Optimization Reformulations of Complementarity Equilibrium Models

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Abstract

We propose a new mathematical model to describe equilibria in competitive markets. Our approach transforms the well-known complementary formulation into a numerically more efficient optimization framework. In complementarity models, the actions of all elastic consumers in the market are implicitly represented by their aggregate demand. Instead, we introduce demand-induced utilities, which can be explicitly constructed individually for each consumer. We then consider the joint optimization of all agents' costs, including the new demand-induced functions for elastic consumers, subject to market-clearing constraints. We prove that the primal-dual solution to this optimization problem yields an equilibrium equivalent to that found by a complementarity model. The computational efficiency of the new methodology is compared to that of the complementarity formulation on a large-scale instance, representing the Brazilian natural gas market. Our model exhibits excellent performance and captures distortions typical of multi-hub network-constrained markets, showing the impact of spatial arbitrage on regional price setting.

Keywords: Complementarity equilibrium models, Explicit optimization reformulations, Multi-hub network-constrained competitive markets.

MSC Classification: 90C33 , 65K5 , 49M29

1 Introduction

We study equilibrium problems for competitive markets where price-taking wholesalers can buy or sell a commodity on multiple distribution hubs connected by a network. In such a context, transactions are constrained not only by the clearing requirements at each hub, but also by the capacity and topology of the transportation network.

The mathematical representation of multi-hub network-constrained markets involves a large number of variables and constraints. Such is the case for the application studied in § 5, which characterizes competition in the Brazilian natural gas market. In 2025, the average daily commodity flow managed in the market was of $100\,\mathrm{Mm}^3$, with transactions involving approximately 40 suppliers and 150 buyers. Among the latter nearly 18 % are Local Distribution Companies (LDCs), whose flexible consumption is strategic and highly specific to each company. Suppliers in the market typically source natural gas from 30 domestic fields, an international link with Bolivia, and various Liquefied Natural Gas (LNG) terminals. A vast network integrates over 200 entry/exit points across $9,400$ km of pipeline infrastructure[1].

From a general modeling perspective, the derivation of equilibrium conditions in competitive markets has been approached with various methodologies. All models consider suppliers as profit-maximizer agents, but they differ in their representation of demand and on the treatement of the coupling constraints, such as the market clearing of each hub or the shared network capacity. Complementarity approaches [13] represent hub-level consumption by aggregating consumer demand functions and consider shared constraints aside of the supply-side maximization problems. We label as $\mathrm{CP}^{\mathrm{imp}}$ this class of equilibrium, to emphasize that individual consumer decisions are implicitly considered within the aggregate hub demand function. An alternative equilibrium paradigm, rooted in game theory, represents individual consumer behavior explicitly, using utility functions. In a Generalized Nash equilibrium model, agents seek to minimize individual costs derived from the utilities, subject to the shared constraints. A third model, the social-welfare equilibrium, minimizes the sum of cost functions of all agents, subject to the shared constraints. In contrast to the $\mathrm{CP}^{\mathrm{imp}}$ framework, both the social-welfare and the Nash models consider consumers as active market players. This is also the case of our proposal, and is a suitable setting to understand how the strategic consumption of LDCs impacts on market-clearing outcomes.

The aforementioned equilibrium formulations lead to different structural and theoretical attributes. Existence of a social-welfare equilibrium follows from standard convex optimization results, while determining unique hub prices (some dual variables) depends on constraint gradients being linearly independent at an optimum. The $\mathrm{CP}^{\mathrm{imp}}$ approach demands a more elaborate mathematical treatment to guarantee equilibrium existence. An abstract viewpoint, relying on existence of solutions to variational inequalities was followed by [21] and [20]; a more recent reference is [12]. Similarly, the thorough analysis in [24] involves generalized equations. For Generalized Nash games, existence results were considered in [11] by means of fixed-point theorems, whereas [16] and [4] employed tools from quasi-variational inequalities theory. In

[1] Source: https://www.gov.br/anp/pt-br/centrais-de-conteudo/publicacoes/anuario-estatistico/anuario-estatistico-brasileiro-do-petroleo-gas-natural-e-biocombustiveis-2025

particular, the reproducible maps in [3] provide sufficient conditions for driving quasi-variational inequalities to a variational inequality setting. In the absence of uniqueness, the variational equilibrium concept transforms the game into a variational inequality ([18, 25]). Finally, to compute specific solutions or the full spectrum of generalized Nash equilibria, the approaches in [8] and [9] can be applied.

In addition to the theoretical complexity of showing existence of equilibria, an imperative of numerical nature must be taken into account. For cases like the Brazilian natural gas market, the development of efficient algorithms is crucial. The equilibrium model must be accompanied by a numerical solver capable of handling large-scale instances typical in a national market. Given that data on natural gas production costs, technological constraints, consumption, and network topology is accessible through public repositories, the $\mathrm{CP}^{\mathrm{imp}}$ framework appears to be a pragmatic modeling choice at first sight. However, in terms of numerical efficiency, a significant bottleneck lies in the low scalability of algorithms for mixed complementarity problems (MCPs), inherent in the $\mathrm{CP}^{\mathrm{imp}}$ formulation. The PATH solver ([7]) is recognized for its excellence when dealing with medium-sized MCPs. Nevertheless, its performance is challenged when solving $\mathrm{CP}^{\mathrm{imp}}$ problems in high dimensional markets, especially when elastic consumers have nonlinear demand functions, see Tables 3.1 and 5.3.

Our proposal aims at mitigating the observed computational limitations by mapping a $\mathrm{CP}^{\mathrm{imp}}$ model into a tractable optimization framework. The solution to the resulting problem provides a new equilibrium model, labeled $\mathrm{CP}^{\mathrm{exp}}$, because it represents the action of elastic consumers in an explicit manner. To achieve our goal, we create a new utility concept for each consumer, built from the respective demand function. The $\mathrm{CP}^{\mathrm{exp}}$ equilibrium is obtained by solving a single optimization problem, a particularly appealing feature for large-scale markets. For comparison, when consumers' demand functions are affine, the $\mathrm{CP}^{\mathrm{imp}}$ model requires solving a bilinear MCP while the $\mathrm{CP}^{\mathrm{exp}}$ formulation reduces to a convex quadratic programming (QP) problem. Our reformulation of the equilibrium model as an optimization problem ensures numerical stability and rapid convergence with off-the-shelf solvers, regardless of the market having large spatial or operational dimensionalities.

This work is organized as follows. Section 2 reviews the mathematical formulation of the considered equilibria for a simple market instance. In Section 3 we show how to build our new demand-induced utility concept for any convex, monotonically decreasing demand function, not necessarily differentiable. The construction is illustrated with three different formats of demand, linear, log-linear, and with exponential decay. Section 4 describes both the $\mathrm{CP}^{\mathrm{imp}}$ and $\mathrm{CP}^{\mathrm{exp}}$ equilibrium concepts for a general market, with different hubs connected by a transportation network. Under mild regularity conditions, we explain how to reformulate $\mathrm{CP}^{\mathrm{imp}}$ as an optimization problem and show in § 4.3 that the equilibria generated by the $\mathrm{CP}^{\mathrm{exp}}$ and $\mathrm{CP}^{\mathrm{imp}}$ models are mathematically equivalent. The excellent performance and scalability of our proposal is illustrated in Section 5, for a very large-scale instance representing the market of natural gas in Brazil.

2 Comparing Equilibrium Models for a Toy Market

To explain the various mathematical formulations of equilibria, we begin with a simple market configuration - the toy market. The setting allows for a clear exposition of the fundamental equilibrium conditions before we extend the analysis to more complex, large-scale network topologies in Section 4.

We assume a unique hub with no transportation network. There is only one supplier offering a volume $x \in \mathbb{R}_+$ of the commodity, with associated cost $\mathbb{C}(x)$, a convex differentiable function of class C^1. If the market pays an exogenous unit price $\bar{p}$ for the commodity, the revenue-maximizer supplier solves the following problem:

$$\begin{cases} \max \ \bar{p}x - \mathbb{C}(x) \\ \text{s.t. } \ x \in X = \{x : Ax \leq b\} \,. \end{cases} \tag{2.1}$$

For our simple case, A is a column matrix and the feasible set X is just a closed box.

