

# Convex Hulls of Binary Reflected Gray Code Intervals

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## Abstract

The binary reflected Gray code orders the vertices of the unit hypercube along a Hamiltonian path in which consecutive vertices differ in exactly one coordinate. While Gray codes have been extensively studied from a combinatorial perspective, much less is known about the polyhedral structure of convex hulls of contiguous subpaths of this order. This paper develops an exact linear description for Gray intervals based on recursive projections on lower-dimensional subintervals and simple lifting operators. We also study the separation problem for the recursive description, which can be performed in polynomial time by a dynamic program over reachable subintervals. Finally, we specialize the recursion to prefix and suffix Gray intervals and derive compact closed-form descriptions in terms of the binary expansions of the endpoints.

Keywords: integer programming, polyhedral theory, binary reflected Gray code

MSC: 90C10, 90C05

## 1 Introduction

Order restrictions on binary and integer vectors are a recurrent source of structured polyhedra in discrete optimization. In the lexicographic setting, a substantial literature has established exact linear descriptions and compact extended formulations for several families of sets, including superincreasing knapsack sets [14], revlex-initial polytopes [8], lexicographically bounded binary vectors [1], two-sided lexicographical polytopes [15, 9, 5], matrices with lexicographically ordered rows or columns [12, 2], orbitopes [13, 6], and symretopes [10]. A common feature behind these results is that lexicographic comparison is governed by the first coordinate at which two vectors differ, which gives rise to a hierarchical structure that is particularly amenable to polyhedral analysis.

The purpose of this paper is to investigate an analogous question for a fundamentally different ordering, namely the Binary Reflected Gray Code (BRGC). The BRGC is the classical ordering of the vectors of  $\{0, 1\}^n$  in which consecutive vectors differ in exactly one coordinate, and therefore trace a Hamiltonian path on the  $n$ -dimensional hypercube. Gray codes have been studied extensively from a combinatorial viewpoint [17, 16]; however, in contrast with the lexicographic case, the literature is centered on generation and adjacency properties rather than on explicit convex-hull descriptions of contiguous intervals in Gray order. This makes Gray intervals a natural family of 0/1-polytopes whose polyhedral structure appears to be largely unexplored. Interestingly, Gray codes have been used as an encoding device to build extended formulations for nonconvex sets so that the adjacency properties of the code translate into efficient branching rules [11].

For each integer  $t \in \{0, 1, \dots, 2^n - 1\}$ , let  $\text{bin}_n(t) = (b_1, \dots, b_n) \in \{0, 1\}^n$  denote the binary representation of  $t$ , where  $b_1$  is the most significant bit. The corresponding Gray vector  $g_n(t) = (x_1, \dots, x_n)$  is given by  $x_1 = b_1$ ,  $x_j = b_{j-1} \oplus b_j$  for  $j = 2, \dots, n$ , where  $\oplus$  denotes addition modulo 2. Equivalently, if  $G_n$  denotes the BRGC sequence in dimension  $n$ , then  $G_1 = (0, 1)$  and, for  $n \geq 2$ , the

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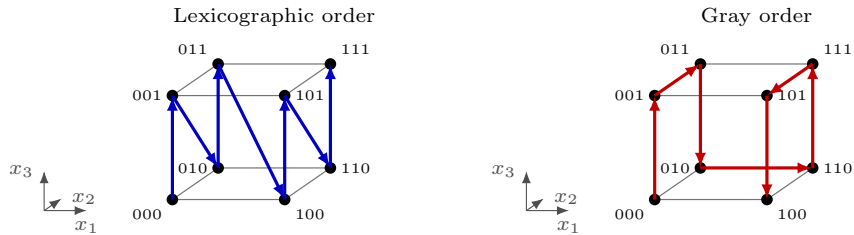


Figure 1: Comparison between the lexicographic order and the reflected Gray order on the vertices of the three-dimensional unit cube.

sequence  $G_n$  is obtained by prefixing a 0 to  $G_{n-1}$  and a 1 to the reversed sequence  $G_{n-1}^R$ , and then concatenating the two lists.

The contrast between the two orders is already visible in dimension three. In lexicographic order, the vertices are listed according to their binary value, and the ordering is organized by prefixes. In Gray order, the same vertices are visited along a Hamiltonian path of the cube: consecutive vertices are adjacent and the recursive reflection reverses the order in the second half of the sequence. Figure 1 compares these two orderings in  $n = 3$ .

We are interested in the convex hulls obtained by selecting consecutive vertices along this Gray path. Thus contiguity is understood with respect to the BRGC index, rather than with respect to lexicographic order or to the coordinatewise order on  $\{0, 1\}^n$ . Given integers  $0 \leq a \leq b \leq 2^n - 1$ , the Gray interval  $[a, b]$  selects the vertices  $g_n(a), g_n(a + 1), \dots, g_n(b)$ , and we define

$$P_n[a, b] := \text{conv}(\{g_n(t) \in \{0, 1\}^n : a \leq t \leq b\}).$$

Equivalently,  $P_n[a, b]$  is the convex hull of the subpath of the BRGC Hamiltonian path induced by the indices  $a, \dots, b$ .

The main contribution of this paper is to exploit a recursive structure of Gray intervals to obtain an exact linear description of  $P_n[a, b]$ . Our construction builds on projection and lifting operations. We develop a polynomial-time separation algorithm for the recursive description and give local criteria that identify which recursively generated inequalities are facet-defining. We also prove that if  $a = 0$  or  $b = 2^n - 1$ , an explicit compact description exists.

The rest of the paper is organized as follows. Section 2 collects the basic polyhedral properties of Gray intervals, including a comparison with the lexicographic ordering, the affine hull, and the projection–fiber representation induced by deleting the last coordinate. Section 3 presents the recursive description, proves validity and exactness of the endpoint-lifting construction, and establishes the replacement arguments needed in the proof. Section 4 studies the separation problem for the recursive system and shows that it can be solved in polynomial time by a dynamic program over reachable intervals and boundary signatures. Section 5 specializes the recursion to prefix and suffix Gray intervals, deriving compact closed-form descriptions directly from the binary expansions of the endpoints. Finally, Section 6 concludes with closing remarks and directions for future research.

## 2 Polyhedral properties

We begin by showing that Gray intervals are not simply lexicographic or knapsack intervals in disguise. We then record the structural properties that drive the recursive description.

### 2.1 Comparison of lexicographic and Gray orders

**Example 2.1** (Linear score). In the lexicographic order on  $\{0, 1\}^n$ , the position of a binary vector is induced by the linear score  $h(x) = \sum_{j=1}^n 2^{n-j} x_j$ . The analogous statement is false for the binary

reflected Gray code. In dimension two, the Gray order is  $g_2(0) = (0, 0)$ ,  $g_2(1) = (0, 1)$ ,  $g_2(2) = (1, 1)$ ,  $g_2(3) = (1, 0)$ . Suppose that this order were induced by a linear score  $h(x) = w_1x_1 + w_2x_2$ . Then we would need  $h(0, 0) < h(0, 1) < h(1, 1) < h(1, 0)$ , that is,  $0 < w_2 < w_1 + w_2 < w_1$ . The first inequality gives  $w_2 > 0$ , whereas the last one gives  $w_2 < 0$ , a contradiction. Hence the Gray order is not the order induced by any linear function.

**Example 2.2** (Two-sided knapsack sets). Gray intervals are not, in general, convex hulls of two-sided knapsack sets of the form  $\{x \in \{0, 1\}^n : \ell \leq w^\top x \leq u\}$ . To see this, consider the three-dimensional Gray sequence  $(0, 0, 0)$ ,  $(0, 0, 1)$ ,  $(0, 1, 1)$ ,  $(0, 1, 0)$ ,  $(1, 1, 0)$ ,  $(1, 1, 1)$ ,  $(1, 0, 1)$ ,  $(1, 0, 0)$ . Then  $P_3[1, 4] = \text{conv}(S)$ , where  $S := \{(0, 0, 1), (0, 1, 1), (0, 1, 0), (1, 1, 0)\}$ .

Suppose, for a contradiction, that  $S = \{x \in \{0, 1\}^3 : \ell \leq \alpha x_1 + \beta x_2 + \gamma x_3 \leq u\}$ . Since  $(0, 0, 0) \notin S$ , we have  $0 \notin [\ell, u]$ . Replacing  $(\alpha, \beta, \gamma, \ell, u)$  by  $(-\alpha, -\beta, -\gamma, -u, -\ell)$ , if necessary, we may assume  $0 < \ell$ . The selected vertices  $(0, 0, 1)$ ,  $(0, 1, 1)$ ,  $(0, 1, 0)$ ,  $(1, 1, 0)$  imply  $\gamma, \beta + \gamma, \beta, \alpha + \beta \in [\ell, u]$ . In particular,  $\beta \geq \ell > 0$ ,  $\gamma \geq \ell > 0$ ,  $\alpha + \beta \geq \ell$ .

Since  $(1, 1, 1) \notin S$ , while  $\alpha + \beta + \gamma > \ell$ , its score cannot lie below the interval. Hence  $\alpha + \beta + \gamma > u$ . But  $(0, 1, 1) \in S$ , so  $\beta + \gamma \leq u$ . Therefore  $\alpha > 0$ . Now  $(1, 0, 0) \notin S$ . Since  $(1, 1, 0) \in S$ , we have  $\alpha + \beta \leq u$ . Because  $\beta > 0$ , this gives  $\alpha < u$ . Thus the excluded score  $\alpha$  cannot lie above the interval, and hence  $\alpha < \ell \leq \beta$ . Therefore  $\alpha < \beta$ . Finally,  $(1, 0, 1) \notin S$ . Since  $\alpha > 0$  and  $\gamma \geq \ell$ , we have  $\alpha + \gamma > \ell$ . Thus this excluded score cannot lie below the interval, so  $\alpha + \gamma > u$ . But  $(0, 1, 1) \in S$  gives  $\beta + \gamma \leq u$ . Hence  $\alpha > \beta$ , a contradiction.

**Example 2.3** (Intersection of one-sided knapsack sets). The identity  $P_n[a, b] = P_n[0, b] \cap P_n[a, 2^n - 1]$  does not hold in general, while an analogous statement for the lexicographic order holds true [15, 9, 5]. In dimension two, the Gray order is  $(0, 0)$ ,  $(0, 1)$ ,  $(1, 1)$ ,  $(1, 0)$ . Thus  $P_2[1, 2] = \text{conv}(\{(0, 1), (1, 1)\})$ . On the other hand,  $P_2[0, 2] = \text{conv}(\{(0, 0), (0, 1), (1, 1)\})$ ,  $P_2[1, 3] = \text{conv}(\{(0, 1), (1, 1), (1, 0)\})$ . Their intersection contains the fractional point  $(1/2, 1/2)$ , since it lies in both triangles. However,  $(1/2, 1/2) \notin P_2[1, 2]$ , because every point in  $P_2[1, 2]$  has second coordinate equal to one. Therefore  $P_2[1, 2] \not\subseteq P_2[0, 2] \cap P_2[1, 3]$ .

We now turn from these negative examples to the recursive structure of Gray intervals.

## 2.2 Affine hull

The affine hull of a Gray interval is determined by the coordinate directions used by the Gray path inside the interval.

For a positive integer  $t$ , let  $\nu(t) := \max\{r \geq 0 : 2^r \text{ divides } t\}$ .

**Lemma 2.4** (Difference of consecutive Gray vectors). *For every  $t = 1, \dots, 2^n - 1$ , we have  $g_n(t) - g_n(t - 1) = \pm e^{n - \nu(t)}$ .*

*Proof.* Fix  $t \in \{1, \dots, 2^n - 1\}$  and let  $r := \nu(t)$ ,  $p := n - r$ . Let  $\text{bin}_n(t - 1) = (y_1, \dots, y_n)$ ,  $\text{bin}_n(t) = (y'_1, \dots, y'_n)$ . Since  $r = \nu(t)$ , when passing from  $t - 1$  to  $t$  in binary, the bit in position  $p = n - r$  changes from 0 to 1, all positions before  $p$  remain unchanged, and all positions after  $p$  change from 1 to 0. That is,  $y'_j = y_j$  for  $j < p$ ,  $y_p = 0$ ,  $y'_p = 1$ , and  $y_j = 1$ ,  $y'_j = 0$  for  $j > p$ .

The Gray encoding is  $x_1 = y_1$ ,  $x_j = y_{j-1} \oplus y_j$  for  $j = 2, \dots, n$ . We claim that only the  $p$ -th Gray coordinate changes. Indeed, if  $j < p$ , then the binary bits involved in the definition of the  $j$ -th Gray coordinate do not change. If  $j > p$ , then the two binary bits involved either both change or both remain unchanged. In either case, their exclusive-or is unchanged. At coordinate  $p$ , exactly one of the binary bits defining the Gray coordinate changes. For  $p = 1$ , this simply means that  $x_1 = y_1$  changes. For  $p \geq 2$ , this means that  $y_{p-1} \oplus y_p$  changes because  $y_{p-1}$  remains fixed while  $y_p$  changes. Hence  $g_n(t) - g_n(t - 1) = \pm e^p = \pm e^{n - \nu(t)}$ .  $\square$

**Proposition 2.5** (Affine hull and dimension of a Gray interval). *Let  $0 \leq a \leq b \leq 2^n - 1$  and define  $J_n(a, b) := \{n - \nu(t) : a + 1 \leq t \leq b\}$ . Then, the affine hull of  $P_n[a, b]$  is*

$$\text{aff}(P_n[a, b]) = g_n(a) + \text{span}\{e^j : j \in J_n(a, b)\} = \{x \in \mathbb{R}^n : x_j = g_n(a)_j \text{ for every } j \notin J_n(a, b)\}.$$

In particular,  $\dim P_n[a, b] = |J_n(a, b)|$ .

Moreover, we also have  $\dim P_n[a, b] = \sum_{r=0}^{n-1} \mathbf{1}\{N_r(a, b) > 0\}$ , where  $N_r(a, b) := \lfloor \frac{b}{2^r} \rfloor - \lfloor \frac{a}{2^r} \rfloor - \lfloor \frac{b}{2^{r+1}} \rfloor + \lfloor \frac{a}{2^{r+1}} \rfloor$ .

*Proof.* Let  $L := \text{lin}(P_n[a, b])$  be the linear space parallel to  $\text{aff}(P_n[a, b])$  and write  $v_t := g_n(t)$ . Then  $L = \text{span}\{v_t - v_a : a + 1 \leq t \leq b\}$ . For every  $t \in \{a + 1, \dots, b\}$ ,  $v_t - v_a = \sum_{q=a+1}^t (v_q - v_{q-1})$ . Therefore  $L \subseteq \text{span}\{v_q - v_{q-1} : a + 1 \leq q \leq b\}$ . The reverse inclusion also holds, because each difference  $v_q - v_{q-1}$ ,  $q = a + 1, \dots, b$ , is the difference of two vertices of  $P_n[a, b]$ , and hence belongs to  $L$ . Thus  $L = \text{span}\{v_q - v_{q-1} : a + 1 \leq q \leq b\}$ . Using Lemma 2.4, we obtain  $L = \text{span}\{e^{n-\nu(q)} : a + 1 \leq q \leq b\} = \text{span}\{e^j : j \in J_n(a, b)\}$ . Consequently,  $\text{aff}(P_n[a, b]) = v_a + L = g_n(a) + \text{span}\{e^j : j \in J_n(a, b)\}$ . This affine space has precisely the coordinates indexed by  $J_n(a, b)$  as free coordinates. Every coordinate outside  $J_n(a, b)$  is fixed to its value at  $g_n(a)$ . Hence  $\text{aff}(P_n[a, b]) = \{x \in \mathbb{R}^n : x_j = g_n(a)_j \text{ for every } j \notin J_n(a, b)\}$ . Since distinct standard unit vectors are linearly independent,  $\dim P_n[a, b] = \dim L = |J_n(a, b)|$ .

We now prove the floor formula. Fix  $r \in \{0, \dots, n - 1\}$ . The number of integers  $t \in \{a + 1, \dots, b\}$  satisfying  $\nu(t) = r$  is equal to the number of multiples of  $2^r$  in that interval minus the number of multiples of  $2^{r+1}$  in that interval. Hence this number is  $N_r(a, b) = \lfloor \frac{b}{2^r} \rfloor - \lfloor \frac{a}{2^r} \rfloor - \lfloor \frac{b}{2^{r+1}} \rfloor + \lfloor \frac{a}{2^{r+1}} \rfloor$ . Thus the valuation  $r$  appears among the integers  $a + 1, \dots, b$  if and only if  $N_r(a, b) > 0$ . Therefore  $\dim P_n[a, b] = \sum_{r=0}^{n-1} \mathbf{1}\{N_r(a, b) > 0\}$ .  $\square$

## 2.3 Projection

The second ingredient is the behavior of Gray intervals under deletion of the last coordinate.

