When do Mixed-Integer Games Admit Rational Equilibria?

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Abstract. We consider mixed-integer linear-quadratic generalized Nash equilibrium problems, i.e., games in which each player solves a mixed-integer program subject to linear constraints in her own and rivals' strategies as well as an objective which is quadratic in her own strategies and bilinear in her own and rivals' strategies. For this class of games, we study the question of the existence of *rational equilibria* assuming rational input data. We distinguish four subclasses according to the presence of player-quadratic terms in the objective and rival-dependent constraints. As our main result, we completely settle the rationality question for all four subclasses, i.e., we show that only player-linear games without player-quadratic terms and without rival-dependent constraints admit rational equilibria—if the game admits equilibria at all. All other three classes contain instances with irrational equilibria only.

1. Introduction

Consider the class of linear-quadratic generalized Nash equilibrium problems (GNEPs) with player set $N = \{1, \dots, n\}$, where each player $i \in N$ solves the optimization problem

$$\begin{aligned} \min_{x_i \in \mathbb{R}^{l_i}_+} \quad & \sum_{j \in N} x_j^\top Q_{ij} x_i + d_i^\top x_i \\ \text{s.t.} \quad & \sum_{j \in N} A_{ij} x_j \geq b_i. \end{aligned}$$

Here, $x_i \in \mathbb{R}^{l_i}_+$ is the strategy of player i, which we assume (w.l.o.g.) to be non-negative. If the matrices Q_i, $i \in N$ are positive semidefinite, we obtain the classic linear-quadratic convex generalized Nash equilibrium problem (GNEP) that contains, among others, the 2-player mixed Nash equilibrium problem (Nash 1950) as a special case. For such convex problems, Nash equilibria can be characterized by the players' KKT conditions, which in turn can be stated as a linear complementarity problem (LCP). For rational input data, i.e., Q_{ij}, A_{ij}, d_i, and b_i are rational for all $i, j \in N$, it follows by basic linear programming arguments that the solution set of an LCP is either empty or it contains a rational point. Indeed, the pivoting algorithm by Lemke (1965) or the Lemke–Howson algorithm (cf. Lemke and Howson (1964)) for bimatrix games would output such a rational solution.

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In this paper, we consider the more general class of *mixed-integer* linear-quadratic generalized Nash equilibrium problems of the form

$$\begin{aligned} \min_{x_i} \quad & \sum_{j \in N} x_j^\top Q_{ij} x_i + d_i^\top x_i \\ \text{s.t.} \quad & \sum_{j \in N} A_{ij} x_j \geq b_i, \\ & x_i \in \mathbb{R}_+^{l_i} \times \mathbb{Z}_+^{k_i}, \end{aligned} \qquad (\mathcal{P}_i(x_{-i}))$$

where we again assume rational data Q_{ij}, A_{ij}, d_i, and b_i for all $i, j \in N$. Despite the presence of integer variables, it is known that mixed-integer linear-quadratic *optimization problems* with convex objective (i.e., the case $n = 1$) always admits a rational solution if the problem admits a solution at all since the KKT conditions for fixed integer variables yield an LCP. For the general case of $n \geq 2$, we investigate in this paper the question whether a *rational* Nash equilibrium exists in case of the game admitting any Nash equilibrium. Note the fundamental nature of this question because rationality of equilibria is a *necessary* prerequisite for the existence of exact finite-time algorithms under the Turing machine model. Moreover, the class of mixed-integer linear-quadratic GNEPs we consider here is highly relevant. The purely continuous sub-class has a long-standing history in game theory (cf. the survey of Facchinei and Kanzow (2010)), while the purely integer sub-class has gained significant interest over the last two decades within the realm of integer programming games (Carvalho et al. 2023; Dragotto and Scatamacchia 2023; Kirst et al. 2024; Schwarze and Stein 2023). Transitioning to a mixed-integer framework represents the natural next step and very recently, a few papers started to study the topic (Duguet et al. 2025a,b; Harks and Schwarz 2025; Sagratella 2017a,b). In particular, no results on the rationality of mixed-integer games are, to the best of our knowledge, known.[1]

In the following, we distinguish four problem sub-classes according to whether or not quadratic terms in the objective are present and whether or not rival-dependent constraints are present:

(i) general player-quadratic GNEPs (PQ-GNEP) as above;
(ii) player-linear GNEPs (PL-GNEP), where $Q_{ii} = 0$ for all $i \in N$;
(iii) player-quadratic NEPs (PQ-NEP), where $A_{ij} = 0$ for all $j \neq i$ and $i \in N$;
(iv) player-linear NEPs (PL-NEP), where $Q_{ii} = 0, A_{ij} = 0$ for all $j \neq i, i \in N$.

Our main result completely classifies all four cases. In particular, we show that only player-linear NEPs admit rational equilibria (if they exist) and no other class does so in general:

	NEP	GNEP
PL	yes (Sec. 3, Thm. 3.5)	no (Sec. 5, Ex. 5.1)
PQ	no (Sec. 4)	no (Sec. 5, Ex. 5.1)

2. Preliminaries

Let us briefly introduce some additional terminology we use throughout the paper. We denote by $\pi_i : \prod_{j \in N} \mathbb{R}^{l_j + k_j} \to \mathbb{R}$ the cost function of player i. Moreover, we use standard game-theoretic notation and write x_{-i} for the vector of strategies of all players except player i and call a vector of strategies $x = (x_i)_{i \in N}$ strategy profile. For any x_{-i}, we define the strategy set of player i by $X_i(x_{-i}) \coloneqq \{x_i \in$

[1]Note that the classic three-player game of Nash (1951) showing a unique irrational mixed Nash equilibrium relies on the fact that the expected payoff of the players is a multilinear *cubic* function so that the equilibrium conditions lead to a quadratic equation. Our model is linear-quadratic and does not include cubic terms.

