# Calmness of the Solution-Set Mapping for Linear Bilevel and Pricing Problems

Henri Lefebvre, Martin Schmidt, Anton Svensson, Johannes Thürauf

Abstract. We study linear bilevel and pricing problems in which the upper- and lower-level constraints' right-hand sides are perturbed. In this setting, it is an important question, also for the validity of numerical solution schemes, if the solution-set mapping of the parametric bilevel problem is calm at the zero-perturbation. We provide the complete picture both for linear bilevel as well as for pricing problems. If the result is positive or not depends on whether the problems have coupling constraints or not and on whether the perturbation is allowed to take place in both levels and if they lead to relaxations or tightenings of the respective constraints. In particular, the solution-set mapping is calm for linear bilevel problems without coupling constraints and for pricing problems if the upper-level problem is not tightened by the perturbation. For the negative results, we provide illustrative counterexamples.

## 1. Introduction

Bilevel optimization problems model two hierarchically interacting decision makers. On the one hand, they allow to model situations in, e.g., energy markets, critical infrastructure defense, or revenue management, which cannot be addressed properly with usual single-level optimization models. On the other hand, the nested structure of these models makes them very challenging to study theoretically and to solve numerically. Nevertheless, many important advances have been made in the field of computational bilevel optimization in the last decades; see Kleinert et al. (2021) for an overview. For linear bilevel optimization problems, this means that we can solve instances today that can be of a size that is relevant for real-world applications (Thürauf et al. 2026).

At the interface of bilevel optimization theory and numerical solution methods, there was, however, a gap in the literature that we close with this paper for the case of linear bilevel problems as well as bilevel pricing problems: The stability of the solution-set under small perturbations of the problem's parameters. This aspect is relevant for at least two reasons. First, if the problem's parameters come from some measured data with controlled measure errors then it is critical to have some certainty on how far the obtained solution can be from the solution of the problem with the true parameters. Second, when a numerical scheme is applied to solve bilevel optimization problems, feasibility is usually not handled in an exact but approximate way, meaning that the upper- and lower-level constraints can be violated by some small tolerance. We theoretically approach this situation by first

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studying linear bilevel problems of the form

$$\begin{aligned} \min_{x,y} \quad & c^\top x + d^\top y \\ \text{s.t.} \quad & Ax + By \geq a - \varepsilon^u, \\ & y \in T_{\varepsilon^l}(x), \end{aligned}$$

where $T_{\varepsilon^l}(x)$ is the set of optimal solutions to the $\varepsilon^l$-perturbed lower-level problem given by

$$T_{\varepsilon^l}(x) = \arg\min_{y'} \left\{ e^\top y' \colon Cx + Dy' \geq b - \varepsilon^l \right\}.$$

Here, $A \in \mathbb{R}^{m \times n_x}$, $B \in \mathbb{R}^{m \times n_y}$, $a, \varepsilon^u \in \mathbb{R}^m$, $C \in \mathbb{R}^{\ell \times n_x}$, $D \in \mathbb{R}^{\ell \times n_y}$, and $b, \varepsilon^l \in \mathbb{R}^\ell$ as well as $c \in \mathbb{R}^{n_x}$ and $d, e \in \mathbb{R}^{n_y}$ are given input parameters. If the upper-level perturbation $\varepsilon^u$ and the lower-level perturbation $\varepsilon^l$ are 0, we obtain the original, i.e., unperturbed, bilevel problem.

We study how solutions to such $\varepsilon := (\varepsilon^u, \varepsilon^l)$-perturbed bilevel problems relate to the exact solutions of the given bilevel problem. In the recent paper by Beck et al. (2023), an example is presented that shows that small feasibility violations can have a significant impact on the solution of the bilevel problem if the latter is nonlinear. For linear bilevel problems, this question has been touched in the latter paper but it has not been answered completely. Moreover, we are not aware of any comprehensive study of this problem in the literature.

Our main contribution is to close this gap for optimistic linear bilevel problems and optimistic pricing problems. For linear bilevel problems with $\varepsilon$-perturbations and the corresponding solution-set mapping $\mathcal{S}(\varepsilon)$, we show that this mapping is calm at 0 (and thus also outer semicontinuous) if the problem does not contain coupling constraints, i.e., if $B = 0$ holds. Calmness at 0 means that there exists a constant $\kappa > 0$ so that for any $(x^*_\varepsilon, y^*_\varepsilon) \in \mathcal{S}(\varepsilon)$ with $\varepsilon$ sufficiently small, there exists $(x^*, y^*) \in \mathcal{S}(0)$ such that

$$\|(x^*_\varepsilon, y^*_\varepsilon) - (x^*, y^*)\| \leq \kappa \|\varepsilon\|.$$

Hence, calmness at 0 is the property of the solution-set mapping that shows that a solution to the $\varepsilon$-perturbed bilevel problem is close to some exact solution if $\varepsilon$ is sufficiently small. For linear bilevel problems with coupling constraints ($B \neq 0$), the situation changes significantly and the results depend on whether the upper- or lower-level constraints are perturbed and if the perturbation leads to a relaxation or tightening of this constraint. For the negative results, we provide illustrative counterexamples. The main results are summarized in Table 1.

Our second main contribution is that we also give the full picture for pricing problems of the type

$$\begin{aligned} \min_{x,y_1,y_2} \quad & -x^\top y_1 \\ \text{s.t.} \quad & Ax \geq a - \varepsilon^u, \\ & y \in T_{\varepsilon^l}(x), \end{aligned}$$

where $T_{\varepsilon^l}(x)$ is the set of optimal solutions to the $\varepsilon^l$-perturbed lower-level problem

$$\begin{aligned} \min_{y_1,y_2} \quad & (c + x)^\top y_1 + d^\top y_2 \\ \text{s.t.} \quad & D_1 y_1 + D_2 y_2 \geq b - \varepsilon^l. \end{aligned}$$

In line with the literature on pricing problems (Bialas and Karwan 1984; Labbé et al. 1998; Labbé and Violin 2013), we focus completely on the case without coupling constraints. If only the lower-level problem's feasible set is perturbed ($\varepsilon^u = 0$), we again prove both calmness at 0 and (hence) upper-semicontinuity of the solution-set mapping. The same holds true if only the upper-level problem is perturbed ($\varepsilon^l = 0$)

TABLE 1. Properties of the solution-set mapping for optimistic linear bilevel problems (LB) with and without coupling constraints, if the upper- or lower-level is perturbed. Here, $\varepsilon^u \geq 0$ indicates relaxations of the upper-level constraints and analogously $\varepsilon^u \leq 0$ restrictions. The same is indicated by $\varepsilon^l$ for the lower-level constraints.

| | Solution-Set Mapping $\varepsilon \mapsto \mathcal{S}(\varepsilon)$ | |
|---|---|---|
| Problem Class | Outer Semicont. at 0 | Calmness at 0 |
| LB without coupling constraints<br>Perturbations: $(\varepsilon^u, \varepsilon^l) \in \mathbb{R}^{m+\ell}$ | ✓ Corollary 1 | ✓ Theorem 3 |
| LB with coupling constraints<br>Perturbations: $\varepsilon^u = 0, \varepsilon^l \leq 0$ | ✗ Example 4 | ✗ Remark 1 |
| LB with coupling constraints<br>Perturbations: $\varepsilon^u = 0, \varepsilon^l \geq 0$ | ✗ Example 5 | ✗ Remark 1 |
| LB with coupling constraints<br>Perturbations: $\varepsilon^u \leq 0, \varepsilon^l = 0$ | ✗ Example 6 | ✗ Remark 1 |
| LB with coupling constraints<br>Perturbations: $\varepsilon^u \geq 0, \varepsilon^l = 0$ | ✓ Remark 1 | ✓ Theorem 7 |

TABLE 2. Properties of the solution-set mapping for optimistic pricing problems without coupling constraints if the upper- or lower-level is perturbed. Here, $\varepsilon^u \geq 0$ indicates relaxations of the upper-level constraints and analogously $\varepsilon^u \leq 0$ restrictions. The same is indicated by $\varepsilon^l$ for the lower-level constraints.

| | Solution-Set Mapping $\varepsilon \mapsto \mathcal{S}(\varepsilon)$ | |
|---|---|---|
| Perturbations | Outer Semicont. at 0 | Calmness at 0 |
| $\varepsilon^u \leq 0, \varepsilon^l = 0$ | ✗ Example 9 | ✗ Remark 1 |
| $\varepsilon^u \geq 0, \varepsilon^l \in \mathbb{R}^\ell$ | ✓ Remark 1 | ✓ Theorem 10 |

so that the perturbation is leading to a relaxation, whereas for tightenings, both positive results fail to hold. The results about pricing problems are summarized in Table 2.

Note that all objective functions are globally Lipschitz continuous on the respective feasible sets, if the latter are compact, which is a classic assumption in bilevel optimization that we make in this paper as well. Hence, all positive results for the solution-set mapping lead to the respective continuity results for the optimal-value function as well.

Let us finally discuss the relation to results on partial calmness in the bilevel literature. In the respective papers, see, e.g., Henrion and Surowiec (2011), Mehlitz et al. (2021), and Ye and Zhu (1995), optimal-value function reformulations are applied to study the single-level problem in which the optimal-value function constraint is penalized. In contrast, we study calmness of the solution-set mapping if the constraints of the original bilevel problem are perturbed. Hence, although the wording is similar, the research papers concerning partial calmness are not strongly related to our results.

The remainder of this paper is structured as follows. In Section 2, we first recap the required basics and the notation from variational analysis. Moreover, we state our main assumptions and prove some preliminary technical results that are used later. Section 3 contains our general results about linear bilevel problems without coupling constraints, followed by Section 4 in which we discuss the case including coupling constraints. Pricing problems are studied in Section 5 before we close this paper with a summary and a sketch of reasonable future research directions in Section 6.

## 2. Preliminaries

We start by recalling some definitions from variational analysis about the continuity of set-valued maps; see, e.g., Section $I^*$ in Rockafellar and Wets (1998). To this end, let us consider a set-valued map $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and a point $x \in \mathbb{R}^n$. We say that $T$ is outer semicontinuous at $x$ if, for any sequence $(x_k, y_k) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $y_k \in T(x_k)$ and $(x_k, y_k) \to (x, y)$ for some $y \in \mathbb{R}^m$, it holds $y \in T(x)$. Informally, this means that if $x_k$ is close to $x$, then any $y_k \in T(x_k)$ is close to some $y \in T(x)$. Moreover, we note that if a set-valued map $T$ is outer semicontinuous at each $x \in \mathbb{R}^n$, then its graph, given by $\operatorname{graph}(T) := \{(x, y) \colon y \in T(x)\}$, is closed.

