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Covering for Set-Valued Mappings in the Absence of Metric Regularity

A. F. Izmailov · **M. Jelitte**

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Abstract Covering properties build the foundation of stability and sensitivity analysis of solutions to a generalized equation and more specific optimization-related stationarity and equilibrium problems. It has been well-understood that metric regularity of the mapping defining the generalized equation is a key to furnish Lipschitzian stability of the solution of interest. With this work, we want to leave the ties of metric regularity, and establish new covering results under a regularity concept introduced by H. Gfrerer, whose relation to notions of 2-regularity has been elaborated recently. Among other things, our findings target a precise description of the set with nonempty interior being covered. Comprehensive comparisons with existing results for constrained equations are included, as well as an application to coupled constraint systems, arising in practically relevant settings, including variational and optimality systems, or complementarity systems over convex cones.

Keywords Generalized Equation · Constrained Equation · Gfrerer Regularity · 2-Regularity · Coupled Constraint System · Cone Complementarity Problem

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1 Introduction

Consider the *generalized equation* (inclusion)

$$S(u) \ni \bar{y}, \tag{1}$$

with a set-valued mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, and given point $\bar{y} \in \mathbb{R}^m$. It has been well-recognized that generalized equations cover a large area of mathematical problems, including optimality systems, complementarity problems, constrained equations, etc.

Alexey F. Izmailov
VMK Faculty, OR Department, Lomonosov Moscow State University, Moscow, Russia
E-mail: izmaf@cs.msu.ru

✉M. Jelitte
Faculty of Mathematics, Technische Universität Dresden, Dresden, Germany
E-mail: Mario.Jelitte@tu-dresden.de

This work targets stability and sensitivity issues under weak (non-Lipschitzian) regularity assumptions on a solution to (1). That said, we aim at weak conditions guaranteeing, for certain y near $\bar{y}$, that the perturbed generalized equation $S(u) \ni y$ admits a solution $u = u(y)$, whose distance to a given solution $\bar{u}$ to (1) can be bounded from above in terms of the distance of y to $\bar{y}$. These properties comprise what we typically understand as *covering*, which, in the standard Lipschitzian case, can be formally introduced through

$$\exists \gamma, \tau > 0: \qquad \bar{y} + \gamma t \mathbb{B} \subset S(\bar{u} + t\mathbb{B}) \quad \forall t \in [0, \tau], \tag{2}$$

where $\mathbb{B}$ is the closed Euclidean unit ball. This property can be seen as stability of a given solution of (1) subject to the right-hand side perturbations, with the Lipschitzian estimate of the distance to the solution set of the perturbed problem.

Among the most central concepts of variational analysis is the *metric regularity* of S at $(\bar{u}, \bar{y})$ *with a constant* $c > 0$,

$$\mathrm{dist}[u, S^{-1}(y)] \leq c \cdot \mathrm{dist}[y, S(u)] \qquad \forall (u, y) \text{ near } (\bar{u}, \bar{y}),$$

with $\mathrm{dist}[\xi, A]$ being the Euclidean distance of a point ξ to a set A, taking the value $+\infty$, when the set A is empty. This concept, although formally stronger than (2), has taken a key role in Lipschitzian stability and sensitivity analysis of set-valued mappings and related variational settings, see [10] for a comprehensive overview. We do not comment on the numerous other applications of metric regularity, both within and outside the field of optimization theory. Instead, we simply refer interested readers to books [7,24,23,27,34,35,36,39] in addition to the latter reference, and also discussions therein.

In this work, we want to leave the realm of Lipschitzian stability and sensitivity, and study covering properties for set-valued mappings in the absence of metric regularity. We mention the papers [16,17] on Hölder metric regularity notions, and the works [4,18,20] on directional metric regularity properties, where not all (u, y) near $(\bar{u}, \bar{y})$ are considered, but only some, out of a directional neighborhood of $(\bar{u}, \bar{y})$. Many other works follow similar lines of analyses. A somewhat different way has been proposed in [2] and the numerous successor works, including [5,6], where *covering-along-a-curve* results have been established for constrained smooth equations. The key to these results are notions of *2-regularity* (cf. Section 6), combining first- and second-order derivatives of the given smooth mapping, which is also why they are frequently called *mixed-order* regularity conditions (see e.g. [8,31]). Many other optimization-related applications of 2-regularity are now known, see references above, and also [1].

An extension of the 2-regularity concept to more general set-valued settings has been introduced in [19] by H. Gfrerer for the purpose of formulating *dual* optimality conditions for *degenerate* optimization problems with constraint-system-constraints, and, more generally, computing limiting normals in [21]. The work [31] aims to embed various 2-regularity notions (including those in [2,3,26]) into the landscape of contemporary variational analysis. Gfrerer's concept has taken a key role there, but also the concept of *metric 2-regularity*, which originated in [28]. The paper [8] addresses some state-of-the-art stationarity conditions, including considerations for degenerate problems, for which Gfrerer's concept and 2-regularity conditions of [3] play an important role. Another optimization-related application of Gfrerer's concept is addressed in [30] on the computation of tangents to the solution set of a generalized equation (1) in terms of second-order Studniarski derivatives (see Section 4 for the definition) in the possible absence of Lipschitzian (metric regularity) assumptions.

With this paper, we want to follow up on the latter line of research, and employ Gfrerer's concept to establish new covering results for set-valued settings in the absence of metric regularity. After presenting some basic notation and definitions in Section 2, we will recall the notions of Gfrerer regularity and metric 2-regularity, as well as their tight relationship and verifiable criteria, in Section 3. Section 4 contains our main covering results. While the key result (Theorem 3) does not provide information under Gfrerer's regularity on the structure of the *possibly empty* set being covered, we will see that, under natural assumptions (transforming Gfrerer's regularity into metric 2-regularity), the set being covered is not thin (has nonempty interior) and can be described quantitatively and qualitatively. In Section 5 we will employ our findings to establish new covering results for nonlinear equations with closed (not necessarily convex) constraints. Under natural assumptions, we will be able to describe the set being covered quantitatively and qualitatively in terms of well-established objects of modern analysis [9,39]. Section 6 is devoted to comparisons with existing covering results for smooth equations with convex constraints under 2-regularity notions. In the final Section 7, we will illustrate how our new covering results can be applied to *coupled constraint systems*, arising e.g. in slack-reformulations of *variational systems* [31, Section 8], *complementarity systems over convex cones* [15, Section 5.1], etc.

2 Basic Definitions

We recall some basic notions of variational analysis [10,35,39] broadly used below. Throughout, $\mathbb{B}$ is the closed (Euclidean) unit ball, and $\operatorname{cone}(A)$ stands for the conic hull of a set A. The *polar* of a cone $\mathscr{C} \subset \mathbb{R}^p$ is the (nonempty) closed convex cone $\mathscr{C}^\circ = \left\{ v \in \mathbb{R}^p \mid v^\top \xi \leq 0 \ \forall \xi \in \mathscr{C} \right\}$. We will often work with the *directional neighborhood* of a point ξ relative to a direction η,

$$K_{\varepsilon,\delta}(\xi;\eta) := \xi + (\operatorname{cone}(\eta + \delta\mathbb{B}) \cap (\varepsilon\mathbb{B})),$$

where $\varepsilon, \delta > 0$.

The (Bouligand/Severi) *tangent (contingent) cone* associated with a set $A \subset \mathbb{R}^p$ at a point $\xi \in A$ is

$$T_A(\xi) := \left\{ \eta \in \mathbb{R}^p \,\middle|\, \exists t_k \searrow 0, \exists \eta^k \to \eta : \ \xi + t_k \eta^k \in A \ \forall k \in \mathbb{N} \right\},$$

while the (Mordukhovich) *limiting (basic) normal cone* to A at ξ is

$$N_A(\xi) := \left\{ v \in \mathbb{R}^p \,\middle|\, \exists \xi^k \to \xi, \exists v^k \to v : \ \xi^k \in A, \ v^k \in T_A(\xi^k)^\circ \ \forall k \right\}.$$

For a set-valued mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, we denote its *graph* by $\mathrm{gph}S = \{(u,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in S(u)\}$, and its *range* by $\mathrm{rge}S = \{y \in \mathbb{R}^m \mid \exists u \in \mathbb{R}^n : (u,y) \in \mathrm{gph}S\}$. The *graphical derivative* of S at $(u,y) \in \mathrm{gph}S$ is the mapping $DS(u|y) : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$,

$$DS(u|y)(w) := \left\{\eta \in \mathbb{R}^m \mid (w,\eta) \in T_{\mathrm{gph}S}(u,y)\right\},$$

and the *coderivative* of S at $(u,y) \in \mathrm{gph}S$ is the mapping $D^*S(u|y) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$,

$$D^*S(u|y)(z) := \left\{v \in \mathbb{R}^n \mid (v,-z) \in N_{\mathrm{gph}S}(u,y)\right\}.$$

The graphical derivative of a single-valued continuously differentiable mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^m$, for instance, is $D\Phi(u|\Phi(u))(w) = \{\Phi'(u)w\}$, while the coderivative is $D^*\Phi(u|\Phi(u))(z) = \{\Phi'(u)^\top z\}$. In the special case of Φ being a linear operator, we will write $\mathrm{im}\Phi$ instead of $\mathrm{rge}\Phi$, and denote the *null space* of such an operator by $\ker\Phi$.

The *subderivative* of a function $\varphi : \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\}$ at a point $\xi \in \mathbb{R}^p$ with $\varphi(\xi) < +\infty$ *relative to a direction* $\omega \neq 0$ is

$$d\varphi(\xi)(\omega) := \liminf_{t \searrow 0,\, v \to \omega} \frac{\varphi(\xi + tv) - \varphi(\xi)}{t}.$$

Other definitions will appear in the following sections as needed.

3 Gfrerer's Regularity and Metric 2-Regularity

This section is devoted to interrelations between two relaxed regularity notions. The first of which specializes a notion coined in [19, Definition 1] with respect to arbitrary orders, and the second notion is stated in [31, Definition 6].

Definition 1 A mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be

(a) *Gfrerer regular* at $(\bar{u},\bar{y}) \in \mathrm{gph}\,S$ relative to a direction $(w,\eta) \in \mathbb{R}^n \times \mathbb{R}^m$ if there are $\varepsilon, \delta, c > 0$ such that

$$\mathrm{dist}\left[u, S^{-1}(y)\right] \cdot \|u - \bar{u}\| \leq c \cdot \mathrm{dist}[y, S(u)]$$

holds for all $(u,y) \in K_{\varepsilon,\delta}\left((\bar{u},\bar{y});(w,\eta)\right)$ with $\mathrm{dist}[y,S(u)] \leq \delta\|u-\bar{u}\|^2$.

(b) *metrically* 2*-regular* at $(\bar{u},\bar{y}) \in \mathrm{gph}\,S$ relative to a direction $w \neq 0$ if there are $\varepsilon, \delta, c, \rho > 0$ with $\rho > \|w\|^{-1} \sup\{\|\eta\| \mid \eta \in DS(\bar{u}|\bar{y})(w)\}$ such that S is metrically regular at each (u,y) with constant $c/\|u-\bar{u}\|$ for $u \in K_{\varepsilon,\delta}(\bar{u};w) \setminus \{\bar{u}\}$ and $y \in S(u) \cap (\bar{y} + \rho\|u-\bar{u}\|\mathbb{B})$.

Both conditions are *robust* in the directional sense, i.e., their fulfillment relative to a direction necessitates fulfillment relative to any neighboring direction and any positive multiples. As has been summarized in [31, Remark 2], metric 2-regularity of S at $(\bar{u},\bar{y})$ relative to $w \neq 0$ yields $\rho > 0$, and sequences $t_k \searrow 0, w^k \to w$ and $y^k \in S(\bar{u} + t_k w^k)$, satisfying $\|y^k - \bar{y}\| \leq \rho t_k \|w^k\|$ with S being metrically regular at $(\bar{u} + t_k w^k, y^k)$ with $c/\|u^k - \bar{u}\|$ for all k. The condition $\rho > \|w\|^{-1} \sup\{\|\eta\| \mid \eta \in DS(\bar{u}|\bar{y})(w)\}$ particularly implies $DS(\bar{u}|\bar{y})(w) \neq \varnothing$, which is key for the existence of *some* $u \neq \bar{u}$, arbitrarily close to $\bar{u}$, in the directional neighborhood of $\bar{u}$, such that the set of corresponding feasible y is nonempty. To sensitize a reader, we emphasize that such y do *not* necessarily exist for all $u \neq \bar{u}$ in the directional neighborhood.

Each of the conditions in Definition 1 certifies metric regularity of S at the special rate $c(u) := c/\|u-\bar{u}\|$ for appropriate points (u,y) near $(\bar{u},\bar{y})$. While their components y (from the range space of S) are described in a qualitative (although possibly empty) way in Gfrerer's concept, this is not so for metric 2-regularity. On the flip side, metric 2-regularity necessitates nonemptyness of the graphical derivative relative to the direction of interest (and with this the existence of nearby y in the range space, see above). Following the discussion below [19, Definition 1], nonemptyness of the graphical derivative is not guaranteed under Gfrerer regularity. The next result sheds more light on the relationship between the two concepts.

Theorem 1 *Let the graph of* $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *be closed around* $(\bar{u},\bar{y}) \in \operatorname{gph} S$*, and let* $w \neq 0$*. Then the following are equivalent:*

(a) S is metrically 2-regular at $(\bar{u},\bar{y})$ relative to w.
(b) S is Gfrerer regular at $(\bar{u},\bar{y})$ relative to (w,η) for all $\eta \in \mathbb{R}^m$, and

$$DS(\bar{u}|\bar{y})(w) \neq \varnothing. \tag{3}$$

(c) S is Gfrerer regular at $(\bar{u},\bar{y})$ relative to $(w,0)$, and (3) holds.

Proof With no loss in generality, we can assume $\|w\| = 1$. Then, the equivalence between (a) and (b) follows by [31, Theorem 12], while the equivalence between (b) and (c) is immediate from the definitions. $\square$

Examples 1 and 2 in the next sections demonstrate the difference between Gfrerer regularity and metric 2-regularity.

We emphasize that graph-closedness is actually not needed to establish $'(b) \Leftrightarrow (c)'$ in the previous theorem, but this is needed in [31] to confirm $'(a) \Leftrightarrow (b)'$ through utilization of the *Least Singular Value* (LSV) function of the coderivative of the mapping S, defined in [31, formula (22)] (and earlier in [28]) for $(u,y) \in \mathbb{R}^n \times \mathbb{R}^m$ as

$$\operatorname{Reg}(u,y;S) := \begin{cases} \inf\limits_{\|\lambda\|=1} \operatorname{dist}[0, D^*S(u|y)(\lambda)] & \text{if } y \in S(u), \\ +\infty & \text{if } y \notin S(u). \end{cases}$$

The result below is [31, Theorem 6], stating a criterion for metric 2-regularity that is based on the LSV function and its subderivative.