$\mathbb{R}^{l_i} \times \mathbb{Z}^{k_i} \colon \sum_{j\in N} A_{ij}x_j \geq b_i\}$ and call the product $X(x) \coloneqq \prod_{i\in N} X_i(x_{-i})$ the joint strategy set at x. We call a strategy profile x feasible if $x \in X(x)$ holds.

A feasible strategy profile $x^* \in X(x^*)$ is called a (generalized) Nash equilibrium if for all $i \in N$ the following holds:

$$\pi_i(x_i^*, x_{-i}^*) \leq \pi_i(x_i, x_{-i}^*) \quad \text{for all } x_i \in X_i(x_{-i}^*).$$

We use the following notation throughout the paper. If nothing else is stated, we assume that a strategy profile is of the form $x = (y, z)$ where $y_i \coloneqq (x_{i,1}, \dots, x_{i,l_i})$ denotes the continuous components of player i's strategy and analogously $z_i \coloneqq (x_{i,l_i+1}, \dots, x_{i,l_i+k_i})$ the integer components. We use the analogue notation for the entire and partial strategy profiles x and x_{-i}, e.g., we abbreviate $z_{-i} \coloneqq (z_j)_{j\neq i}$.

3. Positive Result for Player-Linear NEPs

In this section, we prove—under our working assumption of rational input data—that player-linear NEPs always admit a rational NE if they admit an NE at all. Our proof is constructive and—under the additional condition of the set of feasible strategies being bounded—can be used to implement a finite time algorithm determining whether an equilibrium exists and in case of existence, outputs one, cf. Remark 3.6. To this end, we assume for the entire section that for all $i \in N$, we have $Q_{ii} = 0$ and $A_{ij} = 0$ for $j \neq i$. In order to prove the result, we require the following definitions. For any integer strategy profile z^*, we define the following linear complementary problem (LCP) in the continuous variables y, λ for fixed z^*:

$$\begin{aligned}
\sum_{j\neq i} \left(Q_{ij}^{\text{con}}\right)^\top (y_j, z_j^*) + d_i^{\text{con}} - \left(A_{ii}^{\text{con}}\right)^\top \lambda_i &\geq 0 && \text{for all } i \in N,\\
y_i^\top \left(\sum_{j\neq i} \left(Q_{ij}^{\text{con}}\right)^\top (y_j, z_j^*) + d_i^{\text{con}} - \left(A_{ii}^{\text{con}}\right)^\top \lambda_i \right) &= 0 && \text{for all } i \in N,\\
A_{ii}(y_i, z_i^*) - b_i &\geq 0 && \text{for all } i \in N,\\
\lambda_i^\top \left(A_{ii}(y_i, z_i^*) - b_i \right) &= 0 && \text{for all } i \in N,\\
y_i \in \mathbb{R}_+^{l_i},\ \lambda_i \in \mathbb{R}_+^{m_i} & && \text{for all } i \in N,
\end{aligned} \tag{LCP(z^*)}$$

where m_i is the dimension of b_i and Q_{ij}^{con} as well as A_{ii}^{con} denote the sub-matrices corresponding to the continuous part of x_i. Analogously d_i^{con} denotes the sub-vector corresponding to the continuous part of x_i. This LCP corresponds to the KKT conditions of the continuous game induced by z^*.

In addition, we define the optimal-value function for any player $i \in N$ for fixed integer strategy components z^* as

$$\phi_i(y_{-i}; z^*) \coloneqq \min_{y_i} \left\{ \pi_i(y_i, y_{-i}, z^*) \colon A_{ii}^{\text{con}} y_i \geq b_i - A_{ii}^{\text{int}} z_i^* \right\}, \tag{1}$$

where A_{ii}^{int} denotes the sub-matrix corresponding to the integer part of x_i. Based on this function, we define the system of inequalities

$$\phi_i(y_{-i}; z^*) \leq \phi_i(y_{-i}; (z_i, z_{-i}^*)) \quad \text{for all } z_i \in Z_i \text{ and } i \in N \tag{$\Phi(z^*)$}$$

with $Z_i \coloneqq \{z_i \in \mathbb{Z}^{k_i} \mid \exists y_i \in \mathbb{R}_+^{l_i} : A_{ii}^{\text{con}} y_i \geq b_i - A_{ii}^{\text{int}} z_i\}$. The above system is a necessary condition for a point (y, z^*) to be an NE: For fixed $i \in N$ and z_i, the stated inequality ensures that there exists a continuous strategy corresponding to z_i^* that is optimal compared to all possible unilateral deviations with fixed integer part z_i. The following lemma shows that, together with (LCP(z^*)), we can state a necessary and sufficient condition for the point (y, z^*) to be an NE. In the following, we say that y^* solves (LCP(z^*)) if there exist corresponding λ_i^*, $i \in N$, such that y^* and λ_i^*, $i \in N$, solve (LCP(z^*)).

Lemma 3.1. Consider an arbitrary strategy profile $x^* = (y^*, z^*)$. Then, x^* is an NE if and only if y^* solves (LCP(z^*)) and ($\Phi(z^*)$) simultaneously.

Proof. We start by making two observations.