2.1 The $\mathrm{CP}^{\mathrm{imp}}$ Equilibrium

In the complementarity model, only suppliers act as strategic agents solving problem (2.1). Consumers do not operate under a competitive framework to optimize their own welfare; instead, they function as passive market participants. For the toy configuration, this implies that consumption is determined solely by the demand function $\mathcal{D}$, which maps prices to volumes without any underlying individual problem that is optimized by the consumers. Accordingly, a $\mathrm{CP}^{\mathrm{imp}}$ equilibrium is reached when an optimal production level $\bar{x}$, solving the supplier's problem (2.1), also satisfies the market-clearing condition, that is, the following complementarity relation:

$$0 \leq \bar{x} - \mathcal{D}(\bar{p}) \perp \bar{p} \geq 0 \,. \tag{2.2}$$

At an equilibrium point $(\bar{x}, \bar{p})$, the supplier sells $\bar{x}$ at price $\bar{p}$ and the consumer just buys the quantity $\mathcal{D}(\bar{p})$. This requires solving the optimality conditions associated to (2.1) together with the market-clearing relations (2.2). The $\mathrm{CP}^{\mathrm{imp}}$ equilibrium $(\bar{x}, \bar{p})$, together with a multiplier $\bar{\mu}$, solves the nonlinear complementarity system:

$$\begin{array}{l} 0 \, = \bar{p} - \mathbb{C}'(\bar{x}) - A^\top \bar{\mu} \\ 0 \, \leq \, b - A\bar{x} \perp \bar{\mu} \geq 0 \\ 0 \, \leq \, \bar{x} - \mathcal{D}(\bar{p}) \perp \bar{p} \geq 0 \,. \end{array} \tag{2.3}$$

2.2 Equilibrium with Strategic Consumers

Suppose now that instead of being passive market agents, consumers exhibit strategic behavior. Then y, the volume of consumed commodity, is based on a given concave utility $\mathbb{U}(y)$, that is an increasing function. This yields the consumer problem:

$$\max_y \mathbb{U}(y) - y\bar{p} \,, \tag{2.4}$$

and the new market-clearing constraint is:

$$0 \leq \bar{x} - \bar{y} \perp \bar{p} \geq 0\,, \tag{2.5}$$

for $\bar{x}$ and $\bar{y}$ respectively solving (2.1) and (2.4). Given the convexity of all the elements in the model, an equilibrium $(\bar{x}, \bar{y}, \bar{p})$ solves simultaneously the optimality conditions of (2.1) and (2.4), while satisfying the market-clearing constraint (2.5):

$$\begin{array}{l} 0 = \bar{p} - \mathbb{C}'(\bar{x}) - A^\top \bar{\mu} \\ 0 \leq b - A\bar{x} \perp \bar{\mu} \geq 0 \\ 0 = \mathbb{U}'(\bar{y}) - \bar{p} \\ 0 \leq \bar{x} - \bar{y} \perp \bar{p} \geq 0\,. \end{array} \tag{2.6}$$

Relations (2.6) characterize a social-welfare equilibrium, determined through the maximization of the market surplus (the difference between the consumer utility and the producer costs), while balancing the market and satisfying the producer constraints. Within this framework, a social-welfare equilibrium arises when solving

$$\begin{cases} \max_{x,y} & \mathbb{U}(y) - \mathbb{C}(x) \\ \text{s.t.} & Ax \leq b \\ & 0 \leq x - y \quad (\text{mult. } p)\,. \end{cases} \tag{2.7}$$

Associated with a primal solution $(\bar{x}, \bar{y})$ there is an optimal multiplier $\bar{p}$ of the market-clearing constraint, that defines the social-welfare equilibrium price.

2.3 Toward a $\mathrm{CP}^{\mathbf{exp}}$ Equilibrium

The equilibrium models above adhere to a standard paradigm widely adopted in energy economics, transportation, networks [19], and market models [15]; see also [14]. Strategic agents are represented as independent optimizers whose decisions are linked through shared coupling constraints, exogenous to each agent's internal optimization process. Market equilibrium is attained when all agents reach a simultaneous optimum while satisfying the shared market-clearing constraints.

This is an appealing mechanism because of its flexibility to describe different agent actions and market structures. The numerical solution requires to build a single MCP from the agents' Karush-Kuhn-Tucker conditions and the shared constraints. Modern modeling ecosystems, including AMPL, AIMMS, GAMS and Pyomo, exploit the framework of extended mathematical programming [17] to transform formally described market equilibrium configurations into mathematical formulations that can be interpreted by numerical solvers. Using symbolic differentiation tools, the mechanism automatically generates a MCP that can be solved by calling PATH [7] as an out-of-the-shelf solver. As evidenced by its extensive adoption in diverse applications — ranging from energy markets [22] to global trade equilibrium [5] — PATH is widely regarded as a robust and reliable solver for MCPs. Notwithstanding, its excellent performance is diminished when applied to large-scale market instances, particularly if consumers have nonlinear demand functions, as shown by Table 3.1 below.

Alternatively, being an optimization problem, the equilibrium derived from (2.7) is often much easier to find. Our proposal offers an answer to the following question:

> Is it possible to compute a $\mathrm{CP}^{\mathrm{imp}}$ equilibrium
> for a market with strategic consumers
> by solving an optimization problem similar to (2.7)?

To gain an intuition for our methodology, we compare the optimality conditions derived from the welfare-maximization framework (2.6) against the standard $\mathrm{CP}^{\mathrm{imp}}$ formulation in (2.3). If, for a given demand function $\mathcal{D}$, we were to define a utility $\mathbb{U}_{\mathcal{D}}$ in such a manner that

$$\bar{p} = \mathbb{U}'_{\mathcal{D}}(\bar{y}) \quad \text{and} \quad \bar{y} = \mathcal{D}(\bar{p})\,, \tag{2.8}$$

then the question above would be answered in the affirmative.

3 Demand-induced Utilities for Elastic Agents

In order to transform (2.3) into the form (2.6), so that the $\mathrm{CP}^{\mathrm{imp}}$ equilibrium is a solution to the optimization problem (2.7), we need to build a utility from the demand function, as required by the relations (2.8).

3.1 On Demand Functions

For elastic agents in the market, we consider demand functions $\mathcal{D}(p)$ with domain $dom\,\mathcal{D}$ and the corresponding inverse demand function $\mathcal{D}^{-1}(q)$, expressing price as a function of the quantity q. All along our development, the following condition is required for all demand functions in the market:

Every demand function $\mathcal{D}$ is decreasing and convex on its domain.
These conditions imply continuity for $\mathcal{D}$ (but not differentiability). (3.1)

When demand is strictly decreasing, it is an injective function. Thus, its inverse $\mathcal{D}^{-1}$ is a well defined function that is continuous, decreasing, and concave on its domain.

A peculiarity of the exchanged commodity is that for some agents the demand function may present minimum positive levels (as in (3.3) and (3.4) below). For the natural gas market, these values may correspond to minimal volumes that regional distribution companies are obliged to provide, for example to hospitals. Additionnally, some agents may accept negative values in their demand functions (or in their prices, e.g. (3.2)). This situation corresponds to volumes of stored gas a distribution company is willing to sell in exceptional market situations, or to volumes that sources cannot stop producing, even at a loss, for operational reasons.

Depending on the commodity and whether equilibrium is sought in a short-term or long-term horizon, elastic behavior can be described using different functions. To highlight that our approach for reformulating the $\mathrm{CP}^{\mathrm{imp}}$ equilibrium as an optimization problem is independent of the specific demand function, we review three different alternatives, alluding to their main characteristics.

Linear model

One popular example is the well-known linear demand function, defined by:

$$\mathcal{D}_{\texttt{lin}}(p) = a - bp\,, \quad \text{where } a, b > 0\,, \tag{3.2}$$

whose inverse $\mathcal{D}_{\texttt{lin}}^{-1}(q) = \alpha - \beta q$, where $\alpha = \frac{a}{b}$ and $\beta = \frac{1}{b}$, is also an affine function. The parameter a is interpreted as the demand for the case when the product becomes free ($p = 0$), while b represents the demand sensitivity to price. The simplicity of this function allows simpler models, easier computational implementation, and a more robust numerical behavior of PATH.

Log-linear model

Given $a, b > 0$, the models in [6] for natural gas define the quantities as an affine expression on the natural logarithm of price. In our setting, this approach yields the demand function:

$$\mathcal{D}_{\log}(p) = a + \frac{b}{p} \quad \text{for } p \in (0, \infty)\,, \tag{3.3}$$

which is a convex hyperbolic function of price, the same as the inverse demand function, expressed by $\mathcal{D}_{\log}^{-1}(q) = \frac{b}{q-a}$, defined for $q > a$.

Exponential decay

If finite values for minimum or maximum prices are unknown, the following convex function can be used to describe elastic behavior:

$$\mathcal{D}_{\kappa}(p) = a + \kappa_1 \exp(-\kappa_2 p) \quad \text{where } \kappa_1, \kappa_2 > 0, \text{ defined for all } p \in \mathbb{R}\,. \tag{3.4}$$

The inverse demand function, defined for $q > a$, is $\mathcal{D}_{\kappa}^{-1}(q) = -\frac{1}{\kappa_2} \log(\frac{q-a}{\kappa_1})$.

Impact on Calculation of a $\mathrm{CP}^{\mathbf{imp}}$ Equilibrium

Our framework is applicable to any demand function satisfying the conditions stated in (3.1), including a combination of the linear and exponential decay models:

$$\mathcal{D}_{\texttt{sum}}(p) = a - bp + \kappa_1 \exp(-\kappa_2 p) \text{ for } \kappa_1 \geq 0. \tag{3.5}$$

This demand function becomes less linear for larger values of κ_1, complicating the computational solution of the complementarity framework (2.3). To highlight this effect, Table 3.1 reports the solution times required by PATH when computing $\mathrm{CP}^{\mathrm{imp}}$ for the large Brazilian market described in Section 5, over a planning horizon of 24 months, when LDCs have demand functions as in (3.5). There are six different runs, varying the values of the parameter $\kappa_1 \in \{0, 0.025, 0.050, 0.100, 0.150, 0.200\}$. Because of the exponential decay term, the nonlinearity in $\mathcal{D}_{\texttt{sum}}$ changes decisions mostly for small values of p, which was not an equilibrium price. As a result, for all values of κ_1, the obtained equilibrium was the same point, but the time needed to find the solution was very different.

Table 3.1: Impact of nonlinearity weight κ_1 on PATH.

