Local-to-Global Exactness of SDP Relaxations for Sparse QCQPs

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October 24, 2026

Abstract

We study exact semidefinite programming (SDP) relaxation for a given sparse quadratically constrained quadratic program (QCQP). The SDP relaxation is exact if, whenever it has an optimal solution, it admits a rank-at-most-one optimal solution that corresponds to an optimal solution of the QCQP. Using the maximal cliques of a chordal extension of the aggregate sparsity pattern graph of the data matrices, we formulate the SDP relaxation in terms of clique-wise matrix variables and develop a local-to-global framework for certifying exactness. For each clique-wise matrix variable, we introduce a local sub-SDP with two parameters: a local right-hand-side vector and a consistency matrix specifying the values of entries shared by overlapping clique-wise matrix variables. In the main theorem, these parameters are determined by an optimal solution of the global clique-wise SDP. The theorem shows that if the resulting local sub-SDPs are exact, then the original SDP relaxation is exact. Under the additional assumption that any two distinct cliques intersect in at most one node, we present three classes of local QCQPs that can be incorporated into this framework: convex local QCQPs, local QCQPs characterized by sign-pattern conditions, and separable local QCQPs with a limited number of constraints. Examples illustrate how these different local QCQP classes can be combined in sparse QCQPs.

Key words. Quadratically constrained quadratic program, semidefinite programming relaxation, exact SDP relaxation, sparse optimization, chordal graph, clique-wise formulation, local-to-global exactness, block-clique sparsity, rank-at-most-one optimal solution.

AMS Classification. 90C20, 90C22, 90C25, 90C26.

1 Introduction

Quadratically constrained quadratic programs (QCQPs) form a broad class of nonconvex optimization problems. They include many fundamental models in operations research,

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control, signal processing, and combinatorial optimization. A standard approach to such problems is to lift the quadratic terms to a symmetric matrix variable and to relax the resulting rank-at-most-one constraint, thereby obtaining a semidefinite programming (SDP) relaxation. SDP relaxations of QCQPs have been extensively studied; see, for example, [6, 7, 10, 22, 26] and the references therein. A central question is when such an SDP relaxation is exact, in the sense that it admits a rank-at-most-one optimal solution, which corresponds to an optimal solution of the original QCQP. (We refer to an optimal solution of rank at most one as a rank-at-most-one optimal solution.)

The exactness of SDP relaxations has been studied from several different viewpoints. One line of work derives exactness conditions directly in terms of the data matrices, including convexity [7] and sign-pattern conditions [4, 5, 17, 27]. Another line is based on the number of quadratic constraints and theoretical rank bounds for SDP solutions [13, 22, 24]. A further line studies exactness through geometric properties of the feasible region or of associated cones, such as non-intersecting quadratic constraints (NIQC) conditions [3, 16, 28] and the rank-one-generated (ROG) property [1, 2, 19]. The NIQC and ROG viewpoints are closely related, as discussed in [3], but they are qualitatively different in nature from coefficient-wise or constraint-count conditions: they depend on how the quadratic inequalities jointly shape the feasible region.

This paper develops a local-to-global exactness framework for sparse QCQPs that can be verified locally and then assembled into a global exactness certificate. Many existing exactness results are formulated for a single QCQP with a specific global structure as mentioned above. In sparse problems, however, different parts of the problem may have different structures: one part may correspond to a convex subproblem, another to a sign-pattern class, and another to a separable problem with a limited number of constraints. Convexity and sign-pattern conditions are particularly well suited to such a local treatment, because their exactness guarantees are stable under changes in local right-hand-side values and in the values shared by overlapping local subproblems. Rank-bound arguments based on the limited number of constraints can also be incorporated, although their applicability may depend on the parameters induced by the global problem. Since the ROG and NIQC viewpoints are more global and geometric in nature, they fall outside the scope of the present work.

Sparsity has long played an important role in semidefinite programming. For SDPs whose data matrices have a sparse aggregate pattern, chordal extensions and positive semidefinite matrix completion enable the replacement of a single large positive semidefinite constraint with smaller positive semidefinite constraints on maximal cliques [12, 11, 23]. The resulting clique-wise SDP is equivalent to the original SDP relaxation, but it is formulated in terms of local matrix variables linked by consistency constraints on overlaps. We adopt this clique-wise formulation for the analysis of exactness. We emphasize that the clique-wise formulation is used here as a theoretical tool for certifying exactness, rather than as a computational strategy for solving the sparse SDP relaxation. Once exactness has been certified, the SDP relaxation may be solved by any suitable SDP method. For computational methods for sparse SDPs based on chordal decomposition, see [11, 23].

The main contribution of this paper is the local-to-global exactness framework for sparse QCQPs. For a chordal extension of the aggregate sparsity pattern graph, we introduce a clique-wise SDP formulation and local sub-SDPs associated with the maximal cliques. Each local sub-SDP depends on a pair of parameters: a local right-hand-side vector $\boldsymbol{\delta}$ and a consistency matrix $\boldsymbol{U}$, which specifies the values of entries shared by local matrix variables.

In the main theorem, these parameters are induced by an optimal solution of the global clique-wise SDP. The theorem shows how these induced local certificates can be assembled: if each local sub-SDP with its induced parameters $(\boldsymbol{\delta}, \boldsymbol{U})$ admits a rank-at-most-one optimal solution, then the original SDP relaxation admits a rank-at-most-one optimal solution, and hence is exact. This 'local-to-global exactness' is established for general chordal extensions.

The remaining difficulty is ensuring consistency of local rank-at-most-one solutions on overlaps. For a general chordal extension, two maximal cliques may share more than one node. The resulting consistency constraints may then include off-diagonal entries of local matrix variables, and local rank-at-most-one solutions must satisfy nontrivial product relations on those overlapping off-diagonal entries. These relations are difficult to verify from local exactness alone. Therefore, when applying this framework, we impose a block-clique assumption: any two maximal cliques intersect in at most one node. Under this assumption all consistency constraints are diagonal, and consistency on overlaps reduces to matching squared scalar values. This permits the combination of local exactness mechanisms without imposing additional off-diagonal rank-one consistency conditions.

Under the block-clique assumption, the exactness mechanisms discussed above are recast as local results. Some are independent of the induced parameters $(\boldsymbol{\delta}, \boldsymbol{U})$, while others, especially those based on rank bounds for separable subproblems, depend on them. We further prove a preservation result showing that certain dependent inequality constraints can be added without destroying local exactness. These results provide the local building blocks for the local-to-global exactness certification.

The theoretical significance of the local-to-global exactness framework lies in treating sparsity as part of the exactness analysis. The clique-wise formulation relates global rank-at-most-one attainment to local rank-at-most-one attainment. This provides a route to proving exactness of SDP relaxations of sparse QCQPs in settings where no single global exactness criterion applies to the entire problem. In this sense, the contribution is structural: it shows how sparsity can serve as a mechanism for certifying exact SDP relaxations. The examples in Section 5 illustrate how local exactness certificates can be assigned to different parts of the sparse structure and then assembled through diagonal consistency constraints.

We also mention two related lines of work. First, block-clique graph structures have appeared in doubly nonnegative (DNN) and completely positive (CPP) reformulations of quadratic optimization problems [18]. Although those works concern DNN and CPP reformulations rather than SDP exactness studied here, their use of block-clique structures is closely related to the block-clique assumption mentioned above.

Second, the present framework is complementary to extension results that preserve exact SDP relaxations under the addition of constraints on a fixed variable space [20]. These results may be viewed as a vertical extension of a QCQP, whereas the local-to-global exactness framework developed here gives a horizontal extension: exact sub-QCQPs on different variable subsets are combined through diagonal consistency constraints. A preliminary version of this horizontal viewpoint for separable QCQPs appeared in [21]; the present paper develops a more general sparse framework that combines several local exactness mechanisms through clique-wise decompositions.

The paper is organized as follows. Section 2 formulates the QCQP and its SDP relaxation, introduces the aggregate sparsity pattern graph, and derives the clique-wise formulation based on a chordal extension of the graph. Section 3 develops the local-to-global exactness framework. In particular, it introduces the local sub-SDPs associated with the

clique-wise formulation developed in Section 2, proves the main theorem, and explains the role of diagonal consistency under the block-clique assumption. Section 4 establishes local exactness results for the three classes of sub-QCQPs used in the framework: convex sub-QCQPs, sub-QCQPs satisfying sign-pattern conditions, and separable sub-QCQPs with a limited number of constraints. It also presents a preservation result for dependent inequality constraints. Section 5 gives examples illustrating how these local exactness results can be combined in sparse QCQPs. Section 6 concludes the paper.

2 QCQP, SDP relaxation, and clique-wise reformulation

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space of column vectors $\boldsymbol{x} = (x_1, \ldots, x_n)$, and $\boldsymbol{x}^T$ the transposed row vector of each $\boldsymbol{x} \in \mathbb{R}^n$. Let $\mathbb{S}^n$ denote the linear space of $n \times n$ symmetric matrices equipped with the inner product $\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \sum_{i=1}^n \sum_{j=1}^n [\boldsymbol{A}]_{ij}[\boldsymbol{B}]_{ij}$ for $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{S}^n$, and let $\mathbb{S}^n_+$ be the cone of $n \times n$ symmetric positive semidefinite matrices. For $\boldsymbol{A} \in \mathbb{S}^n$, we often write a quadratic form $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}$ in $\boldsymbol{x} \in \mathbb{R}^n$ as $\langle \boldsymbol{A}, \boldsymbol{x}\boldsymbol{x}^T \rangle$.

2.1 QCQP and its SDP relaxation

Let $\boldsymbol{A}_k \in \mathbb{S}^n$ $(k = 0, 1, \ldots, m)$ and $\boldsymbol{b} \in \mathbb{R}^m$. We consider the following QCQP:

$$\begin{aligned} \zeta &= \inf \left\{ \langle \boldsymbol{A}_0, \boldsymbol{x}\boldsymbol{x}^T \rangle : \boldsymbol{x} \in \mathbb{R}^n,\ \langle \boldsymbol{A}_k, \boldsymbol{x}\boldsymbol{x}^T \rangle \trianglelefteq_k b_k\ (k = 1, \ldots, m) \right\} \\ &= \inf \left\{ \langle \boldsymbol{A}_0, \boldsymbol{X} \rangle : \boldsymbol{X} \in \mathbb{S}^n_+,\ \mathrm{rank}(\boldsymbol{X}) \leq 1,\ \langle \boldsymbol{A}_k, \boldsymbol{X} \rangle \trianglelefteq_k b_k\ (k = 1, \ldots, m) \right\}, \end{aligned} \tag{1}$$

where $\trianglelefteq_k$ denotes either '$\leq$', '$=$', or '$\geq$'. The standard SDP relaxation of (1) is given by

$$\eta = \inf \left\{ \langle \boldsymbol{A}_0, \boldsymbol{X} \rangle : \boldsymbol{X} \in \mathbb{S}^n_+,\ \langle \boldsymbol{A}_k, \boldsymbol{X} \rangle \trianglelefteq_k b_k\ (k = 1, \ldots, m) \right\}. \tag{2}$$

The formulation (1) is written in homogeneous quadratic form. Linear terms in an inhomogeneous QCQP can be represented by including, or adding, a normalization constraint $X_{ii} = 1$ for some index i. For a rank-at-most-one matrix $\boldsymbol{X} = \boldsymbol{x}\boldsymbol{x}^T$, this condition means $x_i = \pm 1$. Since $\boldsymbol{x}\boldsymbol{x}^T = (-\boldsymbol{x})(-\boldsymbol{x})^T$, we may choose the representative with $x_i = 1$. Then the terms $2[\boldsymbol{A}_k]_{ij}x_ix_j$ become linear terms in the remaining variables x_j $(j \neq i, k = 0, \ldots, m)$. The constraint $X_{ii} = 1$ fixes only a diagonal entry of the lifted matrix variable $\boldsymbol{X}$ and therefore does not add any edge to the aggregate sparsity pattern graph defined below. In contrast, the off-diagonal entries used to represent linear terms are treated as part of the data matrices $\boldsymbol{A}_k$ $(k = 0, \ldots, m)$ and are included in the aggregate sparsity pattern in the same way as the other quadratic coefficients.

If QCQP (1) is infeasible, we assume that $\zeta = +\infty$. Throughout this paper, *exactness* of an SDP relaxation is understood in the rank-attainment sense. More precisely, if the SDP relaxation has an optimal solution, then it has an optimal solution of rank at most one. Such a rank-at-most-one optimal solution is also optimal for the corresponding QCQP, because the QCQP feasible region is precisely the rank-at-most-one portion of the SDP feasible region.

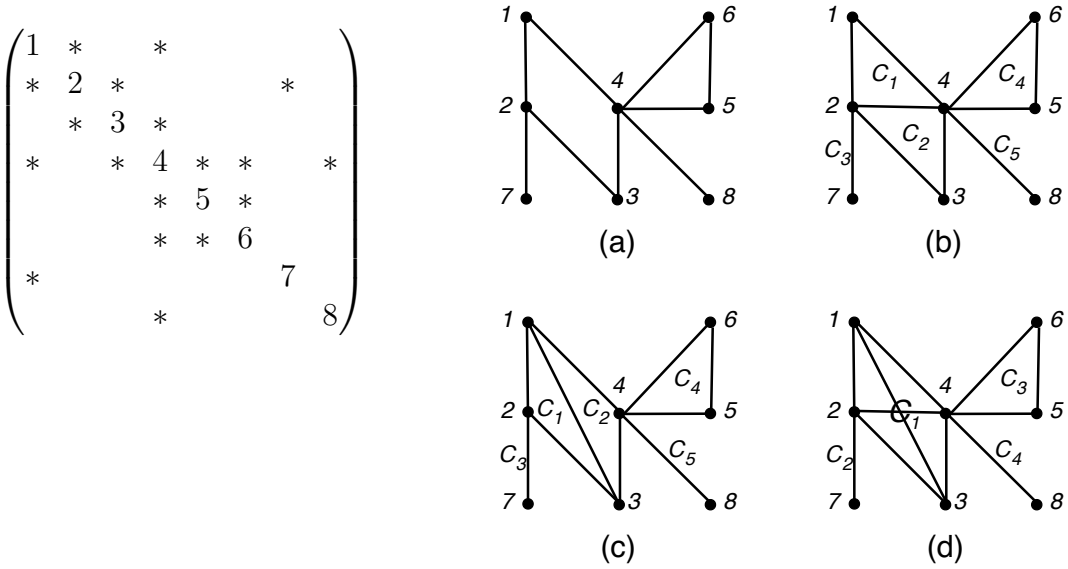


Figure 1: An example of the aggregate sparsity pattern matrix (left), where * denotes nonzero elements. (a) : the associated aggregate sparsity pattern graph $G(N, \mathcal{E}^0)$ with node set $N = \{1, \ldots, 8\}$ and edge set $\mathcal{E}^0 = \{(1,2), (1,4), (2,3), (2,7), (3,4), (4,5), (4,6), (4,8), (5,6)\}$, which is not chordal since the cycle formed by the 4 edges $(1,2), (2,3), (3,4), (4,1)$ is a chordless 4-cycle. (b), (c) and (d): chordal extensions of $G(N, \mathcal{E}^0)$. In (b), the maximal cliques are $C_1 = \{1,2,4\}, C_2 = \{2,3,4\}, C_3 = \{2,7\}, C_4 = \{4,5,6\}$, and $C_5 = \{4,8\}$. In (c), the maximal cliques are $C_1 = \{1,2,3\}, C_2 = \{1,3,4\}, C_3 = \{2,7\}, C_4 = \{4,5,6\}$, and $C_5 = \{4,8\}$. In (d), the maximal cliques are $C_1 = \{1,2,3,4\}, C_2 = \{2,7\}, C_3 = \{4,5,6\}$, and $C_4 = \{4,8\}$.

