

A unified convergence theory for adaptive first-order methods in the nonconvex case, including AdaNorm, full and diagonal AdaGrad, Shampoo and Muon

S. Gratton* and Ph. L. Toint†

15 IV 2026

Abstract

A unified framework for first-order optimization algorithms for nonconvex unconstrained optimization is proposed that uses adaptively preconditioned gradients and includes popular methods such as full and diagonal AdaGrad, AdaNorm, as well as adaptive variants of Shampoo and Muon. This framework also allows combining heterogeneous geometries across different groups of variables while preserving a unified convergence analysis. A fully stochastic global rate-of-convergence analysis is conducted for all methods in the framework, with and without two types of momentum, using reasonable assumptions on the variance of the gradient oracle and without assuming bounded stochastic gradients or small enough stepsize.

Keywords: Unconstrained nonconvex optimization, first-order methods, global rate of convergence, Adam, AdaGrad, Shampoo, Muon.

1 Introduction

We consider the problem of finding a first-order critical point for the optimization problem

$$\min_X \mathbb{E}[f(X, \xi)] \tag{1.1}$$

where f is a smooth, possibly nonconvex function of X and ξ is a suitably defined random variable. Stochastic first-order methods for solving problem (1.1) have surged as a major research theme in recent years, essentially motivated by their very successful use in deep learning applications [9]. Because most of these methods do not evaluate the objective function at all¹, they are very robust in the presence of stochastic noise, a crucial feature in these applications. Starting with stochastic gradient descent [40], this domain has seen major advances in the last twenty years, with landmark proposals such as AdaNorm [42, 13, 48], AdaGrad [13, 36], Adam [27], and, more recently, Shampoo [22] and Muon [25]. The number of their variants (see [43, 37, 49, 48, 4, 45, 41] to cite only a few examples) has grown to such a number that a decent review is today a major undertaking (and beyond the scope of this paper). Their convergence analysis has followed the same trend, with significant contributions in particular in [55, 39, 49, 31, 12, 20, 16, 26, 18, 15, 33, 46, 23, 24, 54, 30, 32, 38].

A major theme in this domain has been preconditioning, as a cure to the known sensitivity of pure gradient methods to (even modest) problem ill-conditioning. The AdaGrad adaptive diagonal

*Université de Toulouse, INP, IRIT, Toulouse, France. Email: serge.gratton@enseeiht.fr. Work partially supported by 3IA Artificial and Natural Intelligence Toulouse Institute (ANITI), French "Investing for the Future - PIA3" program under the Grant agreement ANR-19-PI3A-0004"

†Université de Toulouse, INP, IRIT, Toulouse, France, and NAXYS, University of Namur, Namur, Belgium. Email: philippe.toint@unamur.be

¹They belong to the Objective-Function-Free Optimization (OFFO) class.

preconditioning strategy proposed in [42, 13] and used in Adam [27], has been at the source of many of the proposals mentioned above, first for unstructured methods (e.g. [49, 18, 1, 2]) and, in the past few years, for methods exploiting the tensor structure of deep-learning applications [45, 32, 41, 53]. The convergence of many of these methods has been investigated, but is often specific to the method considered. From our point of view, notable exceptions are [21] whose interesting interpretation of preconditioning in dual norms covers AdaGrad and some variants in the convex case, [12] covering both Adam and AdaGrad in the nonconvex case, [18] covering AdaGrad and divergent series variants in the nonconvex case, [51, 28] covering AdaGrad, AdaNorm and Shampoo in the convex case, and [38] where the idea of preconditioning in dual norm is applied to discuss (non-adaptive) variants of Muon in the nonconvex case.

As it turns out, results for methods for nonconvex problems involving momentum either conclude to non-convergence [44] or to convergence to a mere neighbourhood of a critical point [12], or assume either that gradients are uniformly bounded [39] or (somewhat unrealistically) that the user specifies a stepsize parameter that is sufficiently small compared to the inverse of the problem’s Lipschitz constant [20, 23, 50], or to the termination criterion [29, 54].

As far as the authors are aware, the convergence theory for all these proposals consider that the parameters to optimize are of a single type, either a vector of scalars, or a matrix, each having its own dedicated preconditioner. However, practical applications often mix those types (for instance considering together activation levels and biases, and weight matrices in neural networks). This practice is for instance recommended in [52, 17, 38], but we also note that Shampoo-type methods are typically applied at the level of individual matrix blocks, rather than globally across all parameters, due to their computational cost [3]. Structured second-order methods such as K-FAC [35] approximate curvature information using block-wise Kronecker factorizations, and these methods are typically applied only to selected layers, while simpler and less costly updates are used elsewhere. It is therefore interesting to consider the space of variables as the product space of blocks of parameters of possibly different types. In particular, this has the advantage of allowing a unified view of both vectors and matrix parameters.

Our paper attempts to exploit this observation to derive a general theory. Its main contributions are the following.

1. It provides the first (to the authors’ knowledge) truly unified convergence analysis of adaptive preconditioned gradient methods, covering, in the nonconvex case, full and diagonal AdaGrad, AdaNorm, Shampoo and Muon, with and without momentum, and without assuming boundedness of the gradients or a sufficiently small stepsize parameter.
2. It does so by considering a variable/parameter space structured as the product of blocks of potentially different types and by providing new simple proofs based on non-trivial trace inequalities and the theory of operator monotone functions.
3. To the best of our knowledge, the analysis of existing adaptive methods relies on a single global geometry. The present framework allows combining heterogeneous geometries across different blocks of a Cartesian product space, while preserving a unified complexity analysis and a globally consistent complexity. This is of special interest for adaptive methods because one may fear that the (possibly very) different adaptive stepsizes could generate strong distortions between the rates of convergence across blocks.

The paper is organized as follows. Section 2 introduces the structured parameter space and our general algorithmic framework. Its convergence properties are established in Section 3, while Section 4 details why this framework covers full and diagonal AdaGrad, AdaNorm, Shampoo and Muon. Some conclusions and perspectives are finally outlined in Section 5.

Notations: In what follows, $\|\cdot\|_E$ denotes the Euclidean norm and the symbols \succeq and \preceq respectively denote the “larger or equal” and “less or equal” relations in the Löwner semi-order on positive semidefinite matrices. If $\|\cdot\|$ is a norm on some space \mathcal{S} endowed with an inner product $\langle \cdot, \cdot \rangle$, its dual norm $\|\cdot\|_D$ is defined by $\|x\|_* = \max_{y \in \mathcal{S}} \langle y, x \rangle / \|y\|$.

2 Parameter space structure and algorithmic framework

In order to exploit the possible different types of parameters occurring in problem (1.1), we separate them into L disjoint blocks of uniform type (vectors or matrices), that is

$$X = (X_1, \dots, X_L), \quad X_\ell \in \mathbb{R}^{n_\ell \times m_\ell}.$$

Thus each block contains $d_\ell = n_\ell m_\ell$ parameters, and the total number of parameters (the dimension of our complete optimization space) is

$$N = \sum_{\ell=1}^L n_\ell m_\ell = \sum_{\ell=1}^L d_\ell, \quad (2.1)$$

with $d_\ell = n_\ell m_\ell$. We also define $d_{\max} = \max_{\ell \in \{1, \dots, L\}} d_\ell$. Each block ℓ is equipped with a norm $\|\cdot\|_\ell$ and its dual norm $\|\cdot\|_{*,\ell}$. The canonical pairing in the Cartesian product space $\mathbb{R}^N = \prod_{\ell=1}^L \mathbb{R}^{d_\ell}$ is then defined as

$$\langle U, V \rangle = \sum_{\ell=1}^L \langle U_\ell, V_\ell \rangle_F, \quad \text{where} \quad \langle A, B \rangle_F = \text{tr}(A^T B).$$

The (primal) product space is endowed with the norm

$$\|D\|^2 = \sum_{\ell=1}^L \|D_\ell\|_\ell^2,$$

whose dual norm is

$$\|Z\|_*^2 = \sum_{\ell=1}^L \|Z_\ell\|_{*,\ell}^2. \quad (2.2)$$

(Given a primal norm, the corresponding dual norm is the natural norm to measure gradients or their approximations. The use of dual norms for preconditioning has a long history (e.g. [10, Section 9.4]), and has emerged in the machine learning community following [22, 5] or [38] for instance.)

In this context, our objective is to find a first-order critical point for problem (1.1), that is an X_* such that

$$\mathbb{E}[\|\nabla_X^1 f(X_*, \xi)\|_*] = 0.$$

In order to define our algorithmic framework to achieve this goal, we choose, for each block ℓ , a measurable selector $S_\ell(\cdot)$ such that, for any D ,

$$\|S_\ell(D)\|_\ell = \begin{cases} 1 & \text{if } D \neq 0 \\ 0 & \text{if } D = 0. \end{cases} \quad (2.3)$$

Equivalently,

$$S_\ell(D) \in \arg \max_{\|V\|_\ell \leq 1} \langle U, V \rangle_F.$$

(A similar definition was used in [22] for the single-block convex case.) For notational simplicity, the fact that $S_\ell(0) = 0$ is implicitly assumed in what follows. We also define an iteration-dependent linear operator $\mathcal{L}_{k,\ell}(D) : \mathbb{R}^{n_\ell \times m_\ell} \rightarrow \mathbb{R}^{n_\ell \times m_\ell}$ by

$$\mathcal{L}_{k,\ell}(D)^2 \stackrel{\text{def}}{=} \mathcal{L}_{k,\ell}(D)^* \mathcal{L}_{k,\ell}(D). \quad (2.4)$$

Since computing $G_k = \mathbb{E}[\nabla_X^1 f(X_*, \xi)]$ is in general either impossible or too expensive, we consider an iterative stochastic approach in which, at iterate X_k , G_k is approximated by an oracle \tilde{G}_k depending on a random variable ξ_k , where ξ_k (whose distribution may depend on X_k) is defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (\mathbb{E} denotes the corresponding expectation). The expectation conditioned to knowing $(\xi_0, \dots, \xi_{k-1})$ is denoted by the symbol $\mathbb{E}_k[\cdot]$.

Our simple but general algorithmic ADPREC framework is described on the following page.

Algorithm 2.1: ADPREC

Given: a starting point X_0 and constants $\eta, \varsigma > 0$. Set $\Gamma_{-1,\ell} = \varsigma I_\ell$ for $\ell \in \{1, \dots, L\}$.

For $k = 0, 1, \dots$

1. Draw ξ_k .
2. For $\ell \in \{1, \dots, L\}$, compute

$$\tilde{G}_{k,\ell} \approx G_{k,\ell} = \nabla_{X_\ell}^1 \mathbb{E}[f(X_k), \xi] \quad (2.5)$$

$$\Gamma_{k,\ell} = \Gamma_{k-1,\ell} + \mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2, \quad (2.6)$$

$$Z_{k,\ell} = \Gamma_{k,\ell}^{-1/2} \tilde{G}_{k,\ell}, \quad (2.7)$$

$$X_{k+1,\ell} = X_{k,\ell} - \eta \|Z_{k,\ell}\|_{*,\ell} S_\ell(Z_{k,\ell}). \quad (2.8)$$

This algorithm thus defines a random process $(\tilde{G}_k, Z_k, \Gamma_k, X_k)$, where $\tilde{G}_k = (\tilde{G}_{k,1}, \dots, \tilde{G}_{k,L})$, $Z_k = (Z_{k,1}, \dots, Z_{k,L})$ and $\Gamma_k = (\Gamma_{k,1}, \dots, \Gamma_{k,L})$. Immediately note that the choice $\tilde{G}_{k,\ell} = \nabla_{X_{k,\ell}}^1 f(X_k, \xi_k)$ is possible, but, as we will see later, other choices are possible. At first sight, it may seem that the algorithm performs separate optimization on each block, but this view is deceptive because interaction between the blocks occurs in the computation of the approximate gradient. The name ADPREC alludes to the fact that $\Gamma_{k,\ell}$ may be interpreted as an adaptive preconditioner for block ℓ and (2.7) therefore produces an adaptively preconditioned (approximate) gradient $Z_{k,\ell}$.

Note that (2.4) allows changing the update formula in (2.6) for $\Gamma_{k,\ell}$ at each iteration although, as far as the authors know, this is not done in practice.

3 Convergence analysis

The convergence analysis for ADPREC hinges on the following standard assumptions.

Assumption 1 (Boundedness) *There exists a constant f_{low} such that $\mathbb{E}[f(X, \xi)] \geq f_{\text{low}}$ for all X .*

Assumption 2 (Smoothness) *The objective function f is continuously differentiable and has a Lipschitz continuous gradient, that is there exists a constant $L_G \geq 0$ such that, for all $X, Y \in \mathbb{R}^N$, $\|G(X) - G(Y)\|_* \leq L_G \|X - Y\|$.*

We now introduce conditions that define the class of instances of ADPREC for which we develop our theory. Although seemingly abstract, we will demonstrate in Section 4 that these conditions do hold for several popular first-order methods.

Assumption 3 (Structural identities) *For all $k \geq 0$ and all $\ell \in \{1, \dots, L\}$, we have that*

$$\|Z_{k,\ell}\|_{*,\ell} \left\langle \tilde{G}_{k,\ell}, S_\ell(Z_{k,\ell}) \right\rangle_F = \text{tr}(\Gamma_{k,\ell}^{-1/2} \mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2), \quad (3.1)$$

and

$$\|Z_{k,\ell}\|_{*,\ell}^2 = \text{tr}(\Gamma_{k,\ell}^{-1} \mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2). \quad (3.2)$$

As will become clear in Lemma 3.1 just below, these identities reformulate the terms associated with first-order descent (for (3.1)) and second-order perturbations (for (3.2)) in terms of traces. Note that (3.1) and (3.2) are “iteration-specific” conditions, in that they only involve $\Gamma_{k,\ell}$ and are independent of the value of $\Gamma_{k-1,\ell}$.

Assumption 4 (Gradient-preconditioner compatibility) *There exists a constant $\kappa_\circ > 0$ such that, for each block $\ell \in \{1, \dots, L\}$, each $k \geq 0$ and all G_ℓ ,*

$$\|\tilde{G}_\ell\|_{*,\ell}^2 \leq \kappa_\circ \operatorname{tr}(\mathcal{L}_{k,\ell}(\tilde{G}_\ell)^2). \quad (3.3)$$

Because $\mathcal{L}_{k,\ell}(\cdot)$ is used to build the preconditioners $\Gamma_{k,\ell}$, this last assumption establishes the link between them and the measures of (approximate) optimality $\tilde{G}_{k,\ell}$. (Note that we could choose κ_\circ depending on ℓ , but we use a single constant for simplicity.)

We of course need to be more specific on what we assume on the oracle \tilde{G}_k , but postpone the precise nature of our requirements to the statements of the results where they are needed.

Our first step is to analyze the (expected) descent at iteration k .