κ_1	0	0.025	0.050	0.100	0.150	0.200
CPU Time (s)	1920	1354	2566	6036	5012	11769

The solution times in Table 3.1 confirm the sensitivity of PATH computational efficiency to the weight of the nonlinear term in the demand function. We conjecture that the sharp time increase results not only from a larger number of iterations needed to reach convergence, but also from the more complex Jacobian calculation (that is constant when $\kappa_1 = 0$). The performance of our $\mathrm{CP}^{\mathrm{exp}}$ approach, to be formalized in what follows, has a clear advantage in this respect. Computing a $\mathrm{CP}^{\mathrm{exp}}$ equilibrium for the same market configuration involves the solution of a nonlinear programming problem, efficiently handled by solvers like IPOPT [26]. To obtain the same equilibrium with our model, the hardest instance ($\kappa_1 = 0.200$) was solved in just 4.40 seconds.

3.2 On Utility Functions

To reformulate the $\mathrm{CP}^{\mathrm{imp}}$ equilibrium as an optimization problem, we must construct from $\mathcal{D}$ a utility function that satisfies the relations in (2.8). While our approach bears some ressemblance with the strategic supply-side bidding framework presented by [1] for electricity markets, an important distinction needs to be made. Rather than modeling strategic supply offers, our focus is placed on making a realistic representation of elastic consumers. In natural gas markets, large distribution companies behave as strategic, active market participants rather than passive price-taker agents. As mentioned, they may even change their (buying) role of pure consumers and switch to the offer side, to sell part of their stock in the market. This is the reason why we refer to those market participant as *elastic agents*, instead of just elastic consumers.

Elastic Agent Utility as a Function of Quantity

For the case when the inverse demand function exists, using the antiderivative, the following utility function describes the elastic behavior in the market:

$$\mathbb{U}_{\mathcal{D}}(q) = \int_{q_0}^{q} \mathcal{D}^{-1}(\mu)\, d\mu\,, \text{ for } q_0 \text{ arbitrary in the domain of } \mathcal{D}^{-1}. \tag{3.6}$$

This utility, induced by the inverse demand function, is differentiable, with $\mathbb{U}_{\mathcal{D}}' = \mathcal{D}^{-1}$ decreasing, by (3.1), so $\mathbb{U}_{\mathcal{D}}$ is a concave function satisfying (2.8). For the toy market, the elastic agent optimization problem (2.4) writes down as $\max_y \mathbb{U}_{\mathcal{D}}(y) - y\bar{p}$. The complementarity system (2.3) defining the $\mathrm{CP}^{\mathrm{imp}}$ equilibrium is just (2.6) written with $\mathbb{U} = \mathbb{U}_{\mathcal{D}}$, which is equivalent to the social-welfare optimization problem (2.7).

To realize why the reference point in (3.6) can be arbitrary, note that choosing two different points, say $q_0 \neq \tilde{q}_0$, leads to two utility functions that differ only by a constant, and are therefore indistinguishable for the elastic agent problem (2.4). In particular, with $q_0 \in \{0, a\}$ as reference point, (3.6) is the well-known *willingness-to-pay* function, providing a direct economic interpretation of the utility $\mathbb{U}_{\mathcal{D}}$.

Dropping constant terms, the utilities induced by the inverse demand functions of our three examples are

$$\begin{aligned} \mathbb{U}_{\mathcal{D}_{\mathrm{lin}}}(q) &= -\tfrac{1}{2b}q^2 + \tfrac{a}{b}\, q\,, \\ \mathbb{U}_{\mathcal{D}_{\mathrm{log}}}(q) &= b\log(q-a)\,,\ \text{and} \\ \mathbb{U}_{\mathcal{D}_{\kappa}}(q) &= -\tfrac{(q-a)}{\kappa_2}\Big(\log(\tfrac{q-a}{\kappa_1}) - 1\Big)\,. \end{aligned}$$

Elastic Agent Utility as a Function of Endogenous Price

The computation of the utility $\mathbb{U}_{\mathcal{D}}$ requires knowing an explicit expression for the inverse demand function. It may happen, however, that demand exhibits regions of local inelasticity, in which case there is no inverse demand function, and no utility can be defined. To circumvent this limitation, we introduce an alternative approach derived from the demand function. The new demand-induced utility depends on prices, and not on quantities. However, as explained below, this is not the market price, but rather the perception that the elastic agent has of that price.

Remark 3.1 (Elastic agents have an endogenous perception of price) In real-life markets, the decision-making process of an elastic agent ℓ is dictated not only by commodity volumes but also by the own perception of price, that can be different in different delivery points of the network. Accordingly, the endogenous price, denoted here by a vector p^ℓ, adds to the market price $\bar{p}$ a vector of logistic surcharges and transmission fees associated with receiving the commodity through specific nodes in the transportation network. It is this node-specific vector of prices p^ℓ, rather than $\bar{p}$, that governs the willingness of agent ℓ to pay for the commodity and dictates its strategic behavior within the market. Mathematically, the delivered price p^ℓ is represented as an affine function of the reference market price. In large-scale markets with complex networks, the diversity of transportation fees defines a cost profile that is unique to each elastic agent. To reflect this feature, in (4.7) below we consider agent-specific, endogenous affine relations that map a given market price π to p^ℓ. □

To move from quantity to price, we make the change of variable $\nu = \mathcal{D}^{-1}(\mu)$ in the utility $\mathbb{U}_{\mathcal{D}}$ and compute the antiderivative:

$$\mathbb{U}_{\mathcal{D}}(\mathcal{D}(p)) = \int_{p_0}^{p} \nu \mathcal{D}'(\nu) d\nu\,, \quad \text{where } p_0 = \mathcal{D}^{-1}(q_0) \text{ is arbitrary.}$$

To derive the expression for the new demand-induced utility, suppose first that the demand function $\mathcal{D}$ is differentiable. Then, integrating the right-hand side above by parts, yields

$$\int_{p_0}^{p} \nu \mathcal{D}'(\nu) d\nu = p\mathcal{D}(p) - p_0\mathcal{D}(p_0) - \int_{p_0}^{p} \mathcal{D}(\nu) d\nu\,.$$

Neglecting the constant term, we use the right-hand side expression to define the demand-induced utility function also for non-differentiable demand functions:

$$\mathbb{V}_{\mathcal{D}}(p) = p\mathcal{D}(p) - \int_{p_0}^{p} \mathcal{D}(\nu) d\nu \text{ for } \mathcal{D} \text{ satisfying (3.1) and } p_0 \in dom\,\mathcal{D}. \tag{3.7}$$

The corresponding functions for our three examples are

$$
\begin{array}{llcl}
\text{with a linear demand:} & \mathbb{V}_{\mathcal{D}_{\texttt{lin}}}(p) & = & -\frac{1}{2}bp^2\,, \\
\text{with a log-linear demand:} & \mathbb{V}_{\mathcal{D}_{\log}}(p) & = & -b\log p\,, \\
\text{with an exponential decay demand:} & \mathbb{V}_{\mathcal{D}_{\kappa}}(p) & = & \frac{\kappa_1}{\kappa_2}(\kappa_2 p+1)\exp(-\kappa_2 p)\,, \text{ and} \\
\text{with the demand in (3.5):} & \mathbb{V}_{\mathcal{D}_{\texttt{sum}}}(p) & = & \mathbb{V}_{\mathcal{D}_{\texttt{lin}}}(p)+\mathbb{V}_{\mathcal{D}_{\kappa}}(p)\,.
\end{array}
\tag{3.8}
$$

We see that, while the linear utility is concave, the log-linear demand results in a convex utility, and the utilities associated with the exponential decay demand are quasi-concave [2] functions.

To conclude this section we state some results related to our new demand-induced utility that will be useful to relate the $\mathrm{CP}^{\mathrm{imp}}$ and $\mathrm{CP}^{\mathrm{exp}}$ equilibria in Section 4.

Lemma 3.2 [Technical properties of intermediate function] Consider an demand-induced utility as in (3.7). For any given scalar π^*, the function

$$
\begin{array}{rcl}
\mathbb{C}_{\pi^*} : dom\,\mathcal{D} & \to & \mathbb{R} \\
p & \mapsto & \pi^*\mathcal{D}(p) - \mathbb{V}_{\mathcal{D}}(p)
\end{array}
$$

is continuous and attains its minimum at π^*. Furthermore, if the demand function is such that $0 \notin \partial\mathcal{D}(\pi^*)$, then π^* is the (unique) global strict minimizer of $\mathbb{C}_{\pi^*}$.