2.2 Aggregate sparsity pattern

To describe the structured sparsity of QCQP (1), we introduce the *aggregate sparsity pattern graph* $G(N, \mathcal{E}^0)$ of the data matrices $\boldsymbol{A}_k$ $(k = 0, 1, \ldots, m)$ with $N = \{1, \ldots, n\}$ and

$$\mathcal{E}^0 = \{(i,j) \in N \times N : i \neq j,\ [\boldsymbol{A}_k]_{ij} \neq 0 \text{ for some } k \in \{0, 1, \ldots, m\}\},$$

where (i,j) and (j,i) are identified, since they both represent the same undirected edge between nodes i and j $(i \neq j)$. The sparsity structure can also be encoded by *the aggregate sparsity pattern matrix*, which is an $n \times n$ symmetric symbolic matrix with * at the (i,j)th element for $(i,j) \in \mathcal{E}^0$ and blank elsewhere; * is assigned at (i,j)th element if and only if $[\boldsymbol{A}_k]_{ij} \neq 0$ for some $k \in \{0, \ldots, m\}$. Figure 1 shows an example of the aggregate sparsity pattern matrix and the aggregate sparsity pattern graph $G(N, \mathcal{E}^0)$. This example is used in Section 2.3 to illustrate the definitions introduced below for the clique-wise reformulation of QCQP (1) and its SDP relaxation, and is also revisited in the subsequent discussion.

For every nonempty subset C of N, let $\mathbb{S}^C$ denote the linear space of $|C| \times |C|$ symmetric matrices indexed by $C \times C$, $\mathbb{S}^C_+$ the cone of positive semidefinite matrices in $\mathbb{S}^C$, and $\mathbb{R}^C$ the linear space of $|C|$-dimensional column vectors indexed by C. For every $\boldsymbol{A} \in \mathbb{S}^n$ and every nonempty subset $C \subseteq N$, we denote by $\boldsymbol{A}^C \in \mathbb{S}^C$ the principal submatrix of $\boldsymbol{A}$ indexed by $C \times C$. Similarly, for every $\boldsymbol{x} \in \mathbb{R}^n$, we denote by $\boldsymbol{x}^C \in \mathbb{R}^C$ the subvector indexed by C.

2.3 A clique-wise reformulation of QCQP (1) and its SDP relaxation

The clique-wise reformulation of QCQP (1) and its SDP relaxation developed below relies on the following well-known results on positive semidefinite matrix completions for chordal graphs [8, 12]. An undirected graph is said to be *chordal* if every cycle of length at least four has a chord. For an undirected graph $G(N, \mathcal{E})$, define $\mathcal{E} \cup \{(i,i) : i \in N\}$ by $\overline{\mathcal{E}}$. We denote by $\mathbb{S}^n(\mathcal{E})$ the class of $n \times n$ partial symmetric matrices $\boldsymbol{X}$ such that

$$[\boldsymbol{X}]_{ij} = [\boldsymbol{X}]_{ji} \in \mathbb{R} \quad \text{if } (i,j) \in \overline{\mathcal{E}}, \text{ and } [\boldsymbol{X}]_{ij} = [\boldsymbol{X}]_{ji} \text{ is left unspecified otherwise.}$$

In particular, if C is a clique of $G(N, \mathcal{E})$ and $\boldsymbol{X} \in \mathbb{S}^n(\mathcal{E})$, then the principal submatrix $\boldsymbol{X}^C \in \mathbb{S}^C$ is well-defined since $C \times C \subseteq \overline{\mathcal{E}}$.

Lemma 2.1. *Let C_p $(p = 1, \ldots, \hat{p})$ be the maximal cliques of a chordal graph $G(N, \mathcal{E})$. Assume that $\boldsymbol{X} \in \mathbb{S}^n(\mathcal{E})$. Then the following assertions hold.*

(i) There exists an $\overline{\boldsymbol{X}} \in \mathbb{S}^n_+$ such that $[\boldsymbol{X}]_{ij} = [\overline{\boldsymbol{X}}]_{ij}$ for every $(i,j) \in \overline{\mathcal{E}}$ if and only if $\boldsymbol{X}^{C_p} \in \mathbb{S}^{C_p}_+$ $(p = 1, \ldots, \hat{p})$. We call such an $\overline{\boldsymbol{X}} \in \mathbb{S}^n_+$ a positive semidefinite matrix completion of $\boldsymbol{X} \in \mathbb{S}^n(\mathcal{E})$.

(ii) There exists an $\overline{\boldsymbol{X}} \in \mathbb{S}^n_+$ such that $[\boldsymbol{X}]_{ij} = [\overline{\boldsymbol{X}}]_{ij}$ for every $(i,j) \in \overline{\mathcal{E}}$ and $\mathrm{rank}(\overline{\boldsymbol{X}}) \leq 1$ if and only if $\boldsymbol{X}^{C_p} \in \mathbb{S}^{C_p}_+$ and $\mathrm{rank}(\boldsymbol{X}^{C_p}) \leq 1$ $(p = 1, \ldots, \hat{p})$. We call such an $\overline{\boldsymbol{X}}$ a rank-at-most-one positive semidefinite matrix completion of $\boldsymbol{X} \in \mathbb{S}^n(\mathcal{E})$.

Proof. Assertion (i) is the positive semidefinite matrix completion theorem for chordal graphs; see [12], and also [11, 23] for its use in semidefinite programming. Assertion (ii) follows as a special case of the minimum-rank positive semidefinite matrix completion theorem for chordal graphs; see [9, 14]. □

In the remainder of the paper, we let $G(N, \mathcal{E})$ be a chordal extension of the aggregate sparsity pattern graph $G(N, \mathcal{E}^0)$ associated with QCQP (1). We denote the maximal cliques of $G(N, \mathcal{E})$ by C_p $(p = 1, \ldots, \hat{p})$. Let $\overline{\mathcal{E}} = \mathcal{E} \cup \{(i,i) : i \in N\}$. We note that the aggregate sparsity pattern graph $G(N, \mathcal{E}^0)$ is uniquely determined by the data matrices of QCQP (1), but that there are multiple chordal extensions in general. Figure 1 illustrates an example of the aggregate sparsity pattern matrix, the aggregate sparsity pattern graph $G(N, \mathcal{E}^0)$, and three different chordal extensions of $G(N, \mathcal{E}^0)$.

Since $\mathcal{E}^0 \subseteq \mathcal{E} \subseteq \bigcup_{p=1}^{\hat{p}} (C_p \times C_p)$, we see that $[\boldsymbol{A}_k]_{ij} = 0$ if $(i,j) \notin \bigcup_{p=1}^{\hat{p}} (C_p \times C_p)$; hence

$$\langle \boldsymbol{A}_k, \boldsymbol{X} \rangle = \sum_{(i,j) \in \bigcup_{p=1}^{\hat{p}} (C_p \times C_p)} [\boldsymbol{A}_k]_{ij} [\boldsymbol{X}]_{ij} \text{ for every } \boldsymbol{X} \in \mathbb{S}^n$$

$(k = 0, \ldots, m)$. Let $k \in \{0, \ldots, m\}$ be fixed. We consider decompositions of the coefficient matrix $\boldsymbol{A}_k$, a collection of $\boldsymbol{A}^p_k \in \mathbb{S}^{C_p}$ $(p = 1, \ldots, \hat{p})$ such that

$$[\boldsymbol{A}_k]_{ij} = \sum_{p \in P(i,j)} [\boldsymbol{A}^p_k]_{ij} \text{ for every } (i,j) \text{ such that } P(i,j) \neq \emptyset, \tag{3}$$

where $P(i,j) = \{p : (i,j) \in C_p \times C_p\}$. Namely, the (i,j)th element $[\boldsymbol{A}_k]_{ij}$ of $\boldsymbol{A}_k \in \mathbb{S}^n$ is distributed among $(C_p \times C_p)$s containing (i,j). Then,

$$\langle \boldsymbol{A}_k, \boldsymbol{X} \rangle = \sum_{p=1}^{\hat{p}} \langle \boldsymbol{A}^p_k, \boldsymbol{X}^{C_p} \rangle \text{ for every } \boldsymbol{X} \in \mathbb{S}^n, \tag{4}$$

or equivalently,

$$\sum_{(i,j)\in\bigcup_{p=1}^{\hat{p}}(C_p\times C_p)} [\boldsymbol{A}_k]_{ij}[\boldsymbol{X}]_{ij} = \sum_{p=1}^{\hat{p}} \sum_{(i,j)\in C_p\times C_p} [\boldsymbol{A}_k^p]_{ij}[\boldsymbol{X}]_{ij} \text{ for every } \boldsymbol{X}\in\mathbb{S}^n$$

holds. Comparing the terms of symmetric entries $[\boldsymbol{X}]_{ij}$ with $(i,j)\in(C_p\times C_p)$ on both sides of the identity, we see that the left-hand side contains the term $[\boldsymbol{A}_k]_{ij}[\boldsymbol{X}]_{ij}$, while the right-hand side contains $\sum_{p\in P(i,j)}[\boldsymbol{A}_k^p]_{ij}[\boldsymbol{X}]_{ij}$. The coefficients of $[\boldsymbol{X}]_{ij}$ are equal by the decomposition defined in (3). Therefore, the identity holds for every value of $[\boldsymbol{X}]_{ij}\in\mathbb{R}$. This implies (4). In the case (b) of Figure 1, we see that $P(2,2)=\{1,2,3\}, P(2,4)=\{1,2\}, P(4,4)=\{1,2,4,5\}$. Thus, we can take, for example

$$\begin{aligned}&[\boldsymbol{A}_k^p]_{44} = \frac{[\boldsymbol{A}_k]_{44}}{4} \text{ for every } p\in P(4,4)=\{1,2,4,5\} \text{ or}\\ &[\boldsymbol{A}_k^1]_{44}=[\boldsymbol{A}_k^2]_{44}=[\boldsymbol{A}_k^4]_{44}=0,\ [\boldsymbol{A}_k^5]_{44}=[\boldsymbol{A}_k]_{44}.\end{aligned}$$

By substituting identity (4) into QCQP (1), we obtain an equivalent reformulation of QCQP (1) and its SDP relaxation (2).

$$\zeta_{\rm c} \ = \ \inf\left\{\sum_{p=1}^{\hat{p}}\langle\boldsymbol{A}_0^p,\ \boldsymbol{X}^{C_p}\rangle : \begin{array}{l}\boldsymbol{X}\in\mathbb{S}^n(\mathcal{E}), \boldsymbol{X}^{C_p}\in\mathbb{S}_+^{C_p}, \mathrm{rank}(\boldsymbol{X}^{C_p})\le 1\\ (p=1,\ldots,\hat{p}),\\ \sum_{p=1}^{\hat{p}}\langle\boldsymbol{A}_k^p,\ \boldsymbol{X}^{C_p}\rangle \trianglelefteq_k b_k\ (k=1,\ldots,m)\end{array}\right\}. \tag{5}$$

$$\eta_{\rm c} = \inf\left\{\sum_{p=1}^{\hat{p}}\langle\boldsymbol{A}_0^p,\ \boldsymbol{X}^{C_p}\rangle : \begin{array}{l}\boldsymbol{X}\in\mathbb{S}^n(\mathcal{E}), \boldsymbol{X}^{C_p}\in\mathbb{S}_+^{C_p}\ (p=1,\ldots,\hat{p}),\\ \sum_{p=1}^{\hat{p}}\langle\boldsymbol{A}_k^p,\ \boldsymbol{X}^{C_p}\rangle \trianglelefteq_k b_k\ (k=1,\ldots,m)\end{array}\right\}. \tag{6}$$

We say that a feasible solution $\boldsymbol{X}\in\mathbb{S}^n(\mathcal{E})$ of SDP (6) is rank-at-most-one if

$$\boldsymbol{X}^{C_p}\in\mathbb{S}_+^{C_p},\ \mathrm{rank}(\boldsymbol{X}^{C_p})\le 1\ (p=1,\ldots,\hat{p}).$$

Thus a rank-at-most-one feasible solution of SDP (6) is precisely a feasible solution of QCQP (5); the same terminology is used for optimal solutions. By Lemma 2.1(ii), an $\boldsymbol{X}\in\mathbb{S}^n(\mathcal{E})$ is a feasible solution of QCQP (5) if and only if it admits a rank-at-most-one completion $\overline{\boldsymbol{X}}\in\mathbb{S}_+^n$ that is feasible for the original QCQP (1). The objective values are the same under this correspondence. Hence QCQP (1) and QCQP (5) are equivalent. Similarly, by Lemma 2.1(i), the original SDP (2) and SDP (6) are equivalent. Therefore, $\eta_{\rm c}=\eta\le\zeta_{\rm c}=\zeta$. Consequently, the SDP relaxation (2) of QCQP (1) is exact if and only if the SDP relaxation (6) of QCQP (5) is exact, *i.e.*, the SDP relaxation (6) has a rank-at-most-one optimal solution when it has an optimal solution.

