Automorphisms of hyperbolic polynomials

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Abstract

The pair (p, e) is *hyperbolic* if $p : \mathbb{R}^n \to \mathbb{R}$ is a homogeneous polynomial, if $e \in \mathbb{R}^n$, if $p(e) > 0$, and if the roots of $t \mapsto p(te - x)$ are real for all $x \in \mathbb{R}^n$. In that case, the x for whom these roots are nonnegative form a closed convex cone $\Lambda_{p,e}$ called a hyperbolicity cone. Many cones used in optimization are hyperbolicity cones. For example, all homogeneous and symmetric cones are hyperbolicity cones.

In this setting we borrow a definition of "automorphism" wherein every automorphism of (p, e) is an automorphism of $\Lambda_{p,e}$ satisfying some additional properties. When $\Lambda_{p,e}$ is pointed and p, e are chosen judiciously, these automorphisms are characterized by

$$\mathrm{Aut}(p, e) = \mathrm{Aut}(\Lambda_{p,e})_e = \mathrm{Aut}(\Lambda_{p,e}) \cap \mathrm{Isom}(p, e).$$

Here $\mathrm{Aut}(\Lambda_{p,e})_e$ is the subgroup of $\mathrm{Aut}(\Lambda_{p,e})$ that fixes e, and $\mathrm{Isom}(p, e)$ is the isometry group with respect to a particular norm. This generalizes an important result for symmetric cones, and specializes to homogeneous ones. Subsequently we clarify the relationship between two constructions of homogeneous cones, and find the degree of the hyperbolic polynomial p most commonly used with them.

Keywords: hyperbolicity cone, homogeneous cone, automorphism group, clan, Siegel domain

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1 Introduction

The study of hyperbolic polynomials was initiated by Gårding in connection with partial differential equations [9, 10], and has been continued by optimizers because every hyperbolic polynomial has an associated *hyperbolicity cone* agreeable to interior-point methods [17]. All homogeneous cones—in particular, all symmetric cones—can be realized as hyperbolicity cones [17]. *Hyperbolic programming* therefore encompasses linear programming, second-order cone programming, and semidefinite programming (among others).

A polynomial $p : \mathbb{R}^n \to \mathbb{R}$ is said to be hyperbolic along $e \in \mathbb{R}^n$ if p is homogeneous, if $p(e) > 0$, and if the roots of $t \mapsto p(te - x)$ are real for all $x \in$

$\mathbb{R}^n$. From now on we will say that "(p, e) is hyperbolic" to indicate this situation. When (p, e) is hyperbolic, the univariate polynomial $t \mapsto p(te - x)$ is called the *characteristic polynomial* of x. Accordingly, its roots are the *eigenvalues* of x. We denote by $\lambda_{p,e}(x) \in \mathbb{R}^{\deg(p)}$ the vector of these eigenvalues arranged in nonincreasing order. By taking p to be a polynomial associated with the determinant and e to be the identity matrix, this generalizes in a transparent way the spectral theory of real symmetric matrices.

We define the *hyperbolicity cone* with respect to (p, e) to be

$$\Lambda_{p,e} \coloneqq \{x \in \mathbb{R}^n \mid \lambda_{p,e}(x) \geq 0\}.$$

It is not obvious, but $\Lambda_{p,e}$ is a closed convex cone [9, 28]. In the matrix setting, it is the semidefinite cone. Eigenvalues are continuous in the usual sense of polynomial roots, and those of e are unity, so $\Lambda_{p,e}$ contains a neighborhood of e. As a result, all hyperbolicity cones are solid (have nonempty interior), i.e. $\operatorname{int}(\Lambda_{p,e}) \neq \emptyset$. In contrast, $\Lambda_{p,e}$ is pointed (contains no lines) if and only if $\lambda_{p,e}(x) = 0$ implies $x = 0$. In that case, the pair (p, e) is said to be *complete* [1]. Closed convex cones that are both pointed and solid are called proper. It follows that $\Lambda_{p,e}$ is proper if and only if (p, e) is complete.