Proof Continuity is clear from the function definition. To prove that π^* is a minimizer, we shall show that for any $p \neq \pi^*$, there exists $\tilde{p} = tp + (1-t)\pi^*$ and some $0 < t < 1$, such that

$$
\mathbb{C}_{\pi^*}(p) - \mathbb{C}_{\pi^*}(\pi^*) = |\mathcal{D}(\tilde{p}) - \mathcal{D}(p)||p - \pi^*| \geq 0\,. \tag{3.9}
$$

For this purpose, we introduce the notation $I_{\pi^*}(p) = \displaystyle\int_{\pi^*}^{p}\Big(\mathcal{D}(\nu) - \mathcal{D}(p)\Big)d\nu$. Then, plugging the expression for the utility $\mathbb{V}_{\mathcal{D}}$ in (3.7) in the definition of $\mathbb{C}_{\pi^*}$ yields the identities

$$
\begin{aligned}
\mathbb{C}_{\pi^*}(p) - \mathbb{C}_{\pi^*}(\pi^*) &= \left((\pi^* - p)\mathcal{D}(p) + \int_{p_0}^{p}\mathcal{D}(\nu)d\nu\right) - \int_{p_0}^{\pi^*}\mathcal{D}(\nu)d\nu \\
&= \int_{\pi^*}^{p}\Big(\mathcal{D}(\nu) - \mathcal{D}(p)\Big)d\nu \\
&= I_{\pi^*}(p)\,.
\end{aligned}
\tag{3.10}
$$

By continuity of $\mathcal{D}$ from (3.1), the Mean Value Theorem for integrals applies to the function I_{π^*}. Because $I_{\pi^*}(\pi^*) = 0$, this implies that for any $p \neq \pi^*$ there exists $\tilde{p} = tp + (1-t)\pi^*$ and some $0 < t < 1$, such that

$$
I_{\pi^*}(p) = \Big(\mathcal{D}(\tilde{p}) - \mathcal{D}(p)\Big)(p - \pi^*)\,.
$$

To prove (3.9), in its left-hand side we use the identity (3.10), whereas for the right-hand side we analyze the sign of its two factors. Namely, if $\pi^* < p$, then $p - \pi^* \geq 0$. Additionally, writing $\tilde{p} = p + (1-t)(\pi^* - p)$ yields that $\tilde{p} < p$, so $\mathcal{D}(\tilde{p}) - \mathcal{D}(p) \geq 0$ because $\mathcal{D}$ is decreasing, and (3.9) holds. For the remaining case, $p < \pi^*$, a similar argument, this time writing $\tilde{p} = \pi^* + t(p - \pi^*)$, yields again (3.9), therefore showing that π^* minimizes the function $\mathbb{C}_{\pi^*}$, as stated.

Suppose, for contradiction purposes, that $0 \notin \partial\mathcal{D}(\pi^*)$ and the function $\mathbb{C}_{\pi^*}$ has another minimizer $p^* \neq \pi^*$. Since $\mathbb{C}_{\pi^*}(\pi^*) = \mathbb{C}_{\pi^*}(p^*)$, the left-hand side in (3.9) is zero. Hence,

$\mathcal{D}(\tilde{p}^*) = \mathcal{D}(p^*)$ for the corresponding mean value point $\tilde{p}^* = p^* + (1-t)(\pi^* - p^*)$. By convexity, $\mathcal{D}(\tilde{p}^*) \le t\mathcal{D}(p^*) + (1-t)\mathcal{D}(\pi^*)$, so $(1-t)\mathcal{D}(p^*) \le (1-t)\mathcal{D}(\pi^*)$, which forces $\pi^* \le p^*$ because $\mathcal{D}$ is decreasing by assumption. As $\mathcal{D}(\nu) - \mathcal{D}(p^*) \ge 0$ for all $\nu \in [\pi^*, p^*]$, the zero left-hand side in (3.10) implies that in fact the (continuous) function $\mathcal{D}$ is flat on the non-singleton interval $[\pi^*, p^*]$. This means that $\mathcal{D}$ has a null subgradient in the open interval (π^*, p^*), characterizing all points therein as $\mathcal{D}$-minimizers, leading to $0 \in \partial\mathcal{D}(\pi^*)$, which contradicts our initial assumption and therefore concludes the proof. □

4 Equilibrium of Multi-hub Networked Markets

Consider now an elastic agent ℓ with demand $\mathcal{D}^\ell$ in the toy market. Once the utility $\mathbb{V}^\ell = \mathbb{V}_{\mathcal{D}^\ell}$ is at hand, the explicit form of the agent problem is

$$\begin{cases} \max_{y^\ell, p^\ell} & \mathbb{V}^\ell(p^\ell) - \bar{p}y^\ell \\ \text{s.t.} & \mathcal{D}^\ell(p^\ell) \le y^\ell\,. \end{cases} \tag{4.1}$$

Notice that, in agreement with Remark 3.1, decisions of elastic agents are driven by both the commodity volume and the endogenous perception of price (y^ℓ and p^ℓ).

Given solutions to the individual elastic agents problems, that is $(\bar{y}^\ell, \bar{p}^\ell)$ to (4.1), and $\bar{x}$ to (2.1), the toy market will clear if there is a price $\bar{p}$ satisfying

$$0 \le \bar{x} - \bar{y}^\ell \perp \bar{p} \ge 0\,. \tag{4.2}$$

At equilibrium the consumption must be consistent with the market price, that is, the identity $\bar{y}^\ell = \mathcal{D}^\ell(\bar{p})$ holds.

A $\mathrm{CP}^{\mathrm{exp}}$ equilibrium for the toy market is a primal-dual solution $(\bar{x}, \bar{y}^\ell, \bar{p}^\ell, \bar{p})$ to

$$\begin{cases} \min_{x, y^\ell, p^\ell} & \mathbb{C}(x) - \mathbb{V}(p^\ell) \\ \text{s.t.} & Ax \le b \\ & \mathcal{D}(p^\ell) - y^\ell \le 0 \\ & y^\ell - x \le 0 \qquad (\text{mult. } p)\,. \end{cases} \tag{4.3}$$

We now explain how to build an equilibrium of the complementarity model by solving an optimization problem similar to (4.3). One interesting by-product of our development is related to existence of equilibria. Thanks to our reformulation, the existence of equilibrium is equivalent to the existence of solution of an optimization problem, see Remark 4.6.

Our first step is to extend the equilibria definitions given for the toy market to a general configuration, as the one considered for our numerical assessment in Section 5, representing the competitive natural gas market in Brazil.

4.1 General Market and $\mathrm{CP}^{\mathrm{imp}}$ Equilibrium

In real markets, decisions are made in temporal steps, transactions are carried out through different hubs, by a variety of suppliers and consumers with access to different entry and exit points, connected by a network subject to logistical constraints.

Since our formulation creates demand-induced utilities of the form (3.7) for each elastic agent, we separate market participants into two groups

$$\begin{array}{l} \ell \in \mathcal{L} \text{ refers to agents whose decisions are driven by demand functions, and} \\ i \in \mathcal{I} \text{ gathers other agents in the market, like suppliers and inelastic consumers.} \end{array} \tag{4.4}$$

Agents interact by making transactions in different hubs, and also by sharing logistic resources, when transporting the commodity along a network with limited capacity. These are $m_{\texttt{sh}}$ coupling constraints whose multiplier is denoted by π. In what follows π is referred to as a "price" vector, keeping in mind its components carry diverse meaning. The component will be a price for the commodity in the hub if it corresponds to a multiplier for the clearing of the hub. Multipliers related to shared capacity in the network represent possible congestion fees, not prices. Finally, since inputs can be time-dependent, all vectors involved can span multiple time intervals.

For two vectors v and w of dimension $m_{\texttt{sh}}$, their Euclidean inner product is denoted and defined by $\langle v, w\rangle_{\texttt{sh}} = \sum_{m=1}^{m_{\texttt{sh}}} v_m w_m$. Then, in the extension of (2.1) to the general setting each agent $i \in \mathcal{I}$ solves the problem

$$\begin{cases} \min_{x^i} & \mathbb{C}^i(x^i) - \left\langle \mathtt{H}^i x^i, \bar{\pi} \right\rangle_{\texttt{sh}} \\ \text{s.t.} & x^i \in X^i\,, \end{cases} \quad \text{where} \quad \begin{array}{l} X^i \subset \mathbb{R}^{n_i} \text{ is compact and convex,} \\ \mathbb{C}^i : \mathbb{R}^{n_i} \to \mathbb{R} \text{ is convex,} \\ \mathtt{H}^i \text{ is a matrix of order } m_{\texttt{sh}} \times n_i. \end{array} \tag{4.5}$$

No differentiability assumption is made on the endogenous cost $\mathbb{C}^i$ of the agent.

The format in (4.5) is very general: for our natural gas market model, $\mathcal{I}$ covers suppliers, refineries, and thermoelectric power plants. The agent's decision vector, $x^i \in X^i$ includes the feasible quantities to be supplied, produced, or consumed, as well as their logistical planning; thus requiring X^i to be compact and convex is not a strong assumption. Compactness ensures existence of minimizers in (4.5), so this assumption could be dropped if, for example coercivity of the objective function holds for the market under consideration. The shared interactions resulting from the agent's decision are described by the values $(\mathtt{H}^i x^i)_m$, for $1 \le m \le m_{\texttt{sh}}$. These quantities are valued by the exogenous vector $\bar{\pi}$, recalling that the price provides information on the state of the market, including any network congestion. Finally, for agents i who supply natural gas in our model, the second term in the objective represents revenue and $\mathbb{C}^i$ is a production cost. For inelastic consumers i such as refineries, the second term represents a cost and $\mathbb{C}^i$ is some utility, which may be zero.

Consider now the group $\mathcal{L}$. Each agent ℓ has a demand function $\mathcal{D}^\ell$ satisfying (3.1). The agent makes m_ℓ transactions in the market (along different hubs in the network and/or along different time steps), involving volumes $y^\ell \in \mathbb{R}^{m_\ell}$. For $1 \le m \le m_\ell$, each entry y^ℓ_m may represent consumption of a specific product, at a given physical point in the market, and at a particular time step. The consumption decision y^ℓ leads to market interations with other agents in the market via quantities $\mathtt{F}^\ell y^\ell$ that express the implications of such decision in terms of market state. For a given vector $\mathtt{h} \in \mathbb{R}^{m_{\texttt{sh}}}$, this is modeled by the complementarity condition

$$0 \le \mathtt{h} + \sum_{i\in\mathcal{I}} \mathtt{H}^i \bar{x}^i - \sum_{\ell\in\mathcal{L}} \mathtt{F}^\ell \bar{y}^\ell \perp \bar{\pi} \ge 0 \in \mathbb{R}^{m_{\mathtt{sh}}}\,, \tag{4.6}$$

that must be satisfied by $\bar{x}$ solving (4.5) and $\bar{y}$ as in (4.8). For $1 \le m \le m_{\mathtt{sh}}$, the m-th constraint in (4.6) may model market-clearing conditions expressing that at some points of the market (or network) when transactions take place, demand must meet supply. So $\mathtt{h}_m = 0$ and the corresponding entry $\bar{\pi}_m$ may be interpreted as a market price. A component $\mathtt{h}_m \neq 0$ arises when modeling some network logistical constraints, shared capacity bound of pipeplines, injection or extraction points. On these cases, the entries in the multiplier can be regarded as congestion fees.