Although all constraints in (5) and (6) are expressed in terms of the clique submatrix variables $\boldsymbol{X}^{C_p}\in\mathbb{S}^{C_p}$ $(p=1,\ldots,\hat{p})$, these variables are not independent in general. Indeed, if two sets C_p and C_q overlap, then the corresponding entries of $\boldsymbol{X}^{C_p}$ and $\boldsymbol{X}^{C_q}$ must coincide

on $(C_p \times C_p) \cap (C_q \times C_q)$. To describe these consistency requirements explicitly, we introduce local matrix variables $\boldsymbol{Y}^p \in \mathbb{S}^{C_p}$ $(p = 1, \ldots, \hat{p})$. Let

$$\mathcal{D}_p = \bigcup_{q \neq p} \big((C_p \cap C_q) \times (C_p \cap C_q) \big) \subseteq C_p \times C_p, \ (p = 1, \ldots, \hat{p}),$$

and define

$$\overline{\mathcal{D}} = \bigcup_{p=1}^{\hat{p}} \mathcal{D}_p = \bigcup_{1 \leq p < q \leq \hat{p}} \big((C_p \cap C_q) \times (C_p \cap C_q) \big).$$

Thus, if $\mathcal{D}_p \neq \emptyset$, then $\mathcal{D}_p$ is the set of entries of the local matrix variable $\boldsymbol{Y}^p$ that are shared with at least one other local matrix variable, and $\overline{\mathcal{D}}$ is the set of all such shared entries. Each $\mathcal{D}_p$ is called *a consistency set*. The consistency requirement can then be written as

$$[\boldsymbol{Y}^p]_{ij} = [\boldsymbol{Y}^q]_{ij} \text{ whenever } (i,j) \in (C_p \cap C_q) \times (C_p \cap C_q) \text{ and } p \neq q.$$

Equivalently, we introduce an auxiliary partial symmetric matrix variable $\boldsymbol{U} \in \mathbb{S}^n(\overline{\mathcal{D}})$, called *the consistency matrix*, and write

$$[\boldsymbol{Y}^p]_{ij} = [\boldsymbol{U}]_{ij} \quad ((i,j) \in \mathcal{D}_p, \ p = 1, \ldots, \hat{p}).$$

In this formulation, the clique submatrix variables $\boldsymbol{X}^{C_p}$ in (5) and (6) are replaced by local matrix variables $\boldsymbol{Y}^p$. These local matrix variables are linked through the consistency matrix $\boldsymbol{U}$ on their overlapping entries in the consistency sets $\mathcal{D}_p$ $(p = 1, \ldots, \hat{p})$.

We see in case (b) of Figure 1 that $\mathcal{D}_1 = \mathcal{D}_2 = \{(2,2), (2,4), (4,4)\}, \mathcal{D}_3 = \{(2,2)\}, \mathcal{D}_4 = \mathcal{D}_5 = \{(4,4)\}$ and in case (d) of Figure 1 that $\mathcal{D}_1 = \{(2,2), (4,4)\}, \mathcal{D}_2 = \{(2,2)\}, \mathcal{D}_3 = \mathcal{D}_4 = \{(4,4)\}$. In the former case (b), the consistency constraints are

$$\begin{array}{l} [\boldsymbol{Y}^1]_{ij} = [\boldsymbol{Y}^2]_{ij} \ ((i,j) \in \{(2,2), (2,4), (4,4)\}), \ [\boldsymbol{Y}^1]_{22} = [\boldsymbol{Y}^3]_{22}, \\ [\boldsymbol{Y}^1]_{44} = [\boldsymbol{Y}^2]_{44} = [\boldsymbol{Y}^4]_{44} = [\boldsymbol{Y}^5]_{44}, \end{array}$$

while in the latter case (d), they are

$$[\boldsymbol{Y}^1]_{22} = [\boldsymbol{Y}^2]_{22}, \ [\boldsymbol{Y}^1]_{44} = [\boldsymbol{Y}^3]_{44} = [\boldsymbol{Y}^4]_{44}.$$

In case (b), the two cliques C_1 and C_2 are coupled not only through the diagonal entries $(2,2)$ and $(4,4)$, but also through the off-diagonal entry $(2,4)$. In contrast, in case (d), all consistency constraints are diagonal.

Consequently, SDP (6) can be rewritten in the local matrix variables $\boldsymbol{Y}^p$ $(p = 1, \ldots, \hat{p})$ and the consistency matrix $\boldsymbol{U}$ as follows:

$$\eta_c = \inf \left\{ \sum_{p=1}^{\hat{p}} \langle \boldsymbol{A}_0^p, \boldsymbol{Y}^p \rangle : \begin{array}{l} \boldsymbol{U} \in \mathbb{S}^n(\overline{\mathcal{D}}), \ \boldsymbol{Y}^p \in \mathbb{S}_+^{C_p} \ (p = 1, \ldots, \hat{p}), \\ \displaystyle\sum_{p=1}^{\hat{p}} \langle \boldsymbol{A}_k^p, \boldsymbol{Y}^p \rangle \preceq_k b_k \ (k = 1, \ldots, m), \\ [\boldsymbol{Y}^p]_{ij} = [\boldsymbol{U}]_{ij} \quad ((i,j) \in \mathcal{D}_p, \ p = 1, \ldots, \hat{p}) \end{array} \right\}. \tag{7}$$

We call (7) *the clique-wise formulation* of SDP (2), which was originally proposed in [23]. The problem obtained from SDP (7) by adding a rank-at-most-one condition

$$\operatorname{rank}(\boldsymbol{Y}^p) \leq 1 \quad (p = 1, \ldots, \hat{p}) \tag{8}$$

is called the clique-wise formulation of QCQP (1). Accordingly, a feasible solution $(\boldsymbol{Y}^1, \ldots, \boldsymbol{Y}^{\hat{p}}, \boldsymbol{U})$ of SDP (7) is called *rank-at-most-one* if it satisfies (8). The following two lemmas summarize the relation between the formulations using the partial matrix $\boldsymbol{X}$ and those using the local matrix variables $\boldsymbol{Y}^p$ together with the consistency matrix $\boldsymbol{U}$.

Lemma 2.2. *SDPs (6) and (7) are equivalent. More precisely, the following two assertions hold.*

(i) Suppose that $\boldsymbol{X} \in \mathbb{S}^n(\mathcal{E})$ is a feasible solution of SDP (6) with objective value $\sum_{p=1}^{\hat{p}} \langle \boldsymbol{A}_0^p, \boldsymbol{X}^{C_p} \rangle$. Let

$$\boldsymbol{Y}^p = \boldsymbol{X}^{C_p} \ (p = 1, \ldots, \hat{p}), \ \boldsymbol{U} \in \mathbb{S}^n(\overline{\mathcal{D}}), \ [\boldsymbol{U}]_{ij} = [\boldsymbol{X}]_{ij} \ ((i,j) \in \overline{\mathcal{D}}).$$

Then $(\boldsymbol{Y}^1, \ldots, \boldsymbol{Y}^{\hat{p}}, \boldsymbol{U})$ is a feasible solution of SDP (7) with the same objective value.

(ii) Suppose that $(\boldsymbol{Y}^1, \ldots, \boldsymbol{Y}^{\hat{p}}, \boldsymbol{U})$ is a feasible solution of SDP (7) with objective value $\sum_{p=1}^{\hat{p}} \langle \boldsymbol{A}_0^p, \boldsymbol{Y}^p \rangle$. Then there exists a feasible solution $\boldsymbol{X} \in \mathbb{S}^n(\mathcal{E})$ of SDP (6) satisfying $\boldsymbol{X}^{C_p} = \boldsymbol{Y}^p$ $(p = 1, \ldots, \hat{p})$ with the same objective value.

Lemma 2.3. *QCQP (5) is equivalent to SDP (7) with the additional rank-at-most-one condition (8). More precisely, the following two assertions hold.*

(i) Suppose that $\boldsymbol{X} \in \mathbb{S}^n(\mathcal{E})$ is a feasible solution of QCQP (5) with objective value $\sum_{p=1}^{\hat{p}} \langle \boldsymbol{A}_0^p, \boldsymbol{X}^{C_p} \rangle$. Let

$$\boldsymbol{Y}^p = \boldsymbol{X}^{C_p} \ (p = 1, \ldots, \hat{p}), \ \boldsymbol{U} \in \mathbb{S}^n(\overline{\mathcal{D}}), \ [\boldsymbol{U}]_{ij} = [\boldsymbol{X}]_{ij} \ ((i,j) \in \overline{\mathcal{D}}).$$

Then $(\boldsymbol{Y}^1, \ldots, \boldsymbol{Y}^{\hat{p}}, \boldsymbol{U})$ is a rank-at-most-one feasible solution of the SDP (7) with the same objective value.

(ii) Suppose that $(\boldsymbol{Y}^1, \ldots, \boldsymbol{Y}^{\hat{p}}, \boldsymbol{U})$ is a rank-at-most-one feasible solution of SDP (7) with objective value $\sum_{p=1}^{\hat{p}} \langle \boldsymbol{A}_0^p, \boldsymbol{Y}^p \rangle$. Then there exists a feasible solution $\boldsymbol{X} \in \mathbb{S}^n(\mathcal{E})$ of QCQP (5) satisfying $\boldsymbol{X}^{C_p} = \boldsymbol{Y}^p$ $(p = 1, \ldots, \hat{p})$ with the same objective value.

By Lemmas 2.1, 2.2 and 2.3, we have shown that SDP (2) and QCQP (1) are equivalent to their clique-wise formulations, SDP (7) and SDP (7) with the additional rank-at-most-one condition (8), respectively. Consequently, SDP (2) has a rank-at-most-one optimal solution $\boldsymbol{X}$, which is an optimal solution of QCQP (1), if and only if SDP (7) has a rank-at-most-one optimal solution $(\boldsymbol{Y}^1, \ldots, \boldsymbol{Y}^{\hat{p}}, \boldsymbol{U})$ such that $\boldsymbol{X}^{C_p} = \boldsymbol{Y}^p$ $(p = 1, \ldots, \hat{p})$. This equivalence allows the exactness analysis of the original SDP relaxation (2) to be carried out through the clique-wise formulation (7).

It should be noted that the choice of decompositions of the coefficient matrices $\boldsymbol{A}_k \in \mathbb{S}^n$ into collections $\boldsymbol{A}_k^p \in \mathbb{S}^{C_p}$ $(p = 1, \ldots, \hat{p})$ satisfying (3) does not affect the equivalence of SDP (2) with SDP (7). The choice can, however, affect the local sub-SDP (9) in the single matrix variable $\boldsymbol{Y}^p$, introduced in the next section, where the matrices $\boldsymbol{A}_k^p$ $(k = 0, \ldots, m)$ appear explicitly in the objective and constraint functions $(p = 1, \ldots, \hat{p})$.

3 Local-to-global exactness via clique subproblems

Throughout the remainder of the paper, SDP (7) is referred to as the global SDP. For each clique C_p $(p = 1, \ldots, \hat{p})$, we introduce a local sub-SDP, denoted by SDP (9), which describes the corresponding local component of the global SDP when the local right-hand-side and consistency matrix are fixed. The purpose of this section is to relate the exactness of the global SDP (7) to the exactness of the associated local sub-SDPs.

3.1 Clique-wise subproblems

For $p = 1, \ldots, \hat{p}$, we define a local sub-SDP in the local matrix variable $\boldsymbol{Y}^p$ associated with the clique C_p by fixing a vector $\boldsymbol{\delta}^p = (\delta_1^p, \ldots, \delta_m^p) \in \mathbb{R}^m$ and a consistency matrix $\boldsymbol{U} \in \mathbb{S}^n(\overline{\mathcal{D}})$ as parameters. The parameter δ_k^p represents the distribution of each global constraint right-hand side constant b_k into local constraint right-hand side constants of the sub-SDPs ($k = 1, \ldots, m$), while the other parameter, consistency matrix $\boldsymbol{U}$ specifies the values of the entries of the local matrix variable $\boldsymbol{Y}^p$ shared by different cliques, *i.e.,* it fixes the entries of $\boldsymbol{Y}^p$ on $\mathcal{D}_p$ so as to enforce consistency among overlapping cliques. We now define the corresponding local sub-SDP as follows:

$$\eta_c^p(\boldsymbol{\delta}^p, \boldsymbol{U}) = \inf \left\{ \langle \boldsymbol{A}_0^p, \boldsymbol{Y}^p \rangle : \begin{array}{l} \boldsymbol{Y}^p \in \mathbb{S}_+^{C_p}, \\ \langle \boldsymbol{A}_k^p, \boldsymbol{Y}^p \rangle \trianglelefteq_k \delta_k^p \ (k = 1, \ldots, m), \\ [\boldsymbol{Y}^p]_{ij} = [\boldsymbol{U}]_{ij} \ ((i,j) \in \mathcal{D}_p) \end{array} \right\}. \tag{9}$$

The sub-SDP (9) is the local problem associated with C_p, obtained from the global SDP (7) by fixing parameters $(\boldsymbol{\delta}^p, \boldsymbol{U})$. It will be used to characterize the exactness of SDP (7). If the rank-at-most-one condition $\mathrm{rank}(\boldsymbol{Y}^p) \leq 1$ is added to sub-SDP (9), then we obtain a local sub-QCQP.

In the example illustrated in Figure 1(d), there are four local sub-SDPs associated with C_1, C_2, C_3, C_4. These sub-SDPs are coupled only through the consistency matrix $\boldsymbol{U} \in \mathbb{S}^n(\overline{\mathcal{D}})$ via the consistency constraints $[\boldsymbol{Y}^p]_{ij} = [\boldsymbol{U}]_{ij}$ for $(i,j) \in \mathcal{D}_p$ $(p = 1, \ldots, 4)$, where $\mathcal{D}_1 = \{(2,2),(4,4)\}, \mathcal{D}_2 = \{(2,2)\}, \mathcal{D}_3 = \{(4,4)\}, \mathcal{D}_4 = \{(4,4)\}$.

The connection between the local sub-SDPs (9) $(p = 1, \ldots, \hat{p})$ and the global SDP (7) can be expressed through a bilevel optimization problem. In this representation, the upper level determines the local right-hand-side vectors $\boldsymbol{\delta}^p$ $(p = 1, \ldots, \hat{p})$, and the common consistency matrix $\boldsymbol{U}$, while the lower level consists of the local sub-SDPs (9). The upper-level objective is the sum of the lower-level optimal values $\eta_c^p(\boldsymbol{\delta}^p, \boldsymbol{U})$ $(p = 1, \ldots, \hat{p})$. Define the upper-level problem associated with the global SDP by

$$\widetilde{\eta}_c \ = \ \inf \left\{ \sum_{p=1}^{\hat{p}} \eta_c^p(\boldsymbol{\delta}^p, \boldsymbol{U}) : \begin{array}{l} \eta_c^p(\boldsymbol{\delta}^p, \boldsymbol{U}) < +\infty \quad (p = 1, \ldots, \hat{p}), \\ \displaystyle\sum_{p=1}^{\hat{p}} \delta_k^p \trianglelefteq_k b_k \quad (k = 1, \ldots, m), \\ \boldsymbol{U} \in \mathbb{S}^n(\overline{\mathcal{D}}) \end{array} \right\}. \tag{10}$$

The conditions $\eta_c^p(\boldsymbol{\delta}^p, \boldsymbol{U}) < +\infty$ $(p = 1, \ldots, \hat{p})$ exclude infeasible local subproblems. However, they do not exclude local subproblems with optimal value $-\infty$ at this stage.