**Lemma 2.6** (Last-coordinate projection). *Let  $s \geq 2$ ,  $0 \leq t \leq 2^s - 1$ , and write  $t = 2q + \varepsilon$ ,  $\varepsilon \in \{0, 1\}$ . Then  $\pi_{s-1}(g_s(t)) = g_{s-1}(q)$  and  $g_s(t)_s = (q \bmod 2) \oplus \varepsilon$ , where  $\pi_{s-1}$  keeps the first  $s - 1$  coordinates. Consequently,  $\pi_{s-1}(P_s[a, b]) = P_{s-1}[\lfloor a/2 \rfloor, \lfloor b/2 \rfloor]$ .*

*Proof.* Write  $\text{bin}_s(t) = (b_1, \dots, b_s)$ . If  $t = 2q + \varepsilon$ , then the first  $s - 1$  binary bits encode  $q$ , while the last binary bit is  $\varepsilon$ . The first  $s - 1$  Gray coordinates of  $g_s(t)$  are  $b_1, b_1 \oplus b_2, \dots, b_{s-2} \oplus b_{s-1}$ , thus  $\pi_{s-1}(g_s(t)) = g_{s-1}(q)$ . The last Gray coordinate is  $g_s(t)_s = b_{s-1} \oplus b_s = (q \bmod 2) \oplus \varepsilon$ . The projection identity for intervals follows because linear projection commutes with convex hull.  $\square$

The next lemma turns this projection identity into the product-with-deleted endpoint-fibers representation used throughout the paper.

Let  $s \geq 2$  and let  $0 \leq a < b \leq 2^s - 1$ . Put  $l := \lfloor a/2 \rfloor$ ,  $u := \lfloor b/2 \rfloor$ ,  $\tau_l := (l \bmod 2) \oplus 1$ , and  $\tau_u := u \bmod 2$ .

**Lemma 2.7** (Product representation with deleted endpoint fibers). *Let  $\mathcal{V}_{l,u} := \{(g_{s-1}(q), z) : q = l, \dots, u, z \in \{0, 1\}\}$ . Then  $P_{s-1}[l, u] \times [0, 1] = \text{conv}(\mathcal{V}_{l,u})$  and  $P_s[a, b] = \text{conv}(\mathcal{V}_{l,u} \setminus D(a, b))$ , where  $D(a, b) := D_l(a, b) \cup D_u(a, b)$ , with*

$$D_l(a, b) := \begin{cases} \{(g_{s-1}(l), 1 - \tau_l)\}, & a \text{ is odd,} \\ \emptyset, & a \text{ is even,} \end{cases}$$

and

$$D_u(a, b) := \begin{cases} \{(g_{s-1}(u), 1 - \tau_u)\}, & b \text{ is even,} \\ \emptyset, & b \text{ is odd.} \end{cases}$$

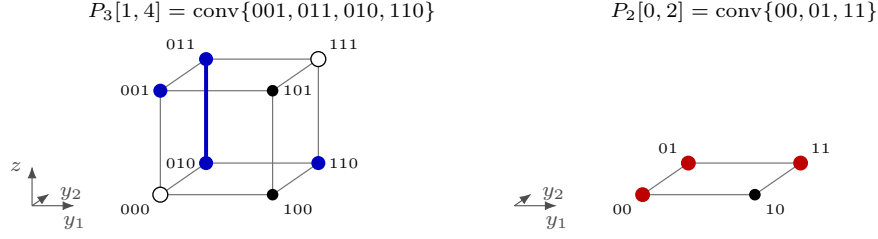


Figure 2: Fiber representation of the Gray interval  $P_3[1, 4]$ .

*Proof.* We have  $P_{s-1}[l, u] \times [0, 1] = \text{conv}(\{g_{s-1}(q) : q = l, \dots, u\}) \times \text{conv}(\{0, 1\}) = \text{conv}(\{(g_{s-1}(q), z) : q = l, \dots, u, z \in \{0, 1\}\}) = \text{conv}(\mathcal{V}_{l,u})$ .

Now, by Lemma 2.6, if  $t = 2q + \varepsilon$ , with  $\varepsilon \in \{0, 1\}$ , then  $g_s(t) = (g_{s-1}(q), (q \bmod 2) \oplus \varepsilon)$ . Thus the vertices of  $P_s[a, b]$  are obtained from the product vertices over  $q = l, \dots, u$ , except possibly at the two projected endpoints. Indeed, for every  $l < q < u$ , both indices  $2q$  and  $2q + 1$  belong to  $[a, b]$ , so the full fiber over  $g_{s-1}(q)$  is present. At the left endpoint, if  $a$  is even then both fiber values over  $g_{s-1}(l)$  are present, whereas if  $a$  is odd only the value  $g_s(a)_s = (l \bmod 2) \oplus 1 = \tau_l$  is present, so the complementary point  $(g_{s-1}(l), 1 - \tau_l)$  is deleted. Similarly, at the right endpoint, if  $b$  is odd then both fiber values over  $g_{s-1}(u)$  are present, whereas if  $b$  is even only the value  $g_s(b)_s = (u \bmod 2) \oplus 0 = \tau_u$  is present, so the complementary point  $(g_{s-1}(u), 1 - \tau_u)$  is deleted. Hence  $P_s[a, b] = \text{conv}(\mathcal{V}_{l,u} \setminus D(a, b))$ .  $\square$

Figure 2 illustrates Lemma 2.7 on  $P_3[1, 4]$ . Its projection onto the first two coordinates is  $P_2[0, 2]$ . Over the interior projected vertex  $(0, 1)$ , the full last-coordinate fiber is present, whereas at the two projected endpoints exactly one endpoint of the fiber is retained. Equivalently,  $P_3[1, 4]$  is obtained from  $P_2[0, 2] \times [0, 1]$  by deleting the complementary endpoint fibers  $(0, 0, 0)$  and  $(1, 1, 1)$ .

We now record the corresponding recursive form of the affine hull.

**Lemma 2.8** (Recursive form of the affine hull). *If  $[a+1, b]$  contains an odd integer, then  $\text{aff}(P_s[a, b]) = \text{aff}(P_{s-1}[l, u]) \times \mathbb{R}$ . Otherwise, necessarily  $a = 2l + 1$ ,  $b = a + 1 = 2l + 2$ ,  $u = l + 1$ , and, with  $\tau := (l \bmod 2) \oplus 1 = (u \bmod 2)$ , we have  $\text{aff}(P_s[a, b]) = \text{aff}(P_{s-1}[l, u]) \times \{\tau\}$ .*

*Proof.* By Proposition 2.5, the free coordinates of  $P_s[a, b]$  are the indices in  $J_s(a, b)$ . For  $j < s$ , a transition in coordinate  $j$  of the  $s$ -dimensional Gray path can only come from an even index  $t = 2q$ . Since  $\nu(2q) = \nu(q) + 1$ , we have  $s - \nu(2q) = (s - 1) - \nu(q)$ . As the even integers in  $[a + 1, b]$  are precisely  $2q$  with  $q = l + 1, \dots, u$ , this gives  $J_s(a, b) \cap \{1, \dots, s - 1\} = J_{s-1}(l, u)$ .

The last coordinate  $s$  belongs to  $J_s(a, b)$  exactly when  $\nu(t) = 0$  for some  $t \in [a + 1, b]$ , that is, exactly when  $[a + 1, b]$  contains an odd integer. Proposition 2.5 then gives the first affine-hull identity.

If  $[a + 1, b]$  contains no odd integer, then the nonempty interval  $[a + 1, b]$  consists of a single even integer. Hence  $b = a + 1$ ,  $a = 2l + 1$ ,  $b = 2l + 2$ , and  $u = l + 1$ . In this case all vertices of  $P_s[a, b]$  have last coordinate  $\tau = (l \bmod 2) \oplus 1 = (u \bmod 2)$ , and Proposition 2.5 again yields  $\text{aff}(P_s[a, b]) = \text{aff}(P_{s-1}[l, u]) \times \{\tau\}$ .  $\square$

### 3 A complete description

We now translate the projection–fiber representation into linear inequalities. The construction keeps, for each interval, both a proposed description and auxiliary normalized rows that can be lifted when endpoint fibers are deleted.

Throughout the section, an inequality is written as an affine row

$$C = (c, \rho), \quad C(x) := c^\top x - \rho \leq 0.$$

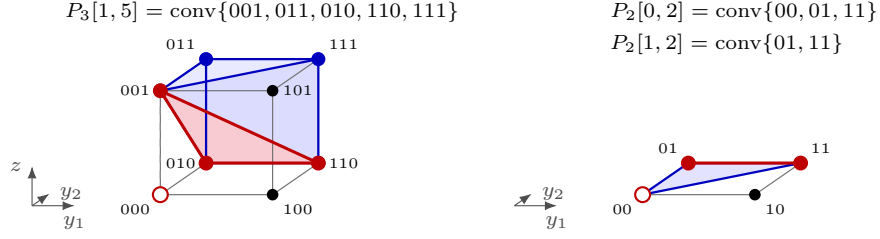


Figure 3: Unilateral lift in the Gray interval  $P_3[1, 5]$ .

For a Gray vertex we define its excess with respect to  $C$  by  $\text{exc}_s(C; t) := C(g_s(t))$ . Thus  $C$  is valid for  $P_s[a, b]$  if and only if  $\text{exc}_s(C; t) \leq 0$  for all  $t = a, \dots, b$ .

### 3.1 Lifting operators

For each  $1 \leq s \leq n$  and  $0 \leq a \leq b \leq 2^s - 1$ , we construct four families of rows:  $\mathcal{H}_s[a, b]$ ,  $\mathcal{L}_s[a, b]$ ,  $\mathcal{R}_s[a, b]$ , and  $\mathcal{B}_s[a, b]$ . The family  $\mathcal{H}_s[a, b]$  is the proposed description of  $P_s[a, b]$ . The auxiliary families store normalized rows that separate deleted endpoints. Whenever the intervals involved are nonempty, define

$$\mathcal{L}_s[a, b] := \{C \in \mathcal{H}_s[a + 1, b] : \text{exc}_s(C; a) = 1\},$$

$$\mathcal{R}_s[a, b] := \{C \in \mathcal{H}_s[a, b - 1] : \text{exc}_s(C; b) = 1\},$$

and

$$\mathcal{B}_s[a, b] := \{C \in \mathcal{H}_s[a + 1, b - 1] : \text{exc}_s(C; a) = \text{exc}_s(C; b) = 1\}.$$

If the indicated interval is empty, the corresponding family is empty.

Given  $\tau \in \{0, 1\}$ , let  $\theta_\tau(z) := (1 - 2\tau)z - (1 - \tau)$  for  $z \in \mathbb{R}$ . Thus  $\theta_0(z) = z - 1$  and  $\theta_1(z) = -z$ . If  $z \in [0, 1]$ , we have  $\theta_\tau(z) \in [-1, 0]$ , and  $\theta_\tau(z) = 0$  if and only if  $z = 1 - \tau$ . We use three lifting operations for rows  $C(y) \leq 0$  defined on  $y \in \mathbb{R}^{s-1}$ .

First, the zero extension of a row  $C(y) \leq 0$  is simply

$$\text{Ext}(C)(y, z) := C(y) \leq 0.$$

Second, consider the interval  $P_{s-1}[l, u]$ , where  $l := \lfloor a/2 \rfloor$ ,  $u := \lfloor b/2 \rfloor$ . Suppose  $C(y) \leq 0$  is valid after deleting a projected endpoint  $q \in \{l, u\}$ , and assume that it is normalized by  $C(g_{s-1}(q)) = 1$ . If the only allowed last-coordinate value above  $g_{s-1}(q)$  is  $\tau \in \{0, 1\}$ , define the unilateral lift

$$\text{Lift}_{q, \tau}(C)(y, z) := C(y) + \theta_\tau(z) = C(y) + (1 - 2\tau)z - (1 - \tau) \leq 0.$$

Third, suppose  $C(y) \leq 0$  is valid for the interior projected interval  $P_{s-1}[l + 1, u - 1]$  and is normalized at both projected endpoints:  $C(g_{s-1}(l)) = C(g_{s-1}(u)) = 1$ . When the two endpoint fibers have the same allowed last-coordinate value  $\tau \in \{0, 1\}$ , we define the bilateral lift by

$$\text{BLift}_{l, u}(C)(y, z) := C(y) + \theta_\tau(z) = C(y) + (1 - 2\tau)z - (1 - \tau) \leq 0.$$

Figure 3 depicts a unilateral lift on  $P_3[1, 5]$ . The projected row  $C(y) = 1 - y_2 \leq 0$ , valid on  $P_2[1, 2]$ , has unit excess at the deleted projected endpoint  $(0, 0)$ . Since  $\tau_l = 1$ , its unilateral lift is  $C(y) - z \leq 0$ , namely  $1 - y_2 - z \leq 0$ , exposing the facet  $\text{conv}(\{(0, 0, 1), (0, 1, 0), (1, 1, 0)\})$ . Similarly, Figure 4 depicts a bilateral lift on  $P_3[1, 6]$ . The projected row  $C(y) = 1 - y_2 \leq 0$ , valid on  $P_2[1, 2]$ , this time has unit excess at both projected endpoints  $(0, 0)$  and  $(1, 0)$ . Since  $\tau_l = \tau_u = 1$ , its bilateral lift is  $C(y) - z \leq 0$ , namely  $1 - y_2 - z \leq 0$ , exposing the facet  $\text{conv}(\{(0, 0, 1), (0, 1, 0), (1, 1, 0), (1, 0, 1)\})$ .

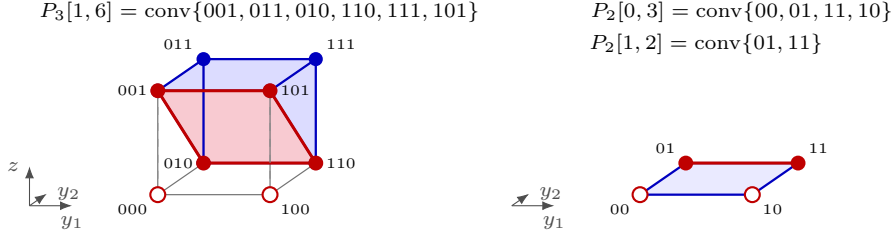


Figure 4: Bilateral lift in the Gray interval  $P_3[1, 6]$ .

### 3.2 The recursive system

The families are defined by induction on  $s$ . The base cases are singleton and edge intervals. Every remaining interval is described from its projection by zero extensions, last-coordinate bounds, and the endpoint lifts introduced above.

For  $s = 1$ , we set  $\mathcal{H}_1[0, 0] := \{-x_1 \leq 0, x_1 \leq 0\}$ ,  $\mathcal{H}_1[1, 1] := \{-x_1 + 1 \leq 0, x_1 - 1 \leq 0\}$ ,  $\mathcal{H}_1[0, 1] := \{-x_1 \leq 0, x_1 - 1 \leq 0\}$ .

Now let  $s \geq 2$ . If  $b = a$ , set  $\mathcal{H}_s[a, a] := \{x_i - g_s(a)_i \leq 0, -x_i + g_s(a)_i \leq 0 : i = 1, \dots, s\}$ . If  $b = a + 1$ , from Lemma 2.4 we have  $g_s(a + 1) - g_s(a) = \pm e^{s-\nu(a+1)}$ . Defining  $r := s - \nu(a + 1)$ , set  $\mathcal{H}_s[a, a + 1] := \{x_i - g_s(a)_i \leq 0, -x_i + g_s(a)_i \leq 0 : i = 1, \dots, s, i \neq r\} \cup \{x_r - 1 \leq 0, -x_r \leq 0\}$ .

Assume  $b \geq a + 2$ . Put  $l := \lfloor a/2 \rfloor$ ,  $u := \lfloor b/2 \rfloor$ . Also define  $\tau_l := (l \bmod 2) \oplus 1$ ,  $\tau_u := u \bmod 2$ . In this case,  $[a + 1, b]$  contains an odd integer. Then, by Lemma 2.8,  $\text{aff}(P_s[a, b]) = \text{aff}(P_{s-1}[l, u]) \times \mathbb{R}$ . The family  $\mathcal{H}_s[a, b]$  consists of the following rows.

1. All zero extensions of rows in  $\mathcal{H}_{s-1}[l, u]$ .
2. The two last-coordinate bounds  $-x_s \leq 0, x_s - 1 \leq 0$ .
3. If  $a$  is odd, all left unilateral lifts  $\text{Lift}_{l, \tau_l}(C)$ ,  $C \in \mathcal{L}_{s-1}[l, u]$ .
4. If  $b$  is even, all right unilateral lifts  $\text{Lift}_{u, \tau_u}(C)$ ,  $C \in \mathcal{R}_{s-1}[l, u]$ .
5. If  $a$  is odd,  $b$  is even, and  $\tau_l = \tau_u$ , all bilateral lifts  $\text{BLift}_{l, u}(C)$ , where either  $C \in \mathcal{B}_{s-1}[l, u]$ , or  $C = C^L + C^R$ ,  $C^L \in \mathcal{L}_{s-1}[l, u]$ ,  $C^R \in \mathcal{R}_{s-1}[l, u]$ , and the normalized sum satisfies  $\text{exc}_{s-1}(C; l) = \text{exc}_{s-1}(C; u) = 1$ .