Lemma 3.1 Suppose that Assumption 2 holds. Then

$$\begin{aligned} \mathbb{E}_k[f(X_{k+1})] &\leq f(X_k) - \eta \sum_{\ell=1}^L \mathbb{E}_k \left[\|Z_{k,\ell}\|_{*,\ell} \left\langle \tilde{G}_{k,\ell}, S_\ell(Z_{k,\ell}) \right\rangle_F \right] \\ &\quad + \eta \sum_{\ell=1}^L \mathbb{E}_k \left[\|Z_{k,\ell}\|_{*,\ell} \left\langle \tilde{G}_{k,\ell} - G_{k,\ell}, S_\ell(Z_{k,\ell}) \right\rangle_F \right] + \frac{LG\eta^2}{2} \sum_{\ell=1}^L \mathbb{E}_k [\|Z_{k,\ell}\|_{*,\ell}^2]. \end{aligned} \quad (3.4)$$

Proof. By Lipschitz continuity of the gradient (Assumption 2),

$$f(X_{k+1}) \leq f(X_k) + \langle G_k, X_{k+1} - X_k \rangle + \frac{LG}{2} \|X_{k+1} - X_k\|^2.$$

Thus, using (2.7) and (2.8), we obtain

$$f(X_{k+1}) \leq f(X_k) - \eta \sum_{\ell=1}^L \|Z_{k,\ell}\|_{*,\ell} \langle G_{k,\ell}, S_\ell(Z_{k,\ell}) \rangle_F + \frac{LG\eta^2}{2} \sum_{\ell=1}^L \|Z_{k,\ell}\|_{*,\ell}^2 \|S_\ell(Z_{k,\ell})\|_\ell^2.$$

Taking conditional expectations, inserting $\tilde{G}_{k,\ell}$ and using (2.3) yields (3.4). \square

3.1 Trace inequalities

Our argument now takes a little linear algebra detour for proving two useful inequalities about the trace of matrices of interest. We first derive an inequality involving the trace of the inverse square root of the sum of two matrices.

Theorem 3.2 Let A and B be symmetric positive semidefinite matrices. Then

$$\operatorname{tr}((A+B)^{-1/2}B) \geq \operatorname{tr}((A+B)^{1/2}) - \operatorname{tr}(A^{1/2}). \quad (3.5)$$

Proof. Set $X = (A+B)^{1/2}$ and $Y = A^{1/2}$. Clearly, $X \succ 0$, $Y \succ 0$, and $X+Y \succ 0$ and is therefore invertible. Moreover

$$B = X^2 - Y^2 = X(X-Y) + (X-Y)Y. \quad (3.6)$$

Multiplying this inequality on the right by $(X+Y)^{-1}$ gives

$$B(X+Y)^{-1} = X(X-Y)(X+Y)^{-1} + (X-Y)Y(X+Y)^{-1}.$$

Taking the trace and using cyclicity, we obtain that

$$\begin{aligned} \operatorname{tr}(B(X+Y)^{-1}) &= \operatorname{tr}((X+Y)^{-1}X(X-Y)) + \operatorname{tr}((X+Y)^{-1}(X-Y)Y) \\ &= \operatorname{tr}\left((X+Y)^{-1}(X(X-Y) + (X-Y)Y)\right) \\ &= \operatorname{tr}\left((X+Y)^{-1}((X-Y)(X+Y) - [X, Y])\right) \end{aligned}$$

where $[X, Y] = XY - YX$ is the commutator between X and Y . Since X and Y are Hermitian, $[X, Y]$ is skew-Hermitian and its trace is zero. Hence, using cyclicity (twice),

$$\begin{aligned} \operatorname{tr}((X+Y)^{-1}B) &= \operatorname{tr}(B(X+Y)^{-1}) = \operatorname{tr}((X+Y)^{-1}(X-Y)(X+Y)) - \operatorname{tr}((X+Y)^{-1}[X, Y]) \\ &= \operatorname{tr}((X+Y)^{-1}(X-Y)(X+Y)) \\ &= \operatorname{tr}(X-Y) \end{aligned}$$

Now, since $Y \succeq 0$, we have that $X+Y \succeq X$. Since both X and $X+Y$ are positive definite and since inversion reverses the Löwner semi-order, we deduce that

$$(X+Y)^{-1} \preceq X^{-1} = (A+B)^{-1/2}.$$

and thus $(A+B)^{-1/2} - (X+Y)^{-1} \succeq 0$. But, since B is symmetric positive definite, we deduce, again using cyclicity, that

$$\operatorname{tr}\left(\left((A+B)^{-1/2} - (X+Y)^{-1}\right)B\right) = \operatorname{tr}\left(B^{1/2}\left((A+B)^{-1/2} - (X+Y)^{-1}\right)B^{1/2}\right) \succeq 0,$$

and therefore that

$$\operatorname{tr}\left((A+B)^{1/2}\right) - \operatorname{tr}\left(A^{1/2}\right) = \operatorname{tr}(X-Y) = \operatorname{tr}\left((X+Y)^{-1}B\right) \leq \operatorname{tr}\left((A+B)^{-1/2}B\right),$$

which is (3.5). \square

The next step of our argument uses the Löwner semi-order \succeq on symmetric positive-semidefinite matrices and the theory of operator monotone functions (see [8, Chapter V], for instance).

Theorem 3.3 Let A and B be symmetric positive semidefinite matrices and let $\phi : (0, \infty) \rightarrow \mathbb{R}$ be a C^1 function. Suppose that ϕ' is operator monotone decreasing on $(0, \infty)$, meaning that for any positive definite matrices X, Y , $X \preceq Y$ implies that $\phi'(X) \succeq \phi'(Y)$. Then

$$\operatorname{tr}(\phi'(A+B)B) \leq \operatorname{tr}(\phi(A+B) - \phi(A)) \leq \operatorname{tr}(\phi'(A)B). \quad (3.7)$$

Proof. Define $g(t) \stackrel{\text{def}}{=} \operatorname{tr}(\phi(A+tB))$ for $t \in [0, 1]$. Using the standard derivative formula for spectral functions under the trace, we see that

$$g'(t) = \operatorname{tr}(\phi'(A+tB)B). \quad (3.8)$$

Since $B \succeq 0$, we also have that

$$A \preceq A+tB \preceq A+B \quad (t \in [0, 1]). \quad (3.9)$$

and hence, because ϕ' is operator monotone decreasing, that

$$\phi'(A+B) \preceq \phi'(A+tB) \preceq \phi'(A).$$

Now if $M \preceq N$ and $B \succeq 0$, then $\operatorname{tr}((N-M)B) = \operatorname{tr}(B^{1/2}(N-M)B^{1/2}) \geq 0$, and thus $\operatorname{tr}(MB) \leq \operatorname{tr}(NB)$. Using this inequality in (3.9) and the identity (3.8), we therefore obtain that

$$\operatorname{tr}(\phi'(A+B)B) \leq g'(t) \leq \operatorname{tr}(\phi'(A)B).$$

Integrating over $t \in [0, 1]$ then yields (3.7). \square

We now use the above and a classical result by K. Löwner to deduce two helpful inequalities on the trace of matrices generated by the ADPREC algorithm.

Lemma 3.4 We have that, for all $k \geq 0$,

$$\sum_{\ell=1}^L \operatorname{tr}(\Gamma_{k,\ell}^{1/2}) - \sum_{\ell=1}^L \operatorname{tr}(\Gamma_{-1,\ell}^{1/2}) \leq \sum_{j=0}^k \sum_{\ell=1}^L \operatorname{tr}(\Gamma_{j,\ell}^{-1/2} \mathcal{L}_{k,\ell}(\tilde{G}_{j,\ell})^2) \quad (3.10)$$

and

$$\sum_{j=0}^k \sum_{\ell=1}^L \operatorname{tr}(\Gamma_{j,\ell}^{-1} \mathcal{L}_{k,\ell}(\tilde{G}_{j,\ell})^2) \leq \sum_{\ell=1}^L \operatorname{tr}(\log(\Gamma_{k,\ell})) - \sum_{\ell=1}^L \operatorname{tr}(\log(\Gamma_{-1,\ell})). \quad (3.11)$$

Proof. Let

$$A = \Gamma_{j-1,\ell} \quad \text{and} \quad B = \mathcal{L}_{j,\ell}(\tilde{G}_{j,\ell})^2. \quad (3.12)$$

Then $\Gamma_{j,\ell} = A + B$ and Theorem 3.2 gives that, for $j \in \{0, \dots, k\}$ and $\ell \in \{1, \dots, L\}$,

$$\operatorname{tr}(\Gamma_{j,\ell}^{1/2}) - \operatorname{tr}(\Gamma_{j-1,\ell}^{1/2}) \leq \operatorname{tr}(\Gamma_{j,\ell}^{-1/2} \mathcal{L}_{j,\ell}(\tilde{G}_{j,\ell})^2).$$

Summing this inequality for $j \in \{0, \dots, k\}$ and $\ell \in \{1, \dots, L\}$ yields (3.10). In order to prove (3.11), let $\phi(t) = \log t$. Then $\phi'(t) = \frac{1}{t}$, which is operator monotone decreasing (see [34] as cited in [8, p. 149]). Applying the first inequality of (3.7) in Theorem 3.3 then yields that

$$\operatorname{tr}((A + B)^{-1}B) \leq \operatorname{tr}(\log(A + B) - \log A),$$

which, with (3.12), ensures that, for $j \in \{0, \dots, k\}$ and $\ell \in \{1, \dots, L\}$,

$$\operatorname{tr}(\Gamma_{j,\ell}^{-1} \mathcal{L}_{j,\ell}(\tilde{G}_{j,\ell})^2) \leq \operatorname{tr}(\log(\Gamma_{j,\ell})) - \operatorname{tr}(\log(\Gamma_{j-1,\ell})).$$

Summing this inequality for $j \in \{0, \dots, k\}$ and $\ell \in \{1, \dots, L\}$ now yields (3.11). \square

3.2 Telescoping inequality and global rate of convergence

We next introduce a variant of the classical telescoping argument, leading to the central inequality in our analysis.

Theorem 3.5 Suppose that Assumption 1 and 3 hold. Suppose also that, for some $\omega \geq 0$ and $\nu_k \geq 0$

$$\sum_{j=0}^k \mathbb{E} \left[\|\tilde{G}_j - G_j\|_*^2 \right] \leq \nu_k^2 + \omega^2 \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E} \left[\|Z_{j,\ell}\|_{*,\ell}^2 \right]. \quad (3.13)$$

Define

$$\Delta_k \stackrel{\text{def}}{=} \mathbb{E} \left[\sum_{\ell=1}^L \text{tr}(\log \Gamma_{k,\ell}) - \sum_{\ell=1}^L \text{tr}(\log \Gamma_{-1,\ell}) \right]. \quad (3.14)$$

Then, for every $k \geq 0$,

$$\eta \mathbb{E} \left[\sum_{\ell=1}^L \text{tr}(\Gamma_{k,\ell}^{1/2}) \right] \leq \kappa_{\text{gap}} + \eta \nu_k \sqrt{\Delta_k} + \left(\eta \omega^2 + \frac{L_G \eta^2}{2} \right) \Delta_k, \quad (3.15)$$

where

$$\kappa_{\text{gap}} \stackrel{\text{def}}{=} \mathbb{E}[f(X_0)] - f_{\text{low}} + \eta \varsigma \sum_{\ell=1}^L d_\ell. \quad (3.16)$$

Proof. Taking the full expectation in the conditional descent inequality (3.4) and summing for $j = 0$ to k gives

$$\begin{aligned} \eta \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E} \left[\|Z_{j,\ell}\|_{*,\ell} \left\langle \tilde{G}_{j,\ell}, S_\ell(Z_{j,\ell}) \right\rangle_F \right] &\leq \mathbb{E}[f(X_0)] - f_{\text{low}} \\ &+ \eta \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E} \left[\|Z_{j,\ell}\|_{*,\ell} \left| \left\langle \tilde{G}_{j,\ell} - G_{j,\ell}, S_\ell(Z_{j,\ell}) \right\rangle_F \right| \right] + \frac{L_G \eta^2}{2} \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E} \left[\|Z_{j,\ell}\|_{*,\ell}^2 \|S_\ell(Z_{j,\ell})\|_\ell^2 \right]. \end{aligned} \quad (3.17)$$

Using successively (3.1) and (3.10), we obtain that

$$\begin{aligned} \eta \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E} \left[\|Z_{j,\ell}\|_{*,\ell} \left\langle \tilde{G}_{j,\ell}, S_\ell(Z_{j,\ell}) \right\rangle_F \right] &= \eta \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E} \left[\text{tr}(\Gamma_{j,\ell}^{-1/2} \mathcal{L}_{j,\ell}(\tilde{G}_{j,\ell})^2) \right] \\ &\geq \eta \mathbb{E} \left[\sum_{\ell=1}^L \text{tr}(\Gamma_{k,\ell}^{1/2}) - \sum_{\ell=1}^L \text{tr}(\Gamma_{-1,\ell}^{1/2}) \right]. \end{aligned} \quad (3.18)$$

Now observe that the Cauchy-Schwarz inequality and (2.3) give that

$$\left| \left\langle \tilde{G}_{j,\ell} - G_{j,\ell}, S_\ell(z_{j,\ell}) \right\rangle_F \right| \leq \|\tilde{G}_{j,\ell} - G_{j,\ell}\|_{*,\ell} \|S_\ell(Z_{j,\ell})\|_\ell \leq \|\tilde{G}_{j,\ell} - G_{j,\ell}\|_{*,\ell}.$$

Therefore

$$\mathbb{E} \left[\|Z_{j,\ell}\|_{*,\ell} \left| \left\langle \tilde{G}_{j,\ell} - G_{j,\ell}, S_\ell(Z_{j,\ell}) \right\rangle_F \right| \right] \leq \sqrt{\mathbb{E} \left[\|\tilde{G}_{j,\ell} - G_{j,\ell}\|_{*,\ell}^2 \right]} \sqrt{\mathbb{E} \left[\|Z_{j,\ell}\|_{*,\ell}^2 \right]}.$$

Observe now that the Cauchy-Schwartz inequality also implies that, for any vectors a and b with nonnegative components,

$$\sum_j \sqrt{a_j} \sqrt{b_j} \leq \|\sqrt{a}\|_E \|\sqrt{b}\|_E = \sqrt{\sum_j a_j} \sqrt{\sum_j b_j}. \quad (3.19)$$

Hence, summing over ℓ and using (2.2), we obtain that

$$\sum_{\ell=1}^L \mathbb{E} \left[\|Z_{j,\ell}\|_{*,\ell} \left| \left\langle \tilde{G}_{j,\ell} - G_{j,\ell}, S_\ell(Z_{j,\ell}) \right\rangle_F \right| \right] \leq \sqrt{\mathbb{E} \left[\|\tilde{G}_j - G_j\|_*^2 \right]} \sqrt{\sum_{\ell=1}^L \mathbb{E} \left[\|Z_{j,\ell}\|_{*,\ell}^2 \right]}.$$

Summing now over j and using (3.19) again, we deduce that

$$\begin{aligned} & \eta \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E} \left[\|Z_{j,\ell}\|_{*,\ell} \left| \left\langle \tilde{G}_{j,\ell} - G_{j,\ell}, S_\ell(z_{j,\ell}) \right\rangle_F \right| \right] \\ & \leq \eta \sqrt{\sum_{j=0}^k \mathbb{E} \left[\|\tilde{G}_j - G_j\|_*^2 \right]} \sqrt{\sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E} \left[\|Z_{j,\ell}\|_{*,\ell}^2 \right]} \\ & \leq \eta \nu_k \sqrt{\sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E} \left[\|Z_{j,\ell}\|_{*,\ell}^2 \right]} + \eta \omega^2 \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E} \left[\|Z_{j,\ell}\|_{*,\ell}^2 \right] \end{aligned}$$

where we used (3.13) to obtain the last inequality. But, by (3.2), (3.11) and (3.14),

$$\sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E} \left[\|Z_{j,\ell}\|_{*,\ell}^2 \right] = \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E} \left[\text{tr}(\Gamma_{j,\ell}^{-1} \mathcal{L}_{j,\ell}(\tilde{G}_{j,\ell})^2) \right] \leq \Delta_k, \quad (3.20)$$

so that

$$\eta \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E} \left[\|Z_{j,\ell}\|_{*,\ell} \left| \left\langle \tilde{G}_{j,\ell} - G_{j,\ell}, S_\ell(Z_{j,\ell}) \right\rangle_F \right| \right] \leq \eta \nu_k \sqrt{\Delta_k} + \eta \omega^2 \Delta_k. \quad (3.21)$$

Similarly, using (2.3) and (3.20), the quadratic term satisfies

$$\frac{L_G \eta^2}{2} \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E} \left[\|Z_{j,\ell}\|_{*,\ell} \|S_\ell(Z_{j,\ell})\|_\ell^2 \right] = \frac{L_G \eta^2}{2} \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E} \left[\|Z_{j,\ell}\|_{*,\ell}^2 \right] \leq \frac{L_G \eta^2}{2} \Delta_k. \quad (3.22)$$

Substituting (3.18), (3.21) and (3.22) into (3.17) then yields that

$$\eta \mathbb{E} \left[\sum_{\ell=1}^L \text{tr}(\Gamma_{k,\ell}^{1/2}) - \sum_{\ell=1}^L \text{tr}(\Gamma_{-1,\ell}^{1/2}) \right] \leq f(X_0) - f_{\text{low}} + \eta \nu_k \sqrt{\Delta_k} + \left(\eta \omega^2 + \frac{L_G \eta^2}{2} \right) \Delta_k, \quad (3.23)$$

which is exactly (3.15)-(3.16) after taking into account that $\Gamma_{-1,\ell} = \varsigma I_\ell$ for all $\ell \in \{1, \dots, L\}$.

