar{g})$ be feasible for the formulation of $\textsc{flow}(\bar{x})$ in Lemma 4. We illustrate these vectors in Figure 2 for a fixed scenario $\xi \in [N]$. Note that the recourse cost captured by the vector $\bar{y}^\xi$ is given by $\sum_{v \in V_+} w_v \bar{y}^\xi_v = 2.4$, which is lower than the scenario-optimal recourse cost $\mathcal{Q}^*_\xi(R) = 4$.

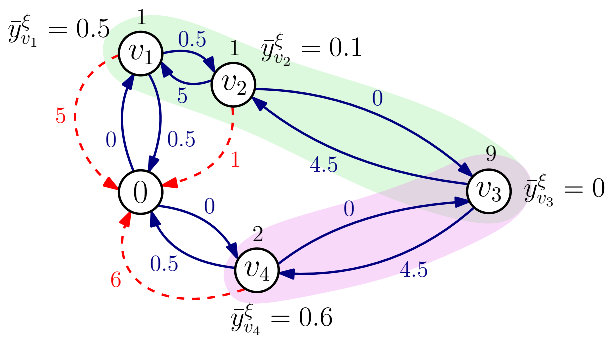


Figure 2: Graphical representation of $(\bar{x}, \bar{y}^{\xi}, \bar{f}^{\xi}, \bar{g}^{\xi})$. Here we have $C = 10$, $w_{v_3} = 6$ and $w_{v_1} = w_{v_2} = w_{v_4} = 2$. The black numbers next to each vertex indicate the scenario demand vector d^{ξ}. The blue and solid arcs represent the vector $\bar{f}^{\xi}$, while the red and dashed arcs represent the vector $\bar{g}^{\xi}$.

Let $S_1 = \{v_1, v_2, v_3\}$ and $S_2 = \{v_3, v_4\}$ (green and violet regions in Figure 2, respectively). Observe that $\bar{y}^{\xi}$ violates the SRIs for S_1 and S_2 (with respect to scenario ξ), since

$$0.6 = \bar{y}^{\xi}(S_i) < \lceil d^{\xi}(S_i)/C \rceil + \bar{x}(E(S_i)) - |S_i| = 1,$$

for all $i \in [2]$.

Interestingly, in this case, the SRIs are sufficient to correctly capture the scenario-optimal recourse cost $\mathcal{Q}^*_{\xi}(R)$. To see this, consider the vector $(y')^{\xi} \in \mathcal{Y}^{\xi}(R)$ defined as $(y')^{\xi}_{v_2} = (y')^{\xi}_{v_4} = 1$ and $(y')^{\xi}_{v_1} = (y')^{\xi}_{v_3} = 0$, and notice that $\mathcal{Q}^*_{\xi}(R) = \sum_{v \in V_+} w_v (y')^{\xi}_v = 4$. Summing the SRIs for S_1 and S_2 (and fixing $x = \bar{x}$) yields $y^{\xi}_{v_1} + y^{\xi}_{v_2} + 2y^{\xi}_{v_3} + y^{\xi}_{v_4} \geq 2$, and multiplying this inequality by 2 we learn that

$$\sum_{v \in V_+} w_v y^{\xi}_v \geq 2y^{\xi}_{v_1} + 2y^{\xi}_{v_2} + 4y^{\xi}_{v_3} + 2y^{\xi}_{v_4} \geq 4.$$

Therefore, $(\bar{x}, (y')^{\xi})$ minimizes $\sum_{v \in V_+} w_v y^{\xi}_v$ subject to the constraints that $y^{\xi} \in [\mathbf{0}, b]$ and $(\bar{x}, y^{\xi})$ satisfies all the SRIs of the form $y^{\xi}(S) \geq k_{\xi}(S) + \bar{x}(E(S)) - |S|$, for every $S \subseteq V_+$. $\square$

It turns out that a consequence of our polyhedral study in Section 3.2 is that the situation illustrated in Example 3 holds more generally, and constraint (7c) can be replaced by the SRIs (and the box constraint $y \in [\mathbf{0}, b]^N$). We prove this result in Appendix E.

Theorem 3. *Fix $b \in \mathbb{Z}^{V_+}_+$. For every $\bar{x} \in \mathcal{X}$, define*

$$\textsc{sri}(\bar{x}) := \left\{ y \in [\mathbf{0}, b]^N : y^{\xi}(S) \geq k_{\xi}(S) + \bar{x}(E(S)) - |S|, \ \forall \emptyset \subsetneq S \subseteq V_+, \xi \in [N] \right\}.$$

Then $\textsc{sri}(\bar{x}) = \operatorname{conv}(\Pi(\bar{x}) \cap [\mathbf{0}, b]^N)$, for every $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$.

Hence, we can formulate $\textsc{vrpsd}(\mathcal{X}, \mathcal{Q}^*)$ as

$$\begin{aligned}
\min \quad & c^{\mathsf{T}} x + \sum_{v \in V_+} \theta_v, & & \\
\text{s.t.} \quad & \theta_v \geq \sum_{\xi \in [N]} p_{\xi} w_v y^{\xi}_v, & \forall v \in V_+, & \quad (9a)\\
& y^{\xi}(S) \geq k_{\xi}(S) + x(E(S)) - |S|, & \forall \emptyset \subsetneq S \subseteq V_+, \xi \in [N], & \quad (9b)\\
& (x, y) \in (\mathcal{X} \cap \mathbb{Z}^E) \times [\mathbf{0}, b]^N. & & \quad (9c)
\end{aligned}$$

Due to the exponential number of SRIs, we describe in Appendix F the heuristics that we use to iteratively separate these inequalities as cutting planes. In these heuristics, whenever a violated inequality is found, we "aggregate" the SRIs using the following simple fact.

Fact 1. *Let* $(\bar{x}, \bar{y}) \in \mathcal{X} \times [\mathbf{0}, b]^N$, $\xi' \in [N]$ *and* $S \subseteq V_+$ *be such that* $\bar{y}^{\xi'}(S) < k_\xi(S) + \bar{x}(E(S)) - |S|$. *Define* $\Xi := \{\xi \in [N] : \bar{y}^\xi(S) < k_\xi(S) + \bar{x}(E(S)) - |S|\}$, *then* $\sum_{\xi \in \Xi} \bar{y}^\xi(S) < \sum_{\xi \in \Xi}(k_\xi(S) + \bar{x}(E(S)) - |S|)$. □

Accordingly, we call inequality $\sum_{\xi \in \Xi} y^\xi(S) \geq \sum_{\xi \in \Xi}(k_\xi(S) + x(E(S)) - |S|)$ an *aggregated SRI* with respect to $\Xi \subseteq [N]$ and $S \subseteq V_+$. Preliminary experiments indicate that separating aggregated SRIs, rather than individual SRIs, significantly improves the overall efficiency of the algorithm.

4.3 LP formulation for the scenario-optimal recourse function

Besides Theorem 3, another consequence of the characterization in Theorem 2 is that the scenario-optimal recourse cost of a route can be formulated as an LP (note that the optimization problem in Definition 2 is an MILP). Specifically, for a given route R, we may write $\mathcal{Q}^*(R) = \sum_{\xi \in [N]} p_\xi \, \mathcal{Q}^*_\xi(R)$, where

$$\begin{aligned} \mathcal{Q}^*_\xi(R) = \min \quad & \sum_{v \in V_+} w_v y^\xi_v, \\ \text{s.t.} \quad & y^\xi(R') \geq k_\xi(R') - 1, && \forall R' \subseteq R, && \text{(10a)} \\ & y^\xi \in [\mathbf{0}, b]. && && \text{(10b)} \end{aligned}$$

As a result, scenario-optimal recourse policies satisfy the following conditions.

Corollary 2. *Let* $\mathcal{Q}^*$ *be a recourse function as in Definition 2 with parameters* $w \in \mathbb{Q}_+^{V_+}$ *and* $b \in \mathbb{Z}_+^{V_+}$. *Suppose we have a disaggregation of* $\mathcal{Q}^*$ *satisfying the following:*

- *for every route* R, *there exists a scenario-optimal recourse policy* $y^* \in \Pi(R)$ *such that* $\mathcal{Q}^*(R, v) = \sum_{\xi \in [N]} p_\xi w_v (y^*)^\xi_v$, *for every customer* $v \in V_+$.

Then, for every route R *and subroute* $R' \subseteq R$, $\sum_{v \in V_+(R')} \mathcal{Q}^*(R', v) \leq \sum_{v \in V_+(R')} \mathcal{Q}^*(R, v)$.

Proof. Let R be a route and $R' \subseteq R$. Let $y^* \in \Pi(R)$ be a scenario-optimal recourse policy such that, for all $v \in V_+$, $\mathcal{Q}^*(R, v) = \sum_{\xi \in [N]} p_\xi w_v (y^*)^\xi_v$. Note that, for each scenario $\xi \in [N]$, $(y^*)^\xi$ is feasible for Formulation (10) with respect to route R', which shows that $\mathcal{Q}^*_\xi(R') \leq \sum_{v \in V_+(R')} p_\xi w_v (y^*)^\xi_v$. Summing this inequality over all scenarios $\xi \in [N]$ yields the desired result. □

Corollary 2 implies that scenario-optimal recourse policies satisfy the *monotonicity/(restricted) superadditivity* properties discussed in the context of the *DL-shaped method* (Parada et al., 2024; Legault et al., 2025; Ota and Fukasawa, 2025). Therefore, rather than using SRIs, one could also solve $\textsc{vrpsd}(\mathcal{X}, \mathcal{Q}^*)$ using the *path* and *set* cuts of the DL-shaped method (path cuts are a special case of the partial route cuts (2) from Section 2.4). However, as we demonstrate next, we can instead leverage Formulation (10) to prove that these ILS cuts are dominated by the SRIs.

5 Projecting out the scenario variables and a branch-and-cut algorithm

In this section, we study the projection of Formulation (9) onto the (x, θ)-space of $\mathcal{F}(\mathcal{X}, \mathcal{Q})$ from both theoretical and computational perspectives. In view of the results in the previous section, it is natural to ask whether it is preferable to formulate $\textsc{vrpsd}(\mathcal{X}, \mathcal{Q}^*)$ using ILS cuts, SRIs, or a combination of both.

From a theoretical standpoint, we show in Section 5.1 that the projection of Formulation (9) onto the (x, θ)-variables can be described by a family of inequalities which we call *projected SRIs*.[1] Section 5.2 then shows that, under certain assumptions on the recourse lower bounds, projected SRIs dominate the ILS cuts from Ota and Fukasawa (2025) with respect to the recourse function $\mathcal{Q}^*$. In other words, these ILS cuts are already satisfied by every feasible solution to the LP relaxation of Formulation (9).

From a computational perspective, we first note that the proofs in Section 5.2 are constructive, and they provide algorithms to generate the dominating projected SRIs. In addition, our preliminary experiments indicate that the LP relaxation of Formulation (9) often provides strong dual bounds, even when the SRIs are separated heuristically (as in Appendix F). However, the large number of y_v^ξ-variables can make solving the formulation to integrality computationally expensive. Motivated by this observation, Section 5.3 shows that solving the LP relaxation of Formulation (9) also provides a single projected SRI that recovers the LP bound of Formulation (9) in the (x, θ)-space. The results of Sections 5.2 and 5.3 are then combined in Section 5.4 to develop a branch-and-cut algorithm for solving $\text{VRPSD}(\mathcal{X}, \mathcal{Q}^*)$ using projected SRIs. Finally, Section 5.5 discusses how we can extend this algorithm to different recourse policies.

In the remainder of this section, we fix the parameters $w \in \mathbb{Q}_+^{V_+}$ and $b \in \mathbb{Z}_+^{V_+}$ of the scenario-optimal recourse function $\mathcal{Q}^*$ (Definition 2). We also fix a disaggregation of $\mathcal{Q}^*$ as specified in Corollary 2.

5.1 Projected scenario recourse inequalities

For every $\bar{x} \in \mathcal{X}$, we augment the set $\text{SRI}(\bar{x})$ as follows:

$$\widehat{\text{SRI}}(\bar{x}) := \left\{ (\theta, y) \in \mathbb{R}_+^{V_+} \times [\mathbf{0}, b]^N : \begin{array}{lr} y^\xi(S) \geq k_\xi(S) + \bar{x}(E(S)) - |S|, & \forall \emptyset \subsetneq S \subseteq V_+,\ \xi \in [N] \\ \theta_v \geq \displaystyle\sum_{\xi \in [N]} p_\xi w_v y_v^\xi, & \forall v \in V_+ \end{array} \right\}. \tag{$\widehat{\text{SRI}}(\bar{x})$}$$

Therefore, we express the projection of the feasible region of the LP relaxation of Formulation (9) as

$$\mathcal{P} := \left\{ (x, \theta) : x \in \mathcal{X},\ \ \theta \in \text{proj}_\theta(\widehat{\text{SRI}}(x)) \right\}. \tag{$\mathcal{P}$}$$

To better understand the strength of the LP relaxation of Formulation (9), we study the polyhedron $\mathcal{P}$, which in turn leads us to analyze the set $\text{proj}_\theta(\widehat{\text{SRI}}(x))$. In this way, one of our goals in this subsection is to explicitly write $\text{proj}_\theta(\widehat{\text{SRI}}(x))$ as the intersection of half-spaces defined by inequalities on the θ-variables.

We start by addressing in Lemma 5 the simpler case of projecting a single inequality defined on the y-variables. This result is not only useful for characterizing $\text{proj}_\theta(\widehat{\text{SRI}}(x))$, but it will also be key in Section 5.2 to construct projected SRIs that dominate certain ILS cuts. The proof is deferred to Appendix G.

Lemma 5. *Let $a^1, \ldots, a^N$ be vectors in $\mathbb{R}^{V_+}$ and let $h \in \mathbb{R}$. Consider the set*

$$\mathcal{H} = \left\{ (\theta, y) \in \mathbb{R}_+^{V_+} \times \mathbb{R}_+^{[N] \times V_+} : \sum_{\xi \in [N]} (a^\xi)^\intercal y^\xi \geq h,\ \theta_v \geq \sum_{\xi \in [N]} p_\xi w_v y_v^\xi, \quad \forall v \in V_+ \right\}.$$

Then the following holds:

[1] In stochastic programming, the term *aggregated cuts* is sometimes used to refer to the inequalities obtained in a single-cut implementation of the Benders decomposition method (Rahmaniani et al., 2017), where the scenario recourse variables are projected onto a single variable given by their sum. Our projected SRIs are similar, in the sense that we project from the (x, y)-variables onto the (x, θ)-space. (Note that the term *aggregated SRI* was already used in Fact 1.)

- *if there exists* $v \in V_+$ *and* $\xi \in [N]$ *such that* $w_v = 0$ *and* $a_v^\xi > 0$, *then* $\mathrm{proj}_\theta(\mathcal{H}) = \mathbb{R}_+^{V_+}$*;*
- *otherwise,* $\mathrm{proj}_\theta(\mathcal{H}) = \left\{\theta \in \mathbb{R}_+^{V_+} : \sum_{v \in V_+ : w_v > 0} \left(\max_{\xi \in [N]} \left\{\frac{a_v^\xi}{p_\xi w_v}\right\}\right)^+ \theta_v \geq h\right\}$.

In what follows, we apply Lemma 5 to the case where the inequality $\sum_{\xi \in [N]} (a^\xi)^\intercal y^\xi \geq h$ is given by a conic combination of SRIs and upper bound constraints on the y-variables. To this end, we reserve the symbol α for a vector of multipliers associated with the SRIs, with components α_S^ξ, for all $S \subseteq V_+$ and $\xi \in [N]$. Similarly, β is reserved for a vector of multipliers associated with the upper bound constraints $y_v^\xi \leq b_v$, with components β_v^ξ, for all $v \in V_+$ and $\xi \in [N]$.

Given $\alpha \geq 0$ and $\beta \leq 0$, we construct the inequalities:

$$\sum_{\xi \in [N]} \sum_{S \subseteq V_+} \alpha_S^\xi y^\xi(S) \geq \sum_{\xi \in [N]} \sum_{S \subseteq V_+} \alpha_S^\xi \left(k_\xi(S) + x(E(S)) - |S|\right), \tag{11}$$

$$\sum_{\xi \in [N]} \sum_{v \in V_+} \beta_v^\xi y_v^\xi \geq \sum_{\xi \in [N]} \sum_{v \in V_+} \beta_v^\xi b_v. \tag{12}$$

Summing (11) with (12) and rearranging the terms gives

$$\sum_{\xi \in [N]} \sum_{v \in V_+} \left(\beta_v^\xi + \sum_{S \subseteq V_+ : v \in S} \alpha_S^\xi\right) y_v^\xi \geq \sum_{\xi \in [N]} \sum_{S \subseteq V_+} \alpha_S^\xi x(E(S)) + \nu(\alpha, \beta), \tag{13}$$

where

$$\nu(\alpha, \beta) \coloneqq \sum_{\xi \in [N]} \sum_{U \subseteq V_+} \alpha_S^\xi (k_\xi(S) - |S|) + \sum_{\xi \in [N]} \sum_{v \in V_+} \beta_v^\xi b_v. \tag{ν}$$

Next, for each $\bar{x} \in \mathcal{X}$, define

$$\Gamma(\bar{x}, \alpha, \beta) \coloneqq \left\{(\theta, y) \in \mathbb{R}_+^{V_+} \times [\mathbf{0}, b]^N : \ (\bar{x}, y) \text{ satisfies (13)}, \ \theta_v \geq \sum_{\xi \in [N]} p_\xi w_v y_v^\xi, \quad \forall v \in V_+\right\}. \tag{Γ}$$

By Lemma 5, if $\left(\beta_v^\xi + \sum_{S \subseteq V_+ : v \in S} \alpha_S^\xi\right) \leq 0$, for all $\xi \in [N]$ and $v \in V_+$ with $w_v = 0$, we have that

$$\mathrm{proj}_\theta(\Gamma(\bar{x}, \alpha, \beta)) = \left\{\theta \in \mathbb{R}_+^{V_+} : \sum_{v \in V_+ : w_v > 0} \phi_v(\alpha, \beta)\, \theta_v \geq \sum_{\xi \in [N]} \sum_{S \subseteq V_+} \alpha_S^\xi\, \bar{x}(E(S)) + \nu(\alpha, \beta)\right\}, \tag{14}$$

where, for each $v \in V_+$ with $w_v > 0$,

$$\phi_v(\alpha, \beta) \coloneqq \left(\max_{\xi \in [N]} \left\{\frac{\beta_v^\xi + \sum_{S \subseteq V_+ : v \in S} \alpha_S^\xi}{p_\xi w_v}\right\}\right)^+. \tag{ϕ_v}$$

We refer to the inequality used to define the set in (14) as a *projected SRI*. For convenience, we also define $\mathcal{A}$ as the set of multipliers (α, β), with $\alpha \geq 0$ and $\beta \leq 0$, such that $\left(\beta_v^\xi + \sum_{S \subseteq V_+ : v \in S} \alpha_S^\xi\right) \leq 0$, for all $\xi \in [N]$ and $v \in V_+$ with $w_v = 0$. The next result builds on Lemma 5 to prove that projected SRIs characterize the set $\mathrm{proj}_\theta(\widehat{\mathrm{SRI}}(\bar{x}))$. The proof is left to Appendix H.