As explained in Remark 3.1, each agent ℓ has an endogenous perception of the market price. The transformation is represented by the following affine function

$$\begin{array}{rcll} \mathbb{F}^\ell : \mathbb{R}^{m_{\mathtt{sh}}} & \to & \mathbb{R}^{m_\ell} & \text{that is increasing } (\pi^1 \le \pi^2 \Rightarrow \mathbb{F}^\ell(\pi^1) \le \mathbb{F}^\ell(\pi^2)) \\ \pi & \mapsto & [\mathtt{F}^\ell]^\top \pi + \mathtt{f}^\ell & \text{and maps non-negative prices into non-negative values.} \end{array} \tag{4.7}$$

The components of the vector $\mathbb{F}^\ell(\pi)$ are denoted by $\mathbb{F}^\ell_m(\pi)$, for $1 \le m \le m_\ell$.

Since an equilibrium if found when the market price coincides with the elastic agents' perceived prices, the following consumption consistency condition is required

$$\forall 1 \le m \le m_\ell \text{ and } \ell \in \mathcal{L} \quad \bar{y}^\ell_m = \mathcal{D}^\ell(\bar{\pi}^\ell_m),\text{ for } \bar{\pi}^\ell_m = \mathbb{F}^\ell_m(\bar{\pi}) \text{ and } \mathbb{F} \text{ from (4.7)}\,. \tag{4.8}$$

Summing up, a CP$^{\rm imp}$ equilibrium for this general model is a point $(\bar{x}, \bar{y}, \bar{\pi})$ such that, for each $i \in \mathcal{I}$ the subvector $\bar{x}^i$ is optimal for (4.5), for each $\ell \in \mathcal{L}$ given a mapping $\mathbb{F}^\ell$ from (4.7), the consumption consistency condition (4.8) holds for each subvector $\bar{y}^\ell$, and all the $m_{\mathtt{sh}}$ shared constraints satisfy (4.6).

The complementarity equilibrium CP$^{\rm imp}$ is related with a solution to an optimization problem for markets satisfying the following conditions.

Assumption 4.1 The market configuration is such that

- For each elastic consumer $\ell \in \mathcal{L}$ the demand function $\mathcal{D}^\ell$ satisfies (3.1).
- For each agent $i \in \mathcal{I}$, the endogenous functions $\mathbb{C}^i$ and $\mathtt{H}^i$ as well as the feasible set X^i satisfy the conditions in (4.5).

4.2 CP$^{\rm exp}$ Equilibrium

By reasoning as for the scalar toy model, given the exogenous market price $\bar{\pi}$, the elastic agent problem (4.1) in our general market setting is

$$\begin{cases} \max_{y^\ell, p^\ell} & \displaystyle\sum_{m=1}^{m_\ell} \Big(\mathbb{V}^\ell(p^\ell_m) - y^\ell_m \mathbb{F}^\ell_m(\bar{\pi})\Big) \\ \text{s.t.} & \mathcal{D}^\ell(p^\ell_m) \le y^\ell_m \quad 1 \le m \le m_\ell\,. \end{cases} \tag{4.9}$$

Note that it is equivalent for agent ℓ to formulate separately m_ℓ problems, each one on variables (y^ℓ_m, p^ℓ_m) for $1 \le m \le m_\ell$, as in (4.11) below. Then, the explicit equilibrium for the general market is given by a solution to the following optimization problem:

$$\begin{cases} \min_{x,y,p} \sum_{i\in\mathcal{I}} \mathbb{C}^i(x^i) + \sum_{\ell\in\mathcal{L}} \sum_{m=1}^{m_\ell} \left(y_m^\ell \mathtt{f}_m^\ell - \mathbb{V}^\ell(p_m^\ell) \right) \\ \text{s.t.} \quad x = (x^i)_{i\in\mathcal{I}}, \text{ and } (y,p) = (y^\ell, p^\ell)_{\ell\in\mathcal{L}} \\ \qquad x^i \in X^i & i \in \mathcal{I} \\ \qquad \mathcal{D}^\ell(p_m^\ell) \le y_m^\ell & 1 \le m \le m_\ell \,, \ell \in \mathcal{L} \\ \qquad \sum_{\ell\in\mathcal{L}} \mathtt{F}^\ell y^\ell \le \mathtt{h} + \sum_{i\in\mathcal{I}} \mathtt{H}^i(x^i) & (\text{mult. } \pi \in \mathbb{R}^{m_{\mathtt{sh}}}) \,. \end{cases} \tag{4.10}$$

The formal definitions of the implicit and explicit equilibria are

- CP$^{\text{imp}}$ is a point $(\bar{x} = (\bar{x}^i)_{i\in\mathcal{I}}, \bar{y}, \bar{\pi})$ solving the system formed by gathering the optimality conditions of (4.5) for $i \in \mathcal{I}$, together with (4.8) and (4.6).
- CP$^{\text{exp}}$ is a primal-dual solution $\Big((\bar{x}, \bar{y}, \bar{p}), \bar{\pi}\Big)$ to problem (4.10).

It should be noted that the CP$^{\text{exp}}$ definition involves finding not only a solution to (4.10), but also a dual value $\bar{\pi}$ with zero duality gap; that is, strong duality must hold for the coupling constraints. For optimization problems to satisfy this property, some regularity condition must hold, see Remark 4.6. For abstract sufficient conditions that ensure strong duality without convexity we refer to [23] and [10].

To connect the two equilibria concepts, we first relate the CP$^{\text{exp}}$ equilibrium with separate problems for the agents, obtained after relaxing the last constraint in (4.10).

Lemma 4.2 (CP$^{\text{exp}}$ equilibria and individual Lagrangian subproblems) Under Assumption 4.1, given an exogenous price $\bar{\pi} \in \mathbb{R}^{m_{\mathtt{sh}}}$, consider the individual problems defined by (4.5) for agents in the $\mathcal{I}$-group and by

$$\begin{cases} \min_{y_m^\ell, p_m^\ell} & y_m^\ell \mathbb{F}_m^\ell(\bar{\pi}) - \mathbb{V}^\ell(p_m^\ell) \\ \text{s.t.} & \mathcal{D}^\ell(p_m^\ell) \le y_m^\ell , \end{cases} \tag{4.11}$$

for each $1 \le m \le m_\ell$ for each agent ℓ in the $\mathcal{L}$-group. The following holds:

i) If $\Big((\bar{x}, \bar{y}, \bar{p}), \bar{\pi}\Big)$ is a CP$^{\text{exp}}$ equilibrium, the $\bar{x}^i$ and $(\bar{y}^\ell, \bar{p}^\ell)$-components of the equilibrium respectively solve (4.5) and (4.11) and satisfy the complementarity condition (4.6).

ii) Reciprocally, a solution x^* and (y^*, p^*) to the relaxed subproblems satisfying

$$0 \le \mathtt{h} + \sum_{i\in\mathcal{I}} \mathtt{H}^i(x^{*i}) - \sum_{\ell\in\mathcal{L}} \mathtt{F}^\ell y^{*\ell} \perp \bar{\pi} \ge 0$$

is a CP$^{\text{exp}}$ equilibrium.

Proof Consider relaxing the last constraint in (4.10), the corresponding objective is

$$\sum_{i\in\mathcal{I}}\mathbb{C}^i(x^i)+\sum_{\ell\in\mathcal{L}}\sum_{m=1}^{m_\ell}\Big(y^\ell_m\mathtt{f}^\ell_m-\mathbb{V}^\ell(p^\ell_m)\Big)+\Big\langle\bar\pi,\sum_{\ell\in\mathcal{L}}\mathtt{F}^\ell y^\ell-\sum_{i\in\mathcal{I}}\mathtt{H}^i x^i-\mathtt{h}\Big\rangle_{\mathtt{sh}}$$
$$=\sum_{i\in\mathcal{I}}\Big(\mathbb{C}^i(x^i)-\Big\langle\mathtt{H}^i x^i,\bar\pi\Big\rangle_{\mathtt{sh}}\Big)+\sum_{\ell\in\mathcal{L}}\sum_{m=1}^{m_\ell}(-\mathbb{V}^\ell(p^\ell_m)+y^\ell_m\mathtt{f}^\ell_m)+\sum_{\ell\in\mathcal{L}}\Big\langle\bar\pi,\mathtt{F}^\ell y^\ell\Big\rangle_{\mathtt{sh}}-\langle\bar\pi,\mathtt{h}\rangle_{\mathtt{sh}}$$