1. Observation: Note that y^* solves (LCP(z^*)) if and only if y_i^* is an optimal solution to

$$\min_{y_i} \quad \pi_i(y_i, y_{-i}, z^*) \quad \text{s.t.} \quad A_{ii}^{\text{con}} y_i \geq b_i - A_{ii}^{\text{int}} z_i^* \tag{2}$$

for each $i \in N$. This is true as the LCP in (LCP(z^*)) for a fixed $i \in N$ contains the KKT conditions for the above problem. These KKT conditions are necessary and sufficient optimality conditions as the above problem is an LP. Moreover, we note that in case that y^* solves (LCP(z^*)), then

$$\pi_i(x^*) = \phi_i(y_{-i}^*; z^*) \quad \text{for all } i \in N. \tag{3}$$

2. Observation: In the situation of (3), y^* solves ($\Phi(z^*)$) if and only if for all $i \in N$, we have

$$\pi_i(x^*) \leq \pi_i(y_i, z_i, y_{-i}^*, z_{-i}^*) \quad \text{for all } (y_i, z_i) \text{ with } A_{ii}(y_i, z_i) \geq b_i,$$

which is exactly the equilibrium condition for x^*.

We now prove both directions of the lemma separately.

"⇒": Let x^* be an NE. Then, every player is optimal w.r.t. unilateral deviations in her continuous variables, i.e., y_i^* solves (2) for each player $i \in N$. Hence, by our first observation, y^* solves (LCP(z^*)). The second observation then shows that y^* solves ($\Phi(z^*)$) since x^* is an NE.

"⇐": Let x^* solve (LCP(z^*)) and ($\Phi(z^*)$). The first observation implies (3). The second observation then shows that x^* is an NE by y^* solving ($\Phi(z^*)$). □

In order to prove the main result of this section, we will make use of the following lemma.

Lemma 3.2. Consider an arbitrary strategy profile $x^* = (y^*, z^*)$. If the system of (in-)equalities given by (LCP(z^*)) and ($\Phi(z^*)$) admits a solution, it admits a rational one as well.

Proof. We first show two claims stating that (LCP(z^*)) and ($\Phi(z^*)$) both have solution sets that are union of rational polyhedra, i.e., polyhedra with rational vertices.

Claim 3.3. The set of solutions of (LCP(z^*)) is a union of rational polyhedra.

Proof. To see this, note that the set of solutions of (LCP(z^*)) is the union over all possible supports $J^i \subseteq \{1, \dots, m_i\}$, $L_i \subseteq \{1, \dots, l_i\}$, $i \in N$ of solutions to

$$\begin{aligned}
\sum_{j \neq i} \left(Q_{ij}^{\text{con}}\right)^\top (y_j, z_j^*) + d_i^{\text{con}} - \left(A_{ii}^{\text{con}}\right)^\top \lambda_i \geq 0 &\quad \text{for all } i \in N,\\
\Big(\sum_{j \neq i} \left(Q_{ij}^{\text{con}}\right)^\top (y_j, z_j^*) + d_i^{\text{con}} - \left(A_{ii}^{\text{con}}\right)^\top \lambda_i\Big)_l = 0 &\quad \text{for all } l \in L^i \text{ and } i \in N,\\
y_{il} = 0 &\quad \text{for all } l \notin L^i \text{ and } i \in N,\\
A_{ii}(y_i, z_i^*) - b_i \geq 0 &\quad \text{for all } i \in N,\\
\Big(A_{ii}(y_i, z_i^*) - b_i\Big)_j = 0 &\quad \text{for all } k \in J^i \text{ and } i \in N,\\
\lambda_{ij} = 0 &\quad \text{for all } j \notin J^i \text{ and } i \in N,\\
y_i \in \mathbb{R}_+^{l_i}, \ \lambda_i \in \mathbb{R}_+^{m_i} &\quad \text{for all } i \in N,
\end{aligned}$$

see Page 144 of Cottle et al. (2009) as well. The above system is linear in y and λ and hence describes a polyhedron. Since all appearing parameters are rational, it follows that any vertex of the polyhedron is rational as well. ■

Claim 3.4. The set of solutions of ($\Phi(z^*)$) is a union of rational polyhedra.

Proof. We start by noting that ($\Phi(z^*)$) is a system of inequalities in which both sides are expressed via a piecewise affine function, where every piece is defined via rational parameters. This is true as $\phi_i(y_{-i}; z^*)$ is the optimal-value function of a rational LP in which the objective function is linearly perturbed via y_{-i}; see Lemma A.4 in the appendix. Such a system is known to admit a solution set describable as union of rational polyhedra; cf. Lemma A.3 in the appendix. ■

Hence, the solution set of the system given by ($\mathrm{LCP}(z^*)$) and ($\Phi(z^*)$) is the intersection of unions of rational polyhedra, i.e.,

$$\left(\bigcup_{l=1}^{L_1} \mathrm{P}_l^{\mathrm{LCP}}\right) \cap \left(\bigcup_{l=1}^{L_2} \mathrm{P}_l^{\Phi}\right) = \bigcup_{l_1 \in L_1, l_2 \in L_2} \left(\mathrm{P}_{l_1}^{\mathrm{LCP}} \cap \mathrm{P}_{l_2}^{\Phi}\right).$$

Using the second representation of the solution set, we can deduce that if there exists a solution to ($\mathrm{LCP}(z^*)$) and ($\Phi(z^*)$), then there need to exist at least one $l_1 \in L_1$ and $l_2 \in L_2$ with $\mathrm{P}_{l_1}^{\mathrm{LCP}} \cap \mathrm{P}_{l_2}^{\Phi}$ not being empty. Since both of these polyhedra are rational, the intersection is again a rational polyhedron. Moreover, we just argued that this polyhedron is non-empty. Hence, it admits a rational vertex, which solves ($\mathrm{LCP}(z^*)$) and ($\Phi(z^*)$). □

Using the above results, we can now easily state and prove the main result of this section.