□

We now start exploiting this result by deriving a more specific bound on the value of Δ_k in (3.14). This requires the following simple technical lemma.

Lemma 3.6 Let $\Gamma \in \mathbb{R}^{d \times d}$ be symmetric positive definite. Then

$$\text{tr}(\log \Gamma) \leq 2d \log(\text{tr}(\Gamma^{1/2})) - d \log(d).$$

Proof. Let $\lambda_1, \dots, \lambda_d > 0$ be the eigenvalues of Γ . Then the concavity of the logarithm ensures that

$$\text{tr}(\log \Gamma) = \sum_{i=1}^d \log(\lambda_i) \leq d \log \left(\frac{1}{d} \sum_{i=1}^d \lambda_i \right) \leq d \log \left(\frac{\left(\sum_{i=1}^d \sqrt{\lambda_i} \right)^2}{d} \right) = 2d \log(\text{tr}(\Gamma^{1/2})) - d \log(d).$$

□

We now derive the desired bound on Δ_k in (3.14).

Lemma 3.7 We have that, for all $k \geq 0$,

$$\Delta_k \leq \kappa_0 + 2N \log \left(\mathbb{E} \left[\sum_{\ell=1}^L \text{tr}(\Gamma_{k,\ell}^{1/2}) \right] \right), \quad (3.24)$$

where

$$\kappa_0 = - \sum_{\ell=1}^L d_\ell \log(d_\ell) - \log(\varsigma)N. \quad (3.25)$$

Proof. Applying Lemma 3.6 to each block gives

$$\sum_{\ell=1}^L \text{tr}(\log \Gamma_{k,\ell}) \leq 2 \sum_{\ell=1}^L d_\ell \log(\text{tr}(\Gamma_{k,\ell}^{1/2})) - \sum_{\ell=1}^L d_\ell \log d_\ell. \quad (3.26)$$

Since $\text{tr}(\Gamma_{k,\ell}^{1/2}) \leq \sum_{r=1}^L \text{tr}(\Gamma_{k,r}^{1/2})$, we have that $\log(\text{tr}(\Gamma_{k,\ell}^{1/2})) \leq \log\left(\sum_{r=1}^L \text{tr}(\Gamma_{k,r}^{1/2})\right)$, and hence, from (3.26),

$$\sum_{\ell=1}^L \text{tr}(\log \Gamma_{k,\ell}) \leq 2 \sum_{\ell=1}^L d_\ell \log \left(\sum_{r=1}^L \text{tr}(\Gamma_{k,r}^{1/2}) \right) - \sum_{\ell=1}^L d_\ell \log d_\ell.$$

We therefore obtain from (3.14) that

$$\Delta_k \leq - \sum_{\ell=1}^L d_\ell \log d_\ell - \sum_{\ell=1}^L \text{tr}(\log \Gamma_{-1,\ell}) + 2 \sum_{\ell=1}^L d_\ell \mathbb{E} \left[\log \left(\sum_{\ell=1}^L \text{tr}(\Gamma_{k,\ell}^{1/2}) \right) \right]. \quad (3.27)$$

But Jensen's inequality for the concave $\log(\cdot)$ function gives that

$$\mathbb{E} \left[\log \left(\sum_{\ell=1}^L \text{tr}(\Gamma_{k,\ell}^{1/2}) \right) \right] \leq \log \left(\mathbb{E} \left[\sum_{\ell=1}^L \text{tr}(\Gamma_{k,\ell}^{1/2}) \right] \right).$$

Inserting this inequality in (3.27) and using the definition $\Gamma_{-1,\ell} = \varsigma I_\ell$ gives (3.24)-(3.25). □

We still need a simple technical result before concluding.

Lemma 3.8 Suppose that $1 \leq t \leq c \log(t)$ for $c \geq 0$ and $t \geq 1$. Then $t \leq 2c \log(2c)$.

Proof. Lemma 3.2 in [6] with $a = 1$ and $b = 0$ gives that $t \leq 2c(\log(2c) - 1)$. □

We now state a first important result on the expected size of the preconditioners $\Gamma_{k,\ell}$.

Theorem 3.9 Suppose that the ADPREC algorithm is applied to problem (1.1). Suppose that (3.13) and Assumptions 1, 2 and 3 hold. Then, for all $k \geq 0$,

$$\mathbb{E} \left[\sum_{\ell=1}^L \text{tr}(\Gamma_{k,\ell}^{1/2}) \right] \leq \Theta_k \stackrel{\text{def}}{=} \max \left[e^{\max[1, \frac{1}{2N}, \frac{\kappa_0}{2N}]}, \frac{3\kappa_{\text{gap}}}{\eta}, T_k, Y_k \right], \quad (3.28)$$

with κ_{gap} defined in (3.16),

$$T_k = 12\sqrt{N} \nu_k \sqrt{\max \left[1, \log \left(12\sqrt{N} \nu_k \right) \right]}, \quad Y_k = 24N \left(\omega^2 + \frac{L_G}{\eta} \right) \log \left(24N \left(\omega^2 + \frac{L_G}{\eta} \right) \right)$$

ν_k being defined in (3.13) and κ_0 in (3.25).

Proof. Define

$$t_k = \mathbb{E} \left[\sum_{\ell=1}^L \text{tr}(\Gamma_{k,\ell}^{1/2}) \right]. \quad (3.29)$$

Substituting (3.24) into (3.15) and using this definition then gives that

$$\eta t_k \leq \kappa_{\text{gap}} + \eta \nu_k \sqrt{\kappa_0 + 2N \log(t_k)} + \left(\eta \omega^2 + \frac{L_G \eta^2}{2} \right) (\kappa_0 + 2N \log(t_k)). \quad (3.30)$$

Suppose first that

$$\log(t_k) < \max \left[\frac{1}{2N}, \frac{\kappa_0}{2N} \right].$$

Then

$$t_k \leq e^{\max[\frac{1}{2N}, \frac{\kappa_0}{2N}]}. \quad (3.31)$$

Alternatively, suppose now that

$$2N \log(t_k) \geq \max[1, \kappa_0]. \quad (3.32)$$

Then

$$\kappa_0 + 2N \log(t_k) \leq 4N \log(t_k)$$

and (3.30) can then be rewritten as

$$t_k \leq a + b \nu_k \sqrt{\log(t_k)} + c \log(t_k) \quad (3.33)$$

with

$$a = \frac{\kappa_{\text{gap}}}{\eta}, \quad b = 2\sqrt{N} \quad \text{and} \quad c = 4N \left(\omega^2 + \frac{L_G}{\eta} \right). \quad (3.34)$$

This formulation is analyzed by distinguishing three cases.

- The first case is when $\max[a, b\nu_k \sqrt{\log(t_k)}, c \log(t_k)] = a$. Then, clearly,

$$t_k \leq 3a. \quad (3.35)$$

- The second case is when

$$\max[a, b\nu_k \sqrt{\log(t_k)}, c \log(t_k)] = b\nu_k \sqrt{\log(t_k)}. \quad (3.36)$$

If $\log(t_k) \leq 1$, then $t_k \leq e$. Alternatively, if $\log(t_k) > 1$ (implying that $t_k > 1$), we deduce that $t_k^2 \leq 9b^2 \nu_k^2 \log(t_k)$, that is

$$h(t_k) \stackrel{\text{def}}{=} \frac{t_k^2}{\tau_k^2} - \log(t_k) \leq 0 \quad \text{with} \quad \tau_k = 3b\nu_k. \quad (3.37)$$

Let $T_k = 2\tau_k \sqrt{\max[1, \log(2\tau_k)]} \geq 2\tau_k$. Then

$$h(T_k) = 4 \max[1, \log(2\tau_k)] - \log(2\tau_k) - \frac{1}{2} \log(\max[1, \log(2\tau_k)]) > 0. \quad (3.38)$$

Moreover, $h'(t) = \frac{2t}{t^2} - \frac{1}{t}$, and therefore, for all $t \geq 2\tau_k$, $h'(t) \geq \frac{4t}{t^2} - \frac{1}{2t} = \frac{4}{t} - \frac{1}{2t} > 0$. As a consequence, $h(t)$ is increasing for $t \geq 2\tau_k$ and thus $h(t) > 0$ for $t \geq T_k$ because of (3.38). We may then deduce from (3.37) that $t_k < T_k$. Thus,

$$t_k \leq \max[e, T_k]. \quad (3.39)$$

• Finally, the third case is when $\max[a, b\nu_k \sqrt{\log(t_k)}, c \log(t_k)] = c \log(t_k)$. Then $t_k \leq 3c \log(t_k)$ and thus, using Lemma 3.8,

$$t_k \leq \max[1, 6c \log(6c)]. \quad (3.40)$$

Combining (3.29), (3.35), (3.39) and (3.40) and including (3.31) then gives (3.28). \square

This result finally gives us all the ingredients to derive convergence and complexity results on the Cartesian product space. The bound (3.28), together with Assumption 4, indeed implies convergence of the criticality measure on each block with a *uniform rate-of-convergence bound*.

Theorem 3.10 Suppose that the ADPREC algorithm is applied to problem (1.1). Suppose that (3.13) and Assumptions 1 to 4 hold. Then

$$\sum_{j=0}^k \mathbb{E} \left[\|\tilde{G}_j\|_* \right] \leq \sqrt{k+1} \kappa_o \Theta_k \quad (3.41)$$

where Θ_k is defined in Theorem 3.9. Moreover, if \tilde{G}_k is an unbiased estimator of G_k , i.e. $\mathbb{E}_k \left[\tilde{G}_k \right] = G_k$ for all $k \geq 0$, then

$$\min_{j \in \{0, \dots, k\}} \mathbb{E} \left[\|G_j\|_* \right] \leq \frac{1}{k+1} \sum_{j=0}^k \mathbb{E} \left[\|G_j\|_* \right] \leq \frac{\kappa_o \Theta_k}{\sqrt{k+1}}. \quad (3.42)$$

Proof. Assumption 4 and Theorem 3.9 directly imply that

$$\sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E} \left[\|\tilde{G}_{j,\ell}\|_{*,\ell} \right] \leq \sqrt{k+1} \kappa_o \sum_{j=0}^k \mathbb{E} \left[\sum_{\ell=1}^L \text{tr}(\Gamma_{k,\ell}^{1/2}) \right] \leq \sqrt{k+1} \kappa_o \Theta_k.$$

Dividing by $k+1$ and using $\mathbb{E}_k \left[\tilde{G}_k \right] = G_k$ together with Jensen's inequality gives (3.42). \square

Observe that the requirement (3.13) defines an upper bound on ν_k , the *cumulative* total variance of the gradient oracle over all past iterates, an approach more general than assuming conditional variance at every iteration. In particular, it allows large variance at early iterations provided later iterations compensate. Trade-offs between different realizations are also theoretically possible. Also note that, since we prove that Γ_k grows very slowly (see (3.28)), and Z_k is the preconditioned approximate gradient, condition (3.13) is akin to a cumulative affine variance condition of the form

$$\sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E}_j \left[\|\tilde{G}_{j,\ell} - G_{j,\ell}\|^2 \right] \leq \nu_k^2 + \omega^2 \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E}_j \left[\|G_{j,\ell}\|_{*,\ell}^2 \right],$$

whose iteration-wise variant has already been used in analysis of first-order methods (see [16, 46] or, for the stronger “affine- $*$ ” version, [4]). In particular, the “strong growth” assumption

motivated by over-parametrized problems and used in [46] to derive an improved convergence rate for AdaGrad is essentially subsumed (at the cumulative level) if “preconditioned cumulative strong growth” (that is $\nu_k = 0$ is assumed in (3.13)).

We now investigate what can be said if one makes a more specific assumption on conditional variance at each iteration.

Corollary 3.11 Suppose that the ADPREC algorithm is applied to problem (1.1). Suppose that Assumptions 1 to 4 hold. Suppose also that

$$\mathbb{E}_k[\tilde{G}_k] = G_k \quad \text{and} \quad \mathbb{E}_k[\|\tilde{G}_{k,\ell} - G_{k,\ell}\|_*^2] \leq \frac{\sigma_\ell^2}{(k+1)^\alpha} + \omega^2 \mathbb{E}_k[\|Z_{k,\ell}\|_*^2] \quad (3.43)$$

for some $\alpha, \omega > 0$ and all $k \geq 0$ and $\ell \in \{1, \dots, L\}$. Then

$$\frac{1}{k+1} \sum_{j=0}^k \mathbb{E}[\|G_j\|_*] = \begin{cases} \mathcal{O}\left(\frac{\sqrt{\log(k+1)}}{(k+1)^{\frac{\alpha}{2}}}\right) & \text{if } \alpha < 1, \\ \mathcal{O}\left(\frac{\sqrt{\log(k+1) \log(\log(k+1))}}{\sqrt{k+1}}\right) & \text{if } \alpha = 1, \\ \mathcal{O}\left(\frac{1}{\sqrt{k+1}}\right) & \text{if } \alpha > 1. \end{cases}$$

Proof. Define $\sigma_{\text{tot}}^2 = \sum_{\ell=1}^L \sigma_\ell^2$. Suppose first that $\alpha < 1$. We have from (3.43) that

$$\begin{aligned} \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E}_j[\|\tilde{G}_{j,\ell} - G_{j,\ell}\|_*^2] &\leq \sigma_{\text{tot}}^2 \sum_{j=0}^k \frac{1}{(j+1)^\alpha} + \omega^2 \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E}_j[\|Z_{j,\ell}\|_*^2] \\ &\leq \frac{\sigma_{\text{tot}}^2}{1-\alpha} (k+1)^{1-\alpha} + \omega^2 \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E}_j[\|Z_{j,\ell}\|_*^2]. \end{aligned}$$

so that (3.13) holds after applying the law of total expectation. We may then apply Theorem 3.10 with $\nu_k = \frac{\sigma_{\text{tot}}}{\sqrt{1-\alpha}} (k+1)^{\frac{1}{2}(1-\alpha)}$, yielding

$$\begin{aligned} T_k &= \frac{12\sqrt{N} \sigma_{\text{tot}}}{\text{sqr}t{1-\alpha}} (k+1)^{\frac{1}{2}(1-\alpha)} \sqrt{\max\left[1, \log\left(\frac{12\sqrt{N} \sigma_{\text{tot}}}{\sqrt{1-\alpha}} (k+1)^{\frac{1}{2}(1-\alpha)}\right)\right]} \\ &= \mathcal{O}\left((k+1)^{\frac{1}{2}(1-\alpha)} \sqrt{\log(k+1)}\right), \end{aligned}$$

and deduce from (3.42) that

$$\frac{1}{k+1} \sum_{j=0}^k \mathbb{E}[\|G_j\|_*] = \mathcal{O}\left(\frac{\sqrt{\log(k+1)}}{(k+1)^{\frac{\alpha}{2}}}\right).$$