Lemma 3.1. *The problem* (10) *has the same optimal value as SDP* (7)*; that is,* $\widetilde{\eta}_c = \eta_c$.

Proof. First, let $(\boldsymbol{Y}^1, \ldots, \boldsymbol{Y}^{\hat{p}}, \boldsymbol{U})$ be any feasible solution of SDP (7). Define

$$\delta_k^p = \langle \boldsymbol{A}_k^p, \boldsymbol{Y}^p \rangle \quad (k = 1, \ldots, m, \ p = 1, \ldots, \hat{p}).$$

Then $(\boldsymbol{\delta}^1, \ldots, \boldsymbol{\delta}^{\hat{p}}, \boldsymbol{U})$ is feasible for problem (10). Since each $\boldsymbol{Y}^p$ is feasible for the corresponding sub-SDP (9),

$$\widetilde{\eta}_c \leq \sum_{p=1}^{\hat{p}} \eta_c^p(\boldsymbol{\delta}^p, \boldsymbol{U}) \leq \sum_{p=1}^{\hat{p}} \langle \boldsymbol{A}_0^p, \boldsymbol{Y}^p \rangle.$$

Taking the infimum over all feasible $(\boldsymbol{Y}^1, \ldots, \boldsymbol{Y}^{\hat{p}}, \boldsymbol{U})$ gives $\widetilde{\eta}_c \leq \eta_c$.

Conversely, fix any feasible solution $(\boldsymbol{\delta}^1, \ldots, \boldsymbol{\delta}^{\hat{p}}, \boldsymbol{U})$ of problem (10). For any feasible solutions $\boldsymbol{Y}^p$ of the corresponding sub-SDPs (9) $(p = 1, \ldots, \hat{p})$, the tuple $(\boldsymbol{Y}^1, \ldots, \boldsymbol{Y}^{\hat{p}}, \boldsymbol{U})$ is feasible for SDP (7). Therefore $\eta_c \leq \sum_{p=1}^{\hat{p}} \langle \boldsymbol{A}_0^p, \boldsymbol{Y}^p \rangle$. Taking the infimum over the feasible sets of the local sub-SDPs yields $\eta_c \leq \sum_{p=1}^{\hat{p}} \eta_c^p(\boldsymbol{\delta}^p, \boldsymbol{U})$. Finally, taking the infimum over all feasible $(\boldsymbol{\delta}^1, \ldots, \boldsymbol{\delta}^{\hat{p}}, \boldsymbol{U})$ in problem (10) gives $\eta_c \leq \widetilde{\eta}_c$. Hence $\widetilde{\eta}_c = \eta_c$. □

3.2 Relation between the global SDP and local sub-SDPs

The following theorem is the main result of the paper. It provides a local-to-global exactness framework for the clique-wise formulation. Specifically, an optimal solution $(\widetilde{\boldsymbol{Y}}^1, \ldots, \widetilde{\boldsymbol{Y}}^{\hat{p}}, \widetilde{\boldsymbol{U}})$ of the global clique-wise SDP (7) induces the local right-hand-side vectors $\tilde{\boldsymbol{\delta}}^p$ $(p = 1, \ldots, \hat{p})$ and the consistency matrix $\widetilde{\boldsymbol{U}}$. The relevant local sub-SDPs (9) are precisely those defined by these induced parameters. If these induced local sub-SDPs admit rank-at-most-one optimal solutions, then exactness of the original SDP relaxation follows.

Theorem 3.2. *Let $(\widetilde{\boldsymbol{Y}}^1, \ldots, \widetilde{\boldsymbol{Y}}^{\hat{p}}, \widetilde{\boldsymbol{U}})$ be an optimal solution of SDP (7). Define*

$$\tilde{\boldsymbol{\delta}}^p \in \mathbb{R}^m, \ \tilde{\delta}_k^p = \langle \boldsymbol{A}_k^p, \widetilde{\boldsymbol{Y}}^p \rangle \ (k = 1, \ldots, m, \ p = 1, \ldots, \hat{p}). \tag{11}$$

Then the following assertions hold:

(i) Each $\widetilde{\boldsymbol{Y}}^p$ is an optimal solution of sub-SDP (9) with the parameters $(\boldsymbol{\delta}^p, \boldsymbol{U}) = (\tilde{\boldsymbol{\delta}}^p, \widetilde{\boldsymbol{U}})$; hence $\langle \boldsymbol{A}_0^p, \widetilde{\boldsymbol{Y}}^p \rangle = \eta_c^p(\tilde{\boldsymbol{\delta}}^p, \widetilde{\boldsymbol{U}})$ $(p = 1, \ldots, \hat{p})$ and $\sum_{p=1}^{\hat{p}} \eta_c^p(\tilde{\boldsymbol{\delta}}^p, \widetilde{\boldsymbol{U}}) = \sum_{p=1}^{\hat{p}} \langle \boldsymbol{A}_0^p, \widetilde{\boldsymbol{Y}}^p \rangle = \eta_c$.

(ii) For $p = 1, \ldots, \hat{p}$, let $\widehat{\boldsymbol{Y}}^p \in \mathbb{S}_+^{C_p}$ be an optimal solution of sub-SDP (9) with the parameters $(\boldsymbol{\delta}^p, \boldsymbol{U}) = (\tilde{\boldsymbol{\delta}}^p, \widetilde{\boldsymbol{U}})$; hence $\langle \boldsymbol{A}_0^p, \widehat{\boldsymbol{Y}}^p \rangle = \eta_c^p(\tilde{\boldsymbol{\delta}}^p, \widetilde{\boldsymbol{U}})$. Then $(\widehat{\boldsymbol{Y}}^1, \ldots, \widehat{\boldsymbol{Y}}^{\hat{p}}, \widetilde{\boldsymbol{U}})$ is an optimal solution of SDP (7), and therefore $\eta_c = \sum_{p=1}^{\hat{p}} \langle \boldsymbol{A}_0^p, \widehat{\boldsymbol{Y}}^p \rangle$.

(iii) Assume that, for $p = 1, \ldots, \hat{p}$, sub-SDP (9) with the parameters $(\boldsymbol{\delta}^p, \boldsymbol{U}) = (\tilde{\boldsymbol{\delta}}^p, \widetilde{\boldsymbol{U}})$ is exact. Then SDP (7) is exact.

Proof. (i) Let $p \in \{1, \ldots, \hat{p}\}$ be fixed. Then $\widetilde{\boldsymbol{Y}}^p \in \mathbb{S}_+^{C_p}$, $\langle \boldsymbol{A}_k^p, \widetilde{\boldsymbol{Y}}^p \rangle \trianglelefteq_k \tilde{\delta}_k^p$, $(k = 1, \ldots, m)$ and $[\widetilde{\boldsymbol{Y}}^p]_{ij} = [\widetilde{\boldsymbol{U}}]_{ij}$ $((i,j) \in \mathcal{D}_p)$ are satisfied. Hence $\widetilde{\boldsymbol{Y}}^p$ is feasible for sub-SDP (9) with the parameters $(\boldsymbol{\delta}^p, \boldsymbol{U}) = (\tilde{\boldsymbol{\delta}}^p, \widetilde{\boldsymbol{U}})$. Assume to the contrary that $\widetilde{\boldsymbol{Y}}^p$ is not optimal. Then there exists a feasible solution $\boldsymbol{Y}^p \in \mathbb{S}_+^{C_p}$ of sub-SDP (9) with the parameters $(\boldsymbol{\delta}^p, \boldsymbol{U}) = (\tilde{\boldsymbol{\delta}}^p, \widetilde{\boldsymbol{U}})$ such that

$$\langle \boldsymbol{A}_0^p, \boldsymbol{Y}^p \rangle < \langle \boldsymbol{A}_0^p, \widetilde{\boldsymbol{Y}}^p \rangle, \ \langle \boldsymbol{A}_k^p, \boldsymbol{Y}^p \rangle \trianglelefteq_k \tilde{\delta}_k^p \ (k = 1, \ldots, m), \ [\boldsymbol{Y}^p]_{ij} = [\widetilde{\boldsymbol{U}}]_{ij} \ ((i,j) \in \mathcal{D}_p).$$

Replacing only the pth block $\widetilde{\boldsymbol{Y}}^p$ in the feasible solution $(\widetilde{\boldsymbol{Y}}^1, \ldots, \widetilde{\boldsymbol{Y}}^{\hat{p}}, \widetilde{\boldsymbol{U}})$ of SDP (7) by $\boldsymbol{Y}^p$, we obtain a feasible solution of SDP (7), which we denote as $(\overline{\boldsymbol{Y}}^1, \ldots, \overline{\boldsymbol{Y}}^{\hat{p}}, \widetilde{\boldsymbol{U}})$, where $\overline{\boldsymbol{Y}}^p = \boldsymbol{Y}^p$ and $\overline{\boldsymbol{Y}}^q = \widetilde{\boldsymbol{Y}}^q$ $(q \neq p)$. Then

$$\sum_{q=1}^{\hat{p}} \langle \boldsymbol{A}_0^q, \overline{\boldsymbol{Y}}^q \rangle = \sum_{q \neq p} \langle \boldsymbol{A}_0^q, \widetilde{\boldsymbol{Y}}^q \rangle + \langle \boldsymbol{A}_0^p, \overline{\boldsymbol{Y}}^p \rangle < \sum_{q=1}^{\hat{p}} \langle \boldsymbol{A}_0^q, \widetilde{\boldsymbol{Y}}^q \rangle$$

holds. This contradicts the optimality of $(\widetilde{\boldsymbol{Y}}^1, \ldots, \widetilde{\boldsymbol{Y}}^{\hat{p}}, \widetilde{\boldsymbol{U}})$ for SDP (7). Therefore $\widetilde{\boldsymbol{Y}}^p$ is an optimal solution of sub-SDP (9) with the parameters $(\boldsymbol{\delta}^p, \boldsymbol{U}) = (\tilde{\boldsymbol{\delta}}^p, \widetilde{\boldsymbol{U}})$.

(ii) By feasibility of each $\widehat{\boldsymbol{Y}}^p \in \mathbb{S}^{C_p}$ for sub-SDP (9), $\langle \boldsymbol{A}_k^p, \widehat{\boldsymbol{Y}}^p \rangle \ \trianglelefteq_k \ \delta_k^p$ $(k = 1, \ldots, m,\ p = 1, \ldots, \hat{p})$. Summing the kth inequalities over $p = 1, \ldots, \hat{p}$, we obtain

$$\sum_{p=1}^{\hat{p}} \langle \boldsymbol{A}_k^p, \widehat{\boldsymbol{Y}}^p \rangle \ \trianglelefteq_k \ \sum_{p=1}^{\hat{p}} \delta_k^p = \sum_{p=1}^{\hat{p}} \langle \boldsymbol{A}_k^p, \widetilde{\boldsymbol{Y}}^p \rangle \trianglelefteq_k \ b_k \quad (k = 1, \ldots, m).$$

We also see that the other consistency constraints $[\widehat{\boldsymbol{Y}}^p]_{ij} = [\widetilde{\boldsymbol{U}}]_{ij}$ $((i,j) \in \mathcal{D}_p, p = 1, \ldots, \hat{p})$ are satisfied by the feasibility of each $\widehat{\boldsymbol{Y}}^p$ for sub-SDP (9) $(p = 1, \ldots, \hat{p})$. Thus $(\widehat{\boldsymbol{Y}}^1, \ldots, \widehat{\boldsymbol{Y}}^{\hat{p}}, \widetilde{\boldsymbol{U}})$ is feasible for SDP (7).

Next, we show its optimality. By Assertion (i), for each $p = 1, \ldots, \hat{p}$, $\widetilde{\boldsymbol{Y}}^p$ is an optimal solution of the same subproblem sub-SDP (9) with the parameters $(\boldsymbol{\delta}^p, \boldsymbol{U}) = (\tilde{\boldsymbol{\delta}}^p, \widetilde{\boldsymbol{U}})$. Since $\widehat{\boldsymbol{Y}}^p$ is also optimal for that subproblem, we have $\langle \boldsymbol{A}_0^p, \widehat{\boldsymbol{Y}}^p \rangle = \langle \boldsymbol{A}_0^p, \widetilde{\boldsymbol{Y}}^p \rangle$ $(p = 1, \ldots, \hat{p})$. Summing these equalities over p yields $\sum_{p=1}^{\hat{p}} \langle \boldsymbol{A}_0^p, \widehat{\boldsymbol{Y}}^p \rangle = \sum_{p=1}^{\hat{p}} \langle \boldsymbol{A}_0^p, \widetilde{\boldsymbol{Y}}^p \rangle$. The right-hand side is the optimal value of SDP (7), because $(\widetilde{\boldsymbol{Y}}^1, \ldots, \widetilde{\boldsymbol{Y}}^{\hat{p}}, \widetilde{\boldsymbol{U}})$ is optimal for SDP (7). Therefore, $(\widehat{\boldsymbol{Y}}^1, \ldots, \widehat{\boldsymbol{Y}}^{\hat{p}}, \widetilde{\boldsymbol{U}})$ is also an optimal solution of SDP (7).

(iii) By assumption, $(\widetilde{\boldsymbol{Y}}^1, \ldots, \widetilde{\boldsymbol{Y}}^{\hat{p}}, \widetilde{\boldsymbol{U}})$ is an optimal solution of SDP (7). It suffices to show that SDP (7) has a rank-at-most-one optimal solution. By Assertion (i), each $\widetilde{\boldsymbol{Y}}^p$ is an optimal solution of sub-SDP (9) with the parameters $(\boldsymbol{\delta}^p, \boldsymbol{U}) = (\tilde{\boldsymbol{\delta}}^p, \widetilde{\boldsymbol{U}})$. By the exactness assumption on this sub-SDP, there exists a rank-at-most-one optimal solution $\widehat{\boldsymbol{Y}}^p$ of this sub-SDP. By Assertion (ii), $(\widehat{\boldsymbol{Y}}^1, \ldots, \widehat{\boldsymbol{Y}}^{\hat{p}}, \widetilde{\boldsymbol{U}})$ is an optimal solution of SDP (7). Since $\mathrm{rank}(\widehat{\boldsymbol{Y}}^p) \leq 1$ $(p = 1, \ldots, \hat{p})$, this optimal solution is rank-at-most-one. Hence, SDP (7) is exact. □

3.3 Diagonal consistency assumption

Theorem 3.2(iii) requires exactness of the local sub-SDPs (9) with the parameters $(\boldsymbol{\delta}^p, \boldsymbol{U}) = (\tilde{\boldsymbol{\delta}}^p, \widetilde{\boldsymbol{U}})$ induced by an optimal solution of the global SDP (7) $(p = 1, \ldots, \hat{p})$. These parameters are not known a priori. Therefore, in order to use the theorem together with local exactness results, it is necessary to identify classes of local sub-SDPs whose exactness can be guaranteed for all admissible parameter values, or under conditions that can be checked without solving the global SDP.