Theorem 3.5. If G admits a Nash equilibrium, it also admits a rational Nash equilibrium.

Proof. Let $x^* = (y^*, z^*)$ be an NE of G. By Lemma 3.1, y^* solves ($\mathrm{LCP}(z^*)$) and ($\Phi(z^*)$). Hence, Lemma 3.2 implies that there exist a rational y' solving ($\mathrm{LCP}(z^*)$) and ($\Phi(z^*)$). Applying Lemma 3.1 shows that (y', z^*) is an NE as well. □

Remark 3.6 (Finite-Time Algorithm)**.** Assume that there exists only finitely many different feasible integer components z, i.e., integer components z for which a corresponding feasible strategy profile $x = (y, z)$ exists. Then, our proof of the above theorem directly leads to a finite-time algorithm that can decide whether the game admits an equilibrium and if so, outputs a (rational) one. The algorithm iterates over all possible feasible integer components z^* and either finds a rational vector solving the corresponding system composed of ($\mathrm{LCP}(z^*)$) and ($\Phi(z^*)$) or determines that the system does not admit any solution. Note that this is possible in finite time as the solution set to this system is the union of solution sets of LCPs, which in turn can be solved in finite time. For the former claim, note that by Claim 3.4, there exist rational polyhedra P_l, $l = 1, \dots, L$, describing the solution set to ($\Phi(z^*)$) and that these polyhedra can be determined in finite time; see Lemmas A.2 to A.4 for their construction. Hence, solving for every $l = 1, \dots, L$ the LCP composed of ($\mathrm{LCP}(z^*)$) augmented with the condition that $y \in P_l$ yields the entire solution set to the system composed of ($\mathrm{LCP}(z^*)$) and ($\Phi(z^*)$).

4. Negative Result for Player-Quadratic NEPs

We now give an example for a player-quadratic NEP with 3 players that admits a unique Nash equilibrium having irrational continuous strategies. Note that in this as well as the following section, we do not represent the problem of a player in the form of ($\mathcal{P}_i(x_{-i})$) but also allow for negative variables. Yet, it can be easily shown that they can be brought into the form of ($\mathcal{P}_i(x_{-i})$) while keeping the property of admitting a single equilibrium having irrational continuous strategies.

The first player solves

$$\begin{aligned} \min_{y_1, z_1} \quad & \pi_1(y_1, z_1; y_2, z_2, y_3, z_3) := y_1^2 - y_3 y_1 + y_3 z_1 - (8z_3 - 1)z_1 \\ \text{s.t.} \quad & -5z_1 \le y_1 \le 5z_1, \\ & y_1 \in \mathbb{R}, \ z_1 \in \{0, 1\}. \end{aligned} \tag{$\mathcal{P}_1(x_2, x_3)$}$$

The second player solves

$$\begin{aligned} \min_{y_2, z_2} \quad & \pi_2(y_2, z_2; y_1, z_1, y_3, z_3) := y_2^2 - y_3 y_2 + y_3 z_2 + (30z_3 + 1)z_2 \\ \text{s.t.} \quad & -5z_2 \le y_2 \le 5z_2, \\ & y_2 \in \mathbb{R}, \ z_2 \in \{0, 1\}. \end{aligned} \tag{$\mathcal{P}_2(x_1, x_3)$}$$

Finally, the third player solves

$$\begin{aligned} \min_{y_3, z_3} \quad & \pi_3(y_3, z_3; y_1, z_1, y_2, z_2) := \left(z_1 - z_2 - \frac{1}{2}\right) z_3 \\ \text{s.t.} \quad & y_3 \in [-5, 4], \ z_3 \in \{0, 1\}. \end{aligned} \tag{$\mathcal{P}_3(x_1, x_2)$}$$

We first make the following three observations.

Claim 4.1. If x^* is an equilibrium, then $z_3^* = 0$ holds.

Proof. Assume for the sake of a contradiction that $z_3^* = 1$. Then, optimality of player 3 implies $z_1^* - z_2^* \le 1/2$ and, hence, either $z_1^* = z_2^* = 1$ or $z_1^* = 0$. Let us consider the first case. If $z_1^* = z_2^* = 1$, then player 2 does not play optimally. Indeed, her costs for $z_2 = 0 = y_2$ are 0 whereas for $z_2^* = 1$, we can bound her costs from below by

$$\pi_2(y_2, z_2^*; y_1, z_1^*, y_3, z_3^*) = y_2^2 - y_3 y_2 + y_3 + (30 + 1) \ge 0 - 5 \cdot 5 - 5 + 31 \ge 1 > 0.$$

for any y_2, where we used $y_2 \in [-5, 5]$ and $y_3 \in [-5, 4]$.

Now consider the case $z_1^* = 0$. Player 1's constraint then implies $y_1^* = 0$ and, hence, $\pi_1(x^*) = 0$. This is not optimal for player 1 as for $y_1 = 0$ and $z_1 = 1$, we have $\pi_1(x_1, x_{-1}^*) = y_3 - (8 - 1) \le -3 < 0$ by $y_3 \le 4$. ∎

Claim 4.2. If x^* is an equilibrium, then $x_2^* = (y_2^*, z_2^*) = 0$ holds.

Proof. Assume for the sake of a contraction that $x_2^* \neq 0$. By the constraint of player 2, this implies that $z_2^* = 1$. Considering the factor multiplied with z_3 in the objective of player 3, $(z_1^* - z_2^* - 0.5) \le -0.5$ gives that $z_1^* \le 1$ and hence the optimality of player 3 implies $z_3^* = 1$ contradicting Claim 4.1. ∎

Claim 4.3. If x^* is an equilibrium, then $x_1^* = (y_1^*, z_1^*) \neq 0$ holds.