The notion of calmness is a quantitative version of outer semicontinuity. More specifically, we say that $T$ is calm at $x$ if $T(x) \neq \emptyset$ and if there exist constants $\kappa > 0$ and $\delta > 0$ such that

$$T(x') \subseteq T(x) + \kappa \|x' - x\| B(0, 1) \tag{1}$$

holds for all $x' \in B(x, \delta)$. In other words, for any $x'$ close enough to $x$ and any $y' \in T(x')$, there exists $y \in T(x)$ such that $\|y' - y\| \leq \kappa \|x' - x\|$. Given a set $X \subset \mathbb{R}^n$ and a point $x \in X$ we also say that $T$ is calm at $x$ relative to $X$ if the inclusion (1) holds for all $x' \in B(x, \delta) \cap X$.

It is clear that calmness implies outer semicontinuity. This is formalized in the following remark.

**Remark 1.** *If a set-valued map $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is calm at a point $x \in \mathbb{R}^n$ and $T(x)$ is closed, then $T$ is outer semicontinuous at $x$. Indeed, let us assume that $T$ is calm at $x$ with constants $\delta > 0$ and $\kappa > 0$, and consider $(x_k, y_k) \to (x, y)$ with $y_k \in T(x_k)$ for each $k \in \mathbb{N}$. Then, for $k$ large enough, we have $x_k \in B(x, \delta)$ and, thus,*

$$T(x_k) \subseteq T(x) + \kappa \|x_k - x\| B(0, 1)$$

*holds. Since $y_k \in T(x_k)$, there exists $\tilde{y}_k \in T(x)$ satisfying*

$$\|y_k - \tilde{y}_k\| \leq \kappa \|x_k - x\|.$$

*In turn, we obtain that*

$$\|y - \tilde{y}_k\| \leq \|y - y_k\| + \|y_k - \tilde{y}_k\| \leq \|y - y_k\| + \kappa \|x_k - x\|$$

*holds. Since $(x_k, y_k) \to (x, y)$, we have that the right-hand side converges to zero, which implies that $\tilde{y}_k \to y$. Consequently, it follows $y \in T(x)$ by the closedness of $T(x)$. This proves that $T$ is outer semicontinuous at $x$.*

Next, we recall a fundamental result due to Berge that gives a sufficient condition for the outer semicontinuity of the solution-set mapping of a parameterized optimization problem. A proof of this result can be found, e.g., in Aliprantis and Border (2006, Section 17.31).

**Theorem 1** (Berge's maximum theorem)**.** *Let $\Theta \subseteq \mathbb{R}^p$ be given and let $C : \Theta \rightrightarrows \mathbb{R}^k$ be a continuous set-valued map such that $C(\vartheta)$ is non-empty and compact for all $\vartheta \in \Theta$. Let $f : \mathbb{R}^k \times \Theta \to \mathbb{R}$ be a given continuous function. Then, the value function $\psi(\vartheta) := \vartheta \mapsto \min \{f(x, \vartheta) \colon x \in C(\vartheta)\}$ is continuous on $\Theta$. Moreover, the solution-set mapping $\{x \in C(\vartheta) \colon f(x, \vartheta) \leq \psi(\vartheta)\}$ is outer semicontinuous with non-empty and compact values for all $\vartheta \in \Theta$.*

We now apply Berge's maximum theorem to linearly constrained bilevel problems. To this end, for given perturbation parameters $\varepsilon^u \in \mathbb{R}^m$ and $\varepsilon^l \in \mathbb{R}^\ell$, we consider the bilevel problem

$$\begin{aligned} \min_{x,y} \quad & F(x,y) \\ \text{s.t.} \quad & Ax + By \geq a - \varepsilon^u, \\ & y \in T_{\varepsilon^l}(x), \end{aligned} \tag{2}$$

where

$$T_{\varepsilon^l}(x) := \arg\min_y \left\{ f(x,y) \colon Cx + Dy \geq b - \varepsilon^l \right\}.$$

Additionally, let $\varphi_{\varepsilon^l}$ denote the follower's optimal-value function, $P_\varepsilon$ denote the shared-constraint set and $\mathcal{F}_\varepsilon$ denote the bilevel-feasible set, i.e.,

$$\begin{aligned} \varphi_{\varepsilon^l}(x) &:= \min \left\{ f(x,y) \colon Cx + Dy \geq b - \varepsilon^l \right\}, \\ P_\varepsilon &:= \left\{ (x,y) \colon Ax + By \geq a - \varepsilon^u,\ Cx + Dy \geq b - \varepsilon^l \right\}, \\ \mathcal{F}_\varepsilon &:= \left\{ (x,y) \in P_\varepsilon \colon f(x,y) \leq \varphi_{\varepsilon^l}(x) \right\}. \end{aligned}$$

We note that the linearly constrained bilevel problem (2) comprises different important classes of bilevel problems. It reduces to a linear bilevel problem by setting $F(x,y) = c^\top x + d^\top y$ and $f(x,y) = e^\top y$. Similarly, it represents a pricing problem by setting $B = 0$, $F(x,y) = -x^\top y_1$ and $f(x,y) = (c+x)^\top y_1 + d^\top y_2$ with $y = (y_1, y_2)$. We make the following standing assumption, which is classic in bilevel optimization.

**Standing Assumption.** *The shared-constraint set of the unperturbed bilevel problem $P_0$ is non-empty and bounded. Moreover, the functions $F$ and $f$ are continuous.*

Our next lemma shows that the bilevel feasible set mapping $\varepsilon \mapsto \mathcal{F}_\varepsilon$ is outer semicontinuous at 0. Before we do so, we define the set of perturbations $\mathcal{E} \subseteq \mathbb{R}^{m+\ell}$ for which the bilevel problem remains feasible, i.e.,

$$\mathcal{E} := \left\{ \varepsilon \in \mathbb{R}^{m+\ell} \colon \mathcal{F}_\varepsilon \neq \emptyset \right\}.$$

Note that our standing assumption implies that $0 \in \mathcal{E}$ holds, i.e., the unperturbed bilevel problem has at least one feasible solution, and that for any $\varepsilon \in \mathcal{E}$ the perturbed shared constraint set $P_\varepsilon$ is bounded. Moreover, when $B = 0$, i.e., when there are no coupling constraints, $\mathcal{E}$ contains a neighborhood of zero if $P_0$ itself has a non-empty interior.

We now state the lemma.

**Lemma 1.** *The set-valued map $\varepsilon \mapsto \mathcal{F}_\varepsilon$ is outer semicontinuous at* 0.

*Proof.* By Theorem 1, the follower's optimal-value function $(\varepsilon^l, x) \mapsto \varphi_{\varepsilon^l}(x)$ is continuous. By continuity of $f$ and $\varphi_{\varepsilon^l}$ and due to the constraints being linear, it follows that the graph of $\varepsilon \mapsto \mathcal{F}_\varepsilon$ is closed. Moreover, $\mathcal{F}_\varepsilon$ is compact for all $\varepsilon$ since $\mathcal{F}_\varepsilon \subseteq P_\varepsilon$. Thus, the set-valued map $\varepsilon \mapsto \mathcal{F}_\varepsilon$ is outer semicontinuous on $\mathcal{E}$. In particular, it is outer semicontinuous at 0. □

This lemma shows that small perturbations of the right-hand side of the constraints lead to small changes in the bilevel feasible set. In the next lemma, we give a sufficient condition for the outer semicontinuity of the solution-set mapping of the bilevel problem in (2). To this end, let $\mathcal{S}(\varepsilon)$ denote the set of solutions to Problem (2) and let

$$v(\varepsilon) := \min_{(x,y) \in \mathcal{F}_\varepsilon} F(x,y)$$

denote its optimal-value function.

**Lemma 2.** *Assume that $v$ is upper semicontinuous at 0, then the solution-set mapping $\varepsilon \mapsto \mathcal{S}(\varepsilon)$ is outer semicontinuous at 0.*

*Proof.* Let $(\varepsilon_k)_k$ be a sequence with $\varepsilon_k \in \mathcal{E}$ for all $k$ and $\|\varepsilon_k\| \to 0$ as $k \to \infty$. Consider a sequence $(x_k^*, y_k^*) \in \mathcal{S}(\varepsilon_k)$, $k \in \mathbb{N}$, such that $(x_k^*, y_k^*)$ converges to some $(x^*, y^*) \in \mathbb{R}^n$. Since $\varepsilon \mapsto \mathcal{F}_\varepsilon$ is outer semicontinuous and $(x_k^*, y_k^*) \in \mathcal{F}_{\varepsilon_k}$ for all $k$, we have $(x^*, y^*) \in \mathcal{F}_0$. Finally, by upper semicontinuity of $v$, we have that

$$v(0) \geq \limsup_{k\to\infty} v(\varepsilon_k) = \lim_{k\to\infty} F(x_k^*, y_k^*) = F(x^*, y^*)$$

holds, which shows $(x^*, y^*) \in \mathcal{S}(0)$. □

**Remark 2.** *Before we start to prove the main results of this paper, let us mention that we consider the solution of a bilevel problem being a point in the $(x, y)$-space, which is known as the conventional formulation of the bilevel problem. If only the $x$-space is considered for the solutions of a given bilevel problem, this is known as the original formulation. These two variants of the problem are known to be equivalent on the level of globally optimal solutions. However, this is not the case on the level of local solutions; see, e.g., Dempe et al. (2012) or Section 5.5 in Dempe (2002). Since we are only studying globally optimal solutions, our calmness results for the solution-set mapping of the conventional formulation immediately imply the calmness of the respective mapping for the original formulation.*

## 3. Linear Bilevel Problems without Coupling Constraints

In this section, we consider linear bilevel problems without coupling constraints, i.e, we set $F(x, y) = c^\top x + d^\top y$, $f(x, y) = e^\top y$, and $B = 0$. We first show that the leader's optimistic objective function value is Lipschitz continuous. To this end, let $\xi$ be defined as

$$\xi(x, \varepsilon^l) := \inf_{y \in T_{\varepsilon^l}(x)} d^\top y.$$

We denote its domain by $\operatorname{dom}(\xi) = \left\{(x, \varepsilon^l) \colon \exists y \in T_{\varepsilon^l}(x)\right\}$. Using this notation, the optimal-value function of the bilevel problem in (2) can be written as

$$v(\varepsilon) = \min_{x \in X_{\varepsilon^u}} \; c^\top x + \xi(x, \varepsilon^l)$$

with $X_{\varepsilon^u} := \{x \colon Ax \geq a - \varepsilon^u\}$. We have the following lemma.