A critical obstruction for identifying such classes of sub-SDPs is caused by the unknown consistency matrix $\widetilde{\boldsymbol{U}}$. To see this, focus only on the consistency constraints in the sub-SDP (9) with the parameter $(\boldsymbol{\delta}^p, \boldsymbol{U}) = (\tilde{\boldsymbol{\delta}}^p, \widetilde{\boldsymbol{U}})$,

$$[\boldsymbol{Y}^p]_{ij} = [\widetilde{\boldsymbol{U}}]_{ij} \quad ((i,j) \in \mathcal{D}_p).$$

For these constraints to be satisfied by an unknown rank-at-most-one optimal solution $\widetilde{\boldsymbol{Y}}^p$ of the sub-SDP (9), the entries of $\widetilde{\boldsymbol{U}}$ must satisfy the necessary condition

$$[\boldsymbol{u}]_i [\boldsymbol{u}]_j = [\widetilde{\boldsymbol{U}}]_{ij} \quad ((i,j) \in \mathcal{D}_p) \quad \text{for some } \boldsymbol{u} \in \mathbb{R}^{C_p}. \tag{12}$$

Thus, when the consistency set $\mathcal{D}_p$ contains off-diagonal entries, the consistency matrix $\widetilde{\boldsymbol{U}}$ must satisfy the nontrivial product relations (12). If (12) is not satisfied, the sub-SDP (9) cannot have a rank-at-most-one feasible solution. Also, (12) cannot be verified in advance since the consistency matrix $\widetilde{\boldsymbol{U}}$ is unknown.

To avoid this obstruction, we impose the following diagonal consistency assumption:

$$\mathcal{D}_p \subseteq \{(i,i) : i \in N\} \quad (p = 1, \ldots, \hat{p}). \tag{13}$$

Equivalently, any two distinct maximal cliques intersect in at most one node,

$$|C_p \cap C_q| \leq 1 \quad (p \neq q).$$

A chordal graph satisfying this assumption is called a block-clique graph [15, 18]. Under the diagonal consistency assumption, (12) reduces to the simpler necessary condition

$$([\boldsymbol{u}]_i)^2 = [\widetilde{\boldsymbol{U}}]_{ii} \quad ((i,i) \in \mathcal{D}_p) \quad \text{for some } \boldsymbol{u} \in \mathbb{R}^{C_p}.$$

Since $[\widetilde{\boldsymbol{U}}]_{ii} = [\widetilde{\boldsymbol{Y}}^p]_{ii} \geq 0$ for every $(i,i) \in \mathcal{D}_p$, this condition is always satisfied, and the obstruction (12) on the unknown consistency matrix $\widetilde{\boldsymbol{U}}$ caused by off-diagonal entries in $\mathcal{D}_p$ has been removed.

It is important to note that the block-clique assumption does not imply exactness of the sub-SDP (9). Its role is to remove the off-diagonal product obstruction on $\widetilde{\boldsymbol{U}}$ caused by (12) when applying Theorem 3.2(iii). Thus the block-clique assumption makes it possible to formulate local exactness results in Section 4 without having to verify unknown off-diagonal product relations (12) on the consistency matrix $\widetilde{\boldsymbol{U}}$.

The diagonal consistency assumption (13) leads to the following three representative cases of the consistency set $\mathcal{D}_p$:

(0) $\mathcal{D}_p = \emptyset$.

(I) $\mathcal{D}_p = \{(i,i)\}$ for some $i \in C_p$.

(II) $\{(i,i) : i \in C_p\} \supseteq \mathcal{D}_p$ with multiple diagonal indices.

In case (0), the local matrix variable $\boldsymbol{Y}^p$ is not subject to any consistency constraints. The corresponding subproblem is therefore independent of the other local sub-SDPs at the level of matrix variables, although coupling may still occur through the right-hand-side parameter vector $\boldsymbol{\delta}^p$.

In case (I), the consistency constraints fix one diagonal entry of the local matrix variable $\boldsymbol{Y}^p$. This occurs when the clique C_p shares a single node with another clique, as illustrated in the cases of $\mathcal{D}_2$, $\mathcal{D}_3$, and $\mathcal{D}_4$ in Figure 1(d), where $[\boldsymbol{Y}^2]_{22} = [\boldsymbol{U}]_{22}$, $[\boldsymbol{Y}^3]_{44} = [\boldsymbol{U}]_{44}$, and $[\boldsymbol{Y}^4]_{44} = [\boldsymbol{U}]_{44}$ are consistency constraints, respectively.

In case (II), multiple diagonal entries of the local matrix variable $\boldsymbol{Y}^p$ are fixed through the consistency matrix $\boldsymbol{U}$. This may occur when C_p intersects several other cliques, each in a single node. This is illustrated by the case of C_1 in Figure 1(d), where the consistency constraints are $[\boldsymbol{Y}^1]_{22} = [\boldsymbol{U}]_{22}$ and $[\boldsymbol{Y}^1]_{44} = [\boldsymbol{U}]_{44}$.

In the next section, we identify three classes of local sub-QCQPs whose SDP relaxations admit rank-at-most-one optimal solutions under this diagonal consistency assumption (13). These classes provide local building blocks for the local-to-global exactness framework.

4 Sub-QCQPs with exact SDP relaxations

In this section, we present three classes of local sub-QCQPs whose corresponding local sub-SDPs are exact. These are: convex sub-QCQPs, sub-QCQPs characterized by sign-pattern conditions, and separable sub-QCQPs with a limited number of constraints. Sections 4.1, 4.2, and 4.3 discuss these three classes, respectively. For example, in Figure 1(d), different classes can be assigned to different clique-wise subproblems. Further examples will be presented in Section 5.

The first two classes have a parameter-independent character: their local SDP relaxations are exact for every right-hand-side vector and consistency matrix. By contrast, the exactness of the third class may depend on the right-hand-side vector induced from an optimal solution of the global SDP. This distinction is important when the local exactness results are combined through Theorem 3.2(iii).

For notational simplicity, we fix a clique C_p throughout this section and suppress the superscript p. Thus we write $C_p = C = \{1, \ldots, \ell\}$, $\boldsymbol{A}_k^p = \boldsymbol{A}_k$, $\boldsymbol{\delta}^p = \boldsymbol{\delta}$, and $\mathcal{D}_p = \mathcal{D}$. Under the diagonal consistency assumption (13), we have $\mathcal{D} \subseteq \{(i,i) : 1 \le i \le \ell\}$. Then the local sub-SDP is written as

$$\eta(\boldsymbol{\delta}, \boldsymbol{U}) = \inf \left\{ \langle \boldsymbol{A}_0, \boldsymbol{Y} \rangle : \begin{array}{l} \boldsymbol{Y} \in \mathbb{S}^\ell_+, \\ \langle \boldsymbol{A}_k, \boldsymbol{Y} \rangle \trianglelefteq_k \delta_k \ (k = 1, \ldots, m), \\ [\boldsymbol{Y}]_{ii} = U_{ii} \ ((i,i) \in \mathcal{D}) \end{array} \right\}. \tag{14}$$

We call the problem obtained by adding $\mathrm{rank}(\boldsymbol{Y}) \le 1$ to SDP (14) the local sub-QCQP.

4.1 Sub-QCQPs characterized by convexity

Theorem 4.1. *Assume that '$\trianglelefteq_k$' = '$\le$' $(1 \le k \le m)$. Suppose that one of the following conditions holds:*

(0) $\mathcal{D} = \emptyset$ and $\boldsymbol{A}_k$ $(k = 0, \ldots, m)$ are positive semidefinite.

(I) $\mathcal{D} = \{(i,i)\}$ for some $i \in C$ and $\boldsymbol{A}_k^{C \setminus \{i\}}$ $(k = 0, \ldots, m)$ are positive semidefinite.

Then, sub-SDP (14) is exact, i.e., sub-SDP (14) has a rank-at-most-one optimal solution whenever it has an optimal solution, for every $\boldsymbol{\delta} \in \mathbb{R}^m$ and $\boldsymbol{U} \in \mathbb{S}^\ell(\mathcal{D})$.

Proof. Let $\boldsymbol{\delta} \in \mathbb{R}^m$ and $\boldsymbol{U} \in \mathbb{S}^\ell(\mathcal{D})$ be fixed arbitrarily. If sub-SDP (14) has no optimal solution, there is nothing to prove. Suppose that it has an optimal solution. In case (I), $U_{ii} \ge 0$ $((i,i) \in \mathcal{D})$ by positive semidefiniteness of $\boldsymbol{Y}$. The QCQP obtained from sub-SDP (14) with adding the rank constraint $\mathrm{rank}(\boldsymbol{Y}) \le 1$ can be written as

$$\zeta(\boldsymbol{\delta}, \boldsymbol{U}) = \inf \left\{ \langle \boldsymbol{A}_0, \boldsymbol{y}\boldsymbol{y}^T \rangle : \begin{array}{ll} \langle \boldsymbol{A}_k, \boldsymbol{y}\boldsymbol{y}^T \rangle \le \delta_k & (k = 1, \ldots, m), \\ y_i = +\sqrt{U_{ii}} \text{ or } y_i = -\sqrt{U_{ii}} & ((i,i) \in \mathcal{D}) \end{array} \right\}. \tag{15}$$

In case (0), $\mathcal{D} = \emptyset$, and the quadratic function $\langle \boldsymbol{A}_k, \boldsymbol{y}\boldsymbol{y}^T \rangle$ is convex in $\boldsymbol{y} \in \mathbb{R}^\ell$ $(k = 0, \ldots, m)$.

In case (I), for simplicity of notation, assume $i = \ell$. Since $\langle \boldsymbol{A}_k, \boldsymbol{y}\boldsymbol{y}^T \rangle = \langle \boldsymbol{A}_k, (-\boldsymbol{y})(-\boldsymbol{y})^T \rangle$ for every $k = 0, \ldots, m$, the two choices $y_\ell = +\sqrt{U_{\ell\ell}}$ and $y_\ell = -\sqrt{U_{\ell\ell}}$ are equivalent up to the global sign change $\boldsymbol{y} \mapsto -\boldsymbol{y}$. Hence, we may fix $y_\ell = +\sqrt{U_{\ell\ell}}$. After this fixing, the quadratic function $\langle \boldsymbol{A}_k, \boldsymbol{y}\boldsymbol{y}^T \rangle$ is convex in the remaining variables $(y_1, \ldots, y_{\ell-1})$ since $\boldsymbol{A}_k^{C \setminus \{\ell\}} \succeq 0$ $(k = 0, \ldots, m)$.

Thus, in both cases, sub-SDP (14) can be viewed as the SDP relaxation of a convex QCQP. It is well-known that the SDP relaxation is exact for such convex QCQPs. See, for example, [26] and [7, Section 4.2]. Therefore, sub-SDP (14) has a rank-at-most-one optimal solution whenever it has an optimal solution. □

We note that, except for the assumptions in (0) and (I), no additional assumption is required on the right-hand-side vector $\boldsymbol{\delta}$ or the consistency matrix $\boldsymbol{U}$.

Remark 4.2. The argument above does not extend directly to the case $|\mathcal{D}| \geq 2$. For example, suppose that $\mathcal{D} = \{(1,1),(2,2)\}$. Fixing $y_1 = +\sqrt{U_{11}}$ and $y_2 = +\sqrt{U_{22}}$ is not, in general, equivalent to fixing $y_1 = +\sqrt{U_{11}}$ and $y_2 = -\sqrt{U_{22}}$. Thus the rank-constrained problem (15) cannot be reduced consistently to a single convex QCQP by fixing the signs of the variables indexed by $\mathcal{D}$. This is why the convex case in Theorem 4.1 is restricted to $\mathcal{D} = \emptyset$ or $|\mathcal{D}| = 1$.

4.2 Sub-QCQPs characterized by sign-pattern conditions

We next consider a class of generally nonconvex sub-QCQPs whose SDP relaxations are exact under suitable sign-pattern conditions on the data matrices $\boldsymbol{A}_k$ $(0 \leq k \leq m)$. Throughout this section, we assume that $\trianglelefteq_k = \text{`}\leq\text{'}$ $(1 \leq k \leq m)$. As in the convex case, the exactness result for this class holds for every right-hand-side vector $\boldsymbol{\delta}$ and consistency matrix $\boldsymbol{U}$. Let $G(L,\mathcal{F})$ denote the aggregate sparsity pattern graph with the node set $L = \{1,\ldots,\ell\}$ and

$$\mathcal{F} \;=\; \{(i,j) \in L \times L : i \neq j,\ [\boldsymbol{A}_k]_{ij} \neq 0 \text{ for some } k \in \{0,1,\ldots,m\}\}\,.$$

We assume that $\mathcal{D} \subseteq \{(i,i) : i \in C\}$ and $0 \leq |\mathcal{D}| \leq \ell$ (case (II)). Note that the equality constraints $[\boldsymbol{Y}]_{ii} = U_{ii}$ $((i,i) \in \mathcal{D})$ in sub-SDP (14) and the corresponding sub-QCQP can be replaced by the inequality constraints $[\boldsymbol{Y}]_{ii} \leq U_{ii}$, $-[\boldsymbol{Y}]_{ii} \leq -U_{ii}$ $((i,i) \in \mathcal{D})$. These inequality constraints, however, do not affect the aggregate sparsity pattern graph $G(L,\mathcal{F})$ of their data matrices.

For every $(i,j) \in \mathcal{F}$, define

$$\sigma_{ij} = \begin{cases} +1 & \text{if } [\boldsymbol{A}_k]_{ij} \geq 0 \text{ for all } k \in \{0,1,\ldots,m\}, \\ -1 & \text{if } [\boldsymbol{A}_k]_{ij} \leq 0 \text{ for all } k \in \{0,1,\ldots,m\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{F_1,\ldots,F_r\}$ denote a cycle basis for $G(L,\mathcal{F})$. The following theorem and its corollary follow directly from [27, Theorem 2 and Corollary 1], respectively.