Proposition 3. *For every $\bar{x} \in \mathcal{X}$,*

$$\mathrm{proj}_\theta(\widehat{\mathrm{SRI}}(\bar{x})) = \left\{ \theta \in \mathbb{R}_+^{V_+} : \sum_{v \in V_+ : w_v > 0} \phi_v(\alpha, \beta)\, \theta_v \geq \sum_{\xi \in [N]} \sum_{S \subseteq V_+} \alpha_S^\xi\, \bar{x}(E(S)) + \nu(\alpha, \beta), \quad \forall (\alpha, \beta) \in \mathcal{A} \right\}.$$

Consequently,

$$\mathcal{P} = \left\{ (x, \theta) \in \mathcal{X} \times \mathbb{R}_+^{V_+} : \sum_{v \in V_+ : w_v > 0} \phi_v(\alpha, \beta)\, \theta_v \geq \sum_{\xi \in [N]} \sum_{S \subseteq V_+} \alpha_S^\xi\, x(E(S)) + \nu(\alpha, \beta), \quad \forall (\alpha, \beta) \in \mathcal{A} \right\}.$$

Example 4. Suppose that $\mathcal{X} = \mathcal{X}_{\text{CVRP}}$ and let $\emptyset \subsetneq S \subseteq V_+$ be such that $w_v > 0$, for all $v \in S$. Let $\Xi = \{\xi \in [N] : k_\xi(S) > \bar{k}(S)\}$ and assume that Ξ is nonempty. Set $\beta = \mathbf{0}$ and $\alpha_{U'}^\xi = \mathbb{I}(U' = U \text{ and } \xi \in \Xi)$, for all $U' \subseteq V_+$ and $\xi \in [N]$. The pair (α, β) yields the following *aggregated SRI (see Fact 1):*

$$\sum_{\xi \in \Xi} y^\xi(S) \geq \sum_{\xi \in \Xi} (k_\xi(S) + x(E(S)) - |S|) = |\Xi| \cdot (x(E(S)) - |S|) + \sum_{\xi \in \Xi} k_\xi(S)$$

By Lemma 5, for any $\bar{x} \in \mathcal{X}$, $\bar{\theta}$ belongs to $\mathrm{proj}_\theta \Gamma(\bar{x}, \alpha, \beta)$ if and only if $\bar{\theta}$ satisfy the projected SRI:

$$\sum_{v \in S} \max_{\xi \in \Xi} \left\{ \frac{1}{p_\xi w_v} \right\} \theta_v \geq |\Xi| \cdot (x(E(S)) - |S|) + \sum_{\xi \in \Xi} k_\xi(S). \tag{15}$$

We henceforth call inequality (15) a *projected aggregated SRI* associated with $\Xi \subseteq [N]$ and $S \subseteq V_+$. □

5.2 Theoretical comparisons with ILS cuts

In this subsection, we show that, under some conditions on the recourse lower bounds, the ILS cuts shown in Section 2.4 (applied to the scenario-optimal recourse function $\mathcal{Q}^*$) are valid for the set $\mathcal{P}$. Furthermore, using Lemma 5 (and Equation (14)), we also construct the corresponding projected SRIs that dominate these cuts. The general proof strategy consists of two main steps: (1) leverage Formulation (10) to express a recourse lower bound as the optimal value of an LP; and (2) use the dual multipliers (α, β) of this LP to construct a dominating projected SRI.

5.2.1 Comparison with set cuts

The approach of Ota and Fukasawa (2025) to compute a recourse lower bound for the set cut (17) can be interpreted as a simple greedy algorithm. Fix $S = \{v_1, \ldots, v_\ell\} \subseteq V_+$ and a scenario $\xi \in [N]$. Let $k' \in \mathbb{Z}_{++}$ be such that inequality $\bar{x}(E(S)) \leq |S| - k'$ is valid for $\mathcal{X} \cap \mathbb{Z}^E$, and let $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$ satisfy this inequality at equality. Additionally, suppose that $0 < k_\xi(S) - k' \leq b(S)$ (see Remark 2).

To obtain a lower bound on the recourse cost, we assign the $k_\xi(S) - k'$ failures observed by $\bar{x}$ to the customers in S in increasing order of their recourse weights. Thus, assume that $w_{v_1} \leq \ldots \leq w_{v_\ell}$, and let $j \in [\ell]$ be the smallest index such that $k_\xi(S) - k' \leq \sum_{i \in [j]} b_{v_i}$. A lower bound on the recourse cost paid in scenario $\xi \in [N]$ is then given by $\mathcal{L}_\xi^*(S, k') = \sum_{i \in [j-1]} b_{v_i} w_{v_i} + ((k_\xi(S) - k') - \sum_{i \in [j-1]} b_{v_i}) w_{v_j}$. (Recall that $[t] = \emptyset$, for every integer $t \leq 0$.)

It is easy to see that this greedy algorithm is equivalent to the LP:

$$\mathcal{L}_\xi^*(S, k') := \min \sum_{v \in V_+} w_v y_v^\xi,$$

$$\text{s.t. } y^\xi(S) \geq k_\xi(S) - k', \qquad (\alpha_S^\xi) \tag{16a}$$

$$y_v^\xi \leq b_v, \qquad \forall v \in V_+, \qquad (\beta_v^\xi) \tag{16b}$$

$$y^\xi \geq 0. \tag{16c}$$

This yields the recourse lower bound $\mathcal{L}^*(S,k') = \sum_{\xi=1}^{N} p_\xi \, \mathcal{L}^*_\xi(S,k')$ (note that $\mathcal{L}^*_\xi(S,k') = 0$ whenever $k_\xi(S) \le k'$).

Replacing this recourse lower bound in inequality (3) we obtain the set cut

$$\theta(S) \ge \mathcal{L}^*(S,k') \cdot W_{DL}(x; \mathcal{X}(S,k')), \tag{17}$$

where we recall from Section 2.4 that $W_{DL}(x;\mathcal{X}(S,k')) = 1 + (x(E(S)) - |S| + k')$.

Remark 2. Observe that Formulation (16) is feasible if and only if $k_\xi(S) - k' \le b(S)$. This condition always holds when $b \ge \mathbf{1}$, since $k_\xi(S) = \lceil d^\xi(S)/C \rceil \le |S| \le b(S)$ (by Assumption 1, $d^\xi(v) \le C$, for all $v \in V_+$). Alternatively, suppose that there exists $\bar{x} \in \mathcal{X}(S,k')$ (so $\bar{x}(E(S)) = |S| - k'$) and let $\mathcal{R}(\bar{x}) = \{R_1, \ldots, R_k\}$. Since b satisfies the condition in Definition 2, for each $i \in [k]$, there exists $\bar{y}^i \in \Pi(R_i) \cap [\mathbf{0}, b]^N$. Applying Lemma 3 to these vectors we obtain $\bar{y} \in \Pi(\bar{x}) \cap [\mathbf{0}, b]^N$, and since SRIs (with x fixed to $\bar{x}$) are valid for $\Pi(\bar{x})$, we get $b(S) \ge \bar{y}^\xi(S) \ge k_\xi(S) + \bar{x}(E(S)) - |S| = k_\xi(S) - k'$. $\square$

One may note that Formulation (16) has a similar structure to the inner optimization problems found in robust optimization problems with budget uncertainty sets (Bertsimas and Sim, 2003, 2004). Thus, analogously to known results in the robust optimization literature, we can write closed-form expressions for an optimal dual solution for Formulation (16). Combining this with the general strategy outlined at the beginning of this subsection yields the desired results. The proofs are deferred to Appendix I, and a concrete example is provided in Appendix J.

Lemma 6. *Let $w \in \mathbb{Q}_+^{V_+}$ and $b \in \mathbb{Z}_+^{V_+}$. Fix a scenario $\xi \in [N]$ and let $S = \{v_1, \ldots, v_\ell\} \subseteq V_+$ be such that $w_{v_1} \le \ldots \le w_{v_\ell}$. Let $k' \in \mathbb{Z}_{++}$ be such that $k_\xi(S) - k' \le b(S)$ and choose $j \in [\ell]$ as the smallest index such that $k_\xi(S) - k' \le \sum_{i \in [j]} b_{v_i}$. Set*

$$\begin{aligned}
\bar{\alpha}^\xi_S &= \mathbb{I}(k_\xi(S) > k') \cdot w_{v_j}, \quad \textit{and} \\
\bar{\beta}^\xi_v &= \mathbb{I}(v \in \{v_i\}_{i \in [j-1]}) \cdot (w_v - w_{v_j}), & \forall v \in V_+.
\end{aligned}$$

Then $(\bar{\alpha}^\xi_S, \bar{\beta}^\xi)$ is optimal for the dual of Formulation (16) and $\bar{\alpha}^\xi_S \le \mathcal{L}^*_\xi(S,k')$.

Theorem 4. *Let $w \in \mathbb{Q}_+^{V_+}$, $b \in \mathbb{Z}_+^{V_+}$, $\emptyset \subsetneq S \subseteq V_+$ and $k' \in \mathbb{Z}_{++}$. Assume that Formulation (16) is feasible for every scenario $\xi \in [N]$ (see Remark 2). For each $\xi \in [N]$, let $(\bar{\alpha}^\xi_S, \bar{\beta}^\xi)$ be as specified in Lemma 6, and set $\hat{\alpha} \ge 0$ and $\hat{\beta} \le 0$ as*

$$\begin{aligned}
\hat{\alpha}^\xi_{S'} &= \mathbb{I}(S' = S) \cdot (p_\xi \, \bar{\alpha}^\xi_S), & \forall \emptyset \subsetneq S' \subseteq V_+, \xi \in [N], \\
\hat{\beta}^\xi_v &= p_\xi \, \bar{\beta}^\xi_v, & \forall v \in V_+, \xi \in [N].
\end{aligned}$$

Then, $(\hat{\alpha}, \hat{\beta}) \in \mathcal{A}$ and for every $\bar{x} \in \{x \in \mathcal{X} : x(E(S)) \le |S| - k'\}$, inequality (17) (with x fixed to $\bar{x}$) is valid for $\mathrm{proj}_\theta(\Gamma(\bar{x}, \hat{\alpha}, \hat{\beta}))$. In particular, if $x(E(S)) \le |S| - k'$ is valid for $\mathcal{P}$, then inequality (17) (with x free) is also valid for $\mathcal{P}$.

5.2.2 Comparisons with partial route cuts

To compute recourse lower bounds for the partial route cuts (1) and (2), we propose an approach based on Formulation (10). To this end, let $H = (S_1, \ldots, S_\ell)$ be a partial route. We write $H' \subseteq H$ to refer to a partial route of the form $H' = (S_i, \ldots, S_j)$ for some $j \in [\ell]$ and $i \in [j]$. Moreover, for every scenario $\xi \in [N]$, we use the shorthands $y^\xi(H') = y^\xi(V_+(H'))$ and $k_\xi(H') =$

$k_\xi(V_+(H'))$. For each scenario $\xi \in [N]$, let $\mathcal{L}^*_\xi(H)$ be the optimal value of the following LP:

$$
\begin{aligned}
\mathcal{L}^*_\xi(H) \coloneqq \min \quad & \sum_{v \in H} w_v y^\xi_v, \\
\text{s.t.} \quad & y^\xi(H') \geq k_\xi(H') - 1, && \forall H' \subseteq H, \quad (\alpha^\xi_{H'}) && \text{(18a)} \\
& y^\xi_v \leq b_v, && \forall v \in V_+, \quad (\beta^\xi_v) && \text{(18b)} \\
& y^\xi_v \geq 0, && \forall v \in V_+, && \text{(18c)}
\end{aligned}
$$

and define $\mathcal{L}^*(H) \coloneqq \sum_{\xi=1}^N p_\xi \, \mathcal{L}^*_\xi(H)$. Observe that, if $S_i = \{v_i\}$, for all $i \in [\ell]$, then $\mathcal{L}^*(H) = \mathcal{Q}^*(R)$, where $R = (v_1, \ldots, v_\ell)$.

Replacing $\mathcal{L}^*(H)$ into the partial route cut (2) yields

$$\theta(V_+(H)) \geq \mathcal{L}^*(H) \cdot W_{OF}(x; \mathcal{X}_{\supseteq}(H)). \tag{19}$$

It is not hard to see that (19) is valid for $\mathcal{F}(\mathcal{X}, \mathcal{Q}^*)$. Indeed, this follows from the monotonicity of the disaggregation of $\mathcal{Q}^*$ (Corollary 2) together with the fact that Formulation (18) provides a relaxation of Formulation (10) for any route R that adheres to H. Rather than providing more details on this argument, we establish validity by proving that (19) is valid for $\mathcal{P} \supseteq \mathcal{F}(\mathcal{X}, \mathcal{Q}^*)$. Specifically, let α^ξ and β^ξ be the dual variables associated with Formulation (18). We prove the following result in Appendix K.

Theorem 5. *Let H be a partial route. There exist $(\alpha, \beta) \in \mathcal{A}$ such that the projected SRI*

$$\sum_{v \in V_+} \phi_v(\alpha, \beta) \, \theta_v \geq \sum_{\xi \in [N]} \sum_{S \subseteq V_+} \alpha^\xi_S \, x(E(S)) + \nu(\alpha, \beta)$$

dominates inequality (19). In particular, inequality (19) is valid for $\mathcal{P}$.

Combining Theorem 5 with the results in Ota and Fukasawa (2025) yields the following consequences.

Corollary 3. *The following inequalities are dominated by projected SRIs (and thus valid for $\mathcal{P}$):*

(a) the partial route cuts of Hoogendoorn and Spliet (2023):

$$\theta(V_+(H)) \geq \mathcal{L}^*(H) \cdot W_{HS}(x; \mathcal{X}_{=}(H)),$$

for every partial route H;

(b) the path cuts of Parada et al. (2024):

$$\theta(V_+(R)) \geq \mathcal{Q}^*(R) \cdot \left(1 + \sum_{e \in E(R) \setminus \delta(0)} (x_e - 1)\right),$$

for every route R;

(c) the ILS cuts of Gendreau et al. (2016):

$$\mathbb{1}^\top \theta \geq \left(\sum_{R \in \mathcal{R}(\bar{x})} \mathcal{Q}^*(R)\right) \cdot \left(1 + \sum_{e \in E \setminus \delta(0) : \bar{x}_e = 1} x_e - |V_+| + k\right),$$

for every $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$.

Proof. Item (a) holds because Ota and Fukasawa (2025) showed that $W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H)) \geq W_{HS}(\bar{x}; \mathcal{X}_{=}(H))$ for all $\bar{x} \in \mathcal{X}$. For item (b), consider a route $R = (v_1, \ldots, v_\ell)$ and apply Theorem 5 to the partial route $H = (\{v_1\}, \ldots, \{v_\ell\})$. In this case, $\mathcal{L}^*(H) = \mathcal{Q}^*(R)$ and $W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H)) = 1 + \sum_{e \in E(R) \setminus \delta(0)} (x_e - 1)$. Finally, Claim 2 of Ota and Fukasawa (2025) shows that the path cuts dominate those of Gendreau et al. (2016), proving item (c). □

5.3 Recovering the higher-dimensional LP relaxation bound with a single projected SRI

Inspired by the theoretical results in Sections 5.1 and 5.2, we now develop a procedure that uses a single projected SRI to recover the LP relaxation bound of Formulation (9), denoted z^*. The approach consists of first solving the LP relaxation of Formulation (9), and then using the optimal dual variables associated with the SRIs and upper bound constraints on the y-variables to construct the desired projected SRI. We remark that the ideas here were inspired by recent advances on recovering the Dantzig–Wolfe bound via cutting planes (Chen et al., 2024; Ota et al., 2025).

We start by replacing every occurrence of variable θ_v in Formulation (9) with the expression $\sum_{\xi\in[N]} p_s w_v y_v^\xi$. We then dualize the SRIs and the upper bound constraints on the y-variables into the objective function of the resulting formulation. This yields the Lagrangian dual:

$$z^* = \max_{\alpha\geq 0,\beta\leq 0}\left\{\min_{x\in\mathcal{X},y\geq 0}\left\{\begin{aligned}&c^\intercal x + \sum_{v\in V_+}\sum_{\xi\in[N]} p_\xi w_v y_v^\xi + \sum_{\xi\in[N]}\sum_{S\subseteq V_+}\alpha_S^\xi(k_\xi(S) + x(E(S)) - |S| - y^\xi(S))\\ &+\sum_{\xi\in[N]}\sum_{v\in V_+}\beta_v^\xi(b_v - y_v^\xi)\end{aligned}\right\}\right\}.$$

Since this problem is separable, we write $z^* = \max_{\alpha\geq 0,\beta\leq 0}\{\sigma_x(\alpha) + \sigma_y(\alpha,\beta) + \nu(\alpha,\beta)\}$, where

$$\sigma_x(\alpha) := \min_{x\in\mathcal{X}}\left\{\sum_{\xi\in[N]}\sum_{uv\in E}\left(c_{uv} + \sum_{S\subseteq V_+:u,v\in S}\alpha_S^\xi\right)x_{uv}\right\} \quad \text{and} \tag{σ_x}$$

$$\sigma_y(\alpha,\beta) := \min_{y\geq 0}\left\{\sum_{\xi\in[N]}\sum_{v\in V_+}\left(p_\xi w_v - \beta_v^\xi - \sum_{S\subseteq V_+:v\in S}\alpha_S^s\right)y_v^\xi\right\}. \tag{σ_y}$$

The following theorem is proved in Appendix L.