If $\alpha > 1$, then $\sum_{j=0}^k \frac{1}{(j+1)^\alpha} \leq \zeta(\alpha) < +\infty$, where $\zeta(\cdot)$ is the Riemann zeta function. Hence

$$\frac{1}{k+1} \sum_{j=0}^k \mathbb{E}[\|G_j\|_*] = \mathcal{O}\left(\frac{1}{\sqrt{k+1}}\right).$$

Suppose now that $\alpha = 1$. Then $\sum_{j=0}^k \frac{1}{(j+1)^\alpha} = \sum_{j=0}^k \frac{1}{j+1} \leq \log(k+1)$ and inequality (3.33) in the proof of Theorem 3.10 can now be written as

$$t_k \leq a + b \sqrt{\log(k)} \sqrt{\log(t_k)} + c \log(t_k).$$

Cases one and three of the subsequent analysis are unchanged. Consider the second case, that is when (3.36) holds. Then, if $\log(t_k) > 1$, $t_k \leq 3b\sqrt{\log(k+1)}\sqrt{\log(t_k)}$, that is

$$t_k^2 \leq 9b^2 \log(k+1) \log(t_k) \leq \frac{9}{2} b^2 \log(k+1) \log(t_k^2)$$

and therefore, using Lemma 3.8, $t_k \leq \max[1, 3b\sqrt{\log(k+1)\log(9b^2\log(k+1))}]$ instead of (3.39). Thus, in this case,

$$\frac{1}{k+1} \sum_{j=0}^k \mathbb{E}[\|G_j\|_*] = \mathcal{O}\left(\frac{\sqrt{\log(k+1)\log(\log(k+1))}}{\sqrt{k+1}}\right).$$

□

Note that the continuity of the bound expressed in Theorem 3.10 as a function of ν_k is lost in the statement of Corollary 3.11, because, as is clear from its proof, the constants hidden in the $\mathcal{O}(\cdot)$ notation depend on α . In particular, the formulation using $\mathcal{O}(\cdot)$ does not allow taking the limit for α tending to one.

Observe that we could replace the second part of (3.43) by

$$\mathbb{E}_k \left[\|\tilde{G}_{k,\ell} - G_{k,\ell}\|^2 \right] \leq \frac{\sigma_\ell^2}{k^\alpha} + \omega^2 \|Z_{k-1,\ell}\|_*^2 \quad (3.44)$$

with the convention that $Z_{-1,\ell} = 0$, since

$$\sum_{j=0}^{k-1} \sum_{\ell=1}^L \mathbb{E}_j [\|Z_{j,\ell}\|_*^2] \leq \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E}_j [\|Z_{j,\ell}\|_*^2].$$

The advantage of (3.44) over the second part of (3.43) is that the right-hand-side of this new condition is now measurable at iteration k .

Although the rates of convergence obtained in Corollary 3.11 under the bias and variance conditions (3.43) are reasonable, they do not quite match, for high-variance regimes, the best obtained so far for (momentum-less) AdaGrad and AdaNorm [46]. However they recover (in order) the best $\mathcal{O}(\sqrt{\log(k+1)\log(\log(k+1))}/(k+1))$ rate in the preconditioned cumulative strong growth context². Importantly, they do so for a large class of algorithms. We however note that, should the restrictive assumption of a bounded gradient oracle be made for AdaGrad, only the first part of (3.43) (unbiased oracle) is necessary to deduce that the average of $\mathbb{E}[\|G_k\|_E^2]$ decreases like $\mathcal{O}(1/(k+1)^{1/4})$ [18], thereby allowing arbitrary variance.

3.3 Adding momentum (twice)

Because of its widespread use, we now consider properties of variants of ADPREC augmented with momentum. Suppose now that, for some $0 \leq \mu_k \leq \mu_{\max} < 1$, all $k \geq 0$ and $\ell \in \{1, \dots, L\}$, (2.7) and (2.6) are replaced by

$$M_{k,\ell} = \mu_k M_{k-1,\ell} + (1 - \mu_k) \tilde{G}_{k,\ell}, \quad (3.45)$$

$$\Gamma_{k,\ell} = \Gamma_{k-1,\ell} + \mathcal{L}_{k,\ell}(M_{k,\ell})^2, \quad (3.46)$$

$$Z_{k,\ell} = \Gamma_{k,\ell}^{-1/2} M_{k,\ell} \quad (3.47)$$

where we have defined, for all ℓ , $M_{-1,\ell} = \tilde{G}_{0,\ell}$ so that $M_{0,\ell} = \tilde{G}_{0,\ell}$. This modification of the algorithm in effect adds a specific type of momentum, although this is not the only possible one.

²That is not only in the case where $\sigma_\ell = 0$ for each ℓ , but, more generally, in the case where the ‘‘inaccuracy budget’’ ν_k^2 is finite.

Another variant where $\mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})$ is used instead of $\mathcal{L}_{k,\ell}(M_{k,\ell})$ will be considered at the end of this section.

We now show that the theory of the previous section can be extended to cover this case. We start by establishing a bound on the norm of the difference between the block-wise momentum $M_{k,\ell}$ and the approximate gradient $\tilde{G}_{k,\ell}$, which we now define as $\tilde{G}_{k,\ell} = \nabla_{X_\ell}^1 f(X_k, \xi_k)$.

Lemma 3.12 Suppose that Assumption 2 holds. Define

$$E_{k,\ell} = M_{k,\ell} - \tilde{G}_{k,\ell} \quad \text{for } k \geq 0 \text{ and } \ell \in \{1, \dots, L\}. \quad (3.48)$$

Then

$$\sum_{j=0}^k \mathbb{E}[\|E_j\|_*^2] \leq \frac{6}{(1 - \mu_{\max})^2} \sum_{j=0}^k \mu_j^2 \mathbb{E}[\|\tilde{G}_j - G_j\|_*^2] + \frac{3L_G^2 \eta^2}{(1 - \mu_{\max})^2} \sum_{j=0}^k \mu_j^2 \mathbb{E}[\|Z_j\|_*^2]. \quad (3.49)$$

Proof. If $k = 0$, (3.49) trivially holds. Suppose that $k \geq 1$ and note that (3.48) gives that

$$\begin{aligned} E_{k,\ell} &= \mu_k E_{k-1,\ell} - \mu_k (\tilde{G}_{k,\ell} - G_{k,\ell}) + \mu_k (G_{k,\ell} - G_{k-1,\ell}) + \mu_k (G_{k-1,\ell} - \tilde{G}_{k-1,\ell}) \\ &= \mu_k (M_{k-1,\ell} - \tilde{G}_{k-1,\ell}) + \mu_k (\tilde{G}_{k-1,\ell} - \tilde{G}_{k,\ell}) \\ &= \mu_k E_{k-1,\ell} + \mu_k (\tilde{G}_{k-1,\ell} - \tilde{G}_{k,\ell}). \end{aligned}$$

Substituting the bound in the previous inequality and using the facts that $\mu_k \leq \mu_{\max}$, that $(a + b)^2 \leq (1 + \tau)a^2 + (1 + 1/\tau)b^2$ for any $\tau > 0$ and that $(a + b + c)^2 \leq 3a^2 + 3b^2 + 3c^2$, we obtain that

$$\begin{aligned} \|E_{k,\ell}\|_{*,\ell}^2 &\leq (1 + \tau)\mu_{\max}^2 \|E_{k-1,\ell}\|_{*,\ell}^2 \\ &\quad + 3\mu_k^2 \left(1 + \frac{1}{\tau}\right) \left(\|G_{k,\ell} - G_{k-1,\ell}\|_{*,\ell}^2 + \|\tilde{G}_{k,\ell} - G_{k,\ell}\|_{*,\ell}^2 + \|\tilde{G}_{k-1,\ell} - G_{k-1,\ell}\|_{*,\ell}^2\right) \end{aligned}$$

Summing over $\ell \in \{1, \dots, L\}$ and using (2.2), we obtain that

$$\begin{aligned} \|E_k\|_*^2 &\leq (1 + \tau)\mu_{\max}^2 \|E_{k-1}\|_*^2 \\ &\quad + 3\mu_k^2 \left(1 + \frac{1}{\tau}\right) \left(\|G_k - G_{k-1}\|_*^2 + \sum_{\ell=1}^L \|\tilde{G}_{k,\ell} - G_{k,\ell}\|_*^2 + \sum_{\ell=1}^L \|\tilde{G}_{k-1,\ell} - G_{k-1,\ell}\|_*^2\right) \end{aligned}$$

Observe now that Assumption 2 with (2.8) and (2.3) imply that

$$\|G_k - G_{k-1}\|_*^2 \leq L_G^2 \sum_{\ell=1}^L \|X_{k,\ell} - X_{k-1,\ell}\|^2 = L_G^2 \eta^2 \sum_{\ell=1}^L \|Z_{k-1,\ell}\|_{*,\ell}^2 \|S_\ell(Z_{k-1,\ell})\|_\ell^2 = L_G^2 \eta^2 \|Z_{k-1}\|_*^2, \quad (3.50)$$

Now let $\tau = (1 - \mu_{\max})/\mu_{\max}$. Then $(1 + \tau)\mu_{\max}^2 = \mu_{\max} < 1$ and $(1 + 1/\tau)\mu_k^2 = \mu_k^2/(1 - \mu_{\max})$. If $\vartheta_k^2 = \sum_{\ell=1}^L \mathbb{E}[\|\tilde{G}_{k,\ell} - G_{k,\ell}\|_{*,\ell}^2] = \mathbb{E}[\|\tilde{G}_k - G_k\|_*^2]$, we have, after taking total expectation and using (3.50) that

$$\mathbb{E}[\|E_k\|_*^2] \leq \mu_{\max} \mathbb{E}[\|E_{k-1}\|_*^2] + \frac{3\mu_k^2}{1 - \mu_{\max}} (L_G^2 \eta^2 \mathbb{E}[\|Z_{k-1}\|_*^2] + \vartheta_k^2 + \vartheta_k^2)$$

Summing this inequality for $j \in \{1, \dots, k\}$ gives that

$$\sum_{j=1}^k \mathbb{E}[\|E_j\|_*^2] \leq \frac{3}{(1 - \mu_{\max})^2} \left(L_G^2 \eta^2 \sum_{j=1}^k \mu_j^2 \mathbb{E}[\|Z_{j-1}\|_*^2] + 2 \sum_{j=0}^k \mu_j^2 \vartheta_j^2 \right).$$

which, given that $\|E_0\|_* = 0$, proves (3.49). \square

We now wish to apply the theory above by considering that the momentum $M_{k,\ell}$ plays the role of the approximate gradient (in Step 1 of ADPREC) in this theory. In particular, this change of perspectives implies that Assumption 3 now becomes

Assumption 5 (Modified structural identities) For all $k \geq 0$ and all $\ell \in \{1, \dots, L\}$, we have that

$$\|Z_{k,\ell}\|_{*,\ell} \langle M_{k,\ell}, S_\ell(Z_{k,\ell}) \rangle_F = \text{tr}(\Gamma_{k,\ell}^{-1/2} \mathcal{L}_{k,\ell}(M_{k,\ell})^2), \quad (3.51)$$

and

$$\|Z_{k,\ell}\|_{*,\ell}^2 = \text{tr}(\Gamma_{k,\ell}^{-1} \mathcal{L}_{k,\ell}(M_{k,\ell})^2). \quad (3.52)$$

Fortunately, using our theory is possible because, as we show below, (3.49) implies that condition (3.13) of Theorem 3.5 holds with suitable constants. This allows us to derive the analog of Theorem 3.10 for the ‘‘momentum’’ variant of ADPREC.

Theorem 3.13 Suppose that the ADPREC algorithm modified by (3.45)–(3.47) is applied to problem (1.1) and that Assumptions 1, 2, 4 and 5 hold. Then

$$\frac{1}{k+1} \sum_{j=0}^k \mathbb{E}[\|G_j\|_*] \leq \frac{1}{\sqrt{k+1}} \left(2\kappa_\circ \Theta_k + \sqrt{2N \log(\Theta_k)} + \omega \sqrt{\max[\kappa_0, 1]} \right) \quad (3.53)$$

where Θ_k is defined in (3.28)–(3.25) using

$$\nu_k^2 = \left(\frac{6\mu_{\max}^2}{(1-\mu_{\max})^2} + 2 \right) \sum_{j=0}^k \mathbb{E}[\|\tilde{G}_j - G_j\|_*^2] \quad \text{and} \quad \omega^2 = \frac{3\mu_{\max}^2 L_G^2 \eta^2}{(1-\mu_{\max})^2}. \quad (3.54)$$

Proof. Observe first that the identity $(a+b)^2 \leq 2a^2 + 2b^2$, (3.49) and the bound $\mu_k \leq \mu_{\max}$ give that

$$\begin{aligned} \sum_{j=0}^k \mathbb{E}[\|M_j - G_j\|_*^2] &\leq 2 \sum_{j=0}^k \mathbb{E}[\|\tilde{G}_j - G_j\|_*^2] + 2 \sum_{j=0}^k \mathbb{E}[\|E_j\|_*^2] \\ &\leq \left(\frac{6\mu_{\max}^2}{(1-\mu_{\max})^2} + 2 \right) \mathbb{E}[\|\tilde{G}_j - G_j\|_*^2] + \frac{3\mu_{\max}^2 L_G^2 \eta^2}{(1-\mu_{\max})^2} \sum_{j=0}^k \mathbb{E}[\|Z_j\|_*^2] \end{aligned} \quad (3.55)$$

which is (3.13) with (3.54). We may therefore apply Theorem 3.10 (under Assumption 5) with these values and deduce that

$$\sum_{j=0}^k \mathbb{E}[\|M_j\|_*] \leq \sqrt{k+1} \kappa_\circ \Theta_k. \quad (3.56)$$

Hence, using the triangle, Cauchy-Schwarz and Jensen inequalities, we obtain that

$$\begin{aligned}
 \sum_{j=0}^k \mathbb{E}[\|G_j\|_*] &\leq \sum_{j=0}^k \mathbb{E}[\|M_j\|_*] + \sum_{j=0}^k \mathbb{E}[\|M_j - G_j\|_*] \\
 &\leq \sum_{j=0}^k \mathbb{E}[\|M_j\|_*] + \sqrt{k+1} \sqrt{\sum_{j=0}^k \mathbb{E}[\|M_j - G_j\|_*]^2} \\
 &\leq \sqrt{k+1} \kappa_o \Theta_k + \sqrt{k+1} \sqrt{\sum_{j=0}^k \mathbb{E}[\|M_j - G_j\|_*^2]} \\
 &= \sqrt{k+1} \kappa_o \Theta_k + \sqrt{k+1} \sqrt{\nu_k^2 + \omega^2 \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E}[\|Z_{j,\ell}\|_*^2]}
 \end{aligned} \tag{3.57}$$

where we have inserted (3.54) in (3.55) to obtain the last equality. But, successively using (3.20), (3.24), (3.29) and (3.28), we deduce that

$$\sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E}[\|Z_{j,\ell}\|_*^2] \leq \Delta_k \leq \kappa_o + 2N \log \left(\mathbb{E} \left[\sum_{\ell=1}^L \text{tr}(\Gamma_{k,\ell}^{1/2}) \right] \right) = \kappa_o + 2N \log(t_k) \leq \kappa_o + 2N \log(\Theta_k),$$

and thus that

$$\sum_{j=0}^k \mathbb{E}[\|M_j - G_j\|_*^2] \leq \nu_k^2 + \omega^2 (\kappa_o + 2N \log(\Theta_k)).$$

Substituting this inequality into (3.57) then yields that

$$\sum_{j=0}^k \mathbb{E}[\|G_j\|_*] \leq \sqrt{k+1} \kappa_o \Theta_k + \sqrt{k+1} \sqrt{\nu_k^2 + \omega^2 (\kappa_o + 2N \log(\Theta_k))}.$$

But the definition of Θ_k in (3.28) gives that $\nu_k \leq \Theta_k$, and (3.53) then follows by dividing by $k+1$ and using $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$. \square

The difference between (3.42) and (3.53) is only a constant and a logarithmic term in factor of $(k+1)^{-1/2}$. As a consequence, the global rate of convergence of the momentum variant of ADPREC is essentially identical to that of the version without momentum, and Corollary 3.11 still applies to the momentum variant.