Theorem 4.3. *([27, Theorem 2]) Assume that*

(i) $\sigma_{ij} \in \{-1,1\}$ *for every* $(i,j) \in \mathcal{F}$,

(ii) $\prod_{(i,j)\in F_s} \sigma_{ij} = (-1)^{|F_s|}$ *for every* $s = 1,\ldots,r$.

Then sub-SDP (14) is exact for every $\boldsymbol{\delta} \in \mathbb{R}^m$ *and* $\boldsymbol{U} \in \mathbb{S}^\ell(\mathcal{D})$.

Corollary 4.4. *([27, Corollary 1]) Assume that one of the following conditions holds:*

(i) The graph $G(L,\mathcal{F})$ *is arbitrary and* $\sigma_{ij} = -1$ *for every* $(i,j) \in \mathcal{F}$ *(or equivalently all off-diagonal elements of* $\boldsymbol{A}_k$ *are nonpositive, i.e.,* $\boldsymbol{A}_k$ *is a symmetric Z-matrix* $(0 \leq k \leq m)$*).*

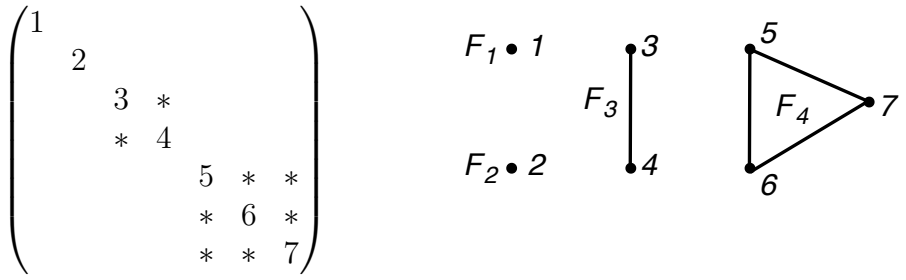


Figure 2: An example of the aggregate sparsity pattern matrix of a block diagonal (or separable) QCQP and its aggregate sparsity pattern graph $G(L,\mathcal{F})$ with node set $L = \{1,\ldots,7\}$ and edge set $\mathcal{F} = \{(3,4),(5,6),(5,7),(6,7)\}$. The maximal cliques are $F_1 = \{1\}, F_2 = \{2\}, F_3 = \{3,4\}$ and $F_4 = \{5,6,7\}$, which are disjoint.

(ii) The graph $G(L,\mathcal{F})$ is a forest and $\sigma_{ij} \in \{-1,1\}$ for every $(i,j) \in \mathcal{F}$.

(iii) The graph $G(L,\mathcal{F})$ is bipartite and $\sigma_{ij} = 1$ for every $(i,j) \in \mathcal{F}$.

Then sub-SDP (14) is exact for every $\boldsymbol{\delta} \in \mathbb{R}^m$ and $\boldsymbol{U} \in \mathbb{S}^\ell(\mathcal{D})$. The condition (i) originally was proposed in [17] for SDP exactness.

The theorem and corollary above establish exactness for every local sub-SDP in this sign-pattern class. Thus, under the diagonal consistency assumption (13), this class is parameter-independent: once the sign-pattern conditions are satisfied, no further assumption is required on the right-hand-side vector $\boldsymbol{\delta}$ or the consistency matrix $\boldsymbol{U}$.

4.3 Separable sub-QCQPs characterized by a limited number of constraints

We now consider a class of separable sub-QCQPs whose objective and constraint functions share a separable structure. In contrast to the classes considered in Sections 4.1 and 4.2, exactness of the corresponding SDP relaxations is not guaranteed by structural properties. It depends on the number of constraints and the right-hand-side vector $\boldsymbol{\delta}$ induced by the global problem. Exactness of the SDP relaxation for this class follows from a rank argument ([24, Theorem 2.2]) that depends on the number of constraints relative to the number of separable blocks. This dependence explains why the class must be treated separately: when a separable subproblem appears as a clique-wise component of the global SDP, the relevant right-hand-side vector $\boldsymbol{\delta}$ is not chosen independently, but is induced by an optimal solution of the global problem.

To describe the sub-SDP corresponding to a separable sub-QCQP, we impose the following condition on sub-SDP (14):

(A) All the data matrices $\boldsymbol{A}_k$ $(k = 0,\ldots,m)$ share a common block-diagonal structure. We denote the qth diagonal block of each $\boldsymbol{A}_k \in \mathbb{S}^\ell$ by $\boldsymbol{B}_k^q = \boldsymbol{A}_k^{F_q} \in \mathbb{S}^{F_q}$ and the corresponding qth diagonal block of the variable matrix $\boldsymbol{Y} \in \mathbb{S}^\ell$ by $\boldsymbol{W}^q = \boldsymbol{Y}^{F_q} \in \mathbb{S}^{F_q}$ $(q = 1,\ldots,\hat{q})$, where $\{F_q : q = 1,\ldots,\hat{q}\}$ is a partition of $C = \{1,\ldots,\ell\}$ such that $i < j$ if $i \in F_q$, $j \in F_r$ and $q < r$. We simply write $\boldsymbol{A}_k = \mathrm{diag}(\boldsymbol{B}_k^1,\ldots,\boldsymbol{B}_k^{\hat{q}})$ and $\boldsymbol{Y} = \mathrm{diag}(\boldsymbol{W}^1,\ldots,\boldsymbol{W}^{\hat{q}})$.

In this case, sub-SDP (14) possesses a block-diagonal sparsity structure. Its aggregate sparsity pattern graph itself is chordal and consists of $\hat{q}$ disjoint cliques F_q $(q = 1, \ldots, \hat{q})$. See Figure 2, where $\ell = 7$, $\hat{q} = 4$, $F_1 = \{1\}, F_2 = \{2\}, F_3 = \{3, 4\}$ and $F_4 = \{5, 6, 7\}$.

Let $\mathcal{D}_q = \mathcal{D} \cap (F_q \times F_q)$, $(q = 1, \ldots, \hat{q})$, where the consistency set $\mathcal{D} \subseteq \{(i, i) : 1 \leq i \leq \ell\}$ is given for sub-SDP (14). Then we can convert sub-SDP (14) further into a clique-wise SDP as follows.

$$\eta(\boldsymbol{\delta}, \boldsymbol{U}) = \inf \left\{ \sum_{q=1}^{\hat{q}} \langle \boldsymbol{B}_0^q, \boldsymbol{W}^q \rangle : \begin{array}{l} \boldsymbol{W}^q \in \mathbb{S}_+^{F_q} \ (q = 1, \ldots, \hat{q}), \\ \sum_{q=1}^{\hat{q}} \langle \boldsymbol{B}_k^q, \boldsymbol{W}^q \rangle \preceq_k \delta_k \ (k = 1, \ldots, m), \\ [\boldsymbol{W}^q]_{ii} = U_{ii} \ ((i, i) \in \mathcal{D}_q, \ q = 1, \ldots, \hat{q}) \end{array} \right\}. \tag{16}$$

Under Condition (A), Lemma 2.1 shows that sub-SDP (14) is equivalent to SDP (16). SDP (16) is interpreted as a separable problem because the matrix variables $\boldsymbol{W}^q \in \mathbb{S}^{F_q}$ $(q = 1, \ldots, \hat{q})$ do not overlap, *i.e.*, $F_q \cap F_r = \emptyset$ $(q \neq r)$.

Theorem 4.5. *Assume that Condition (A) is satisfied, so that sub-SDP (14) is equivalent to SDP (16). Let $\boldsymbol{\delta} \in \mathbb{R}^m$ and $\boldsymbol{U} \in \mathbb{S}^\ell(\mathcal{D})$. Define*

$$\mu = \textit{the number of elements in } \{(i, i) \in \mathcal{D} : U_{ii} > 0\}.$$

Assume that

(B) For every optimal solution $(\boldsymbol{W}^1, \ldots, \boldsymbol{W}^{\hat{q}})$ of SDP (16), at least $m + \mu - 1$ members of the following collection are nonzero:

$$\{\boldsymbol{W}^q : q = 1, \ldots, \hat{q}\} \ \cup \ \left\{ \delta_k - \sum_{q=1}^{\hat{q}} \langle \boldsymbol{B}_k^q, \boldsymbol{W}^q \rangle : k = 1, \ldots, m \right\}.$$

Then SDP (14) is exact.

Proof. By the definition of exactness, it suffices to show that if sub-SDP (14) has an optimal solution, then it has a rank-at-most-one optimal solution. Assume that sub-SDP (14) has an optimal solution. Under Condition (A), sub-SDP (14) is equivalent to SDP (16); hence SDP (16) also has an optimal solution. Introducing auxiliary variables $s_k \in \mathbb{R}$ $(k = 1, \ldots, m)$, we rewrite SDP (16) as

$$\eta(\boldsymbol{\delta}, \boldsymbol{U}) = \inf \left\{ \sum_{q=1}^{\hat{q}} \langle \boldsymbol{B}_0^q, \boldsymbol{W}^q \rangle : \begin{array}{l} \boldsymbol{W}^q \in \mathbb{S}_+^{F_q} \quad (q = 1, \ldots, \hat{q}), \\ 0 \preceq_k s_k \quad (k = 1, \ldots, m), \\ \sum_{q=1}^{\hat{q}} \langle \boldsymbol{B}_k^q, \boldsymbol{W}^q \rangle + s_k = \delta_k \quad (k = 1, \ldots, m), \\ [\boldsymbol{W}^q]_{ii} = U_{ii} \quad ((i, i) \in \mathcal{D}_q, \ q = 1, \ldots, \hat{q}) \end{array} \right\}. \tag{17}$$

Here $s_k = \delta_k - \sum_{q=1}^{\hat{q}} \langle \boldsymbol{B}_k^q, \boldsymbol{W}^q \rangle$ $(k = 1, \ldots, m)$.

If $U_{ii} = 0$ for some equality constraint $[\boldsymbol{W}^q]_{ii} = U_{ii}$, then positive semidefiniteness of $\boldsymbol{W}^q$ implies that the ith row and column of $\boldsymbol{W}^q$ are zero. Thus we may eliminate the ith row and column from $\boldsymbol{W}^q$ and $\boldsymbol{B}_k^q$ $(k = 0, \ldots, m)$, and also eliminate (i, i) from $\mathcal{D}$. After

carrying out this reduction for all such diagonal constraints, the remaining SDP (17) has m linear equations and μ positive diagonal equality constraints.

Since the reduced SDP has an optimal solution, we can apply Theorem 2.2 of [24], which provides a rank bound for feasible solutions of an SDP, for the existence of an optimal solution $(\widetilde{\boldsymbol{W}}^1, \ldots, \widetilde{\boldsymbol{W}}^{\hat{q}}, \bar{s}_1, \ldots, \bar{s}_m)$ satisfying

$$\sum_{q \in Q} \frac{\mathrm{rank}(\widetilde{\boldsymbol{W}}^q)(\mathrm{rank}(\widetilde{\boldsymbol{W}}^q)+1)}{2} + \sum_{k \in K} 1 \leq m + \mu, \tag{18}$$

where $Q = \{q : \widetilde{\boldsymbol{W}}^q \neq \boldsymbol{O}\}$ and $K = \{k : \bar{s}_k \neq 0\}$. Suppose, to the contrary, that $\mathrm{rank}(\widetilde{\boldsymbol{W}}^r) \geq 2$ for some $r \in Q$. Then

$$\frac{\mathrm{rank}(\widetilde{\boldsymbol{W}}^r)(\mathrm{rank}(\widetilde{\boldsymbol{W}}^r)+1)}{2} \geq 3.$$

By Condition (B), at least $m + \mu - 1$ elements among the matrices $\widetilde{\boldsymbol{W}}^q$ and the residuals $\bar{s}_k = \delta_k - \sum_{q=1}^{\hat{q}} \langle \boldsymbol{B}_k^q, \widetilde{\boldsymbol{W}}^q \rangle$ $(k = 1, \ldots, m)$ are nonzero. Hence $|Q \setminus \{r\}| + |K| \geq m + \mu - 2$. Therefore,

$$\begin{aligned} m + \mu + 1 &= 3 + (m + \mu - 2) \\ &\leq \frac{\mathrm{rank}(\widetilde{\boldsymbol{W}}^r)(\mathrm{rank}(\widetilde{\boldsymbol{W}}^r)+1)}{2} + \sum_{q \in Q \setminus \{r\}} \frac{\mathrm{rank}(\widetilde{\boldsymbol{W}}^q)(\mathrm{rank}(\widetilde{\boldsymbol{W}}^q)+1)}{2} + \sum_{k \in K} 1, \end{aligned}$$

which contradicts (18). Thus $\mathrm{rank}(\widetilde{\boldsymbol{W}}^q) \leq 1$ $(q = 1, \ldots, \hat{q})$.

By Lemma 2.1(ii), there exists a rank-at-most-one completion $\widetilde{\boldsymbol{Y}} \in \mathbb{S}^\ell_+$ satisfying $\widetilde{\boldsymbol{Y}}^{F_q} = \widetilde{\boldsymbol{W}}^q$ $(q = 1, \ldots, \hat{q})$. Since SDP (16) is equivalent to sub-SDP (14) under Condition (A), this $\widetilde{\boldsymbol{Y}}$ is a rank-at-most-one optimal solution of sub-SDP (14). Hence SDP (14) is exact. □

It should be noted that Condition (B) involves $\boldsymbol{U} \in \mathbb{S}^\ell(\mathcal{D})$, which determines μ, and $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_m) \in \mathbb{R}^m$. Thus, exactness of sub-SDP (14) depends not only on the number m of constraints, but also on the right-hand-side vector $\boldsymbol{\delta} \in \mathbb{R}^m$ and the consistency matrix $\boldsymbol{U} \in \mathbb{S}^\ell(\mathcal{D})$. In Theorem 3.2, these are not free parameters: they are induced by an optimal solution $(\widetilde{\boldsymbol{Y}}^1, \ldots, \widetilde{\boldsymbol{Y}}^{\hat{p}}, \widetilde{\boldsymbol{U}})$ of the global SDP (7), with $\boldsymbol{\delta}$ corresponding to one of the $\tilde{\boldsymbol{\delta}}^p$ $(p = 1, \ldots, \hat{p})$, and $\boldsymbol{U}$ corresponding to $\widetilde{\boldsymbol{U}}$. Consequently, when SDP (16) appears as a local sub-SDP within the global SDP, its exactness may depend on the behavior of the other local sub-SDPs. This dependence distinguishes the present separable class from the parameter-independent classes based on convexity or sign-pattern conditions.