Theorem 6. *For every $\alpha\geq 0$ and $\beta\leq 0$,*

$$\min\{c^\intercal x + \mathbb{1}^\intercal\theta : x\in\mathcal{X},\ \theta\in \text{proj}_\theta(\Gamma(x,\alpha,\beta))\} \geq \sigma_x(\alpha) + \sigma_y(\alpha,\beta) + \nu(\alpha,\beta).$$

Corollary 4. *Suppose that we solve the LP relaxation of Formulation (9) to optimality. Let $\alpha^*\geq 0$ and $\beta^*\leq 0$ be optimal dual solutions associated with the SRIs and upper bound constraints on the y-variables, respectively. Then*

$$z^* = \min\{c^\intercal x + \mathbb{1}^\intercal\theta : x\in\mathcal{X},\ \theta\in\text{proj}_\theta(\Gamma(x,\alpha^*,\beta^*))\}.$$

Proof. Let $z' = \min\{c^\intercal x + \mathbb{1}^\intercal\theta : x\in\mathcal{X},\ \theta\in\text{proj}_\theta(\Gamma(x,\alpha^*,\beta^*))\}$. By Proposition 3, the feasible region of the problem in the RHS is a relaxation of $\mathcal{P}$, meaning that $z^*\geq z'$. For the converse, apply Theorem 6 to obtain $z'\geq \sigma_x(\alpha^*) + \sigma_y(\alpha^*,\beta^*) + \nu(\alpha^*,\beta^*)$. We are now done by the fact that (α^*,β^*) is optimal for Formulation (5.3) (Frangioni (2005) has a nice proof via "partial dualization"). $\square$

Since the separation of SRIs and RCIs is performed heuristically, the LP relaxation of Formulation (9) is not solved to optimality in practice, and we instead obtain a bound $\tilde{z}\leq z^*$. Nevertheless, we can still apply Theorem 6 to learn that, by querying the duals of the separated SRIs and the upper bound constraints on the y-variables, we obtain a projected SRI that implies the bound $\tilde{z}$ in the (x,θ)-space.

5.4 Branch-and-cut algorithm for the VRPSD under a scenario-optimal recourse policy

Inspired by the previous theoretical results, we propose a two-phase branch-and-cut algorithm for solving problem $\text{VRPSD}(\mathcal{X}, \mathcal{Q}^*)$. First, we solve the LP relaxation of Formulation (9) and construct projected SRIs using the results from Sections 5.1 and 5.3. This allows us to recover strong bounds in the (x, θ)-space. In the second step, we solve the problem to integrality by separating the projected SRIs developed in Sections 5.1 and 5.2. Since these projected SRIs dominate the ILS path cuts (Corollary 3), it follows from Theorem 1 of Ota and Fukasawa (2025) that this procedure yields an exact algorithm for solving $\text{VRPSD}(\mathcal{X}, \mathcal{Q}^*)$. We describe the procedure in detail next.

Step 1: Solving the high-dimensional LP to obtain a strong root node relaxation. We first solve the LP relaxation of Formulation (9) with a time limit of 60 seconds, using the separation of SRIs described in Appendix F. Let $\tilde{z}$ be the resulting LP bound and let $(\tilde{\alpha}, \tilde{\beta})$ be the corresponding optimal dual multipliers associated with the SRIs and upper bound constraints on the y-variables.

Recall that we use aggregated SRIs of the form $\sum_{\xi\in\Xi} y^{\xi}(S) \geq \sum_{\xi\in\Xi}(k_{\xi}(S) + x(E(S)) - |S|)$, for $\Xi \subseteq [N]$ and $\emptyset \subsetneq S \subseteq V_+$ (see Fact 1). For each such constraint, we have a corresponding dual multiplier $\tilde{\iota}_S^{\Xi}$. To build $\tilde{\alpha}$ from $\tilde{\iota}$, we may set $\tilde{\alpha}_S^{\xi} = \sum_{\Xi\subseteq[N]} \tilde{\iota}_S^{\Xi}$, for each $\xi \in [N]$ and $S \subseteq V_+$. We emphasize that we present this construction only for conceptual purposes; our algorithm works directly with $\tilde{\iota}$ and builds the projected SRI associated with $(\tilde{\alpha}, \tilde{\beta})$ without explicitly constructing $\tilde{\alpha}$.

Let $\tilde{\mathcal{X}}$ be the relaxation of $\mathcal{X}$ obtained by considering only the RCIs (or SECs, if $\mathcal{X} = \mathcal{X}_{\text{SUB}}$) that were separated when solving the LP relaxation of Formulation (9). Moreover, let $\mathcal{S} := \{(S, \Xi) : \tilde{\iota}_S^{\Xi} > 0 \text{ and } w_v > 0 \text{ for all } v \in S\}$. We then solve the following relaxation:

$$
\begin{aligned}
\min \quad & c^{\mathsf{T}} x + \mathbf{1}^{\mathsf{T}} \theta, \\
\text{s.t.} \quad & \sum_{v\in S} \max_{\xi\in[N]} \left\{ \frac{1}{p_{\xi} w_v} \right\} \theta_v \geq |\Xi| \cdot (x(E(S)) - |S|) + \sum_{\xi\in\Xi} k_{\xi}(S), \qquad \forall (S, \Xi) \in \mathcal{S}, && \text{(20a)} \\
& (x, \theta) \in \tilde{\mathcal{X}} \times \mathbb{R}_+^{V_+}. && \text{(20b)}
\end{aligned}
$$

As observed in Example 4, inequalities (20a) are projected aggregated SRIs. Preliminary computational results show that it is better to use projected aggregated SRIs than the inequalities developed in Theorem 4.

If the optimal value of Formulation (20) is at least $\tilde{z}$, we proceed to Step 2. Otherwise, we add the projected SRI $\sum_{v\in V_+ : w_v > 0} \phi_v(\tilde{\alpha}, \tilde{\beta})\, \theta_v \geq \sum_{\xi\in[N]} \sum_{S\subseteq V_+} \tilde{\alpha}_S^{\xi}\, x(E(S)) + \nu(\tilde{\alpha}, \tilde{\beta})$ which, by Theorem 6, guarantees the bound $\tilde{z}$.

One may wonder why we add inequalities (20a) if Theorem 6 already guarantees that one projected SRI is enough to recover the bound $\tilde{z}$. Our reasoning here is the same as highlighted in (Chen et al., 2024; Gamrath and Lübbecke, 2010; Ota et al., 2025): using a single inequality to recover a strong bound can be detrimental to the MIP solver performance, since it leads to a high-dimensional optimal face. By adding several projected SRIs, we try to mitigate this issue.

Step 2: Branch-and-cut algorithm based on the separation of projected SRIs. After constructing a strong root-node relaxation in Step 1, we proceed similarly to the branch-and-cut algorithm of Ota and Fukasawa (2025), which separates ILS partial route and set cuts. However, whenever a violated ILS partial route cut is identified, we leverage the results from Section 5.2 to separate the corresponding dominating projected SRI instead.

Specifically, given a candidate solution $(\bar{x}, \bar{\theta})$, we use the separation algorithm of Ota and Fukasawa (2025) to find partial routes $H = (S_1, \ldots, S_\ell)$ such that $W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H)) > 0$. We then construct a single LP with $N \cdot |V_+(H)|$ variables to simultaneously solve the LPs (18) for every scenario $\xi \in [N]$. To improve the efficiency, for each scenario $\xi \in [N]$, we only consider constraints of type (18a) corresponding to *minimal* partial routes $H' \subseteq H$, meaning that $k_\xi(H'') < k_\xi(H')$, for every $H'' \subseteq H'$ with $H'' \neq H'$. In this way, for each scenario $\xi \in [N]$, our formulation only requires $\ell(k_\xi(H) - 1)$ constraints of type (18a), rather than ℓ^2.

By solving the previous LP, we obtain dual multipliers (α, β) and a corresponding projected SRI that we add to the formulation. Theorem 5 implies that this projected SRI dominates the ILS partial route cut $\theta(V_+(H)) \geq \mathcal{L}^*(H) \cdot W_{OF}(x; \mathcal{X}_{\supseteq}(H))$. Consequently, it follows from Theorem 1 of Ota and Fukasawa (2025) that this procedure already yields a correct branch-and-cut algorithm for VRPSD$(\mathcal{X}, \mathcal{Q}^*)$.

To improve performance, the algorithm of Ota and Fukasawa (2025) also separates set cuts. Whenever a violated set cut is detected, we instead add a corresponding projected aggregated SRI (15). Preliminary experiments indicate that these inequalities are more effective in practice than the dominating projected SRI described in Theorem 4. Further details on our separation procedures are provided in Appendix N.

5.5 Extension to other recourse functions

Suppose now that $\mathcal{Q} \geq \mathcal{Q}^*$ is a recourse function satisfying Assumptions 2 and 3 (with the same parameters $w \in \mathbb{Q}_+^{V_+}$ and $b \in \mathbb{Z}_{++}^{V_+}$ used for defining $\mathcal{Q}^*$). By Proposition 1, we know that projected SRIs are valid for $\mathcal{F}(\mathcal{X}, \mathcal{Q}) \subseteq \mathcal{P}$. Consequently, we can solve VRPSD$(\mathcal{X}, \mathcal{Q})$ by separating additional inequalities on top of the algorithm described in Section 5.4.

Specifically, for any candidate solution $(\bar{x}, \bar{\theta}) \in (\mathcal{X} \cap \mathbb{Z}^E) \times \mathbb{R}_+^{V_+}$, we can verify if every route $R \in \mathcal{R}(\bar{x})$ satisfies $\sum_{v \in V_+(R)} \bar{\theta}_v \geq \mathcal{Q}(R)$. If not, we add the (partial) route cut $\theta(V_+(R)) \geq \mathcal{Q}(R) \cdot W_{HS}(x; \mathcal{X}_{=}(R))$ to the formulation. As shown by Ota and Fukasawa (2025), these cuts suffice to recover the recourse cost of a solution. In our implementation, we also incorporate additional partial route cuts developed in Ota and Fukasawa (2025) for the recourse function $\mathcal{Q}_C$. Appendix N contains more details.

6 Computational experiments

We conducted computational experiments on the VRPSD with scenarios under both the scenario-optimal and classical recourse policies, with and without the assumption of CVRP feasibility. In other words, we considered problems VRPSD$(\mathcal{X}, \mathcal{Q})$, for $\mathcal{X} \in \{\mathcal{X}_{\text{CVRP}}, \mathcal{X}_{\text{SUB}}\}$ and $\mathcal{Q} \in \{\mathcal{Q}^*, \mathcal{Q}_C\}$. Our goal here is twofold. First, we complement the theoretical findings in Section 5.2 by empirically comparing the performance of projected SRIs and ILS cuts with respect to $\mathcal{Q}^*$. Second, since Proposition 1 ensures that inequalities valid for $\mathcal{Q}^*$ are also valid for $\mathcal{Q}_C$, we also investigate whether incorporating projected SRIs can improve the performance of the ILS-based algorithm for VRPSD$(\mathcal{X}, \mathcal{Q}_C)$ proposed by Ota and Fukasawa (2025).

6.1 Experimental setup

Our approaches were evaluated on the instances developed by Ota and Fukasawa (2025) for the VRPSD with scenarios, which in turn are based on the 270 instances of Jabali et al. (2014) and the 20 instances of Dinh et al. (2018). We implemented three algorithmic approaches:

- ILS: the ILS-based method proposed by Ota and Fukasawa (2025), which we applied to both $\mathcal{Q}^*$ and $\mathcal{Q}_C$;
- SRI: the SRI-based algorithm discussed in Section 5.4, which only applies to $\mathcal{Q}^*$;
- ILS+SRI: the extension combining projected SRIs with ILS cuts described in Section 5.5, which we applied only to $\mathcal{Q}_C$.

All algorithms were implemented in C++ using Gurobi 12 as the LP/MIP solver and the Lemon library (Dezső et al., 2011) for basic graph operations. Experiments were executed in single-thread mode with a time limit of 1800 seconds per instance on a machine equipped with an Intel(R) Xeon(R) Gold 6142 CPU @ 2.60 GHz.

6.2 Numerical results

We first consider in Table 1 the scenario-optimal recourse function $\mathcal{Q}^*$. Each row represents a different choice of first-stage feasibility region ($\mathcal{X} \in \{\mathcal{X}_{\text{CVRP}}, \mathcal{X}_{\text{SUB}}\}$) and instance set ("Dinh" or "Jabali"). The "Total" column indicates the total number of instances in each set. For each algorithm, we report the number of instances solved to optimality within the time limit ("Solved"), the average solution time in seconds ("T(s)"), the average optimality gap for unsolved instances ("G(%)"), and the average root node gap ("RG(%)"). The average solution times were computed considering only the instances that were solved to optimality by both algorithms. Moreover, the optimality and root gaps were measured relative to the best primal bound found across the algorithms.

				ILS				SRI			
$\mathcal{X}$	$\mathcal{Q}$	Instance set	Total	Solved	T(s)	G(%)	RG(%)	Solved	T(s)	G(%)	RG(%)
$\mathcal{X}_{\text{CVRP}}$	$\mathcal{Q}^*$	Dinh	20	9	204.08	8.15	7.75	9	117.53	7.71	6.65
$\mathcal{X}_{\text{CVRP}}$	$\mathcal{Q}^*$	Jabali	270	267	54.02	1.56	1.73	267	92.34	0.70	1.23
$\mathcal{X}_{\text{SUB}}$	$\mathcal{Q}^*$	Dinh	20	0	–	29.00	34.39	4	–	9.39	12.44
$\mathcal{X}_{\text{SUB}}$	$\mathcal{Q}^*$	Jabali	270	77	211.58	6.74	8.00	249	67.99	2.40	2.41

Table 1: Comparison of ILS and SRI algorithms on VRPSD instances with $\mathcal{Q} = \mathcal{Q}^*$.

We see that algorithm SRI consistently achieved smaller root gaps than ILS, which validates our theoretical findings from Section 5.2, where we proved that the inequalities used by algorithm SRI dominate the ILS cuts used by ILS (since $\mathcal{Q} = \mathcal{Q}^*$). Although this improved relaxation does not translate into an overall better performance when $\mathcal{X} = \mathcal{X}_{\text{CVRP}}$, it yields a substantial improvement when $\mathcal{X} = \mathcal{X}_{\text{SUB}}$, with SRI solving 176 more instances than ILS in total.

These different behaviors can be explained by the fact that enforcing CVRP feasibility (i.e., $x \in \mathcal{X}_{\text{CVRP}} \cap \mathbb{Z}^E$) may significantly restrict the feasible region, strengthening the ILS set cuts and leaving little room for improvement via better recourse approximation. On the other hand, when $\mathcal{X} = \mathcal{X}_{\text{SUB}}$, the feasible region is typically much larger, which makes the quality of the recourse approximation critical. As discussed in (Hoogendoorn and Spliet, 2025), while the CVRP-feasibility assumption is often reasonable, it can be arbitrary and undesirable in certain settings. In such situations, our experiments highlight that projected SRIs provide a tighter recourse approximation without artificially restricting the problem formulation.

Table 2 shows that a similar situation occurs for the classical recourse policy ($\mathcal{Q} = \mathcal{Q}_C$). The hybrid approach ILS+SRI consistently achieved stronger root gaps than ILS, and this improvement translated into an overall better performance of the algorithm. As with the scenario-optimal recourse policy, the gains were much more pronounced for $\mathcal{X} = \mathcal{X}_{\text{SUB}}$, but algorithm ILS+SRI still solved 5 more instances than ILS when $\mathcal{X} = \mathcal{X}_{\text{CVRP}}$. These results highlight the value of our generic treatment and Proposition 1, which enables the use of SRIs to recourse functions besides the scenario-optimal one. In this case, we successfully applied (projected) SRIs to the recourse function $\mathcal{Q}_C$.

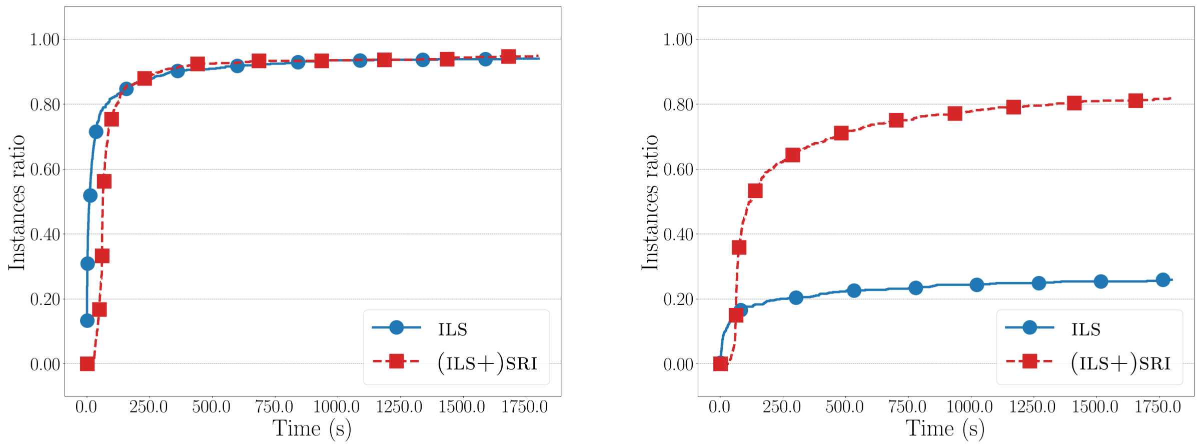
Lastly, to complement the results from Tables 1 and 2, we show in Figure 3 the empirical cumulative distribution of execution times for both algorithms. Specifically, consider Figure 3a and take as an example a point $p = (p_1, p_2) \in \mathbb{R}^2$ on the line of algorithm ILS. Let $\mathcal{I}$ be the union of the instance sets of Jabali et al. (2014) and Dinh et al. (2018), so $|\mathcal{I}| = 290$. For a given

				ILS				ILS+SRI			
$\mathcal{X}$	$\mathcal{Q}$	Instance set	Total	Solved	T(s)	G(%)	RG(%)	Solved	T(s)	G(%)	RG(%)
$\mathcal{X}_{\text{CVRP}}$	$\mathcal{Q}_C$	Dinh	20	8	246.56	7.34	7.48	11	125.14	8.67	6.38
$\mathcal{X}_{\text{CVRP}}$	$\mathcal{Q}_C$	Jabali	270	261	47.72	1.13	1.81	263	90.26	0.96	1.32
$\mathcal{X}_{\text{SUB}}$	$\mathcal{Q}_C$	Dinh	20	0	–	30.59	35.80	4	–	12.56	14.89
$\mathcal{X}_{\text{SUB}}$	$\mathcal{Q}_C$	Jabali	270	73	203.90	7.21	8.42	217	90.24	2.87	2.83

Table 2: Comparison of ILS and SRI algorithms on VRPSD instances.

recourse function $\mathcal{Q} \in \{\mathcal{Q}_C, \mathcal{Q}^*\}$, let $\hat{\mathcal{I}}(\mathcal{Q}) \subseteq \mathcal{I}$ be the subset of instances that algorithm ILS solved within p_1 seconds using first-stage feasible region $\mathcal{X}_{\text{CVRP}}$ and recourse function $\mathcal{Q}$. Point p_2 is then given by the ratio $p_2 = \frac{|\hat{\mathcal{I}}(\mathcal{Q}^*)|+|\hat{\mathcal{I}}(\mathcal{Q}_C)|}{2|\mathcal{I}|}$. Figure 3b was constructed similarly for $\mathcal{X} = \mathcal{X}_{\text{SUB}}$.