Proof. Assume for the sake of a contraction that $x_1^* = 0$ holds. By Claim 4.2, we have $x_2^* = 0$ as well. But this implies by optimality of player 3 that $z_3^* = 1$ in contradiction to Claim 4.1. ∎

We show in the following that only one NE exists, which has an irrational component. To this end, we compute the best-response maps $\text{BR}_i(x_{-i}^*)$ and show that the fixed point condition $x^* \in \text{BR}(x^*) := \times_{i \in N} \text{BR}_i(x_{-i}^*)$ is fulfilled by a single x^* admitting at least one irrational component. By the above claims, we already know that $z_3^* = 0$ holds in an NE. Moreover, y_3 does not have an impact on the costs of player 3 and hence x^* fulfills the fixed-point condition if and only if $x_i^* \in \text{BR}_i(x_{-i}^*)$ for $i = 1, 2$ and $z_3^* = 0$. Hence, it is enough to calculate the best response maps $\text{BR}_1(x_2, y_3, 0)$ and $\text{BR}_2(x_1, y_3, 0)$.

Claim 4.4. The best response map of player 1 and 2 fulfill $\mathrm{BR}_1(x_2, y_3, 0) = \mathrm{BR}_2(x_1, y_3, 0) = \mathrm{BR}(y_3, 0)$ for all x_1, x_2, y_3 with

$$\mathrm{BR}(y_3,0) = \begin{cases} (y_3/2, 1), & \text{if } y_3 \in \mathbb{R} \setminus \left[2-2\sqrt{2}, 2+2\sqrt{2}\right] \\ \{(y_3/2, 1)\} \cup \{(0,0)\}, & \text{if } y_3 \in \{2-2\sqrt{2}, 2+2\sqrt{2}\} \\ (0,0), & \text{if } y_3 \in \left]2-2\sqrt{2}, 2+2\sqrt{2}\right[. \end{cases}$$

Proof. Observe that it is enough to show that the best response mapping of player 1 is of the claimed form as player 2 solves the exact same problem as player 1 for $z_3 = 0$. Let us consider player 1 in the following. For z_1, player 1 has two options: If he plays $z_1 = 0$, then $y_1 = 0$ and he gets costs of 0. Let us calculate in the following the optimal y_1 for $z_1 = 1$ and the corresponding objective value. For $z_1 = 1$, the optimal y_1 solves

$$\begin{aligned} \min_{y_1} \quad & \pi_1(y_1, 1; y_2, z_2, y_3, 0) = y_1^2 - y_3 y_1 + y_3 + 1 \\ \text{s.t.} \quad & -5 \le y_1 \le 5, \\ & y_1 \in \mathbb{R}. \end{aligned}$$

The objective is an upward opening parabola and its minimal point is at $y_1 = y_3/2$ which is contained in $[-5, 5]$ by $y_3 \in [-5, 4]$. The corresponding objective value is given by

$$\pi_1(y_3/2, 1, x_2, y_3, 0) = \frac{1}{4} y_3^2 - \frac{1}{2} y_3^2 + y_3 + 1 = -\frac{1}{4} y_3^2 + y_3 + 1.$$

Hence, player 1 plays $z_1 = 1$ and $y_1 = y_3/2$ in a best-response if and only if the above value is smaller or equal to 0, which are the costs when playing $z_1 = 0$. Now $\pi_1(y_3/2, 1, x_2, y_3, 0)$ is a downwards opening parabola and thus is exactly 0 at the roots and below zero in case that y_3 is outside of the interval between the roots of the parabola. The roots are

$$\frac{-1 \pm \sqrt{1 - 4 \cdot \frac{-1}{4} \cdot 1}}{2 \cdot \frac{-1}{4}} = 2 \pm 2\sqrt{2}$$

which results in the claimed best-response sets. ■

We are now in the position to show that there is exactly one irrational equilibrium. Assume that x^* is an equilibrium. We know $x_2^* = 0$ by Claim 4.2 and Claim 4.4 for player 2 implies that $y_3^* \in [2-2\sqrt{2}, 2+2\sqrt{2}]$ has to hold. Using Claim 4.3 and Claim 4.4 for player 1, we further get $y_3^* \in \mathbb{R} \setminus]2-2\sqrt{2}, 2+2\sqrt{2}[$. Thus, we have $y_3^* \in \{2-2\sqrt{2}, 2+2\sqrt{2}\} \cap [-5, 4] = \{2-2\sqrt{2}\}$ and by Claim 4.3 and Claim 4.4 for player 1, we further get $x_1^* = (y_3^*/2, 1)$. Hence, we have shown that there is a single candidate for an equilibrium x^* with an irrational component. Moreover, it is now an immediate consequence of Claim 4.4 (and the argumentation preceding Claim 4.4) that x^* is indeed an NE.

5. Negative Result for Player-Quadratic and Player-Linear GNEPs

The following example describes a rational mixed-integer GNEP with player-linear costs that admits a unique GNE having irrational components. As the class of rational mixed-integer GNEPs with player-linear costs is a sub-class of rational mixed-integer GNEPs with player-quadratic costs, the example also proves that the latter class does not necessarily admit a rational equilibrium if an equilibrium exists.