We finally note that our momentum definition and convergence analysis do not make the assumption that the stepsize parameter η is sufficiently small, in contrast with previous proofs of convergence where results assume that η is small enough, in particular smaller than a multiple of the (usually unknown) Lipschitz constant L_G [20, 23, 50].

The momentum approach we have described above uses (3.45)–(3.47). Another version could be considered, using (3.45) and (3.47) but keeping the “pure gradient” technique (2.6) to accumulate the preconditioner, that is

$$\Gamma_{k,\ell} = \Gamma_{k-1,\ell} + \mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2, \tag{3.58}$$

$$M_{k,\ell} = \mu_k M_{k-1,\ell} + (1 - \mu_k) \tilde{G}_{k,\ell}, \tag{3.59}$$

$$Z_{k,\ell} = \Gamma_{k,\ell}^{-1/2} M_{k,\ell}. \tag{3.60}$$

As it turns out, it is also possible to derive a unified convergence theory for this choice, albeit this requires stronger assumptions and leads to a worse rate of convergence if μ_k is kept bounded away from zero. However, selecting the sequence $\{\mu_k\}$ adequately allows recovering the rate of the momentum-less variant. This alternative theory hinges on the fact that $\mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})$ can be viewed

as a perturbation of $\mathcal{L}_{k,\ell}(M_{k,\ell})$, and therefore that the structural relations of Assumption 5 are themselves perturbed. Fortunately, it remains possible to bound these perturbations by a mix of variance-related and second-order terms, the first of which potentially affecting the resulting global convergence rate. The final outcome is given by the following theorem and corollary (whose detailed proofs can be found in appendix).

Theorem 3.14 Suppose that the ADPREC algorithm with (2.7) replaced by (3.59) and (3.60) is applied to problem (1.1). Suppose that Assumptions 1, 2, 4 and 5 hold. If $\mu > 0$, suppose also that there exists constants $\kappa_{\square}, \kappa_{\diamond} > 0$ such that

$$\mathcal{L}_{k,\ell}(U + V)^2 \preceq \kappa_{\square} \mathcal{L}_{k,\ell}(U)^2 + \kappa_{\square} \mathcal{L}_{k,\ell}(V)^2 \quad \text{and} \quad \text{tr}\left(\mathcal{L}_{k,\ell}(U)^2\right) \leq \kappa_{\diamond} \|U\|_{*,\ell}^2 \quad (3.61)$$

for all $\ell \in \{1, \dots, L\}$, $k \geq 0$ and all $U, V \in \mathbb{R}^{n_{\ell} \times m_{\ell}}$, and that

$$\text{either } \eta \leq \frac{1 - \mu_{\max}}{\mu_{\max} L_G} \sqrt{\frac{\varsigma}{6\kappa_{\square}\kappa_{\diamond}}} \quad \text{or} \quad \sum_{j=0}^{\infty} \mu_j^2 \|Z_j\|_*^2 \leq \kappa_{\mu Z} \quad (3.62)$$

for some $\kappa_{\mu Z} \geq 0$. Then

$$\sum_{j=0}^k \mathbb{E} \left[\|\tilde{G}_j\|_* \right] \leq \sqrt{k+1} \kappa_{\circ} \Theta_k \quad (3.63)$$

where

$$\Theta_k = \max \left[e^{\max\left[1, \frac{1}{2N}, \frac{\kappa_0}{2N}\right]}, \frac{3(\kappa_{\text{gap}} + \kappa_{\nu\nu}\theta_k^2)}{\eta}, T_k, Y_k \right], \quad (3.64)$$

with

$$\theta_k^2 = \sum_{j=0}^k \sum_{\ell=1}^L \mu_j^2 \mathbb{E} \left[\|\tilde{G}_{j,\ell} - G_{j,\ell}\|_{*,\ell}^2 \right], \quad T_k = 12\sqrt{N} \kappa_{\nu\Delta} \theta_k \sqrt{\max \left[1, \log \left(12\sqrt{N} \kappa_{\nu\Delta} \theta_k \right) \right]}, \quad (3.65)$$

$$Y_k = 24N\kappa_{\Delta} \left(\omega^2 + \frac{L_G}{\eta} \right) \log \left(24N\kappa_{\Delta} \left(\omega^2 + \frac{L_G}{\eta} \right) \right)$$

κ_{\circ} being defined in Assumption 4, κ_0 in (3.25), and $\kappa_{\text{gap}}, \kappa_{\nu\nu}, \kappa_{\nu\Delta}$ and κ_{Δ} in (A.13)–(A.14).

Moreover, if \tilde{G}_k is an unbiased estimator of G_k , i.e. $\mathbb{E}_k[\tilde{G}_k] = G_k$ for all $k \geq 0$, then

$$\min_{j \in \{0, \dots, k\}} \mathbb{E}[\|G_j\|_*] \leq \frac{1}{k+1} \sum_{j=0}^k \mathbb{E}[\|G_j\|_*] \leq \frac{\kappa_{\circ} \Theta_k}{\sqrt{k+1}}. \quad (3.66)$$

As it turns out, assuming (3.61) is not restrictive in the applications considered in this paper, as we briefly discuss below in appendix. Supposing (3.62) is definitely less desirable because the value of the Lipschitz constant L_G or that of $\kappa_{\mu Z}$ (if it exists) is usually unknown. It is however common practice in the literature [20, 23, 50]. Note the term $\kappa_{\nu\nu}\theta_k^2$ in the middle term of (3.64), which is the crucial difference between (3.64) and (3.28) and that induces the modified convergence rate expressed by the following corollary.

Corollary 3.15 Suppose that the ADPREC algorithm with (2.7) replaced by (3.59) and (3.60) is applied to problem (1.1). Suppose that Assumptions 1, 2, 4 and 5 hold. If $\mu > 0$, suppose also that (3.61) and (3.62) hold, that

$$\mathbb{E}_k[\tilde{G}_k] = G_k \quad \text{and} \quad \mathbb{E}_k[\|\tilde{G}_{k,\ell} - G_{k,\ell}\|_*^2] \leq \frac{\sigma_\ell^2}{(k+1)^\alpha} \quad (3.67)$$

for some $\alpha > 0$ and all $k \geq 0$ and $\ell \in \{1, \dots, L\}$, and that

$$\mu_k = \frac{\mu_{\max}}{(k+1)^\beta}$$

for some $\mu_{\max} < 1$ and some $\beta \geq 0$. Then

$$\frac{1}{k+1} \sum_{j=0}^k \mathbb{E}[\|G_j\|_*] = \begin{cases} \mathcal{O}\left(\frac{1}{(k+1)^{\alpha+2\beta-\frac{1}{2}}}\right) & \text{if } \alpha + 2\beta < 1, \\ \mathcal{O}\left(\frac{\sqrt{\log(k+1)\log(\log(k+1))}}{\sqrt{k+1}}\right) & \text{if } \alpha + 2\beta = 1, \\ \mathcal{O}\left(\frac{1}{\sqrt{k+1}}\right) & \text{if } \alpha + 2\beta > 1. \end{cases}$$

Again, be aware that the constants hidden in the $\mathcal{O}(\cdot)$ depend on α , preventing taking limits for α tending to one. The rate of convergence now depends on the value of $\alpha + 2\beta$, indicating how acting on the momentum parameter μ_k can alleviate the effect of a high gradient-oracle variance. Because the convergence rate is, in this case, determined by ν_k as defined in (3.65), choosing a small momentum parameter in effect reduces the propagation of larger errors in the gradient oracle across iterations. In particular, setting μ_k to a multiple of $k^{-\frac{1}{4}\alpha}$ recovers the rate of the momentum-less variant also for the high-variance regime ($\alpha < 1$). If however μ_k is kept constant (or bounded away from zero), that is if $\beta = 0$, then (3.67) is stronger than (3.43) and the rate of convergence for $\alpha < 1$ is now $\mathcal{O}\left((k+1)^{\frac{1}{2}-\alpha}\right)$ instead of $\mathcal{O}\left((k+1)^{-\frac{\alpha}{2}}\right)$, requiring in particular that $\alpha > \frac{1}{2}$.

One should however be careful not to identify better theoretical convergence properties with better practical performance on specific classes of problems. Indeed, limited numerical experiments suggest that the second momentum variant might outperform the first.

4 Application to existing algorithms

We now consider four popular first-order algorithms and show that their convergence behaviour is covered by the above results by considering a single block ($L = 1$). But, we also show that this is also true if they are applied blockwise in a more complex setting. This is achieved by clarifying the correspondence between ADPREC and each algorithm and verifying that (3.1), (3.2) and (3.3) hold in each case for all blocks concerned. In all cases considered here, the preconditioner update $\mathcal{L}_{k,\ell}$ is independent of k . For simplicity of exposition, we focus on the situation where there is no momentum ($\mu_k = 0$). We also use lower case symbols for vectors.

4.1 AdaNorm [42, 13, 48]

Taking the AdaNorm update on a given block ℓ , corresponds to choosing an isotropic scalar geometry. More specifically, if block ℓ has dimension n_ℓ , one considers

$$\Gamma_{-1,\ell} = \varsigma I_{n_\ell} \quad \text{and} \quad \Gamma_{k,\ell} = \Gamma_{k-1,\ell} + \frac{\|\tilde{g}_{k,\ell}\|_\ell^2}{n_\ell} I_{n_\ell} \quad (k \geq 0),$$

so that the contribution of this block to the step is

$$s_{k,\ell} = -\eta \Gamma_{k,\ell}^{-1/2} \tilde{g}_{k,\ell} = -\frac{\eta}{\sqrt{\gamma_{k,\ell}}} \tilde{g}_{k,\ell}$$

when $\Gamma_{k,\ell} = \gamma_{k,\ell} I_{n_\ell}$. This is exactly the AdaNorm update on block ℓ .

In ADPREC, this corresponds to choosing

$$m_\ell = 1, \quad \|\cdot\|_{*,\ell} = \|\cdot\|_\ell, \quad S_\ell(\tilde{G}_{k,\ell}) = \frac{\tilde{G}_{k,\ell}}{\|\tilde{G}_{k,\ell}\|_\ell}, \quad \text{and} \quad \mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2 = \frac{\|\tilde{G}_{k,\ell}\|_\ell^2}{n_\ell} I_{n_\ell}. \quad (4.1)$$

With these choices, we have that

$$Z_{k,\ell} = \Gamma_{k,\ell}^{-1/2} \tilde{G}_{k,\ell} \quad \text{and} \quad \text{tr} \left(\Gamma_{k,\ell}^{-1} \mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2 \right) = \text{tr} \left(\frac{\|\tilde{G}_{k,\ell}\|_\ell^2}{n_\ell} \Gamma_{k,\ell}^{-1} \right).$$

If $\Gamma_{k,\ell} = \gamma_{k,\ell} I_{n_\ell}$, this becomes

$$\text{tr} \left(\Gamma_{k,\ell}^{-1} \mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2 \right) = \frac{\|\tilde{G}_{k,\ell}\|_\ell^2}{\gamma_{k,\ell}} = \|Z_{k,\ell}\|_\ell^2,$$

which is (3.2). Similarly,

$$\text{tr}(\Gamma_{k,\ell}^{-1/2} \mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2) = \frac{\|\tilde{G}_{k,\ell}\|_\ell^2}{\sqrt{\gamma_{k,\ell}}} = \|Z_{k,\ell}\|_\ell \left\langle \tilde{G}_{k,\ell}, S_\ell(Z_{k,\ell}) \right\rangle,$$

proving (3.1). Moreover, Assumption 4 also holds on block ℓ . Indeed, for any \tilde{G}_ℓ , the second and last parts of (4.1) give

$$\text{tr}(\mathcal{L}_{k,\ell}(\tilde{G}_\ell)^2) = \|\tilde{G}_\ell\|_\ell^2 = \|\tilde{G}_\ell\|_{*,\ell}^2,$$

and thus (3.3) holds with $\kappa_\circ = 1$.

Thus, whenever a block ℓ is endowed with this isotropic scalar geometry, Theorem 3.10 and Corollary 3.11 apply to that block without modification. The standard ‘‘one block’’ AdaNorm is obtained when there is only one block ($L = 1, n_1 = d_1 = N$).

4.2 Full AdaGrad [13]

Taking the ‘‘full’’ version of AdaGrad [13] update on block ℓ corresponds to choosing, on a given block ℓ , a full matrix-valued geometry. More precisely, if block ℓ has dimension n_ℓ , one considers

$$\Gamma_{-1,\ell} = \varsigma I_{n_\ell} \quad \text{and} \quad \Gamma_{k,\ell} = \Gamma_{k-1,\ell} + \tilde{g}_{k,\ell} \tilde{g}_{k,\ell}^T \quad (k \geq 0),$$

so that the contribution of this block to the step is $s_{k,\ell} = -\eta \Gamma_{k,\ell}^{-1/2} \tilde{g}_{k,\ell}$. This is exactly the full AdaGrad update on block ℓ .

In ADPREC, this corresponds to choosing

$$m_\ell = n_\ell, \quad \|\cdot\|_{*,\ell} = \|\cdot\|_\ell = \|\cdot\|_E, \quad S_\ell(\tilde{G}_{k,\ell}) = \tilde{G}_{k,\ell} / \|\tilde{G}_{k,\ell}\|_E, \quad \text{and} \quad \mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2 = \tilde{G}_{k,\ell} \tilde{G}_{k,\ell}^T. \quad (4.2)$$

With these choices, we have that

$$Z_{k,\ell} = \Gamma_{k,\ell}^{-1/2} \tilde{G}_{k,\ell}, \quad \text{and} \quad \text{tr}(\Gamma_{k,\ell}^{-1} \mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2) = \text{tr}(\Gamma_{k,\ell}^{-1/2} \tilde{G}_{k,\ell} \tilde{G}_{k,\ell}^T \Gamma_{k,\ell}^{-1/2}) = \|Z_{k,\ell}\|_E^2,$$

which is (3.2). Similarly,

$$\text{tr}(\Gamma_{k,\ell}^{-1/2} \mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2) = \text{tr}(\Gamma_{k,\ell}^{-1/2} \tilde{G}_{k,\ell} \tilde{G}_{k,\ell}^T) = \left\langle \tilde{G}_{k,\ell}, Z_{k,\ell} \right\rangle = \|Z_{k,\ell}\|_E \left\langle \tilde{G}_{k,\ell}, S_\ell(Z_{k,\ell}) \right\rangle,$$

proving (3.1). Moreover, Assumption 4 also holds on block ℓ . Indeed, for any G_ℓ , the second and last part of (4.2) give

$$\mathrm{tr}(\mathcal{L}_{k,\ell}(\tilde{G}_\ell)^2) = \mathrm{tr}(\tilde{G}_\ell \tilde{G}_\ell^T) = \|\tilde{G}_\ell\|_E^2 = \|\tilde{G}_\ell\|_{*,\ell}^2$$

so that (3.3) holds with $c_\circ = 1$.