**Lemma 3.** *The function $\xi$ is Lipschitz continuous on $\operatorname{dom}(\xi)$, i.e., there exists a constant $L_\xi > 0$ such that for any $(x_1, \varepsilon_1^l), (x_2, \varepsilon_2^l) \in \operatorname{dom}(\xi)$, it holds*

$$|\xi(x_1, \varepsilon_1^l) - \xi(x_2, \varepsilon_2^l)| \leq L_\xi \left\|(x_1, \varepsilon_1^l) - (x_2, \varepsilon_2^l)\right\|.$$

*Proof.* By Corollary 5.2 from Still (2018), it holds that the follower's optimal-value function $(x, \varepsilon^l) \mapsto \varphi_{\varepsilon^l}(x)$ is Lipschitz continuous. Moreover, we may write $\xi(x, \varepsilon^l) = \bar{\xi}(t(x, \varepsilon^l))$, where $\bar{\xi}(u) := \min_y \left\{d^\top y \colon Ey \geq u\right\}$ for $u \in \mathbb{R}^{\ell+1}$,

$$E = \begin{pmatrix} D \\ -e^\top \end{pmatrix}, \quad \text{and} \quad t(x, \varepsilon^l) = \begin{bmatrix} b - \varepsilon^l - Cx \\ -\varphi_{\varepsilon^l}(x) \end{bmatrix}.$$

Clearly, $t$ is Lipschitz continuous. Again, by Corollary 5.2 from Still (2018), we obtain that $\bar{\xi}$ is Lipschitz continuous. Finally, the composition $\xi = \bar{\xi} \circ t$ is Lipschitz continuous, which completes the proof. □

Next, we bound the distance between two (projected) bilevel-feasible points for the $\varepsilon_1$- and $\varepsilon_2$-perturbed problem for any two $\varepsilon_1, \varepsilon_2 \in \mathcal{E}$.

**Lemma 4.** *Let* $(x_1, \varepsilon_1) \in \operatorname{dom}(\xi)$ *with* $x_1 \in X_{\varepsilon_1^u}$ *be given. Then, for any* $\varepsilon_2 \in \mathcal{E}$, *there exists* $x_2 \in X_{\varepsilon_2^u}$ *such that* $(x_2, \varepsilon_2^l) \in \operatorname{dom}(\xi)$ *and*

$$\|x_1 - x_2\| \leq H\|\varepsilon_1 - \varepsilon_2\|$$

*holds, where* $H > 0$ *is a constant that only depends on* $A, C, D, b$, *and* $a$.

*Proof.* First, we briefly argue that $\operatorname{dom}(\xi)$ is a polyhedron. Indeed, our standing assumption implies that if the lower-level problem is feasible, then also a solution exists. Thus,

$$\operatorname{dom}(\xi) = \left\{(x, \varepsilon^l) \colon \exists y \in T_{\varepsilon^l}(x)\right\} = \left\{(x, \varepsilon^l) \colon \exists y : Cx + Dy \geq b - \varepsilon^l\right\}$$

holds. Hence, $\operatorname{dom}(\xi)$ can be written as the projection of a polyhedron. By Fourier–Motzkin elimination, it can therefore be expressed as

$$\operatorname{dom}(\xi) = \left\{(x, \varepsilon^l) \colon Mx + N\varepsilon^l \geq r\right\},$$

for some matrices $M$, $N$, and a vector $r$ of appropriate sizes.

Now, let $\varepsilon_1, \varepsilon_2$ and $x_1$ be as described in the lemma. It follows that $x_1$ is a solution to the linear inequality system

$$Mx \geq r - N\varepsilon_1^l, \quad Ax \geq a - \varepsilon_1^u.$$

By the main theorem from Hoffman (1952) on approximate solutions to linear systems, it follows that there exists a constant $H > 0$ (independent of $x_1$) such that we can find a solution $x_2$ to the (perturbed) linear inequality system

$$Mx \geq r - N\varepsilon_2^l, \quad Ax \geq a - \varepsilon_2^u,$$

such that $\|x_1 - x_2\| \leq H\|\varepsilon_1 - \varepsilon_2\|$ holds. □

Lemma 3 and 4 are the key ingredients to show that the optimal-value function of a linear bilevel problem is Lipschitz continuous. This is formalized in the next theorem.

**Theorem 2.** *The optimal-value function* $v$ *is globally Lipschitz continuous on* $\mathcal{E}$, *i.e., there exists a constant* $L_v > 0$ *such that*

$$|v(\varepsilon_1) - v(\varepsilon_2)| \leq L_v\|\varepsilon_1 - \varepsilon_2\|$$

*holds for any* $\varepsilon_1, \varepsilon_2 \in \mathcal{E}$.

*Proof.* Let $\varepsilon_1, \varepsilon_2 \in \mathcal{E}$ be given and let $x^* \in X_{\varepsilon_1^u}$ be such that $v(\varepsilon_1) = c^\top x^* + \xi(x^*, \varepsilon_1^l)$, i.e., $(x^*, y^*) \in \mathcal{S}(\varepsilon_1)$ holds for some $y^*$. By Lemma 4, there exists $x_2 \in X_{\varepsilon_2^u}$ such that $(x_2, \varepsilon_2^l) \in \operatorname{dom}(\xi)$ holds and satisfies $\|x^* - x_2\| \leq H\|\varepsilon_1 - \varepsilon_2\|$ for some constant $H > 0$. By feasibility of $x_2$ for the $\varepsilon_2$-perturbed bilevel problem, we get that $v(\varepsilon_2) \leq c^\top x_2 + \xi(x_2, \varepsilon_2^l)$ holds. Moreover, Lemma 3 shows that $\xi$ is Lipschitz continuous with some constant $L_\xi > 0$. Thus, we obtain

$$\begin{aligned}
v(\varepsilon_2) - v(\varepsilon_1) &\leq \left(c^\top x_2 + \xi(x_2, \varepsilon_2^l)\right) - \left(c^\top x^* + \xi(x^*, \varepsilon_1^l)\right) \\
&\leq \|c\|\|x^* - x_2\| + L_\xi\|(x^*, \varepsilon_1^l) - (x_2, \varepsilon_2^l)\| \\
&\leq \left(\|c\| + L_\xi\right)\|(x^*, \varepsilon_1^l) - (x_2, \varepsilon_2^l)\| \\
&\leq \left(\|c\| + L_\xi\right)\left(\|x^* - x_2\| + \|\varepsilon_1 - \varepsilon_2\|\right) \\
&\leq \left(\|c\| + L_\xi\right)(H + 1)\|\varepsilon_1 - \varepsilon_2\|.
\end{aligned}$$

An analogue analysis for $v(\varepsilon_1) - v(\varepsilon_2)$ leads to

$$|v(\varepsilon_1) - v(\varepsilon_2)| \leq \left(\|c\| + L_\xi\right)(H + 1)\|\varepsilon_1 - \varepsilon_2\|,$$

which completes the proof by taking $L_v := \left(\|c\| + L_\xi\right)(H + 1)$. □

Theorem 2 shows that the optimal-value function $v$ of a linear bilevel problem is Lipschitz continuous. By Lemma 2, we obtain the following corollary.

**Corollary 1.** *The solution-set mapping* $\varepsilon \mapsto \mathcal{S}(\varepsilon)$ *is outer semicontinuous at* $0$.

We conclude this section by showing that the solution-set mapping is calm at $0$.

**Theorem 3.** *Let the standing assumption hold. Then the solution-set mapping* $\varepsilon \mapsto \mathcal{S}(\varepsilon)$ *is calm at* $0$ *relative to* $\mathcal{E}$*, i.e., there exist constants* $\kappa > 0$ *and* $\eta > 0$ *so that for all* $\varepsilon \in \mathcal{E}$ *with* $\|\varepsilon\| \leq \eta$ *and for any* $(x^*_\varepsilon, y^*_\varepsilon) \in \mathcal{S}(\varepsilon)$*, there exists* $(x^*, y^*) \in \mathcal{S}(0)$ *with*

$$\|(x^*_\varepsilon, y^*_\varepsilon) - (x^*, y^*)\| \leq \kappa \|\varepsilon\|.$$

*Proof.* Let us assume by contradiction that the statement is not true. Then, there exists a sequence $(\varepsilon_k)_k \subset \mathcal{E}$ such that $\varepsilon_k \to 0$ along with $(x^*_k, y^*_k) \in \mathcal{S}(\varepsilon_k)$ such that for all $k$ sufficiently large, we have

$$\|(x^*_k, y^*_k) - (x^*, y^*)\| > k\|\varepsilon_k\| \quad \forall (x^*, y^*) \in \mathcal{S}(0). \tag{3}$$

Let $I_l(x^*_k, y^*_k)$ denote the set of active constraints in the respective lower-level problem, i.e., let

$$I_l(x^*_k, y^*_k) = \left\{ i \in [\ell] \colon (Cx^*_k + Dy^*_k)_i = (b - \varepsilon^l_k)_i \right\}.$$

Because the number of possible active constraints is finite, we may assume (by otherwise passing to a subsequence) that $I_l(x^*_k, y^*_k) = \bar{I}_l$ holds for all $k$.