These plots confirm that projected SRIs are particularly effective when no CVRP-feasibility assumption is imposed. We also point out that the "(ILS+) SRI" curve exhibits slow progress in the first seconds. This is due to the procedure described in Step 1 of Section 5.4, where we solve a high-dimensional formulation in the (x, y)-space for 60 seconds. Overall, after 200 seconds and with $\mathcal{X} = \mathcal{X}_{\text{CVRP}}$, we see no significant difference between the performance of ILS and SRI-based methods. More detailed computational results can be found in Appendix O.



(a) Execution time for $\mathcal{X}_{\text{CVRP}}$.

(b) Execution time for $\mathcal{X}_{\text{SUB}}$.

Figure 3: Empirical cumulative distribution of the execution times. The legend (ILS+) SRI refers to algorithm SRI when $\mathcal{Q} = \mathcal{Q}^*$, and to algorithm ILS+SRI when $\mathcal{Q} = \mathcal{Q}_C$.

7 Conclusion and future work

In this paper, we introduced a framework for the VRPSD with scenarios in which recourse policies are represented as feasible solutions of a MIP. We characterized the convex hull of such recourse policies and introduced a new class of inequalities termed scenario recourse inequalities (SRIs). We showed that SRIs yield a formulation for the VRPSD under a scenario-optimal recourse policy, and we further demonstrated that they are valid for other recourse policies and dominate certain ILS cuts.

Our computational results show that, compared to ILS cuts, SRIs provide a better approximation of the recourse function and are particularly useful when CVRP feasibility is not imposed. Since we can now better approximate the recourse function, there is less need to rely on the assumption of CVRP feasibility, which imposes a somewhat arbitrary hard capacity constraint on the sum of expected demands (see Hoogendoorn and Spliet (2025)). Using our results, one could instead consider more flexible approaches, such as assigning larger costs to recourse actions to reduce the occurrence of failures. We leave this direction for future work.

Overall, this work shows that modeling recourse decisions via a network flow formulation can be useful and yields a polyhedral structure that can be exploited. For the future, it would be interesting to study the separation of projected SRIs and to explore similar ideas in other stochastic routing problems where the recourse policy involves other types of resource collection/consumption along routes.

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A Proof of Lemma 1

Lemma 1. *Let R be a route and $\xi \in [N]$. For any vector $y^\xi \in \mathbb{Z}^{V_+}$, we have that $y^\xi \in \mathcal{Y}^\xi(\vec{R})$ if and only if there exist $f^\xi \in \mathbb{R}^A_+$ and $g^\xi \in \mathbb{R}^{V_+}_+$ such that*

$$f^\xi_{(v_{i-1},v_i)} + f^\xi_{(v_{i+1},v_i)} + d^\xi(v_i) = f^\xi_{(v_i,v_{i+1})} + f^\xi_{(v_i,v_{i-1})} + g^\xi_{v_i}, \qquad \forall i \in [\ell], \tag{5a}$$

$$f^\xi_{(v_{i-1},v_i)} \le \frac{C}{2}, \qquad \forall i \in [\ell+1], \tag{5b}$$

$$f^\xi_{(v_i,v_{i-1})} \le \frac{C}{2}, \qquad \forall i \in [\ell+1], \tag{5c}$$

$$g^\xi_{v_i} \le C \cdot y^\xi_{v_i}, \qquad \forall i \in [\ell]. \tag{5d}$$

In particular, this implies that $\mathcal{Y}^\xi(\vec{R}) = \mathcal{Y}^\xi(\overleftarrow{R})$.

Proof. Let $\bar{y}^\xi \in \mathcal{Y}^\xi(\vec{R})$. We first show that $\bar{y}^\xi$ also belongs to $\mathcal{Y}^\xi(\overleftarrow{R})$. Using Definition 1, we know that there exist $\vec{f}^\xi$ and $\bar{g}^\xi$ satisfying (4). Construct vector $\overleftarrow{f}^\xi \in \mathbb{R}^A_+$ by setting $\overleftarrow{f}^\xi_a = C - \vec{f}^\xi_{(v_i,v_{i+1})}$ if $a = (v_{i+1}, v_i)$ and $\overleftarrow{f}^\xi_a = 0$ otherwise. For each $i \in [\ell]$, we have that

$$\begin{aligned} &\vec{f}^\xi_{(v_{i-1},v_i)} + d^\xi(v_i) = \vec{f}^\xi_{(v_i,v_{i+1})} + g^\xi_{v_i} \\ \iff &C - \overleftarrow{f}^\xi_{(v_i,v_{i+1})} + d^\xi(v_i) = C - \overleftarrow{f}^\xi_{(v_{i-1},v_i)} + g^\xi_{v_i} \\ \iff &\overleftarrow{f}^\xi_{(v_{i+1},v_i)} + d^\xi(v_i) = \overleftarrow{f}^\xi_{(v_i,v_{i-1})} + g^\xi_{v_i}. \end{aligned}$$

So the pair $(\overleftarrow{f}^\xi, \bar{g}^\xi)$ verifies that $\bar{y}^\xi \in \mathcal{Y}^\xi(\overleftarrow{R})$. Take $\bar{f}^\xi = (\vec{f}^\xi + \overleftarrow{f}^\xi)/2$ and observe that $(\bar{f}^\xi, \bar{g}^\xi)$ is feasible for Formulation (5).

Suppose now that $(\bar{f}^\xi, \bar{g}^\xi)$ is an arbitrary pair that is feasible for Formulation (5) with $y^\xi = \bar{y}^\xi$ integer. We use $\bar{f}^\xi$ to construct a vector $\vec{f}^\xi \in \mathbb{R}^A_+$ feasible for Formulation (4) by setting $\vec{f}^\xi_a = \bar{f}^\xi_{(v_i,v_{i+1})} + \left(\frac{C}{2} - \bar{f}^\xi_{(v_{i+1},v_i)}\right)$ if $a = (v_i, v_{i+1})$, and $\vec{f}^\xi_a = 0$ otherwise. Clearly $0 \le \vec{f}^\xi_a \le C$, for all $a \in A$. Moreover, for each $i \in [\ell]$,

$$\begin{aligned} &\bar{f}^\xi_{(v_{i-1},v_i)} + \bar{f}^\xi_{(v_{i+1},v_i)} + d^\xi(v_i) = \bar{f}^\xi_{(v_i,v_{i+1})} + \bar{f}^\xi_{(v_i,v_{i-1})} + \bar{g}^\xi_{v_i} \\ \iff &\bar{f}^\xi_{(v_{i-1},v_i)} + \left(\frac{C}{2} - \bar{f}^\xi_{(v_i,v_{i-1})}\right) + d^\xi(v_i) = f^\xi_{(v_i,v_{i+1})} + \left(\frac{C}{2} - \bar{f}^\xi_{(v_{i+1},v_i)}\right) + \bar{g}^\xi_{v_i} \\ \iff &\vec{f}^\xi_{(v_{i-1},v_i)} + d^\xi(v_i) = \vec{f}^\xi_{(v_i,v_{i+1})} + \bar{g}^\xi_{v_i}, \end{aligned}$$

which shows that $\bar{y}^\xi \in \mathcal{Y}^\xi(\vec{R})$. ◻

B Proof of Lemma 3

Lemma 3. *Let $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$ and $\mathcal{R}(\bar{x}) = \{R_1, \dots, R_k\}$. Let $\bar{y} \in \mathbb{R}^{[N]\times V_+}$ and $\bar{y}^1, \dots, \bar{y}^k \in \mathbb{R}^{[N]\times V_+}$ be such that*

$$(\bar{y}^i)^\xi_v = \mathbb{I}(v \in V_+(R_i)) \cdot \bar{y}^\xi_v, \qquad \forall i \in [k], \xi \in [N], v \in V_+.$$

Then $\bar{y} \in \Pi(\bar{x})$ if and only if $(\bar{y}^1, \dots, \bar{y}^k) \in \Pi(R_1) \times \dots \times \Pi(R_k)$.

Proof. Suppose $\bar{y} \in \Pi(\bar{x})$ and fix an arbitrary $i \in [k]$. Clearly $\bar{y}^i$ is integral. Additionally, Formulation (4) (with respect to route R_i) only has constraints for the y-variables associated with customers $v \in V_+(R_i)$, in which case $\bar{y}_v^\xi = (\bar{y}^i)_v^\xi$, for all $\xi \in [N]$. This implies that $\bar{y}^i \in \Pi(R_i)$. Conversely, whenever $(\bar{y}^1, \ldots, \bar{y}^k) \in \Pi(R_1) \times \ldots \times \Pi(R_k)$, we have that $\bar{y}$ is integral and $(\bar{y}^i)_v^\xi \le \bar{y}_v^\xi$, for all $i \in [k]$, $\xi \in [N]$ and $v \in V_+$. By Definition 1 (and Formulation (4)), we conclude that $\bar{y}$ is a recourse policy for routes $R_1, \ldots, R_k$, meaning that $\bar{y} \in \cap_{i=1}^k \Pi(R_i) = \Pi(\bar{x})$. ☐

C Proof of Lemma 4

Lemma 4. *Fix $b \in \mathbb{Z}_+^{V_+}$. For every $\bar{x} \in \mathcal{X}$, define*

$$\textsc{flow}(\bar{x}) := \left\{ y \in [\mathbf{0}, b]^N : \begin{array}{ll} f^\xi(\delta^-(v)) + d^\xi(v) = f^\xi(\delta^+(v)) + g_v^\xi, & \forall v \in V_+, \xi \in [N] \\ f_{(u,v)}^\xi \le \dfrac{C}{2} \bar{x}_{\{u,v\}}, & \forall uv \in A, \xi \in [N] \\ g_v^\xi \le C \cdot y_v^\xi, & \forall v \in V_+, \xi \in [N] \\ f \in \mathbb{R}_+^{[N]\times A}, g \in \mathbb{R}_+^{[N]\times V_+} & \end{array} \right\}.$$

Then $\textsc{flow}(\bar{x}) \cap \mathbb{Z}^{[N]\times V_+} = \Pi(\bar{x}) \cap [\mathbf{0}, b]^N$, for every $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$.

Proof. Fix $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$ and let $\mathcal{R}(\bar{x}) = \{R_1, \ldots, R_k\}$. Let $(\bar{y}, \bar{f}, \bar{g})$ be feasible for the formulation of $\textsc{flow}(\bar{x}) \cap \mathbb{Z}^{[N]\times V_+}$ and set $\bar{y}^1, \ldots, \bar{y}^k$ as in Lemma 3. To show that $\bar{y} \in \Pi(\bar{x})$ it suffices to show that $\bar{y}^i \in \Pi(R_i)$, for each $i \in [k]$. Fix an arbitrary $i \in [k]$. Suppose that R_i has a single customer v. By Assumption 1, $d^\xi(v) \le C$, for all $\xi \in [N]$, so $\Pi(R_i)$ contains the zero vector and $\mathbb{Z}_+^{[N]\times V_+} \cap [\mathbf{0}, b]^N \subseteq \Pi(R_i)$. So assume $|V_+(R_i)| \ge 2$. Set $\bar{f}^i \in \mathbb{R}_+^{[N]\times A}$ and $\bar{g}^i \in \mathbb{R}_+^{[N]\times V_+}$ as follows.

$$(\bar{f}^i)_a^\xi = \mathbb{I}(a \in A(\vec{R}_i) \cup A(\overleftarrow{R}_i)) \cdot \bar{f}_a^\xi, \qquad \forall a \in A, \xi \in [N],$$
$$(\bar{g}^i)_v^\xi = \mathbb{I}(v \in V_+(R_i)) \cdot \bar{g}_v^\xi, \qquad \forall v \in V_+, \xi \in [N].$$

Then one can check that $(\bar{y}_i, \bar{f}_i, \bar{g}_i)$ is feasible for Formulation (5).

A similar reasoning is applied for the reverse direction. Let $\hat{y} \in \Pi(\bar{x}) \cap [\mathbf{0}, b]^N$ and apply Lemma 3 to get vectors $\hat{y}^1, \ldots, \hat{y}^k$. For every $i \in [k]$, there exists $\hat{f}^i \in \mathbb{R}_+^{[N]\times A}$ and $\hat{g}^i \in \mathbb{R}_+^{[N]\times V_+}$ such that $(\hat{y}^i, \hat{f}^i, \hat{g}^i)$ is feasible for Formulation (5). Moreover, without loss of generality, we may assume that, for every $\xi \in [N]$, we have $(\bar{f}_i)_a^\xi = 0$, for $a \in A \setminus (A(\vec{R}_i) \cup A(\overleftarrow{R}_i))$, and $(\bar{g}_i)_v^\xi = 0$, for $v \notin V_+(R_i)$. Set $\hat{f} = \sum_{i=1}^k \hat{f}^i$ and $\hat{g} = \sum_{i=1}^k \hat{g}^i$ and note that $(\hat{y}, \hat{f}, \hat{g})$ is feasible for the formulation defining $\textsc{flow}(\bar{x})$. ☐

D Proof of Proposition 2

Proposition 2. *Fix $b \in \mathbb{Z}_+^{V_+}$. For every $\bar{x} \in \mathcal{X}$,*

$$\textsc{flow}(\bar{x}) = \left\{ y \in [\mathbf{0}, b]^N : y^\xi(S) \ge \frac{d^\xi(S)}{C} + \bar{x}(E(S)) - |S|, \ \forall \emptyset \subsetneq S \subseteq V_+, \ \xi \in [N] \right\}.$$

Proof. Let $\bar{x} \in \mathcal{X}$ and $\bar{y} \in [\mathbf{0}, b]^N$. Fix a scenario $\xi \in [N]$ and create a digraph $D' = (V, A')$ obtained from D by duplicating the arcs $(v, 0)$, for each $v \in V_+$. Next, construct a vector of arc capacities $h \in \mathbb{R}^{A'}$ by setting $h_a = (C/2)\, \bar{x}_{\{u,v\}}$, for each $a = uv \in A$; and $h_a = C\bar{y}_v^\xi$, for each $a = (v, 0) \in A' \setminus A$. Additionally, set $d'_0 = d^\xi(V_+)$ and $d'_v = -d^\xi(v)$, for all $v \in V_+$.

From this construction, apply Theorem 1 to learn that a tuple $(\bar{y}^\xi, f^\xi, g^\xi)$ satisfying the constraints in $\text{FLOW}(\bar{x})$ with respect to scenario $\xi \in [N]$ exists if and only if

$$
\begin{aligned}
& h(\delta^+(S)) \geq d'(V \setminus S), && \forall \emptyset \subsetneq S \subseteq V, \\
\iff & h(\delta^+(S)) \geq d'(V \setminus S), && \forall \emptyset \subsetneq S \subseteq V_+, \quad (\text{as } d'_v \leq 0, \text{ for all } v \in V_+) \\
\iff & C\left(\bar{y}^\xi(S) + \bar{x}(\delta(S))/2\right) \geq d^\xi(S) && \forall \emptyset \subsetneq S \subseteq V_+.
\end{aligned}
$$

Since $\bar{x} \in \mathcal{X}$, we close the proof by applying $\bar{x}(\delta(S)) = 2|S| - 2\bar{x}(E(S))$ to the last inequality. $\square$

E Proof of Theorem 3

We first show a simple reformulation of the set $\text{conv}(\Pi(\bar{x}) \cap [\mathbf{0}, b]^N)$:

Lemma 7. *Let $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$. Then*

$$\text{conv}(\Pi(\bar{x}) \cap [\mathbf{0}, b]^N) = \bigcap_{R \in \mathcal{R}(\bar{x})} \text{conv}(\Pi(R) \cap [\mathbf{0}, b]^N).$$

Proof. Call $\mathcal{H}$ the set in the RHS of the statement. One direction of the inclusion follows from standard arguments:

$$\text{conv}(\Pi(\bar{x}) \cap [\mathbf{0}, b]^N) = \text{conv}\left(\bigcap_{R \in \mathcal{R}(\bar{x})} (\Pi(R) \cap [\mathbf{0}, b]^N)\right) \subseteq \text{conv}(\mathcal{H}) = \mathcal{H},$$

where the last equality follows because the intersection of convex sets is convex.

For the other direction, let $\mathcal{R}(\bar{x}) = \{R_1, \ldots, R_k\}$ and let $\bar{y}$ be an extreme point of $\mathcal{H}$. Set $\bar{y}^1, \ldots, \bar{y}^k$ according to Lemma 3. We show that $\bar{y}^i \in \Pi(R_i)$, for all $i \in [k]$. Suppose by contradiction that, for some $i \in [k]$, $\bar{y}^i \notin \Pi(R_i)$. Without loss of generality, assume $i = k$. Then $\bar{y}^k$ is a (strict) convex combination of points in $\Pi(R_k) \cap [\mathbf{0}, b]^N$, i.e., there exists $\{r^1, \ldots r^p\} \subseteq \Pi(R_k) \cap [\mathbf{0}, b]^N$ and $\lambda \in \mathbb{R}^p_{++}$ such that $p \geq 2$, $\bar{y}^k = \sum_{j=1}^p \lambda_j r^j$ and $\mathbb{1}^\intercal \lambda = 1$. For each $\xi \in [N]$ and $v \notin V_+(R_k)$, $(\bar{y}^k)^\xi_v = 0$ implies $(r^j)^\xi_v = 0$, for all $j \in [p]$. Hence, for each $j \in [p]$, $\hat{y}^j = (\sum_{i=1}^{k-1} \bar{y}^i) + r^j$ belongs to $\mathcal{H}$. Therefore, $\bar{y} = \sum_{j=1}^p \lambda_j \hat{y}^j$, contradicting the extremality of $\bar{y}$. $\square$

Theorem 3. *Fix $b \in \mathbb{Z}_+^{V_+}$. For every $\bar{x} \in \mathcal{X}$, define*

$$\text{SRI}(\bar{x}) := \left\{ y \in [\mathbf{0}, b]^N : y^\xi(S) \geq k_\xi(S) + \bar{x}(E(S)) - |S|, \ \forall \emptyset \subsetneq S \subseteq V_+, \xi \in [N] \right\}.$$

Then $\text{SRI}(\bar{x}) = \text{conv}(\Pi(\bar{x}) \cap [\mathbf{0}, b]^N)$, for every $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$.