Theorem 2 *For a closed-graph mapping* $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *and* $(\bar{u},\bar{y}) \in \operatorname{gph} S$*, assume that* $DS(\bar{u}|\bar{y})(w) \neq \varnothing$ *holds with some* $w \neq 0$*. The following are equivalent:*

(a) S is metrically 2-regular at $(\bar{u},\bar{y})$ relative to w.
(b) It holds that

$$\operatorname{Reg}(\bar{u},\bar{y};S) = 0 \quad \Longrightarrow \quad \inf_{\eta \in DS(\bar{u}|\bar{y})(w,\eta)} d\operatorname{Reg}(\bar{u},\bar{y};S)(w,\eta) > 0.$$

It can be difficult to use the criterion without knowledge of the subderivative $d\mathrm{Reg}(\bar{u},\bar{y};S)(w,\eta)$. For this reason, [28,31] address several estimates for that subderivative with respect to specific set-valued mappings S. We will revisit some of them in an appropriate way in this article. For the sake of getting an intuition for the LSV function, we would like to mention for a (single-valued) locally Lipschitzian mapping $\Phi:\mathbb{R}^n\to\mathbb{R}^m$, that the estimates

$$\min_{\substack{\|\lambda\|=1\\ A\in\partial_C\Phi(u)}}\left\|A^\top\lambda\right\| \le \mathrm{Reg}(u,\Phi(u);\Phi) \le \min_{\substack{\|\lambda\|=1\\ A\in\partial\Phi(u)}}\left\|A^\top\lambda\right\|$$

are always true, due to relations in [22, Propositions 4.6, 5.3], in which $\partial\Phi$ stands for the *B-differential* and $\partial_C\Phi$ for Clarke's *generalized Jacobian* of Φ, cf. e.g. [27, Section 1.4.1] for their definitions and basic properties. Assuming that Φ is continuously differentiable, then the generalized Jacobian and B-differential coincide with the derivative of Φ, and we have

$$\mathrm{Reg}(u,\Phi(u);\Phi) = \min_{\|\lambda\|=1}\left\|\Phi'(u)^\top\lambda\right\|.$$

As is well-known, the expression on the right-hand side of the above equality is the least singular value of the rectangular matrix $\Phi'(u)$, hence the name LSV function. In particular, the criterion in Theorem 2 for metric 2-regularity of Φ now requires that either the matrix $\Phi'(\bar{u})$ has full-rank, or, if not, then we have that the directional derivative at $\bar{u}$ relative to $w\neq 0$ of the LSV function $u\mapsto \min\limits_{\|\lambda\|=1}\left\|\Phi'(u)^\top\lambda\right\|$ is positive.

4 Main Covering Results

This section contains our main covering results for set-valued mappings under Gfrerer regularity relative to $(w,0)$ for a direction w, and later also under metric 2-regularity.

Now we present the main result of this work. This originated in the thesis [28] with another proof.

Theorem 3 *Let $S:\mathbb{R}^n\rightrightarrows\mathbb{R}^m$, a point $(\bar{u},\bar{y})\in\mathrm{gph}S$ around which $\mathrm{gph}S$ is closed, and a direction $w\neq 0$ be given. If S is Gfrerer regular at $(\bar{u},\bar{y})$ relative to $(w,0)$, then for any $\alpha>0$, there are $\sigma,\rho,\gamma,\tau>0$, so that*

$$S(\bar{u}+t(w+\sigma\mathbb{B}))\cap(\bar{y}+\rho t\mathbb{B})+t^2\gamma\mathbb{B}\subset S(K_{t,\alpha}(\bar{u};w)) \quad \forall t\in[0,\tau]. \tag{4}$$

Proof Let $\varepsilon,\delta,c>0$ be the numbers with which S is Gfrerer regular at $(\bar{u},\bar{y})$ relative to $(w,0)$. We prove the claim by contradiction. Then, there exist a constant $\alpha_0>0$, and sequences $\gamma_k\searrow 0, t_k\searrow 0$, $\eta^k\to 0$, and $\{x^k\},\{v^k\}\subset\mathbb{R}^m$, so that

$$\left.\begin{aligned} v^k &\in S\left(\bar{u}+t_k(w+\eta^k)\right)\cap(\bar{y}+\gamma_k t_k\mathbb{B}),\\ x^k &\in \gamma_k\mathbb{B},\\ v^k+t_k^2x^k &\notin S\left(K_{t_k,\alpha_0}(\bar{u};w)\right)\end{aligned}\right\}\quad \forall k\in\mathbb{N}. \tag{5}$$

The first of these inclusions yields $\|v^k - \bar{y}\| \leq \gamma_k t_k$ for all k, while with $\gamma_k \searrow 0$ in mind, it also follows from the second inclusion that $x^k \to 0$. Therefore, setting $u^k := \bar{u} + t_k(w + \eta^k)$ and $y^k := v^k + t_k^2 x^k$ for $k \in \mathbb{N}$, we can assume without loss of generality that $(u^k, y^k) \in K_{\varepsilon,\delta}((\bar{u},\bar{y});(w,0))$ holds for all k. The first inclusion in (5) tells us that $v^k \in S(u^k)$ holds for all k. Furthermore, $w \neq 0$ and $\eta^k \to 0$ allow us to assume $w + \eta^k \neq 0$ for all k. Combining these things with $x^k \to 0$, and using the definitions of u^k and y^k, then we get

$$\operatorname{dist}\left[y^k, S(u^k)\right] \leq \|y^k - v^k\| = t_k^2 \|x^k\| = \frac{\|x^k\|}{\|w+\eta^k\|^2} \cdot \|u^k - \bar{u}\|^2 \leq \delta \|u^k - \bar{u}\|^2 \quad (6)$$

for all large k. Hence, Gfrerer regularity of S at $(\bar{u},\bar{y})$ relative to $(w,0)$ brings us to $\operatorname{dist}\left[u^k, S^{-1}\left(y^k\right)\right] \cdot \left\|u^k - \bar{u}\right\| \leq c \cdot \operatorname{dist}\left[y^k, S\left(u^k\right)\right]$ for all large k. Combining the latter with the first estimate in (6) yields

$$\operatorname{dist}\left[u^k, S^{-1}\left(y^k\right)\right] \cdot \left\|u^k - \bar{u}\right\| \leq c \cdot t_k^2 \|x^k\| \quad (7)$$

for all large k. Since gph$S$ is closed near $(\bar{u},\bar{y})$, we can find $\{\xi^k\} \subset \mathbb{R}^n$ satisfying $\xi^k \in S^{-1}\left(y^k\right)$ and $\operatorname{dist}\left[u^k, S^{-1}\left(y^k\right)\right] = \left\|u^k - \xi^k\right\|$ for all k. With this, $u^k - \bar{u} = t_k(w + \eta^k)$, $w \neq 0$, $\eta^k \to 0$, and $x^k \to 0$, it follows by (7) that $\left\|u^k - \xi^k\right\| = o(t_k)$ as $k \to +\infty$. Thus, by the construction of the sequence $\{u^k\}$, we observe that $\xi^k \in K_{t_k,\alpha_0}(u^*, w)$ must be fulfilled for infinitely many k. At the same time, we deduce $v^k + t_k^2 x^k = y^k \in S(\xi^k)$ for all $k \in \mathbb{N}$ from the definition of the sequences $\{y^k\}, \{\xi^k\}$. This, however, contradicts the last condition in (5). Hence, the claim of the theorem is true. □

For the arguments in the proof to go through, it would be enough to assume that the inverse S^{-1} is closed-valued for all neighboring points of $\bar{y}$. To guarantee this, we impose the standard assumption that gph$S$ is locally closed, which is important for Theorems 1–2, and equivalent to local outer semicontinuity of a truncation of S, cf. [39, Theorem 5.7].

The estimate in (4) is only meaningful, when its left-hand side is *not* empty. Observe, however, that this set can happen to be empty even in the case of a non-metrically regular single-valued continuous mapping S:

Example 1 Consider $S(u) := \sqrt{\max\{0,u\}}$, $\bar{u} := 0$, and $w := 1$. This S is not metrically regular around $(\bar{u},0)$, but is continuous at $\bar{u}$, and all the assumptions of Theorem 3 are fulfilled. However, the left-hand side of (4) is empty, as the intersection it contains is empty for sufficiently small $t > 0$, regardless of $\sigma, \rho > 0$. □

More generally, we can argue in virtue of the definition of graphical derivatives that the intersection in the left-hand side of (4) is empty, if the constants $\sigma, \rho > 0$, obtained in the theorem, satisfy the condition $\rho < \inf\{\|\eta\| \mid \eta \in DS(\bar{u}|\bar{y})(v),\ v \in w + \sigma\mathbb{B}\}$. Having in mind that $T_{\mathrm{gph}S}(\bar{u},\bar{y})$ is closed, we can see that the latter inequality holds for sufficiently small $\sigma > 0$, and any $\rho > 0$, when $DS(\bar{u}|\bar{y})(w) = \varnothing$, i.e., (3) is violated. This is exactly the point that justifies the use of metric 2-regularity in what follows, which will come in through Theorem 1.

In the rest of this section, we introduce specializations of Theorem 3, for which the set being covered by S is nonempty, i.e., the conclusion becomes nontrivial.

Lemma 1 *Let $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, a point $(\bar{u}, \bar{y}) \in \operatorname{gph} S$, and a direction $w \neq 0$ with $0 \in DS(\bar{u}|\bar{y})(w)$ be given. Then for any $\sigma, \rho > 0$, there is a sequence $t_k \searrow 0$, so that for all $t \in \{t_k\}_{k \in \mathbb{N}}$,*

$$S(\bar{u} + t(w + \sigma\mathbb{B})) \cap (\bar{y} + \rho t\mathbb{B}) \neq \varnothing. \tag{8}$$

If, moreover, it holds that

$$\forall t_k \searrow 0 \exists (w^k, x^k) \to (w, 0) : \quad \bar{y} + t_k x^k \in S(\bar{u} + t_k w^k) \forall k \in \mathbb{N}, \tag{9}$$

then for any $\sigma, \rho > 0$, there exists $\tau > 0$, so that (8) holds for all $t \in [0, \tau]$.

Proof The condition $0 \in DS(\bar{u}|\bar{y})(w)$ yields $t_k \searrow 0$, $w^k \to w$, and $x^k \to 0$, so that $\|w^k - w\| \leq \sigma$, $\|x^k\| \leq \rho$, and $\bar{y} + t_k x^k \in S(\bar{u} + t_k w^k)$ hold for all k. Hence, we have $\bar{y} + t_k x^k \in S(\bar{u} + t_k(w + \sigma\mathbb{B})) \cap (\bar{y} + \rho t_k \mathbb{B})$ for all k, so the first claim is true.

Suppose from now on that (9) is in force, and assume by contradiction that there are $\sigma, \rho > 0$, along with a sequence $t_k \searrow 0$, so that (8) is violated for each $t \in \{t_k\}$. Thanks to (9), we can find $(w^k, x^k) \to (w, 0)$, satisfying $\bar{y} + t_k x^k \in S(\bar{u} + t_k w^k)$ for each $k \in \mathbb{N}$. But now we see that (8) must actually hold for all $t \in \{t_k\}$, as k is sufficiently large. This is absurd. □

Remark 1 The condition in (9) is inspired by the condition in [39, Section 8H] (or [38]) terming S *proto-differentiable* at $(\bar{u}, \bar{y})$ relative to $w \neq 0$, when

$$\forall x \in DS(\bar{u}|\bar{y})(w) \forall t_k \searrow 0 \exists (w^k, x^k) \to (w, x) : \quad \bar{y} + t_k x^k \in S(\bar{u} + t_k w^k) \forall k \in \mathbb{N}.$$

Evidently, proto-differentiability of S at $(\bar{u}, \bar{y})$ relative to w is stronger than (9). □

Proposition 1 *Let $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, a point $(\bar{u}, \bar{y}) \in \operatorname{gph} S$ around which $\operatorname{gph} S$ is closed, and a direction $w \neq 0$ with $0 \in DS(\bar{u}|\bar{y})(w)$ be given. If S is metrically 2-regular at $(\bar{u}, \bar{y})$, then for any $\alpha > 0$, there are $\sigma, \rho, \gamma > 0$ and a sequence $t_k \searrow 0$, so that the inclusion in (4) holds with its left-hand side nonempty for all $t \in \{t_k\}_{k \in \mathbb{N}}$. If, moreover, the condition in (9) is satisfied (as is the case, when S is proto-differentiable at $(\bar{u}, \bar{y})$ relative to w), then for any $\alpha > 0$, there are $\sigma, \rho, \gamma, \tau > 0$, so that (4) holds with its left-hand side nonempty.*

Proof Thanks to $0 \in DS(\bar{u}|\bar{y})(w)$, it follows by Theorem 1 that metric 2-regularity of S relative to w becomes equivalent to Gfrerer regularity relative to $(w, 0)$. Therefore, and with Lemma 1 in mind, the claims in this proposition are immediate consequences of Theorem 3. □

Looking at the proof of Lemma 1, we can rewrite Proposition 1 in the following way: If closed-graph S is metrically 2-regular at $(\bar{u}, \bar{y})$ relative to $w \neq 0$ with $0 \in DS(\bar{u}|\bar{y})(w)$, then for any small $\sigma > 0$, there exists a curve $\Gamma_\sigma : \mathbb{R} \to \mathbb{R}^m$ with $(\Gamma_\sigma)'_+(0) = 0$ (i.e., the right-sided derivative is 0), along with a sequence $t_k \searrow 0$, so that for any $t \in \{t_k\}_{k \in \mathbb{N}}$,

$$\begin{aligned} \Gamma_\sigma(t) &\in S(\bar{u} + t(w + \sigma\mathbb{B})) \cap (\bar{y} + o(t)\mathbb{B}) \quad (\text{as } t \searrow 0), \\ \Gamma_\sigma(t) + t^2\gamma\mathbb{B} &\subset S(K_{t,\alpha}(\bar{u}; w)). \end{aligned} \tag{10}$$

Under (9), we even know for some $\tau > 0$ that (10) holds for all $t \in [0,\tau]$. This interpretation of Proposition 1 follows the *covering-along-a-curve* paradigm in [2]. Below, in Proposition 3, we will state a related result with an explicit description of the curve. Before we come to this, we introduce another specialization of Theorem 3 that concerns the case where $S:\mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *Lipschitz-like* (has the *Aubin property*) around $(\bar{u},\bar{y}) \in \operatorname{gph} S$ at some rate $c > 0$ relative to a direction $w \neq 0$,

$$\exists \varepsilon,\delta,\kappa > 0: \qquad S(u)\cap(\bar{y}+\kappa\mathbb{B}) \subset S(v)+c\|u-v\|\mathbb{B} \qquad \forall u,v \in K_{\varepsilon,\delta}(\bar{u};w).$$

It can be easily seen that the latter holds for any $w \neq 0$, if S is Lipschitz-like in the common sense of [35,39], etc.