Now, consider the single-level reformulation of the $\varepsilon_k$-perturbed bilevel problem that we obtain by using the Karush–Kuhn–Tucker conditions of the lower-level problem, i.e., consider

$$\begin{aligned}
\min_{x,y,\lambda} \quad & c^\top x + d^\top y \\
\text{s.t.} \quad & Ax \geq a - \varepsilon^u_k, \\
& Cx + Dy \geq b - \varepsilon^l_k, \\
& D^\top \lambda = e, \\
& \lambda \geq 0, \\
& \lambda^\top (Cx + Dy - b + \varepsilon^l_k) = 0.
\end{aligned} \tag{4}$$

We know that there exists $\lambda^*_k$ such that $(x^*_k, y^*_k, \lambda^*_k)$ solves Problem (4). Because the active constraints are fixed, it follows that $(\lambda^*_k)_i = 0$ holds for all $i \in [\ell] \setminus \bar{I}_l$. Thus, we may rewrite Problem (4) as

$$\begin{aligned}
\min_{x,y,\lambda} \quad & c^\top x + d^\top y \\
\text{s.t.} \quad & Ax \geq a - \varepsilon^u_k, \\
& Cx + Dy \geq b - \varepsilon^l_k, \\
& (Cx + Dy - b + \varepsilon^l_k)_{\bar{I}_l} = 0, \\
& D^\top \lambda = e, \\
& \lambda_i = 0 \quad \text{for all } i \in [\ell] \setminus \bar{I}_l, \\
& \lambda_i \geq 0 \quad \text{for all } i \in \bar{I}_l.
\end{aligned}$$

Note that the dual variables $\lambda$ are completely decoupled from $x$ and $y$ and do not appear in the objective function value. Thus, they can be effectively removed from the model and we obtain that $(x^*_k, y^*_k)$ also solves

$$\begin{aligned}
\min_{x,y} \quad & c^\top x + d^\top y \\
\text{s.t.} \quad & Ax \geq a - \varepsilon^u_k, \\
& Cx + Dy \geq b - \varepsilon^l_k, \\
& (Cx + Dy - b + \varepsilon^l_k)_{\bar{I}_l} = 0.
\end{aligned} \tag{5}$$

Problem (5) can be seen as the perturbed version of the linear problem

$$\begin{aligned} \min_{x,y} \quad & c^\top x + d^\top y \\ \text{s.t.} \quad & Ax \geq a, \\ & Cx + Dy \geq b, \\ & (Cx + Dy - b)_{\bar{I}_l} = 0. \end{aligned} \tag{6}$$

Applying Proposition 12 in the Appendix to Problem (5) and (6), we obtain that there exists a constant $\kappa > 0$ and $k_0 \in \mathbb{N}$ so that for all $k \geq k_0$ we can find a solution $(x_k, y_k)$ to Problem (6) that satisfies

$$\|(x_k^*, y_k^*) - (x_k, y_k)\| \leq \kappa \|\varepsilon_k\|. \tag{7}$$

To get a contradiction, it remains to show that $(x_k, y_k) \in \mathcal{S}(0)$ holds. By our standing assumption, the sequence $(x_k^*, y_k^*)_k$ is bounded and we may assume without loss of generality that the sequence $(x_k^*, y_k^*)$ converges to some point $(\bar{x}, \bar{y})$. Hence, using Corollary 1 we have that $(\bar{x}, \bar{y}) \in \mathcal{S}(0)$, implying $F(\bar{x}, \bar{y}) = v(0)$. Moreover, using (7) we also obtain that $(x_k, y_k)$ converges to $(\bar{x}, \bar{y})$ and since the solution set of (6) is closed then also $(\bar{x}, \bar{y})$ is an optimal solution of (6). Therefore, we have $F(\bar{x}, \bar{y}) = F(x_k, y_k) = v(0)$, from which we conclude that $(x_k, y_k) \in \mathcal{S}(0)$. We have encountered a contradiction between (7) and (3) and hence the proof is complete. □

**Remark 3.** *We remark that, except for the outer semicontinuity of the solution-set mapping $\varepsilon \mapsto \mathcal{S}(\varepsilon)$ used in the final step, the proof of Theorem 3 does not rely on the assumption $B = 0$.*

## 4. Linear Bilevel Problems with Coupling Constraints

In the previous section, we have seen that linear bilevel problems without coupling constraints are stable under small perturbations. We now turn our attention to problems that include coupling constraints. In this setting, we show through a series of examples that the results obtained in Section 3 do not generalize directly. More formally, we assume again that $F(x,y) = c^\top x + d^\top y$ and $f(x,y) = e^\top y$ holds but $B$ is not necessarily equal to zero.

We consider different ways to perturb the bilevel problem in (2). We begin by considering perturbations at the lower level and show that the optimal set-valued map $\varepsilon \mapsto \mathcal{S}(\varepsilon)$ may fail to be outer semicontinuous. This is established for both lower-level restrictions ($\varepsilon^u = 0$ and $\varepsilon^l < 0$) and lower-level relaxations ($\varepsilon^u = 0$ and $\varepsilon^l > 0$). We then turn to upper-level perturbations. In the case of upper-level restrictions ($\varepsilon^u < 0$ and $\varepsilon^l = 0$), we again show that the optimal set-valued map is not outer semicontinuous.

The only positive result in this section concerns upper-level relaxations. Under such perturbations, we show that the optimal set-valued map is in fact calm at 0; see Theorem 7. However, we also show that the value function $\varepsilon \mapsto v(\varepsilon)$ is not globally Lipschitz continuous, contrasting with Theorem 2.

We conclude this introductory part with a brief remark on the recent literature regarding coupling constraints in linear bilevel optimization.

**Remark 4.** *In the recent papers by Henke et al. (2024, 2026), the authors study the impact of coupling constraints in linear bilevel optimization. In particular, they show that coupling constraints can be penalized exactly, yielding equivalent formulations without coupling constraints. At first sight, this might appear to contradict the negative results established in this section. However, the exact penalization result is derived for a penalty parameter computed only at $\varepsilon = 0$. Consequently, this*

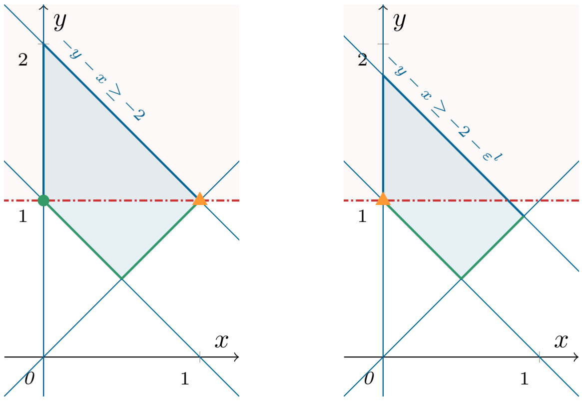


FIGURE 1. Both figures illustrate the bilevel problem in Example 4 for different values of $\varepsilon_l$. Left: the unperturbed problem with $\varepsilon^l = 0$. Right: the perturbed problem for some $\varepsilon^l < 0$. The shaded blue area is the graph of the follower's feasible set mapping. The shaded red area delimited by the dashed red line is the halfspace defined by the coupling constraint. The green lines represent the optimal response of the follower, while the green dot and the orange triangles are the bilevel-feasible points that satisfy the coupling constraints. The orange triangles are the optimal point of the respective bilevel problem. We see that restricting the lower-level feasible region leads to a jump in the optimal points.

*parameter need not remain exact on a neighborhood of* $0$*, and thus these results do not contradict the phenomena observed here.*

4.1. **Lower-Level Perturbations.** We focus now on perturbations of the lower-level problem. To this end, we fix $\varepsilon^u = 0$ as allowing $\varepsilon^u$ to vary would only broaden the class of perturbations. Our first example concerns lower-level restrictions, i.e., $\varepsilon^l < 0$, and shows that the optimal set-valued map is not outer semicontinuous at 0.

**Example 4** (Lower-Level Restriction)**.** *Consider the linear bilevel problem*

$$\begin{aligned} \min_{x,y} \quad & -x \\ \text{s.t.} \quad & x \geq 0, \\ & y \geq 1, \\ & y \in \arg\min_{y'} \left\{ y' \colon y' \geq 1 - x,\ y' \geq x,\ -y' - x \geq -2 - \varepsilon^l \right\}. \end{aligned}$$

*An illustration of the problem is given in Figure 1 for the case* $\varepsilon^l = 0$ *(left) and* $\varepsilon^l < 0$ *(right). First, observe that the shared constraint set is non-empty, compact, and full-dimensional for all* $\varepsilon_l \in (-1, 0]$*.*

*We start by analyzing the case* $\varepsilon^l = 0$*. For any feasible upper-level decision* $x$*, the lower-level problem has a unique solution given by* $y(x) = \max\{1 - x, x\}$*. The coupling constraint "*$y \geq 1$*" therefore forces the leader to choose either* $x = 0$ *or* $x = 1$*. Since the upper level minimizes* $-x$ *(equivalently, maximizes* $x$*), the unique optimal point is* $(x^*, y^*) = (1, 1)$*.*

*Now, consider any perturbation* $\varepsilon^l < 0$*. The perturbed lower-level constraint* $y + x \leq 2 + \varepsilon^l$ *becomes more restrictive. As a consequence, any point* $x \in (1 - \varepsilon^l, 1]$ *becomes infeasible. Thus, the only feasible upper-level decision is* $x = 0$*, and the unique solution to the bilevel problem is* $(x^*_\varepsilon, y^*_\varepsilon) = (0, 1)$*.*

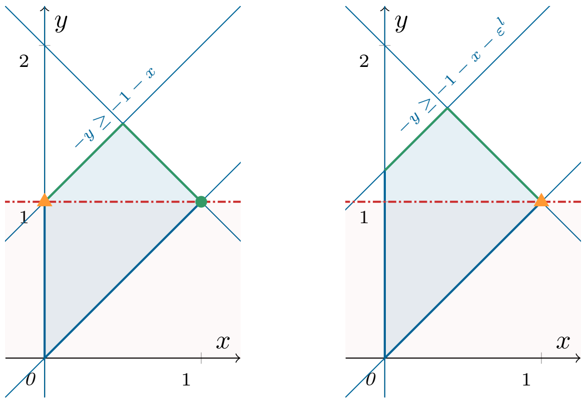


FIGURE 2. Both figures illustrate the bilevel problem in Example 5 for different values of $\varepsilon^l$. Left: the unperturbed problem with $\varepsilon^l = 0$. Right: the perturbed problem for some $\varepsilon^l > 0$. We see that relaxing the lower-level feasible region leads to a jump in the optimal points.

*Combining both cases, we have that the optimal set-valued map of this bilevel problem is given by*

$$\mathcal{S}(\varepsilon) = \begin{cases} \{(1,1)\}, & \text{if } \varepsilon^l = 0, \\ \{(0,1)\}, & \text{if } \varepsilon^l \in [-1,0). \end{cases}$$

*Clearly, $\mathcal{S}$ is not outer semicontinuous.*

The next example concerns lower-level relaxations ($\varepsilon^l > 0$) and exhibits a similar behavior.