Proof. Fix $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$ and let $\mathcal{R}(\bar{x}) = \{R_1, \ldots, R_k\}$. Define

$$\mathcal{H} := \left\{ y \in [\mathbf{0}, b]^N : y^\xi(R') \geq k_\xi(R') - 1, \ \ \forall i \in [k], R' \subseteq R_i, \xi \in [N] \right\}.$$

Then

$$\mathcal{H} \overset{\text{Corollary 1}}{=} \bigcap_{i=1}^k \text{conv}(\Pi(R_i) \cap [\mathbf{0}, b]^N) \overset{\text{Lemma 7}}{=} \text{conv}(\Pi(\bar{x}) \cap [\mathbf{0}, b]^N)$$

Since SRIs are valid for $\Pi(\bar{x}) \cap [\mathbf{0}, b]^N$, we have $\text{conv}(\Pi(\bar{x}) \cap [\mathbf{0}, b]^N) \subseteq \text{SRI}(\bar{x})$. Conversely, for every $\xi \in [N]$, $i \in [k]$ and $R' \subseteq R_i$, inequality

$$y^\xi(R') \geq k_\xi(R') - 1 = k_\xi(R') + \bar{x}(V_+(R')) - |V_+(R')|$$

is an SRI, so $\text{SRI}(\bar{x}) \subseteq \mathcal{H} = \text{conv}(\Pi(\bar{x}) \cap [\mathbf{0}, b]^N)$. $\square$

F Heuristics for separating scenario recourse inequalities

Before discussing our separation routines, we remark that the separation of SRIs is strongly $\mathcal{NP}$-hard in general. Indeed, suppose that $(\bar{x}, \bar{y}) \in \mathcal{X} \times [\mathbf{0}, b]^N$ is a candidate solution with $\bar{y}^\xi = \mathbf{0}$, for some $\xi \in [N]$. Since for every $\emptyset \subsetneq S \subseteq V_+$, we have that $\bar{y}^\xi(S) \geq k_\xi(S) + \bar{x}(E(S)) - |S|$ is equivalent to $\bar{x}(E(S)) \leq |S| - \lceil d^\xi(S)/C \rceil$, it follows that separating a violated SRI reduces to finding a violated rounded-capacity inequality (RCI). As the separation of RCIs is strongly $\mathcal{NP}$-hard (Diarrassouba, 2017), we do not expect to separate the SRIs exactly in pseudo-polynomial time.

Instead, we propose two separation heuristics: one based on the CVRPSEP package of Lysgaard et al. (2004), and another one based on an MILP formulation. By ordering the scenarios by their total demands, we assume henceforth that $d^1(V_+) \geq \ldots \geq d^N(V_+)$.

F.1 Heuristic based on the CVRPSEP package

This separation heuristic simply calls CVRPSEP, for each scenario $\xi = 1, \ldots, N$, and checks whether the returned candidate sets correspond to violated SRIs. The overall procedure is summarized in Algorithm 1. We make a few brief observations regarding this algorithm. First, we limit CVRPSEP to return at most 10 sets, so $|\mathcal{S}| \leq 10$ in line 4. Second, line 6 effectively constructs the aggregated SRIs discussed in Fact 1. Finally, we stop the algorithm earlier in line 10 to avoid calling CVRPSEP for every scenario.

Algorithm 1 SRI-CVRPSEP

Input: $(\bar{x}, \bar{y}) \in \mathcal{X} \times [\mathbf{0}, b]^N$.

Output: A set of pairs (S, Ξ), with $\emptyset \subsetneq S \subseteq V_+$ and $\Xi \subseteq [N]$, such that $\sum_{\xi \in \Xi} \bar{y}^\xi(S) < \sum_{\xi \in \Xi}(k_\xi(S) + \bar{x}(E(S)) - |S|)$. Each of these violated inequalities will be later added to a relaxation of Formulation (9).

1: **procedure** SRI-CVRPSEP$(\bar{x}, \bar{y})$
2: $\quad \mathcal{C} \leftarrow \emptyset$
3: $\quad$ **for** $\xi = 1, \ldots, N$ **do**
4: $\quad\quad$ Call CVRPSEP with demand vector d^ξ to get a family of sets $\mathcal{S} \subseteq 2^{V_+}$ such that $\bar{x}(E(S)) > |S| - k_\xi(S)$, for every $S \in \mathcal{S}$.
5: $\quad\quad$ **for** $S \in \mathcal{S}$ **do**
6: $\quad\quad\quad \Xi \leftarrow \{\xi' \in [N] : \bar{y}^{\xi'}(S) < k_{\xi'}(S) + \bar{x}(E(S)) - |S|\}$
7: $\quad\quad\quad$ **if** $\Xi \neq \emptyset$ **then**
8: $\quad\quad\quad\quad \mathcal{C} \leftarrow \mathcal{C} \cup \{(S, \Xi)\}$
9: $\quad\quad$ **if** $\mathcal{C} \neq \emptyset$ **then**
10: $\quad\quad\quad$ **break**
11: $\quad$ **return** $\mathcal{C}$

F.2 MILP formulation

We also adapt the mixed-integer linear programming model of Pavlikov et al. (2024), originally proposed for the exact separation of RCIs, to separate SRIs.

To explain this model, let $(\bar{x}, \bar{y}) \in \mathcal{X} \times [\mathbf{0}, b]^N$ be a candidate solution. This solution violates an SRI if and only if there exists a scenario $\xi \in [N]$ and a set $S \subseteq V_+$ such that

$$\left\lceil \frac{d^\xi(S)}{C} \right\rceil + \bar{x}(E(S)) - |S| - \bar{y}^\xi(S) > 0. \tag{21}$$

Fix an arbitrary $\varepsilon \in (0, 1)$. The problem of maximizing the LHS of (21) can be solved with the following formulation:

$$
\begin{aligned}
\max \quad & (\gamma+1) - \sum_{v \in V_+} (\bar{y}_v + 1) q_v + \sum_{e \in E \setminus \delta(0)} \bar{x}_e h_e, \\
\text{s.t.} \quad & \gamma C \leq \sum_{v \in V_+} d_v^\xi q_v - \varepsilon, && && \text{(22a)} \\
& h_{uv} \leq q_u, && \forall uv \in E \setminus \delta(0), && \text{(22b)} \\
& h_{uv} \leq q_v, && \forall uv \in E \setminus \delta(0), && \text{(22c)} \\
& h_{uv} \geq q_u + q_v - 1, && \forall uv \in E \setminus \delta(0), && \text{(22d)} \\
& \gamma \in \mathbb{Z}, \ q \in \{0,1\}^{V_+}, \ h \in \mathbb{R}_+^{E \setminus \delta(0)}. && && \text{(22e)}
\end{aligned}
$$

The binary variables q_v, for each $v \in V_+$, indicate whether customer v belongs to the set S. Constraints (22b)-(22d) enforce that $h_{uv} = 1$ if and only if $q_u = q_v = 1$, meaning that h_{uv} indicates whether $uv \in E$ belongs to $E(S)$. Finally, since we maximize $\gamma \in \mathbb{Z}$ in the objective and $\varepsilon \in (0,1)$, inequality (22a) determines that, at an optimal solution $(\bar{\gamma}, \bar{q}, \bar{h})$ with $S = \{v \in V_+ : \bar{q}_v = 1\}$, we have that $\bar{\gamma} = \lceil (\sum_{v \in V_+} d_v^\xi \bar{q}_v)/C \rceil - 1 = \lceil d^\xi(S)/C \rceil - 1$.

Although Formulation (22) exactly separates SRIs, our implementation yields a heuristic for the following reasons. First, rather than maximizing the original objective function, we change the objective function to zero and enforce the constraint $(\gamma+1) - \sum_{v \in V_+} (\bar{y}_v + 1) q_v + \sum_{e \in E \setminus \delta(0)} \bar{x}_e h_e \geq 0.01$. Second, for each scenario $\xi = 1, \ldots, N$, we solve Formulation (22) with a time limit of 30 seconds.

Whenever we find a violated SRI of the form $y^\xi(S) < k_\xi(S) + \bar{x}(E(S)) - |S|$, we apply Fact 1 and add the corresponding aggregated SRI. As in line 10 of Algorithm 1, we stop the separation procedure whenever a violated (aggregated) SRI is found.

G Proof of Lemma 5

Lemma 5. *Let $a^1, \ldots, a^N$ be vectors in $\mathbb{R}^{V_+}$ and let $h \in \mathbb{R}$. Consider the set*

$$
\mathcal{H} = \left\{ (\theta, y) \in \mathbb{R}_+^{V_+} \times \mathbb{R}_+^{[N] \times V_+} : \sum_{\xi \in [N]} (a^\xi)^\intercal y^\xi \geq h, \ \theta_v \geq \sum_{\xi \in [N]} p_\xi w_v y_v^\xi, \quad \forall v \in V_+ \right\}.
$$

Then the following holds:

- *if there exists $v \in V_+$ and $\xi \in [N]$ such that $w_v = 0$ and $a_v^\xi > 0$, then $\operatorname{proj}_\theta(\mathcal{H}) = \mathbb{R}_+^{V_+}$;*
- *otherwise, $\operatorname{proj}_\theta(\mathcal{H}) = \left\{ \theta \in \mathbb{R}_+^{V_+} : \sum_{v \in V_+ : w_v > 0} \left(\max_{\xi \in [N]} \left\{ \frac{a_v^\xi}{p_\xi w_v} \right\} \right)^+ \theta_v \geq h \right\}$.*

Proof. It is clear that $\operatorname{proj}_\theta(\mathcal{H}) \subseteq \mathbb{R}_+^{V_+}$. Now take any $\theta' \in \mathbb{R}_+^{V_+}$ and suppose that $v \in V_+$ and $\xi \in [N]$ are such that $w_v = 0$ and $a_v^\xi > 0$. Choosing $\bar{y} \in \mathbb{R}_+^{[N] \times V_+}$ so that $\bar{y}_v^\xi$ is arbitrary large, we can guarantee that $\sum_{\xi \in [N]} (a^\xi)^\intercal \bar{y}^\xi \geq h$ and $\theta'_u \geq \sum_{\xi' \in [N]} p_{\xi'} w_u \bar{y}_u^{\xi'}$, for all $u \in V_+$ (as $w_v = 0$). Therefore, $\theta' \in \operatorname{proj}_\theta(\mathcal{H})$ and we assume for the rest of the proof that $a_v^\xi \leq 0$, for every $v \in V_+$ with $w_v = 0$ and $\xi \in [N]$.

Let $\bar{\theta}$ be an arbitrary vector $\mathbb{R}_+^{V_+}$. Then $\bar{\theta}$ belongs to $\operatorname{proj}_\theta(\mathcal{H})$ if and only if the optimal value of the LP below is zero:

$$\begin{aligned}
\min \quad & 0 \\
\text{s.t.} \quad & \sum_{\xi\in[N]} p_\xi w_v y_v^\xi \le \bar\theta_v, && \forall v \in V_+ \text{ with } w_v > 0, && (\gamma_v)\\
& \sum_{\xi\in[N]} (a^\xi)^\intercal y^\xi \ge h, && && (\eta)\\
& y_v^\xi \ge 0, && \forall v \in V_+, \xi \in [N].
\end{aligned}$$

The dual formulation is:

$$\begin{aligned}
\max \quad & \sum_{v\in V_+} \gamma_v \bar\theta_v + \eta h, && && \text{(24a)}\\
\text{s.t.} \quad & p_\xi w_v \gamma_v + \eta\, a_v^\xi \le 0, && \forall v \in V_+ \text{ with } w_v > 0 \text{ and } \xi \in [N], && \text{(24b)}\\
& \eta \ge 0 && && \text{(24c)}\\
& \gamma_v \le 0, && \forall v \in V_+ \text{ with } w_v > 0. && \text{(24d)}
\end{aligned}$$

Note that, for every $\xi \in [N]$, we omitted the constraints $\eta\, a_v^\xi \le 0$, for $v \in V_+$ with $w_v = 0$, as these are already satisfied by our assumption that $a_v^\xi \le 0$.

By LP duality, $\bar\theta \in \operatorname{proj}_\theta(\mathcal{H})$ if and only if $(\gamma, \eta) = (\mathbf{0}, 0)$ is optimal for Formulation (24). Since $\bar\theta$ is nonnegative, we know that at optimality, for every $v \in V_+$ with $w_v > 0$,

$$\gamma_v = \min_{\xi\in[N]} \left\{ -\eta \left(\frac{a_v^\xi}{p_\xi w_v} \right)^+ \right\} = -\eta \cdot \left(\max_{\xi\in[N]} \left\{ \frac{a_v^\xi}{p_\xi w_v} \right\} \right)^+.$$

Therefore, $\bar\theta \in \mathbb{R}_+^{V_+}$ belongs to $\operatorname{proj}_\theta(\mathcal{H})$ if and only if

$$\max_{\eta\ge 0} \left\{ \eta \left(- \sum_{v\in V_+ : w_v > 0} \left(\max_{\xi\in[N]} \left\{ \frac{a_v^\xi}{p_\xi w_v} \right\} \right)^+ \bar\theta_v + h \right) \right\} \le 0.$$

Normalizing η, we may assume that $\eta = 1$. □

H Proof of Proposition 3

Proposition 3. *For every $\bar x \in \mathcal{X}$,*

$$\operatorname{proj}_\theta(\widehat{\text{SRI}}(\bar x)) = \left\{ \theta \in \mathbb{R}_+^{V_+} : \sum_{v\in V_+ : w_v > 0} \phi_v(\alpha,\beta)\, \theta_v \ge \sum_{\xi\in[N]} \sum_{S\subseteq V_+} \alpha_S^\xi\, \bar x(E(S)) + \nu(\alpha,\beta), \quad \forall (\alpha,\beta) \in \mathcal{A} \right\}.$$

Consequently,

$$\mathcal{P} = \left\{ (x,\theta) \in \mathcal{X} \times \mathbb{R}_+^{V_+} : \sum_{v\in V_+ : w_v > 0} \phi_v(\alpha,\beta)\, \theta_v \ge \sum_{\xi\in[N]} \sum_{S\subseteq V_+} \alpha_S^\xi\, x(E(S)) + \nu(\alpha,\beta), \quad \forall (\alpha,\beta) \in \mathcal{A} \right\}.$$

Proof. Let $\bar\theta \in \mathbb{R}_+^{V_+}$. The vector $\bar\theta$ belongs to $\operatorname{proj}_z(\widehat{\text{SRI}}(\bar x))$ if and only if the optimal value of the linear program below is zero:

$$\begin{aligned}
\min \quad & 0 \\
\text{s.t.} \quad & \sum_{\xi\in[N]} p_\xi w_v y_v^\xi \le \bar\theta_v, && \forall v \in V_+ \\
& y^\xi(S) \ge k_\xi(S) + \bar x(E(S)) - |S|, && \forall S \subseteq V_+, \xi \in [N], && (\alpha_S^\xi)\\
& y_v^\xi \le b_v, && \forall v \in V_+, \xi \in [N], && (\beta_v^\xi)\\
& y_v^\xi \ge 0, && \forall v \in V_+, \xi \in [N].
\end{aligned}$$

By Lagrangian duality, this holds if and only if, for every $\alpha \geq 0$ and $\beta \leq 0$,

$$
\begin{aligned}
0 \quad \geq \quad \min \quad & \nu(\alpha,\beta) - \sum_{\xi\in[N]}\sum_{v\in V_+}\left(\beta_v^\xi + \sum_{S\subseteq V_+: v\in S}\alpha_S^\xi\right) y_v^\xi + \sum_{\xi\in[N]}\sum_{S\subseteq V_+}\alpha_S^\xi\, \bar{x}(E(S)) \\
\text{s.t.} \quad & \sum_{\xi\in[N]} p_\xi w_v y_v^\xi \leq \bar{\theta}_v, && \forall v\in V_+ \\
& y_v^\xi \geq 0, && \forall v\in V_+, \xi\in[N].
\end{aligned}
$$

which is equivalent to stating that $\bar{\theta} \in \mathrm{proj}_\theta(\Gamma(\bar{x},\alpha,\beta))$. By Lemma 5, $\bar{\theta} \in \mathrm{proj}_\theta(\Gamma(\bar{x},\alpha,\beta)) = \mathbb{R}_+^{V_+}$ whenever $(\alpha,\beta)\notin\mathcal{A}$. Otherwise, it follows from Lemma 5 (and Equation (14)) that $\bar{\theta} \in \mathrm{proj}_\theta(\Gamma(\bar{x},\alpha,\beta))$ if and only if

$$
\sum_{v\in V_+}\phi_v(\alpha,\beta)\,\bar{\theta}_v \geq \sum_{\xi\in[N]}\sum_{S\subseteq V_+}\alpha_S^\xi\,\bar{x}(E(S)) + \nu(\alpha,\beta).
$$

□

I Proofs of Lemma 6 and Theorem 4

Lemma 6. *Let $w\in\mathbb{Q}_+^{V_+}$ and $b\in\mathbb{Z}_+^{V_+}$. Fix a scenario $\xi\in[N]$ and let $S=\{v_1,\ldots,v_\ell\}\subseteq V_+$ be such that $w_{v_1}\leq\ldots\leq w_{v_\ell}$. Let $k'\in\mathbb{Z}_{++}$ be such that $k_\xi(S)-k'\leq b(S)$ and choose $j\in[\ell]$ as the smallest index such that $k_\xi(S)-k'\leq\sum_{i\in[j]}b_{v_i}$. Set*

$$
\begin{aligned}
\bar{\alpha}_S^\xi &= \mathbb{I}(k_\xi(S)>k')\cdot w_{v_j}, \quad \textit{and} \\
\bar{\beta}_v^\xi &= \mathbb{I}(v\in\{v_i\}_{i\in[j-1]})\cdot(w_v - w_{v_j}), && \forall v\in V_+.
\end{aligned}
$$

Then $(\bar{\alpha}_S^\xi,\bar{\beta}^\xi)$ is optimal for the dual of Formulation (16) and $\bar{\alpha}_S^\xi\leq\mathcal{L}_\xi^*(S,k')$.

Proof. The condition $k_\xi(S)-k'\leq b(S)$ guarantees that Formulation (16) is feasible, so the following dual formulation is bounded and attains the same objective value of $\mathcal{L}_\xi^*(S,k')$.

$$
\begin{aligned}
\max \quad & \alpha_S^\xi\,(k_\xi(S)-k') + \sum_{v\in V_+}\beta_v^\xi b_v && \text{(27a)}\\
\text{s.t.:} \quad & \alpha_S^\xi + \beta_v^\xi \leq w_v, \quad \forall v\in S, && \text{(27b)}\\
& \beta_v^\xi \leq w_v, \quad \forall v\in V_+\setminus S, && \text{(27c)}\\
& \alpha_S^\xi\geq 0,\ \beta^\xi\leq 0. && \text{(27d)}
\end{aligned}
$$

One can check that $(\bar{\alpha}_S^\xi,\bar{\beta}^\xi)$ is feasible for Formulation (27). If $k_\xi(S)\leq k'$, then $(\bar{\alpha}_S^\xi,\bar{\beta}^\xi)=(0,\mathbf{0})$, so we may safely assume that $k_\xi(S)>k'$. In this case, since

$$
\bar{\alpha}_S^\xi(k_\xi(S)-k') + \sum_{v\in V_+}\bar{\beta}_v^\xi b_v = \sum_{i\in[j-1]} b_{v_i}w_{v_i} + \left((k_\xi(S)-k') - \sum_{i\in[j-1]} b_{v_i}\right) w_{v_j} = \mathcal{L}_s^*(S,k'),
$$

we have that $(\bar{\alpha}_S^\xi,\bar{\beta}^\xi)$ is optimal for Formulation (27).