Lemma 2 *If $S:\mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is Lipschitz-like around $(\bar{u},\bar{y}) \in \operatorname{gph} S$ at a rate $c > 0$ relative to $w \neq 0$, then there are $\varepsilon,\delta > 0$, so that $DS(\bar{u}|\bar{y})(w) \neq \varnothing$ holds in addition to $S(u)\cap(\bar{y}+c\|u-\bar{u}\|\mathbb{B}) \neq \varnothing$ for all $u \in K_{\varepsilon,\delta}(\bar{u};w)$.*

Proof Let $\varepsilon,\delta,\kappa > 0$ be the numbers with which S is Lipschitz-like at rate $c > 0$ relative to w. To confirm $DS(\bar{u}|\bar{y})(w) \neq \varnothing$, pick $t_k \searrow 0$ and $w^k \to w$, so that $v^k := \bar{u}+t_k w^k \in K_{\varepsilon,\delta}(\bar{u};w)$ holds for all $k \in \mathbb{N}$. Taking $u = \bar{u}$, then the Lipschitz-like property yields $\bar{y} \in S(v^k)+ct_k\|w^k\|\mathbb{B}$ for all $k \in \mathbb{N}$. Hence, we can find some $\{\xi^k\} \subset \mathbb{B}$ satisfying $\bar{y}+t_k(c\|w^k\|)\xi^k \in S(v^k) = S(\bar{u}+t_k w^k)$ for all $k \in \mathbb{N}$. Passing to subsequences, when needed, then we can assume that $\{(c\|w^k\|)\xi^k\}$ tends to some η as k tends to $+\infty$. This requires $\eta \in DS(\bar{u}|\bar{y})(w)$.

The remaining proof is even simpler: Fix $u = \bar{u}$ in the definition of the Lipschitz-like property, and find $\bar{y} \in S(v)+c\|v-\bar{u}\|\mathbb{B}$ for all $v \in K_{\varepsilon,\delta}(\bar{u};w)$. □

Proposition 2 *Let $S:\mathbb{R}^n \rightrightarrows \mathbb{R}^m$, a point $(\bar{u},\bar{y}) \in \operatorname{gph} S$ around which $\operatorname{gph} S$ is closed, and a direction $w \neq 0$ be given, so that S is Gfrerer regular at $(\bar{u},\bar{y})$ relative to $(w,0)$. Fix $\alpha > 0$, and let $\sigma,\rho,\gamma,\tau > 0$ be the constants, guaranteed by Theorem 3. If S is Lipschitz-like around $(\bar{u},\bar{y})$ at a rate $c \in (0,\rho/(\|w\|+\sigma))$ relative to w, then S is metrically 2-regular at $(\bar{u},\bar{y})$ relative to w, and there are $\widehat{\sigma},\widehat{\tau} > 0$, satisfying*

$$\varnothing \neq S\left(\bar{u}+t(w+\widehat{\sigma}\mathbb{B})\right)\cap(\bar{y}+\rho t\mathbb{B})+t^2\gamma\mathbb{B} \subset S\left(K_{t,\alpha}(\bar{u};w)\right) \quad \forall t \in [0,\widehat{\tau}]. \tag{11}$$

Proof Combining Lemma 2 with Theorem 1 yields metric 2-regularity relative to w under the imposed Gfrerer regularity assumption. Furthermore, with Lemma 2 in mind, the Lipschitz-like assumption at rate $c \in (0,\rho/(\|w\|+\sigma))$ yields $\varepsilon,\delta > 0$, so that $S(u)\cap(\bar{y}+c\|u-\bar{u}\|\mathbb{B}) \neq \varnothing$ holds for all $u \in K_{\varepsilon,\delta}(\bar{u};w)$. With $\widehat{\sigma} := \min\{\delta,\sigma\}$ and $\widehat{\tau} := \min\{\tau,\varepsilon/(\|w\|+\sigma)\}$, we get with any $t \in [0,\widehat{\tau}]$ and $\xi \in \mathbb{B}$ for a point $u := \bar{u}+t(w+\widehat{\sigma}\xi)$ that $u \in \bar{u}+\operatorname{cone}(w+\widehat{\sigma}\mathbb{B}) \subset \bar{u}+\operatorname{cone}(w+\delta\mathbb{B})$, and also $\|u-\bar{u}\| \leq t(\|w\|+\widehat{\sigma}\|\xi\|) \leq \widehat{\tau}(\|w\|+\sigma) \leq \varepsilon$, i.e., we have $u \in K_{\varepsilon,\delta}(\bar{u};w)$ for any so constructed u. Therefore, our considerations entail

$$\begin{aligned}
\varnothing &\neq S(\bar{u}+t(w+\widehat{\sigma}\mathbb{B}))\cap(\bar{y}+ct(\|w\|+\widehat{\sigma})\mathbb{B})\\
&\subset S(\bar{u}+t(w+\sigma\mathbb{B}))\cap(\bar{y}+ct(\|w\|+\sigma)\mathbb{B})\\
&\subset S(\bar{u}+t(w+\sigma\mathbb{B}))\cap(\bar{y}+\rho t\mathbb{B}) \qquad\qquad \forall t \in [0,\widehat{\tau}].
\end{aligned}$$

From here, the desired condition (11) follows immediately through Theorem 3. □

The role of condition $0 \in DS(\bar{u}|\bar{y})(w)$ in Proposition 1 (or the usual Lipschitz-like property) for the *nontrivial* (i.e., nonempty) covering has been noticed in [28, Theorem 6.19], but the use of proto-differentiability (Proposition 1), or directional Lipschitz-like property (Proposition 2), has not yet been addressed in this context.

Under assumptions of Proposition 2, we can say that, for any small $\sigma > 0$, there exists a curve $\Gamma_\sigma : \mathbb{R} \to \mathbb{R}^m$ with $(\Gamma_\sigma)'_+(0) \in DS(\bar{u}|\bar{y})(w)$, so that for some $\tau > 0$,

$$\begin{aligned} \Gamma_\sigma(t) &\in S(\bar{u} + t(w + \sigma\mathbb{B})) \cap (\bar{y} + \rho t\mathbb{B}) && \forall t \in [0, \tau], \\ \Gamma_\sigma(t) + t^2\gamma\mathbb{B} &\subset S(K_{t,\alpha}(\bar{u}; w)) && \forall t \in [0, \tau]. \end{aligned}$$

This follows again the *covering-along-a-curve* paradigm in [2], but in contrast to the interpretation below Proposition 1, we now do *not* necessarily have $(\Gamma_\sigma)'_+(0) = 0$.

Remark 2 It is challenging to use the proposition, when the constants $\sigma, \rho > 0$ therein are not known. However, if the condition

$$\operatorname{rge} D^*S(\bar{u}|\bar{y}) = \{0\} \quad \text{(full coderivative singularity)} \tag{12}$$

holds, then [35, Theorem 3.3] tells us that S is Lipschitz-like around $(\bar{u}, \bar{y})$ at any rate $c > 0$. Thus, it is Lipschitz-like at any rate $c > 0$, relative to any direction $w \neq 0$. $\square$

By construction of the coderivative, it follows by [35, Proposition 1.2] that the condition in (12) requires $(\bar{u}, \bar{y}) \in \operatorname{int}(\operatorname{gph} S)$, hence, $0 \in DS(\bar{u}|\bar{y})(w)$ holds for any direction $w \neq 0$. Therefore, the conclusion of Proposition 1 can be extracted under (12) from Proposition 2. At the same time, the assumptions of the former can be fulfilled even in the absence of the Lipschitz-like assumption in the latter proposition, as can be illustrated at $\bar{u} := 0$ for $w := 1$ by any (single-valued) continuous but non-Lipschitzian function *zig-zagging* appropriately between $u \mapsto u^2$ and $u \mapsto 2u^2$.

In our last specialization of Theorem 3, we want to invoke the *second-order Studniarski derivative* of a mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at $(\bar{u}, \bar{y}) \in \operatorname{gph} S$ relative to a direction $w \neq 0$, which is given e.g. in [40, Definition 4] and elsewhere through

$$\mathscr{D}^2 S(\bar{u}|\bar{y})(w) := \left\{ x \in \mathbb{R}^m \mid \exists t_k \searrow 0, \exists (w^k, x^k) \to (w, x) : \bar{y} + t_k^2 x^k \in S(\bar{u} + t_k w^k) \forall k \right\}.$$

It can be seen that $\mathscr{D}^2 S(\bar{u}|\bar{y})(w) \neq \varnothing$ requires $0 \in DS(\bar{u}|\bar{y})(w)$. So we can expect that the results yet to come will be more specific than Proposition 1. Next we introduce a counterpart to Lemma 1, which will help us to specialize our general covering Theorem 3 afterwards.

Lemma 3 *Let $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, a point $(\bar{u}, \bar{y}) \in \operatorname{gph} S$, and a direction $w \neq 0$ be given. Then for any $\sigma, \rho, \gamma > 0$, and any $x \in \mathscr{D}^2 S(\bar{u}|\bar{y})(w)$, there are $\delta > 0$ and $t_k \searrow 0$, so that for all $t \in \{t_k\}_{k \in \mathbb{N}}$,*

$$\bar{y} + t^2(x + \delta\mathbb{B}) \subset S(\bar{u} + t(w + \sigma\mathbb{B})) \cap (\bar{y} + \rho t\mathbb{B}) + \gamma t^2\mathbb{B}. \tag{13}$$

If, moreover, $x \in \mathscr{D}^2 S(\bar{u}|\bar{y})(w)$ satisfies

$$\forall t_k \searrow 0, \exists (w^k, x^k) \to (w, x) : \ \bar{y} + t_k^2 x^k \in S(\bar{u} + t_k w^k) \forall k \in \mathbb{N}, \tag{14}$$

then for any $\sigma, \rho, \gamma > 0$, there exist $\delta, \tau > 0$, so that (13) holds for all $t \in [0, \tau]$.

Proof Fix $\sigma,\rho,\gamma > 0$, and pick $x \in \mathscr{D}^2 S(\bar{u}|\bar{y})(w)$ arbitrarily. Then there are sequences $t_k \searrow 0$, $(w^k,x^k) \to (w,x)$, so that $\bar{y}+t_k^2 x^k \in S(\bar{u}+t_k w^k)$ holds for all $k \in \mathbb{N}$. Without loss of generality, we can assume $\|w^k - w\| \le \sigma$ and $t_k\|x^k\| \le \rho$ for all k. Furthermore, we can assume that there is $\varepsilon \in (0,\gamma)$, satisfying $\|x^k - x\| \le \varepsilon$ for all k. Put $\delta := \gamma - \varepsilon$, and pick $\eta \in \delta\mathbb{B}$ arbitrarily. Then we get $\|x - x^k + \eta\| \le \|x - x^k\| + \|\eta\| \le \varepsilon + \delta \le \gamma$ for all k. With this, the definition of x, and the aforementioned properties of σ,ρ, we observe $\bar{y}+t_k^2(x+\eta) = \bar{y}+t_k^2 x^k + t_k^2(x - x^k + \eta) \in S(\bar{u}+t_k w^k) \cap (\bar{y}+\rho t_k\mathbb{B}) + \gamma t_k^2\mathbb{B}$ for all k, so the first claim of the lemma is true.

It remains to establish the second claim under (14). Having $\sigma,\rho,\gamma > 0$ and $x \in \mathscr{D}^2 S(\bar{u}|\bar{y})(w)$ fixed (with the latter satisfying (14)), then (14) necessitates $\bar{y}+t^2 x \in S(\bar{u}+t(w+\sigma\mathbb{B})) \cap (\bar{y}+\rho t\mathbb{B}) + o(t^2)\mathbb{B}$ as $t \searrow 0$. In other words, for any $\varepsilon > 0$, we can find $\tau = \tau(\varepsilon) > 0$, such that

$$\bar{y}+t^2 x \in S(\bar{u}+t(w+\sigma\mathbb{B})) \cap (\bar{y}+\rho t\mathbb{B}) + \varepsilon t^2\mathbb{B} \qquad \forall t \in [0,\tau]. \tag{15}$$

So let us fix $\varepsilon \in (0,\gamma)$, and find $\tau = \tau(\varepsilon) > 0$ satisfying (15). Put $\delta := \gamma - \varepsilon$, and pick $\eta \in \delta\mathbb{B}$ arbitrarily. Then $\eta + \varepsilon\mathbb{B} \subset \gamma\mathbb{B}$ follows by construction, and (15) entails

$$\begin{aligned}\bar{y}+t^2(x+\eta) &\in S(\bar{u}+t(w+\sigma\mathbb{B})) \cap (\bar{y}+\rho t\mathbb{B}) + t^2(\eta+\varepsilon\mathbb{B})\\ &\subset S(\bar{u}+t(w+\sigma\mathbb{B})) \cap (\bar{y}+\rho t\mathbb{B}) + t^2\gamma\mathbb{B} \qquad \forall t \in [0,\tau],\end{aligned}$$

as we wanted. The proof is complete. □

Remark 3 Just as in Remark 1 above, we can say that (14) can be regarded as relaxation of the *second-order proto-differentiability* condition in [40, Remark 1] of the mapping S at $(\bar{u},\bar{y})$ relative to $w \neq 0$,

$$\forall x \in \mathscr{D}^2 S(\bar{u}|\bar{y})(w) \forall t_k \searrow 0, \exists (w^k,x^k) \to (w,x):\ \bar{y}+t_k^2 x^k \in S(\bar{u}+t_k w^k) \forall k. \tag{16}$$

This concept has not received much attention by now. Our observations in Remark 4 and Lemma 8 below will yield novel sufficient conditions for it with respect to special choices of S. □

Proposition 3 *Let $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, a point $(\bar{u},\bar{y}) \in \operatorname{gph} S$ around which $\operatorname{gph} S$ is closed, a direction $w \neq 0$, and $x \in \mathscr{D}^2 S(\bar{u}|\bar{y})(w)$ be given. If S is metrically 2-regular at $(\bar{u},\bar{y})$ relative to w, then for any $\alpha > 0$, there are $\sigma,\rho,\gamma,\delta > 0$ and a sequence $t_k \searrow 0$, so that for all $t \in \{t_k\}_{k\in\mathbb{N}}$,*

$$\bar{y}+t^2(x+\delta\mathbb{B}) \subset S(\bar{u}+t(w+\sigma\mathbb{B})) \cap (\bar{y}+\rho t\mathbb{B}) + \gamma t^2\mathbb{B} \subset S(K_{t,\alpha}(\bar{u};w)). \tag{17}$$

If, moreover, condition in (14) is satisfied, then for any $\alpha > 0$, there are $\sigma,\rho,\gamma,\delta,\tau > 0$, so that (17) holds for all $t \in [0,\tau]$.