Remark 4.6. Theorem 4.5 can be compared with the result in [22], where the following assumptions were imposed:

$$\begin{aligned} &m \leq \hat{q} + 1,\ \mathcal{D}_q = \emptyset\ (q = 1, \ldots, \hat{q})\ \text{(hence } \mu = 0), \\ &\boldsymbol{W}^q \neq \boldsymbol{O}\ (1 \leq q \leq \hat{q})\ \text{for every optimal solution } (\boldsymbol{W}^1, \ldots, \boldsymbol{W}^{\hat{q}}). \end{aligned}$$

Under these assumptions, (18) implies

$$m - 1 \leq \hat{q} \leq \sum_{q=1}^{\hat{q}} \frac{\mathrm{rank}(\widetilde{\boldsymbol{W}}^q)(\mathrm{rank}(\widetilde{\boldsymbol{W}}^q)+1)}{2} \leq m,$$

and hence either $\hat{q} = m - 1$ or $\hat{q} = m$, that is, either $m = \hat{q} + 1$ or $m = \hat{q}$. In contrast, Condition (B) of Theorem 4.5 considerably relaxes this restriction.

Even in case $\mathcal{D}_q = \emptyset$ $(q = 1, \ldots, \hat{q})$, whether Condition (B) holds depends on the right-hand-side vector $\boldsymbol{\delta}$ and the relations '$\trianglelefteq_k$' $(1 \leq k \leq m)$. When $m + |\mathcal{D}| \leq 2$, the conclusion of Theorem 4.5 holds without any additional assumptions on $\boldsymbol{\delta}$, the relations '$\trianglelefteq_k$', and $\boldsymbol{B}_k^q$ $(q = 1, \ldots, \hat{q},\ k = 0, \ldots, m)$, and the number $\hat{q}$ of the separable blocks as shown below. (This result is known; see [2, 25, 29].)

Corollary 4.7. *Assume that* $m + |\mathcal{D}| \leq 2$.

(i) SDP (16) is exact.

(ii) SDP (14) is exact.

Proof. Since (ii) follows from (i) by taking the single block $F_1 = C$, we only prove (i). If Condition (B) of Theorem 4.5 holds, then the conclusion follows immediately. Suppose that Condition (B) fails. Since $\mu \leq |\mathcal{D}|$ and $m + |\mathcal{D}| \leq 2$, we have $m + \mu \leq 2$. If $m + \mu \leq 1$, then Condition (B) cannot fail. Hence the failure of Condition (B) implies $m + \mu = 2$, and there exists an optimal solution $(\widetilde{\boldsymbol{W}}^1, \ldots, \widetilde{\boldsymbol{W}}^{\hat{q}})$ of SDP (16) for which at most $m + \mu - 2 = 0$ elements among the matrices $\widetilde{\boldsymbol{W}}^q$ and the residuals are nonzero. Therefore $\widetilde{\boldsymbol{W}}^q = \boldsymbol{O}$ $(q = 1, \ldots, \hat{q})$, and hence $\mathrm{rank}(\widetilde{\boldsymbol{W}}^q) = 0 \leq 1$ $(q = 1, \ldots, \hat{q})$. This proves the result. □

4.4 Preserving the exactness of sub-SDP (14) by adding dependent inequality constraints

The following theorem establishes a preservation result for a family of sub-SDPs with varying right-hand-side vectors. Assume that sub-SDP (14) has a rank-at-most-one optimal solution for every right-hand-side vector whenever its optimal value is finite and attained. Then inequality constraints whose coefficient matrices lie in the conic hull of the coefficient matrices of the existing constraints may be added without destroying exactness. Thus, the resulting modified sub-SDP is exact.

Theorem 4.8. *Assume that* $\trianglelefteq_k =$ '$\leq$' $(k = 1, \ldots, m)$ *and* $\boldsymbol{U} \in \mathbb{S}^\ell(\mathcal{D})$. *Suppose that, for every* $\boldsymbol{\delta}' \in \mathbb{R}^m$, *if* $-\infty < \eta(\boldsymbol{\delta}', \boldsymbol{U}) < \infty$, *then SDP (14) with* $\boldsymbol{\delta} = \boldsymbol{\delta}'$ *has a rank-at-most-one optimal solution. Let* $\boldsymbol{\delta} \in \mathbb{R}^m$ *and* $\delta_j \in \mathbb{R}$ $(j = m + 1, \ldots, m')$ *be fixed, where* $m < m'$. *Assume that*

$$\boldsymbol{A}_j \in \mathrm{cone}\{\boldsymbol{A}_k : k = 1, \ldots, m\} \equiv \left\{ \sum_{k=1}^{m} \alpha_k \boldsymbol{A}_k : \alpha_k \geq 0 \ (k = 1, \ldots, m) \right\} \quad (j = m + 1, \ldots, m').$$

Then the modified SDP obtained from SDP (14) by adding the constraints

$$\langle \boldsymbol{A}_j, \boldsymbol{Y} \rangle \leq \delta_j \ (j = m + 1, \ldots, m')$$

is exact.

Proof. Let $\widetilde{\boldsymbol{Y}} \in \mathbb{S}^{\ell}_{+}$ be an optimal solution of the modified SDP. Define

$$\delta'_k = \langle \boldsymbol{A}_k, \widetilde{\boldsymbol{Y}} \rangle \ (k = 1, \ldots, m), \ \boldsymbol{\delta}' = (\delta'_1, \ldots, \delta'_m).$$

Then $\delta'_k \leq \delta_k$ $(k = 1, \ldots, m)$, and $\widetilde{\boldsymbol{Y}}$ is feasible for SDP (14) with right-hand-side vector $\boldsymbol{\delta}'$. For each $j = m+1, \ldots, m'$, take coefficients $\alpha_{jk} \geq 0$ $(k = 1, \ldots, m)$ such that $\boldsymbol{A}_j = \sum_{k=1}^{m} \alpha_{jk} \boldsymbol{A}_k$. We show that all the added constraints are redundant for SDP (14) with right-hand-side vector $\boldsymbol{\delta}'$. Indeed, if $\boldsymbol{Y} \in \mathbb{S}^{\ell}_{+}$ satisfies $\langle \boldsymbol{A}_k, \boldsymbol{Y} \rangle \leq \delta'_k \quad (k = 1, \ldots, m)$, then, for every $j = m+1, \ldots, m'$,

$$\langle \boldsymbol{A}_j, \boldsymbol{Y} \rangle = \sum_{k=1}^{m} \alpha_{jk} \langle \boldsymbol{A}_k, \boldsymbol{Y} \rangle \leq \sum_{k=1}^{m} \alpha_{jk} \delta'_k = \sum_{k=1}^{m} \alpha_{jk} \langle \boldsymbol{A}_k, \widetilde{\boldsymbol{Y}} \rangle = \langle \boldsymbol{A}_j, \widetilde{\boldsymbol{Y}} \rangle \leq \delta_j.$$

Thus, after replacing $\boldsymbol{\delta}$ by $\boldsymbol{\delta}'$, the added constraints are redundant. Consequently, SDP (14) with right-hand-side vector $\boldsymbol{\delta}'$ and the modified SDP have the same optimal value. Indeed, every feasible solution of the former is feasible for the modified SDP by the redundancy shown above, while $\widetilde{\boldsymbol{Y}}$ is feasible for the former and attains the optimal value of the modified SDP. Therefore $-\infty < \eta(\boldsymbol{\delta}', \boldsymbol{U}) < \infty$. By the assumption, SDP (14) with right-hand-side vector $\boldsymbol{\delta}'$ has a rank-at-most-one optimal solution. This solution is also feasible for the modified SDP and has the same optimal value. Hence it is a rank-at-most-one optimal solution of the modified SDP. □

The proof above shows that, although the additional constraints may shrink the feasible region, they become redundant after replacing the right-hand-side vector by the values attained at an optimal solution of the modified SDP. This preservation result is particularly useful when the resulting SDP appears as a local sub-SDP in the clique-wise relaxation (7). In that setting, the relevant local right-hand side vector is induced by an optimal solution of the global SDP. Combining Theorem 4.8 with Corollary 4.7(ii), we obtain the following extension.

Corollary 4.9. *Assume that $m \in \{1, 2\}$ and $m + |\mathcal{D}| \leq 2$ in SDP (14). Let*

$$\boldsymbol{A}_j \in \mathrm{cone}\{\boldsymbol{A}_k : k = 1, \ldots, m\}, \ \delta_j \in \mathbb{R} \ (j = m+1, \ldots, m'),$$

where $m < m'$. Then the modified SDP obtained from SDP (14) by adding the constraints $\langle \boldsymbol{A}_j, \boldsymbol{Y} \rangle \leq \delta_j$ $(j = m+1, \ldots, m')$ is exact.

We call the inequalities $\langle \boldsymbol{A}_j, \boldsymbol{Y} \rangle \leq \delta_j$ $(j = m+1, \ldots, m')$ *dependent inequalities* if their coefficient matrices satisfy $\boldsymbol{A}_j \in \mathrm{cone}\{\boldsymbol{A}_k : k = 1, \ldots, m\}$ $(j = m+1, \ldots, m')$. The dependence occurs only at the level of the coefficient matrices; the right-hand-side constants δ_j $(j = m+1, \ldots, m')$ are arbitrary real numbers. Here $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m$ serve as the *base coefficient matrices*, and the remaining matrices $\boldsymbol{A}_j$ $(j = m+1, \ldots, m')$ are dependent on them in the sense that they lie in the conic hull of the base coefficient matrices. For notational convenience, we call all inequalities $\langle \boldsymbol{A}_j, \boldsymbol{Y} \rangle \leq \delta_j$ $(j = 1, \ldots, m')$ *a system of dependent inequalities* generated by $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m$.

5 Examples

In this section, we present three examples that illustrate how the exactness of the SDP relaxation (2) of QCQP (1) can be derived from local sub-SDPs by combining Theorem 3.2 with

the local exactness results in Section 4. These local results include sub-QCQPs characterized by convexity, sign-pattern conditions and a system of dependent inequalities. The final example shows how heterogeneous classes of sub-QCQPs including separable sub-QCQPs characterized by a limited number of constraints can be incorporated within the local-to-global exactness framework while preserving exactness of the SDP relaxation.

Example 5.1. This example illustrates the simplest case in which the clique-wise reformulation becomes completely separable and no consistency constraints are required. We assume that $\trianglelefteq_k = \text{`}\leq\text{'}$ $(k = 1, \ldots, m)$ and that each $\boldsymbol{A}_k$ in QCQP (1) and its SDP relaxation (2) is of the following block diagonal form:

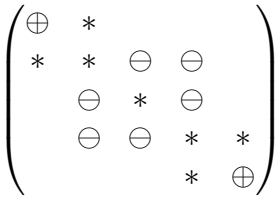
$$\boldsymbol{A}_k = \begin{pmatrix} \boldsymbol{A}_k^1 & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{A}_k^2 & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_k^3 \end{pmatrix} \in \mathbb{S}^n, \quad \boldsymbol{A}_k^p \in \mathbb{S}^{C_p} \quad (p = 1, 2, 3,\ k = 0, \ldots, m),$$

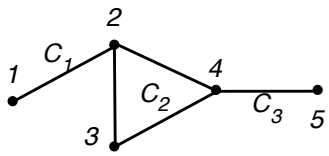
where $C_1 = \{1, \ldots, n_1\}$, $C_2 = \{n_1 + 1, \ldots, n_2\}$, $C_3 = \{n_2 + 1, \ldots, n_3\}$, and $n_3 = n$. Since $C_p \cap C_q = \emptyset$ $(p \neq q)$, we have $\mathcal{D}_p = \emptyset$ $(p = 1, 2, 3)$ and $\overline{\mathcal{D}} = \emptyset$. The aggregate sparsity pattern graph consists of three disconnected cliques C_p $(p = 1, 2, 3)$. Hence the clique-wise formulation (7) of SDP (2) becomes separable.

If we assign to each clique C_p a sub-QCQP characterized by convexity (Theorem 4.1), sign-pattern conditions (Theorem 4.3 and Corollary 4.4), or a system of dependent inequalities (Corollary 4.9), then SDP (7), and hence SDP (2), is exact. For example, we can take $\boldsymbol{A}_k^1 \in \mathbb{S}_+^{C_1}$ $(k = 0, \ldots, m)$ for a convex sub-QCQP on C_1, and $\boldsymbol{A}_k^2 \in \mathbb{S}^{C_2}$ with all off-diagonal elements nonpositive $(k = 0, \ldots, m)$ for a sub-QCQP characterized by sign-pattern conditions (Corollary 4.4). For the sub-QCQP on C_3 with dependent inequalities, assume $m \geq 2$ and take two base coefficient matrices $\boldsymbol{O} \neq \boldsymbol{A}_1^3 \in \mathbb{S}^{C_3}$, $\boldsymbol{O} \neq \boldsymbol{A}_2^3 \in \mathbb{S}^{C_3}$. Assume that the remaining coefficient matrices satisfy $\boldsymbol{A}_k^3 \in \mathrm{cone}\{\boldsymbol{A}_1^3, \boldsymbol{A}_2^3\}$ $(k = 3, \ldots, m)$, while $\boldsymbol{A}_0^3 \in \mathbb{S}^{C_3}$ is arbitrary. Since $\mathcal{D}_3 = \emptyset$, Corollary 4.9 applies to this local sub-SDP.

The discussion after SDP (2) also applies to the present example: linear terms can be incorporated by including, or adding, a normalization constraint $X_{ii} = 1$ for some $i \in \{1, \ldots, n_2\}$. The same observation applies to Examples 5.2 and 5.3. If this normalization is added as an extra constraint, however, the number of equality constraints increases. This must be taken into account when applying results whose assumptions depend on the number of equality and/or inequality constraints, such as Theorem 4.1, Theorem 4.5 and Corollary 4.9.

Example 5.2. This example illustrates how exactness can be preserved under a clique-wise coupling of sub-QCQPs characterized by convexity and those characterized by sign-pattern conditions, with a nonempty consistency set $\overline{\mathcal{D}}$. We consider QCQP (1) with $n = 5$, $m \geq 1$, $\trianglelefteq_k = \text{`}\leq\text{'}$ $(k = 1, \ldots, m)$ and $\boldsymbol{A}_k$ $(k = 0, \ldots, m)$ whose aggregate sparsity pattern matrix and graph $G(N, \mathcal{E}^0)$ are given by the following.