Example 5.1. We study the Nash equilibria of the following 3-player game. Player 1 solves the problem

$$\begin{aligned}\min_{y_1,z_1} \quad & \pi_1(y_1,z_1;y_2,z_2,z_3) = -(2y_2+4z_2)y_1 + \left(1-3y_2+\frac{14}{5}z_2-2z_3\right)z_1 \\ \text{s.t.} \quad & 0 \le y_1 \le z_1, \\ & y_1+y_2 \le 1, \\ & y_1 \in \mathbb{R}, z_1 \in \{0,1\},\end{aligned} \qquad (\mathcal{P}_1(x_2,z_3))$$

player 2 solves

$$\begin{aligned}\min_{y_2,z_2} \quad & \pi_2(y_2,z_2;y_1,z_1,z_3) = (2y_1-4z_1)y_2 + \left(1-3y_1+\frac{11}{5}z_1-2z_3\right)z_2 \\ \text{s.t.} \quad & 0 \le y_2 \le z_2, \\ & y_1+y_2 \le 1, \\ & y_2 \in \mathbb{R}, z_2 \in \{0,1\},\end{aligned} \qquad (\mathcal{P}_2(x_1,z_3))$$

and player 3 faces the problem

$$\begin{aligned}\min_{z_3} \quad & \pi_3(z_3;z_1,z_2) = \left(z_1+z_2-\frac{1}{2}\right)z_3 \\ \text{s.t.} \quad & z_3 \in \{0,1\}.\end{aligned} \qquad (\mathcal{P}_3(x_1,x_2))$$

Using the following two claims we show that in an equilibrium, $z_3^* = 0$ and $z_1^* = z_2^* = 1$ holds.

Claim 5.2. If x^* is an equilibrium, then $z_3^* = 0$.

Proof. Assume for the sake of a contradiction that $z_3^* = 1$ holds. By the optimality of player 3, this implies that $z_1^* + z_2^* \le 1/2$. This in turn implies that $z_1^* = z_2^* = 0$ and by the constraints of player 1 and 2, we also have $y_1^* = y_2^* = 0$. Hence, we get $\pi_1(x^*) = 0$. This, however, contradicts the optimality condition for player 1 as $(y_1,z_1) = (0,1)$ yields costs of $\pi_1(0,1;x_{-1}^*) = -1 < 0$, where we used that $x_2^* = 0$ and $z_3^* = 1$. ∎

Claim 5.3. If x^* is an NE, then $z_1^* = 1$ and $z_2^* = 1$.

Proof. We start by arguing that $z_2^* = 1$ holds. Assume for the sake of a contradiction that $z_2^* = 0$ holds. By the constraint of player 2, we have $y_2^* = 0$. By Claim 5.2, we also have $z_3^* = 0$. Hence, the objective of player 1 in this case is $\pi_1(x^*) = z_1^*$. By optimality of player 1, we thus get that $x_1^* = 0$. However, this now contradicts the optimality of player 3 as $\pi_3^*(z_3;0) = -z_3/2$ and hence $z_3^* = 0$ is not optimal.

The proof for $z_1^* = 1$ works analogously. Assume for the sake of a contradiction that $z_1^* = 0$ holds. By the constraint of player 1, we have $y_1^* = 0$. By Claim 5.2, we also have $z_3^* = 0$. Hence, the objective of player 2 in this case is $\pi_2(x^*) = z_2^*$. By optimality of player 1, we hence get that $x_2^* = 0$. However, this now contradicts the optimality of player 3 as before. ∎

By the above claims, we know that we have in an equilibrium x^* that $z_3^* = 0$ and $z_1^* = z_2^* = 1$. For this case, we now describe the best-response mappings of player 1 and 2.

Claim 5.4. For $z_2^* = 1$ and $z_3^* = 0$, the best-response mapping of player 1 for any $y_2 \in [0,1]$ is given by

$$\mathrm{BR}_1(y_2,1,0) = \begin{cases} \{(1-y_2,1)\}, & \text{if } y_2 \in \left[0, \frac{1+\sqrt{13/5}}{4}\right[, \\ \{(1-y_2,1)\} \cup \{(0,0)\}, & \text{if } y_2 = \frac{1+\sqrt{13/5}}{4}, \\ \{(0,0)\}, & \text{if } y_2 \in \left]\frac{1+\sqrt{13/5}}{4}, 1\right]. \end{cases}$$

Proof. For z_1, player 1 has two options: If she plays $z_1 = 0$, then $y_1 = 0$ and she gets a cost of 0. Let us calculate in the following the optimal y_1 for $z_1 = 1$ and the corresponding objective value. For $z_1 = 1$, the optimal y_1 solves

$$\begin{aligned} \min_{y_1} \quad & \pi_1(y_1,1;y_2,1,0) = -(2y_2+4)y_1 - 3y_2 + \frac{19}{5} \\ \text{s.t.} \quad & 0 \le y_1 \le 1, \quad y_1 \le 1-y_2, \quad y_1 \in \mathbb{R}. \end{aligned}$$

Since $y_2 \ge 0$, the term $-(2y_2+4)$ is negative and, thus, the optimal y_1 is as large as possible, i.e., $y_1 = 1 - y_2$. The corresponding objective value is given by

$$\pi_1(1-y_2,1;y_2,1,0) = 2y_2^2 - y_2 - \frac{1}{5}.$$

Hence, $z_1 = 1$ and $y_1 = 1 - y_2$ is a best-response if and only if the above value is smaller or equal to 0, which are the costs when playing $z_1 = 0$. The above value is an upwards opening parabola in y_2 with roots

$$\frac{1-\sqrt{13/5}}{4} < 0 \quad \text{and} \quad \frac{1+\sqrt{13/5}}{4} \in [0,1].$$

Subsequently, for $y_2 \in [0,1]$, the cost $\pi_1(1-y_2,1;y_2,1,0)$ is smaller or equal to zero if and only if

$$y_2 \in \left[0, \frac{1+\sqrt{13/5}}{4}\right],$$

leading to the claimed best-response mapping. ∎

A similar proof shows the following claim.