To show that $\bar{\alpha}_S^\xi\leq\mathcal{L}_s^*(S,k')$, suppose first that $j=1$. Then $\sum_{i\in[j-1]}b_{v_i}w_{v_i}=0$ and we are done. On the other hand, if $j\geq 2$, the choice of j implies that $k_\xi(S)-k'>\sum_{i\in[j-1]}b_{v_i}$, meaning that $\mathcal{L}_s^*(S,k')\geq\sum_{i\in[j-1]}b_{v_i}w_{v_i}+w_{v_j}\geq\bar{\alpha}_S^\xi$. □

Theorem 4. *Let $w \in \mathbb{Q}_+^{V_+}$, $b \in \mathbb{Z}_+^{V_+}$, $\emptyset \subsetneq S \subseteq V_+$ and $k' \in \mathbb{Z}_{++}$. Assume that Formulation (16) is feasible for every scenario $\xi \in [N]$ (see Remark 2). For each $\xi \in [N]$, let $(\bar{\alpha}_S^\xi, \bar{\beta}^\xi)$ be as specified in Lemma 6, and set $\hat{\alpha} \geq 0$ and $\hat{\beta} \leq 0$ as*

$$\begin{aligned}
\hat{\alpha}_{S'}^\xi &= \mathbb{I}(S' = S) \cdot (p_\xi\, \bar{\alpha}_S^\xi), && \forall \emptyset \subsetneq S' \subseteq V_+, \xi \in [N], \\
\hat{\beta}_v^\xi &= p_\xi\, \bar{\beta}_v^\xi, && \forall v \in V_+, \xi \in [N].
\end{aligned}$$

Then, $(\hat{\alpha}, \hat{\beta}) \in \mathcal{A}$ and for every $\bar{x} \in \{x \in \mathcal{X} : x(E(S)) \leq |S| - k'\}$, inequality (17) (with x fixed to $\bar{x}$) is valid for $\mathrm{proj}_\theta(\Gamma(\bar{x}, \hat{\alpha}, \hat{\beta}))$. In particular, if $x(E(S)) \leq |S| - k'$ is valid for $\mathcal{P}$, then inequality (17) (with x free) is also valid for $\mathcal{P}$.

Proof. Fix $\bar{x} \in \{x \in \mathcal{X} : x(E(S)) \leq |S| - k'\}$. We first rewrite the inequality defining the set $\Gamma(\bar{x}, \hat{\alpha}, \hat{\beta})$:

$$\begin{aligned}
&\sum_{\xi \in [N]} \sum_{v \in V_+} \left(\hat{\beta}_v^\xi + \sum_{\emptyset \subsetneq S' \subseteq V_+ : v \in S'} \hat{\alpha}_{S'}^\xi \right) y_v^\xi \geq \sum_{\xi \in [N]} \sum_{\emptyset \subsetneq S' \subseteq V_+} \hat{\alpha}_{S'}^\xi\, \bar{x}(E(S)) + \nu(\hat{\alpha}, \hat{\beta}) \\
\iff &\sum_{\xi \in [N]} p_\xi \sum_{v \in S} \left(\bar{\beta}_v^\xi + \bar{\alpha}_S^\xi \right) y_v^\xi \geq \sum_{\xi \in [N]} p_s \left(\bar{\alpha}_S^\xi \left(k_\xi(S) + \bar{x}(E(S)) - |S| \right) + \sum_{v \in S} \bar{\beta}_v^\xi b_v \right).
\end{aligned}$$

Fix a scenario $\xi \in [N]$. It follows from Lemma 6 and the choice of $(\bar{\alpha}_S^\xi, \bar{\beta}^\xi)$ that $\bar{\beta}_v^\xi + \bar{\alpha}_S^\xi \leq w_v$, for all $v \in S$. In particular, whenever $w_v = 0$, we have that $\hat{\beta}_v^\xi + \hat{\alpha}_S^\xi \leq 0$, so $(\hat{\alpha}, \hat{\beta}) \in \mathcal{A}$. Moreover,

$$\bar{\alpha}_S^\xi \left(k_\xi(S) + \bar{x}(E(S)) - |S| \right) + \sum_{v \in V_+} \bar{\beta}_v^\xi b_v = \underbrace{\bar{\alpha}_S^\xi \left(k_\xi(S) - k' \right) + \sum_{v \in S} \bar{\beta}_v^\xi\, b_v}_{= \mathcal{L}_\xi^*(S, k')} + \underbrace{\bar{\alpha}_S^\xi (x(E(S)) - |S| + k')}_{\geq \mathcal{L}_\xi^*(S,k')(x(E(S)) - |S| + k')}.$$

Summing over all scenarios $\xi \in [N]$, we learn that inequality

$$\theta(S) \geq \sum_{\xi \in [N]} \sum_{v \in S} p_\xi w_v y_v^\xi \geq \sum_{\xi \in [N]} p_\xi \sum_{v \in S} \left(\bar{\beta}_v^\xi + \bar{\alpha}_S^\xi \right) y_v^\xi \geq \mathcal{L}^*(S, k') \left(1 + \bar{x}(E(S)) - |S| + k' \right)$$

is valid for $\mathrm{proj}_\theta(\Gamma(\bar{x}, \hat{\alpha}, \hat{\beta}))$. To prove the second part of the statement, it suffices to observe that, if $x(E(S)) \leq |S| - k'$ is valid for $\mathcal{X}$, then the above inequality (with $\bar{x}$ replaced by a free variable x) is valid for $\mathcal{P}$. ☐

J Example of a projected SRI dominating a set cut

Let $S = \{v_1, v_2, v_3\} \subseteq V_+$, and suppose that $w_{v_1} = 2$, $w_{v_2} = 3$, $w_{v_3} = 4$, and $b = \mathbf{1}$. Assume that $x(E(S)) \leq |S| - 1$ is valid for $\mathcal{X}$, so $k' = 1$ in Formulation (16). Suppose that $\xi \in [N]$ is such that $k_\xi(S) = 3$, while for every other $\xi' \in [N] \setminus \xi$ we have $k_{\xi'}(S) \leq 1$. Since $\mathcal{L}^*(S, 1) = p_\xi\,(2 + 3) = 5p_\xi$, the set cut (17) becomes

$$\theta_{v_1} + \theta_{v_2} + \theta_{v_3} \geq 5p_\xi + 5p_\xi \cdot (x(E(S)) - |S| + 1). \tag{28}$$

Applying Lemma 6, we obtain a dual solution $(\bar{\alpha}_S^\xi, \bar{\beta}^\xi)$ with $\bar{\alpha}_S^\xi = 3$, $\bar{\beta}_{v_1}^\xi = -1$, and $\bar{\beta}_v^\xi = 0$ for all $v \in V_+ \setminus \{v_1\}$. Note that $\mathcal{L}_\xi^*(S, 1) = \bar{\alpha}_S^\xi (k_\xi(S) - k') + \sum_{v \in V_+} \bar{\beta}_v^\xi b_v = 3 \cdot 2 - 1 = 5$.

Following the construction in the proof of Theorem 4, we sum the following two inequalities:

$$\begin{aligned}
&(3p_\xi) \cdot y^\xi(S) \geq (3p_\xi) \cdot (k_\xi(S) + x(E(S)) - |S|) = (3p_\xi) \cdot (2 + x(E(S)) - |S| + 1) \\
&(-1p_\xi) \cdot y_{v_1}^\xi \geq (-1p_\xi) \cdot 1,
\end{aligned}$$

which yields

$$(2p_\xi)y^\xi_{v_1} + (3p_\xi)y^\xi_{v_2} + (3p_\xi)y^\xi_{v_3} \geq 5p_\xi + 3p_\xi \cdot (x(E(S)) - |S| + 1).$$

By Lemma 5, the projection of the above inequality onto the (x, θ)-space is given by the projected SRI

$$\frac{2}{2}\theta_{v_1} + \frac{3}{3}\theta_{v_2} + \frac{3}{4}\theta_{v_3} \geq 5p_\xi + 3p_\xi \cdot (x(E(S)) - |S| + 1),$$

which indeed dominates the set cut (28).

K Proof of Theorem 5

We first need a few technical lemmas.

Lemma 8. *For every partial route H and scenario $\xi \in [N]$, there exists an optimal dual solution $(\bar{\alpha}^\xi, \bar{\beta}^\xi)$ associated with (18) such that*

- $\sum_{H' \subseteq H} \bar{\alpha}^\xi_{H'} \leq \mathcal{L}^*_\xi(H)$; *and*
- $\sum_{H' \subseteq H : v \in V_+(H')} \bar{\alpha}^\xi_{H'} + \bar{\beta}^\xi_v \leq 0$, *for every $v \in V_+$ with $w_v = 0$.*

Proof. Consider the dual of (18):

$$\max \sum_{H' \subseteq H} \alpha^\xi_{H'} (k_\xi(H') - 1) + \sum_{v \in V_+} \beta^\xi_v b_v \tag{29a}$$

$$\text{s.t.:} \sum_{H' \subseteq H : v \in V_+(H')} \alpha^\xi_{H'} + \beta^\xi_v \leq w_v, \qquad \forall v \in V_+(H), \tag{29b}$$

$$\alpha^\xi \geq 0, \tag{29c}$$

$$\beta^\xi \leq 0. \tag{29d}$$

Let $(\bar{\alpha}^\xi, \bar{\beta}^\xi)$ be an optimal solution to Formulation (29) and let $S = \{v \in V_+ : \bar{\beta}^\xi_v < 0\}$. Assume that we choose $(\bar{\alpha}^\xi, \bar{\beta}^\xi)$ so that the size of its support (i.e., $|\{H' \subseteq H : \bar{\alpha}^\xi_{H'} > 0\}| + |S|$) is minimum. By optimality of $(\bar{\alpha}^\xi, \bar{\beta}^\xi)$, we may also assume that $\bar{\beta}^\xi_v = \min\{w_v - \sum_{H' \subseteq H : v \in V_+(H')} \bar{\alpha}^\xi_{H'}, 0\}$, for all $v \in V_+$.

Rewriting the objective value of $(\bar{\alpha}^\xi, \bar{\beta}^\xi)$ we get

$$\begin{aligned}
\mathcal{L}^*_\xi(H) &= \sum_{H' \subseteq H} \bar{\alpha}^\xi_{H'} (k_\xi(H') - 1) + \sum_{v \in V_+} \bar{\beta}^\xi_v b_v \\
&= \sum_{H' \subseteq H} \bar{\alpha}^\xi_{H'} (k_\xi(H') - 1) + \sum_{v \in S} \left(w_v - \sum_{H' \subseteq H : v \in V_+(H')} \bar{\alpha}^\xi_{H'} \right) b_v \\
&= \sum_{v \in S} w_v b_v + \sum_{H' \subseteq H} \bar{\alpha}^\xi_{H'} (k_\xi(H') - b(V_+(H') \cap S) - 1).
\end{aligned} \tag{30}$$

We now build on the support minimality to prove the following claim.

Claim 1. *For every $H' \subseteq H$ with $\bar{\alpha}^\xi_{H'} > 0$, we have that $k_\xi(H') - b(V_+(H') \cap S) \geq 2$.*

Proof. Suppose by contradiction that there exists $H' \subseteq H$ such that $\bar{\alpha}^{\xi}_{H'} > 0$ and $k_\xi(H') - b(V_+(H') \cap S) \leq 1$. Consider the direction $r = (-\mathbb{1}_{H'}, \sum_{v \in V_+(H') \cap S} \mathbb{1}_v)$ and take $\varepsilon = \min\left\{\bar{\alpha}^{\xi}_{H'}, \min_{v \in V_+(H') \cap S}\{-\bar{\beta}^{\xi}_v\}\right\} > 0$. Construct $(\hat{\alpha}^\xi, \hat{\beta}^\xi) = (\bar{\alpha}^\xi, \bar{\beta}^\xi) + \varepsilon\, r$ and observe that, by the choice of ε, $(\hat{\alpha}^\xi, \hat{\beta}^\xi)$ is feasible for Formulation (29), has smaller support than $(\bar{\alpha}^\xi, \bar{\beta}^\xi)$, and has cost

$$\mathcal{L}^*_\xi(H) + \varepsilon\,(-k_\xi(H') + b(V_+(H') \cap S) + 1) \geq \mathcal{L}^*_\xi(H),$$

since $-k_\xi(H') + b(V_+(H') \cap S) + 1 \geq 0$ by the choice of H'. Therefore, $(\hat{\alpha}^\xi, \hat{\beta}^\xi)$ is optimal for Formulation (29), contradicting the minimality of the support of $(\bar{\alpha}^\xi, \bar{\beta}^\xi)$. □

Applying Claim 1 to (30), we conclude that $\mathcal{L}^*_\xi(H) \geq \sum_{v \in S} w_v b_v + \sum_{H' \subseteq H} \bar{\alpha}^{\xi}_{H'} \geq \sum_{H' \subseteq H} \bar{\alpha}^{\xi}_{H'}$. To conclude the proof, note that, by the feasibility of $(\bar{\alpha}^\xi, \bar{\beta}^\xi)$ to Formulation (29), it follows that $\sum_{H' \subseteq H : v \in V_+(H')} \bar{\alpha}^{\xi}_{H'} + \bar{\beta}^{\xi}_v \leq 0$, for every $v \in V_+$ with $w_v = 0$. □

Recall that $H = (S_1, \ldots, S_\ell)$. The activation function proposed by Ota and Fukasawa (2025) is shown below:

$$W_{OF}(x; \mathcal{X}_{\supseteq}(H)) = 1 + (x(H) - |V_+(H)| + 1) + \sum_{i \in \{2, \ell-1\} \cap [\ell]} (x(E(S_i)) - |S_i| + 1), \tag{31}$$

where $x(H) := \sum_{i \in [\ell]} x(E(S_i)) + \sum_{i \in [\ell-1]} x(E(S_i, S_{i+1}))$.

To prove Theorem 5, we use the following property of $W_{OF}(x; \mathcal{X}_{\supseteq}(H))$.

Lemma 9. *For every $H' \subseteq H$ and $\bar{x} \in \mathcal{X}$, $\bar{x}(H') - |V_+(H')| + 1 \geq W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H)) - 1$.*

Proof. Suppose by contradiction that $H' = (S_i, \ldots, S_j) \subseteq H$ and $\bar{x} \in \mathcal{X}$ is such that $\bar{x}(H') - |V_+(H')| + 1 < W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H)) - 1$. Furthermore, assume that H' is a maximal counterexample, meaning that, for every $H'' \supseteq H'$ with $H'' \neq H'$, we have $\bar{x}(H'') - |V_+(H'')| + 1 \geq W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H)) - 1$.

Claim 2. *$j \geq \ell - 1$ and $i \leq 2$.*

Proof. By symmetry, we only show that $j \geq \ell - 1$. Suppose by contradiction that $j \leq \ell - 2$ (so $\ell \geq 3$). We first consider the case $|S_{j+1}| > 1$. By the definition of partial routes, we know that $|S_j| = 1$, so $\bar{x}(E(S_j, S_{j+1})) + \bar{x}(E(S_{j+1})) = \bar{x}(E(S_j \cup S_{j+1})) \leq |S_j \cup S_{j+1}| - 1 = |S_{j+1}|$. Hence, for $H'' = (S_i, \ldots, S_{j+1})$,

$$\begin{aligned}
\bar{x}(H'') - |V_+(H'')| + 1 &= (\bar{x}(H') + \bar{x}(E(S_j, S_{j+1})) + \bar{x}(E(S_{j+1}))) - (|V_+(H')| + |S_{j+1}|) + 1 \\
&= (\bar{x}(H') - |V_+(H')| + 1) + (\bar{x}(E(S_j \cup S_{j+1})) - |S_{j+1}|) \\
&\leq \bar{x}(H') - |V_+(H')| + 1.
\end{aligned}$$

On the other hand, if $S_{j+1} = \{v\}$, set $H'' = (S_i, \ldots, S_{j+2})$ and note that

$$\begin{aligned}
&\bar{x}(H'') - |V_+(H'')| + 1 \\
&\quad = (\bar{x}(H') + \bar{x}(E(v, S_j \cup S_{j+2})) + \bar{x}(E(S_{j+2})) - (|V_+(H')| + |S_{j+2}| + 1) + 1 \\
&\quad = (\bar{x}(H') + \bar{x}(E(v, S_j \cup S_{j+2})) + \bar{x}(E(S_{j+2})) - (|V_+(H')| + |S_{j+2}| + 2) + 2 \\
&\quad = (\bar{x}(H') - |V_+(H')| + 1) + (\bar{x}(E(v, S_j \cup S_{j+2})) - 2) + (\bar{x}(E(S_{j+2})) - |S_{j+2}| + 1) \\
&\quad \leq \bar{x}(H') - |V_+(H')| + 1.
\end{aligned}$$

□

By Claim 2, it remains to check the cases where $i \in \{1,2\}$ and $j \in \{\ell-1,\ell\}$. Note that $\bar{x}(H) - |V_+(H)| + 1 \geq W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H)) - 1$, so we cannot have $i = 1$ and $j = \ell$, which implies that $\ell \geq 2$. Assume first that $j = \ell - 1$ and $i = 1$ (so $\ell \geq 2$). Using the formula in (31) we get

$$\begin{aligned} W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H)) - 1 &\leq (\bar{x}(H) - |V_+(H)| + 1) + (\bar{x}(E(S_{\ell-1})) - |S_{\ell-1}| + 1) \\ &= (\bar{x}(H') - |V_+(H')| + 1) + (\bar{x}(E(S_{\ell-1} \cup S_\ell)) - |S_{\ell-1} \cup S_\ell| + 1) \\ &\leq \bar{x}(H') - |V_+(H')| + 1. \end{aligned}$$

By symmetry, this also handles the case $i = 2$ and $j = \ell$.