Thus, whenever a block ℓ is endowed with this full matrix geometry, Theorem 3.10, and Corollary 3.11 apply to that block without modification.

4.3 Diagonal AdaGrad [13, 36]

We now turn to the use of the diagonal/component-wise AdaGrad update on a given block ℓ . If block ℓ has dimension n_ℓ , this geometry is defined by

$$\Gamma_{-1,\ell} = \varsigma I_{n_\ell} \quad \text{and} \quad \Gamma_{k,\ell} = \Gamma_{k-1,\ell} + \mathrm{Diag}((\tilde{g}_{k,\ell})_i^2) \quad (k \geq 0) \quad \text{and} \quad x_{k+1,\ell} = x_{k,\ell} - \eta \Gamma_{k,\ell}^{-1/2} \tilde{g}_{k,\ell}.$$

In our framework, this corresponds to choosing

$$m_\ell = n_\ell, \quad \|\cdot\|_{*,\ell} = \|\cdot\|_\ell = \|\cdot\|_E, \quad \mathcal{L}_{k,\ell}(\tilde{g}_{k,\ell}) = \mathrm{Diag}(|(\tilde{g}_{k,\ell})_i|), \quad \text{and} \quad S_\ell(\tilde{g}_{k,\ell}) = \tilde{g}_{k,\ell} / \|\tilde{g}_{k,\ell}\|_E, \quad (4.3)$$

so that

$$\mathcal{L}_{k,\ell}(\tilde{g}_{k,\ell})^2 = \mathrm{Diag}((\tilde{g}_{k,\ell})_i^2).$$

We then have that

$$Z_{k,\ell} = \Gamma_{k,\ell}^{-1/2} \tilde{G}_{k,\ell}, \quad \text{and} \quad \mathrm{tr}(\Gamma_{k,\ell}^{-1} \mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2) = \sum_{i=1}^{n_\ell} \frac{(\tilde{G}_{k,\ell})_i^2}{(\Gamma_{k,\ell})_{ii}} = \|Z_{k,\ell}\|_E^2,$$

which proves (3.2). Moreover,

$$\mathrm{tr}(\Gamma_{k,\ell}^{-1/2} \mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2) = \sum_{i=1}^{n_\ell} \frac{(\tilde{G}_{k,\ell})_i^2}{\sqrt{(\Gamma_{k,\ell})_{ii}}} = \langle \tilde{G}_{k,\ell}, Z_{k,\ell} \rangle = \|Z_{k,\ell}\|_E \langle \tilde{G}_{k,\ell}, S_\ell(Z_{k,\ell}) \rangle,$$

which proves (3.1). Moreover, Assumption 4 also holds in this case. Indeed, for any $G_\ell \in \mathbb{R}^{n_\ell}$,

$$\mathcal{L}_{k,\ell}(G_\ell) = \mathrm{Diag}(|(G_\ell)_i|), \quad \mathcal{L}_{k,\ell}(G_\ell)^2 = \mathrm{Diag}((G_\ell)_i^2),$$

and therefore

$$\mathrm{tr}(\mathcal{L}_{k,\ell}(G_\ell)^2) = \sum_{i=1}^{n_\ell} (G_\ell)_i^2 = \|G_\ell\|_E^2 = \|G_\ell\|_{*,\ell}^2,$$

so that (3.3) holds with $\kappa_\circ = 1$.

Thus, as above, Theorem 3.10 and Corollary 3.11 apply without modification to this diagonal AdaGrad geometry on block ℓ .

Observe that we could also view the diagonal AdaGrad algorithm as a Cartesian product of scalar AdaNorm. Also note that, as discussed in Section 3.3, momentum in the form (3.45)-(3.47) or (3.58)-(3.60) may be introduced on this block, resulting in an ADPREC variant somewhat similar to Adam [27]. It is theoretically interesting that this variant enjoys the convergence behaviour described by Theorem 3.13 and Corollary 3.11, in contrast with Adam.

4.4 Adaptive Shampoo [22]

Shampoo [22] is a stochastic preconditioned optimization method designed for tensor parameters and in particular matrix blocks in neural networks. We consider here an adaptive variant of this algorithm. We assume a block structure with L blocks, each associated with matrix parameter

$X_{k,\ell} \in \mathbb{R}^{n_\ell \times m_\ell}$, implying that $d_\ell = n_\ell m_\ell$. Given a stochastic gradient $\tilde{G}_{k,\ell}$, the Adaptive Shampoo algorithm maintains the Gram matrices

$$L_{k,\ell} = L_{k-1,\ell} + \tilde{G}_{k,\ell} \tilde{G}_{k,\ell}^T, \quad R_{k,\ell} = R_{k-1,\ell} + \tilde{G}_{k,\ell}^T \tilde{G}_{k,\ell},$$

initialized with $L_{-1,\ell} = R_{-1,\ell} = \varsigma I$. The Shampoo preconditioned gradient is

$$\hat{Z}_{k,\ell} = L_{k,\ell}^{-1/4} \tilde{G}_{k,\ell} R_{k,\ell}^{-1/4} \quad \text{and} \quad X_{k+1,\ell} = X_{k,\ell} - \frac{\eta_{\text{shampoo}}}{\sqrt{d_\ell}} \hat{Z}_{k,\ell}. \quad (4.4)$$

We now express Shampoo in the framework of ADPREC. On $\mathbb{R}^{n_\ell \times m_\ell}$, we choose

$$\|\cdot\|_\ell = \|\cdot\|_F, \quad \|\cdot\|_{*,\ell} = \|\cdot\|_F, \quad S_\ell(Z) = Z/\|Z\|_F.$$

Let $\tilde{g}_{k,\ell} = \text{vec}(\tilde{G}_{k,\ell})$ and $z_{k,\ell} = \text{vec}(Z_{k,\ell})$. Using $\text{vec}(AMB) = (B^T \otimes A) \text{vec}(M)$, we obtain that

$$\text{vec}(\hat{Z}_{k,\ell}) = (R_{k,\ell}^{-1/4} \otimes L_{k,\ell}^{-1/4}) \tilde{g}_{k,\ell}. \quad (4.5)$$

Define

$$\Gamma_{k,\ell}^{-1/2} = R_{k,\ell}^{-1/4} \otimes L_{k,\ell}^{-1/4}, \quad \Gamma_{k,\ell} = R_{k,\ell}^{1/2} \otimes L_{k,\ell}^{1/2}.$$

Then $z_{k,\ell} = \Gamma_{k,\ell}^{-1/2} \tilde{g}_{k,\ell}$, and thus $Z_{k,\ell} = \hat{Z}_{k,\ell}$. The ADPREC update becomes

$$X_{k+1,\ell} = X_{k,\ell} - \eta Z_{k,\ell},$$

and choosing $\eta = \eta_{\text{shampoo}}/\sqrt{d_\ell}$ recovers (4.4). We now define

$$\mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2 = \tilde{g}_{k,\ell} \tilde{g}_{k,\ell}^T, \quad \text{so that} \quad \mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2 = \tilde{g}_{k,\ell} \tilde{g}_{k,\ell}^T.$$

Then

$$\|Z_{k,\ell}\|_{*,\ell}^2 = \|z_{k,\ell}\|_2^2 = \tilde{g}_{k,\ell}^T \Gamma_{k,\ell}^{-1} \tilde{g}_{k,\ell} = \text{tr}(\Gamma_{k,\ell}^{-1} \mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2),$$

which proves (3.2). Moreover,

$$\|Z_{k,\ell}\|_{*,\ell} \langle \tilde{G}_{k,\ell}, S_\ell(Z_{k,\ell}) \rangle_F = \langle \tilde{G}_{k,\ell}, Z_{k,\ell} \rangle_F = \tilde{g}_{k,\ell}^T \Gamma_{k,\ell}^{-1/2} \tilde{g}_{k,\ell} = \text{tr}(\Gamma_{k,\ell}^{-1/2} \mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2),$$

which proves (3.1). Moreover, Assumption 4 also holds. Indeed, for any \tilde{G}_ℓ ,

$$\mathcal{L}_{k,\ell}(\tilde{G}_\ell)^2 = \tilde{g}_\ell \tilde{g}_\ell^T, \quad \text{so that} \quad \text{tr}(\mathcal{L}_{k,\ell}(\tilde{G}_\ell)^2) = \|\tilde{g}_\ell\|_2^2 = \|\tilde{G}_\ell\|_F^2 = \|\mathcal{G}_\ell\|_{*,\ell}^2,$$

and thus (3.3) holds with $\kappa_\circ = 1$.

Therefore Theorem 3.10 and Corollary 3.11 apply without modification to Shampoo.

4.5 Adaptive MUON/AdaGO [25, 41, 53]

We now consider an AdaGrad-inspired adaptive version of MUON [25], as described as AdaGO for the single block case in [53]. In our framework, this corresponds to assigning, for each block ℓ , a geometry adapted to the structure of neural network parameters. We distinguish two types of blocks. For $\ell \in \mathcal{M}$, $G_{k,\ell}$ is a matrix of size $n_\ell \times m_\ell$, while for $\ell \in \mathcal{A} = \{1, \dots, L\} \setminus \mathcal{M}$, $G_{k,\ell}$ is a vector in \mathbb{R}^{n_ℓ} . For $\ell \in \mathcal{A}$, we use

$$\|\cdot\|_\ell = \|\cdot\|_E = \|\cdot\|_{*,\ell},$$

while for $\ell \in \mathcal{M}$,

$$\|\cdot\|_\ell = \|\cdot\|_s, \quad \|\cdot\|_{*,\ell} = \|\cdot\|_*.$$

The MUON normalization is defined by

$$S_\ell(\tilde{G}_{k,\ell}) = \begin{cases} U_{k,\ell} V_{k,\ell}^T & \text{if } \ell \in \mathcal{M} \text{ and } \tilde{G}_{k,\ell} = U_{k,\ell} \Sigma_{k,\ell} V_{k,\ell}^T, \\ \tilde{G}_{k,\ell} / \|\tilde{G}_{k,\ell}\|_E & \text{if } \ell \in \mathcal{A}. \end{cases}$$

We then define, for each block ℓ ,

$$\Gamma_{k,\ell} = \gamma_{k,\ell} I_{d_\ell}, \quad \text{where } \gamma_{-1,\ell} = \varsigma \quad \text{and} \quad \gamma_{k,\ell} = \begin{cases} \gamma_{k-1,\ell} + \|\tilde{G}_{k,\ell}\|_*^2 / d_\ell & \text{if } \ell \in \mathcal{M}, \\ \gamma_{k-1,\ell} + \|\tilde{G}_{k,\ell}\|_E^2 / d_\ell & \text{if } \ell \in \mathcal{A}. \end{cases} \quad (4.6)$$

This corresponds to choosing, for each block ℓ ,

$$\mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell}) = \begin{cases} \frac{\|\tilde{G}_{k,\ell}\|_*}{\sqrt{d_\ell}} I_{d_\ell} & \text{if } \ell \in \mathcal{M}, \\ \frac{\|\tilde{G}_{k,\ell}\|_E}{\sqrt{d_\ell}} I_{d_\ell} & \text{if } \ell \in \mathcal{A}, \end{cases} \quad \text{so that} \quad \mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2 = \begin{cases} \frac{\|\tilde{G}_{k,\ell}\|_*^2}{d_\ell} I_{d_\ell} & \text{if } \ell \in \mathcal{M}, \\ \frac{\|\tilde{G}_{k,\ell}\|_E^2}{d_\ell} I_{d_\ell} & \text{if } \ell \in \mathcal{A}. \end{cases}$$

We also have that $Z_{k,\ell} = \frac{1}{\sqrt{\gamma_{k,\ell}}} \tilde{G}_{k,\ell}$.

With these choices, the ADPREC algorithm corresponds to a block-wise adaptive MUON method, with stepsize $\eta_{\text{muon},\ell} = \eta / \sqrt{d_\ell}$. For $\ell \in \mathcal{A}$, the verification of (3.1), (3.2) and (3.3) is identical to that of AdaNorm. For $\ell \in \mathcal{M}$, we have

$$\text{tr}(\Gamma_{k,\ell}^{-1} \mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2) = \frac{\|\tilde{G}_{k,\ell}\|_*^2}{\gamma_{k,\ell}} = \|Z_{k,\ell}\|_*^2,$$

which proves (3.2). Therefore,

$$\|Z_{k,\ell}\|_* \left\langle \tilde{G}_{k,\ell}, S_\ell(Z_{k,\ell}) \right\rangle = \frac{\|\tilde{G}_{k,\ell}\|_*^2}{\sqrt{\gamma_{k,\ell}}} = \text{tr}(\Gamma_{k,\ell}^{-1/2} \mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2),$$

which proves (3.1). Moreover, Assumption 4 also holds in this case. Indeed, for any block ℓ and any \tilde{G}_ℓ ,

$$\text{tr}(\mathcal{L}_{k,\ell}(\tilde{G}_\ell)^2) = \begin{cases} \text{tr} \left(\frac{\|\tilde{G}_\ell\|_*^2}{d_\ell} I_{d_\ell} \right) = \|\tilde{G}_\ell\|_*^2 & \text{if } \ell \in \mathcal{M}, \\ \text{tr} \left(\frac{\|\tilde{G}_\ell\|_E^2}{d_\ell} I_{d_\ell} \right) = \|\tilde{G}_\ell\|_E^2 = \|\tilde{G}_\ell\|_*^2 & \text{if } \ell \in \mathcal{A}. \end{cases}$$

Thus (3.3) holds with $\kappa_\circ = 1$ for all $\ell \in \{1, \dots, L\}$. Therefore Theorem 3.10 and Corollary 3.11 apply without modification to this adaptive MUON geometry.

5 Conclusions

We have provided a general framework for first-order optimization algorithms using adaptively preconditioned gradients but no function values. We have shown that this framework covers a few of the popular adaptive first-order methods, including AdaGrad, full AdaGrad, AdaNorm, Shampoo and Muon, as well as any Cartesian mix of these. We have provided a fully stochastic rate-of-convergence analysis for all methods in the framework, with and without momentum, using standard assumptions on the variance of the gradient oracle and without assuming bounded stochastic gradients or small enough stepsizes.

Our results open several theoretical research issues. We anticipate that, as is the case for AdaGrad, the framework can be adapted to handle bounds [6] or more general convex [22, 11, 1] or nonconvex [14, 7, 19, 47] constraints, as well as second-order information (should it be available), but this requires verifications that are beyond the scope of the present paper. Other questions also remain. Is approximate normalization (i.e. approximate $S_\ell(\cdot)$) acceptable? Can the generality of (2.4) be exploited to derive new preconditioners? Does the Cartesian framework allow more general algorithmic settings? Can the freedom to choose a different preconditioner update at each iteration be used to advantage? All these topics are the subject of ongoing investigation.