### 3.3 Validity and endpoint excess

The next two propositions establish the basic invariants of the construction: validity on the target interval and unit excess at adjacent exterior vertices.

**Proposition 3.1** (Validity). *Every row in  $\mathcal{H}_s[a, b]$  is valid for  $P_s[a, b]$ .*

*Proof.* We argue by induction on  $s$ . The initial cases are immediate. Let  $l := \lfloor a/2 \rfloor$ ,  $u := \lfloor b/2 \rfloor$ , and write a point of  $\mathbb{R}^s$  as  $(y, z)$ , where  $y \in \mathbb{R}^{s-1}$  and  $z = x_s$ .

First consider the rows inherited from the projected interval. If  $C(y) \leq 0$  is valid for  $P_{s-1}[l, u]$ , then its extension  $C(y) \leq 0$ , viewed in the variables  $(y, z)$ , is valid for  $P_s[a, b]$ , because the projection of  $P_s[a, b]$  onto the first  $s - 1$  coordinates is contained in  $P_{s-1}[l, u]$ . The last-coordinate bounds  $-z \leq 0, z - 1 \leq 0$  are valid because  $P_s[a, b] \subseteq [0, 1]^s$ .

Now consider a unilateral lift. Let  $q$  be the projected endpoint restored by the lift, and let  $\tau \in \{0, 1\}$  be the last coordinate of the restored endpoint. The row being lifted is valid for the projected interval with  $q$  deleted and is normalized so that  $C(g_{s-1}(q)) = 1$ . Every point of the lifted Gray interval is either in the corresponding product over the projected interval with  $q$  deleted, or is the restored

endpoint  $(g_{s-1}(q), \tau)$ . If  $\tau = 0$ , the lifted row is  $C(y) + z - 1 \leq 0$ . On the product part we have  $C(y) \leq 0$  and  $z \leq 1$ , so  $C(y) + z - 1 \leq 0$ . At the restored endpoint,  $C(g_{s-1}(q)) + 0 - 1 = 1 - 1 = 0$ . If  $\tau = 1$ , the lifted row is  $C(y) - z \leq 0$ . On the product part we have  $C(y) \leq 0$  and  $z \geq 0$ , so  $C(y) - z \leq 0$ . At the restored endpoint,  $C(g_{s-1}(q)) - 1 = 1 - 1 = 0$ . Thus every unilateral lift generated by the recursion is valid.

It remains to consider bilateral lifts. Let  $C$  be the row from which the bilateral lift is generated. In every bilateral branch, the construction ensures that  $C(y) \leq 0$  for all  $y \in P_{s-1}[l+1, u-1]$ , and that the two exterior endpoints have unit excess,  $C(g_{s-1}(l)) = C(g_{s-1}(u)) = 1$ . Indeed, if  $C \in \mathcal{B}_{s-1}[l, u]$ , this is part of the definition of  $\mathcal{B}_{s-1}[l, u]$ . If  $C = C^L + C^R$ ,  $C^L \in \mathcal{L}_{s-1}[l, u]$ ,  $C^R \in \mathcal{R}_{s-1}[l, u]$ , then both  $C^L$  and  $C^R$  are valid on the common interior projected interval  $P_{s-1}[l+1, u-1]$ , and hence so is their sum. The unit-excess conditions  $C(g_{s-1}(l)) = C(g_{s-1}(u)) = 1$  are precisely the defining conditions required for adding the normalized sum to the bilateral branch.

The vertices of  $P_s[a, b]$  are contained in the product  $P_{s-1}[l+1, u-1] \times [0, 1]$  together with the two restored endpoints  $(g_{s-1}(l), \tau_l)$ ,  $(g_{s-1}(u), \tau_u)$ . For the bilateral lifts generated by the recursion, the restored endpoints have the same last coordinate. If  $\tau_l = \tau_u = 0$ , the lifted row is  $C(y) + z - 1 \leq 0$ . It is valid on the product part because  $C(y) \leq 0$  and  $z \leq 1$ , and it is tight at both restored endpoints because  $C(g_{s-1}(l)) + 0 - 1 = C(g_{s-1}(u)) + 0 - 1 = 0$ . If  $\tau_l = \tau_u = 1$ , the lifted row is  $C(y) - z \leq 0$ . It is valid on the product part because  $C(y) \leq 0$  and  $z \geq 0$ , and it is tight at both restored endpoints because  $C(g_{s-1}(l)) - 1 = C(g_{s-1}(u)) - 1 = 0$ . Therefore every bilateral lift generated by the recursion is valid.  $\square$

**Proposition 3.2** (Unit exterior excess). *Every row generated by the recursion has integral coefficients and integral right-hand side. Moreover, let  $C \in \mathcal{H}_s[a, b]$ . If  $a > 0$  and  $\text{exc}_s(C; a-1) > 0$ , then  $\text{exc}_s(C; a-1) = 1$ . Similarly, if  $b < 2^s - 1$  and  $\text{exc}_s(C; b+1) > 0$ , then  $\text{exc}_s(C; b+1) = 1$ . Consequently,  $C \in \mathcal{H}_s[a+1, b]$  and  $\text{exc}_s(C; a) > 0$  imply  $C \in \mathcal{L}_s[a, b]$ , and  $C \in \mathcal{H}_s[a, b-1]$  and  $\text{exc}_s(C; b) > 0$  imply  $C \in \mathcal{R}_s[a, b]$ .*

*Proof.* Integrality is immediate from the initial rows and from the fact that zero extension, unilateral lifting, bilateral lifting, and row addition preserve integrality. We prove the exterior-excess statement by induction over the recursive construction of the rows.

The case  $s = 1$  is immediate. For singleton and segment intervals, the rows are coordinate equalities. At a binary vertex, every positive violation of such a row is equal to one.

Let  $s \geq 2$ ,  $b \geq a + 2$ , and put  $l := \lfloor a/2 \rfloor$ ,  $u := \lfloor b/2 \rfloor$ . We check each possible type of generated row.

First consider a zero extension  $C = \text{Ext}(D)$ . Then, for every  $0 \leq t \leq 2^s - 1$ ,  $\text{exc}_s(C; t) = \text{exc}_{s-1}(D; \lfloor t/2 \rfloor)$ . At the left exterior endpoint  $a - 1$ , if  $a$  is odd then  $a - 1$  projects to  $l$ , which belongs to the projected interval  $[l, u]$ ; hence no positive violation is possible. If  $a$  is even, then  $a - 1$  projects to  $l - 1$ , and the induction hypothesis gives unit excess whenever the violation is positive. At the right exterior endpoint  $b + 1$ , if  $b$  is even then  $b + 1$  projects to  $u$ , which belongs to  $[l, u]$ ; hence no positive violation is possible. If  $b$  is odd, then  $b + 1$  projects to  $u + 1$ , and the induction hypothesis again gives unit excess whenever the violation is positive.

The bound rows  $-x_s \leq 0$ ,  $x_s - 1 \leq 0$  never have positive violation at binary vertices.

Suppose now that  $C$  is a left unilateral lift. Then  $a = 2l + 1$ , and  $C(y, z) = D(y) + \theta_{\tau_l}(z)$ , where  $D \in \mathcal{L}_{s-1}[l, u]$  and  $D(g_{s-1}(l)) = 1$ . At the left exterior endpoint  $a - 1 = 2l$ , the projected point is  $g_{s-1}(l)$ , and the last coordinate is  $1 - \tau_l$ . Therefore  $\text{exc}_s(C; a-1) = D(g_{s-1}(l)) + \theta_{\tau_l}(1 - \tau_l) = 1 + 0 = 1$ . At the right exterior endpoint  $b + 1$ , if the projected point lies in  $[l + 1, u]$ , then  $D \leq 0$ , and since  $\theta_{\tau_l} \leq 0$ , no positive violation is possible. If the projected point is  $u + 1$ , then any positive violation of  $C$  implies a positive violation of  $D$  at  $u + 1$ . By induction this projected excess is equal to one. Since  $\theta_{\tau_l}(z) \leq 0$ , positivity of the lifted excess forces  $\theta_{\tau_l}(z) = 0$ , and hence the lifted excess is exactly one.

The case of a right unilateral lift is symmetric. Indeed, if  $C(y, z) = D(y) + \theta_{\tau_u}(z)$ ,  $D \in \mathcal{R}_{s-1}[l, u]$ ,  $D(g_{s-1}(u)) = 1$ , then the right exterior endpoint  $b + 1$  has projected point  $g_{s-1}(u)$  and last coordinate  $1 - \tau_u$ , giving unit excess. At the left exterior endpoint, any positive lifted violation

must come from a positive projected violation of  $D$ , which is unit by induction; since the lifting term is nonpositive, the lifted excess is again one.

Finally consider a bilateral lift. Then  $a = 2l + 1$ ,  $b = 2u$ , and  $\tau_l = \tau_u =: \tau$ . The lifted row has the form  $C(y, z) = D(y) + \theta_\tau(z)$ , where the base row  $D$  has endpoint excess  $D(g_{s-1}(l)) = D(g_{s-1}(u)) = 1$ . At the left exterior endpoint  $a - 1 = 2l$ , the projected point is  $g_{s-1}(l)$  and the last coordinate is  $1 - \tau$ . Thus  $\text{exc}_s(C; a - 1) = 1 + \theta_\tau(1 - \tau) = 1$ . At the right exterior endpoint  $b + 1 = 2u + 1$ , the projected point is  $g_{s-1}(u)$  and the last coordinate is again  $1 - \tau$ . Therefore  $\text{exc}_s(C; b + 1) = 1 + \theta_\tau(1 - \tau) = 1$ .

The last assertions follow from the definitions of the auxiliary families. Indeed, if  $C \in \mathcal{H}_s[a + 1, b]$  has positive excess at the deleted left endpoint  $a$ , then the result applied to the interval  $[a + 1, b]$  gives  $\text{exc}_s(C; a) = 1$ , and hence  $C \in \mathcal{L}_s[a, b]$ . The argument for  $\mathcal{R}_s[a, b]$  is symmetric.  $\square$

### 3.4 Normalization and replacement lemmas

We next analyze arbitrary facet-defining inequalities and show how to replace them by normalized rows generated by the recursion. We assume  $s \geq 2$  and  $b \geq a + 2$ . The first case occurs when the facet inequality is already valid for the full product relaxation.

**Lemma 3.3** (Product-valid facets). *Let  $Q := P_{s-1}[l, u]$ ,  $\bar{P} := Q \times [0, 1]$ , and  $P := P_s[a, b]$ . Let  $W(y, z) := \alpha^\top y + \eta z - \beta \leq 0$  be valid for  $\bar{P}$ . If  $G := \{(y, z) \in P : W(y, z) = 0\}$  is a facet of  $P$ , then  $G$  is exposed on  $P$  by either  $-z \leq 0$ ,  $z - 1 \leq 0$ , or by the zero extension of a facet of  $Q$ .*

*Proof.* Since  $[a + 1, b]$  contains an odd integer, by Lemma 2.8,  $\text{aff}(P) = \text{aff}(Q) \times \mathbb{R} = \text{aff}(\bar{P})$ . Let  $\bar{G} := \{(y, z) \in \bar{P} : W(y, z) = 0\}$ . Since  $W$  is valid for  $\bar{P}$ ,  $\bar{G}$  is a face of  $\bar{P}$ , and since  $P \subseteq \bar{P}$ ,  $G = P \cap \bar{G}$ . Because  $G$  is a facet of  $P$ , and  $P$  and  $\bar{P}$  have the same affine hull,  $\dim G = \dim P - 1 = \dim \bar{P} - 1$ . Moreover,  $\bar{G}$  is a proper face of  $\bar{P}$ ; otherwise  $W$  would be tight on all of  $\bar{P}$ , hence on all of  $P$ , contradicting that  $G$  is a facet of  $P$ . Therefore  $\dim \bar{G} \leq \dim \bar{P} - 1$ . Since  $G \subseteq \bar{G}$ , we get  $\dim \bar{G} = \dim P - 1$ . Thus  $\bar{G}$  is a facet of the product  $\bar{P} = Q \times [0, 1]$ .

If  $\eta = 0$ , then the face of  $\bar{P}$  exposed by  $W$  is  $F \times [0, 1]$ , where  $F := \{y \in Q : \alpha^\top y - \beta = 0\}$ . Since  $\bar{G}$  is a facet of  $Q \times [0, 1]$ , the set  $F$  is a facet of  $Q$ . Hence the face  $G$  of  $P$  is exposed by the zero extension of the row defining  $F$ .

If  $\eta > 0$ , then tightness on the product can occur only at  $z = 1$ . Hence  $\bar{G} = \{y \in Q : \alpha^\top y + \eta - \beta = 0\} \times \{1\}$ . Since  $\bar{G}$  has dimension  $\dim Q$ , the first factor must have full relative dimension in  $Q$ . Therefore the row  $\alpha^\top y + \eta - \beta$  is constant on  $\text{aff}(Q)$ , and  $\bar{G} = Q \times \{1\}$ . Thus the product facet is exposed by the bound  $z - 1 \leq 0$ .

Similarly, if  $\eta < 0$ , tightness can occur only at  $z = 0$ , and the product facet is  $Q \times \{0\}$ , which is exposed by the bound  $-z \leq 0$ .  $\square$

The second case occurs when an endpoint fiber is deleted, in which case a lifted inequality has a very restricted set of tight vertices and the endpoint excess is forced to be exactly one.

**Lemma 3.4** (Tight vertices of unilateral lifts). *Assume that  $a = 2l + 1$ . Let  $Q^L := P_{s-1}[l + 1, u]$ ,  $\ell := g_{s-1}(l)$ , and  $\tau := \tau_l$ . Let  $C(y) \leq 0$  be valid for  $Q^L$ , with  $C(\ell) = 1$ , and let  $F := \{y \in Q^L : C(y) = 0\}$ . If  $G$  is the face of  $P_s[a, b]$  exposed by  $\text{Lift}_{l, \tau}(C) \leq 0$ , then  $G \subseteq \text{conv}(\{(\ell, \tau)\} \cup (F \times \{1 - \tau\}))$ . In particular, if  $G$  is a facet and  $F \neq \emptyset$ , then  $\text{codim}_{Q^L}(F) \leq 1$ . The analogous statement holds for right unilateral lifts.*

*Proof.* We prove the left case. The right case is symmetric.

First, note that  $Q^L \neq \emptyset$ . Let  $W(y, z) := \text{Lift}_{l, \tau}(C)(y, z) = C(y) + \theta_\tau(z)$  be the corresponding left unilateral lift, and write  $G = \{(y, z) \in P_s[a, b] : W(y, z) = 0\}$ .

If  $\tau = 0$ , then  $W(y, z) = C(y) + z - 1$ . The restored endpoint  $(\ell, 0)$  is tight because  $C(\ell) = 1$ . For every other allowed vertex with projected component in  $Q^L$ , we have  $C(y) \leq 0$ . Hence equality can hold only if  $z = 1$  and  $C(y) = 0$ . Thus all non-endpoint tight vertices lie in  $F \times \{1\} = F \times \{1 - \tau\}$ . If  $\tau = 1$ , then  $W(y, z) = C(y) - z$ . The restored endpoint  $(\ell, 1)$  is tight. For every other allowed vertex,

equality can hold only if  $z = 0$  and  $C(y) = 0$ . Thus the same containment holds. Note that in either case, we have  $\dim G \leq \dim F + 1$ .

Since  $b \geq a + 2$ , the fiber above  $g_{s-1}(l+1)$  is full, so  $P_s[a, b]$  contains the last-coordinate direction. Since the projection of  $P_s[a, b]$  contains  $Q^L$ , we have  $\dim P_s[a, b] \geq \dim Q^L + 1$ . Finally, if  $G$  is a facet, then  $\dim F + 1 \geq \dim G = \dim P_s[a, b] - 1 \geq \dim Q^L$ , and therefore  $\text{codim}_{Q^L}(F) \leq 1$ .  $\square$

**Lemma 3.5** (Unilateral facet normalization). *Assume that  $a = 2l + 1$ . Let  $Q^L := P_{s-1}[l + 1, u]$ ,  $\ell := g_{s-1}(l)$ , and  $\tau := \tau_l$ . Let  $C(y) \leq 0$  be valid for  $Q^L$ , and suppose  $0 < C(\ell) \leq 1$ . If  $\text{Lift}_{l, \tau}(C) \leq 0$  defines a facet of  $P_s[a, b]$ , then  $C(\ell) = 1$ . The analogous statement holds for right unilateral lifts.*

*Proof.* We prove the left case. The right case is symmetric.