**Example 5** (Lower-Level Relaxation)**.** *Consider the linear bilevel problem*

$$\begin{aligned} \min_{x,y} \quad & x \\ \text{s.t.} \quad & x \geq 0, \\ & y \leq 1, \\ & y \in \arg\min_{y'} \left\{ -y' : \; -y' \geq -1 - x - \varepsilon^l, \; -y' \geq -2 + x, \; y' \geq x \right\}. \end{aligned}$$

*An illustration of the problem is given in Figure 2 for the case $\varepsilon^l = 0$ (left) and $\varepsilon^l > 0$ (right). Observe that the shared constraint set is non-empty, compact, and full-dimensional for all $\varepsilon^l \geq 0$.*

*We start by analyzing the case $\varepsilon^l = 0$. For any feasible upper-level decision $x$, the lower-level problem has a unique solution given by $y(x) = \min\{1 + x, 2 - x\}$. The coupling constraint "$y \leq 1$" therefore forces the leader to choose either $x = 0$ or $x = 1$. Since the upper level minimizes $x$, the unique optimal point is $(x^*, y^*) = (0, 1)$.*

*Now, consider any perturbation $\varepsilon^l > 0$. The perturbed lower-level constraint $-y' \geq -1 - x - \varepsilon^l$ becomes less restrictive, so the follower can select a larger $y$ for a given $x$. In particular, for $x = 0$, the follower chooses $y = 1 + \varepsilon^l$, which violates the coupling constraint $y \leq 1$. Consequently, the only feasible upper-level decision is $x = 1$, yielding the unique solution $(x^*_\varepsilon, y^*_\varepsilon) = (1, 1)$.*

*Combining both cases, we have that the optimal set-valued map of this bilevel problem is given by*

$$\mathcal{S}(\varepsilon) = \begin{cases} \{(0,1)\}, & \text{if } \varepsilon^l = 0, \\ \{(1,1)\}, & \text{if } \varepsilon^l > 0. \end{cases}$$

*Again, $\mathcal{S}$ is not outer semicontinuous.*

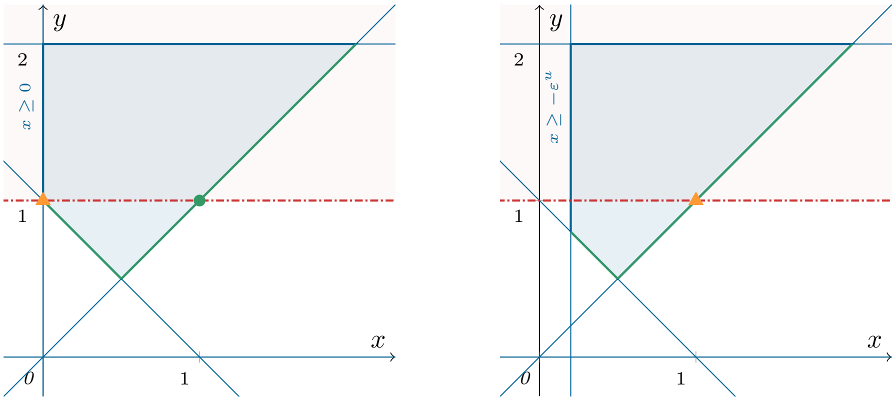


FIGURE 3. Both figures illustrate the bilevel problem in Example 6 for different values of $\varepsilon^u$. Left: the unperturbed problem with $\varepsilon^u = 0$. Right: the perturbed problem for some $\varepsilon^u < 0$. We see that restricting the upper-level feasible space lead to a jump in the optimal points.

The two previous examples show that, in general, linear bilevel problems with coupling constraints cannot be expected to be calm under lower-level perturbations.

4.2. **Upper-Level Perturbations.** We now consider perturbations of the upper-level problem, i.e., we fix $\varepsilon^l = 0$. Our first example concerns upper-level restrictions, i.e., $\varepsilon^u < 0$, and shows that the solution-set mapping $\mathcal{S}$ is not outer semicontinuous.

**Example 6** (Upper-Level Restriction)**.** *Consider the linear bilevel problem*

$$\begin{aligned} \min_{x,y} \quad & x \\ \text{s.t.} \quad & x \geq -\varepsilon^u, \\ & y \geq 1, \\ & y \in \arg\min_{y'} \left\{ y' \colon y' \geq 1 - x,\ y' \geq x,\ 0 \leq y' \leq 2 \right\}. \end{aligned}$$

*An illustration of the problem is given in Figure 3 for the case $\varepsilon^u = 0$ (left) and $\varepsilon^u < 0$ (right). Observe again that the shared constraint set is non-empty, compact, and full dimensional for all $\varepsilon^u \in (-1, 0]$.*

*We start by analyzing the case $\varepsilon^u = 0$. For any feasible upper-level decision $x$, the lower-level problem has a unique solution given by $y(x) = \max\{1 - x, x\}$. The coupling constraint "$y \geq 1$" therefore forces the leader to choose either $x = 0$ or $x = 1$. Since the upper level minimizes $x$, the unique optimal point is $(x^*, y^*) = (0, 1)$.*

*Now, consider any perturbation $\varepsilon^u \in [0, 1)$. Then, any upper-level decision $x < -\varepsilon^u$ becomes infeasible. Consequently, the only feasible upper-level decision is $x = 1$, yielding the unique solution $(x^*_\varepsilon, y^*_\varepsilon) = (1, 1)$.*

*Combining both cases, we have that the optimal set-valued map of this bilevel problem is given by*

$$\mathcal{S}(\varepsilon) = \begin{cases} \{(0, 1)\}, & \text{if } \varepsilon^u = 0, \\ \{(1, 1)\}, & \text{if } \varepsilon^u \in [-1, 0). \end{cases}$$

*As in the other examples, $\mathcal{S}$ is not outer semicontinuous.*

Example 6 shows that, also for upper-level restrictions, the solution-set mapping may fail to be outer semicontinuous. Our next result concerns upper-level relaxations. In this setting, we can show calmness of the solution-set mapping.

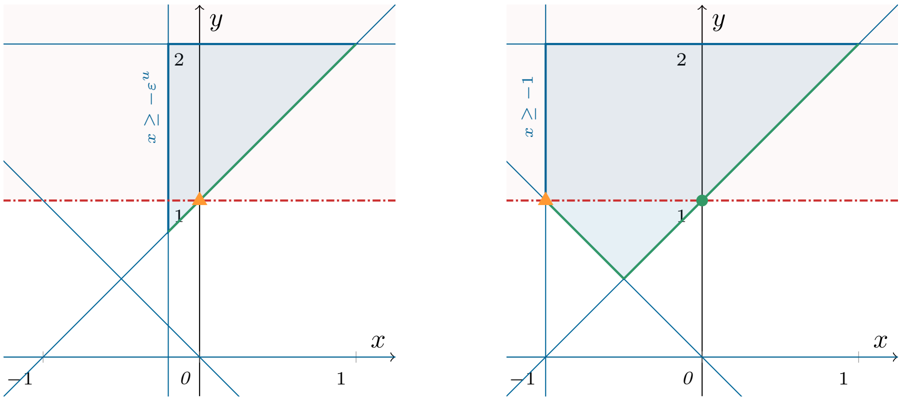


FIGURE 4. Both figures illustrate the bilevel problem in Example 8 for different values of $\varepsilon^u$. On the left, the problem after a small perturbation $\varepsilon^u$. On the right, the problem after a larger perturbation $\varepsilon^u = 1$. We see that relaxing the upper-level feasible region leads to a jump in the optimal points and in the optimal value for large perturbations.

**Theorem 7** (Upper-Level Relaxation)**.** *Let the standing assumption hold. Then, the solution-set mapping $\varepsilon \mapsto \mathcal{S}(\varepsilon)$ is calm at 0 relative to $\mathcal{E}^{\text{ur}} := \mathcal{E} \cap (\mathbb{R}^m_{\geq 0} \times \{0\}^\ell)$, i.e., there exist constants $\kappa > 0$ and $\eta > 0$ so that for all $\varepsilon \in \mathcal{E}^{\text{ur}}$ with $\|\varepsilon\| \leq \eta$ and for any $(x^*_\varepsilon, y^*_\varepsilon) \in \mathcal{S}(\varepsilon)$, there exists $(x^*, y^*) \in \mathcal{S}(0)$ with*

$$\|(x^*_\varepsilon, y^*_\varepsilon) - (x^*, y^*)\| \leq \kappa \|\varepsilon\|.$$

*Proof.* We first show that the value function $v$ is upper semicontinuous at 0 if restricted to $\mathcal{E}^{\text{ur}}$. Indeed, because any $\varepsilon \in \mathcal{E}^{\text{ur}}$ leads to a relaxation of the unperturbed bilevel problem, it holds $\mathcal{F}_0 \subseteq \mathcal{F}_{\varepsilon^u}$. This implies that $v(0) \geq v(\varepsilon)$. In turn, this leads to

$$v(0) \geq \limsup_{\mathcal{E}^{\text{ur}} \ni \varepsilon \to 0} v(\varepsilon),$$

which shows that $v$ is upper semicontinuous at 0 relative to $\mathcal{E}^{\text{ur}}$. In turn, it follows by Lemma 2 that $\varepsilon \mapsto \mathcal{S}(\varepsilon)$ is outer semicontinuous. Finally, a small adaptation of the proof of Theorem 3 leads to the desired result; see Remark 3. □

The result of Theorem 7 is in line with the results obtained for problems without coupling constraints. However, the next example shows that the value function $v$ is not globally Lipschitz continuous in the presence of coupling constraints, contrasting with Theorem 2 for bilevel problems without coupling constraints.

**Example 8** (Upper-Level Relaxation)**.** *We consider the perturbed linear bilevel problem*

$$\begin{aligned} \min_{x,y} \quad & x \\ \text{s.t.} \quad & x \geq -\varepsilon^u, \\ & y \geq 1, \\ & y \in \arg\min_{y'} \{y' \colon y' \geq 1 + x,\ y' \geq -x,\ 0 \leq y' \leq 2\}. \end{aligned}$$

*We show that the value function $v$ is not globally Lipschitz continuous on* $[0, 1]$. *In fact, we show that it has a discontinuity at* $\varepsilon^u = 1$.

*An illustration of the problem is given in Figure 4 for the case* $\varepsilon^u = 0.1$ *(left) and* $\varepsilon^u = 1$ *(right). Like in the previous examples, the shared constraint set is non-empty, compact, and full-dimensional for all* $\varepsilon^u \geq 0$.