Proof As seen earlier, the condition $x \in \mathscr{D}^2 S(\bar{u}|\bar{y})(w)$ yields $0 \in DS(\bar{u}|\bar{y})(w)$. Thus, under the imposed assumptions, Theorem 1 tells us once again that S is equivalently Gfrerer regular at $(\bar{u},\bar{y})$ relative to $(w,0)$. With this, and Lemma 3 in mind, the claims in this proposition are immediate consequences of Theorem 3. □

Corollary 1 *Let $S:\mathbb{R}^n \rightrightarrows \mathbb{R}^m$, a point $(\bar{u},\bar{y}) \in \operatorname{gph} S$ around which $\operatorname{gph} S$ is closed, a direction $w \neq 0$, and $x \in \mathscr{D}^2 S(\bar{u}|\bar{y})(w)$ be given, so that (14) holds. If S is metrically 2-regular at $(\bar{u},\bar{y})$ relative to w, then for any $\alpha > 0$, there are $\varepsilon,\beta,c,\tau > 0$, satisfying*

$$\operatorname{dist}\left[\bar{u}, S^{-1}(z) \cap K_{\tau,\alpha}(\bar{u};w)\right] \leq c \cdot \sqrt{\|z-\bar{y}\|} \qquad \forall z \in K_{\varepsilon,\beta}(\bar{y};x). \tag{18}$$

Proof Let $\sigma,\rho,\gamma,\delta,\tau > 0$ denote the constants in Proposition 3 satisfying (17). We distinct between two cases: First we assume $x = 0$. Put $\varepsilon := \tau^2\delta$ and $c := 1/\sqrt{\delta}$, and fix any $\beta > 0$. Pick $z \in K_{\varepsilon,\beta}(\bar{y};0) = \bar{y} + \varepsilon\mathbb{B}$, and find $t \in [0,\tau]$ together with a unit vector η, so that $z = \bar{y} + t^2\delta\eta$ holds true. The desired estimate in (18) is trivial, when $z = \bar{y}$ is valid, so we assume $z \neq \bar{y}$ from now on, which implies $t = \sqrt{\|z-\bar{y}\|/\delta} > 0$ by construction. The inclusions in (17) yield existence of $u(z) \in K_{t,\alpha}(\bar{u};w) \subset K_{\tau,\alpha}(\bar{u};w)$ with $z \in S(u(z))$. Combining these facts entails

$$\operatorname{dist}\left[\bar{u}, S^{-1}(z) \cap K_{\tau,\alpha}(\bar{u};w)\right] \leq \|\bar{u} - u(z)\| \leq t \leq c\sqrt{\|z-\bar{y}\|}, \tag{19}$$

as we wanted. Now we consider the case $x \neq 0$, and assume with no loss in generality that $0 \notin x + \delta\mathbb{B}$. Put $c := 1/\sqrt{\operatorname{dist}[0, x+\delta\mathbb{B}]} > 0$. There clearly are $\varepsilon,\beta > 0$ with $K_{\varepsilon,\beta}(\bar{u};w) \subset \cup_{t\in[0,\tau]}(\bar{y} + t^2(x+\delta\mathbb{B}))$. Having this in mind, we pick $z \in K_{\varepsilon,\beta}(\bar{y};x)$ arbitrarily, and find $\eta \in \delta\mathbb{B}$ and $t \in [0,\tau]$, such that $z = \bar{y} + t^2(x+\eta)$. The desired estimate in (18) is clearly fulfilled for $z = \bar{y}$, so we can assume $z \neq \bar{y}$ from now on. This gives us $t \neq 0$. More precisely, we have $t = \sqrt{\|z-\bar{y}\|/\|x+\eta\|} \neq 0$, which yields not only $t \leq c \cdot \sqrt{\|z-\bar{y}\|}$, but also, through (17), the existence of $u(z) \in K_{t,\alpha}(\bar{u};w) \subset K_{\tau,\alpha}(\bar{u};w)$ with $z \in S(u(z))$. Combining these facts brings us to (19), which completes the proof. □

The corollary particularly tells us for $0 = x \in \mathscr{D}^2 S(\bar{u}|\bar{y})(w)$ that the whole ball around $\bar{y}$ with radius $\varepsilon > 0$ is covered. In the case when $x = 0$ is not admissible, it may be worth further investigating, e.g. as in [14] (or also Section 6), whether the covered region can be enlarged by: taking the union over all appropriate x, and using directional robustness of metric 2-regularity and varying the direction w. Considerations of that kind were also exploited in [28, Theorem 6.20] on covering in the absence of metric regularity of a mapping defined as the sum of a smooth single-valued and a set-valued mapping, but without approximating the set-valued component.

5 Covering on a Closed Set for a Single-Valued Mapping

We want to utilize our observations to guarantee stability under metric 2-regularity of a solution to the constrained equation

$$F(u) = 0, \qquad u \in \Omega, \tag{20}$$

when it underlies small perturbations, where $F:\mathbb{R}^n \to \mathbb{R}^m$ and $\Omega \subset \mathbb{R}^n$. More specifically, we are interested in stability of a solution $\bar{u}$ to (20) in the absence of the metric regularity condition,

$$\exists c > 0: \qquad \operatorname{dist}[u, F^{-1}(y) \cap \Omega] \leq c\|y - F(u)\| \qquad \forall (u,y) \in \Omega \times \mathbb{R}^m \text{ near } (\bar{u},0).$$

If F is smooth and Ω is closed, then the latter is characterized by the standard first-order condition (cf. [15, Proposition 3.7] and subsequent discussions),

$$F'(\bar{u})^\top \lambda \in N_\Omega(\bar{u}) \quad \Longrightarrow \quad \lambda = 0, \tag{21}$$

and, as has also been mentioned in the latter reference and elsewhere: If Ω is convex, then (21) reduces to Robinson's regularity condition from [37], see also e.g. [9],

$$0 \in \operatorname{int}\left(F'(\bar{u})(\Omega - \bar{u})\right). \tag{22}$$

The papers [2,6] target stability issues for solutions of (20) with convex Ω under the constrained *2-regularity* condition (cf. Section 6 below) in the absence of Robinson's regularity, hence, in the absence of (21). The goal of this section is to extend their results to the more general nonconvex case under metric 2-regularity. Before doing so, we need to recall a few things.

In virtue of [39, Section 8H], we call a mapping $\mathscr{A} : \mathbb{R}^n \to \mathbb{R}^{m\times n}$ *semidifferentiable* at a point $\bar{u} \in \mathbb{R}^n$ relative to $w \neq 0$, if the following limit exists in $\mathbb{R}^{m\times n}$:

$$\mathscr{A}'(\bar{u}; w) := \lim_{t \searrow 0, v \to w} \frac{\mathscr{A}(\bar{u} + tv) - \mathscr{A}(\bar{u})}{t}.$$

When considering sufficient conditions for metric 2-regularity, we will frequently assume that F is continuously differentiable with its derivative F' semidifferentiable relative to some direction w. In this case, we will write $F''(\bar{u}; w)$ instead of $(F')'(\bar{u}; w)$. Observe that the latter smoothness is particularly guaranteed, when F is twice differentiable, but also beyond, cf. [29]. In what follows, we will also work with the indicator mapping [35] (with range space $\mathbb{R}^m$) for the set Ω,

$$\Delta_\Omega(u) := \begin{cases} \{0\} & \text{if } u \in \Omega, \\ \varnothing & \text{if } u \notin \Omega. \end{cases}$$

The mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ of interest in this section will therefore be

$$S(u) := F(u) + \Delta_\Omega(u),$$

for which we can combine sum-rules for graphical derivatives and coderivatives in [39, Exercise 10.43] with e.g. [31, Lemma 2] (on generalized derivatives of the indicator) in order to observe for any $(u, w, \lambda) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^m$ that

$$DS(u|F(u))(w) = D(F + \Delta_\Omega)(u|F(u))(w) = F'(u)w + \Delta_{T_\Omega(u)}(w), \tag{23}$$

$$D^*S(u|F(u))(\lambda) = D^*(F + \Delta_\Omega)(u|F(u))(\lambda) = F'(u)^\top \lambda + N_\Omega(u), \tag{24}$$

provided F is sufficiently smooth and Ω is closed. With this, we particularly notice that the condition in (3) with $(\bar{u}, \bar{y} = F(\bar{u}))$ reduces to $w \in T_\Omega(\bar{u})(w)$.

The subsequent result is a specialization of Theorem 2:

Theorem 4 *Given a continuously differentiable mapping $F : \mathbb{R}^n \to \mathbb{R}^m$, a nonempty closed set $\Omega \subset \mathbb{R}^n$, and a solution $\bar{u} \in \Omega$ to the constrained equation (20), consider a nonzero direction $w \in T_\Omega(\bar{u})$, and define the function*

$$\varphi(u) := \begin{cases} \min\limits_{\|\lambda\|=1} \operatorname{dist}[0, F'(u)^\top \lambda + N_\Omega(u)] & \text{if } u \in \Omega, \\ +\infty & \text{if } u \notin \Omega. \end{cases}$$

The following are equivalent:

a) $S = F + \Delta_\Omega$ is metrically 2-regular at $(\bar{u}, 0)$ relative to w.
b) The condition $\varphi(\bar{u}) = 0$ necessitates $d\varphi(\bar{u})(w) > 0$.

Proof Imposed assumptions allow us to apply Theorem 2 with the mapping $S = F + \Delta_\Omega$. So all we have to do is to compare statements b) of the two theorems. In particular, and with the formula in (24) at hand, we know that the LSV function in Section 2 now takes the form

$$\operatorname{Reg}(u, y; F + \Delta_\Omega) = \begin{cases} \inf\limits_{\|\lambda\|=1} \operatorname{dist}[0, F'(u)^\top \lambda + N_\Omega(u)] & \text{if } u \in \Omega \text{ and } y = F(u), \\ +\infty & \text{if } u \notin \Omega \text{ or } y \neq F(u). \end{cases}$$

Hence, we can immediately observe that

$$\operatorname{Reg}(u, y; F + \Delta_\Omega) = \begin{cases} \varphi(u) & \text{if } y = F(u), \\ +\infty & \text{if } y \neq F(u). \end{cases} \tag{25}$$

This readily implies with $\bar{y} := 0$ that $\operatorname{Reg}(\bar{u}, \bar{y}; F + \Delta_\Omega) = 0$ in Theorem 2 b) reduces to $\varphi(\bar{u}) = 0$ in part b) of this theorem. Assuming $\varphi(\bar{u}) = 0$, then we can further argue, with (23), definition of the subderivative, and (25) in mind, that

$$\begin{aligned} \inf_{\eta \in D(F+\Delta_\Omega)(\bar{u}|\bar{u})(w)} d\operatorname{Reg}(\bar{u}, \bar{y}; F + \Delta_\Omega)(w, \eta) &= d\operatorname{Reg}(\bar{u}, \bar{y}; F + \Delta_\Omega)\big(w, F'(\bar{u})w\big) \\ &= \liminf_{\substack{t \searrow 0 \\ v \to w \\ \xi \to F'(\bar{u})w}} \frac{\operatorname{Reg}(\bar{u} + tv, \bar{y} + t\xi; F + \Delta_\Omega)}{t} \\ &= \liminf_{\substack{t \searrow 0 \\ v \to w}} \frac{\varphi(\bar{u} + tv)}{t} = d\varphi(\bar{u})(w). \end{aligned}$$

This confirms that statements b) of the two theorems are same. □

Computing $d\varphi(\bar{u})(w)$ in the theorem can be a challenging task. For this reason, we recall sufficient conditions for the metric 2-regularity of interest, while emphasizing that sharper sufficient conditions for this can possibly be derived from Theorem 4.

Proposition 4 *Given a continuously differentiable $F : \mathbb{R}^n \to \mathbb{R}^m$, a nonempty closed set $\Omega \subset \mathbb{R}^n$, and a solution $\bar{u} \in \Omega$ to the constrained equation (20), assume that the derivative F' is semidifferentiable at $\bar{u}$ relative to a nonzero direction $w \in T_\Omega(\bar{u})$. Each of the following conditions (a)–(b) is sufficient for the metric 2-regularity of $S = F + \Delta_\Omega$ at $(\bar{u}, 0)$ relative to w:*

(a) The set Ω is a finite union of polyhedral sets, and it holds that

$$\left[F'(\bar{u})^\top\lambda \in N_\Omega(\bar{u}),\ F''(\bar{u};w)^\top\lambda \in \mathrm{im}F'(\bar{u})^\top + N_\Omega(\bar{u})\right] \implies \lambda = 0. \tag{26}$$

(b) The following holds in addition to (26):

$$\mathrm{im}\left(F'(\bar{u})^\top\right)\bigcap N_\Omega(\bar{u}) = \{0\}. \tag{27}$$

Proof The claim under the assumptions in (a) follows by [31, Corollary 2], while under the assumptions in (b), it is a consequence of [31, Theorem 12]. □

The condition in (27) also plays a role in [15, Lemma 3.6] on sufficient conditions for Lipschitzian error bounds for the constrained equation (20), and elsewhere. If $N_\Omega(\bar{u})$ is convex (as is the case when Ω itself is convex), the condition (27) means that the two cones therein, $\mathrm{im}F'(\bar{u})^\top$ and $N_\Omega(\bar{u})$, can be separated by a hyperplane.