First, note that  $Q^L \neq \emptyset$  and let  $F := \{y \in Q^L : C(y) = 0\}$ . By the same tight-vertex analysis as in Lemma 3.4, the face  $G$  is contained in  $\text{conv}(\{(\ell, \tau) : C(\ell) = 1\} \cup (F \times \{1 - \tau\}))$ .

Suppose, for contradiction, that  $C(\ell) < 1$ . Then the restored endpoint  $(\ell, \tau)$  is not tight, and therefore  $G \subseteq F \times \{1 - \tau\}$ . Hence  $\dim G \leq \dim F \leq \dim Q^L$ .

Since  $a = 2l + 1$ ,  $b \geq a + 2$  implies  $b \geq 2l + 3$ . Hence both indices  $2(l + 1)$  and  $2(l + 1) + 1$  belong to  $[a, b]$ . Therefore the fiber above  $g_{s-1}(l+1)$  is full, so  $P_s[a, b]$  contains the last-coordinate direction. Its projection contains  $Q^L$ .

If  $\ell \notin \text{aff}(Q^L)$ , then adding the restored endpoint  $(\ell, \tau)$  contributes one further independent projected direction, and therefore  $\dim P_s[a, b] \geq \dim Q^L + 2$ . Since  $G$  is a facet, this would imply  $\dim G = \dim P_s[a, b] - 1 \geq \dim Q^L + 1$ , contradicting  $\dim G \leq \dim Q^L$ .

It remains to consider the case  $\ell \in \text{aff}(Q^L)$ . If  $F \neq Q^L$ , then  $\dim F \leq \dim Q^L - 1$ , and hence  $\dim G \leq \dim Q^L - 1$ . But the full fibers over  $Q^L$  give  $\dim P_s[a, b] \geq \dim Q^L + 1$ , so a facet has dimension at least  $\dim Q^L$ , again a contradiction.

Thus we must have  $F = Q^L$ . But then  $C$  vanishes on all of  $Q^L$ . Since  $C$  is affine, it also vanishes on  $\text{aff}(Q^L)$ . As  $\ell \in \text{aff}(Q^L)$ , this gives  $C(\ell) = 0$ , contradicting  $0 < C(\ell)$ . Therefore  $C(\ell) < 1$  is impossible, and so  $C(\ell) = 1$ .  $\square$

The bilateral case is similar, but the tight face may involve both restored endpoints.

**Lemma 3.6** (Bilateral facet normalization). *Assume that  $a = 2l + 1$ ,  $b = 2u$ , and  $\tau_l = \tau_u =: \tau$ . Let  $Q := P_{s-1}[l + 1, u - 1]$ , and assume  $Q \neq \emptyset$  and that every facet of  $Q$  is exposed by a row generated in  $\mathcal{H}_{s-1}[l + 1, u - 1]$ . Put  $\ell := g_{s-1}(l)$  and  $v := g_{s-1}(u)$ . Let  $C(y) \leq 0$  be valid for  $Q$ , with  $0 < C(\ell) \leq 1$  and  $0 < C(v) \leq 1$ , and define  $F := \{y \in Q : C(y) = 0\}$ . If  $\text{BLift}_{l, u}(C) \leq 0$  defines a facet of  $P_s[a, b]$ , then  $F \neq \emptyset$ ,  $\text{codim}_Q(F) \leq 2$ , and  $C(\ell) = C(v) = 1$ .*

*Proof.* Let  $W(y, z) := \text{BLift}_{l, u}(C)(y, z) = C(y) + \theta_\tau(z)$  be the corresponding bilateral lift and write  $G := \{(y, z) \in P_s[a, b] : W(y, z) = 0\}$ . Define  $T := \{l : C(\ell) = 1\} \cup \{u : C(v) = 1\}$ .

On every full fiber above a projected vertex of  $Q$ , we have  $C(y) \leq 0$ . Since  $\theta_\tau(z) \leq 0$  for  $z \in [0, 1]$ ,  $W(y, z) = 0$  can hold over  $Q$  only when  $C(y) = 0$  and  $\theta_\tau(z) = 0$ . Hence all tight vertices over  $Q$  lie in  $F \times \{1 - \tau\}$ . The restored endpoint  $(\ell, \tau)$  is tight if and only if  $C(\ell) = 1$ , and the restored endpoint  $(v, \tau)$  is tight if and only if  $C(v) = 1$ . Consequently,  $G \subseteq \text{conv}(F \times \{1 - \tau\} \cup \{(\ell, \tau) : l \in T\} \cup \{(v, \tau) : u \in T\})$ . Thus  $\dim G \leq \dim F + |T|$ .

We next exclude  $F = \emptyset$  in the facet case. If  $F = \emptyset$ , then the containment above gives  $G \subseteq \text{conv}(\{(\ell, \tau) : l \in T\} \cup \{(v, \tau) : u \in T\})$ , and therefore  $\dim G \leq 1$ . Since  $Q \neq \emptyset$ , we have  $u - l \geq 2$ . Moreover,  $\tau_l = \tau_u$  means  $(l \bmod 2) \oplus 1 = u \bmod 2$ , so  $u - l$  is odd. Hence  $u - l \geq 3$ . Thus the projected interval  $P_{s-1}[l, u]$  contains the three consecutive Gray vertices  $g_{s-1}(l)$ ,  $g_{s-1}(l+1)$ ,  $g_{s-1}(l+2)$ . The two consecutive edge directions of a Gray path are distinct coordinate directions, so these three vertices are affinely independent. Hence  $\dim P_{s-1}[l, u] \geq 2$ . Also, the interval  $[a + 1, b] = [2l + 2, 2u]$  contains an odd integer. By Lemma 2.8,  $\text{aff}(P_s[a, b]) = \text{aff}(P_{s-1}[l, u]) \times \mathbb{R}$ . Therefore  $\dim P_s[a, b] \geq 3$ . A facet of  $P_s[a, b]$  has dimension at least 2, whereas  $\dim G \leq 1$ . Hence  $G$  cannot be a facet when  $F = \emptyset$ . Thus, in the facet case,  $F \neq \emptyset$ .

Assume now that  $G$  is a facet. Since  $Q \neq \emptyset$ , the lifted interval contains the last-coordinate direction above the full fibers over  $Q$ , and its projection contains  $Q$ . Hence  $\dim P_s[a, b] \geq \dim Q + 1$ . Using

the containment above,  $\dim F + |T| \geq \dim G = \dim P_s[a, b] - 1 \geq \dim Q$ . Therefore  $\text{codim}_Q(F) = \dim Q - \dim F \leq |T| \leq 2$ . In particular,  $\text{codim}_Q(F) \leq 2$ .

Let  $r := \text{codim}_Q(F)$ . We prove that  $C(\ell) = C(v) = 1$ , considering the possible values  $r = 0, 1, 2$ .

If  $r = 2$ , then  $2 = r \leq |T| \leq 2$ , so  $|T| = 2$ . Hence both restored endpoints are active:  $C(\ell) = C(v) = 1$ .

Now suppose  $r = 0$ . Then  $F = Q$ , so  $C$  vanishes on  $Q$ . Let  $R := P_{s-1}[l, u]$ . Since  $G$  is a facet of  $P_s[a, b]$ ,  $\dim G = \dim P_s[a, b] - 1$ . As above,  $\text{aff}(P_s[a, b]) = \text{aff}(R) \times \mathbb{R}$ , so  $\dim G = \dim R$ . On the other hand, because  $F = Q$ , the containment of tight vertices gives  $\dim G \leq \dim Q + |T|$ . Thus  $|T| \geq \dim R - \dim Q$ .

Because  $C$  vanishes on  $Q$  but satisfies  $C(\ell) > 0$ ,  $C(v) > 0$ , both endpoints  $\ell, v$  lie outside  $\text{aff}(Q)$ . Hence  $\dim R - \dim Q \geq 1$ . Since  $R$  is obtained from  $Q$  by adding only the two endpoint vertices  $\ell$  and  $v$ , we also have  $\dim R - \dim Q \leq 2$ .

If  $\dim R - \dim Q = 2$ , then  $|T| \geq 2$ , so  $|T| = 2$ , and hence  $C(\ell) = C(v) = 1$ .

It remains in the case  $r = 0$  to consider  $\dim R - \dim Q = 1$ . Then  $|T| \geq 1$ , so at least one of  $C(\ell)$  and  $C(v)$  is equal to one. We claim that in this codimension-one enlargement,  $C(\ell) = C(v)$ . Indeed, let  $J_Q := J_{s-1}(l+1, u-1)$  and  $J_R := J_{s-1}(l, u)$ , where the sets  $J_{s-1}(\cdot, \cdot)$  are those from Proposition 2.5. Since  $Q \subseteq R$  and  $\dim R = \dim Q + 1$ , we have  $J_R \setminus J_Q = \{j^*\}$  for a unique coordinate  $j^*$ . Choose a vertex  $\bar{y}$  of  $Q$ . Since  $C$  vanishes on  $\text{aff}(Q)$ , it can be written as  $C(y) = \sum_{j \notin J_Q} \lambda_j (y_j - \bar{y}_j)$ . For every  $j \notin J_R$ , the coordinate  $y_j$  is fixed on  $R$ , and hence  $\ell_j = v_j = \bar{y}_j$ . Therefore only  $j^*$  can contribute to the endpoint values:  $C(\ell) = \lambda_{j^*} (\ell_{j^*} - \bar{y}_{j^*})$ ,  $C(v) = \lambda_{j^*} (v_{j^*} - \bar{y}_{j^*})$ . Since all three points are binary and both endpoint values are positive, the two nonzero differences have the same sign and belong to  $\{-1, 0, 1\}$ . They are therefore equal, and so  $C(\ell) = C(v)$ . Since at least one of these two values is one, both are one.

Finally suppose  $r = 1$ . Then  $F$  is a facet of  $Q$ . Since  $G$  is a facet of  $P_s[a, b]$ , the dimension bound gives  $1 = r \leq |T|$ . Assume, for contradiction, that  $|T| = 1$ . Then  $\dim G \leq \dim F + 1 = \dim Q$ . As above,  $\text{aff}(P_s[a, b]) = \text{aff}(P_{s-1}[l, u]) \times \mathbb{R}$ , so  $\dim G = \dim P_s[a, b] - 1 = \dim P_{s-1}[l, u]$ . Since  $\dim G = \dim R$  and the assumption  $|T| = 1$  gives  $\dim G \leq \dim Q$ , we obtain  $\dim R \leq \dim Q$ . Because  $Q \subseteq R$ , the reverse inequality  $\dim Q \leq \dim R$  is automatic. Hence  $\dim R = \dim Q$ , and therefore  $\text{aff}(R) = \text{aff}(Q)$ . In particular,  $\ell, v \in \text{aff}(Q)$ .

By the assumed facet-completeness of  $Q$ , the facet  $F$  is exposed by a generated row  $D \in \mathcal{H}_{s-1}[l+1, u-1]$ . Since  $C$  and  $D$  expose the same facet of  $Q$ , their restrictions to  $\text{aff}(Q)$  are positive multiples of each other. Thus there exists  $\lambda > 0$  such that  $C(y) = \lambda D(y)$  for all  $y \in \text{aff}(Q)$ . Because  $\ell, v \in \text{aff}(Q)$ , we get  $C(\ell) = \lambda D(\ell)$ ,  $C(v) = \lambda D(v)$ . The assumptions  $C(\ell) > 0$  and  $C(v) > 0$  imply  $D(\ell) > 0$ ,  $D(v) > 0$ . By Proposition 3.2, these positive exterior excesses are equal to one. Therefore  $C(\ell) = C(v) = \lambda$ . But  $|T| = 1$  means that exactly one of  $C(\ell)$  and  $C(v)$  is equal to one, contradicting  $C(\ell) = C(v)$ . Hence  $|T| = 2$ , and therefore  $C(\ell) = C(v) = 1$ .  $\square$

The remaining replacement steps use Farkas' lemma to select generated rows that are active on a prescribed face.

**Lemma 3.7** (Unilateral replacement). *Assume that  $\mathcal{H}_s[I]$  describes  $P_s[I]$  exactly, where  $I$  is a nonempty interval. Let  $q$  be an endpoint immediately outside  $I$ . Let  $C$  be valid for  $P_s[I]$  and active on a nonempty face  $F$  of  $P_s[I]$ . If  $C(g_s(q)) > 0$ , then there exists a row  $E \in \mathcal{H}_s[I]$  such that  $E$  is active on all of  $F$  and  $E(g_s(q)) = 1$ .*

*Proof.* Write the rows of  $\mathcal{H}_s[I]$  as affine functions  $E_i \leq 0$ . Since the system is exact and  $C \leq 0$  is valid for  $P_s[I]$ , Farkas' lemma gives multipliers  $\lambda_i \geq 0$  and a scalar  $\delta \geq 0$  such that  $\sum_i \lambda_i E_i = C + \delta$ . Evaluating at any point of  $F$ , where  $C = 0$ , gives  $\sum_i \lambda_i E_i = \delta$ . The left-hand side is nonpositive on  $F$ , while  $\delta \geq 0$ . Hence  $\delta = 0$ . Thus every row with  $\lambda_i > 0$  is active on all of  $F$ .

Evaluating at  $g_s(q)$  yields  $C(g_s(q)) = \sum_i \lambda_i E_i(g_s(q)) > 0$ . Hence at least one active row  $E_i$  has positive exterior excess at  $q$ . By Proposition 3.2, this positive excess is equal to one.  $\square$

**Lemma 3.8** (Bilateral replacement). *Assume that the recursive descriptions are exact for  $P_s[a+1, b-1]$ ,  $P_s[a+1, b]$ , and  $P_s[a, b-1]$ , where  $Q := P_s[a+1, b-1]$  is nonempty. Let  $C \leq 0$  be valid for  $Q$ , active on a nonempty face  $F$  of  $Q$ , and normalized at both deleted endpoints:  $C(g_s(a)) = C(g_s(b)) = 1$ . Then there exists a row  $\widehat{C}$ , active on  $F$ , with unit excess at both deleted endpoints, such that either  $\widehat{C} \in \mathcal{B}_s[a, b]$ , or  $\widehat{C} = E^L + E^R$ ,  $E^L \in \mathcal{L}_s[a, b]$ ,  $E^R \in \mathcal{R}_s[a, b]$ .*

*Proof.* Write the rows of  $\mathcal{H}_s[a+1, b-1]$  as affine functions  $E_i \leq 0$ . Since  $\mathcal{H}_s[a+1, b-1]$  describes  $Q$  exactly and  $C \leq 0$  is valid for  $Q$ , Farkas' lemma gives multipliers  $\lambda_i \geq 0$  and a scalar  $\delta \geq 0$  such that  $\sum_i \lambda_i E_i = C + \delta$ . Evaluating this identity at any point of  $F$ , where  $C = 0$ , gives  $\sum_i \lambda_i E_i = \delta$ . The left-hand side is nonpositive on  $F$ , while  $\delta \geq 0$ . Hence  $\delta = 0$ . Therefore every row with  $\lambda_i > 0$  is active on all of  $F$ .

For a row  $E$ , write its endpoint excess vector as  $e(E) := (E(g_s(a)), E(g_s(b)))$ . Evaluating  $C = \sum_i \lambda_i E_i$  at the two deleted endpoints gives  $(1, 1) = \sum_i \lambda_i e(E_i)$ .

If some row  $E_i$  with  $\lambda_i > 0$  has positive excess at both endpoints, then Proposition 3.2 gives  $e(E_i) = (1, 1)$ . Thus  $E_i \in \mathcal{B}_s[a, b]$ , and we may take  $\widehat{C} := E_i$ .

We may therefore assume that no active generated row has positive excess at both endpoints. Since the positive combination above equals  $(1, 1)$ , there exists at least one active row with positive left endpoint excess and at least one active row with positive right endpoint excess. By Proposition 3.2, such rows have endpoint excess vectors of the form  $(1, \beta)$ ,  $\beta \leq 0$ , and  $(\alpha, 1)$ ,  $\alpha \leq 0$ , respectively.

We next show that the left and right rows can be chosen so that at least one crossed excess is zero. Let  $\mathcal{I}_L := \{i : \lambda_i > 0, E_i(g_s(a)) > 0\}$ ,  $\mathcal{I}_R := \{i : \lambda_i > 0, E_i(g_s(b)) > 0\}$ . Both sets are nonempty, because  $(1, 1) = \sum_i \lambda_i e(E_i)$ . By the standing assumption, no active generated row has positive excess at both deleted endpoints. Hence, for every  $i \in \mathcal{I}_L$ ,  $e(E_i) = (1, \beta_i)$ ,  $\beta_i \leq 0$ , and, for every  $i \in \mathcal{I}_R$ ,  $e(E_i) = (\alpha_i, 1)$ ,  $\alpha_i \leq 0$ .