It remains to check when $i = 2$ and $j = \ell - 1$. Since $i \leq j$, this implies that $\ell \geq 3$. If $\ell = 3$, we have that $H' = (S_2)$, so

$$W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H)) - 1 = (\bar{x}(H) - |V_+(H)| + 1) + (\bar{x}(H') - |V_+(H')| + 1) \leq \bar{x}(H') - |V_+(H')| + 1.$$

On the other hand, if $\ell \geq 4$, we rearrange (31) to obtain

$$\begin{aligned} W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H)) - 1 = {} & (\bar{x}(H') - |V_+(H')| + 1) \\ & + (\bar{x}(E(S_1 \cup S_2)) - |S_1 \cup S_2| + 1) + (\bar{x}(E(S_{\ell-1} \cup S_\ell)) - |S_{\ell-1} \cup S_\ell| + 1) \\ \leq {} & \bar{x}(H') - |V_+(H')| + 1. \end{aligned}$$

□

Theorem 5. *Let H be a partial route. There exist $(\alpha, \beta) \in \mathcal{A}$ such that the projected SRI*

$$\sum_{v \in V_+} \phi_v(\alpha, \beta)\, \theta_v \geq \sum_{\xi \in [N]} \sum_{S \subseteq V_+} \alpha_S^\xi\, x(E(S)) + \nu(\alpha, \beta)$$

dominates inequality (19). In particular, inequality (19) is valid for $\mathcal{P}$.

Proof. Take an arbitrary scenario $\xi \in [N]$. We first show that inequality

$$\sum_{v \in V_+(H)} w_v y_v^\xi \geq \mathcal{L}_\xi^*(H) \cdot W_{OF}(x; \mathcal{X}_{\supseteq}(H)) \tag{32}$$

is valid for the LP relaxation of Formulation (9).

Let $(\bar{\alpha}^\xi, \bar{\beta}^\xi)$ be an optimal dual solution chosen according to Lemma 8. We construct an inequality from $(\bar{\alpha}^\xi, \bar{\beta}^\xi)$ as follows. For each $H' \subseteq H$, we multiply the SRI $y^\xi(H') \geq (k_\xi(H') - 1) + x(E(V_+(H'))) - |H'| + 1$ by $\bar{\alpha}_{H'}^\xi \geq 0$, and for each $v \in V_+(H)$, we multiply $y_v^\xi \leq b_v$ by $\bar{\beta}_v^\xi \leq 0$. Summing these inequalities we get an inequality (valid for the LP relaxation of (9)) of the form $A^\xi(y) \geq B^\xi(x)$, where

$$\begin{aligned} A^\xi(y) &= \sum_{v \in V_+(H)} \left(\sum_{H' \subseteq H : v \in V_+(H')} \bar{\alpha}_{H'}^\xi + \bar{\beta}_v^\xi \right) y_v^\xi, \quad \text{and} \\ B^\xi(x) &= \sum_{H' \subseteq H} \bar{\alpha}_{H'}^\xi \left[(k_\xi(H') - 1) + x(E(V_+(H'))) - |H'| + 1 \right] + \sum_{v \in V_+(H)} \bar{\beta}_v^\xi b_v. \end{aligned}$$

Feasibility of $(\bar{\alpha}^\xi, \bar{\beta}^\xi)$ to the dual of (18) implies $A^\xi(y) \leq \sum_{v \in V_+(H)} w_v y_v^\xi$. Moreover, for every $\bar{x} \in \mathcal{X}$,

$$\begin{aligned} B^\xi(\bar{x}) &= \mathcal{L}_\xi^*(H) + \sum_{H' \subseteq H} \bar{\alpha}_{H'} \cdot (\bar{x}(E(V_+(H'))) - |H'| + 1) && \text{(by optimality of } (\bar{\alpha}^\xi, \bar{\beta}^\xi)) \\ &\geq \mathcal{L}_\xi^*(H) + \sum_{H' \subseteq H} \bar{\alpha}_{H'}^\xi \cdot (W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H)) - 1) && \text{(by Lemma 9)} \\ &\geq \mathcal{L}_\xi^*(H) \cdot W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H)), && \text{(by Lemma 8 and since } W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H)) - 1 \leq 0) \end{aligned}$$

proving that (32) is indeed valid for the LP relaxation of Formulation (9).

To obtain the projected SRI that dominates inequality (32), construct $(\hat{\alpha}, \hat{\beta})$ by concatenating the optimal dual solutions obtained from Lemma 8 and multiplying by the corresponding probabilities, i.e., $\hat{\alpha} = (p_1 \bar{\alpha}^1, \dots, p_N \bar{\alpha}^N)$ and $\bar{\beta} = (p_1 \bar{\beta}^1, \dots, p_N \bar{\beta}^N)$. Observe that, by Lemma 8, $\bar{\beta}^\xi_v + \sum_{S \subseteq V_+ : v \in S} \bar{\alpha}^\xi_S$ is nonpositive for every $\xi \in [N]$ and $v \in V_+$ with $w_v = 0$, meaning that $(\hat{\alpha}, \hat{\beta}) \in \mathcal{A}$.

Applying the argument used to show that (32) is valid for the LP relaxation of Formulation (9), it follows that, for every $\bar{x} \in \mathcal{X}$, inequality

$$\theta(V_+(H)) \geq \sum_{\xi \in [N]} \sum_{v \in V_+(H)} p_\xi w_v y^\xi_v \geq \sum_{\xi \in [N]} p_\xi \, A^\xi(y) \geq \sum_{\xi \in [N]} p_\xi \, B^\xi(x) \geq \mathcal{L}^*(H) \cdot W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H))$$

is valid for $\Gamma(\bar{x}, \hat{\alpha}, \hat{\beta})$. Lemma 5 then yields the desired projected SRI, and since $\mathcal{P}$ satisfies all projected SRIs (Proposition 3), inequality (19) is valid for $\mathcal{P}$. $\square$

L Proof of Theorem 6

Theorem 6. *For every $\alpha \geq 0$ and $\beta \leq 0$,*

$$\min\{c^\intercal x + \mathbb{1}^\intercal \theta : x \in \mathcal{X},\ \theta \in \mathrm{proj}_\theta(\Gamma(x, \alpha, \beta))\} \geq \sigma_x(\alpha) + \sigma_y(\alpha, \beta) + \nu(\alpha, \beta).$$

Proof. Fix $\alpha \geq 0$ and $\beta \leq 0$ and consider

$$\begin{aligned} z' &:= \min\{c^\intercal x + \mathbb{1}^\intercal \theta : x \in \mathcal{X},\ \theta \in \mathrm{proj}_\theta(\Gamma(x, \alpha, \beta))\} \\ &= \min\left\{ c^\intercal x + \sum_{v \in V_+} \sum_{\xi \in [N]} p_\xi w_v y^\xi_v : x \in \mathcal{X},\ y \in \Gamma(x, \alpha, \beta) \right\}. \end{aligned}$$

Multiplying the inequality defining $\Gamma(x, \alpha, \beta)$ by $\eta \in \mathbb{R}_+$ and dualizing into the objective function yields

$$z' = \max_{\eta \geq 0} \left\{ \min_{x \in \mathcal{X}, y \geq 0} \left\{ \begin{aligned} & c^\intercal x + \sum_{v \in V_+} \sum_{s=1}^N p_s w_v y^s_v + \eta \sum_{s=1}^N \sum_{S \subseteq V_+} \alpha^s_S x(E(S)) + \eta\, \nu(\alpha, \beta) \\ & - \eta \sum_{s=1}^N \sum_{v \in V_+} \left(\beta^s_v + \sum_{S \subseteq V_+ : v \in S} \alpha^s_S \right) y^s_v \end{aligned} \right\} \right\}.$$

Taking $\eta = 1$ we conclude that $z' \geq \sigma_x(\alpha) + \sigma_y(\alpha, \beta) + \nu(\alpha, \beta)$. $\square$

M Extension to Assumption 4

Fix $w \in \mathbb{Q}_+^{V_+ \times V_+}$ and $b \in \mathbb{Z}_+^{V_+}$ according to Assumption 4. Let $R = (v_1, \dots, v_\ell)$ and consider the recourse function

$$\tilde{\mathcal{Q}}^*(R) := \min \left\{ \sum_{\xi \in [N]} \sum_{i \in [\ell]} p_\xi \min\{w_{v_i, u} : u \in \{v_i, v_{i-1}, v_{i+1}\} \cap V_+\} \cdot (\bar{y}_R)^\xi_{v_i} :\ y \in \Pi(R) \cap [\mathbf{0}, b]^N \right\}. \tag{$\tilde{\mathcal{Q}}^*$}$$

Recall that $\mathcal{X} \in \{\mathcal{X}_{\text{SUB}}, \mathcal{X}_{\text{CVRP}}\}$. We first show that the SRI-based formulation developed in Section 4 can be adapted to the recourse function $\tilde{\mathcal{Q}}^*$. To achieve this, we introduce extra variables s_v, for $v \in V_+$, and r_{vu}, for every pair $(v, u) \in V_+ \times V_+$. To gain an intuition for the following formulation, suppose that $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$ is a first-stage solution such that $R = (v_1, \dots, v_\ell) \in \mathcal{R}(\bar{x})$ and let $i \in \{2, \dots, \ell-1\}$. For this solution, variable s_{v_i} represents the expected number of failures that route R observes at customer v_i. This value is then "distributed" as $s_v = r_{v_i,v_i} + r_{v_i,v_{i-1}} + r_{v_i,v_{i+1}}$, where $r_{v_i,v_i}, r_{v_i,v_{i-1}}$ and $r_{v_i,v_{i+1}}$ are associated with edges $0v_i, v_{i-1}v_i$ and v_iv_{i+1}. Although variables s_v could be replaced by the expression $\sum_{\xi\in[N]} p_\xi y^\xi_v$, keeping these variables now will be useful later on.

Corollary 5. *Problem* VRPSD$(\mathcal{X}, \tilde{\mathcal{Q}}^*)$ *can be formulated as*

$$
\begin{aligned}
\min\ & c^\intercal x + \sum_{v\in V_+} \theta_v, \\
\text{s.t.}\ & y^\xi(S) \geq k_\xi(S) + x(E(S)) - |S|, && \forall \emptyset \subsetneq S \subseteq V_+, \xi \in [N], && \text{(33a)}\\
& \theta_v \geq \sum_{u\in V_+} w_{vu} r_{vu}, && \forall v \in V_+, && \text{(33b)}\\
& s_v \geq \sum_{\xi\in[N]} p_\xi y^\xi_v, && \forall v \in V_+, && \text{(33c)}\\
& s_v = \sum_{u\in V_+} r_{vu}, && \forall v \in V_+, && \text{(33d)}\\
& r_{vu} \leq b_v \cdot x_{vu}, && \forall vu \in E \setminus \delta(0), && \text{(33e)}\\
& (x, \theta, y, s, r) \in (\mathcal{X} \cap \mathbb{Z}^E) \times \mathbb{R}^{V_+}_+ \times [\mathbf{0}, b]^N \times \mathbb{R}^{V_+}_+ \times \mathbb{R}^{V_+\times V_+}_+. && && \text{(33f)}
\end{aligned}
$$

Proof. We first simplify the formulation to

$$
\begin{aligned}
\min\ & c^\intercal x + \sum_{v\in V_+} \theta_v, \\
\text{s.t.}\ & y^\xi(S) \geq k_\xi(S) + x(E(S)) - |S|, && \forall \emptyset \subsetneq S \subseteq V_+, \xi \in [N], && \text{(34a)}\\
& \theta_v \geq \sum_{u\in V_+} w_{vu} r_{vu}, && \forall v \in V_+, && \text{(34b)}\\
& \sum_{u\in V_+} r_{vu} \geq \sum_{\xi\in[N]} p_\xi y^\xi_v, && \forall v \in V_+, && \text{(34c)}\\
& r_{vu} \leq b_v \cdot x_{vu}, && \forall vu \in E \setminus \delta(0), && \text{(34d)}\\
& (x, \theta, y, r) \in (\mathcal{X} \cap \mathbb{Z}^E) \times \mathbb{R}^{V_+}_+ \times [\mathbf{0}, b]^N \times \mathbb{R}^{V_+\times V_+}_+. && && \text{(34e)}
\end{aligned}
$$

Now fix $x = \bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$ in the above formulation. For each $R = (v_1, \dots, v_\ell) \in \mathcal{R}(\bar{x})$ and $i \in [\ell]$, set $\tilde{w}_{v_i} = \min\{w_{v_i,v_i}, w_{v_i,v_{i-1}}, w_{v_i,v_{i+1}}\}$. Note that we may assume that whenever variable $r_{v_i,u}$ attains a positive value, then $u \in \{v_i, v_{i-1}, v_{i+1}\} \cap V_+$ and $w_{v_i,u} = \tilde{w}_{v_i}$. Moreover, since $\sum_{\xi\in[N]} p_\xi = 1$, we know that $\sum_{\xi\in[N]} p_\xi y^\xi_v \leq b_v$, for all $v \in V_+$. Hence, fixing $x = \bar{x}$ we rewrite Formulation (34) as:

$$
\begin{aligned}
\min\ & c^\intercal \bar{x} + \sum_{v\in V_+} \theta_v, \\
\text{s.t.}\ & y^\xi(S) \geq k_\xi(S) + \bar{x}(E(S)) - |S|, && \forall \emptyset \subsetneq S \subseteq V_+, \xi \in [N], \\
& \theta_v \geq \tilde{w}_v \sum_{\xi\in[N]} p_\xi y^\xi_v, && \forall v \in V_+, \\
& y \in [\mathbf{0}, b]^N.
\end{aligned}
$$

By Theorem 2, this is equivalent to

$$\min\left\{c^\intercal \bar{x} + \sum_{\xi\in[N]}\sum_{v\in V_+} p_\xi \tilde{w}_v y_v^\xi : y \in \operatorname{conv}(\Pi(\bar{x}) \cap [\mathbf{0}, b]^N)\right\},$$

as desired. □

Let $\bar{x} \in \mathcal{X} \cap \mathbb{Z}^E$. Similarly to $\widehat{\text{SRI}}(\bar{x})$, let us now define

$$\widetilde{\text{SRI}}(\bar{x}) := \left\{(s, y) \in \mathbb{R}_+^{V_+} \times [\mathbf{0}, b]^N : \begin{array}{lr} y^\xi(S) \geq k_\xi(S) + \bar{x}(E(S)) - |S|, & \forall \emptyset \subsetneq S \subseteq V_+,\ \xi \in [N] \\ s_v \geq \sum_{\xi\in[N]} p_\xi y_v^\xi, & \forall v \in V_+ \end{array}\right\}. \tag{35}$$

In this way, $\widetilde{\text{SRI}}(\bar{x})$ is equivalent to $\widehat{\text{SRI}}(\bar{x})$ in the special case that $w_v = 1$, for every customer $v \in V_+$.

This implies that Proposition 3 can be naturally extended to $\widetilde{\text{SRI}}(\bar{x})$. To this end, for any $\alpha \geq 0$, $\beta \leq 0$ and $v \in V_+$, define

$$\tilde{\phi}_v(\alpha, \beta) := \left(\max_{\xi\in[N]}\left\{\frac{\beta_v^\xi + \sum_{S\subseteq V_+ : v\in S} \alpha_S^\xi}{p_\xi}\right\}\right)^+. \tag{36}$$

Corollary 6. *For every $\bar{x} \in \mathcal{X}$,*

$$\operatorname{proj}_s(\widetilde{\text{SRI}}(\bar{x})) = \left\{s \in \mathbb{R}_+^{V_+} : \sum_{v\in V_+} \tilde{\phi}_v(\alpha, \beta)\, s_v \geq \sum_{\xi\in[N]}\sum_{S\subseteq V_+} \alpha_S^\xi\, \bar{x}(E(S)) + \nu(\alpha, \beta), \quad \forall \alpha \geq 0, \beta \leq 0\right\}$$

Consequently, Formulation (33) can be reformulated as

$$\begin{aligned}
\min\ & c^\intercal x + \sum_{v\in V_+} \theta_v, \\
\textit{s.t.}\ & \tilde{\phi}_v(\alpha, \beta)\, s_v \geq \sum_{\xi\in[N]}\sum_{S\subseteq V_+} \alpha_S^\xi\, x(E(S)) + \nu(\alpha, \beta), && \forall \alpha \geq 0, \beta \leq 0, && (37a)\\
& \theta_v \geq \sum_{u\in V_+} w_{vu} r_{vu}, && \forall v \in V_+, && (37b)\\
& s_v = \sum_{u\in V_+} r_{vu}, && \forall v \in V_+, && (37c)\\
& r_{vu} \leq b_v \cdot x_{vu}, && \forall vu \in E \setminus \delta(0), && (37d)\\
& (x, \theta, s, r) \in (\mathcal{X} \cap \mathbb{Z}^E) \times \mathbb{R}_+^{V_+} \times \mathbb{R}_+^{V_+} \times \mathbb{R}_+^{V_+\times V_+}. && && (37e)
\end{aligned}$$

To simplify further, we replace s_v by $\sum_{u\in V_+} r_{vu}$ to obtain the following formulation for $\textsc{vrpsd}(\mathcal{X}, \tilde{\mathcal{Q}}^*)$:

$$\begin{aligned}
\min\ & c^\intercal x + \sum_{v\in V_+} \theta_v, \\
\text{s.t.}\ & \tilde{\phi}_v(\alpha, \beta)\left(\sum_{u\in V_+} r_{vu}\right) \geq \sum_{\xi\in[N]}\sum_{S\subseteq V_+} \alpha_S^\xi\, x(E(S)) + \nu(\alpha, \beta), && \forall \alpha \geq 0, \beta \leq 0, && (38a)\\
& \theta_v \geq \sum_{u\in V_+} w_{vu} r_{vu}, && \forall v \in V_+, && (38b)\\
& r_{vu} \leq b_v \cdot x_{vu}, && \forall vu \in E \setminus \delta(0), && (38c)\\
& (x, \theta, r) \in (\mathcal{X} \cap \mathbb{Z}^E) \times \mathbb{R}_+^{V_+} \times \mathbb{R}_+^{V_+\times V_+}. && && (38d)
\end{aligned}$$

N Separation algorithms

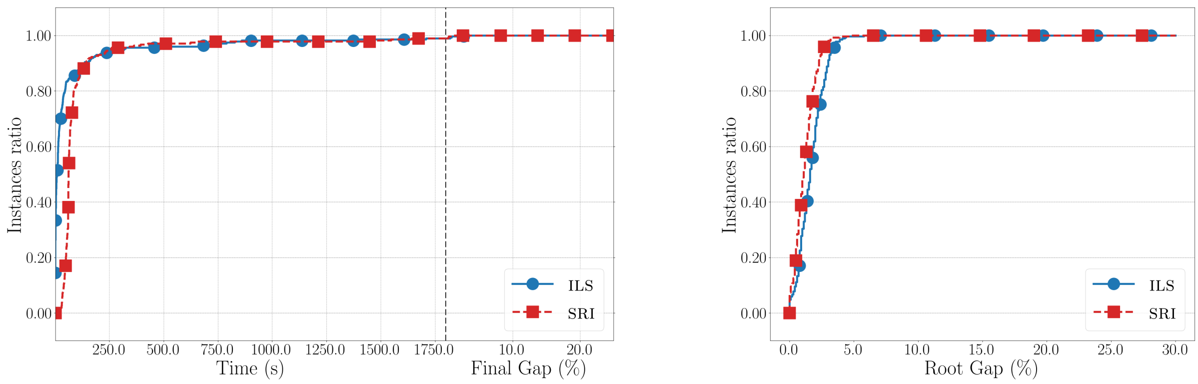
The following are the pseudocodes for our separation of ILS cuts and SRIs. Algorithm GetPartialRoutes refers to the heuristic described in (Ota and Fukasawa, 2025).