If $\langle v,w\rangle_\ell=\sum_{m=1}^{m_\ell}v_m w_m$ denotes the inner product of v and $w\in\mathbb{R}^{m_\ell}$, by (4.7) the identities

$$\Big\langle y^\ell,\mathtt{f}^\ell\Big\rangle_\ell+\Big\langle\mathtt{F}^\ell y^\ell,\bar\pi\Big\rangle_\ell=\Big\langle y^\ell,\mathbb{F}^\ell(\bar\pi)\Big\rangle_\ell=\sum_{m=1}^{m_\ell}y^\ell_m\mathbb{F}^\ell_m(\bar\pi)$$

hold. Therefore, the relaxed objective is nothing but the sum of the objective functions in (4.5) and (4.11). Moreover, once the coupling constraint is relaxed, the feasible set yields separate constraints on variables x^i and (y^ℓ_m,p^ℓ_m), as in (4.5) and (4.11):

$$x^i\in X^i\quad\text{and}\quad\mathcal{D}^\ell(y^\ell_m)\le p^\ell_m\text{ for }1\le m\le m_\ell\,.$$

Therefore, i) follows, noting that the complementarity relation therein results from the optimality conditions characterizing $\Big((\bar x,\bar y,\bar p),\bar\pi\Big)$ as a solution to (4.10). Item ii) is straighforward from feasibility of the relaxed constraint in (4.10) together with the complementarity relations assumed in this case. □

4.3 CP$^{\text{exp}}$ and CP$^{\text{imp}}$ Equilibria are Mathematically Equivalent

In order to explain how to recover a solution to (4.10) from a CP$^{\text{imp}}$ equilibrium, some consequences of the properties required for $\mathbb{F}^\ell$ are useful. More precisely, the associated m_ℓ-dimensional vector $\mathtt{f}^\ell$ and $m_\ell\times m_{\mathtt{sh}}$-matrix $\mathtt{F}^\ell$ are such that

$$(4.7)\text{ implies }\begin{cases}\mathtt{f}^\ell\ge0\in\mathbb{R}^{m_\ell}, & \\ \forall\pi\in\mathbb{R}^{m_{\mathtt{sh}}} & \pi\ge0\Longrightarrow[\mathtt{F}^\ell]^\top\pi\ge0\in\mathbb{R}^{m_\ell}\text{ or, equivalently,}\\ \forall y\in\mathbb{R}^{m_\ell} & y\ge0\Longrightarrow\mathtt{F}^\ell y\ge0\in\mathbb{R}^{m_{\mathtt{sh}}}\,.\end{cases}\tag{4.12}$$

Lemma 4.3 (from CP$^{\text{imp}}$ to CP$^{\text{exp}}$) Under Assumption 4.1, a CP$^{\text{imp}}$ equilibrium yields the CP$^{\text{exp}}$ equilibrium $\Big((\bar x,\bar y,\bar p),\bar\pi\Big)$ with $\bar p^\ell=\mathbb{F}^\ell(\bar\pi)$ for $\ell\in\mathcal{L}$.

Proof Given a CP$^{\text{imp}}$ equilibrium $(\bar x,\bar y,\bar\pi)$, we shall apply Lemma 4.2ii) to $x^*=\bar x$ and $(y^*,p^*)=(\bar y,\bar p)$ where $\bar p^\ell=\mathbb{F}^\ell(\bar\pi)$. First notice that (4.12) implies that $\bar p^\ell\ge0$. Second, the market-clearing condition (4.6) is just the complementarity relation in Lemma 4.2ii). Thus, we just need to prove that each $\bar x^i$ and $(\bar y^\ell_m,\bar p^\ell_m=\mathbb{F}^\ell_m(\bar\pi))$ solve (4.5) and (4.11), respectively. That $\bar x^i$ solves problem (4.5) for each $\mathcal{I}$-agent is straightforward. To continue, we claim that for each elastic agent $\ell\in\mathcal{L}$ and $1\le m\le m_\ell$ the pair $\Big(\bar y^\ell_m,\bar p^\ell_m=\mathbb{F}^\ell_m(\bar\pi)\Big)$ is optimal for problem (4.11). Feasibility is straightforward from (4.8) noting that $\bar\pi^\ell$ therein is just $\bar p^\ell$. For notational convenience we shorten the objective function in (4.11) to

$$f(y,p)=y\mathbb{F}^\ell_m(\bar\pi)-\mathbb{V}^\ell(p)\,,$$

and consider the intermediate function in Proposition 3.2, written with $\pi^* = \mathbb{F}^\ell_m(\bar\pi) = \bar p^\ell_m$, for demand and utilities $\mathcal{D}^\ell$ and $\mathbb{V}^\ell$, that is

$$\mathbb{C}^\ell_{\mathbb{F}^\ell_m(\bar\pi)}(p) = \mathbb{F}^\ell_m(\bar\pi)\mathcal{D}^\ell(p) - \mathbb{V}^\ell(p)\,.$$

The consistency condition (4.8) implies that $\bar y^\ell_m = \mathcal{D}^\ell(\bar p^\ell_m)$, so $f(\bar y^\ell_m, \bar p^\ell_m) = \mathbb{C}^\ell_{\mathbb{F}^\ell_m(\bar\pi)}(\bar p^\ell_m)$. Moreover, by Proposition 3.2 a minimizer for this function is $\bar p^\ell_m = \pi^*$, which implies that

$$f(\bar y^\ell_m, \bar p^\ell_m) \le \mathbb{C}^\ell_{\mathbb{F}^\ell_m(\bar\pi)}(p) = f(y,p) + (\mathcal{D}^\ell(p) - y)\mathbb{F}^\ell_m(\bar\pi)\,.$$

The result follows, combining the non-negativity of $\mathbb{F}^\ell_m(\bar\pi)$ with the inequality $y \ge \mathcal{D}^\ell(p)$ satisfied by any feasible pair (y,p) in (4.11). □

Before deriving a $\mathrm{CP}^{\mathrm{imp}}$ equilibrium from (4.10), note that in a $\mathrm{CP}^{\mathrm{exp}}$ equilibrium each optimal consumption can be taken equal to the demand function at the endogenous price perceived by the elastic consumer. Specifically,

$$\left(\bar x, \bar y, \bar p\right) \text{ solves (4.10)} \Longrightarrow \left(\bar x, \left(\mathcal{D}^\ell(\bar p^\ell_m)\right)_{1\le m\le m_\ell}, \bar p\right) \text{ also solves (4.10).} \tag{4.13}$$

To prove this relation, combining the third relation in (4.12) with the inequality $\mathcal{D}^\ell(\bar p^\ell_m) \le \bar y^\ell_m$ ensures that the new point is feasible for (4.10). Optimality follows by computing the difference betwee the objective value at the new point and the optimal one. The difference amounts to summing over ℓ and m the terms $(\mathcal{D}^\ell(\bar p^\ell_m) - \bar y^\ell_m)\mathtt{f}^\ell_m$, which are all non-positive, by the first relation in (4.12).

Building a $\mathrm{CP}^{\mathrm{imp}}$ equilibrium from a solution to (4.10) depends on the market price, as perceived by elastic consumers ($\mathbb{F}^\ell(\bar\pi)$), coinciding with their endogenous optimal prices ($\bar p^\ell$), an identity formalized in (4.14) below.

Lemma 4.4 (from $\mathrm{CP}^{\mathrm{exp}}$ to $\mathrm{CP}^{\mathrm{imp}}$) Under Assumption 4.1, a $\mathrm{CP}^{\mathrm{exp}}$ equilibrium $\left((\bar x, \bar y, \bar p), \bar\pi\right)$ in which all price perceptions coincide, i.e., satisfying

$$\bar p^\ell = \mathbb{F}^\ell(\bar\pi) \text{ for all } \ell \in \mathcal{L}\,, \tag{4.14}$$

gives $(\bar x, \bar y, \bar\pi)$ as a $\mathrm{CP}^{\mathrm{imp}}$ equilibrium.

Proof The optimality conditions stating that $\left((\bar x, \bar y, \bar p), \bar\pi\right)$ solves (4.10) yields the optimality conditions for (4.5) as well as the market-clearing relations in (4.6). Next, by (4.13) the $\mathrm{CP}^{\mathrm{exp}}$ equilibrium takes the form $\left((\bar x, \bar y, \bar p), \bar\pi\right)$ with $\bar y^\ell_m = \mathcal{D}^\ell(\bar p^\ell_m)$. Combining this identity with (4.14) we see that $\bar y^\ell_m = \mathcal{D}^\ell(\bar\pi^\ell_m)$ so the relation of price consistency in (4.8) is satisfied. This is the last condition required for $(\bar x, \bar y, \bar p)$ to be a $\mathrm{CP}^{\mathrm{imp}}$ equilibrium. □

We are now in a position to state the equivalence between a complementarity equilibrium and solving the optimization problem (4.10). This is achieved by associating with a $\mathrm{CP}^{\mathrm{exp}}$ equilibrium a new equilibrium point that satisfies (4.14).

Theorem 4.5 (Equivalence of equilibria) Under Assumption 4.1 the $\mathrm{CP}^{\mathrm{imp}}$ and $\mathrm{CP}^{\mathrm{exp}}$ equilibria are equivalent.