*We first consider* $\varepsilon^u \in [0, 1)$ *and show that* $v(\varepsilon^u) = 0$ *holds. Indeed, for any* $x \in [-\varepsilon^u, 1]$*, the follower's optimal response is given by* $y(x) = \max\{1+x, -x\}$*. The coupling constraint "*$y \geq 1$*" then forces the leader to choose* $x \geq 0$*. Since the leader minimizes* $x$*, the optimal choice is* $x = 0$ *and it holds* $v(\varepsilon^u) = 0$ *for all* $\varepsilon^u \in [0, 1)$.

*Next, we consider* $\varepsilon^u = 1$*. As before, the follower's optimal response is* $y(x) = \max\{1 + x, -x\}$*. However, both* $x = 0$ *and* $x = -1$ *lead to a follower's optimal response of* $y = 1$*, which satisfies the coupling constraint. Since the leader minimizes* $x$*, the unique optimal choice is* $x = -1$*, resulting in* $v(1) = -1$.

*Combining both cases, the value function is given by*

$$v(\varepsilon^u) = \begin{cases} -1 & \text{if } \varepsilon^u = 1, \\ 0 & \text{if } \varepsilon^u \in [0, 1). \end{cases}$$

*Thus, the optimal-value function* $v$ *is discontinuous at* $\varepsilon^u = 1$ *and, therefore, it cannot be (globally) Lipschitz continuous.*

## 5. Pricing Problems

In this section, we focus on linear pricing problems, i.e., bilevel problems with $F(x, y) = -x^\top y_1$ and $f(x, y) = (c + x)^\top y_1 + d^\top y_2$ as well as $y = (y_1, y_2)$. Moreover, it holds $B = 0$ and $C = 0$. Hence, these problems are of the form

$$\begin{aligned} \min_{x, y_1, y_2} \quad & -x^\top y_1 \\ \text{s.t.} \quad & Ax \geq a - \varepsilon^u, \\ & (y_1, y_2) \in T_{\varepsilon^l}(x), \end{aligned} \tag{8}$$

where $T_{\varepsilon^l}(x)$ is the set of optimal solutions to the $\varepsilon^l$-perturbed lower-level problem

$$\begin{aligned} \min_{y_1, y_2} \quad & (c + x)^\top y_1 + d^\top y_2 \\ \text{s.t.} \quad & D_1 y_1 + D_2 y_2 \geq b - \varepsilon^l. \end{aligned}$$

We start with an example that shows that the solution-set mapping $\varepsilon \mapsto \mathcal{S}(\varepsilon)$ of Problem (8) is not outer semicontinuous (and hence it is not calm) at 0 if one allows for upper-level restrictions, i.e., for $\varepsilon^u < 0$.

**Example 9** (Upper-Level Restriction)**.** *We consider the linear pricing problem*

$$\begin{aligned} \min_{x, y} \quad & -xy_1 \\ \text{s.t.} \quad & 1 - \varepsilon^u \leq x \leq 42, \\ & (y_1, y_2) \in \operatorname*{arg\,min}_{y_1', y_2'} \left\{ xy_1' + y_2' \colon y_1' \geq 0,\ y_2' \geq 0,\ y_1' + y_2' \geq 1 \right\}. \end{aligned}$$

*We start by analyzing the unperturbed problem, i.e.,* $\varepsilon^u = 0$*. For any upper-level decision* $x > 1$*, the unique lower-level solution is given by* $y = (0, 1)$*. Hence, the resulting objective function value is* 0*. For* $x = 1$*, the lower-level solutions are given by all convex combinations of* $y = (1, 0)$ *and* $y = (0, 1)$*. Since the upper level is maximizing* $xy_1$ *and we consider the optimistic variant of bilevel optimization, the unique solution to the bilevel problem is* $(x, y_1, y_2) = (1, 1, 0)$ *and has value* $-1$.

*We now consider restrictions of the upper level, i.e.,* $\varepsilon^u < 0$*. In this setting, the upper-level decision* $x = 1$ *is no longer feasible. Hence, all feasible and, at the same time, optimal points are of the form* $(x^*, 0, 1)$ *for some* $x^* \in [1 - \varepsilon^u, 42]$ *and have the value* 0.

*Combining both cases, the solution-set mapping is given by*

$$\mathcal{S}(\varepsilon) = \begin{cases} \{(1,1,0)\}, & \text{if } \varepsilon^u = 0, \\ [1-\varepsilon^u, 42] \times \{(0,1)\}, & \text{if } \varepsilon^u \in [-42, 0). \end{cases}$$

*Clearly, $\mathcal{S}$ is not outer semicontinuous. Note also that the optimal-value function $v$ is lower semicontinuous but not upper semicontinuous at* 0*, which is consistent with Lemma 2.*

This example shows that when the perturbation restricts the upper-level feasible region, the solution-set mapping is not necessarily outer semicontinuous at 0 in general. For that reason, we focus on upper-level relaxations, i.e., $\varepsilon^u \geq 0$. In this setting, we show in the next lemma that the optimal-value function $v$ is continuous.

**Lemma 5.** *Let our standing assumption hold. Then, the optimal-value function $v$ is continuous at 0 relative to $\mathcal{E}^{\mathrm{ur}} := \mathcal{E} \cap (\mathbb{R}^m_{\geq 0} \times \mathbb{R}^\ell)$, i.e., for any sequence $(\varepsilon_k)_k$ with $\|\varepsilon_k\| \to 0$ and $\varepsilon_k \in \mathcal{E}^{\mathrm{ur}}$ for each $k$, it holds $v(\varepsilon_k) \to v(0)$.*

*Proof.* Let $(\varepsilon_k)_k$ be a sequence with $\varepsilon_k \in \mathcal{E}^{\mathrm{ur}}$ and $\|\varepsilon_k\| \to 0$ as $k \to \infty$. Let $(x^*, y^*)$ be a solution to the unperturbed bilevel problem, i.e., $(x^*, y^*) \in \mathcal{S}(0)$. In particular, $y^*$ solves the unperturbed $x^*$-parameterized lower-level problem

$$\min_y \quad (c + x^*)^\top y_1 + d^\top y_2 \quad \text{s.t.} \quad D_1 y_1 + D_2 y_2 \geq b.$$

Hence, by Proposition 12 in the appendix, there exists $\hat{y}_k$ for each $k$ that solves the $\varepsilon_k^l$-perturbed $x^*$-parameterized lower-level problem

$$\min_y \quad (c + x^*)^\top y_1 + d^\top y_2 \quad \text{s.t.} \quad D_1 y_1 + D_2 y_2 \geq b - \varepsilon_k^l,$$

such that $\|y^* - \hat{y}_k\| \leq L\|\varepsilon_k\|$ holds for some $L > 0$. Hence, $\hat{y}_k \in T_{\varepsilon_l^k}(x^*)$. Moreover, because $\varepsilon_k^u \geq 0$, we have that $x^* \in X_{\varepsilon_u^k}$ holds. In turn, this shows that $(x^*, \hat{y}_k)$ is feasible for the $\varepsilon_k$-perturbed bilevel problem, i.e., $(x^*, \hat{y}_k) \in \mathcal{F}_{\varepsilon_k}$. From this, we conclude that

$$v(\varepsilon_k) \leq -(x^*)^\top \hat{y}_{k,1},$$

leading to

$$v(\varepsilon_k) - v(0) \leq -(x^*)^\top \hat{y}_{k,1} + (x^*)^\top y_1^* \leq \|x^*\| \, \|y_1^* - \hat{y}_{k,1}\| \leq L \, \|x^*\| \, \|\varepsilon_k\|.$$

Taking the upper limit on both sides leads to

$$\limsup_{k \to \infty} \; v(\varepsilon_k) \leq v(0).$$

To conclude, it remains to show that

$$\liminf_{k \to \infty} \; v(\varepsilon_k) \geq v(0). \tag{9}$$

Recall that, by Lemma 1, the set-valued map $\varepsilon \mapsto \mathcal{F}_\varepsilon$ is outer semicontinuous. Then, by Theorem 1.4.16 from Aubin and Frankowska (2009), since the upper-level objective function $F(x, y) = -x^\top y_1$ is (jointly) continuous in $(x, y, \varepsilon)$, we have that $v$ is lower semicontinuous. Thus, in particular, the inequality in (9) holds. □

Using Lemma 2 and 5, we directly obtain the following corollary.

**Corollary 2.** *Let our standing assumption hold. Then, the solution-set mapping $\varepsilon \mapsto \mathcal{S}(\varepsilon)$ is outer semicontinuous at* 0 *relative to $\mathcal{E}^{\mathrm{ur}} := \mathcal{E} \cap (\mathbb{R}^m_{\geq 0} \times \mathbb{R}^\ell)$.*

Next, we further show that the solution-set mapping is calm at 0 if only upper-level relaxations are considered.

**Theorem 10.** *Suppose that our standing assumption holds. Then, $\mathcal{S}$ is calm at 0 relative to $\mathcal{E}^{\text{ur}} := \mathcal{E} \cap (\mathbb{R}^m_{\geq 0} \times \mathbb{R}^\ell)$, i.e. there exist constants $\kappa > 0$ and $\eta > 0$ such that for all $\varepsilon \in \mathcal{E}^{\text{ur}}$ with $\|\varepsilon\| \leq \eta$ and for any $(x^*_\varepsilon, y^*_\varepsilon) \in \mathcal{S}(\varepsilon)$, there exists $(x^*, y^*) \in \mathcal{S}(0)$ such that*

$$\|(x^*_\varepsilon, y^*_\varepsilon) - (x^*, y^*)\| \leq \kappa\|\varepsilon\|.$$

*Proof.* Let $(\varepsilon_k)_k$ be a sequence with $\varepsilon_k \in \mathcal{E}^{\text{ur}}$ and $\|\varepsilon_k\| \to 0$ as $k \to \infty$. For each $k$, let $(x^*_k, y^*_k) \in \mathcal{S}(\varepsilon_k)$. It is enough to prove, similarly to what is done in the proof of Theorem 3, that there exists a constant $\kappa > 0$ (independent of the sequence) such that up to a subsequence of $(x^*_k, y^*_k)_k$ we can find $(x_k, y_k) \in \mathcal{S}(0)$ satisfying

$$\|(x^*_k, y^*_k) - (x_k, y_k)\| \leq \kappa\|\varepsilon_k\| \quad \forall k \in \mathbb{N}. \tag{10}$$

Given $y \in \mathbb{R}^{n_y}$ and $\varepsilon^l \in \mathbb{R}^\ell$ let us denote by $I(y, \varepsilon^l)$ the set of active constraints in the respective lower-level problem, i.e., let

$$I(y, \varepsilon^l) := \left\{i \in [\ell] \colon (D_1 y_1 + D_2 y_2 - b + \varepsilon^l)_i = 0\right\}.$$

Let us now consider the sequence of index sets $(I(y^*_k, \varepsilon^l_k))_k$. Since the number of possible active constraints is finite, it follows (up to passing to a suitable subsequence) that there exists a fixed index set $\bar{I} \subseteq [\ell]$ such that $I(y^*_k, \varepsilon^l_k) = \bar{I}$ holds for all $k$.