Now we specialize Theorem 3:

Theorem 5 *Given a mapping $F : \mathbb{R}^n \to \mathbb{R}^m$, a nonempty closed set $\Omega \subset \mathbb{R}^n$, a solution $\bar{u} \in \Omega$ to the constrained equation (20), and a direction $w \neq 0$, assume that* gph$F$ *is closed around $(\bar{u}, 0)$. If $S = F + \Delta_\Omega$ is Gfrerer regular at $(\bar{u}, 0)$ relative to $(w, 0)$, then for any $\alpha > 0$, there are $\sigma, \rho, \gamma, \tau > 0$, so that*

$$(F + \Delta_\Omega)(\bar{u} + t(w + \sigma\mathbb{B})) \cap (\rho t\mathbb{B}) + t^2\gamma\mathbb{B} \subset F(K_{t,\alpha}(\bar{u};w) \cap \Omega) \quad \forall t \in [0, \tau]. \tag{28}$$

Proof We have $\mathrm{gph}S = \mathrm{gph}F \cap (\Omega \times \{0\})$, so gph$S$ is closed near $(\bar{u}, 0)$ under the stated assumptions. The claim thus follows by an application of Theorem 3 with $S = F + \Delta_\Omega$, noticing for any $t \geq 0$ that $S(K_{t,\alpha}(\bar{u};w)) = F(K_{t,\alpha}(\bar{u};w) \cap \Omega)$. □

The assumptions on F in the theorem do not imply its continuity, but merely its *localized* outer semicontinuity, see also discussions below Theorem 3. In particular, in conjunction with [39, Corollary 5.20], we can say that the assumptions of the theorem allow F to be unbounded near $\bar{u}$:

Example 2 Consider the example with $\Omega = \mathbb{R}$, and

$$F(u) := \begin{cases} \frac{1}{u} & \text{if } u < 0, \\ u^2 & \text{if } u \geq 0. \end{cases}$$

This F is discontinuous at $\bar{u} := 0$, and unbounded in any neighborhood around it. However, gph$F$ is closed around $(\bar{u}, 0)$, and we find for $w := 1$ that F is Gfrerer regular at $(\bar{u}, 0)$ relative to $(w, 0)$, i.e., Theorem 5 is applicable. Furthermore, F is semi-differentiable at $\bar{u}$ relative to w. More precisely, we have $DF(\bar{u}|0)(w) = \{F'(\bar{u};w)\} = \{0\}$, so F is even metrically 2-regular at $(\bar{u}, 0)$ relative to w, as follows by Theorem 1. At the same time, for $\omega := -1$, we have $DF(\bar{u}|0)(\omega) = \varnothing$, implying that F can not be metrically 2-regular at $\bar{u}$ relative to ω. In the same breath, and in contrast to the above, emptyness of $DF(\bar{u}|0)(\omega)$ implies $(\omega, 0) \notin T_{\mathrm{gph}F}(\bar{u}, 0)$, so discussions below [19, Definition 1] yield Gfrerer regularity of F at $(\bar{u}, 0)$ relative to $(\omega, 0)$. □

In the context of the above discussion, we emphasize that differentiability is *not* used for the covering property per se, but for the formulation of tractable sufficient conditions for Gfrerer regularity, and for the description of the set being covered. The latter will be the subject of interest in the sequel.

Yet again, nonemptyness of the left-hand side of (28) does not follow from the imposed Gfrerer regularity assumption alone. To achieve this, we will impose additional assumptions, which, through Theorem 1, lead to an equivalence between Gfrerer regularity and the corresponding metric 2-regularity. First, we present a specialization of Proposition 1. For this purpose, we follow [39, Definition 6.2] and say for $\bar{u} \in \Omega$ that $w \in T_\Omega(\bar{u})$ is *derivable*, if the subsequent condition holds:

$$\limsup_{t \searrow 0} \frac{\mathrm{dist}[\bar{u}+tw, \Omega]}{t} = 0. \tag{29}$$

There is no harm in replacing the lim sup by a lim. We also mention that the set of derivable tangents forms the *inner tangent cone* in [9, Definition 2.54]. Any tangent to a closed convex set is derivable, as follows by [39, Theorem 6.9] (or [9, Proposition 2.55]). Furthermore, any tangent to a (possibly nonconvex) polyhedral set is derivable, as can be extracted from [23, Proposition 8.24].

Lemma 4 *Given a mapping $F : \mathbb{R}^n \to \mathbb{R}^m$, a nonempty closed set $\Omega \subset \mathbb{R}^n$, a solution $\bar{u} \in \Omega$ to the constrained equation (20), and a direction $w \neq 0$, assume that F is differentiable at $\bar{u}$, and*

$$w \in \ker F'(\bar{u}) \bigcap T_\Omega(\bar{u}), \tag{30}$$

which is equivalent to the condition $0 \in DS(\bar{u}|0)(w)$ for $S = F + \Delta_\Omega$. If w is derivable, then the condition in (9) is satisfied with $S = F + \Delta_\Omega$.

Proof Equivalence between (30) and $0 \in DS(\bar{u}|0)(w)$ is a direct consequence of (23). If $w \in \ker F'(\bar{u}) \bigcap T_\Omega(\bar{u})$ is derivable, then we can find for any $t_k \searrow 0$, a sequence $w^k \to w$, such that $\bar{u}+t_k w^k \in \Omega$ holds for all $k \in \mathbb{N}$. Combining this with differentiability of F at $\bar{u}$, then we get $(F+\Delta_\Omega)(\bar{u}+t_k w^k) = F(\bar{u}+t_k w^k) = F(\bar{u}) + t_k F'(\bar{u}) w^k + \xi^k$ for all $k \in \mathbb{N}$, where the sequence $\{\xi^k\}$ satisfies $\|\xi^k\| = o(t_k)$ as k tends to $+\infty$. This readily brings us to the desired condition in (9) with $S = F + \Delta_\Omega$ by taking the sequence $\{x^k\}$ defined through $x^k := t_k^{-1} \xi^k + F'(\bar{u}) w^k$ for $k \in \mathbb{N}$. □

Proposition 5 *Given a mapping $F : \mathbb{R}^n \to \mathbb{R}^m$, a nonempty closed set $\Omega \subset \mathbb{R}^n$, a solution $\bar{u} \in \Omega$ to the constrained equation (20), and $w \neq 0$, assume that F is differentiable at $\bar{u}$, $\mathrm{gph} F$ is closed around $(\bar{u}, 0)$, and (30) is in force. If $S = F + \Delta_\Omega$ is metrically 2-regular at $(\bar{u}, 0)$ relative to w, then for any $\alpha > 0$, there are $\sigma, \rho, \gamma > 0$ and a sequence $t_k \searrow 0$, so that the inclusion in (28) holds with its left-hand side nonempty for all $t \in \{t_k\}_{k \in \mathbb{N}}$. If, moreover, the direction w is derivable, then for any $\alpha > 0$, there are $\sigma, \rho, \gamma, \tau > 0$, so that (28) holds with its left-hand side nonempty.*

Proof Imposed assumption guarantee $\mathrm{gph} S = \mathrm{gph}(F + \Delta_\Omega)$ is closed, so the claim follows by combining Lemma 4 and Proposition 1 applied with $S = F + \Delta_\Omega$. □

Let us point out that a counterpart to Lemma 4 and Proposition 5 can actually be established, replacing the differentiability assumption by semidifferentiability.

We do not formulate a counterpart of Proposition 2, as the needed Lipschitz-like property of $S = F + \Delta_\Omega$ at sufficiently small rate can certainly only be guaranteed under rather restrictive assumptions. For instance, with regard to (24) (assuming F is smooth), the natural sufficient condition in (12) for the Lipschitz-like property of $S = F + \Delta_\Omega$ at small rate reduces to the combination of $F'(\bar{u}) = 0$ and $\bar{u} \in \operatorname{int}\Omega$.

Next, we want to present a specialization of Proposition 3. For this purpose, we recall formulas for the second-order Studniarski derivative of $S = F + \Delta_\Omega$, where we will be using the wording in [29,30] that F has a *semi-quadratic expansion* at $\bar{u}$ relative to $w \neq 0$, if F is differentiable at $\bar{u}$, and the following limit exists:

$$\mathscr{E}F(\bar{u};w) := \lim_{t\searrow 0, v\to w} \frac{F(\bar{u}+tv) - F(\bar{u}) - tF'(\bar{u})v}{\frac{1}{2}t^2}.$$

If F is differentiable near $\bar{u}$ with its derivative F' semidifferentiable at $\bar{u}$ relative to w, then F has a semi-quadratic expansion there, satisfying $\mathscr{E}F(\bar{u};w) = F''(\bar{u};w)w$, cf. [29, Lemma 2]. We moreover recall that the *second-order tangent set* to Ω at $\bar{u} \in \Omega$ relative to $w \in T_\Omega(\bar{u})$ is given, cf. e.g. [39, Definition 13.11], as

$$T^2_\Omega(\bar{u}|w) := \left\{ \eta \in \mathbb{R}^n \,\middle|\, \exists t_k \searrow 0, \exists \eta^k \to \eta :\ \bar{u} + t_k w + \frac{1}{2}t_k^2\eta^k \in \Omega\ \forall k \in \mathbb{N} \right\}.$$

The aforementioned definition also calls Ω *parabolically derivable* at $\bar{u} \in \Omega$ relative to $w \in T_\Omega(\bar{u})$, if for any $\eta \in T^2_\Omega(\bar{u}|w)$,

$$\limsup_{t\searrow 0} \frac{\operatorname{dist}\left[\bar{u} + tw + \frac{1}{2}t^2\eta, \Omega\right]}{\frac{1}{2}t^2} = 0 \tag{31}$$

holds, again with the understanding that lim sup therein can be replaced by lim. Equivalently, we can say that $w \in T_\Omega(\bar{u})$ is parabolically derivable, if the *inner second-order tangent set* in [9, Definition 3.28] coincides with the second-order tangent set defined above. The latter agrees with the concept in [9, Definition 3.32] of *second-order directional differentiability* of Ω at $\bar{u}$ for w, provided *every* tangent to Ω at $\bar{u}$ is derivable. While generic convex sets are *not* necessarily second-order directionally differentiable (see [9, Example 3.35]), polyhedral (possibly nonconvex) sets are, as follows e.g. by [39, Proposition 13.12] in conjunction with [30, Lemma 5].

Lemma 5 *Given a mapping $F : \mathbb{R}^n \to \mathbb{R}^m$, a nonempty closed set $\Omega \subset \mathbb{R}^n$, a solution $\bar{u} \in \Omega$ to the constrained equation (20), and a direction $w \neq 0$, assume that F has a semi-quadratic expansion at $\bar{u}$ relative to w. If (30) is valid, then*

$$\frac{1}{2}\left(\mathscr{E}F(\bar{u};w) + \operatorname{cl}\left(F'(\bar{u})T^2_\Omega(\bar{u}|w)\right)\right) \subset \mathscr{D}^2(F + \Delta_\Omega)(\bar{u}|0)(w), \tag{32}$$

and this inclusion holds with equality under any of the following conditions:

(a) The set Ω is a finite union of polyhedral sets.
(b) It holds that $F'(\bar{u}) = 0$.

(c) The set Ω is convex, and it holds that $0 \in T^2_\Omega(\bar{u}|w)$.

If, in addition to (30), Ω is parabolically derivable at $\bar{u}$ relative to w, then (14) is satisfied with $S = F + \Delta_\Omega$ for any $x \in \frac{1}{2}\left(\mathscr{E}F(\bar{u};w) + F'(\bar{u})T^2_\Omega(\bar{u}|w)\right)$.

Proof The lower estimate for $\mathscr{D}^2(F+\Delta_\Omega)(\bar{u}|0)(w)$ is given in [30, Theorem 1]. That this inclusion holds with equality under the assumptions in (b)–(c) is a consequence of [30, Corollary 1], whereas Corollary 2 of the aforementioned paper yields equality in the case that Ω is finite union of polyhedral sets.

The remaining claim under parabolic derivability follows by direct computations: Pick $\eta \in T^2_\Omega(\bar{u}|w)$ arbitrarily, and consider any $t_k \searrow 0$. Then (31) yields $\eta^k \to \eta$, satisfying $\bar{u} + t_k w + \frac{1}{2}t_k^2\eta^k \in \Omega$ for each $k \in \mathbb{N}$. Combining this with semi-quadratic expansion property, $F(\bar{u}) = 0$, and (30), then we get

$$\begin{aligned}(F+\Delta_\Omega)\left(\bar{u}+t_k w+\tfrac{1}{2}t_k^2\eta^k\right) &= F\left(\bar{u}+t_k w+\tfrac{1}{2}t_k^2\eta^k\right)\\ &= F(\bar{u}) + t_k F'(\bar{u})\left(w+\frac{t_k^2}{2}\eta^k\right) + \frac{t_k^2}{2}\mathscr{E}F(\bar{u};w)+\xi^k\\ &= \tfrac{1}{2}t_k^2\left(2t_k^{-2}\xi^k + F'(\bar{u})\eta^k + \mathscr{E}F(\bar{u};w)\right)\end{aligned}$$

for all $k \in \mathbb{N}$, where $\{\xi^k\}_{k\in\mathbb{N}}$ satisfies $\|\xi^k\| = o(t_k^2)$ as k tends to $+\infty$. The latter brings us to the desired condition in (14) for $S = F+\Delta_\Omega$ by taking the sequences defined through $w^k := w + \frac{1}{2}t_k\eta^k$ and $x^k := 2t_k^{-2}\xi^k + F'(\bar{u})\eta^k + \mathscr{E}F(\bar{u};w)$. □

Remark 4 Conditions (a)–(b) in the lemma allow to remove the closure in (32). Hence, under the assumptions of the lemma, we can say that (30), parabolic derivability of Ω relative to w, and any of the conditions (a)–(b) held together entail second-order proto-differentiability of $S = F + \Delta_\Omega$ at $(\bar{u}, 0)$ relative to w in the sense of (16) in Remark 3. □

Proposition 6 *Given a mapping $F : \mathbb{R}^n \to \mathbb{R}^m$, a nonempty closed set $\Omega \subset \mathbb{R}^n$, a solution $\bar{u} \in \Omega$ to the constrained equation (20), a direction $w \neq 0$, and $x \in \mathscr{D}^2(F+\Delta_\Omega)(\bar{u}|0)(w)$, assume that* gph$F$ *is closed around $(\bar{u},0)$, F has a semi-quadratic expansion at $\bar{u}$ relative to w, and (30) is in force. If $S = F+\Delta_\Omega$ is metrically 2-regular at $(\bar{u},0)$ relative to w, then for any $\alpha > 0$, there are $\sigma,\rho,\gamma,\delta > 0$ and a sequence $t_k \searrow 0$, so that for all $t \in \{t_k\}_{k\in\mathbb{N}}$,*

$$t^2(x+\delta\mathbb{B}) \subset (F+\Delta_\Omega)(\bar{u}+t(w+\sigma\mathbb{B}))\cap(\rho t\mathbb{B}) + \gamma t^2\mathbb{B} \subset F(K_{t,\alpha}(\bar{u};w)\cap\Omega). \quad (33)$$

If, moreover, the condition (14) is satisfied with $S = F+\Delta_\Omega$ (as is the case, when Ω is parabolically derivable at $\bar{u}$ relative to w, and $x \in \frac{1}{2}\left(\mathscr{E}F(\bar{u};w) + F'(\bar{u})T^2_\Omega(\bar{u}|w)\right)$), then for any $\alpha > 0$, there are $\sigma,\rho,\gamma,\delta,\tau > 0$, so that (33) holds for all $t \in [0,\tau]$.