Proof Lemma 4.3 shows that a $\mathrm{CP}^{\mathrm{imp}}$ equilibrium yields a $\mathrm{CP}^{\mathrm{exp}}$ one. The reciprocal result in Lemma 4.4 depends on condition (4.14) being satisfied. We start with a $\mathrm{CP}^{\mathrm{exp}}$ equilibrium point $\bar{z} = \Big((\bar{x}, \bar{y}, \bar{p}), \bar{\pi}\Big)$ with $\bar{y}^\ell_m = \mathcal{D}^\ell(\bar{p}^\ell_m)$, by (4.13). Then we define a new point $\bar{\mathtt{Z}} = \Big((\bar{x}, \bar{y}, \bar{\mathtt{P}}), \bar{\pi}\Big)$ where $\bar{\mathtt{P}}^\ell_m = \mathbb{F}^\ell_m(\bar{\pi})$. From its definition, $\bar{\mathtt{Z}}$ satisfies condition (4.14), written with $\bar{p} = \bar{\mathtt{P}}$. Thus, by Lemma 4.4, showing that $\bar{\mathtt{Z}}$ is a $\mathrm{CP}^{\mathrm{exp}}$ equilibrium is enough to ensure that $(\bar{x}, \bar{y}, \bar{\pi})$ is a $\mathrm{CP}^{\mathrm{imp}}$ equilibrium. In turn, to prove $\bar{\mathtt{Z}}$ is a $\mathrm{CP}^{\mathrm{exp}}$ equilibrium, we will apply Lemma 4.2ii). Because $\bar{z}$ is a $\mathrm{CP}^{\mathrm{exp}}$ equilibrium, by Lemma 4.2i) for all $i \in \mathcal{I}$, the subvector $\bar{x}^i$ solves (4.5), the triplet $(\bar{x}, \bar{y}, \bar{\pi})$ satisfies the complementarity condition (4.6), and the pairs $(\mathcal{D}^\ell(\bar{p}^\ell_m), \bar{p}^\ell_m)$ solve (4.11) for all ℓ and m. Hence, we only need to show that $(\mathcal{D}^\ell(\bar{p}^\ell_m), \bar{\mathtt{P}}^\ell_m)$ also solves (4.11). Feasibility of this point is clear. If $f(y, p)$ denotes the objective function in (4.11), the optimal value is is $f^\ell_m = f(\mathcal{D}^\ell(\bar{p}^\ell_m), \bar{p}^\ell_m))$. Moreover, the definition of the intermediate function in Proposition 3.2 with $\pi^* = \mathbb{F}^\ell_m(\bar{\pi})$ gives that $f^\ell_m = \mathbb{C}^\ell_{\mathbb{F}^\ell_m(\bar{\pi})}(\bar{p}^\ell_m)$. However, since $\bar{\mathtt{P}}$ satisfies (4.14), $\mathbb{C}^\ell_{\mathbb{F}^\ell_m(\bar{\pi})} = \mathbb{C}^\ell_{\bar{\mathtt{P}}^\ell_m}$, by Proposition 3.2, the point $p = \bar{\mathtt{P}}^\ell_m$ is a minimizer, so $f^\ell_m = \mathbb{C}^\ell_{\mathbb{F}^\ell_m(\bar{\pi})}(\bar{p}^\ell_m) \geq \mathbb{C}^\ell_{\bar{\mathtt{P}}^\ell_m}(\bar{\mathtt{P}}^\ell_m)$. Since, by definition, $\mathbb{C}^\ell_{\bar{\mathtt{P}}^\ell_m}(\bar{\mathtt{P}}^\ell_m) = f(\mathcal{D}^\ell(\bar{p}^\ell_m), \bar{\mathtt{P}}^\ell_m))$, the point $(\mathcal{D}^\ell(\bar{p}^\ell_m), \bar{\mathtt{P}}^\ell_m)$ solves (4.11), as claimed, therefore concluding the proof. □

Remark 4.6 (On existence of equilibrium) Thanks to Theorem 4.5, proving existence of $\mathrm{CP}^{\mathrm{imp}}$ or $\mathrm{CP}^{\mathrm{exp}}$ is equivalent. For the latter, suppose the feasible set in (4.10) is bounded. The objective function will be convex if the demand-induced utilities are concave (a property which, in view of (3.8), is satisfied when demand functions are linear). Concavity of the utilities together with Assumption 4.1 imply that problem (4.10) is convex. Then, if a Slater's-like constraint qualification holds, the problem is also feasible, which suffices to guarantee there exists a primal-dual solution with zero duality gap.

5 Application to Brazil's Natural Gas Market

In order to assess the computational performance and scalability of our approach, we consider a model representing the competitive market of natural gas in Brazil. The considered instances, described below, were created from

5.1 Market Description

As illustrated in Figure 5.1, prepared by the authors based on data from the National Agency of Petroleum, Natural Gas and Biofuels and the Energy Research Office, natural gas in Brazil circulates through a network with three hubs: NTS (South-Southeast), TAG (North-Northeast), and TBG (Midwest-South).

The network in Figure 5.1 reflects the radial topology of the Brazilian infrastructure, modeled as directed arcs, where each interconnection consists of an exit point from the origin hub and an entry point into the destination hub. Specifically, bidirectional flows connect the Southeast and Northeast through the NTS ↔ TAG link, while the NTS ↔ TBG connection links the Southeast to the Midwest and South. Because

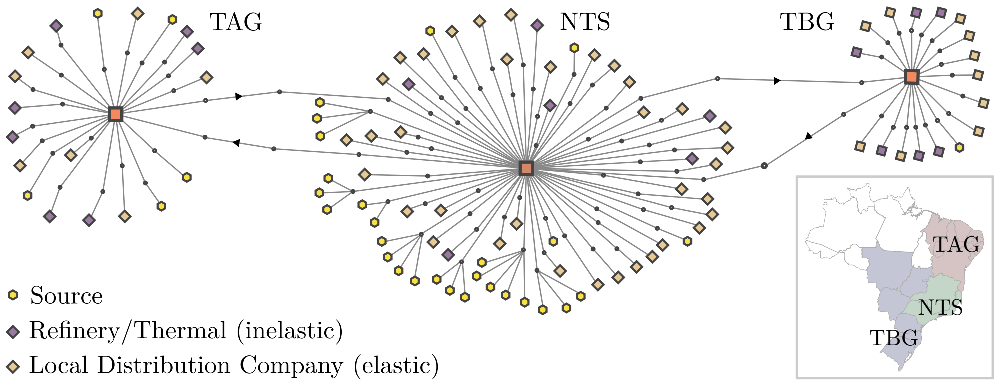


Fig. 5.1: Schematic diagram of the Brazilian natural gas transportation network. Nodes represent strategic aggregation points for market-clearing analysis rather than individual physical delivery stations.

there is no direct link between TAG and TBG, all gas volumes exchanged between these hubs must transit through NTS. An interconnection between hubs i and j has a transport capacity $\mathtt{h}_{i,j}$ and a tariff (the exit fee of hub i added to the entry fee of hub j). These parameters define spatial arbitrage limits in the market: price differentials between hubs are bounded by the fees, except when capacity constraints bind and trigger a scarcity rent (the corresponding multiplier in the shared constraints (4.6)).

In the notation (4.4), the group $\mathcal{L}$ has 19 elastic agents, primarily representing the largest local distribution companies, in charge of supplying the industrial, commercial, and residential sectors in the country. For each $\ell \in \mathcal{L}$ a linear demand function with intercept a^ℓ and slope b^ℓ was estimated from historical data in a manner that the resulting market behavior was consistent with national energy balance reports.

On the supply side of group $\mathcal{I}$, there are 12 producers getting natural gas from the Bolivia-Brazil pipeline, 24 domestic sources, and 8 LNG terminals. Four of those producers can import LNG directly into both NTS and TBG (these agents represent the market's "backstop technology", offering high-volume availability at a premium price, which effectively caps the equilibrium price in periods of extreme scarcity).

The group $\mathcal{I}$ also comprises 54 inelastic agents distributed across the network: 7 within the TAG hub, 11 in TBG, and 36 in NTS. Physically, these consumers correspond to 13 refineries with a constant demand and 41 thermoelectric plants featuring deterministic time-varying demand profiles. Unlike refineries, thermoelectric consumption is highly seasonal; in the Brazilian power system, dispatch of thermal power plants is closely coupled with hydrology. During the dry season, gas-fired generation increases to compensate for depleted hydroelectric reservoirs. This dynamic is incorporated through time-dependent demand parameters that mimick historical seasonal peaks. The 13 modeled refineries were treated as synthetic agents whose locations and demand magnitudes were created preserving the spatial density and consumption scale of the actual infrastructure while maintaining anonymity. Finally, to simulate the hydrologically driven dispatch of the Brazilian power sector, seasonal profiles were assigned to the 41 thermoelectric agents. These profiles mimic elevated gas demand during the dry season (May-September), inducing the numerical stress required by the temporal coupling constraints to rigorously evaluate the system's reliance on LNG imports during peak periods.

In (4.5), the cost term $\mathbb{C}^i(x^i)$ includes not only production costs but also the transportation fees for each $i \in \mathcal{I}$. By contrast, transportation fees incurred by elastic agents determine their price perception and are incorporated in the vector $\mathtt{f}^\ell$ from (4.7). The matrices $\mathtt{H}^i$ and $\mathtt{F}^\ell$ in the shared constraints (4.6) represent the access points allocated by the regulatory agency to each market participant.

Continuing with (4.5), the feasibility sets X^i include production ramp limits to reflect the technical inertia of gas extraction and processing facilities. Domestic sources and the Bolivia pipeline are subject to a 10% and 50% ramp-up and ramp-down limit per period. No ramp is imposed to LNG terminals, given their high regasification flexibility. From a computational perspective, ramp constraints introduce a temporal coupling across the planning horizon, preventing the decomposition of the problem into independent monthly subproblems. As the solver must optimize the entire horizon simultaneously, the problem difficulty increases (there are more non-zero elements in the MCP Jacobian or the optimization Hessian).