We claim that there exist  $i_L \in \mathcal{I}_L$  and  $i_R \in \mathcal{I}_R$  such that at least one of  $\beta_{i_L}$  and  $\alpha_{i_R}$  is equal to zero. Suppose not. Then every active row with positive left endpoint excess has strictly negative right endpoint excess, and every active row with positive right endpoint excess has strictly negative left endpoint excess. Since endpoint excesses are integral, every active row  $E_i$  with  $\lambda_i > 0$  satisfies  $E_i(g_s(a)) + E_i(g_s(b)) \leq 0$ . Taking the nonnegative combination defining  $C$ , we obtain  $2 = C(g_s(a)) + C(g_s(b)) = \sum_i \lambda_i (E_i(g_s(a)) + E_i(g_s(b))) \leq 0$ , a contradiction. Therefore such indices  $i_L, i_R$  exist.

Set  $D^L := E_{i_L}$ ,  $D^R := E_{i_R}$ . Then  $e(D^L) = (1, \beta)$ ,  $e(D^R) = (\alpha, 1)$ , with  $\alpha, \beta \leq 0$ , and with at least one of  $\alpha, \beta$  equal to zero. We now distinguish cases.

First suppose  $\alpha = \beta = 0$ . Then  $D^L$  is valid for  $P_s[a+1, b]$ : it is valid on  $Q$ , and at  $g_s(b)$  its value is zero. Moreover,  $D^L$  is active on  $F$ , active at  $g_s(b)$ , and has left exterior excess one. Let  $F^L$  be the face of  $P_s[a+1, b]$  exposed by  $D^L$ . This face contains both  $F$  and  $g_s(b)$ . Applying Lemma 3.7 to the interval  $P_s[a+1, b]$ , to the outside endpoint  $a$ , and to the face  $F^L$ , we obtain a row  $E^L \in \mathcal{H}_s[a+1, b]$  active on  $F^L$  and satisfying  $E^L(g_s(a)) = 1$ . Since  $g_s(b) \in F^L$ , we also have  $E^L(g_s(b)) = 0$ . Thus  $E^L \in \mathcal{L}_s[a, b]$ ,  $e(E^L) = (1, 0)$ , and  $E^L$  is active on  $F$ .

Similarly,  $D^R$  is valid for  $P_s[a, b-1]$ , active on  $F$ , active at  $g_s(a)$ , and has right exterior excess one. Applying Lemma 3.7 to  $D^R$  on  $P_s[a, b-1]$ , we obtain  $E^R \in \mathcal{R}_s[a, b]$ ,  $e(E^R) = (0, 1)$ , with  $E^R$  active on  $F$ . Therefore  $\widehat{C} := E^L + E^R$  has the required properties.

Now suppose  $\alpha = 0$ ,  $\beta = -k < 0$ ,  $k \geq 1$ . Define  $S^L := D^L + kD^R$ . Then  $S^L$  is valid for  $P_s[a+1, b]$ : it is valid on  $Q$ , and at  $g_s(b)$  its value is  $-k + k = 0$ . Moreover,  $S^L$  is active on  $F$ , active at  $g_s(b)$ , and has left exterior excess one. Applying Lemma 3.7 to the face of  $P_s[a+1, b]$  exposed by  $S^L$ , we obtain  $E^L \in \mathcal{L}_s[a, b]$ ,  $e(E^L) = (1, 0)$ , with  $E^L$  active on  $F$ .

Since  $\alpha = 0$ , the row  $D^R$  is valid for  $P_s[a, b-1]$ , active on  $F$ , active at  $g_s(a)$ , and has right exterior excess one. Applying Lemma 3.7 to  $D^R$  on  $P_s[a, b-1]$ , we obtain  $E^R \in \mathcal{R}_s[a, b]$ ,  $e(E^R) = (0, 1)$ , with  $E^R$  active on  $F$ . Hence  $\widehat{C} := E^L + E^R$  has the required properties.

The remaining case is symmetric. If  $\beta = 0$  and  $\alpha = -k < 0$ , define  $S^R := D^R + kD^L$ . Then  $S^R$  is valid for  $P_s[a, b-1]$ , active on  $F$ , active at  $g_s(a)$ , and has right exterior excess one. Applying Lemma 3.7 gives  $E^R \in \mathcal{R}_s[a, b]$  with endpoint vector  $(0, 1)$ , active on  $F$ . Since  $\beta = 0$ , applying the

same argument to  $D^L$  on  $P_s[a+1, b]$  gives  $E^L \in \mathcal{L}_s[a, b]$  with endpoint vector  $(1, 0)$ , active on  $F$ . Their sum is the desired row.  $\square$

### 3.5 Exactness

We now state the main result of the recursion. Facets are understood relative to the affine hull of the corresponding interval polytope.

**Lemma 3.9** (Facets plus affine hull imply exactness). *Let  $P \subseteq \mathbb{R}^d$  be a nonempty polytope, and let  $Q := \{x \in \mathbb{R}^d : C(x) \leq 0 \text{ for all } C \in \mathcal{H}\}$ , where every row in  $\mathcal{H}$  is valid for  $P$ . Suppose that  $Q \subseteq \text{aff}(P)$ , and that every facet of  $P$  is exposed by at least one row in  $\mathcal{H}$ . Then  $Q = P$ .*

*Proof.* Since every row in  $\mathcal{H}$  is valid for  $P$ , we have  $P \subseteq Q$ . Let  $E := \text{aff}(P)$ . Inside  $E$ , the polytope  $P$  is the intersection of its facet-defining halfspaces. By assumption, each of these facet halfspaces is present among the inequalities in  $\mathcal{H}$ . Hence  $Q \cap E \subseteq P$ . Since  $Q \subseteq E$ , this gives  $Q \subseteq P$ , and therefore  $Q = P$ .  $\square$

**Theorem 3.10** (Exact description). *For every  $s \geq 1$  and every interval  $0 \leq a \leq b \leq 2^s - 1$ , we have  $P_s[a, b] = \{x \in \mathbb{R}^s : C(x) \leq 0 \text{ for all } C \in \mathcal{H}_s[a, b]\}$ .*

*Proof.* We prove the following two assertions simultaneously by induction on  $s$ . First, every facet of  $P_s[a, b]$ , relative to its affine hull, is exposed by a row generated in  $\mathcal{H}_s[a, b]$ . Second, for every affine-hull fixing  $x_j = \gamma_j$  of  $P_s[a, b]$ , the family  $\mathcal{H}_s[a, b]$  contains both rows  $x_j - \gamma_j \leq 0$ ,  $-x_j + \gamma_j \leq 0$ . Together with validity, these two assertions imply exactness. Indeed, let  $Q_s[a, b] := \{x \in \mathbb{R}^s : C(x) \leq 0 \text{ for all } C \in \mathcal{H}_s[a, b]\}$ . By Proposition 3.1, every row in  $\mathcal{H}_s[a, b]$  is valid for  $P_s[a, b]$ . The first assertion says that every facet of  $P_s[a, b]$  is present in the recursive system, while the second gives  $Q_s[a, b] \subseteq \text{aff}(P_s[a, b])$ . Hence Lemma 3.9 implies  $Q_s[a, b] = P_s[a, b]$ .

The case  $s = 1$  is immediate, and for every  $s$ , the cases  $b = a$  and  $b = a + 1$  follow directly from the definitions. Therefore, we assume  $s \geq 2$  and  $b \geq a + 2$ . Put  $l := \lfloor a/2 \rfloor$ ,  $u := \lfloor b/2 \rfloor$ . Also set  $\tau_l := (l \bmod 2) \oplus 1$ ,  $\tau_u := u \bmod 2$ . By the induction hypothesis, all lower-dimensional intervals used below have exact and facet-complete recursive descriptions, and contain the required affine-hull fixing rows.

Validity of  $\mathcal{H}_s[a, b]$  follows from Proposition 3.1. We now prove facet-completeness.

Let  $G$  be a facet of  $P_s[a, b]$ . Choose a valid inequality  $\alpha^\top y + \eta z \leq \beta$ ,  $z = x_s$ , that exposes  $G$ , that is,  $G = \{(y, z) \in P_s[a, b] : \alpha^\top y + \eta z = \beta\}$ . We prove that  $G$  is also exposed by a row generated in  $\mathcal{H}_s[a, b]$ .

Let  $\bar{P} := P_{s-1}[l, u] \times [0, 1]$ . By Lemma 2.7,  $\bar{P} = \text{conv}(\mathcal{V}_{l,u})$  and  $P_s[a, b] = \text{conv}(\mathcal{V}_{l,u} \setminus D(a, b))$ . The left deleted point exists exactly when  $a$  is odd, and is  $d_l := (g_{s-1}(l), 1 - \tau_l)$ . The right deleted point exists exactly when  $b$  is even, and is  $d_u := (g_{s-1}(u), 1 - \tau_u)$ .

If the inequality exposing  $G$  is valid for  $\bar{P}$ , then Lemma 3.3 applies. Hence the exposed facet of  $P_s[a, b]$  is exposed either by one of the generated bounds  $-z \leq 0$ ,  $z - 1 \leq 0$ , or by the zero extension of a facet of  $P_{s-1}[l, u]$ . In the latter case, the induction hypothesis gives a generated row in  $\mathcal{H}_{s-1}[l, u]$  exposing that projected facet, and its zero extension is generated in  $\mathcal{H}_s[a, b]$ . Thus this case is covered.

Therefore we may assume that the inequality exposing  $G$  is not valid for  $\bar{P}$ . Then it is violated by at least one vertex of  $\mathcal{V}_{l,u}$ . Since the inequality is valid for  $P_s[a, b]$ , no point of  $\mathcal{V}_{l,u} \setminus D(a, b)$  can violate it. Therefore some violated vertex must belong to  $D(a, b)$ .

Suppose first that exactly one deleted point is cut. We treat the left case; the right case is symmetric. Thus  $a = 2l + 1$  and the inequality cuts  $d_l = (g_{s-1}(l), 1 - \tau_l)$ .

If  $\tau_l = 0$ , cutting  $d_l$  forces  $\eta > 0$ . Define  $C(y) := \frac{\alpha^\top y + \eta - \beta}{\eta}$ . Then the original inequality is the positive multiple  $\eta$  of  $C(y) + z - 1 \leq 0$ .

If  $\tau_l = 1$ , cutting  $d_l$  forces  $\eta < 0$ . Define  $C(y) := \frac{\alpha^\top y - \beta}{-\eta}$ . Then the original inequality is the positive multiple  $-\eta$  of  $C(y) - z \leq 0$ .

In both cases,  $C(y)$  is the original left-hand side evaluated at the fiber value  $z = 1 - \tau_l$ , divided by the positive scalar  $|\eta|$ . Since no product vertex other than  $d_l$  is cut in the present case, this value is nonpositive for all projected vertices  $g_{s-1}(q)$ ,  $q = l + 1, \dots, u$ . Hence  $C \leq 0$  is valid for  $P_{s-1}[l + 1, u]$ . Moreover, cutting the deleted endpoint gives  $C(g_{s-1}(l)) > 0$ , while validity at the restored endpoint gives  $C(g_{s-1}(l)) \leq 1$ . Hence  $0 < C(g_{s-1}(l)) \leq 1$ . By Lemma 3.5,  $C(g_{s-1}(l)) = 1$ .

Let  $Q^L := P_{s-1}[l + 1, u]$  and  $F := \{y \in Q^L : C(y) = 0\}$ . We claim that  $F \neq \emptyset$ . Indeed, if  $F = \emptyset$ , then the tight-vertex description of Lemma 3.4 implies that the lifted face is contained in the single restored endpoint  $(g_{s-1}(l), \tau_l)$ , and therefore has dimension zero. Since  $b \geq a + 2$ , the interval contains the three consecutive Gray vertices  $g_s(a)$ ,  $g_s(a + 1)$ ,  $g_s(a + 2)$ . By Lemma 2.4, the transition  $a + 1$  changes a coordinate different from  $s$ , because  $a + 1$  is even, whereas the transition  $a + 2$  changes coordinate  $s$ , because  $a + 2$  is odd. Hence the two edge directions are distinct. Hence  $\dim P_s[a, b] \geq 2$ . A zero-dimensional face cannot be a facet of such a polytope. This contradiction proves that  $F \neq \emptyset$ .

By Lemma 3.4, the face  $F$  has relative codimension at most one in  $Q^L$ . Applying Lemma 3.7 to  $C \leq 0$  on  $Q^L$ , active on  $F$ , with exterior excess  $C(g_{s-1}(l)) = 1$ , we obtain a generated row  $E \in \mathcal{H}_{s-1}[l + 1, u]$  active on  $F$  and satisfying  $E(g_{s-1}(l)) = 1$ . Thus  $E \in \mathcal{L}_{s-1}[l, u]$ . Let  $\ell := g_{s-1}(l)$ , and let  $G_E$  be the face of  $P_s[a, b]$  exposed by the generated row  $\text{Lift}_{l, \tau_l}(E)$ . By the tight-vertex containment in Lemma 3.4, applied to the original row  $C$ , we have  $G \subseteq \text{conv}(\{(\ell, \tau_l)\} \cup (F \times \{1 - \tau_l\}))$ . Since  $E(\ell) = 1$ , the row  $\text{Lift}_{l, \tau_l}(E)$  is tight at the restored endpoint  $(\ell, \tau_l)$ . Moreover, since  $E$  is active on  $F$ , the same lifted row is tight on  $F \times \{1 - \tau_l\}$ . Hence  $G \subseteq G_E$ . Since  $G$  is a facet, any face containing  $G$  is either  $G$  itself or the whole polytope. The latter is impossible: the interval contains a full last-coordinate fiber, while a unilateral lift cannot be tight at both points of any full fiber. Therefore  $G_E = G$ , and the generated unilateral lift exposes the same facet  $G$ .

It remains to treat the case in which both deleted points are cut. Then  $a = 2l + 1$ ,  $b = 2u$ . Cutting the left deleted point requires  $\eta(1 - 2\tau_l) > 0$ , and cutting the right deleted point requires  $\eta(1 - 2\tau_u) > 0$ . Hence necessarily  $\tau_l = \tau_u$ .

The interval  $[a + 1, b] = [2l + 2, 2u]$  contains an odd integer. Hence  $u \geq l + 2$ , and therefore  $Q := P_{s-1}[l + 1, u - 1]$  is nonempty.

If  $\tau_l = \tau_u = 0$ , then  $\eta > 0$ , and define  $C(y) := \frac{\alpha^\top y + \eta - \beta}{\eta}$ . If  $\tau_l = \tau_u = 1$ , then  $\eta < 0$ , and define  $C(y) := \frac{\alpha^\top y - \beta}{-\eta}$ . In both cases  $C \leq 0$  is valid for  $Q$ . Since the original inequality cuts both deleted complementary endpoint vertices and is valid at the corresponding restored endpoint vertices,  $0 < C(g_{s-1}(l)) \leq 1$ ,  $0 < C(g_{s-1}(u)) \leq 1$ .

Let  $F := \{y \in Q : C(y) = 0\}$ . The facet-completeness assumption required in Lemma 3.6 holds here by the induction hypothesis applied to the lower-dimensional interval  $P_{s-1}[l + 1, u - 1]$ . By Lemma 3.6,  $F \neq \emptyset$ ,  $\text{codim}_Q(F) \leq 2$ , and  $C(g_{s-1}(l)) = C(g_{s-1}(u)) = 1$ .

Apply Lemma 3.8 to the face  $F$  of  $Q$ . We obtain a row  $\widehat{C}$ , active on  $F$ , with endpoint excess vector  $(1, 1)$ , such that either  $\widehat{C} \in \mathcal{B}_{s-1}[l, u]$ , or  $\widehat{C} = C^L + C^R$ ,  $C^L \in \mathcal{L}_{s-1}[l, u]$ ,  $C^R \in \mathcal{R}_{s-1}[l, u]$ . Thus  $\widehat{C}$  is one of the projected rows, or normalized sums of projected rows, admitted by the bilateral branch of the recursion. Therefore the bilateral lift  $\text{BLift}_{l, u}(\widehat{C})$  is generated. Let  $\widehat{G}$  be the face of  $P_s[a, b]$  exposed by this generated row, and put  $\ell := g_{s-1}(l)$ ,  $v := g_{s-1}(u)$ ,  $\tau := \tau_l = \tau_u$ . By the tight-vertex containment established in Lemma 3.6 for the original row  $C$ ,  $G \subseteq \text{conv}((F \times \{1 - \tau\}) \cup \{(\ell, \tau), (v, \tau)\})$ . Since  $\widehat{C}$  is active on  $F$  and has endpoint excess vector  $(1, 1)$ , its bilateral lift is tight on  $F \times \{1 - \tau\}$  and on the two restored endpoint vertices  $(\ell, \tau)$ ,  $(v, \tau)$ . Hence  $G \subseteq \widehat{G}$ . The generated bilateral lift is valid for  $P_s[a, b]$ , so  $\widehat{G}$  is a face of  $P_s[a, b]$ . Since  $G$  is a facet, any valid face containing  $G$  is either  $G$  itself or the whole polytope. The latter is impossible. Indeed,  $F \neq \emptyset$ , and for any  $y \in F$  the lifted interval contains the full fiber over  $y$ . The generated bilateral lift is tight at  $(y, 1 - \tau)$ , but at the other point of the same fiber,  $(y, \tau)$ , its value is  $-1$ . Hence it is not identically tight on  $P_s[a, b]$ . Therefore  $\widehat{G} = G$ , and the generated bilateral lift exposes the same facet.