Now, consider the KKT reformulation of the $\varepsilon_k$-perturbed bilevel problem, i.e., consider

$$\begin{aligned} \min_{x, y_1, y_2, \lambda} \quad & -x^\top y_1 \\ \text{s.t.} \quad & Ax \geq a - \varepsilon^u_k, \\ & D_1 y_1 + D_2 y_2 \geq b - \varepsilon^l_k, \\ & \lambda \geq 0, \\ & c + x - D_1^\top \lambda = 0, \\ & d - D_2^\top \lambda = 0, \\ & \lambda^\top (D_1 y_1 + D_2 y_2 - b + \varepsilon^l_k) = 0. \end{aligned} \tag{11}$$

We know that there exist $\lambda^*_k$ such that $(x^*_k, y^*_k, \lambda^*_k)$ is an optimal solution to (11). Because the follower's constraints are active for all $i \in \bar{I}$ at $y^*_k$, it follows that $\lambda^*_{k,i} = 0$ holds for all $i \in [\ell] \setminus \bar{I}$. Thus, we may rewrite Problem (11) as

$$\begin{aligned} \min_{x, y_1, y_2, \lambda} \quad & -x^\top y_1 \\ \text{s.t.} \quad & Ax \geq a - \varepsilon^u_k, \\ & D_1 y_1 + D_2 y_2 \geq b - \varepsilon^l_k, \\ & (D_1 y_1 + D_2 y_2 - b + \varepsilon^l_k)_{\bar{I}} = 0, \\ & \lambda \geq 0, \\ & \lambda_{[\ell] \setminus \bar{I}} = 0, \\ & c + x - D_1^\top \lambda = 0, \\ & d - D_2^\top \lambda = 0. \end{aligned} \tag{12}$$

Note that both problems (11) and (12) have the same optimal value $v(\varepsilon_k)$, which is the value of the $\varepsilon_k$-perturbed pricing problem.

By construction, $(x^*_k, y^*_k, \lambda^*_k)$ is optimal to (12) and we deduce that $y^*_k$ is also optimal for the linear problem

$$\min_y \left\{-(x^*_k)^\top y_1 \colon Dy \geq b - \varepsilon^l_k,\ (Dy - b + \varepsilon^l_k)_{\bar{I}} = 0\right\}. \tag{13}$$

Let us denote by $R(x_k^*, \varepsilon_k^l)$ the solution set of Problem (13). The optimal value of Problem (13) is equal to $v(\varepsilon_k)$ and for any $y \in R(x_k^*, \varepsilon_k^l)$ we have that $(x_k^*, y) \in \mathcal{S}(\varepsilon_k)$. By our standing assumption, it follows that $R(x_k^*, \varepsilon_k^l)$ is a polytope. Since $y_k^* \in R(x_k^*, \varepsilon_k^l)$, it can be written as a convex combination of the vertices of $R(x_k^*, \varepsilon_k^l)$, i.e., there exist weights $(\alpha_{t,k})_{t\in[\tau_k]} \in \mathbb{R}_+^{\tau_k}$ with $\sum_{t=1}^{\tau_k} \alpha_{t,k} = 1$ such that $y_k^* = \sum_{t=1}^{\tau_k} \alpha_{t,k} u_{t,k}$ with $(u_{t,k})_{t\in[\tau_k]}$ being the vertices of $R(x_k^*, \varepsilon_k^l)$. As $\tau_k$ is the number of vertices of $R(x_k^*, \varepsilon_k^l)$, which is bounded by $\binom{\ell}{n_y}$, we may w.l.o.g. assume (otherwise we pass to a suitable chosen subsequence) that $\tau_k$ is equal to a constant $\tau \in \mathbb{N}$ for all $k \in \mathbb{N}$. By potentially passing again to a suitable chosen subsequence, we may further assume that for each $t \in [\tau]$, the set of indices corresponding to active lower level constraints at the point $u_{t,k}$ are constant for $k$, i.e., there exists a fixed subset $\bar{I}_t \subset [\ell]$ such that $I(u_{t,k}, \varepsilon_k^l) = \bar{I}_t$ for each $k$.

Note that each $u_{t,k}$ is also a vertex of $\{y \colon Dy \geq b - \varepsilon_k^l\}$. Thus, there exists a set $I_t \subseteq \bar{I}_t$ corresponding to $n_y$ linearly independent rows (not depending on $k$) of $D$ such that we can write

$$u_{t,k} = N_t(b - \varepsilon_k^l) \quad \forall k \in \mathbb{N},$$

where $N_t$ is the $|I_t| \times \ell$ matrix with $D_{I_t}^{-1}$ in the columns associated to $I_t$ and the zero vector in the columns associated to $[\ell] \setminus I_t$. Thus,

$$y_k^* = \sum_{t\in[\tau]} \alpha_{t,k} u_{t,k} = \sum_{t\in[\tau]} \alpha_{t,k} N_t(b - \varepsilon_k^l).$$

Let us define for each $t \in [\tau]$ the points $u_t := N_t b$ and $y_k := \sum_{t\in[\tau]} \alpha_{t,k} u_t$. We directly get

$$\|y_k^* - y_k\| = \left\| \sum_{t\in[\tau]} \alpha_{t,k} N_t \varepsilon_k^l \right\| \leq K \left\|\varepsilon_k^l\right\| \tag{14}$$

with $K := \sum_{t\in[\tau]} \|N_t\|$ and $u_{t,k} \to u_t$ for each $t \in [\tau]$.

We now define the point $\bar{u}_k := \frac{1}{\tau} \sum_{t\in[\tau]} u_{t,k}$. Since it is a convex combination of solutions of (13), $\bar{u}_k$ is also a solution to (13) and, thus, $(x_k^*, \bar{u}_k) \in \mathcal{S}(\varepsilon_k)$. By substituting $\bar{u}_k = (\bar{u}_{k,1}, \bar{u}_{k,2})$ into the perturbed problem (12), we see that $(x_k^*, \lambda_k^*)$ solves the linear problem

$$\begin{aligned} \min_{x,\lambda} \quad & -x^\top \bar{u}_{k,1} \\ \text{s.t.} \quad & Ax \geq a - \varepsilon_k^u, \\ & \lambda \geq 0, \\ & \lambda_{[\ell]\setminus \bar{I}} = 0, \\ & c + x - D_1^\top \lambda = 0, \\ & d - D_2^\top \lambda = 0. \end{aligned}$$

Let us define $\bar{u} := \frac{1}{\tau} \sum_{t=1}^{\tau} u_t$. Since $\bar{u} = (\bar{u}_1, \bar{u}_2) = \lim_{k\to\infty} (\bar{u}_{k,1}, \bar{u}_{k,2})$, we get $\bar{u}_{k,1} \to \bar{u}_1$ and, by Proposition 12, there exists a constant $L > 0$ such that for each $k$ large enough, we can find a solution $(x_k, \lambda_k)$ to

$$\begin{aligned} \min_{x,\lambda} \quad & -x^\top \bar{u}_1 \\ \text{s.t.} \quad & Ax \geq a, \\ & \lambda \geq 0, \\ & \lambda_{[\ell]\setminus \bar{I}} = 0, \\ & c + x - D_1^\top \lambda = 0, \\ & d - D_2^\top \lambda = 0, \end{aligned} \tag{15}$$

that satisfies

$$\|x_k^* - x_k\| \le \|(x_k^*, \lambda_k^*) - (x_k, \lambda_k)\| \le L\,\|\varepsilon_k^u\|\,. \tag{16}$$

Using (14) and (16), it is clear that for $k$ large enough, Inequality (10) holds with $\kappa := L + K$. Therefore, it only remains to prove that $(x_k, y_k) \in \mathcal{S}(0)$.

By our standing assumption, the sequence $(x_k)_k$ is bounded and we can, w.l.o.g., assume that $x_k$ converges to some point $x^*$. Using (16), we see that also $x_k^*$ converges to $x^*$ and so we have $(x_k^*, \bar{u}_k) \to (x^*, \bar{u})$. Recall that $(x_k^*, \bar{u}_k) \in \mathcal{S}(\varepsilon_k)$ for all $k \in \mathbb{N}$. Thus, using Corollary 2, we deduce that $(x^*, \bar{u}) \in \mathcal{S}(0)$ and hence $v(0) = F(x^*, \bar{u})$.

A bit less obvious is that the sequence $(\lambda_k)_k$ can be chosen to be bounded. Indeed, the constraints of Problem (15) can be written as $(x, \lambda) \in Q$, where $Q$ is defined by the conditions

$$Ax \ge a, \quad \lambda \ge 0, \quad \lambda_{[\ell]\setminus \bar{I}} = 0, \quad c + x - D_1^\top \lambda = 0, \quad d - D_2^\top \lambda = 0.$$

We observe that $Q$ is a pointed polyhedron, as each coordinate is bounded from below. Thus, by the Minkowski–Weyl Theorem (see, e.g., Theorem 3.13 and Proposition 3.15 in Conforti et al. (2014)) there exists a non-empty polytope $Q_0$ such that

$$Q = Q_0 + \operatorname{rec}(Q),$$

where $\operatorname{rec}(Q)$ is the recession cone of $Q$, which can be computed as

$$\operatorname{rec}(Q) = \left\{(0, \beta) \colon D_1^\top \beta = 0, D_2^\top \beta = 0\right\}.$$

We know that $(x_k, \lambda_k) \in Q$ and so there exist $\tilde{\lambda}_k$ and $\beta_k$ such that $(x_k, \tilde{\lambda}_k) \in Q_0$, $(0, \beta_k) \in \operatorname{rec}(Q)$. Since $Q_0 \subseteq Q$, we deduce that $(x_k, \tilde{\lambda}_k) \in Q$ and, hence, it is feasible for Problem (15). Additionally, since the objective function in Problem (15) does not depend on $\lambda$, we see that $(x_k, \tilde{\lambda}_k)$ is also optimal for this problem. As $(x_k, \tilde{\lambda}_k)_k$ is in the (bounded) polytope $Q_0$, we see that there also is no loss of generality by assuming that $(\lambda_k)$ is bounded and, further, that $\lambda_k$ converges to some $\lambda^*$.