Proof Combine Lemma 5 and Proposition 3 applied with $S = F+\Delta_\Omega$. □

The conclusion of Proposition 6 improves the outcome of [28, Example 6.24 (c)], which is tailored to *radial* unit directions only, and uses *feasible parabola* instead of second-order tangents to describe the set being covered by F.

Corollary 2 *Given a mapping $F:\mathbb{R}^n\to\mathbb{R}^m$, a nonempty closed set $\Omega\subset\mathbb{R}^n$, a solution $\bar u\in\Omega$ to the constrained equation* (20)*, a direction $w\neq 0$, and $x\in\mathscr{D}^2(F+\Delta_\Omega)(\bar u|0)(w)$, assume that* gph$F$ *is closed around $(\bar u,0)$, F has a semi-quadratic expansion at $\bar u$ relative to w, and* (30) *is in force. If $S=F+\Delta_\Omega$ is metrically 2-regular at $(\bar u,0)$ relative to w, and* (14) *holds with this S (as is the case, when Ω is parabolically derivable at $\bar u$ relative to w, and $x\in\frac{1}{2}\left(\mathscr{E}F(\bar u;w)+F'(\bar u)T^2_\Omega(\bar u|w)\right)$), then for any $\alpha>0$, there are $\varepsilon,\beta,\tau,c>0$, satisfying*

$$\operatorname{dist}\left[\bar u,F^{-1}(z)\cap\Omega\cap K_{\tau,\alpha}(\bar u;w)\right]\le c\cdot\sqrt{\|z-\bar y\|}\qquad\forall z\in K_{\varepsilon,\beta}(0;x).\tag{34}$$

Proof Similar to the proof of Corollary 1, this can be extracted from Proposition 6. However, we take the short path here and derive the statement from Lemma 5 and Corollary 1 with $S=F+\Delta_\Omega$, noticing that $S^{-1}(z)=F^{-1}(z)\cap\Omega$. □

We emphasize that, under parabolic derivability of Ω at $\bar u$ relative to w, and $0\in\mathscr{E}F(\bar u;w)+F'(\bar u)T^2_\Omega(\bar u|w)$, the corollary ensures covering of the entire ball centered at 0 with some radius $\varepsilon>0$.

6 Discussions

The goal of this section is to compare our results with some existing covering results on closed convex sets, established in the absence of Robinson's regularity (22) under variants of the *2-regularity* condition.

The following lemma deals with the *primal form* of our sufficient condition (26) for metric 2-regularity of $F+\Delta_\Omega$. To present the lemma, we rely on [39, Corollary 6.29], and say that a closed set Ω is *Clarke regular* at $\bar u\in\Omega$, if the equality

$$N_\Omega(\bar u)=T_\Omega(\bar u)^\circ\tag{35}$$

holds true. Observe that the cited corollary also characterizes Clarke regularity of Ω at $\bar u$ through the equality $T_\Omega(\bar u)=N_\Omega(\bar u)^\circ$. Closed convex sets, for example, are everywhere Clarke regular, see [39, Theorem 6.9].

Lemma 6 *Given a continuously differentiable mapping $F:\mathbb{R}^n\to\mathbb{R}^m$, a non-empty closed set $\Omega\subset\mathbb{R}^n$, and a solution $\bar u\in\Omega$ to the constrained equation* (20)*, assume that Ω is Clarke regular at $\bar u$, and the derivative F' is semidifferentiable at $\bar u$ relative to a direction $w\in T_\Omega(\bar u)\setminus\{0\}$. The following condition is sufficient for* (26)*, and it is also necessary, if, in addition, the set* im$F'(\bar u)^\top+N_\Omega(\bar u)$ *is closed (as is true, when $N_\Omega(\bar u)$ is polyhedral, or* (27) *holds):*

$$F'(\bar u)T_\Omega(\bar u)+F''(\bar u;w)\left(\ker F'(\bar u)\bigcap T_\Omega(\bar u)\right)=\mathbb{R}^m.\tag{36}$$

Proof First we explain that im$F'(\bar u)^\top+N_\Omega(\bar u)$ is closed, provided $N_\Omega(\bar u)$ is polyhedral, or the condition (27) is satisfied: If $N_\Omega(\bar u)$ is polyhedral, then it is actually convex polyhedral by the Clarke regularity of Ω at $\bar u$. Hence, [39, Proposition 3.55] tells us that im$F'(\bar u)^\top+N_\Omega(\bar u)$ is polyhedral as well, so, in particular, it is a closed set. Now we demonstrate that the latter sum is closed under the condition (27). To see this, we can consider any convergent sequence $\zeta^k \to \zeta$ with $\zeta^k = F'(\bar{u})^\top \eta^k + v^k$ for some $\{\eta^k\} \subset \mathbb{R}^m$ and $\{v^k\} \subset N_\Omega(\bar{u})$. If $\{v^k\}$ is bounded, then we get $\zeta \in \operatorname{im} F'(\bar{u})^\top + N_\Omega(\bar{u})$. If it is unbounded, then we have $\|v^k\|^{-1}(F'(\bar{u})^\top \eta^k + v^k) \to 0$. With no loss in generality, we can assume that $\{\|v^k\|^{-1} v^k\}$ converges to some unit vector v, and because $N_\Omega(\bar{u})$ is a closed cone, this v must belong to $N_\Omega(\bar{u})$, giving a contradiction to (27).

Now we explain that (36) implies (26): To see this, assume the contrary. That said, we suppose (36), and assume:

$$\exists \overline{\lambda} \neq 0: \qquad F'(\bar{u})^\top \overline{\lambda} \in N_\Omega(\bar{u}), \qquad F''(\bar{u}; w)^\top \overline{\lambda} \in \operatorname{im} F'(\bar{u})^\top + N_\Omega(\bar{u}). \tag{37}$$

With [9, formula (2.32)], [39, Example 6.23], and Clarke regularity of Ω at $\bar{u}$, we find $\operatorname{im} F'(\bar{u})^\top + N_\Omega(\bar{u}) \subset \operatorname{cl}\left(\operatorname{im} F'(\bar{u})^\top + N_\Omega(\bar{u})\right) = (\ker F'(\bar{u}) \cap T_\Omega(\bar{u}))^\circ$. Therefore, and again with Clarke regularity of Ω in mind, we see that (37) implies $F'(\bar{u})^\top \overline{\lambda} \in T_\Omega(\bar{u})^\circ$ and $F''(\bar{u}; w)^\top \overline{\lambda} \in (\ker F'(\bar{u}) \cap T_\Omega(\bar{u}))^\circ$ for some $\overline{\lambda} \neq 0$, which, in turn, is nothing else than

$$\exists \overline{\lambda} \neq 0: \qquad \overline{\lambda}^\top \left(F'(\bar{u})v + F''(\bar{u}; w)\xi\right) \leq 0 \quad \forall v \in T_\Omega(\bar{u}), \forall \xi \in \ker F'(\bar{u}) \cap T_\Omega(\bar{u}).$$

According to [39, Exercise 6.48], we see that the latter is equivalent to the violation of (36), so we have the desired contradiction. Assuming that $\operatorname{im} F'(\bar{u})^\top + N_\Omega(\bar{u})$ is closed, we can reverse arguments above and confirm necessity of (36) for (26). □

In conjunction with Proposition 4, the above lemma says that (36) is another (possibly less sharp) sufficient condition for metric 2-regularity of $F + \Delta_\Omega$. Observe that $N_\Omega(\bar{u})$ is convex polyhedral, if Ω itself is convex polyhedral, but also beyond, e.g., when Ω is a smooth manifold, or defined by inequalities, satisfying some CQ.

From now on, we suppose that Ω is closed convex, and F is twice differentiable at a solution $\bar{u} \in \Omega$ to the constrained equation (20). It has been pointed out in [28, Remark 4.25] that (26) (i.e., our sufficient condition for metric 2-regularity of $F + \Delta_\Omega$) is in connection with the *2-regularity* of F at $\bar{u}$ *with respect to* Ω in a direction $w \in \Omega - \bar{u}$, the condition characterized in [6, Theorem 5.1] through the equation

$$F'(\bar{u})\operatorname{span}(\Omega - \bar{u}) + F''(\bar{u})[w, \ker F'(\bar{u}) \cap \operatorname{cone}(\Omega - \bar{u})] = \mathbb{R}^m. \tag{38}$$

In the following, we comment on relations with existing results in [6]. All the needed conclusions are achieved under a condition in [6, Theorem 5.2], which calls F *weakly 2-regular with respect to* Ω *at* $\bar{u}$ *in a direction* $w \in \Omega - \bar{u}$ if

$$F'(\bar{u})\operatorname{span}(\Omega - \bar{u}) + F''(\bar{u})[w, \ker F'(\bar{u}) \cap T_{\operatorname{cone}(\Omega - \bar{u})}(w)] = \mathbb{R}^m. \tag{39}$$

This condition is significantly weaker than (38) and (36), even in the case where Ω is (convex) polyhedral, and it might be of interest for future research to check whether this condition is also sufficient for metric 2-regularity of $F + \Delta_\Omega$.

We want to compare Proposition 6 and Corollary 2 with the corresponding results obtained in [6]. This comparison is not that straightforward because of several nuances discussed below.

The assertion in [6, Theorem 5.1] in a simplified (less sharp) form, and under the additional assumption $w \in \ker F'(\bar{u})$, also appearing in Proposition 6 and Corollary 2 through (30), is the same as in Proposition 6, but with the requirement

$$x \in \mathscr{D}^2(F+\Delta_\Omega)(\bar{u}|0)(w) = \frac{1}{2}\left(F''(\bar{u})[w, w] + F'(\bar{u})T^2_\Omega(\bar{u}|w)\right) \tag{40}$$

replaced by

$$x \in \frac{1}{2}F''(\bar{u})[w, w] + \operatorname{rint} F'(\bar{u})(\Omega - \bar{u}). \tag{41}$$

Furthermore, the combined simplified assertions of [6, Corollaries 6.3, 6.4] agree with the assertion of Corollary 2, but with x satisfying (41) instead of (40).

In [6, Theorem 5.1], it is also assumed that $w \in \Omega - \bar{u}$, and this assumption (and even its weaker *homogeneous* counterpart $w \in \operatorname{cone}(\Omega - \bar{u})$) is stronger than the assumption $w \in T_\Omega(\bar{u})$ appearing in Proposition 6 and Corollary 2 through (30). That said, for $w \notin T_\Omega(\bar{u}) \setminus \operatorname{cone}(\Omega - \bar{u})$, the set $T^2_\Omega(\bar{u}|w)$ can be empty.

According to [9, p. 168], a convex Ω is parabolically derivable at $\bar{u}$ in any direction $w \in T_\Omega(\bar{u})$ such that the corresponding *inner* second-order tangent set contains 0, which is obviously the case when $w \in \operatorname{cone}(\Omega - \bar{u})$, and in particular, when $w \in \Omega - \bar{u}$. Moreover, in this case

$$T^2_\Omega(\bar{u}|w) = T_{T_\Omega(\bar{u})}(w) = \operatorname{cl}(T_\Omega(\bar{u}) + \operatorname{span}\{w\}). \tag{42}$$

This set is a cone, and we have $T_\Omega(\bar{u}) \subset T^2_\Omega(\bar{u}|w)$. On the other hand, for any $x \in F'(\bar{u})\operatorname{cl}(T_\Omega(\bar{u}) + \operatorname{span}\{w\})$, there exist sequences $\{w^k\} \subset T_\Omega(\bar{u})$ and $\{t_k\} \subset \mathbb{R}$ such that the sequence $\{w^k + t_k w\}$ converges, and

$$x \in F'(\bar{u}) \lim_{k\to\infty}(w^k + t_k w) = \lim_{k\to\infty} F'(\bar{u})(w^k + t_k w) = \lim_{k\to\infty} F'(\bar{u})w^k \in \operatorname{cl} F'(\bar{u})T_\Omega(\bar{u}),$$

where we have used the assumption $w \in \ker F'(\bar{u})$. Hence, we have the following relations for the sets appearing in the right-hand sides of (40) and (41):

$$\begin{aligned}
\operatorname{rint} F'(\bar{u})(\Omega - \bar{u}) &\subset F'(\bar{u})(\Omega - \bar{u}) \\
&\subset F'(\bar{u})\operatorname{cone}(\Omega - \bar{u}) \\
&\subset F'(\bar{u})T_\Omega(\bar{u}) \\
&\subset F'(\bar{u})T^2_\Omega(\bar{u}|w) \\
&= F'(\bar{u})\operatorname{cl}(T_\Omega(\bar{u}) + \operatorname{span}\{w\}) \\
&\subset \operatorname{cl} F'(\bar{u})T_\Omega(\bar{u}) = \operatorname{cl} F'(\bar{u})\operatorname{cone}(\Omega - \bar{u}) \\
&= \operatorname{cl}\operatorname{rint} F'(\bar{u})\operatorname{cone}(\Omega - \bar{u}).
\end{aligned} \tag{43}$$

In particular, the set in the right-hand sides of (40), defining the set guaranteed to be covered, is in general larger than in (41) with the difference arising from taking the conic hull and the closure in the right-hand side of the last formula. The conic hull obviously makes this set larger, thus making the assertion of Proposition 6 and Corollary 2 formally stronger than those in [6]. The set being covered in [6] can be further enlarged by varying the norm of w, as we discuss next.

One extra assumption appearing in [6] is $\|w\| < 1$. Observe that the fulfillment of (26), (36), (38), and (39), does not depend on the norm of w. Moreover, unlike the development in this paper, the case of $w = 0$ is also allowed in [6], thus covering some *regular cases* [6, Corollary 6.1]. In our paper, the case of Robinson's regularity is covered through the first-order condition (21) that appears indirectly in Theorem 4 b) through $\varphi(\bar{u}) \neq 0$, and in a more direct form in (26).