We have proved facet-completeness for  $P_s[a, b]$ . We now verify the affine-fixing assertion for the current interval.

Since  $[a + 1, b]$  contains an odd integer, Lemma 2.8 gives  $\text{aff}(P_s[a, b]) = \text{aff}(P_{s-1}[l, u]) \times \mathbb{R}$ . Thus the

only affine-hull fixings of  $P_s[a, b]$  are the inherited fixings from  $P_{s-1}[l, u]$ . By the induction hypothesis, these fixings are represented in  $\mathcal{H}_{s-1}[l, u]$  by both inequalities, and their zero extensions belong to  $\mathcal{H}_s[a, b]$ . Therefore  $\mathcal{H}_s[a, b]$  contains both inequalities associated with every affine-hull fixing of  $P_s[a, b]$ .

Having proved both inductive assertions for the current interval, Proposition 3.1 together with Lemma 3.9 give  $Q_s[a, b] = P_s[a, b]$ . The induction is complete.  $\square$

### 3.6 Minimality of the recursive description

In this section we refine the recursive construction so as to keep only facet-defining inequalities, together with the bounds needed to impose affine-hull fixings. We assume that the cases  $b = a$  and  $b = a + 1$  are handled as initial cases. Hence, throughout this section, the nontrivial recursive step satisfies  $s \geq 2$  and  $b \geq a + 2$ . Put  $l := \lfloor a/2 \rfloor$ ,  $u := \lfloor b/2 \rfloor$ ,  $\tau_l := (l \bmod 2) \oplus 1$ ,  $\tau_u := u \bmod 2$ , and write  $z = x_s$ . We also set  $R := P_{s-1}[l, u]$  and  $d := \dim R$ .

We assume inductively that, for every interval in dimension  $s - 1$ , we have a complete, minimal, and normalized description. The rows in such a description are of two types. First, for each affine-hull fixing  $y_j = \gamma_j$ , we keep both bounds, namely  $y_j - \gamma_j \leq 0$  and  $-y_j + \gamma_j \leq 0$ . Second, every remaining stored row is a facet-defining inequality, including nonredundant cube bounds. Notice that a cube bound never has positive excess at a binary endpoint, and therefore cannot belong to any of the auxiliary families  $\mathcal{L}$ ,  $\mathcal{R}$ , or  $\mathcal{B}$ .

For an interval  $I = [p, q]$  in dimension  $s - 1$ , we write  $d(I) := \dim P_{s-1}[p, q]$  and  $J(I) := J_{s-1}(p, q)$ . If  $p > q$ , we set  $P_{s-1}[p, q] = \emptyset$  and  $d(I) = -1$ .

We shall use the following intervals. The projected indices carrying a full last-coordinate fiber in  $P_s[a, b]$  form  $I^{\text{full}} := [l + \mathbf{1}_{\{a \text{ odd}\}}, u - \mathbf{1}_{\{b \text{ even}\}}]$ . For  $\gamma \in \{0, 1\}$ , the projected indices carrying the fiber value  $z = \gamma$  form  $I^\gamma := [l + \mathbf{1}_{\{a \text{ odd}, \gamma \neq \tau_l\}}, u - \mathbf{1}_{\{b \text{ even}, \gamma \neq \tau_u\}}]$ . For a left unilateral lift, the projected indices available in the tight fiber  $z = 1 - \tau_l$  are  $I^L := [l + 1, u - \mathbf{1}_{\{b \text{ even}, \tau_u = \tau_l\}}]$ , and, symmetrically, for a right unilateral lift,  $I^R := [l + \mathbf{1}_{\{a \text{ odd}, \tau_l = \tau_u\}}, u - 1]$ . Finally, in the bilateral branch we use  $Q := P_{s-1}[l + 1, u - 1]$ ,  $d_Q := \dim Q$ ,  $\ell := g_{s-1}(l)$ ,  $v := g_{s-1}(u)$ .

We give necessary and sufficient conditions for each possible recursive insertion to define a facet.

**Proposition 3.11** (Facet criteria for one recursive step). *Assume  $s \geq 2$  and  $b \geq a + 2$ . The following statements hold.*

- (i) *Let  $C(y) \leq 0$  be a stored row valid for  $R = P_{s-1}[l, u]$ . Its zero extension  $\text{Ext}(C)(y, z) := C(y) \leq 0$  defines a facet of  $P_s[a, b]$  if and only if  $C$  defines a facet of  $R$  and  $P_{s-1}[I^{\text{full}}] \cap \{y \in \mathbb{R}^{s-1} : C(y) = 0\} \neq \emptyset$ .*
- (ii) *For  $\gamma \in \{0, 1\}$ , the coordinate face  $z = \gamma$  is a facet of  $P_s[a, b]$  if and only if  $\dim P_{s-1}[I^\gamma] = d$ .*
- (iii) *Assume  $a$  is odd. Let  $C(y) \leq 0$  be valid for  $P_{s-1}[l + 1, u]$  and normalized by  $C(\ell) = 1$ . The left unilateral lift  $\text{Lift}_{l, \tau_l}(C)(y, z) = C(y) + (1 - 2\tau_l)z - (1 - \tau_l) \leq 0$  defines a facet of  $P_s[a, b]$  if and only if  $\dim\{y \in P_{s-1}[I^L] : C(y) = 0\} = d - 1$ . The symmetric right unilateral statement is obtained by replacing  $P_{s-1}[l + 1, u]$ ,  $\ell$ , and  $I^L$  by  $P_{s-1}[l, u - 1]$ ,  $v$ , and  $I^R$ , respectively.*
- (iv) *Assume  $a$  is odd,  $b$  is even, and  $\tau_l = \tau_u =: \tau$ . Let  $C(y) \leq 0$  be valid for  $Q = P_{s-1}[l + 1, u - 1]$  and normalized by  $C(\ell) = C(v) = 1$ . Let  $F_C := \{y \in Q : C(y) = 0\}$ . Then  $\text{BLift}_{l, u}(C)(y, z) = C(y) + (1 - 2\tau)z - (1 - \tau) \leq 0$  defines a facet of  $P_s[a, b]$  if and only if  $F_C \neq \emptyset$  and  $\dim F_C + 1 + \chi(F_C) = d$ , where*

$$\chi(F_C) := \begin{cases} 0, & v - \ell \in \text{lin}(F_C), \\ 1, & v - \ell \notin \text{lin}(F_C). \end{cases}$$

*Proof.* Since  $b \geq a + 2$ , the interval  $[a + 1, b]$  contains an odd integer. Therefore, by Lemma 2.8,  $\text{aff}(P_s[a, b]) = \text{aff}(R) \times \mathbb{R}$  and  $\dim P_s[a, b] = d + 1$ . Thus an exposed proper face of  $P_s[a, b]$  is a facet if and only if it has dimension  $d$ .

We prove the four statements in order.

(i) Let  $F := \{y \in R : C(y) = 0\}$ . The face exposed by  $\text{Ext}(C)$  on  $P_s[a, b]$  is the convex hull of all available fibers above the vertices of  $F$ . Its projection is  $F$ . It contains the vertical direction if and only if  $F$  contains at least one projected point whose last-coordinate fiber is full, which is equivalent to  $P_{s-1}[I^{\text{full}}] \cap F \neq \emptyset$ . If  $C$  defines a facet of  $R$ , then  $\dim F = d - 1$ . Hence the zero extension exposes a face of dimension  $d$  exactly when the vertical direction is present. Conversely, if the zero extension exposes a facet of  $P_s[a, b]$ , then its projection must have dimension  $d - 1$ , so  $C$  defines a facet of  $R$ , and the vertical direction must be present.

(ii) The face of  $P_s[a, b]$  contained in  $z = \gamma$  is  $P_{s-1}[I^\gamma] \times \{\gamma\}$ . Therefore it has dimension  $\dim P_{s-1}[I^\gamma]$ . Since a facet of  $P_s[a, b]$  has dimension  $d$ , the result follows.

(iii) We prove the left case; the right case is symmetric. Let  $W(y, z) := C(y) + (1 - 2\tau_l)z - (1 - \tau_l)$ . The restored endpoint  $(\ell, \tau_l)$  is tight. Every other tight vertex must have  $z = 1 - \tau_l$  and must satisfy  $C(y) = 0$ . The projected indices for which this tight fiber is available are precisely the indices in  $I^L$ . Hence the exposed face is  $\text{conv}(\{(\ell, \tau_l)\} \cup (\{y \in P_{s-1}[I^L] : C(y) = 0\} \times \{1 - \tau_l\}))$ . The restored endpoint has last coordinate  $\tau_l$ , whereas all other tight points have last coordinate  $1 - \tau_l$ . Therefore it adds exactly one affine dimension whenever the projected tight set is nonempty. Thus the lifted face has dimension  $1 + \dim\{y \in P_{s-1}[I^L] : C(y) = 0\}$ . It is a facet if and only if this dimension is  $d$ , which gives the claimed condition.

(iv) In the bilateral branch, the two restored endpoints are  $(\ell, \tau)$ ,  $(v, \tau)$ , and every tight point over  $Q$  must lie in the opposite fiber  $z = 1 - \tau$ . Hence the face exposed by the bilateral lift is  $G = \text{conv}(F_C \times \{1 - \tau\} \cup \{(\ell, \tau), (v, \tau)\})$ . If  $F_C = \emptyset$ , then  $G$  is contained in the segment joining the two restored endpoints and cannot be a facet in the present non-base bilateral case. Suppose  $F_C \neq \emptyset$ . Adding one restored endpoint gives one new independent direction, because its last coordinate differs from  $1 - \tau$ . The second restored endpoint contributes one additional dimension exactly when the projected chord  $v - \ell$  is not in  $\text{lin}(F_C)$ . Hence  $\dim G = \dim F_C + 1 + \chi(F_C)$ . Since  $G$  is a facet if and only if  $\dim G = d$ , the first statement follows.  $\square$

## 4 Separation for the recursive description

We now discuss the separation problem associated with the recursive description of Section 3. Given  $\bar{x} \in \mathbb{R}^n$  and an interval  $0 \leq a \leq b \leq 2^n - 1$ , the goal is either to certify that  $\bar{x} \in P_n[a, b]$ , or to return a row  $C \in \mathcal{H}_n[a, b]$  such that  $C(\bar{x}) > 0$ . Below we present a dynamic program that solves the separation problem.

### 4.1 Boundary signatures

The dynamic program only needs to remember how a generated row behaves at the two interval endpoints and at the two adjacent exterior vertices.

For an interval  $I = [p, q] \subseteq \{0, \dots, 2^s - 1\}$ , write  $\partial I := (p^-, p, q, q^+)$ , where  $p^- := p - 1$ , if  $p > 0$ , and  $q^+ := q + 1$ , if  $q < 2^s - 1$ . If one of these exterior indices does not exist, we use the symbol  $\star$ .

For a row  $C \in \mathcal{H}_s[p, q]$ , define its boundary signature  $\sigma_{s,p,q}(C) = (\sigma^-(C), \sigma^L(C), \sigma^R(C), \sigma^+(C)) \in \{\star, 1, 0, -\}^4$  as follows. At an existing boundary index  $r \in \{p^-, p, q, q^+\}$ , set

$$\sigma_{s,p,q}(C, r) := \begin{cases} 1, & C(g_s(r)) = 1, \\ 0, & C(g_s(r)) = 0, \\ -, & C(g_s(r)) < 0. \end{cases}$$

If the index does not exist, the corresponding signature component is  $\star$ .

This definition is well posed for generated rows. Indeed, rows in  $\mathcal{H}_s[p, q]$  are valid for  $P_s[p, q]$  by Proposition 3.1, so their values at the two interval endpoints  $p$  and  $q$  are nonpositive. At the two

exterior endpoints, Proposition 3.2 implies that any positive excess is exactly one. Hence the three categories 1, 0,  $-$  capture all possible signs relevant to membership in the auxiliary families  $\mathcal{L}_s, \mathcal{R}_s, \mathcal{B}_s$ .

For each state  $(s, p, q)$  and each signature  $\sigma$ , define

$$M_s[p, q, \sigma] := \max\{C(\bar{x}^{(s)}) : C \in \mathcal{H}_s[p, q], \sigma_{s,p,q}(C) = \sigma\},$$

where  $\bar{x}^{(s)} := (\bar{x}_1, \dots, \bar{x}_s)$ , and where the maximum over an empty set is  $-\infty$ . Along with the value  $M_s[p, q, \sigma]$ , the algorithm stores one row attaining the maximum, or a pointer from which such a row can be reconstructed.

For a collection  $\mathcal{S}$  of signatures, write  $M_s[p, q; \mathcal{S}] := \max_{\sigma \in \mathcal{S}} M_s[p, q, \sigma]$ . We use the following shorthand. The family  $\mathcal{L}_s[p, q]$  is represented by rows in  $\mathcal{H}_s[p+1, q]$  whose left exterior excess is one, namely  $M(\mathcal{L}_s[p, q]) := M_s[p+1, q; \sigma^- = 1]$ . Similarly,  $M(\mathcal{R}_s[p, q]) := M_s[p, q-1; \sigma^+ = 1]$ , and  $M(\mathcal{B}_s[p, q]) := M_s[p+1, q-1; \sigma^- = 1, \sigma^+ = 1]$ . The two filtered families needed for normalized sums are  $M(\mathcal{L}_s^0[p, q]) := M_s[p+1, q; \sigma^- = 1, \sigma^R = 0]$ , and  $M(\mathcal{R}_s^0[p, q]) := M_s[p, q-1; \sigma^L = 0, \sigma^+ = 1]$ . Thus  $\mathcal{L}_s^0[p, q]$  consists of rows in  $\mathcal{L}_s[p, q]$  that are also tight at the right endpoint  $g_s(q)$ , and  $\mathcal{R}_s^0[p, q]$  consists of rows in  $\mathcal{R}_s[p, q]$  that are also tight at the left endpoint  $g_s(p)$ .

## 4.2 Dynamic recursion

We now propagate these maxima through the same operations that define the recursive system. For the separation oracle associated with the target interval  $[a, b]$ , the dynamic program is not built over all intervals in all dimensions. We first define the set of reachable states.

Let  $\mathcal{K}(a, b)$  be the smallest collection of triples  $(s, p, q)$  such that  $(n, a, b) \in \mathcal{K}(a, b)$ , and, whenever  $(s, p, q) \in \mathcal{K}(a, b)$ ,  $s \geq 2$ , and  $p < q$ , with  $l := \lfloor p/2 \rfloor$ ,  $u := \lfloor q/2 \rfloor$ , then  $\mathcal{K}(a, b)$  also contains every nonempty triple among  $(s-1, l, u)$ ,  $(s-1, l+1, u)$ ,  $(s-1, l, u-1)$ , and  $(s-1, l+1, u-1)$ . Here nonempty means that the projected interval has left endpoint at most its right endpoint. These are precisely the intervals that can be queried by the zero-extension branch, the unilateral-lift branches, the bilateral branch, and the normalized-sum branch of the recursion.

The table  $M_s[p, q, \sigma]$  is defined only for triples  $(s, p, q) \in \mathcal{K}(a, b)$ . Equivalently, the algorithm constructs only the part of the recursive system that is reachable from the target interval. In particular, singleton and segment states are initialized only when they belong to  $\mathcal{K}(a, b)$ .

The states are evaluated in any topological order in which all nontrivial predecessors of  $(s, p, q)$  have already been processed. Since every non-initial transition used below comes from dimension  $s-1$ , processing the reachable states by increasing dimension gives such an order.

The initial reachable states are explicit. If  $(s, p, p) \in \mathcal{K}(a, b)$ , then  $\mathcal{H}_s[p, p] = \{x_i - g_s(p)_i \leq 0, -x_i + g_s(p)_i \leq 0 : i = 1, \dots, s\}$ . If  $(s, p, p+1) \in \mathcal{K}(a, b)$ , then  $\mathcal{H}_s[p, p+1] := \{x_i - g_s(p)_i \leq 0, -x_i + g_s(p)_i \leq 0 : i = 1, \dots, s, i \neq r\} \cup \{x_r - 1 \leq 0, -x_r \leq 0\}$ , where  $r := s - \nu(p+1)$ . The corresponding table entries are obtained by evaluating these  $\mathcal{O}(s)$  rows at  $\bar{x}^{(s)}$  and assigning each row to its boundary signature.