References

- [1] A. Alacaoglu and H. Lyu. Convergence of first-order methods for constrained nonconvex optimization with dependent data. In *International Conference on Machine Learning. PMLR*, pages 458–489, 2023.
- [2] A. Alacaoglu, Y. Malitsky, and V. Cevher. Convergence of adaptive algorithms for constrained weakly-convex optimization. In *Proceedings of the 35th Conference on Neural Information Processing Systems (NEURIPS 2021)*, 2021.
- [3] R. Anil, V. Gupta, T. Koren, K. Regan, and Y. Singer. Scalable second order optimization for deep learning. In *Proceedings of the 37th International Conference on Machine Learning (ICML)*, volume 119 of *Proceedings of Machine Learning Research*, pages 459–469. PMLR, 2020.
- [4] A. Attia and T. Koren. SGD with AdaGrad stepsizes: Full adaptivity with high probability to unknown parameters, unbounded gradients and affine variance. arXiv:2302.08783, 2023.
- [5] L. Balles, F. Pedregosa, and N. Le Roux. The geometry of sign gradient descent. arXiv:2002.08056, 2020.
- [6] S. Bellavia, G. Gratton, B. Morini, and Ph. L. Toint. Fast stochastic Adagrad for nonconvex bound-constrained optimization. arXiv:2505.06374, 2025.
- [7] S. Bellavia, G. Gratton, B. Morini, and Ph. L. Toint. An objective-function-free algorithm for general smooth constrained optimization. arXiv:2602.11770, 2026.
- [8] R. Bhatia. *Matrix Analysis*. Number 169 in Graduate Texts in Mathematics. Springer Verlag, Heidelberg, Berlin, New York, 1996.
- [9] L. Bottou, F. E. Curtis, and J. Nocedal. Optimization methods for large-scale machine learning. *SIAM Review*, 60(2):223–311, 2018.
- [10] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, Cambridge, England, 2004.
- [11] D. Davis and D. Drusvyatskiy. Stochastic model-based minimization of weakly convex functions. *SIAM Journal on Optimization*, 29(1):207–239, 2019.
- [12] A. Défossez, L. Bottou, F. Bach, and N. Usunier. A simple convergence proof for Adam and Adagrad. *Transactions on Machine Learning Research*, October 2022.
- [13] J. Duchi, E. Hazan, and Y. Singer. Adaptive subgradient methods for online learning and stochastic optimization. *Journal of Machine Learning Research*, 12, July 2011.
- [14] Y. Fang, S. Na, M. Mahoney, and M. Kolar. Fully stochastic trust-region sequential quadratic programming for equality constrained optimization problems. *SIAM Journal on Optimization*, 34(2):2007–2037, 2024.
- [15] M. Faw, L. Rout, C. Caramanis, and S. Shakkottai. Beyond uniform smoothness: A stopped analysis of adaptive SGD. arXiv:2302.06570, 2023.
- [16] M. Faw, I. Tziotis, C. Caramanis, A. Mokhtari, S. Shakkottai, and R. Ward. The power of adaptivity in SGD: Self-tuning step sizes with unbounded gradients and affine variance. In *Proceedings of 35th Conference on Learning Theory*, volume 178, pages 313–355, 2022.
- [17] B. Ginsburg, P. Castonguay, O. Hrinchuk, O. Kuchaiev, R. Leary, V. Lavrukhin, J. Li, H. Nguyen, Y. Zhang, and J. M. Cohen. Training deep networks with stochastic gradient normalized by layerwise adaptive second moments. arXiv:1905.11286v3, 2029.
- [18] S. Gratton, S. Jerad, and Ph. L. Toint. Complexity and performance for two classes of noise-tolerant first-order algorithms. *Optimization Methods and Software*, 40(6):1473–1499, 2025.
- [19] S. Gratton and Ph. L. Toint. An objective-function-free algorithm for nonconvex stochastic optimization with deterministic equality and inequality constraints. arXiv:2603.29685, 2026.
- [20] Z. Guo, W. Yin, R. Jin, and T. Yang. A novel convergence analysis for algorithms of the Adam family. In *Proceedings of OPT2021: 13th Annual Workshop on Optimization for Machine Learning*, 2021.
- [21] V. Gupta, T. Koren, and Y. Singer. A unified approach to adaptive regularization in online and stochastic optimization. arXiv:1706.06569, 2017.
- [22] V. Gupta, T. Koren, and Y. Singer. Shampoo: Preconditioned stochastic tensor optimization. In *Proceedings of the International Conference on Machine Learning*, pages 1842–1850, 2018.
- [23] Y. Hong and J. Lin. Revisiting convergence of Adagrad with relaxed assumptions. arXiv:2403.13794v2, 2024.
- [24] R. Jiang, D. Maladkar, and A. Mokhtari. Convergence analysis of adaptive gradient methods under refined smoothness and noise assumptions. arXiv:2406.04592, 2024.
- [25] K. Jordan, Y. Jin, V. Boza, Y. Jiacheng, F. Cecista, L. Newhouse, and J. Bernstein. URL: <https://kellerjordan.github.io.posts/muon>, 2024.
- [26] A. Kavis, K.-Y. Levy, and V. Cevher. High probability bounds for a class of nonconvex algorithms with AdaGrad stepsize. In *International Conference on Learning Representations*, 2022.
- [27] D. Kingma and J. Ba. Adam: A method for stochastic optimization. In *Proceedings in the International Conference on Learning Representations (ICLR)*, 2015.

- [28] D. Kovalev. SGD with adaptive preconditioning: Unified analysis and momentum acceleration. arXiv:2506.23804, 2025.
- [29] H. Li, Y. Dong, and Z. Lin. On the $\mathcal{O}(\sqrt{d}/t^{1/4})$ convergence rate for RMSProp and its momentum extension measured in ℓ_1 norm. *Journal of Machine Learning Research*, 26:1–25, 2025.
- [30] J. Li and M. Hong. A note on the convergence of Muon. arXiv:2502.02900, 2025.
- [31] X. Li and F. Orabona. On the convergence of stochastic gradient descent with adaptive stepsizes. In *The 22nd International Conference on Artificial Intelligence and Statistics*, page 983–992, 2019.
- [32] L. Liu, Z. Xu, Z. Zhang, H. Kang, Z. Li, C. Liang, W. Chen, and T. Zhao. COSMOS: A hybrid adaptive optimizer for memory-efficient training of LLMs. arXiv:2502.17410, 2025.
- [33] Z. Liu, T. D. Nguyen, A. Ene, and H. Nguyen. On the convergence of Adagrad(Norm) on \mathbb{R}^d : Beyond convexity, non-asymptotic rate and acceleration. In *International Conference on Learning Representations*, 2023.
- [34] K. Löwner. Über monotone Matrixfunktionen. *Mathematisch Zeitung*, 38:177–216, 1934.
- [35] J. Martens and R. Grosse. Optimizing neural networks with Kronecker-factored approximate curvature. In *International Conference on Machine Learning*, 2015.
- [36] B. McMahan and M. Streeter. Adaptive bound optimization for online convex optimization. In *Conference on Learning Theory*, page 244sq, 2010.
- [37] M. C. Mikkamala and M. Hein. Variants of RMSProp and Adagrad with logarithmic regret bounds. In *Proceedings of the 34th International Conference on Machine Learning*, page 2545–2553, 2017.
- [38] Th. Pethick, W. Xie, K. Antonakopoulos, Z. Zhu, A. Silveti-Falls, and V. Cevher. Training deep learning models with norm-constrained LMOs. arXiv:2502.07529v2, 2025.
- [39] S. Reddi, S. Kale, and S. Kumar. On the convergence of Adam and beyond. In *Proceedings in the International Conference on Learning Representations (ICLR)*, 2018.
- [40] H. Robbins and S. Monroe. A stochastic approximation method. *The Annals of Mathematical Statistics*, 22(3):400–407, 1951.
- [41] C. Si, D. Zhang, and W. Shen. AdaMuon: Adaptive Muon optimizer. arXiv:2507.11005, 2025.
- [42] M. Streeter and B. McMahan. Less regret via online conditioning, 2010.
- [43] T. Tieleman and G. Hinton. Lecture 6.5-RMSPROP. COURSE: Neural Networks for Machine Learning, 2012.
- [44] Ph. L. Toint. Divergence of the ADAM algorithm with fixed-stepsize: a (very) simple example. In K. Fackeldey, A. Kannan, S. Pokutta, K. Sharma, D. Walter, A. Walther, and M. Weiser, editors, *Mathematical Optimization for Machine Learning: Proceedings of the MATH+ Thematic Einstein Semester 2023*, pages 195–199. De Gruyter Brill, 2025.
- [45] N. Vyas, D. Morwani, R. Zhao, M. Kwun, I. Shapira, B. Brandfonbrener, L. Janson, and S. Kakade. SOAP: Improving and stabilizing Shampoo using Adam. arXiv:2409.11321, 2024.
- [46] B. Wang, H. Zhang, Z. Ma, and W. Chen. Convergence of Adagrad for non-convex objectives: simple proofs and relaxed assumptions. In *Proceedings of the Thirty-Sixth Conference on Learning Theory*, pages 161–190, 2023.
- [47] Q. Wang, Ch. Peirmarini, Y. Zhu, and F. E. Curtis. Projected stochastic momentum methods for nonlinear equality-constrained optimization for machine learning. arXiv:2601.11785, 2026.
- [48] R. Ward, X. Wu, and L. Bottou. Adagrad stepsizes: sharp convergence over nonconvex landscapes. In *Proceedings in the International Conference on Machine Learning (ICML2019)*, 2019.
- [49] X. Wu, R. Ward, and L. Bottou. WNGRAD: Learn the learning rate in gradient descent. arXiv:1803.02865, 2018.
- [50] N. Xiao, X. Hu, X. Lin, and K.-C. Toh. Adam family methods for nonsmooth optimization with convergence guarantees. *Journal of Machine Learning Research*, 25, 2024.
- [51] S. Xie, T. Wang, S. Reddi, S. Kumar, and Z. Li. Structured preconditioners in adaptive optimization: A unified analysis. arXiv:2503.10537, 2025.
- [52] A. W. Yu, L. Huang, Q. Lin, R. Salakhutdinov, and J. Carbonell. Block-normalized gradient method; an empirical study for training deep neural network. arXiv:1707.04822v2, 2017.
- [53] M. Zhang, Y. Liu, and H. Schaeffer. AdaGrad meets Muon: Adaptive stepsizes for orthogonal updates. arXiv:2509.02981v2, 2025.
- [54] Q. Zhang, Y. Zhou, and S. Zou. Convergence guarantees for RMSProp and Adam in generalized-smooth non-convex optimization with affine noise variance, 2025.
- [55] T. Zhang and Z. Xu. Convergence of stochastic gradient descent for non-convex problems. In *Proceedings for the COLT Conference*, 2012.

Proofs for Theorem 3.14 and Corollary 3.15

The definition (2.6) immediately implies the following bounds.

Lemma A.1 We have that, for all $k \geq 0$ and all $\ell \in \{1, \dots, L\}$, $\Gamma_{k,\ell}$ is symmetric positive definite and

$$\Gamma_{k,\ell}^{-1/2} \preceq \frac{1}{\sqrt{\varsigma}} I_\ell \quad \text{and} \quad \Gamma_{k,\ell}^{-1} \preceq \frac{1}{\varsigma} I_\ell \quad (\text{A.1})$$

Proof. Directly result from (2.6), (2.4) and $\Gamma_{k,\ell} = \Gamma_{-1,\ell} + \sum_{j=0}^k \mathcal{L}_{j,\ell}(\tilde{G}_{j,\ell})^2 \succeq \Gamma_{-1,\ell} = \varsigma I_\ell$.
□

We now consider the impact of using momentum (i.e. $\mu > 0$ and $Z_{k,\ell}$ is a multiple of $M_{k,\ell}$) on the structural identities of Assumption 5.

Lemma A.2 Suppose that $\mu > 0$ and that Assumptions 2 and 5 and (3.61) hold. Then

$$\|Z_{k,\ell}\|_{*,\ell} \langle G_{k,\ell}, S_\ell(Z_{k,\ell}) \rangle_F \geq \frac{1}{\kappa_\square} \text{tr}(\Gamma_{k,\ell}^{-1/2} \mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2) - \text{tr}(\Gamma_{k,\ell}^{-1/2} \mathcal{L}_{k,\ell}(E_{k,\ell})^2) - \|Z_{k,\ell}\|_{*,\ell} \|E_{k,\ell}\|_{*,\ell} \quad (\text{A.2})$$

and

$$\|Z_{k,\ell}\|_{*,\ell}^2 \leq \kappa_\square \text{tr}(\Gamma_{k,\ell}^{-1} \mathcal{L}_{k,\ell}(G_{k,\ell})^2) + \kappa_\square \text{tr}(\Gamma_{k,\ell}^{-1} \mathcal{L}_{k,\ell}(E_{k,\ell})^2) \quad (\text{A.3})$$

Proof. Using (3.48), the linearity of the inner product and (3.51), we have that

$$\begin{aligned} \|Z_{k,\ell}\|_{*,\ell} \langle \tilde{G}_{k,\ell}, S_\ell(Z_{k,\ell}) \rangle_F &= \|Z_{k,\ell}\|_{*,\ell} \langle M_{k,\ell}, S_\ell(Z_{k,\ell}) \rangle_F - \|Z_{k,\ell}\|_{*,\ell} \langle E_{k,\ell}, S_\ell(Z_{k,\ell}) \rangle_F \\ &= \text{tr}(\Gamma_{k,\ell}^{-1/2} \mathcal{L}_{k,\ell}(M_{k,\ell})^2) - \|Z_{k,\ell}\|_{*,\ell} \langle E_{k,\ell}, S_\ell(Z_{k,\ell}) \rangle_F. \end{aligned} \quad (\text{A.4})$$

Now observe that (3.61) and (3.48) give that

$$\mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2 = \mathcal{L}_{k,\ell}(M_{k,\ell} - E_{k,\ell})^2 \preceq \kappa_\square \mathcal{L}_{k,\ell}(M_{k,\ell})^2 + \kappa_\square \mathcal{L}_{k,\ell}(E_{k,\ell})^2$$

Because $\Gamma_{k,\ell}^{-1/2}$ is positive definite (Lemma A.1) and the trace is monotone for the Löwner semi-order, we have that

$$\text{tr}(\Gamma_{k,\ell}^{-1/2} \mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2) \leq \kappa_\square \text{tr}(\Gamma_{k,\ell}^{-1/2} \mathcal{L}_{k,\ell}(M_{k,\ell})^2) + \kappa_\square \text{tr}(\Gamma_{k,\ell}^{-1/2} \mathcal{L}_{k,\ell}(E_{k,\ell})^2).$$

Substituting the resulting lower bound on $\Gamma_{k,\ell}^{-1/2} \mathcal{L}_{k,\ell}(M_{k,\ell})^2$ into (A.4), we deduce that

$$\begin{aligned} \|Z_{k,\ell}\|_{*,\ell} \langle \tilde{G}_{k,\ell}, S_\ell(Z_{k,\ell}) \rangle_F &\geq \frac{1}{\kappa_\square} \text{tr}(\Gamma_{k,\ell}^{-1/2} \mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2) - \text{tr}(\Gamma_{k,\ell}^{-1/2} \mathcal{L}_{k,\ell}(E_{k,\ell})^2) \\ &\quad - \|Z_{k,\ell}\|_{*,\ell} \langle E_{k,\ell}, S_\ell(Z_{k,\ell}) \rangle_F \end{aligned}$$

and (A.2) then results from the fact that, by duality of the primal and dual norms and (2.3),

$$\langle E_{k,\ell}, S_\ell(Z_{k,\ell}) \rangle_F \leq \|E\|_{*,\ell} \|S_\ell(Z_{k,\ell})\|_\ell = \|E\|_{*,\ell}.$$

In order to prove (A.3), we first note that (3.48) and (3.61) yield that

$$\mathcal{L}_{k,\ell}(M_{k,\ell})^2 = \mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell} + E_{k,\ell})^2 \preceq \kappa_\square \mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2 + \kappa_\square \mathcal{L}_{k,\ell}(E_{k,\ell})^2$$

and thus, again using the monotonicity of the trace for the Löwner semi-order, that

$$\mathrm{tr}(\Gamma_{k,\ell}^{-1} \mathcal{L}_{k,\ell}(M_{k,\ell})^2) \preceq \kappa_{\square} \mathrm{tr}(\Gamma_{k,\ell}^{-1} \mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2) + \kappa_{\square} \mathrm{tr}(\Gamma_{k,\ell}^{-1} \mathcal{L}_{k,\ell}(E_{k,\ell})^2).$$

Substituting this inequality in (3.52) then gives (A.3). \square

Observe that (A.2) is (up to a constant factor) a perturbation of a version of (3.51) for block ℓ at iteration k , the perturbation term being $\mathrm{tr}(\Gamma_{k,\ell}^{-1/2} \mathcal{L}_{k,\ell}(E_{k,\ell})^2) + \|Z_{k,\ell}\|_{*,\ell} \|E_{k,\ell}\|_{*,\ell}$. The same is true for (A.3), in which case the perturbation of (3.52) is $\mathrm{tr}(\Gamma_{k,\ell}^{-1} \mathcal{L}_{k,\ell}(E_{k,\ell})^2)$. We now show that the terms occurring in these perturbations can be bounded.