Algorithm 2 AddSetCutOrSRI

procedure AddSetCutOrSRI($\bar{x}, \bar{\theta}, \texttt{use\_sri}, S, k'$)
 Compute $\mathcal{L}^*(S, k')$ using the greedy algorithm mentioned in Section 5.2.1.
 if $\texttt{use\_sri} = \texttt{true}$ **then**
  $\Xi \leftarrow \{\xi \in [N] : k_\xi(S) + \bar{x}(E(S)) - |S| > 0\}$
  if $\mathcal{X} = \mathcal{X}_{\text{CVRP}}$ **then**
   $\Xi \leftarrow \{\xi \in \Xi : k_\xi(S) > \bar{k}(S)\}$
  if $\sum_{v \in S} \max_{\xi \in [N]} \left\{\frac{1}{p_\xi w_v}\right\} \bar{\theta}_v < |\Xi| \cdot (\bar{x}(E(S)) - |S|) + \sum_{\xi \in \Xi} k_\xi(S)$ **then**
   Add projected SRI $\sum_{v \in S} \max_{\xi \in [N]} \left\{\frac{1}{p_\xi w_v}\right\} \theta_v \geq |\Xi| \cdot (x(E(S)) - |S|) + \sum_{\xi \in \Xi} k_\xi(S)$.
   return `true`
 else
  if $\bar{\theta}(S) < \mathcal{L}^*(S, k') \cdot (\bar{x}(E(S)) - |S| + k')$ **then**
   Add set cut $\theta(S) \geq \mathcal{L}^*(S, k') \cdot (x(E(S)) - |S| + k')$.
   return `true`
 return `false`

Algorithm 3 AddPartialRouteCutOrSRI

procedure AddPartialRouteCutOrSRI($\bar{x}, \bar{\theta}, \texttt{use\_sri}, H$)
 Compute $\mathcal{L}^*(H)$ using Formulation (18).
 if $\texttt{use\_sri} = \texttt{true}$ **then**
  Compute (α, β) using Theorem 5.
  if $\sum_{v \in V_+} \phi_v(\alpha, \beta)\, \bar{\theta}_v < \sum_{\xi \in [N]} \sum_{S \subseteq V_+} \alpha_S^\xi\, \bar{x}(E(S)) + \nu(\alpha, \beta)$ **then**
   Add projected SRI $\sum_{v \in V_+} \phi_v(\alpha, \beta)\, \theta_v \geq \sum_{\xi \in [N]} \sum_{S \subseteq V_+} \alpha_S^\xi\, x(E(S)) + \nu(\alpha, \beta)$.
   return `true`
 else
  if $\bar{\theta}(V_+(H)) < \mathcal{L}^*(H) \cdot W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H))$ **then**
   Add partial route cut $\theta(V_+(H)) \geq \mathcal{L}^*(H) \cdot W_{OF}(x; \mathcal{X}_{\supseteq}(H))$.
   return `true`
 return `false`



(a) Execution time and final optimality gaps.

(b) Root gaps.

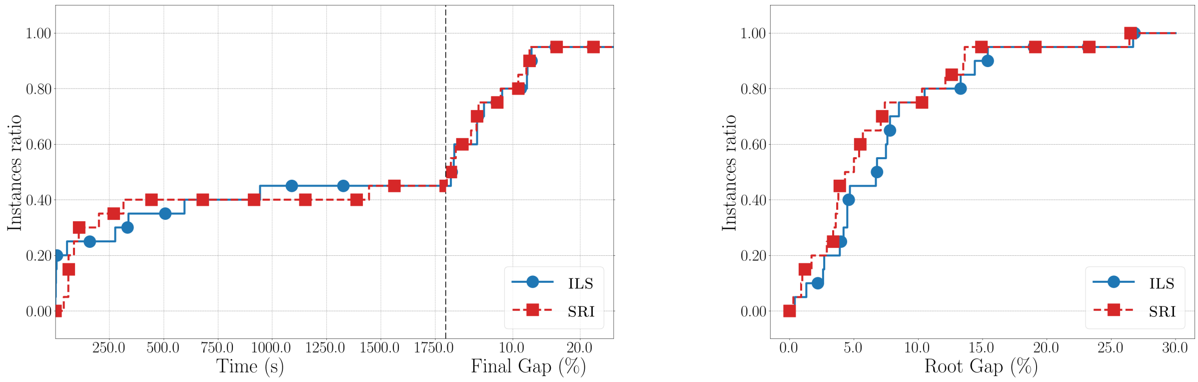
Figure 4: Empirical cumulative distribution of the execution times and root gaps for Jabali et al. (2014) instances, with $\mathcal{X} = \mathcal{X}_{\text{CVRP}}$ and $\mathcal{Q} = \mathcal{Q}^*$.

Algorithm 4 SeparationVRPSD

Input: A candidate solution $(\bar{x}, \bar{\theta}) \in \mathbb{R}^E \times \mathbb{Q}_+^{V_+}$ and a boolean flag `use_sri`, which indicates if we use projected SRIs or not.

procedure SeparationVRPSD($\bar{x}, \bar{\theta}$, `use_sri`)
- Call CVRPSEP to get a family of customer sets $\mathcal{S} \subseteq 2^{V_+}$.
- **for** $S \in \mathcal{S}$ **do**
  - $k' \leftarrow 1 + (\bar{k}(S) - 1) \cdot \mathbb{I}(\mathcal{X} = \mathcal{X}_{\text{CVRP}})$
  - Add inequality $x(E(S)) \leq |S| - k'$.
  - AddSetCutOrSRI($\bar{x}, \bar{\theta}$, `use_sri`, S, k')
- **if** $\mathcal{S} \neq \emptyset$ **then**
  - **return**
- $\mathcal{H} \leftarrow$ GetPartialRoutes($\bar{x}, \bar{\theta}$)
- **for** $H \in \mathcal{H}$ **do**
  - $k' \leftarrow 1 + (\bar{k}(S) - 1) \cdot \mathbb{I}(\mathcal{X} = \mathcal{X}_{\text{CVRP}})$
  - **if** AddSetCutOrSRI($\bar{x}, \bar{\theta}$, `use_sri`, $V_+(H), k'$) $=$ `true` **then**
    - **continue**
  - **if** AddPartialRouteCutOrSRI($\bar{x}, \bar{\theta}$, `use_sri`, H) $=$ `true` **then**
    - **continue**
  - **if** $\mathcal{Q} = \mathcal{Q}_C$ **and** $\bar{\theta}(V_+(H)) < \mathcal{L}_C(H) \cdot W_{OF}(\bar{x}; \mathcal{X}_{\supseteq}(H))$ **then**
    - Add inequality $\theta(V_+(H)) \geq \mathcal{L}_C(H) \cdot W_{OF}(x; \mathcal{X}_{\supseteq}(H))$.

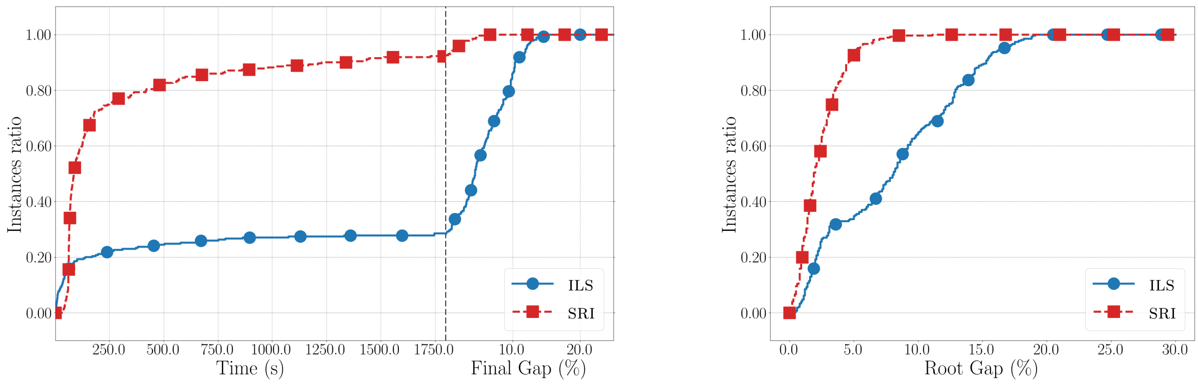
O Detailed computational experiments



(a) Execution time and final optimality gaps.

(b) Root gaps.

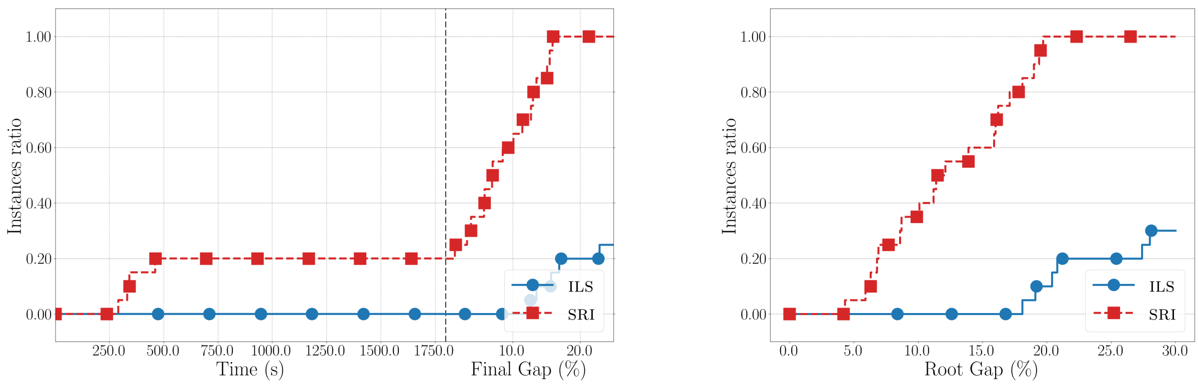
Figure 5: Empirical cumulative distribution of the execution times and root gaps for Dinh et al. (2018) instances, with $\mathcal{X} = \mathcal{X}_{\text{CVRP}}$ and $\mathcal{Q} = \mathcal{Q}^*$.



(a) Execution time and final optimality gaps.

(b) Root gaps.

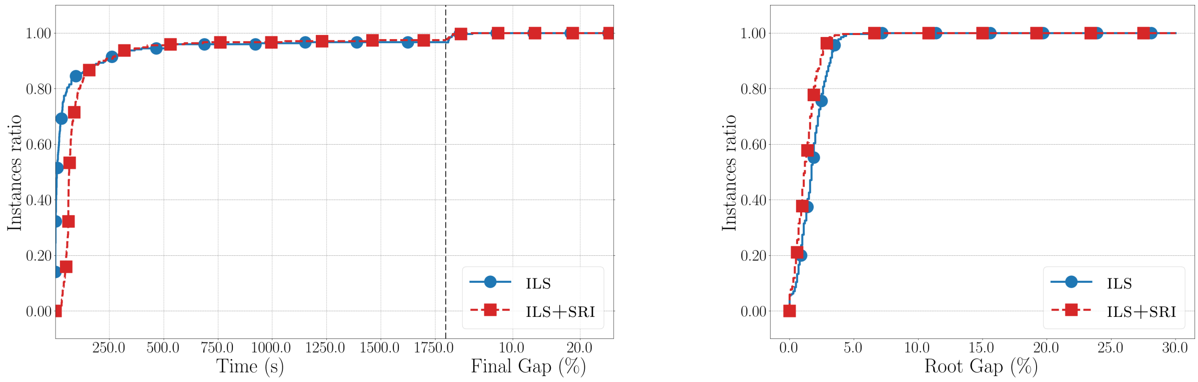
Figure 6: Empirical cumulative distribution of the execution times and root gaps for Jabali et al. (2014) instances, with $\mathcal{X} = \mathcal{X}_{\text{SUB}}$ and $\mathcal{Q} = \mathcal{Q}^*$.



(a) Execution time and final optimality gaps.

(b) Root gaps.

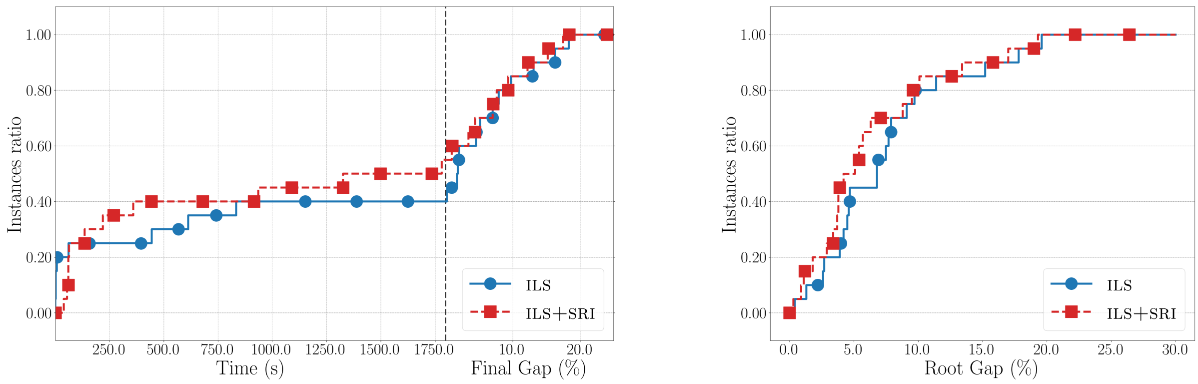
Figure 7: Empirical cumulative distribution of the execution times and root gaps for Dinh et al. (2018) instances, with $\mathcal{X} = \mathcal{X}_{\text{SUB}}$ and $\mathcal{Q} = \mathcal{Q}^*$.



(a) Execution time and final optimality gaps.

(b) Root gaps.

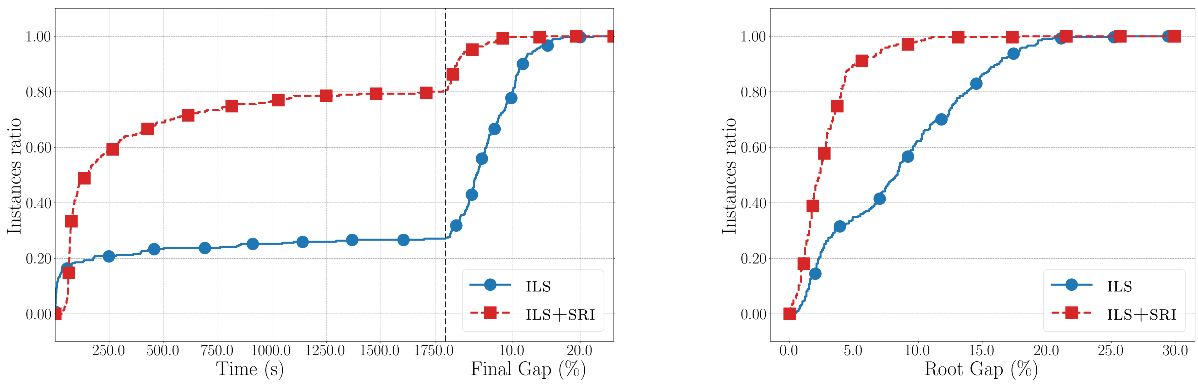
Figure 8: Empirical cumulative distribution of the execution times and root gaps for Jabali et al. (2014) instances, with $\mathcal{X} = \mathcal{X}_{\text{CVRP}}$ and $\mathcal{Q} = \mathcal{Q}_C$.



(a) Execution time and final optimality gaps.

(b) Root gaps.

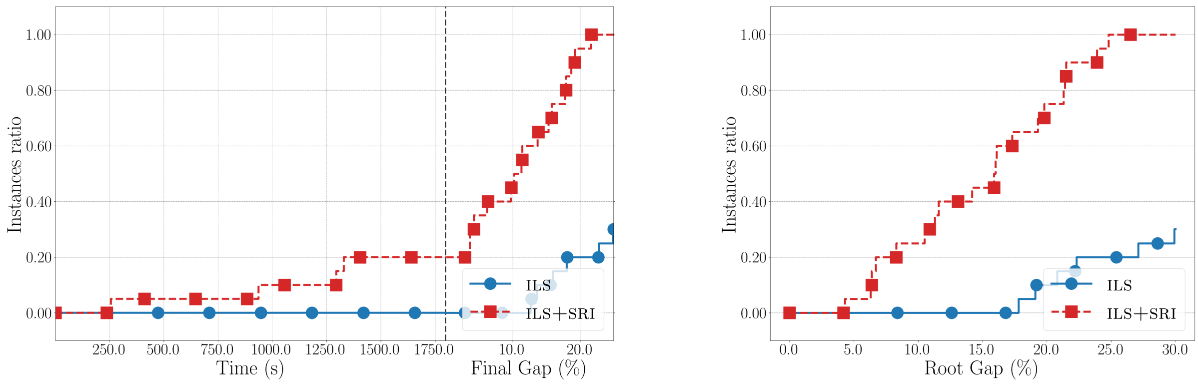
Figure 9: Empirical cumulative distribution of the execution times and root gaps for Dinh et al. (2018) instances, with $\mathcal{X} = \mathcal{X}_{\text{CVRP}}$ and $\mathcal{Q} = \mathcal{Q}_C$.



(a) Execution time and final optimality gaps.

(b) Root gaps.

Figure 10: Empirical cumulative distribution of the execution times and root gaps for Jabali et al. (2014) instances, with $\mathcal{X} = \mathcal{X}_{\text{SUB}}$ and $\mathcal{Q} = \mathcal{Q}_C$.



(a) Execution time and final optimality gaps.

(b) Root gaps.

Figure 11: Empirical cumulative distribution of the execution times and root gaps for Dinh et al. (2018) instances, with $\mathcal{X} = \mathcal{X}_{\text{SUB}}$ and $\mathcal{Q} = \mathcal{Q}^*$.