Now let  $(s, p, q) \in \mathcal{K}(a, b)$  be a non-initial state, so that  $s \geq 2$  and  $q \geq p+2$ . Put  $l := \lfloor p/2 \rfloor$ ,  $u := \lfloor q/2 \rfloor$ , and  $\tau_l := (l \bmod 2) \oplus 1$ ,  $\tau_u := u \bmod 2$ . Also recall, for  $\tau \in \{0, 1\}$ ,  $\theta_\tau(z) = (1 - 2\tau)z - (1 - \tau)$ . The candidate rows are generated as follows.

First, we insert zero extensions. The table entries are obtained from the candidates  $\text{Ext}(D)(y, z) = D(y)$ ,  $D \in \mathcal{H}_{s-1}[l, u]$ , with value  $\text{Ext}(D)(\bar{x}^{(s)}) = D(\bar{x}^{(s-1)})$ . The boundary signature of an extended row is obtained from  $\text{exc}_s(\text{Ext}(D); t) = \text{exc}_{s-1}(D; \lfloor \frac{t}{2} \rfloor)$ , evaluated at the at most four boundary indices  $t \in \{p^-, p, q, q^+\}$ .

Second, we insert the two last-coordinate bounds  $-x_s \leq 0$ ,  $x_s - 1 \leq 0$ .

Third, if  $p$  is odd, then the left unilateral branch contributes  $\text{Lift}_{l, \tau_l}(D)$ ,  $D \in \mathcal{L}_{s-1}[l, u]$ . For each signature class of rows  $D \in \mathcal{L}_{s-1}[l, u]$ , the value of the lifted row at  $\bar{x}^{(s)} = (\bar{x}^{(s-1)}, \bar{x}_s)$  is  $D(\bar{x}^{(s-1)}) + \theta_{\tau_l}(\bar{x}_s)$ . The boundary signature is computed from  $\text{exc}_s(\text{Lift}_{l, \tau_l}(D); t) = \text{exc}_{s-1}(D; \lfloor \frac{t}{2} \rfloor) + \theta_{\tau_l}(g_s(t)_s)$ , for  $t \in \{p^-, p, q, q^+\}$ . Since  $\theta_{\tau_l}(g_s(t)_s) \in \{0, -1\}$  at binary vertices, the signature of the lifted row is determined by the signature of  $D$ .

Fourth, if  $q$  is even, the right unilateral branch contributes  $\text{Lift}_{u, \tau_u}(D)$ ,  $D \in \mathcal{R}_{s-1}[l, u]$ , with value  $D(\bar{x}^{(s-1)}) + \theta_{\tau_u}(\bar{x}_s)$ , and with boundary signature computed in the same way.

Finally, suppose that  $p$  is odd,  $q$  is even, and  $\tau_l = \tau_u =: \tau$ . Then the bilateral branch contributes two types of candidates. The first type is  $\text{BLift}_{l, u}(D)$ ,  $D \in \mathcal{B}_{s-1}[l, u]$ . Each such candidate has value  $D(\bar{x}^{(s-1)}) + \theta_\tau(\bar{x}_s)$ . The second type comes from normalized sums  $D = D^L + D^R$ ,  $D^L \in \mathcal{L}_{s-1}[l, u]$ ,  $D^R \in \mathcal{R}_{s-1}[l, u]$ , satisfying  $\text{exc}_{s-1}(D; l) = \text{exc}_{s-1}(D; u) = 1$ . For  $D^L \in \mathcal{L}_{s-1}[l, u]$ , the row belongs to  $\mathcal{H}_{s-1}[l+1, u]$ , whose right endpoint is  $u$ . Therefore  $D^L(g_{s-1}(u)) = 0$  if and only if  $\sigma^R(D^L) = 0$ . Similarly, for  $D^R \in \mathcal{R}_{s-1}[l, u]$ , the row belongs to  $\mathcal{H}_{s-1}[l, u-1]$ , whose left endpoint is  $l$ , and hence  $D^R(g_{s-1}(l)) = 0$  if and only if  $\sigma^L(D^R) = 0$ . Thus the normalization conditions restrict  $D^L$  and  $D^R$  independently. Therefore the best normalized sum separates additively:

$$\begin{aligned} & \max \left\{ (D^L + D^R)(\bar{x}^{(s-1)}) : D^L \in \mathcal{L}_{s-1}[l, u], D^R \in \mathcal{R}_{s-1}[l, u], \right. \\ & \qquad \qquad \qquad \left. D^L(g_{s-1}(u)) = 0, D^R(g_{s-1}(l)) = 0 \right\} \\ & = M(\mathcal{L}_{s-1}^0[l, u]) + M(\mathcal{R}_{s-1}^0[l, u]). \end{aligned}$$

Consequently, the best candidate of this type has value  $M(\mathcal{L}_{s-1}^0[l, u]) + M(\mathcal{R}_{s-1}^0[l, u]) + \theta_\tau(\bar{x}_s)$ , provided both terms are finite. The corresponding generated row is  $\text{BLift}_{l, u}(D^L + D^R)$ .

For every bilateral candidate, whether it comes from  $\mathcal{B}_{s-1}[l, u]$  or from a normalized sum, the boundary values at the four boundary vertices of  $[p, q]$  are fixed. Indeed, let  $p = 2l + 1$ ,  $q = 2u$ , and assume  $\tau_l = \tau_u =: \tau$ . Let  $C$  be any projected row admitted by the bilateral branch, either  $C \in \mathcal{B}_{s-1}[l, u]$  or a normalized sum. In both cases,  $C(g_{s-1}(l)) = C(g_{s-1}(u)) = 1$ . The bilateral lift has the form  $\widehat{C}(y, z) = C(y) + \theta_\tau(z)$ ,  $\theta_\tau(z) := (1 - 2\tau)z - (1 - \tau)$ . Since  $\theta_\tau(1 - \tau) = 0$ ,  $\theta_\tau(\tau) = -1$ , and since the four boundary vertices of  $[p, q]$  are  $(g_{s-1}(l), 1 - \tau)$ ,  $(g_{s-1}(l), \tau)$ ,  $(g_{s-1}(u), \tau)$ ,  $(g_{s-1}(u), 1 - \tau)$ , we obtain  $\text{exc}_s(\widehat{C}; p^-) = 1$ ,  $\text{exc}_s(\widehat{C}; p) = 0$ , and  $\text{exc}_s(\widehat{C}; q) = 0$ ,  $\text{exc}_s(\widehat{C}; q^+) = 1$ . Thus every bilateral candidate has boundary signature  $(1, 0, 0, 1)$ .

**Theorem 4.1** (Polynomial-time separation). *There is a polynomial-time separation algorithm for the recursive description  $\mathcal{H}_n[a, b]$ . More precisely, given  $\bar{x} \in \mathbb{R}^n$ , the algorithm either returns a row  $C \in \mathcal{H}_n[a, b]$  with  $C(\bar{x}) > 0$ , or certifies that  $\bar{x} \in P_n[a, b]$ .*

*Proof.* By construction of the dynamic program, the following invariant holds for every reachable state  $(s, p, q)$  and every boundary signature  $\sigma$ : the table entry  $M_s[p, q, \sigma]$  is the maximum value of  $C(\bar{x}^{(s)})$  over all rows  $C \in \mathcal{H}_s[p, q]$  with signature  $\sigma_{s, p, q}(C) = \sigma$ , with value  $-\infty$  if no such row exists. Moreover, whenever the value is finite, the stored predecessor pointers reconstruct a row attaining it. Indeed, the initialization enumerates exactly the coordinate rows in the singleton and segment cases, while the recursive transitions are in one-to-one correspondence with the zero extensions, the last-coordinate bounds, the unilateral lifts, and the bilateral lifts defining  $\mathcal{H}_s[p, q]$ . For normalized bilateral sums, the two normalization conditions involve  $D^L$  and  $D^R$  separately, so the best value is obtained by adding the two independent maxima over  $\mathcal{L}_{s-1}^0[l, u]$  and  $\mathcal{R}_{s-1}^0[l, u]$ . Thus the invariant follows by induction over the topological order of the reachable states.

Let  $M^* := \max_\sigma M_n[a, b, \sigma]$ . If  $M^* > 0$ , the invariant reconstructs a row  $C \in \mathcal{H}_n[a, b]$  such that  $C(\bar{x}) = M^* > 0$ . This row is valid for  $P_n[a, b]$  by Proposition 3.1, and hence it separates  $\bar{x}$ . If  $M^* \leq 0$ , then every row in  $\mathcal{H}_n[a, b]$  is satisfied by  $\bar{x}$ . By Theorem 3.10,  $P_n[a, b] = \{x \in \mathbb{R}^n : C(x) \leq 0 \text{ for all } C \in \mathcal{H}_n[a, b]\}$ . Therefore  $\bar{x} \in P_n[a, b]$ .

It remains to bound the running time. Recall that the dynamic program is restricted to the reachable triples in  $\mathcal{K}(a, b)$ . The computation can be viewed as a longest-path problem on an acyclic directed hypergraph. Its nodes are  $(s, p, q, \sigma)$  for  $(s, p, q) \in \mathcal{K}(a, b)$ , where  $\sigma$  is a boundary signature. Zero extensions, unilateral lifts, and simple bilateral lifts give ordinary arcs from lower-dimensional states to higher-dimensional states. The normalized bilateral-sum operation gives a directed hyperarc with two tails, corresponding to the independent choices of rows in  $\mathcal{L}_{s-1}^0[l, u]$  and  $\mathcal{R}_{s-1}^0[l, u]$ . Since every arc or hyperarc goes from dimension  $s - 1$  to dimension  $s$ , the hypergraph is acyclic. Hence the

longest-path labels can be computed in topological order in time linear in the number of nodes plus arcs and hyperarcs; see, for example, [7] for optimal path algorithms in directed hypergraphs.

We now bound the size of this hypergraph. Consider a reachable state obtained after  $k$  projections from the target interval  $[a, b]$ , so that its dimension is  $n - k$ . Its left endpoint is of the form  $\lfloor a/2^k \rfloor + \delta$ ,  $0 \leq \delta \leq k$ . Indeed, at each projection step the left endpoint is transformed either into  $\lfloor p/2 \rfloor$  or into  $\lfloor p/2 \rfloor + 1$ , so the accumulated shift can increase by at most one per level. Similarly, every reachable right endpoint is of the form  $\lfloor b/2^k \rfloor - \delta'$ ,  $0 \leq \delta' \leq k$ . Thus, after  $k$  projections, there are at most  $k + 1$  possible left endpoints and at most  $k + 1$  possible right endpoints. Therefore the number of reachable intervals at depth  $k$  is  $O(k^2)$ , and  $|\mathcal{K}(a, b)| \leq \sum_{k=0}^n O(k^2) = O(n^3)$ . For each reachable interval there are only constantly many boundary signatures, and each state has only constantly many incident arcs or constant-size hyperarcs. Therefore the associated acyclic hypergraph has polynomial size. The separation labels, together with predecessor pointers that give a compact recursive representation of a violated row, can therefore be computed in polynomial time.  $\square$

## 5 Compact descriptions of prefix and suffix intervals

For prefix and suffix intervals  $P_n[0, b]$  and  $P_n[a, 2^n - 1]$ , the endpoint-lifting recursion collapses to a one-sided construction. This allows the generated rows to be written directly from the binary expansion of the endpoint.

### 5.1 The interval $P_n[0, b]$

Unlike the general recursion, the prefix specialization has no left deleted endpoint and no bilateral interaction. Thus the only nontrivial operation that can create new inequalities is a right unilateral lift. We encode the resulting inequalities by grouping the relevant bit positions into maximal consecutive blocks.

For  $c \in \{0, \dots, 2^s - 1\}$ , write  $\text{bin}_s(c) = (c_1, \dots, c_s)$ . Let  $Z_s(c) := \{k \in \{1, \dots, s\} : c_k = 0\}$ . For each  $k \in Z_s(c)$ , define the dependency set  $V_s^k(c) := \{j < k : c_j = 1\} \cup \{k\}$ . Let  $\mathcal{I}_s^k(c) = \{I_1, \dots, I_{p_k}\}$  be the partition of  $V_s^k(c)$  into maximal contiguous intervals, ordered from left to right. For an interval  $I_q$ , write  $f_q := \min I_q$ . The corresponding prefix block row is

$$\Gamma_s^k(c)(x) := \sum_{q=1}^{p_k} \left( x_{f_q} - \sum_{j \in I_q \setminus \{f_q\}} x_j \right) - (p_k - 1).$$

Thus the prefix block inequality is  $\Gamma_s^k(c)(x) \leq 0$ . These inequalities are closely related to the minimal covers that suffice to describe the convex hull in the lexicographic ordering case [14], although negative signs appear in the Gray ordering setting.

We first record the elementary binary interpretation of one block.

**Lemma 5.1** (Block tightness criterion). *Let  $I = [f, l] \subseteq \{1, \dots, s\}$ , and define  $\phi_I(x) := x_f - \sum_{j=f+1}^l x_j$ . Let  $x \in \{0, 1\}^s$ , and let  $w \in \{0, 1\}^s$  be the associated binary vector  $w_i = x_1 \oplus x_2 \oplus \dots \oplus x_i$ ,  $i = 1, \dots, s$ , with the convention  $w_0 = 0$ . Then  $\phi_I(x) \leq 1$ . Moreover,  $\phi_I(x) = 1$  if and only if  $w_f = w_{f+1} = \dots = w_l = 1 - w_{f-1}$ .*

*Proof.* Since  $x \in \{0, 1\}^s$ ,  $\phi_I(x) = x_f - \sum_{j=f+1}^l x_j \leq x_f \leq 1$ . Equality holds if and only if  $x_f = 1$  and  $x_j = 0$  for every  $j = f + 1, \dots, l$ . Since  $x_f = w_{f-1} \oplus w_f$ , the condition  $x_f = 1$  is equivalent to  $w_f = 1 - w_{f-1}$ . Similarly, the equalities  $x_j = 0$ ,  $j = f + 1, \dots, l$ , are equivalent to  $w_j = w_{j-1}$  for  $j = f + 1, \dots, l$ . Hence equality holds precisely when  $w_f = w_{f+1} = \dots = w_l = 1 - w_{f-1}$ .  $\square$

The next lemma identifies the unique block row that separates the next Gray vertex after a prefix.

**Lemma 5.2** (Successor block row). *For  $0 \leq c < 2^s - 1$ , let  $r := \max Z_s(c)$  be the rightmost zero in the  $s$ -bit binary expansion of  $c$ . Then, among the prefix block rows  $\Gamma_s^k(c)$ ,  $k \in Z_s(c)$ , the only row with positive excess at the next Gray vertex  $g_s(c+1)$  is  $\Gamma_s^r(c)$ . Moreover,  $\Gamma_s^r(c)(g_s(c+1)) = 1$ .*

*Proof.* Let  $w = \text{bin}_s(c+1)$ . Since  $r$  is the rightmost zero of  $\text{bin}_s(c)$ , the binary addition  $c \mapsto c+1$  leaves all positions before  $r$  unchanged, turns position  $r$  from 0 to 1, and turns all positions after  $r$  from 1 to 0. Thus  $w_j = c_j$  for  $j < r$ ,  $w_r = 1$ , and  $w_j = 0$  for  $j > r$ .

Consider first the row  $\Gamma_s^r(c)$ . If  $I = [f, l]$  is a maximal block of  $V_s^r(c)$ , then all positions in  $I$  have  $w$ -value equal to 1. Moreover, if  $f > 1$ , then  $f-1 \notin V_s^r(c)$  and  $f-1 < r$ , so  $c_{f-1} = 0$ , and hence  $w_{f-1} = 0$ . If  $f = 1$ , we use the convention  $w_0 = 0$ . Therefore Lemma 5.1 gives  $x_f - \sum_{j=f+1}^l x_j = 1$  for every block  $I = [f, l] \in \mathcal{I}_s^r(c)$ , evaluated at  $x = g_s(c+1)$ . Hence  $\Gamma_s^r(c)(g_s(c+1)) = p_r - (p_r - 1) = 1$ .