Lemma A.3 Suppose that $\mu > 0$ and that Assumption 2 and 5 and (3.61) hold. Then

$$\sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E}[\|Z_{j,\ell}\|_{*,\ell} \|E_{j,\ell}\|_{*,\ell}] \leq \frac{12\theta_k^2}{(1-\mu_{\max})^2} + \frac{6L_G^2 \eta^2}{(1-\mu_{\max})^2} \sum_{j=0}^k \mu_j^2 \mathbb{E}[\|Z_j\|_*^2] + 2 \sum_{j=0}^k \mathbb{E}[\|Z_j\|_*^2] \quad (\text{A.5})$$

$$\sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E}[\mathrm{tr}(\Gamma_{j,\ell}^{-1/2} \mathcal{L}_{k,\ell}(E_{j,\ell})^2)] \leq \frac{6\kappa_{\diamond} \theta_k^2}{(1-\mu_{\max})^2 \sqrt{\varsigma}} + \frac{3\kappa_{\diamond} L_G^2 \eta^2}{(1-\mu_{\max})^2 \sqrt{\varsigma}} \sum_{j=0}^k \mu_j^2 \mathbb{E}[\|Z_j\|_*^2] \quad (\text{A.6})$$

and

$$\sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E}[\mathrm{tr}(\Gamma_{k,\ell}^{-1} \mathcal{L}_{k,\ell}(E_{k,\ell})^2)] \leq \frac{6\kappa_{\diamond} \theta_k^2}{(1-\mu_{\max})^2 \varsigma} + \frac{3\kappa_{\diamond} L_G^2 \eta^2}{(1-\mu_{\max})^2 \varsigma} \sum_{j=0}^k \mu_j^2 \mathbb{E}[\|Z_j\|_*^2]. \quad (\text{A.7})$$

Proof. For each block $\ell \in \{1, \dots, L\}$ and each iteration $j \in \{0, \dots, k\}$, we have, using $ab \leq 2a^2 + 2b^2$, that

$$\|Z_{k,\ell}\|_{*,\ell} \|E_{k,\ell}\|_{*,\ell} \leq 2\|Z_{k,\ell}\|_{*,\ell}^2 + 2\|E_{k,\ell}\|_{*,\ell}^2$$

and thus

$$\sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E}[\|Z_{j,\ell}\|_{*,\ell} \|E_{j,\ell}\|_{*,\ell}] \leq 2 \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E}[\|Z_{j,\ell}\|_{*,\ell}^2] + 2 \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E}[\|E_{j,\ell}\|_{*,\ell}^2]$$

Substituting now (3.49) in this inequality gives (A.5). Consider now the trace terms. Because of (A.1), the monotonicity of the trace for the Löwner semi-order and Assumption 3.61, we have that

$$\mathrm{tr}(\Gamma_{k,\ell}^{-1/2} \mathcal{L}_{k,\ell}(E_{k,\ell})^2) \leq \frac{1}{\sqrt{\varsigma}} \mathrm{tr}(\mathcal{L}_{k,\ell}(E_{k,\ell})^2) \leq \frac{\kappa_{\diamond}}{\sqrt{\varsigma}} \|E_{k,\ell}\|_{*,\ell}^2$$

Taking full expectation, summing for $j \in \{0, \dots, k\}$ and $\ell \in \{1, \dots, L\}$ and substituting (3.49) then gives (A.6). The proof of (A.7) is entirely similar, using ς instead of $\sqrt{\varsigma}$. \square

We may now substitute the bounds of Lemma A.3 into those of Lemma A.2.

Lemma A.4 Suppose that $\mu > 0$ and that Assumption 2 and 5 and (3.61) hold. Suppose in addition that (3.62) is satisfied. Then

$$\begin{aligned} \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E}[\|Z_{j,\ell}\|_{*,\ell} \langle G_{j,\ell}, S_\ell(Z_{j,\ell}) \rangle_F] &\geq \frac{1}{\kappa_\square} \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E}[\text{tr}(\Gamma_{k,\ell}^{-1/2} \mathcal{L}_{k,\ell}(\tilde{G}_{k,\ell})^2)] \\ &\quad - \kappa_{1,\nu} \theta_k^2 - \kappa_{1,z} \sum_{j=0}^k \mathbb{E}[\|Z_j\|_*^2] \end{aligned} \quad (\text{A.8})$$

with

$$\kappa_{1,\nu} = \frac{6\kappa_\diamond}{(1-\mu_{\max})^2\sqrt{\varsigma}} + \frac{12}{(1-\mu_{\max})^2} \quad \text{and} \quad \kappa_{1,z} = \frac{3\kappa_\diamond\mu_{\max}^2 L_G^2 \eta^2}{(1-\mu_{\max})^2\sqrt{\varsigma}} + \frac{6\mu_{\max}^2 L_G^2 \eta^2}{(1-\mu_{\max})^2} + 2, \quad (\text{A.9})$$

and

$$\sum_{j=0}^k \mathbb{E}[\|Z_j\|_*^2] \leq 2 \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E}[\text{tr}(\Gamma_{k,\ell}^{-1} \mathcal{L}_{k,\ell}(G_{k,\ell})^2)] + \kappa_{2,\nu} \theta_k^2 + \kappa_{2,z}, \quad (\text{A.10})$$

where

$$\kappa_{2,\nu} = \frac{6\kappa_\square\kappa_\diamond}{(1-\mu_{\max})^2\varsigma} \quad \text{and} \quad \kappa_{2,z} = \begin{cases} 0 & \text{if the first part of (3.62) holds,} \\ \frac{3\kappa_\square\kappa_\diamond L_G^2 \eta^2}{(1-\mu_{\max})^2\varsigma} \kappa_{\mu Z} & \text{if the second part of (3.62) holds.} \end{cases} \quad (\text{A.11})$$

Proof. The inequality (A.8) is obtained by substituting (A.5) and (A.6) into (A.2). Moreover, taking the expectation in (A.3), summing for $j \in \{0, \dots, k\}$ and $\ell \in \{1, \dots, L\}$ and substituting (A.7) gives that

$$\begin{aligned} \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E}[\|Z_{k,\ell}\|_{*,\ell}^2] &\leq \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E}[\text{tr}(\Gamma_{k,\ell}^{-1} \mathcal{L}_{k,\ell}(G_{j,\ell})^2)] + \frac{6\kappa_\square\kappa_\diamond}{(1-\mu_{\max})^2\varsigma} \theta_k^2 \\ &\quad + \frac{3\kappa_\square\kappa_\diamond L_G^2 \eta^2}{(1-\mu_{\max})^2\varsigma} \sum_{j=0}^k \mu_j^2 \mathbb{E}[\|Z_j\|_*^2]. \end{aligned}$$

Suppose first that the first part of (3.62) holds. Then, since $\mu_j \leq \mu_{\max}$,

$$\frac{3\kappa_\square\kappa_\diamond\mu_{\max}^2 L^2 \eta^2}{(1-\mu_{\max})^2\varsigma} \leq \frac{1}{2}$$

and (A.10) follows with $\kappa_{2,z} = 0$. Alternatively, if the second part of (3.62) holds, then (A.10) follows with

$$\kappa_{2,z} = \frac{3\kappa_\square\kappa_\diamond L_G^2 \eta^2}{(1-\mu_{\max})^2\varsigma} \kappa_{\mu Z}.$$

□

From this point on, the analysis follows the lines of the argument of Section 3 with Lemma A.4 providing an alternate set of structural inequalities. Lemmas ?? to 3.4 are unchanged. The modifications to Theorem 3.5 are minor. It is restated as follows.

Theorem A.5 Suppose that Assumption 1 and 5 and (3.62) hold. Define Δ_k as in (3.14). Then, for every $k \geq 0$,

$$\eta \mathbb{E} \left[\sum_{\ell=1}^L \text{tr}(\Gamma_{k,\ell}^{1/2}) \right] \leq \kappa_{\text{gap}} + \kappa_{\nu\nu} \theta_k^2 + \eta \kappa_{\nu\Delta} \theta_k \sqrt{\Delta_k} + \eta \kappa_{\Delta} \Delta_k, \quad (\text{A.12})$$

where

$$\kappa_{\text{gap}} \stackrel{\text{def}}{=} \mathbb{E}[f(X_0)] - f_{\text{low}} + \eta \varsigma N + \eta \sqrt{\kappa_{2,z}} + \left(\eta \kappa_{1,z} + \frac{L_G \eta^2}{2} \right) \kappa_{2,z} \quad (\text{A.13})$$

$$\kappa_{\nu\nu} = \eta \left(\kappa_{1,\nu} + \sqrt{\kappa_{2,\nu}} + \kappa_{1,z} \kappa_{2,\nu} + \frac{L_G \eta}{2} \right), \quad \kappa_{\nu\Delta} = \sqrt{2} \quad \text{and} \quad \kappa_{\Delta} = 2\kappa_{1,z} + L_G \eta. \quad (\text{A.14})$$

Proof. In the proof of Theorem 3.5, a perturbation $\eta \kappa_{1,\nu} \theta_k^2 + \eta \kappa_{1,z} \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E} \left[\|Z_{k,\ell}\|_{*,\ell}^2 \right]$ is subtracted from the right-hand side of (3.18) in order to reflect (A.8). The proof then goes on unmodified, up to the substitution leading to (3.23), in which the perturbed (3.18) then gives that

$$\eta \mathbb{E} \left[\sum_{\ell=1}^L \text{tr}(\Gamma_{k,\ell}^{1/2}) - \sum_{\ell=1}^L \text{tr}(\Gamma_{-1,\ell}^{1/2}) \right] \leq f(X_0) - f_{\text{low}} + \eta \kappa_{1,\nu} \theta_k^2 + \eta \theta_k \sqrt{\zeta_k} + \left(\eta \kappa_{1,z} + \frac{L_G \eta^2}{2} \right) \zeta_k.$$

where $\zeta_k = \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E} \left[\|Z_{j,\ell}\|_{*,\ell}^2 \right]$. But this definition, (A.10) and (3.11) give that

$$\zeta_k \leq \kappa_{2,z} + \kappa_{2,\nu} \theta_k^2 + 2 \sum_{j=0}^k \sum_{\ell=1}^L \mathbb{E} \left[\text{tr}(\Gamma_{j,\ell}^{-1} \mathcal{L}_{j,\ell}(G_{j,\ell})^2) \right] \leq \kappa_{2,z} + \kappa_{2,\nu} \theta_k^2 + 2\Delta_k,$$

and thus, using $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$,

$$\begin{aligned} \eta \mathbb{E} \left[\sum_{\ell=1}^L \text{tr}(\Gamma_{k,\ell}^{1/2}) - \sum_{\ell=1}^L \text{tr}(\Gamma_{-1,\ell}^{1/2}) \right] &= f(X_0) - f_{\text{low}} + \eta \sqrt{\kappa_{2,z}} + \eta \left(\kappa_{1,\nu} + \sqrt{\kappa_{2,\nu}} + \kappa_{1,z} \kappa_{2,\nu} + \frac{L_G \eta}{2} \right) \theta_k^2 \\ &\quad + \sqrt{2} \eta \theta_k \sqrt{\Delta_k} + 2 \left(\eta \kappa_{1,z} + \frac{L_G \eta^2}{2} \right) \Delta_k + \left(\eta \kappa_{1,z} + \frac{L_G \eta^2}{2} \right) \kappa_{2,z} \end{aligned}$$

yielding (A.12)-(A.14) after taking into account that $\Gamma_{-1,\ell} = \varsigma I_\ell$ for all $\ell \in \{1, \dots, L\}$. \square

Note that the form of bound (A.12) differs very little from that of (3.15): besides using different constants, (A.12) now involves a term in $\kappa_{\nu\nu} \theta_k^2$.

Lemmas 3.6 and 3.7 are again unmodified. Because of the similarity between (A.12) and (3.15), the proof of Theorem 3.14 is then identical to those of Theorems 3.9 and 3.10, except that the crucial inequality (3.33) in Theorem 3.9 now holds with (3.34) replaced by

$$a = \frac{1}{\eta} (\kappa_{\text{gap}} + \kappa_{\nu\nu} \theta_k^2), \quad b = 2\kappa_{\nu\Delta} \sqrt{N} \quad \text{and} \quad c = 4N\kappa_{\Delta}.$$

Finally, the proof of Corollary 3.15 is very similar to that of Corollary 3.11, except that the simplified (3.67) now implies that, for $\alpha < 1$, $\theta_k^2 = \frac{\sigma_{\text{tot}}^2}{1-\alpha} k^{1-\alpha-2\beta}$ when applying Theorem 3.14. Since $k^{\frac{1}{2}(1-\alpha-2\beta)} \sqrt{\log(k+1)} = \mathcal{O}(k^{1-\alpha-2\beta})$, the term in θ_k^2 in (3.64) dominates and we conclude from (3.66) that

$$\frac{1}{k+1} \sum_{j=0}^k \mathbb{E}[\|G_j\|_*] = \mathcal{O} \left(\frac{1}{(k+1)^{\alpha-\frac{1}{2}}} \right).$$

The case $\alpha = 1$ is also similar to that of Corollary 3.11, except that one has now to consider what happens when $\kappa_{\text{gap}} < \kappa_{\nu\nu}\theta_k^2$. Should this be the case, we obtain from (3.33) in Theorem 3.10 that $t_k \leq 6\kappa_{\nu\Delta} \log(k+1)$, and thus that

$$\frac{1}{k+1} \sum_{j=0}^k \mathbb{E}[\|G_j\|_*] = \mathcal{O}\left(\frac{\log(k+1)}{k+1}\right), \quad (\text{A.15})$$

which does not dominate the rate obtained from the other terms.

We conclude by noting that (3.51) and (3.52) are satisfied for all methods considered because the results from Section 4 still hold when the approximate gradient \tilde{G}_k is set to the momentum M_k . Moreover (3.61) can easily be verified for all methods, as replacing $\tilde{G}_{k,\ell}$ by a general matrix U of suitable size in the derivations involving $\mathcal{L}_{k,\ell}(\cdot)$ implies that (3.61) holds with $\kappa_{\square} = 2$ and $\kappa_{\diamond} = 1$ for all these methods, except that $\kappa_{\diamond} = 1/N$ for AdaNorm.