At the same time, the set in the right-hand side of (41) depends on the norm of w. In particular, if (39) holds with some $w \in \ker F'(\bar{u}) \cap \operatorname{cone}(\Omega - \bar{u})$ (with any norm), the statements in [6, Theorem 5.1, Corollaries 6.3, 6.4] apply with w in (41) replaced by θw for *every* $\theta > 0$ such that $\theta\|w\| < 1$ and $\theta w \in \Omega - \bar{u}$. This observation actually enriches the covered set, as to every such w it associates the enlarged collection of appropriate x that can appear in (33) and (34): they must satisfy

$$x \in \frac{\theta^2}{2} F''(\bar{u})[w, w] + \operatorname{rint} F'(\bar{u})(\Omega - \bar{u}) \tag{44}$$

with *some* $\theta > 0$ such that $\theta\|w\| < 1$ and $\theta w \in \Omega - \bar{u}$. Needless to say, constants appearing in the corresponding assertions do depend on θ.

Furthermore, for fixed w, take $z = \frac{1}{2} F''(\bar{u})[w, w] + \xi$ with $\xi \in \operatorname{rint} F'(\bar{u}) \operatorname{cone}(\Omega - \bar{u})$. Then there exists $\widetilde{\theta} > 0$ such that $\widetilde{\theta}\xi \in \operatorname{rint} F'(\bar{u})(\Omega - \bar{u})$ and $\sqrt{\widetilde{\theta}}\|w\| < 1$, $\sqrt{\widetilde{\theta}} w \in \Omega - \bar{u}$. Take $x = \widetilde{\theta} z$, then x satisfies (44) with $\theta = \sqrt{\widetilde{\theta}}$. According to the discussion above, this implies that such z belongs to the interior of the cone of feasible directions at 0 to the covered set.

We finally discuss the important case when (40) or (41) allow to take $x = 0$. In the former case, the covered set contains the whole ball around 0, of radius δ in Proposition 6, and of radius ε in Corollary 2. In the latter case, the same conclusion is obtained in [6, Corollary 6.4]. According to the discussion above, condition (40) with $x = 0$ is in general weaker than (41) with $x = 0$. However, the latter can in fact be replaced by

$$-\frac{1}{2} F''(\bar{u})[w, w] \in \operatorname{rint} F'(\bar{u}) \operatorname{cone}(\Omega - \bar{u}) = \operatorname{rint} F'(\bar{u}) T_\Omega(\bar{u}) \tag{45}$$

(and without any restrictions on the norm of w), by application of [6, Corollary 6.4] with w replaced by θw with appropriate $\theta > 0$: taking it small enough ensures (41) with $x = 0$, as well as the other requirements on thus modified w. Therefore, taking into account (43), the essential difference between (40) and (41) with $x = 0$ consists of the relative interior appearing in the right-hand side of (45). As the next example demonstrates, this can still be an important difference that may allow to ensure covering of a whole neighborhood of 0 by means of Proposition 6 when [6, Corollary 6.4] is not applicable.

Example 3 Let $n = 4$, $m = 2$, $F(u) = (u_1 + u_2 u_4,\, u_2^2 + u_3^2 - u_4^2)$, $\Omega = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_-$, $\bar{u} = 0$. Then, we have $T_\Omega(\bar{u}) = \Omega = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_-$, $F'(\bar{u}) T_\Omega(\bar{u}) = \mathbb{R}_+ \times \{0\}$, and $\ker F'(\bar{u}) = \{0\} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Thus, $w, v \in \ker F'(\bar{u}) \cap T_\Omega(\bar{u})$ have the form $w = (0, w_2, w_3, w_4)$, $v = (0, v_2, v_3, v_4)$, with $w_2, v_2 \geq 0$, arbitrary w_3, v_3, and $w_4, v_4 \leq 0$, and for them, $F''(\bar{u})[w, v] = (w_4 v_2 + w_2 v_4,\, 2(w_2 v_2 + w_3 v_3 - w_4 v_4))$ holds. In the case of a polyhedral Ω (as in this example), the cone $F'(\bar{u}) T_\Omega(\bar{u})$ is closed, and from (42) and intermediate relations in (43) it then follows that $F'(\bar{u}) T_\Omega^2(\bar{u}|w) = F'(\bar{u}) T_\Omega(\bar{u})$.

Since $F''(\bar{u})[w,\, w] = 2(w_2w_4,\, w_2^2 + w_3^2 - w_4^2)$ is valid, we now conclude that for (40) to hold with $x = 0$, we must have $w_2^2 + w_3^2 - w_4^2 = 0$ and $w_2w_4 \geq 0$. Thanks to the restrictions on the signs of w_2 and w_4, this is only possible when $w_2 = 0$ and $w_3^2 = w_4^2$ are fulfilled, and for such w, (45) does not hold. Therefore, we further consider $w = (0, 0, t, -t)$ with $t > 0$. Then $F''(\bar{u})[w,\, \ker F'(\bar{u}) \cap T_\Omega(\bar{u})]$ consists of $z \in \mathbb{R}^2$ satisfying $z_1 = -tv_2$ and $z_2 = 2t(v_3 + v_4)$ with some $v_2 \geq 0$, arbitrary v_3, and $v_4 \leq 0$. Evidently, this set is $\mathbb{R}_- \times \mathbb{R}$, and hence, (36) holds. It can be directly verified that the actually covered set in this example is indeed the entire $\mathbb{R}^2$, with the square-root estimate. □

We complete this section by a short discussion of the unconstrained case, i.e., let $\Omega = \mathbb{R}^n$. Both 2-regularity (38) and weak 2-regularity (39) with respect to Ω reduce to the usual 2-regularity of F at $\bar{u}$ in a direction w,

$$\operatorname{im} F'(\bar{u}) + F''(\bar{u})[w, \ker F'(\bar{u})] = \mathbb{R}^m, \tag{46}$$

and the requirement $w \in \Omega - \bar{u}$ is no more a restriction. Moreover, (36) also reduces to the same condition, and according to Lemma 6 (taking into account that $N_\Omega(\bar{u}) = \{0\}$), this condition is equivalent to (26), and the requirement $w \in T_\Omega(\bar{u})$ is no more a restriction. Interestingly, [28, Example 6.4] states that (46) and metric 2-regularity of F coincide in this case as well. Requirements (40) and (41) on admissible choices of x both take the form $x \in \frac{1}{2}F''(\bar{u})[w,\, w] + \operatorname{im} F'(\bar{u})$. Finally, parabolic derivability of such Ω holds trivially.

Therefore, Proposition 6, Corollary 2, and [6, Corollaries 6.3, 6.4] essentially (up to weaker smoothness assumptions and some sharper estimates) reduce in this case, e.g., to the combined results of [14, Theorems 4.1, 4.2] (see also references therein to earlier related works, and comments in [14, p. 1026]). Furthermore, (40) and (41) with $x = 0$ both take the form $F''(\bar{u})[w,\, w] \in \operatorname{im} F'(\bar{u})$, and when this holds, Proposition 6, Corollary 2, and [6, Corollary 6.4] agree with [14, Theorem 4.1].

7 Application to Coupled Constraint Systems

This section deals with stability of a solution to the *coupled constraint system*

$$\Phi(\chi, \sigma) = 0, \quad \chi \in \mathscr{X}, \quad 0 \in \Sigma(\sigma), \tag{47}$$

where $\Phi : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^m$ is continuously differentiable, $\mathscr{X} \subset \mathbb{R}^k$ is closed, and the set-valued mapping $\Sigma : \mathbb{R}^l \rightrightarrows \mathbb{R}^q$ has closed graph. Many problems can be written as coupled constraint system, such as slack-reformulations of *variational systems* [31, Section 8] or *complementarity systems over convex cones* [15, Section 5.1], etc. The former includes *M-stationarity* and more specific *KKT-type* systems [35,36], among others. The latter is a common equilibrium problem, which, in a way slightly extending the standard form in [32] by additional geometric constraints, reads as

$$f(\chi) \in \mathscr{C}, \quad g(\chi) \in \mathscr{C}^\circ, \quad f(\chi)^\top g(\chi) = 0, \quad \chi \in \mathscr{X} \tag{48}$$

with smooth mappings f, g, and a nonempty closed convex cone $\mathscr{C}$. This system can be reformulated as (47), e.g., if we take

$$\Phi(\chi, \sigma) = \Psi(\chi) + \sigma, \tag{49}$$

and

$$\Psi(\chi)=\begin{pmatrix} f(\chi)\\ g(\chi)\end{pmatrix},\quad \Sigma(\sigma)=\Sigma(\sigma^1,\sigma^2)=\left\{\begin{pmatrix}\sigma^1+\alpha^1\\ \sigma^2+\alpha^2\end{pmatrix}\,\middle|\,\begin{array}{l}\alpha^1\in\mathscr{C},\alpha^2\in\mathscr{C}^\circ,\\ (\alpha^1)^\top\alpha^2=0\end{array}\right\}.$$

With this reformulation, covering results can still be interpreted as stability results subject to perturbations *of the original problem data f and g*, which is not automatic, e.g., for reformulations of complementarity problems via complementarity functions, see corresponding discussions on [14, p. 1028] or [25, p. 10].

System (47) fits the constrained equation setting (20) with

$$F(u):=F(\chi,\sigma):=\Phi(\chi,\sigma),\qquad \Omega:=\mathscr{X}\times\Sigma^{-1}(0). \tag{50}$$

For this particular choice, it follows by [39, Proposition 6.41] that for any $(\chi,\sigma)\in\Omega$,

$$T_\Omega(\chi,\sigma)\subset T_{\mathscr{X}}(\chi)\times T_{\Sigma^{-1}(0)}(\sigma),\quad N_\Omega(\chi,\sigma)=N_{\mathscr{X}}(\chi)\times N_{\Sigma^{-1}(0)}(\sigma). \tag{51}$$

With the partial derivatives of Φ denoted by $\partial_\chi\Phi$ and $\partial_\sigma\Phi$, we have $\Phi'=(\partial_\chi\Phi\ \partial_\sigma\Phi)$, and the first-order regularity condition (21) with respect to a solution $(\bar\chi,\bar\sigma)$ of the coupled constraint system (47) transforms into

$$\left[\partial_\chi\Phi(\bar\chi,\bar\sigma)^\top\lambda\in N_{\mathscr{X}}(\bar\chi),\quad \partial_\sigma\Phi(\bar\chi,\bar\sigma)^\top\lambda\in N_{\Sigma^{-1}(0)}(\bar\sigma)\right]\quad\Longrightarrow\quad\lambda=0. \tag{52}$$

If Φ is determined by (49) with smooth Ψ, then $\partial_\sigma\Phi$ is the identity, and (52) becomes

$$\left[\Psi'(\bar\chi)^\top\lambda\in N_{\mathscr{X}}(\bar\chi),\quad \lambda\in N_{\Sigma^{-1}(0)}(\bar\sigma)\right]\quad\Longrightarrow\quad\lambda=0, \tag{53}$$

a condition that plays a crucial role e.g. in [15, Section 5.1], dealing with Lipschitzian stability properties and error bounds for the cone complementarity system (48).

The criterion in Theorem 4 for the metric 2-regularity of $S=F+\Delta_\Omega$ underlies no significant changes in the setting of this section, so we do not present an explicit counterpart to the theorem. Instead, we formulate a counterpart to Proposition 4:

Proposition 7 *In the setting of this section, let $(\bar\chi,\bar\sigma)$ be a solution to the coupled constraint system (47), and assume that the derivative Φ' is semidifferentiable at $(\bar\chi,\bar\sigma)$ relative to a direction $(v,\omega)\in T_\Omega(\bar\chi,\bar\sigma)\setminus\{(0,0)\}$. Each of the following conditions (a)–(b) is sufficient for the metric 2-regularity of $S=F+\Delta_\Omega=\Phi+\Delta_{\mathscr{X}\times\Sigma^{-1}(0)}$ at $((\bar\chi,\bar\sigma),0)$ relative to (v,ω):*

(a) The sets $\mathscr{X},\Sigma^{-1}(0)$ are each a finite union of polyhedral sets, and

$$\left.\begin{array}{r}\partial_\chi\Phi(\bar\chi,\bar\sigma)^\top\lambda\in N_{\mathscr{X}}(\bar\chi),\\ \partial_\sigma\Phi(\bar\chi,\bar\sigma)^\top\lambda\in N_{\Sigma^{-1}(0)}(\bar\sigma),\\ (\partial_\chi\Phi)'((\bar\chi,\bar\sigma);(v,\omega))^\top\lambda\in\partial_\chi\Phi(\bar\chi,\bar\sigma)^\top\xi+N_{\mathscr{X}}(\bar\chi),\\ (\partial_\sigma\Phi)'((\bar\chi,\bar\sigma);(v,\omega))^\top\lambda\in\partial_\sigma\Phi(\bar\chi,\bar\sigma)^\top\xi+N_{\Sigma^{-1}(0)}(\bar\sigma)\end{array}\right\}\Rightarrow\lambda=0. \tag{54}$$

(b) The following holds in addition to (54):

$$\mathrm{im}\left(\Phi'(\bar\chi,\bar\sigma)^\top\right)\bigcap\left(N_{\mathscr{X}}(\bar\chi)\times N_{\Sigma^{-1}(0)}(\bar\sigma)\right)=\{0\}. \tag{55}$$

In the case of (49) with smooth Ψ, we know that $\partial_\sigma \Phi$ is the identity, so (55) turns into (53), indicating that Proposition 7 only plays a role for this special setting, when $\mathscr{X}, \Sigma^{-1}(0)$ are polyhedral. The condition (54), in turn, reduces to

$$\left[\Psi'(\bar\chi)^\top\lambda \in N_{\mathscr{X}}(\bar\chi), \lambda \in N_{\Sigma^{-1}(0)}(\bar\sigma), \Psi''(\bar\chi;v)^\top\lambda \in \mathrm{im}\Psi'(\bar\chi)^\top + N_{\mathscr{X}}(\bar\chi)\right] \Rightarrow \lambda = 0,$$

which is independent of ω. This condition agrees with the one in [28, formula (6.31)] on the special instance of (48) with $\mathscr{C} = \mathbb{R}^p_+$ and $\mathscr{X} = \mathbb{R}^k$. That said, our sufficient conditions for metric 2-regularity in Proposition 7 are new.