Every nontrivial hyperbolicity cone is shared by multiple hyperbolic pairs. If (p, e) is hyperbolic and if $\tilde{e}$ belongs to $\operatorname{int}(\Lambda_{p,e})$, then $(p, \tilde{e})$ is hyperbolic and their hyperbolicity cones coincide [28]. We say that $\tilde{p}$ *has minimal degree* with respect to (p, e) if $(\tilde{p}, e)$ is hyperbolic, if $\Lambda_{\tilde{p},e} = \Lambda_{p,e}$, and if no polynomial of lower degree would work. Such a $\tilde{p}$ is essentially unique [19]. It is easy to construct examples where $\deg(p) \neq \deg(\tilde{p})$, so neither the polynomial p nor the point e are unique in general.

Below we collect standard information about hyperbolic pairs [17, 1, 28]. When it is clear from the context, we omit "with respect to (p, e)."

Lemma 1. *If (p, e) is hyperbolic, then*

1. $\lambda_{p,e}(e) = (1, 1, \ldots, 1)^T$.
2. $\lambda_{p,e}(x + \alpha e) = \lambda_{p,e}(x) + \alpha \lambda_{p,e}(e)$ *for all* $x \in \mathbb{R}^n$ *and* $\alpha \in \mathbb{R}$.
3. *If* $\tilde{e} \in \operatorname{int}(\Lambda_{p,e})$, *then* $\Lambda_{p,\tilde{e}} = \Lambda_{p,e}$.
4. *The eigenvalue map* $\lambda_{p,e} : \mathbb{R}^n \to \mathbb{R}^{\deg(p)}$ *is continuous.*
5. *The cone* $\Lambda_{p,e}$ *is solid and* e *lies in its interior.*
6. *The cone* $\Lambda_{p,e}$ *is pointed (and thus proper) if and only if* (p, e) *is complete.*
7. *If* (p, e) *is complete and if* $\tilde{e} \in \operatorname{int}(\Lambda_{p,e})$, *then* $(p, \tilde{e})$ *is complete.*
8. *If* (p, e) *is complete and if* $\tilde{p}$ *has minimal degree with respect to* (p, e), *then* $(\tilde{p}, e)$ *is complete.*

2 Automorphisms

The automorphisms of a convex cone are uncontroversial. If K is a convex cone, the automorphism group $\operatorname{Aut}(K)$ consists of all invertible linear maps A on the ambient space such that $A(K) = K$. But in many examples of hyperbolicity cones, we find some additional structure that we would like to preserve; for instance, the cone of positive-semidefinite matrices. The PSD cone is the "cone of squares" in the algebra of real symmetric matrices when it is equipped with the product $X \circ Y \coloneqq (XY + YX)/2$. This algebra has its own group of invertible endomorphisms, and both are known explicitly: the algebra automorphisms are the automorphisms of the cone that fix the identity matrix [27]. Either group can additionally be characterized as isometric automorphisms of the cone, or as spectrum-preserving maps.

Definition 1. If G is a group of linear operators on a vector space V and if $e \in V$, then $G_e \coloneqq \{g \in G \mid ge = e\}$ will denote the stabilizer of e in G.

The PSD cone is but one example. *All* symmetric cones are the cone of squares in some Euclidean Jordan algebra [8]. If $\mathcal{E}$ denotes a Euclidean Jordan algebra on V with multiplication $x \circ y$, cone of squares $K \coloneqq \{x \circ x \mid x \in V\}$, and algebra automorphism group $\operatorname{Aut}(\mathcal{E})$, then it was recently shown that the characterization for the PSD cone in the algebra of real symmetric matrices extends to K in $\mathcal{E}$ [26].

Theorem 1 (Orlitzky, Theorem 3.9)**.** *If $\mathcal{E} = (V, \circ, \langle\cdot,\cdot\rangle)$ is a Euclidean Jordan algebra with cone of squares K and unit e, then*

$$\operatorname{Aut}(\mathcal{E}) = \operatorname{Aut}(K)_e = \operatorname{Aut}(K) \cap \operatorname{Isom}(\mathcal{E})$$

if and only if every algebra isomorphism between simple subalgebras of $\mathcal{E}$ is an isometry with respect to $\langle\cdot,\cdot\rangle$.

Euclidean Jordan algebra automorphisms preserve the spectrum, and the uniqueness of the spectral decomposition implies that a spectrum-preserving map on $\mathcal{E}$ must belong to $\operatorname{Aut}(K)_e$. So under the stated conditions, any of the groups in Theorem 1 is moreover the group of spectrum-preserving maps. Our goal is to generalize these ideas to hyperbolic pairs and hyperbolicity cones. In that setting, we no longer have access to algebra automorphisms (there may be no algebra), but we do have spectrum-preserving maps.