Claim 5.5. For $z_1^* = 1$ and $z_3^* = 0$, the best-response mapping of player 2 for any $y_1 \in [0,1]$ is given by

$$\mathrm{BR}_2(y_1,1,0) = \begin{cases} \{(1-y_1,1)\}, & \text{if } y_1 \in \left[0, \frac{3-\sqrt{13/5}}{4}\right[, \\ \{(1-y_1,1)\} \cup \{(0,0)\}, & \text{if } y_1 = \frac{3-\sqrt{13/5}}{4}, \\ \{(0,0)\}, & \text{if } y_1 \in \left]\frac{3-\sqrt{13/5}}{4}, 1\right]. \end{cases}$$

Proof. For z_2, player 2 has two options: If she plays $z_2 = 0$, then $y_2 = 0$ follows and the cost is 0. Let us now calculate the optimal y_2 for $z_2 = 1$ and the corresponding objective value. For $z_2 = 1$, the optimal y_2 solves

$$\begin{aligned} \min_{y_2} \quad & \pi_2(y_2,1;y_1,1,0) = (2y_1-4)y_2 - 3y_1 + \frac{16}{5} \\ \text{s.t.} \quad & 0 \le y_2 \le 1, \quad y_2 \le 1-y_1, \quad y_2 \in \mathbb{R}. \end{aligned}$$

Since $y_1 \le 1$, the term $(2y_1 - 4)$ is negative and thus the optimal y_2 is as large as possible, i.e., $y_2 = 1 - y_1$. The corresponding objective value is given by

$$\pi_2(1-y_1,1;y_1,1,0) = -2y_1^2 + 3y_1 - \frac{4}{5}.$$

Hence, $z_1 = 1$ and $y_1 = 1 - y_2$ is a best-response if and only if the above value is smaller or equal to 0, which are the costs when playing $z_1 = 0$. The above is a downwards opening parabola in y_1 with roots

$$\frac{3-\sqrt{13/5}}{4} \in [0,1] \quad \text{and} \quad \frac{3+\sqrt{13/5}}{4} > 1.$$

Hence, for $y_1 \in [0,1]$, the cost is smaller or equal to zero if and only if

$$y_1 \in \left[0, \frac{3-\sqrt{13/5}}{4}\right],$$

leading to the claimed best-response mapping. ■

We can now show that there exists only one equilibrium x^* and that this equilibrium admits at least one irrational component. For what follows, assume that x^* is an NE. Claims 5.2 and 5.3 imply $z_3^* = 0$ and $z_1^* = z_2^* = 1$. This together with the best-response maps shown in the last two claims leads to

$$y_1^* \in \left[0, \frac{3-\sqrt{13/5}}{4}\right], \quad y_2^* \in \left[0, \frac{1+\sqrt{13/5}}{4}\right], \quad y_1^* = 1 - y_2^*.$$

These conditions can only be satisfied for

$$y_1^* = \frac{3-\sqrt{13/5}}{4} \quad \text{and} \quad y_2^* = \frac{1+\sqrt{13/5}}{4}.$$

Hence, we have argued that there can only exist a single NE given by

$$x^* = (y_1^*, z_1^*, y_2^*, z_2^*, z_3^*) = \left(\frac{3-\sqrt{13/5}}{4}, 1, \frac{1+\sqrt{13/5}}{4}, 1, 0\right),$$

which is irrational. One can easily verify that this is indeed an NE using the above best-response maps.

6. Conclusion

Let us close this paper with a brief discussion about the computational consequences of our results. First, the results of Section 3 immediately lead to a finite time algorithm for computing an equilibrium of player-linear NEPs (that admit equilibria at all); see Remark 3.6 for a more detailed discussion. Second, the existence of instances of player-quadratic NEPs and player-linear or player-quadratic GNEPs that only admit irrational equilibria makes it impossible to derive finite-time algorithms in general.

This implication directly leads to the question of approximation schemes for these classes of problems. One could, for instance, try to design finite-time algorithms that compute points that are ε-close to irrational equilibria if only those exist. To the best of our knowledge, methods like this have not been studied in the past and are an interesting topic of future research.

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Appendix A. Technical Auxiliary Results

We prove in the following that a system of inequalities with piecewise affine functions and rational parameters admits a solution set describable as the union of rational polyhedra. Moreover, we show that the optimal value function of an LP with respect to change in the objective vector is a piecewise affine function with rational parameters. Here, we use the following definition:

Definition A.1. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is a *piecewise affine function with rational parameters* if there exists a finite set of polyhedra $P_i := \{x \in \mathbb{R}^n \colon A_i x \leq b_i\}$, $i = 1, \dots, k$, with rational A_i, b_i together with rational $c_i \in \mathbb{Q}^n$ and $d_i \in \mathbb{Q}$ such that $\bigcup_{i=1}^k P_i = \mathbb{R}^n$ and

$$f(x) = c_i^\top x + d_i \quad \text{if } x \in P_i.$$

We first need the following intermediate lemma.

Lemma A.2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a piecewise affine function with rational parameters. Then, the set of solutions of $f(x) \leq 0$ is the union of rational polyhedra.

Proof. The set of solutions of $f(x) \leq 0$ is the union over $i \in \{1, \dots, k\}$ of P_i intersected with $\{x \in \mathbb{R}^n \colon c_i^\top x + d_i \leq 0\}$. The latter intersection is again a rational polyhedron as it is the intersection of two rational polyhedra. Hence, the lemma is proven. □

Lemma A.3. Consider a finite set of piecewise affine functions f_i, $i = 1, \dots, k$, with rational parameters and a corresponding system of inequalities of the form $f_i(x) \leq 0$. Then, the solution set of this system is the union of rational polyhedra.