Conditions above target metric regularity and metric 2-regularity of the mapping $S = F + \Delta_\Omega$ with F and Ω defined according to (50). In contrast to this, we would like to mention [31, Section 7] on metric regularity and metric 2-regularity properties of the (set-valued) *natural residual mapping for* (47),

$$\mathscr{R}(\chi,\sigma) := \left\{ \begin{pmatrix} \Phi(\chi,\sigma) \\ \chi - \xi \\ \eta \end{pmatrix} \,\middle|\, \xi \in \mathscr{X},\ \eta \in \Sigma(\sigma) \right\}.$$

Combining Lemma 7 below with (52) and [31, Proposition 5], we can see that metric regularity of $S = F + \Delta_\Omega$ is implied (under additional assumptions on Σ) by the corresponding metric regularity of $\mathscr{R}$. Therefore, one can expect that metric 2-regularity of $S = F + \Delta_\Omega$ is also implied (up to additional assumptions) by the corresponding metric 2-regularity of $\mathscr{R}$, and a comparison of the sufficient conditions in Proposition 7 and [31, Propositions 6,7] consolidates this expectation.

We do not present specific covering results in this section, as their conclusions are the ones obtained in Section 5. Instead, we will specialize the key assumptions of that section. To this end, we want to compute the cones in (51) first, for which we employ proto-differentiability of Σ (see Remark 1 for a definition).

Lemma 7 *In the setting of this section, let $(\bar\chi,\bar\sigma)$ be a solution to the coupled constraint system (47). If each $(v,\omega) \in T_{\mathscr{X}}(\bar\chi) \times T_{\Sigma^{-1}(0)}(\bar\sigma)$ is derivable, then each element of $T_\Omega(\bar\chi,\bar\sigma)$ is derivable as well, and $T_\Omega(\bar\chi,\bar\sigma) = T_{\mathscr{X}}(\bar\chi) \times T_{\Sigma^{-1}(0)}(\bar\sigma)$ holds true. In particular, under the condition*

$$\exists c > 0: \qquad \mathrm{dist}[\sigma, \Sigma^{-1}(0)] \le c \cdot \mathrm{dist}[0, \Sigma(\sigma)] \qquad \forall \sigma \text{ near } \bar\sigma \tag{56}$$

(which is satisfied, when Σ is metrically regular at $(\bar\sigma,0)$, or its graph is a finite union of polyhedral sets), we have $T_{\Sigma^{-1}(0)}(\bar\sigma) = \left\{\omega \in \mathbb{R}^l \mid 0 \in D\Sigma(\bar\sigma|0)(\omega)\right\}$ and $N_{\Sigma^{-1}(0)}(\bar\sigma) \subset \mathrm{rge}D^*\Sigma(\bar\sigma|0)$. If, in addition to (56), Σ is proto-differentiable at $(\bar\sigma,0)$ for any $\omega \ne 0$ (as is true, if its graph is convex, or a finite union of polyhedral sets), then every element of $T_{\Sigma^{-1}(0)}(\bar\sigma)$ is derivable.

Proof It is due to the definitions that derivability of the elements of $T_\Omega(\bar\chi,\bar\sigma)$ follows from derivability of elements of $T_{\mathscr{X}}(\bar\chi)$ and $T_{\Sigma^{-1}(0)}(\bar\sigma)$. The formula $T_\Omega(\bar\chi,\bar\sigma) = T_{\mathscr{X}}(\bar\chi) \times T_{\Sigma^{-1}(0)}(\bar\sigma)$ results from the latter, and direct computations.

The equality for $T_{\Sigma^{-1}(0)}(\bar\sigma)$ under (56) is well-known, and can be extracted e.g. from [13, Theorem 1] or [12]. The upper estimate for $N_{\Sigma^{-1}(0)}(\bar\sigma)$ under (56) is also known, see [23, Theorem 7.16] for instance.

It is obvious that metric regularity implies (56). That this condition also holds, when $\mathrm{gph}\Sigma$ is polyhedral, follows e.g. by [10, Proposition 3H.1 and Theorem 3H.3].

The zero vector is always derivable. Under proto-differentiability and (56), we have for a nonzero $\bar{\omega} \in T_{\Sigma^{-1}(0)}(\bar{\sigma}) = \{\omega \in \mathbb{R}^l \mid 0 \in D\Sigma(\bar{\sigma}|0)(\omega)\}$ that

$$\limsup_{t\searrow 0} \frac{\mathrm{dist}[\bar{\sigma}+t\bar{\omega}, \Sigma^{-1}(0)]}{t} \leq c \cdot \limsup_{t\searrow 0} \frac{\mathrm{dist}[0, \Sigma(\bar{\sigma}+t\bar{\omega})]}{t} = 0,$$

which yields derivability of elements of $T_{\Sigma^{-1}(0)}(\bar{\sigma})$. That Σ is proto-differentiable, whenever its graph is convex or polyhedral, follows from $\mathrm{gph}D\Sigma(\bar{\sigma}|0) = T_{\mathrm{gph}\Sigma}(\bar{\sigma}, 0)$, and the derivability of any tangent to such sets (cf. discussions above Lemma 4). □

Hence, if all tangents to $\mathscr{X}$ at $\bar{\chi}$, and to $\Sigma^{-1}(0)$ at $\bar{\sigma}$, are derivable, then condition (30) with $\bar{u} = (\bar{\chi}, \bar{\sigma})$ (for a solution $(\bar{\chi}, \bar{\sigma})$ to (47)) and $w = (v, \omega)$ transforms into

$$\partial_\chi \Phi(\bar{\chi}, \bar{\sigma})v + \partial_\sigma \Phi(\bar{\chi}, \bar{\sigma})\omega = 0, \qquad v \in T_{\mathscr{X}}(\bar{\chi}), \qquad \omega \in T_{\Sigma^{-1}(0)}(\bar{\sigma}), \tag{57}$$

or, assuming additionally (56), into

$$\partial_\chi \Phi(\bar{\chi}, \bar{\sigma})v + \partial_\sigma \Phi(\bar{\chi}, \bar{\sigma})\omega = 0, \qquad v \in T_{\mathscr{X}}(\bar{\chi}), \qquad 0 \in D\Sigma(\bar{\sigma}|0)(\omega). \tag{58}$$

The condition (57) always implies (58), but the converse does not hold without (56), in general. Under the aforementioned derivability assumption, any nonzero solution (v, ω) to (57) is a derivable element of $T_\Omega(\bar{x}, \bar{\sigma})$, and thus a potential candidate for the *nontrivial* (i.e., nonempty) covering in Proposition 5. Let us also mention that, in the special case of Φ being defined as in (49) with smooth Ψ, the conditions in (57) reduce to the combination of $v \in T_{\mathscr{X}}(\bar{\chi})$ and $-\Psi'(\bar{\chi})v \in T_{\Sigma^{-1}(0)}(\bar{\sigma})$, while the ones in (58) become $v \in T_{\mathscr{X}}(\bar{\chi})$ and $0 \in D\Sigma(\bar{\sigma}|0)(-\Psi'(\bar{\chi})v)$, using $w = (v, \omega) = (v, -\Psi'(\bar{\chi})v)$.

In order to specialize assumptions of Proposition 6, let us compute the second-order tangent set to Ω next, and ensure parabolic derivability. For this purpose, we utilize the *second-order contingent derivative* of Σ at $(\bar{\sigma}, 0) \in \mathrm{gph}\Sigma$ relative to $(\omega, \zeta) \in T_{\mathrm{gph}\Sigma}(\bar{\sigma}, 0)$ for $\eta \in \mathbb{R}^l$, which is defined in [33, Definition 11.7.1] through

$$C^2\Sigma\left((\bar{\sigma}, 0)|(\omega, \zeta)\right)(\eta) := \left\{ \nu \in \mathbb{R}^q \,\middle|\, (\eta, \nu) \in T^2_{\mathrm{gph}\Sigma}\left((\bar{\sigma}, 0)|(\omega, \zeta)\right) \right\}.$$

Some relations between these derivatives and the second-order Studniarski derivative (cf. Section 4) have been explored recently in [30]. Whenever the set $\mathrm{gph}\Sigma$ is parabolically derivable at $(\bar{\sigma}, 0)$ relative to some $(\omega, \zeta) \in T_{\mathrm{gph}\Sigma}(\bar{\sigma}, 0)$ (as is the case, when $\mathrm{gph}\Sigma$ is a finite union of polyhedral sets, cf. discussions above Lemma 8), then we say that the mapping Σ is *parabolically derivable at* $(\bar{\sigma}, 0)$ *relative to* (ω, ζ).

The following lemma can be proved similarly as Lemma 7 above, relying on formulas on the representation of tangents and second-order tangents under (56), established e.g. in [11, 12]. We omit details for brevity.

Lemma 8 *In the setting of this section, let $(\bar{\chi}, \bar{\sigma})$ be a solution to the coupled constraint system (47). If $\mathscr{X}$ is parabolically derivable at $\bar{\chi}$ relative to $v \in T_{\mathscr{X}}(\bar{\chi})$, and $\Sigma^{-1}(0)$ is parabolically derivable at $\bar{\sigma}$ relative to $\omega \in T_{\Sigma^{-1}(0)}(\bar{\sigma})$, then $(v, \omega) \in$*

$T_\Omega(\bar{x},\bar{\sigma})$, Ω is parabolically derivable at $(\bar{\chi},\bar{\sigma})$ relative to (v,ω), and it holds that $T^2_\Omega((\bar{\chi},\bar{\sigma})|(v,\omega)) = T^2_{\mathscr{X}}(\bar{\chi}|v)\times T^2_{\Sigma^{-1}(0)}(\bar{\sigma}|\omega)$. Furthermore, under (56), we have $T_{\Sigma^{-1}(0)}(\bar{\sigma}) = \{\omega\in\mathbb{R}^l \mid 0\in D\Sigma(\bar{\sigma}|0)(\omega)\}$, and

$$T^2_{\Sigma^{-1}(0)}(\bar{\sigma}|\omega) = \left\{\eta\in\mathbb{R}^l \,\middle|\, 0\in C^2\Sigma((\bar{\sigma},0)|(\omega,0))(\eta)\right\} \quad \forall\omega\in T_{\Sigma^{-1}(0)}(\bar{\sigma}).$$

If, in addition to (56), the mapping Σ is parabolically derivable at $(\bar{\sigma},0)$ relative to $(\omega,0)$ with some $\omega\in T_{\Sigma^{-1}(0)}(\bar{\sigma})$ (as is true, if gphΣ *is a finite union of polyhedral sets), then the set $\Sigma^{-1}(0)$ is parabolically derivable at $\bar{\sigma}$ relative to ω.*

From now on, we assume that Φ' is semidifferentiable at a solution $(\bar{\chi},\bar{\sigma})$ to (47) relative to nonzero (v,ω). Then Φ has a semi-quadratic expansion there, which reads as $\mathscr{E}\Phi((\bar{\chi},\bar{\sigma});(v,\omega)) = (\partial_\chi\Phi)'((\bar{\chi},\bar{\sigma});(v,\omega))v + (\partial_\sigma\Phi)'((\bar{\chi},\bar{\sigma});(v,\omega))\omega$.

Lemma 8 states that if $\mathscr{X}$ is parabolically derivable at $\bar{\chi}$ relative to $v\in T_{\mathscr{X}}(\bar{\chi})$, and $\Sigma^{-1}(0)$ is parabolically derivable at $\bar{\sigma}$ relative to $\omega\in T_{\Sigma^{-1}(0)}(\bar{\sigma})$, then Ω is parabolically derivable at $\bar{u}=(\bar{\chi},\bar{\sigma})$ relative to $w=(v,\omega)\in T_\Omega(\bar{\chi},\bar{\sigma})$. Under these assumptions, the condition $x\in\frac{1}{2}\left(\mathscr{E}F(\bar{u};w)+F'(\bar{u})T^2_\Omega(\bar{u}|w)\right)$ of interest in Proposition 6 and Corollary 2 transforms into the system

$$\begin{array}{l} x = \frac{1}{2}((\partial_\chi\Phi)'((\bar{\chi},\bar{\sigma});(v,\omega))v + (\partial_\sigma\Phi)'((\bar{\chi},\bar{\sigma});(v,\omega))\omega \\ \qquad +\partial_\chi\Phi(\bar{\chi},\bar{\sigma})\xi + \partial_\sigma\Phi(\bar{\chi},\bar{\sigma})\eta), \\ \quad \xi\in T^2_{\mathscr{X}}(\bar{\chi}|v), \qquad \eta\in T^2_{\Sigma^{-1}(0)}(\bar{\sigma}|\omega), \end{array} \tag{59}$$

or, assuming additionally (56), into

$$\begin{array}{l} x = \frac{1}{2}((\partial_\chi\Phi)'((\bar{\chi},\bar{\sigma});(v,\omega))v + (\partial_\sigma\Phi)'((\bar{\chi},\bar{\sigma});(v,\omega))\omega \\ \qquad +\partial_\chi\Phi(\bar{\chi},\bar{\sigma})\xi + \partial_\sigma\Phi(\bar{\chi},\bar{\sigma})\eta), \\ \quad \xi\in T^2_{\mathscr{X}}(\bar{\chi}|v), \qquad 0\in C^2\Sigma((\bar{\sigma},0)|(\omega,0))(\eta). \end{array} \tag{60}$$

Yet again, (59) always implies (60), but the converse does not hold without (56), in general. Under the above parabolic derivability assumptions, particularly guaranteeing Ω to be parabolically derivable at any nonzero $(v,\omega)\in T_\Omega(\bar{\chi},\bar{\sigma})$, we can say that any such (v,ω) and x, for which we have (ξ,η) satisfying (59), is a potential candidate for the *nontrivial* covering in Proposition 6 and Corollary 2.

Finally, let us also mention that in the special case of Φ being defined as in (49) with sufficiently smooth Ψ, the system in (59) reduces to

$$x = \frac{1}{2}\left(\Psi''(\bar{\chi};v)v + \Psi'(\bar{\chi})\xi + \eta\right), \quad \xi\in T^2_{\mathscr{X}}(\bar{\chi}|v), \quad \eta\in T^2_{\Sigma^{-1}(0)}(\bar{\sigma}|-\Psi'(\bar{\chi})v),$$

while the system in (60) becomes

$$x = \frac{1}{2}\left(\Psi''(\bar{\chi};v)v + \Psi'(\bar{\chi})\xi + \eta\right), \xi\in T^2_{\mathscr{X}}(\bar{\chi}|v), 0\in C^2\Sigma\left((\bar{\sigma},0)|(-\Psi'(\bar{\chi})v,0)\right)(\eta),$$

using again $w=(v,\omega)=(v,-\Psi'(\bar{\chi})v)$. Covering results that can now be extracted from Corollary 2 for the special instance of the complementarity system (48) with $\mathscr{C}=\mathbb{R}^p_+$ and $\mathscr{X}=\mathbb{R}^k$ agree with the outcome of [28, Remark 6.26]. Related results for a more special problem setting were also obtained in [25].

Data Availability No data was generated or analyzed.

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