5.2 Numerical Setup and Experimental Design

Both the $\mathrm{CP}^{\mathrm{exp}}$ and $\mathrm{CP}^{\mathrm{imp}}$ models were formulated in Python via Pyomo and solved on an AMD Ryzen 7 4700U machine with 16 GB of RAM. The MCP version of each instance is derived directly from the optimality conditions of (4.10), to ensure that all solvers explore the exact same equilibrium point. No specific heuristics or warm-starts were utilized, and all solvers were executed with default convergence parameters.

As shown in (3.8), different demand functions yield different demand-induced utilities, which yield different types of optimization problems in (4.10). To find a $\mathrm{CP}^{\mathrm{exp}}$ equilibrium this involves solving a quadratic programming problem (QP) or a nonlinear programming problem (NLP), the solvers details is given in Table 5.1.

Table 5.1: Solvers setup in the benchmarking study.

Problem Type	Demand Nature	Primary Solver	Role in the Study
QP	Linear	Gurobi 11.0	Benchmark for the reformulation
NLP	Nonlinear	IPOPT	Robustness with nonlinear utility
MCP	Any	PATH	Baseline (State-of-the-art)

To analyse how dimensionality and temporal coupling (ramp constraints) affect convergence, we generated 5 instances varying the planning horizon T, in the short-term ($T \in \{1, 3, 6\}$ months) and in the medium to long-term ($T \in \{12, 24\}$ months). Table 5.2 reports the dimensions involved when solving the two equilibrium models, as the planning horizon T expands. With $\mathrm{CP}^{\mathrm{exp}}$, the size of the problem scales linearly, reaching 6648 primal variables for the 24-month instance. In the $\mathrm{CP}^{\mathrm{imp}}$ formulation, the MCP square system has 12 718 nonlinear complementarity constraints. As reported in Table 5.3, this dimensionality poses a significant challenge for PATH, due to the large number of nonzero elements in the Jacobian and dense intertemporal coupling.

Table 5.2: Model dimensions and planning horizons.

	CP^{exp}		CP^{imp}
T (months)	**Variables**	**Constraints**	**# Equations**
1	277	224	482
3	831	772	1546
6	1662	1594	3142
12	3324	3238	6334
24	6648	6526	12 718

5.3 Computational Performance and Scalability

Table 5.3 compares CP^{imp} and CP^{exp} computational performances, distinguishing between **Build**, the formulation build time needed by by Pyomo's symbolic differentiation and model assembly and **Solve**, the solver execution time.

Table 5.3: Performance comparison of the two models.

	CP^{imp} **(PATH)** [s]		CP^{exp} **(QP)** [s]		**Total**
T (months)	**Build**	**Solve**	**Build**	**Solve**	**Speedup**
1	0.26	0.26	0.15	0.03	3×
3	3.29	6.41	0.17	0.06	42×
6	9.67	46.60	0.21	0.11	176×
12	33.00	190.60	0.29	0.38	334×
24	115.60	2123.00	0.68	0.46	1964×

We notice a stark contrast in computational effort between the two equilibrium formulations. The difference in performance is noticeable not only in the intrinsic better scalability of CP^{exp} solution algorithm, but also in the mathematical overhead required from Pyomo when dealing with CP^{imp} . In fact, a significant portion of the total computational burden with CP^{imp} is incurred prior to solver invocation. Mounting the MCP model scales from 0.26 s to more than 115 s. The bottleneck seems to be the symbolic differentiation required to generate the MCP: differentiating each constraint to construct the Jacobian matrix becomes computationally more and more intensive as the instance size grows. CP^{exp} , by contrast, exhibits a near-linear scaling, solving the largest instance in only 0.68 s (to be compared with about 35 min needed by PATH).

5.4 Market Equilibrium Analysis

We consider finding a CP^{exp} equilibrium solution for the Brazilian natural gas market over a 24-month horizon. Simulating the network over two years allows us to track how decentralized strategic choices by market agents converge toward a global equilibrium under fluctuating demand and rigid infrastructure constraints. The results on consumption and supply are reported in the top and bottom graphs in Figure 5.2, which show, respectively, the monthly demand from refineries, elastic LDCs, and thermal plants, and the the percentage contribution of the supply mix.

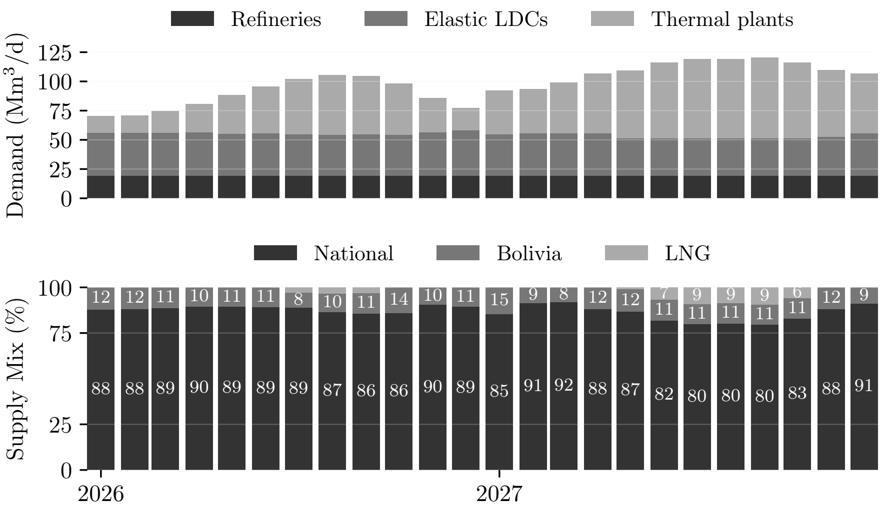
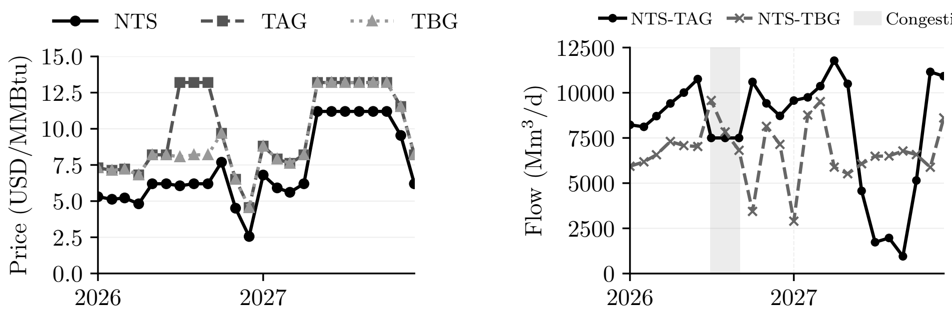


Fig. 5.2: Demand segmentation in $Mm^3\,d^{-1}$ (a) and Supply mix contribution (b).

Figure 5.2 shows that, at equilibrium, the CP^{exp} model prioritizes domestic supply, consistently contributing 80 % to 92 % of the total volume due to its lower marginal extraction costs. Volatility on the demand is driven by the seasonality of the demand of thermal power plants. The thermal demand exhibits a sharp peak in the second half of 2027, beyond the domestic production and Bolivia imports capacity. To clear the market, the CP^{exp} model resorted to (the expensive) LNG as the marginal flexible source, resulting in a peak share of LNG supply of 15 %. During the same period, the price-responsiveness of elastic LDCs is clear. Under normal market conditions, this segment exhibits stable consumption. But in the second half of 2027, when the LNG supply increased the spot price, the LDC demand contracted visibly.

The regional interplay between hubs is shown in Figure 5.3, contrasting the spot prices across the three main hubs (Figure 5.3a) against physical flows and localized congestion events within the main transmission corridors (Figure 5.3b).

On the left graph in Figure 5.3, we note that, in agreement with the observed thermal demand spike, a generalized price surge occurs across all hubs during the second half of 2027, with equilibrium prices stabilizing around 13 USD/MMBtu. The



(a) NTS, TAG, and TBG Spot prices.

(b) Transmission flows and congestion.

Fig. 5.3: Hub pricing dynamics (a) and corresponding infrastructure utilization (b).

same graph also shows that a significant spatial price decoupling emerges at the TAG hub over a three-month window in 2026. Prices at the NTS and TBG hubs remain low, but the TAG price spikes past 13 USD/MMBtu. This locational divergence stems from the congestion event highlighted as a vertical bar on the right plot in Figure 5.3, where the NTS → TAG interconnection hits its physical capacity ceiling of 7500 Mm3/d.

In optimization terms, the shared capacity constraint becomes active and its dual variable is positive. The physical bottleneck prevents lower-cost gas in the NTS hub from reaching the TAG hub, effectively isolating the region. The TAG hub is then forced to satisfy local demand using more expensive regional sources, or LNG. This behavior confirms the CP^{exp} model's capacity to capture locational marginal pricing and the economic consequences of infrastructure scarcity.

Concluding Remarks

For multi-hub network-constrained competitive markets having consumers with strategic behavior, complementarity models face numerical drawbacks when the equilibrium must be found over many time periods, even when employing PATH, the state-of-the-art MCP solver. We presented an alternative model, that through the solution of an optimization problem, provides the equilibrium with much less computational effort.

The mathematical equivalence of both formulations was shown assuming concavity for the demand-induced utilities introduced for elastic consumers. An interesting topic of future research could focus on relating the CP^{exp} and CP^{imp} equilibria when the assumption is relaxed to quasi-concavity of the utilities.

Declarations

Contribution. All authors contributed equally to this research.
Conflicts of Interest. There is no conflict of interest of any kind related to the manuscript.
Data and Code Availability. All data and code is available from the authors upon request.
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