Here $\oplus$ denotes a nonnegative real number, $\ominus$ a nonpositive real number and $*$ an arbitrary real number. The graph $G(N, \mathcal{E}^0)$ itself is a block-clique graph with $\hat{p} = 3$ maximal cliques $C_1 = \{1, 2\}, C_2 = \{2, 3, 4\}$ and $C_3 = \{4, 5\}$. The corresponding consistency sets $\mathcal{D}_1 = \{(2, 2)\}$, $\mathcal{D}_2 = \{(2, 2), (4, 4)\}$ and $\mathcal{D}_3 = \{(4, 4)\}$ are diagonal. Define

$$\begin{aligned}
&\boldsymbol{A}_k^p = \boldsymbol{A}_k^{C_p} \in \mathbb{S}^{C_p} \ (k = 0, 1, \ldots, m, p = 1, 3), \\
&\boldsymbol{A}_k^2 = \begin{pmatrix} 0 & [\boldsymbol{A}_k]_{23} & [\boldsymbol{A}_k]_{24} \\ [\boldsymbol{A}_k]_{32} & [\boldsymbol{A}_k]_{33} & [\boldsymbol{A}_k]_{34} \\ [\boldsymbol{A}_k]_{42} & [\boldsymbol{A}_k]_{43} & 0 \end{pmatrix} \in \mathbb{S}^{C_2} \ (k = 0, 1, \ldots, m),
\end{aligned}$$

so that $\boldsymbol{A}_k^p$ $(k = 0, \ldots, m, p = 1, 2, 3)$ satisfy (3).

Applying the clique-wise reformulation discussed in Section 2.3 to the SDP relaxation (2) of QCQP (1), we obtain SDP (7). The sub-QCQPs on C_1 and C_3 fall into the convex class in Theorem 4.1, case (I). The sub-QCQP on C_2 falls into the sign-pattern class of Corollary 4.4, case (i), because all its off-diagonal coefficients are nonpositive. Therefore, for the parameters $\boldsymbol{\delta}^p \in \mathbb{R}^m$ $(p = 1, 2, 3)$ and the consistency matrix $\boldsymbol{U} \in \mathbb{S}^n(\overline{\mathcal{D}})$ induced by any optimal solution of SDP (7), all local sub-SDPs are exact. It follows from Theorem 3.2(iii) that SDP (7), and consequently SDP (2), is exact.

This example illustrates that exactness can be preserved even when different local exactness mechanisms are assembled through diagonal consistency constraints.

Example 5.3. This example illustrates how the local-to-global exactness framework can incorporate three heterogeneous classes of QCQPs with exact SDP relaxations, including a separable sub-QCQP characterized by a limited number of constraints discussed in Theorem 4.5. The main point of this example is the treatment of the separable sub-QCQP. In contrast to sub-QCQPs characterized by convexity or sign-pattern conditions, the exactness of the associated sub-SDP (16) is parameter-dependent; it depends on the parameters $(\boldsymbol{\delta}, \boldsymbol{U})$ induced by the global SDP (7). We show that suitable conditions on the other sub-SDPs can force the sub-SDP (16) to satisfy Condition (B) of Theorem 4.5.

We construct an instance of QCQP (5) by combining five sub-QCQPs on $C_1 = \{1, \ldots, 6\}$, $C_2 = \{5, 7\}, C_3 = \{6, 7, 8\}, C_4 = \{7, 9\}$ and $C_5 = \{6, 10\}$. See Figure 3. The node set $C_1 = \{1, \ldots, 6\}$ involves four cliques F_1, F_2, F_3, F_4, while C_2, C_3, C_4, C_5 are cliques. We assign a separable QCQP to C_1, QCQPs satisfying sign-pattern conditions to C_2 and C_3, convex QCQPs to C_4 and C_5. Each sub-QCQP has $m = 5$ inequality constraints, including some redundant constraints. Table 1 provides the details of those sub-QCQPs.

For a separable sub-QCQP on C_1, we assume that C_1 consists of 4 disjoint cliques $F_1 = \{1\}, F_2 = \{2\}, F_3 = \{3, 5\}, F_4 = \{4, 6\}$ as shown in Figure 3. The sub-QCQP on C_1 contains 5 inequality constraints, where the last two constraints, the 4th and 5th constraints are redundant. These two redundant constraints are included solely to match the number $m = 5$ of inequality constraints of the QCQP (5), which also embeds the sub-QCQPs on $C_2, \ldots, C_5$. Hence, when applying Theorem 4.5 to the separable sub-SDP on C_1, we first remove them and use the reduced formulation with the three effective inequalities with $k = 1, 2, 3$. For Condition (B), we impose

$$\boldsymbol{A}_k^p \in \mathbb{S}_+^{C_p}, \ b_k < 0 \ (p = 2, 3, 4, 5, \ k = 1, 2), \tag{19}$$

$$\boldsymbol{B}_1^q \in \mathbb{S}_+^{F_q} \ (q = 2, 3, 4), \ \boldsymbol{B}_2^q \in \mathbb{S}_+^{F_q} \ (q = 1, 3, 4). \tag{20}$$

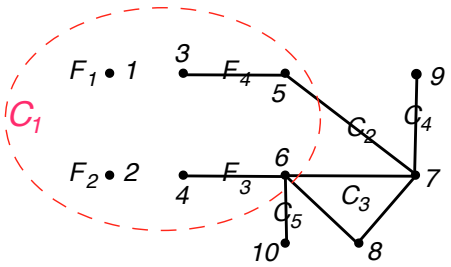


Figure 3: The aggregate sparsity pattern graph $G(N, \mathcal{E}^0)$ of QCQP (1) of Example 5.3. Here F_q $(q = 1, \ldots, 4)$ and C_p $(p = 2, \ldots, 5)$ are cliques of $G(N, \mathcal{E}^0)$, whereas C_1 is a clique of a chordal extension of the graph. Separable QCQPs with a limited number of constraints are assigned to $C_1 = F_1 \cup F_2 \cup F_3 \cup F_4$, QCQPs characterized by sign-pattern conditions to C_2 and C_3, and convex QCQPs to C_4 and C_5.

		$p = 1 : C_1 = \{1,2,3,4,5,6\}$ Separable Sect.4.3				$p = 2,3$ Sign Pat. Sect.4.2	$p = 4,5$ Convex Sect.4.1		
		$q = 1$ $F_1 = \{1\}$	$q = 2$ $F_2 = \{2\}$	$q = 3$ $F_3 = \{3,5\}$	$q = 4$ $F_4 = \{4,6\}$	$C_2 = \{5,7\}$ $C_3 = \{6,7,8\}$	$C_4 = \{7,9\}$ $C_5 = \{6,10\}$		
	$\mathcal{D}$			$(5,5) \in \mathcal{D}_1$	$(6,6) \in \mathcal{D}_1$	$(5,5), (7,7) \in \mathcal{D}_2$ $(6,6), (7,7) \in \mathcal{D}_3$	$(7,7) \in \mathcal{D}_4$ $(6,6) \in \mathcal{D}_5$		
	k	$\boldsymbol{B}_k^1$	$\boldsymbol{B}_k^2$	$\boldsymbol{B}_k^3$	$\boldsymbol{B}_k^4$	$\boldsymbol{A}_k^2$ and $\boldsymbol{A}_k^3$	$\boldsymbol{A}_k^4$ and $\boldsymbol{A}_k^5$	$\trianglelefteq_k$	b_k
obj.	0	$\forall$sym	$\forall$sym	$\forall$sym	$\forall$sym	off-diag$\ominus$	convex		
const.	1	$\forall$sym	$\oplus$	$\oplus$	$\oplus$	$\oplus$ & off-diag$\ominus$	$\oplus$	$\leqq$	–
	2	$\oplus$	$\forall$sym	$\oplus$	$\oplus$	$\oplus$ & off-diag$\ominus$	$\oplus$	$\leqq$	–
	3	$\forall$sym	$\forall$sym	$\forall$sym	$\forall$sym	off-diag$\ominus$	convex	$\leqq$	$\forall$
	4	$\boldsymbol{O}$	$\boldsymbol{O}$	$\boldsymbol{O}$	$\boldsymbol{O}$	off-diag$\ominus$	convex	$\leqq$	$\forall$
	5	$\boldsymbol{O}$	$\boldsymbol{O}$	$\boldsymbol{O}$	$\boldsymbol{O}$	off-diag$\ominus$	convex	$\leqq$	$\forall$

Table 1: Summary of data matrices and conditions imposed on the five sub-SDPs on $C_1, \ldots, C_5$. $\oplus$: a positive semidefinite symmetric matrix. off-diag$\ominus$: a symmetric matrix with all off-diagonal elements nonpositive. convex: a symmetric matrix $\boldsymbol{A}$ such that $\langle \boldsymbol{A}, \boldsymbol{y}\boldsymbol{y}^T \rangle$ is convex in $\boldsymbol{y}$ with y_i fixed for some i (see case (I) in Section 4.1). $\forall$sym: an arbitrary symmetric matrix. $\boldsymbol{O}$: a zero matrix.

The condition (19) ensures that, for every optimal solution $(\boldsymbol{Y}^1, \ldots, \boldsymbol{Y}^5, \boldsymbol{U})$ of SDP (7),

$$\delta_k^1 = \langle \boldsymbol{A}_k^1, \boldsymbol{Y}^1 \rangle \leq b_k - \sum_{p=2}^{5} \langle \boldsymbol{A}_k^p, \boldsymbol{Y}^p \rangle < 0 \ (k = 1, 2).$$

Then the condition (20) ensures that $\boldsymbol{W}^1 \neq \boldsymbol{O}$ and $\boldsymbol{W}^2 \neq \boldsymbol{O}$ for every optimal solution $(\boldsymbol{W}^1, \ldots, \boldsymbol{W}^4)$ of the separable SDP (16) for these $\delta_1^1 < 0$, $\delta_2^1 < 0$ and any $\delta_3^1 \in \mathbb{R}$.

If both U_{55} and U_{66} are positive, then $\mu = 2$ in the reduced separable sub-SDP. Since the reduced formulation has three effective inequality constraints, Condition (B) requires $3 + \mu - 1 = 4$ nonzero elements. The conditions above imply $\boldsymbol{W}^1 \neq \boldsymbol{O}$ and $\boldsymbol{W}^2 \neq \boldsymbol{O}$, while $U_{55} > 0$ and $U_{66} > 0$ imply $\boldsymbol{W}^3 \neq \boldsymbol{O}$ and $\boldsymbol{W}^4 \neq \boldsymbol{O}$. Hence Condition (B) is satisfied.

If $U_{55} = 0$ and/or $U_{66} = 0$, the corresponding row and column can be eliminated by positive semidefiniteness, as described in Theorem 4.5. The number μ decreases accordingly, and the required number of nonzero elements in Condition (B) decreases by the same

amount. Therefore, the reduced separable sub-SDP on C_1 is exact.

For the sub-QCQPs on C_2 and C_3, we impose the sign-pattern conditions. Specifically, all off-diagonal elements of $\boldsymbol{A}_k^p$ $(k = 0, \ldots, 5,\ p = 2, 3)$ are assumed to be nonpositive. By Corollary 4.4(i), the corresponding sub-SDPs are exact for every $\boldsymbol{\delta} \in \mathbb{R}^m$ and $\boldsymbol{U}$. In addition, to ensure that the separable sub-QCQP on C_1 satisfies Condition (B), we have assumed that $\boldsymbol{A}_k^p \in \mathbb{S}_+^{C_p}$ $(k = 1, 2, p = 2, 3)$.

For the sub-QCQPs on C_4 and C_5, we use the convex QCQPs discussed in Section 4.1. Since $\mathcal{D}_4 = \{(i_4, i_4)\} = \{(7, 7)\}$ and $\mathcal{D}_5 = \{(i_5, i_5)\} = \{(6, 6)\}$, case (I) of Theorem 4.1 applies to both sub-QCQPs. Thus, we impose the convexity condition that the principal submatrix of $\boldsymbol{A}_k^p$ indexed by $C_p \backslash \{i_p\}$ is positive semidefinite $(k = 0, \ldots, 5, p = 4, 5)$, or equivalently, $[\boldsymbol{A}_k^4]_{99} \geq 0,\ [\boldsymbol{A}_k^5]_{10,10} \geq 0$ $(k = 0, \ldots, 5)$. It follows that the sub-SDPs (14) on C_4 and C_5 are exact for every $\boldsymbol{\delta} \in \mathbb{R}^m$ and $\boldsymbol{U}$. In order for the separable sub-QCQP on C_1 to satisfy Condition (B), we have assumed that $\boldsymbol{A}_k^p \in \mathbb{S}_+^{C_p}$ $(k = 1, 2, p = 4, 5)$.

Consequently, each local sub-SDP induced by an optimal solution of SDP (7) is exact. It follows from Theorem 3.2(iii) that SDP (7), and therefore SDP (2), is exact. This example highlights a feature that does not arise in the purely convex or sign-pattern cases: the exactness of one local sub-SDP may be certified using parameter information imposed by the other local sub-SDPs through the global constraints.

6 Concluding remarks

We have developed a local-to-global exactness framework for SDP relaxations of sparse QCQPs. Using a chordal extension of the aggregate sparsity pattern graph, the SDP relaxation is reformulated in clique-wise matrix variables associated with the maximal cliques. This equivalent reformulation induces local sub-SDPs linked by consistency constraints on clique overlaps. The main result shows that exactness of the original SDP relaxation can be certified by rank-at-most-one attainability of the local sub-SDPs with the local right-hand-side vectors and consistency matrix induced by an optimal solution of the clique-wise formulation.

For the applications developed in this paper, the block-clique assumption is imposed in handling the consistency of rank-at-most-one optimal solutions of the local sub-SDPs. Under this assumption, all consistency constraints on clique overlaps are diagonal, so local rank-at-most-one optimal solutions can be combined without imposing additional off-diagonal rank-one consistency conditions. This allows different local exactness mechanisms to be used within a single sparse QCQP. In particular, we have identified three mechanisms based, respectively on convexity, sign-pattern conditions, and separability with a limited number of constraints. The examples in Section 5 illustrate how these heterogeneous local exactness mechanisms can be assembled to prove exactness of the original SDP relaxation.

An important issue for future work is the choice of chordal extension and clique-wise decomposition of the data matrices. Different chordal extensions, and different decompositions over their maximal cliques, may lead to different local sub-SDPs. Consequently, they may affect whether the local exactness certificates developed in this paper can be applied.

It would also be valuable to further enlarge the list of local sub-QCQP classes with exact SDP relaxations. The framework is modular: any new local exactness result that is stable under the induced right-hand side and consistency matrix can be incorporated into

the global certification scheme. This suggests that sparse exactness analysis may serve as a way to combine otherwise separate exactness criteria for nonconvex QCQPs.

Data availability statement

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Competing interests The authors have no competing interests to declare that are relevant to the content of this article.

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