Proof. By Lemma A.2, we know that for any $i \in \{1, \dots, k\}$, the set of solutions of $f_i(x) \leq 0$ is the union of rational polyhedra. Let us denote these polyhedra via P_l^i with $l \in \{1, \dots, L_i\}$ and $L_i \in \mathbb{N}$. Hence, the set of solutions of the entire system is given by

$$\bigcap_{i=1}^{k} \bigcup_{l=1}^{L_i} P_l^i = \bigcup_{(l_i)_i \in \prod_{i=1}^k \{1, \dots, L_i\}} \bigcap_{i=1}^{k} P_{l_i}^i$$

and since $\bigcap_{i=1}^k P_{l_i}^i$ is a rational polyhedron (as it is the intersection of rational polyhedra), the proof is complete. □

We now state and proof the second promised result.

Lemma A.4. Consider the optimal-value function function

$$\varphi(c) := \min \left\{ c^\top x \colon Ax \leq b \right\}$$

of a linear optimization problem with rational constraint data $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$. Then, φ is a piecewise linear function with rational parameters.

Proof. For any $c \in \mathbb{R}^n$, a solution to the linear problem $\{c^\top x \colon Ax \leq b\}$ is attained at one of the finite vertices of the polyhedron given by $Ax \leq b$. Let us denote these vertices with $x^1, \dots, x^s$. These vertices are all rational vectors as A and b are rational. Hence, the value function can be written as

$$\varphi(c) = \min \left\{ c^\top x \colon x \in \left\{ x^1, \dots, x^s \right\} \right\}.$$

In particular, for any $i \in \{1, \dots, s\}$, we have $\varphi(c) = c^\top x^i$ if $c^\top x^i \leq c^\top x^j$ for all $j \in \{1, \dots, s\}$. The latter condition is equivalent to $c \in P_i$ where P_i is the rational polyhedron given by $P_i := \{c \in \mathbb{R}^n : D_i c \le 0\}$ with

$$D_i := \begin{bmatrix} (x^i)^\top - (x^1)^\top \\ \vdots \\ (x^i)^\top - (x^s)^\top \end{bmatrix}$$

Hence, $\varphi(c)$ is of the form as in Definition A.1 and, thus, the proof is finished. □

References

Carvalho, M., G. Dragotto, A. Lodi, and S. Sankaranarayanan (2023). "Integer Programming Games: A Gentle Computational Overview." In: *Tutorials in Operations Research: Advancing the Frontiers of OR/MS: From Methodologies to Applications*. INFORMS, pp. 31–51. DOI: 10.1287/educ.2023.0260.

Cottle, R., J. Pang, and R. Stone (2009). *The Linear Complementarity Problem*. Society for Industrial and Applied Mathematics. DOI: 10.1137/1.9780898719000.

Dragotto, G. and R. Scatamacchia (2023). "The zero regrets algorithm: Optimizing over pure Nash equilibria via integer programming." In: *INFORMS Journal on Computing* 35.5, pp. 1143–1160. DOI: 10.1287/ijoc.2022.0282.

Duguet, A., M. Carvalho, G. Dragotto, and S. U. Ngueveu (2025a). "Computing Approximate Nash Equilibria for Integer Programming Games." In: *Optimization Letters*. DOI: 10.1007/s11590-025-02221-5.

Duguet, A., T. Harks, M. Schmidt, and J. Schwarz (2025b). *Branch-and-Cut for Mixed-Integer Generalized Nash Equilibrium Problems*. URL: https://arxiv.org/abs/2506.02520.

Facchinei, F. and C. Kanzow (2010). "Generalized Nash equilibrium problems." In: *Annals of Operations Research* 175.1, pp. 177–211. DOI: 10.1007/s10479-009-0653-x.

Harks, T. and J. Schwarz (2025). "Generalized Nash equilibrium problems with mixed-integer variables." In: *Mathematical Programming* 209, pp. 231–277. DOI: 10.1007/s10107-024-02063-6.

Kirst, P., S. Schwarze, and O. Stein (2024). "A Branch-and-Bound Algorithm for Nonconvex Nash Equilibrium Problems." In: *SIAM Journal on Optimization* 34.4, pp. 3371–3398. DOI: 10.1137/23M1548189.

Lemke, C. E. (1965). "Bimatrix Equilibrium Points and Mathematical Programming." In: *Management Science* 11.7, pp. 681–689. URL: http://www.jstor.org/stable/2627771.

Lemke, C. E. and J. T. Howson (1964). "Equilibrium Points of Bimatrix Games." In: *Journal of the Society for Industrial and Applied Mathematics* 12.2, pp. 413–423. URL: http://www.jstor.org/stable/2946376.

Nash, J. (1951). "Non-Cooperative Games." In: *Annals of Mathematics* 54.2, pp. 286–295.

Nash, J. F. (1950). "Equilibrium points in N-person games." In: *Proceedings of the National Academy of Sciences* 36.1, pp. 48–49. DOI: 10.1073/pnas.36.1.48.

Sagratella, S. (2017a). "Algorithms for generalized potential games with mixed-integer variables." In: *Computational Optimization and Applications* 68, pp. 689–717. DOI: 10.1007/s10589-017-9927-4.

– (2017b). "Computing equilibria of Cournot oligopoly models with mixed-integer quantities." In: *Mathematical Methods of Operations Research* 86.3, pp. 549–565.

Schwarze, S. and O. Stein (2023). "A branch-and-prune algorithm for discrete Nash equilibrium problems." In: *Computational Optimization and Applications* 86.2, pp. 491–519. DOI: 10.1007/s10589-023-00500-4.

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