Let us now consider the KKT reformulation

$$\begin{aligned}
\min_{x, y_1, y_2, \lambda} \quad & -x^\top y_1 \\
\text{s.t.} \quad & Ax \ge a, \\
& D_1 y_1 + D_2 y_2 \ge b, \\
& \lambda \ge 0, \\
& c + x - D_1^\top \lambda = 0, \\
& d - D_2^\top \lambda = 0, \\
& \lambda^\top (D_1 y_1 + D_2 y_2 - b) = 0
\end{aligned} \tag{17}$$

of the unperturbed bilevel problem, which has the optimal value $v(0)$. We first observe that since $(x^*, \bar{u}) \in \mathcal{S}(0)$, we get that $(x^*, \bar{u}, \lambda^*)$ solves (17). Moreover, $\bar{I} \subseteq I(\bar{u})$ implies that $(x_k, \bar{u}, \lambda_k)$ is feasible for (17). Second, since $(x_k, \lambda_k)$ belongs to the closed set $Q$, its limit $(x^*, \lambda^*)$ also belongs to $Q$. Thus, the limit is feasible for (15). By further recalling that $(x_k, \lambda_k)$ is optimal for (15), we deduce that

$$F(x_k, \bar{u}) \le F(x^*, \bar{u}) = v(0).$$

This proves that $(x_k, \bar{u}, \lambda_k)$ solves (17), i.e., $(x_k, \bar{u}) \in \mathcal{S}(0)$.

By substituting $x_k$ in (17), we see that $\bar{u} \in R(x_k, 0)$, where

$$R(x_k, 0) := \arg\min_y \left\{-(x_k)^\top y_1 \colon Dy \ge b, (Dy - b)_{\bar{I}} = 0\right\}.$$

By the linearity of the objective $y \mapsto -(x_k)^\top y_1$ and by recalling that $\bar{u} = \frac{1}{\tau}\sum_{t \in [\tau]} u_t$ holds, we deduce that $u_t \in R(x_k, 0)$ for each $t \in [\tau]$. Indeed, let us note that the optimal value of the problem that defines $R(x_k, 0)$ is $v(0)$. We see that all the vectors $u_t$, $t \in [\tau]$, are feasible for the LP that define $R(x_k, 0)$. Hence, $F(x_k, u_t) \ge$

$v(0)$ holds while by the optimality of $\bar{u}$, we have the equality $F(x_k, \bar{u}) = v(0)$. By the definition of $\bar{u}$, if we take $t \in [\tau]$, we obtain

$$u_t = \tau\bar{u} - \sum_{s\in[\tau]\setminus\{t\}} u_s.$$

Plugging $u_t$ into the objective we get

$$F(x_k, u_t) = \tau F(x_k, \bar{u}) - \sum_{s\in[\tau]\setminus\{t\}} F(x_k, u_s) \leq \tau v(0) - (\tau - 1)v(0) = v(0).$$

Therefore, $u_t$ is optimal, too, i.e., $u_t \in R(x_k, 0)$.

Finally, as $R(x_k, 0)$ is convex, we conclude that $y_k = \sum_{t\in[\tau]} \alpha_{t,k} u_t \in R(x_k, 0)$, leading to $(x_k, y_k) \in \mathcal{S}(0)$. □

We close this section with the following proposition.

**Proposition 11.** *Suppose that our standing assumption holds. Then, the optimal-value function $v$ is calm at $0$ relative to $\mathcal{E}^{\text{ur}} := \mathcal{E} \cap (\mathbb{R}^m_+ \times \mathbb{R}^\ell)$, i.e., there exists $L > 0$ such that if $\varepsilon$ is sufficiently small with $\varepsilon^u \geq 0$ it holds*

$$|v(0) - v(\varepsilon)| \leq L\|\varepsilon\|. \tag{18}$$

*Proof.* First note note that by our standing assumption, we can find a constant $L_1 > 0$ such that $\mathcal{S}(\varepsilon) \subset P_\varepsilon \subset B(0, L_1)$ for all $\varepsilon$ small enough. Given $\varepsilon$, we can take $(x^*_\varepsilon, y^*_\varepsilon) \in \mathcal{S}(\varepsilon)$. By Theorem 10, there exists $(x^*, y^*) \in \mathcal{S}(0)$ such that

$$\|(x^*_\varepsilon, y^*_\varepsilon) - (x^*, y^*)\| \leq L_2 \|\varepsilon\|$$

holds. Finally, we obtain

$$\begin{aligned}
|v(\varepsilon) - v(0)| &= |F(x^*_\varepsilon, y^*_\varepsilon) - F(x^*, y^*)| \\
&= |(x^*)^\top y^*_\varepsilon - (x^*_\varepsilon)^\top y^*_\varepsilon + (x^*)^\top y^* - (x^*)^\top y^*_\varepsilon| \\
&\leq \|x^* - x^*_\varepsilon\|\|y^*_\varepsilon\| + \|x^*\|\|y^* - y^*_\varepsilon\| \\
&\leq L_1\|(x^*_\varepsilon, y^*_\varepsilon) - (x^*, y^*)\| \\
&\leq L_1 L_2 \|\varepsilon\|.
\end{aligned}$$

so that (18) applies with $L := L_1 L_2$. □

## 6. Conclusion

We studied linear bilevel and pricing problems with right-hand side perturbations of both the upper- and lower-level constraints. For the theoretical validity of many numerical approaches to solve bilevel problems, it is important to know if the solution-set mapping is calm at the zero-perturbation. We proved that this is true for linear bilevel problems without coupling constraints (Theorem 3). For the case of coupling constraints, a positive result only holds for the case of no lower-level perturbations and upper-level relaxations only (Theorem 7). For all other cases, we show by simple counterexamples that calmness—and even outer semicontinuity—does not necessarily hold. In the case of pricing problems, we prove the calmness property for lower-level perturbation as well as for upper-level relaxations only (Theorem 10).

Our future research will contain the analogue study of nonlinear bilevel problems. While the results by Beck et al. (2023) seem to indicate that general nonlinear problems might behave badly, there is still hope that positive results can be obtained under suitably chosen convexity and continuity assumptions.

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## Appendix A. Stability in Linear Optimization Problems

Let a matrix $A$ be given and fixed. For additionally given vectors $b$ and $c$, we consider the linear problem

$$\min \quad c^\top x \quad \text{s.t.} \quad Ax \geq b$$

and write $S(b, c)$ for its solution set and $v(b, c)$ its optimal value. Given a fixed $c$, it is well-known, see, e.g., Li (1993), that $b \mapsto S(b, c)$ is (globally) calm. We show next that the optimal-set mapping $S$ is (locally) calm jointly on $(b, c)$.

**Proposition 12.** *Let $A, b, c$ be given and suppose that the feasible set $\{x \colon Ax \geq b\}$ is a non-empty and compact set. Then, there exist constants $L$ and $\varepsilon > 0$ such that for any $b' \in B(b, \varepsilon)$ and $c' \in B(c, \varepsilon)$ as well as for any $x' \in S(b', c')$, there exists $x \in S(b, c)$ with*

$$\|x - x'\| \leq L\|b - b'\|.$$

*Proof.* Let us assume by contradiction that there exist sequences $b_k$ and $c_k$ converging to $b$ and $c$, respectively, and $x_k \in S(b_k, c_k)$ for each $k$ such that for all $x \in S(b, c)$ it holds

$$\|x - x_k\| > L\|b - b_k\|. \tag{19}$$

By the assumption on the feasible set we know that $S(b, c)$ is non-empty. Hence, strong duality holds, and so there exists $\lambda_k \geq 0$ such that

$$\begin{aligned} c_k &= A^\top \lambda_k, \\ Ax_k - b_k &\geq 0, \\ \lambda_k^\top (Ax_k - b_k) &= 0. \end{aligned}$$

Let us denote the active index set of the problem associated to $(b_k, c_k)$ as $I_k := \{i \colon (Ax_k - b_k)_i = 0\}$. We may assume without loss of generality (by otherwise passing to a suitable subsequence) that $x_k$ converges to some $\bar{x}$ and that there exists a fixed set of indices $\bar{I}$ such that $\bar{I} = I_k$ for all $k \in \mathbb{N}$.

Let $K$ be the cone generated by the rows of $A$ (columns of $A^\top$) that are indexed by $\bar{I}$. The cone $K$ is closed as it is finitely generated; see, e.g., Conforti et al. (2014). Then, the first optimality condition reads $c_k \in K$, so that by passing to the limit we obtain $c \in K$. Therefore, there exist $\lambda_i \geq 0$ for $i \in \bar{I}$ such that

$$c = \sum \lambda_i (A^\top)_{.i} = A^\top \lambda,$$

where we set $\lambda_i := 0$ for $i \notin \bar{I}$.

We also see that $\lambda^\top (Ax_k - b_k) = 0$, from which we deduce that $x_k \in S(b_k, c)$. It is well-known the optimal-set mapping is calm with respect to right-hand side perturbations only, see, e.g., Li (1993), say with a constant $L$. Thus, for each $k$

there exists an $x \in S(b, c)$ such that $\|x - x_k\| \leq L\|b - b_k\|$, which is a contradiction to (19). $\square$

(H. Lefebvre) LIRMM, University of Montpellier, CNRS, Montpellier, France
*Email address*: henri.lefebvre@cnrs.fr

(M. Schmidt) Trier University, Department of Mathematics, Universitätsring 15, 54296 Trier, Germany
*Email address*: martin.schmidt@uni-trier.de

(A. Svensson) Universidad de O'Higgins, Av Libertador Bernardo O'Higgins 611, Rancagua, Chile
*Email address*: anton.svensson@uoh.cl

(J. Thürauf) University of Technology Nuremberg (UTN), Department Liberal Arts and Social Sciences, Discrete Optimization, Dr.-Luise-Herzberg-Str. 4, 90461 Nuremberg, Germany
*Email address*: johannes.thuerauf@utn.de