Now let  $k \in Z_s(c)$  with  $k \neq r$ . Since  $r$  is the rightmost zero, we have  $k < r$ . Then  $w_j = c_j$  for every  $j \leq k$ , and in particular  $w_k = c_k = 0$ . Let  $I$  be the block of  $V_s^k(c)$  containing  $k$ . If  $\phi_I(g_s(c+1))$  were equal to 1, then Lemma 5.1 would force the  $w$ -values to be constant on  $I$ . This is impossible if  $k$  is attached to a preceding run of ones, and if  $I = \{k\}$ , then the equality condition would require  $w_k = 1 - w_{k-1}$ , which also fails because either  $k = 1$  and  $w_0 = 0$ , or  $c_{k-1} = 0$  by maximality of the singleton block. Thus the block containing  $k$  contributes strictly less than 1, while all other blocks contribute at most 1. Therefore  $\Gamma_s^k(c)(g_s(c+1)) \leq 0$ .  $\square$

We now show that the block prefix rows are exactly the rows generated by the endpoint-lifting recursion in the special case  $a = 0$ .

**Lemma 5.3** (Prefix recursion generates block rows). *For every  $s \geq 1$  and every  $c \in \{0, \dots, 2^s - 1\}$ , each row generated by the prefix specialization of the recursion for  $\mathcal{H}_s[0, c]$  is either a cube bound, that is, one of  $-x_j \leq 0$  or  $x_j - 1 \leq 0$ , or one of the prefix block rows  $\Gamma_s^k(c)(x) \leq 0$ ,  $k \in Z_s(c)$ . Moreover, every prefix block row  $\Gamma_s^k(c)(x) \leq 0$ ,  $k \in Z_s(c)$ , is generated by the recursion.*

*Proof.* We argue by induction on  $s$ , where the case  $s = 1$  is covered by  $c = 0$  and  $c = 1$ .

If  $c = 0$ , then  $P_s[0, 0]$  is the singleton  $\{g_s(0)\} = \{0\}$ . The initialization gives the rows  $x_j \leq 0$  and  $-x_j \leq 0$  for  $j = 1, \dots, s$ . Since all bits of  $c = 0$  are zero, each  $V_s^j(c)$  is the singleton  $\{j\}$ , and the corresponding prefix block row is exactly  $x_j \leq 0$ . Thus the claim holds.

If  $c = 1$ , then  $P_s[0, 1]$  is the segment  $\text{conv}(\{g_s(0), g_s(1)\}) = \text{conv}(\{0, e^s\})$ . The initialization gives the rows  $x_j \leq 0$  and  $-x_j \leq 0$  for  $j = 1, \dots, s-1$ ,  $x_s - 1 \leq 0$ , and  $-x_s \leq 0$ . Thus the claim holds as above.

Now assume  $s \geq 2$  and  $c \geq 2$ . Write  $c = 2u + \varepsilon$ ,  $\varepsilon \in \{0, 1\}$ . In the recursion for  $\mathcal{H}_s[0, c]$ , we have  $l := 0$  and  $u := \lfloor c/2 \rfloor \geq 1$ . Because the left endpoint is 0, no left unilateral lift and no bilateral lift can occur. Therefore the recursion consists only of zero extensions, the last-coordinate bounds, and, when  $c$  is even, right unilateral lifts.

Suppose first that  $\varepsilon = 1$ , so  $c = 2u + 1$ . Then  $\text{bin}_s(c) = (\text{bin}_{s-1}(u), 1)$ . Hence the zero positions of  $\text{bin}_s(c)$  are exactly the zero positions of  $\text{bin}_{s-1}(u)$ , and for each such zero the dependency set and its block decomposition are unchanged. Thus the zero extensions of the rows generated for  $\mathcal{H}_{s-1}[0, u]$  are exactly the block rows  $\Gamma_s^k(c)$  with  $k < s$ , along with cube bounds. Since  $c_s = 1$ , there is no new block row indexed by  $s$ . The only additional rows are the bounds  $-x_s \leq 0$ ,  $x_s - 1 \leq 0$ . The claim follows in this case.

Now suppose that  $\varepsilon = 0$ , so  $c = 2u$ . In addition to cube bounds, the zero extensions of  $\mathcal{H}_{s-1}[0, u]$  generate exactly the block rows associated with the zero positions of  $\text{bin}_{s-1}(u)$ , by the induction hypothesis. These are precisely the block rows  $\Gamma_s^k(c)$  with  $k < s$ , because  $\text{bin}_s(c) = (\text{bin}_{s-1}(u), 0)$ .

It remains to identify the right unilateral lifts. By definition,  $\mathcal{R}_{s-1}[0, u] = \{C \in \mathcal{H}_{s-1}[0, u-1] : \text{exc}_{s-1}(C; u) = 1\}$ . By the induction hypothesis, the rows of  $\mathcal{H}_{s-1}[0, u-1]$  are cube bounds and prefix block rows for the prefix endpoint  $u-1$ . Cube bounds cannot have positive excess at a binary point. Therefore Lemma 5.2, applied in dimension  $s-1$  to the prefix endpoint  $u-1$ , shows that the only row in  $\mathcal{H}_{s-1}[0, u-1]$  with unit excess at  $g_{s-1}(u)$  is the successor block row  $\Gamma_{s-1}^r(u-1)$ ,  $r := \max Z_{s-1}(u-1)$ .

We claim that its right unilateral lift is exactly the new block row  $\Gamma_s^s(c)$ . Let  $\text{bin}_{s-1}(u-1) = (d_1, \dots, d_{s-1})$ . Since  $r$  is the rightmost zero of  $u-1$ , the binary expansion of  $u$  satisfies  $\text{bin}_{s-1}(u)_j = d_j$  for  $j < r$ ,  $\text{bin}_{s-1}(u)_r = 1$ , and  $\text{bin}_{s-1}(u)_j = 0$  for  $j > r$ . Thus  $V_s^s(c) = V_{s-1}^r(u-1) \cup \{s\}$ .

There are two cases. If  $r = s-1$ , then  $u$  is odd, so the recursion uses  $\tau_u = 1$ , and the unilateral lift is  $\text{Lift}_{u,1}(\Gamma_{s-1}^r(u-1))(y, x_s) = \Gamma_{s-1}^r(u-1)(y) - x_s$ . In this case the new index  $s$  is appended to the last block of  $V_{s-1}^r(u-1)$ , so this row is exactly  $\Gamma_s^s(c)(x)$ .

If  $r < s-1$ , then  $u$  is even, so the recursion uses  $\tau_u = 0$ , and the unilateral lift is  $\text{Lift}_{u,0}(\Gamma_{s-1}^r(u-1))(y, x_s) = \Gamma_{s-1}^r(u-1)(y) + x_s - 1$ . Here the new index  $s$  forms a new singleton block of  $V_s^s(c)$ , increasing the number of blocks by one. Hence the right-hand side increases by one, and the row above is again exactly  $\Gamma_s^s(c)(x)$ .

Therefore the only right unilateral lift generated in the prefix specialization is precisely the new block row indexed by the last coordinate. The induction is complete.  $\square$

The compact prefix description is an immediate consequence of the exact recursive description and the identification in Lemma 5.3.

**Theorem 5.4** (Compact prefix description). *For every  $b \in \{0, \dots, 2^n - 1\}$ , the prefix Gray interval satisfies  $P_n[0, b] = \{x \in [0, 1]^n : \Gamma_n^k(b)(x) \leq 0 \text{ for all } k \in Z_n(b)\}$ .*

*Proof.* Let  $Q$  denote the polyhedron described by the right-hand side. By Lemma 5.3, every row generated by the prefix specialization of the recursion for  $\mathcal{H}_n[0, b]$  is either a cube bound or one of the displayed prefix block rows. Hence every point of  $Q$  satisfies all rows in  $\mathcal{H}_n[0, b]$ . By Theorem 3.10, this gives  $Q \subseteq P_n[0, b]$ . Conversely, the cube bounds are valid for  $P_n[0, b]$ , and every displayed prefix block row is generated by the recursion, hence is valid by Proposition 3.1. Therefore  $P_n[0, b] \subseteq Q$ . Thus  $Q = P_n[0, b]$ .  $\square$

## 5.2 The interval $P_n[a, 2^n - 1]$

The suffix case is obtained from the prefix case by the reflection symmetry of the binary reflected Gray code. Thus the inequalities are indexed by the ones of the lower endpoint  $a$ , rather than by the zeros of the upper endpoint  $b$ .

**Lemma 5.5** (Gray reflection symmetry). *For every  $i \in \{0, \dots, 2^n - 1\}$ , we have  $g_n(2^n - 1 - i) = g_n(i) \oplus e^1$ .*

*Proof.* Let  $y = \text{bin}_n(i) = (y_1, \dots, y_n)$ , and let  $\bar{y} := \text{bin}_n(2^n - 1 - i)$ . Since  $2^n - 1$  is the integer whose binary expansion is  $(1, \dots, 1)$ , we have  $\bar{y}_j = 1 - y_j$  for  $j = 1, \dots, n$ . For a binary vector  $w = (w_1, \dots, w_n)$ , denote its Gray encoding by  $G(w)_1 = w_1$ ,  $G(w)_j = w_{j-1} \oplus w_j$  for  $j = 2, \dots, n$ . Then  $g_n(i) = G(\text{bin}_n(i))$ . Therefore  $G(\bar{y})_1 = \bar{y}_1 = 1 - y_1 = G(y)_1 \oplus 1$ , and, for every  $j \geq 2$ ,  $G(\bar{y})_j = \bar{y}_{j-1} \oplus \bar{y}_j = (1 - y_{j-1}) \oplus (1 - y_j) = y_{j-1} \oplus y_j = G(y)_j$ . Hence  $G(\bar{y}) = G(y) \oplus e^1$ . Since  $g_n(i) = G(y)$  and  $g_n(2^n - 1 - i) = G(\bar{y})$ , the result follows.  $\square$

**Corollary 5.6** (Prefix-suffix reflection). *For every  $a \in \{0, \dots, 2^n - 1\}$ , we have  $P_n[a, 2^n - 1] = T(P_n[0, 2^n - 1 - a])$ , where  $T(x_1, x_2, \dots, x_n) := (1 - x_1, x_2, \dots, x_n)$ .*

*Proof.* Observe that  $T$  is an affine bijection of  $\mathbb{R}^n$  that coincides with the  $\oplus e^1$  operator on  $\{0, 1\}^n$ .

By Lemma 5.5, for every  $i \in \{0, \dots, 2^n - 1 - a\}$ ,  $T(g_n(i)) = g_n(i) \oplus e^1 = g_n(2^n - 1 - i)$ . As  $i$  ranges from 0 to  $2^n - 1 - a$ , the index  $2^n - 1 - i$  ranges from  $2^n - 1$  down to  $a$ . Therefore  $T(\{g_n(i) : 0 \leq i \leq 2^n - 1 - a\}) = \{g_n(j) : a \leq j \leq 2^n - 1\}$ . Since  $T$  is affine, it commutes with convex hull, and thus  $T(P_n[0, 2^n - 1 - a]) = T(\text{conv}\{g_n(i) : 0 \leq i \leq 2^n - 1 - a\}) = \text{conv}(\{g_n(j) : a \leq j \leq 2^n - 1\}) = P_n[a, 2^n - 1]$ .  $\square$

It remains only to write the image of the prefix block rows under this reflection. Let  $a \in \{0, \dots, 2^n - 1\}$ , and write  $\text{bin}_n(a) = (a_1, \dots, a_n)$ . Define  $O_n(a) := \{k \in \{1, \dots, n\} : a_k = 1\}$ . For each  $k \in O_n(a)$ ,

define  $V_n^k(a) := \{j < k : a_j = 0\} \cup \{k\}$ . Let  $\mathcal{I}_n^k(a) = \{I_1, \dots, I_{p_k}\}$  be the partition of  $V_n^k(a)$  into maximal contiguous intervals, ordered from left to right, and let  $f_q := \min I_q$ .

If  $1 \notin V_n^k(a)$ , set

$$\Delta_n^k(a)(x) := \sum_{q=1}^{p_k} \left( x_{f_q} - \sum_{j \in I_q \setminus \{f_q\}} x_j \right) - (p_k - 1).$$

If  $1 \in V_n^k(a)$ , set

$$\Delta_n^k(a)(x) := \left( -x_1 - \sum_{j \in I_1 \setminus \{1\}} x_j \right) + \sum_{q=2}^{p_k} \left( x_{f_q} - \sum_{j \in I_q \setminus \{f_q\}} x_j \right) - (p_k - 2).$$

These reflected rows give the compact suffix description.

**Theorem 5.7** (Compact suffix description). *For every  $a \in \{0, \dots, 2^n - 1\}$ , the suffix Gray interval satisfies  $P_n[a, 2^n - 1] = \{x \in [0, 1]^n : \Delta_n^k(a)(x) \leq 0 \text{ for all } k \in O_n(a)\}$ .*

*Proof.* Let  $c := 2^n - 1 - a$ . Then  $\text{bin}_n(c) = (1 - a_1, \dots, 1 - a_n)$ . Hence  $Z_n(c) = \{k : c_k = 0\} = \{k : a_k = 1\} = O_n(a)$ . Moreover, for  $k \in O_n(a)$ , the dependency set used in the prefix description of  $P_n[0, c]$  is  $V_n^k(c) = \{j < k : c_j = 1\} \cup \{k\} = \{j < k : a_j = 0\} \cup \{k\} = V_n^k(a)$ .

By Corollary 5.6,  $P_n[a, 2^n - 1] = T(P_n[0, c])$ , where  $T(x_1, x_2, \dots, x_n) = (1 - x_1, x_2, \dots, x_n)$ . By Theorem 5.4, the prefix polytope  $P_n[0, c]$  is described by the hypercube bounds and the prefix block inequalities indexed by  $k \in Z_n(c)$ . Applying the substitution  $x_1 \leftarrow 1 - x_1$  and  $x_j \leftarrow x_j$  for  $j \geq 2$  preserves the hypercube bounds.

Fix  $k \in O_n(a) = Z_n(c)$ , and let  $\mathcal{I}_n^k(a) = \{I_1, \dots, I_{p_k}\}$  be the corresponding block decomposition. If  $1 \notin V_n^k(a)$ , then the prefix block inequality does not involve  $x_1$ , so it is unchanged. This gives  $\sum_{q=1}^{p_k} \left( x_{f_q} - \sum_{j \in I_q \setminus \{f_q\}} x_j \right) \leq p_k - 1$ .

If  $1 \in V_n^k(a)$ , then 1 is the first element of the first block  $I_1$ . The first block contribution  $x_1 - \sum_{j \in I_1 \setminus \{1\}} x_j$  is transformed into  $1 - x_1 - \sum_{j \in I_1 \setminus \{1\}} x_j$ . Moving the constant 1 to the right-hand side changes  $p_k - 1$  into  $p_k - 2$ , yielding  $\left( -x_1 - \sum_{j \in I_1 \setminus \{1\}} x_j \right) + \sum_{q=2}^{p_k} \left( x_{f_q} - \sum_{j \in I_q \setminus \{f_q\}} x_j \right) \leq p_k - 2$ . Thus the stated system is exactly the image of the compact prefix description under  $T$ , and hence describes  $P_n[a, 2^n - 1]$ .  $\square$

## 6 Closing remarks

This paper studies the convex hulls of contiguous subpaths of the binary reflected Gray code. The main result is an exact recursive description of  $P_n[a, b]$ , together with a polynomial-time separation algorithm for the resulting system. We also showed that the recursion collapses to compact closed-form descriptions for prefix and suffix intervals, whose inequalities can be read directly from the binary expansions of the endpoints.

Beyond the intrinsic polyhedral structure of Gray intervals, these descriptions suggest an alternative way of partitioning and handling vertices in the unit cube. For instance, given a set  $X \subseteq \{0, 1\}^n$  of forbidden vertices, let  $\mathcal{G}(X) := \{t \in \{0, \dots, 2^n - 1\} : g_n(t) \in X\}$  be the corresponding set of forbidden Gray indices. If the elements of  $\mathcal{G}(X)$  are sorted increasingly, then the complement  $\{0, 1\}^n \setminus X$  decomposes into a disjoint union of maximal Gray intervals. Consequently,  $\text{conv}(\{0, 1\}^n \setminus X) = \text{conv} \left( \bigcup_{[a, b] \in \mathcal{I}(X)} P_n[a, b] \right)$ , where  $\mathcal{I}(X)$  is the collection of maximal intervals in the Gray order that avoid the forbidden set. This gives a Gray-order analogue of the interval-based constructions used for the forbidden-vertices problem in the unit cube [4]. Since forbidden-vertex structures arise naturally in decomposition methods for stochastic integer programming, including the integer L-shaped method and related computational approaches [3], it would be interesting to compare lexicographic and Gray-interval formulations computationally.

Several other questions remain open. The recursive description gives a polynomial-time separation algorithm, but the worst-case size and redundancy pattern of the full recursively generated system deserve further study. It would also be valuable to identify broader classes of Gray intervals admitting compact nonrecursive descriptions, extending the prefix and suffix cases developed here. Finally, the local facet criteria suggest that the recursive construction contains substantial structure beyond validity and exactness; a more explicit classification of the facet-defining branches could lead to smaller descriptions and more efficient separation routines.

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