Definition 2. We say that $A \in \mathrm{GL}_n(\mathbb{R})$ is an automorphism of the hyperbolic pair (p, e) if $\lambda_{p,e}(Ax) = \lambda_{p,e}(x)$ for all $x \in \mathbb{R}^n$. The group of all such automorphisms we denote by $\operatorname{Aut}(p, e)$. When (p, e) is complete, the presumption of invertibility may be omitted.

This definition appears obvious in hindsight, but it must be credited to Gowda, Jeong, and Sossa who have used it in (semi-)FTvN systems [15, 16]. These are systems where "eigenvalues" and "commutativity" are defined in great generality. For us it suffices to note that every complete hyperbolic pair (p, e)

is associated with such a system, and that its automorphisms are as we have defined them.

We begin in earnest with an important result of Ito and Lourenço [23], strengthening its conclusion when the automorphism happens to fix the point e. In much of what follows, it will be helpful to note that the automorphism group of a proper cone is the same as that of its interior—not all authors agree that hyperbolicity cones are closed.

Proposition 1 (Ito and Lourenço, 2.6)**.** *Suppose (p, e) is hyperbolic, that p has minimal degree, and that $A \in \mathrm{GL}_n(\mathbb{R})$. Then $A \in \mathrm{Aut}(\Lambda_{p,e})$ if and only if $Ae \in \mathrm{int}(\Lambda_{p,e})$ and there exists some $\kappa > 0$ such that $p = \kappa(p \circ A)$.*

Corollary 1. *If (p, e) is hyperbolic, if p has minimal degree, and if $A \in \mathrm{Aut}(\Lambda_{p,e})_e$, then $p \circ A = p$.*

Proof. Use Proposition 1 to write $p = \kappa(p \circ A)$, and then note that $p(e) = \kappa p(Ae) = \kappa p(e)$, implying that $\kappa = 1$. □

Theorem 2. *If (p, e) is complete and if p has minimal degree, then*

$$\mathrm{Aut}(p, e) = \mathrm{Aut}(\Lambda_{p,e})_e .$$

Proof. Any $A \in \mathrm{Aut}(p, e)$ is obviously in $\mathrm{Aut}(\Lambda_{p,e})$ because it preserves eigenvalues. That any such A fixes e is a consequence of completeness: let $x \coloneqq Ae$ in Item 2 of Lemma 1, and notice that $\lambda_{p,e}(Ae - e) = \lambda_{p,e}(Ae) - \lambda_{p,e}(e) = 0$. The completeness of p now implies that $Ae - e = 0$.

For the other inclusion we must show that any $A \in \mathrm{Aut}(\Lambda_{p,e})_e$ preserves eigenvalues. Recall that the eigenvalues of any $x \in \mathbb{R}^n$ are the roots of $t \mapsto p(te - x)$. The eigenvalues of Ax are thus the roots of $t \mapsto p(te - Ax)$. But $A^{-1}e = e$, so the eigenvalues of Ax are in fact the roots of $t \mapsto p(A[te - x])$. Now as p has minimal degree, we apply Corollary 1 to eliminate the A. □

If p does not have minimal degree, this result can fail. Example 2.7 of Ito and Lourenço shows that the number of nonzero eigenvalues an element has may not be preserved by $\mathrm{Aut}(\Lambda_{p,e})$ when p does not have minimal degree [23]. Their counterexample also fixes e, implying that $\mathrm{Aut}(\Lambda_{p,e})_e \not\subseteq \mathrm{Aut}(p, e)$.

This brings us to isometry groups. Every complete hyperbolic pair induces an inner product due to Bauschke, Güler, Lewis, and Sendov (BGLS) that will play a central role in what follows [1].

Definition 3 (BGLS inner product)**.** If $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ is the usual inner product on $\mathbb{R}^d$ and if (p, e) is a complete hyperbolic pair, then

$$\langle x, y \rangle_{p,e} \coloneqq \frac{1}{4} \|\lambda_{p,e}(x + y)\|^2_{\mathbb{R}^{\deg(p)}} - \frac{1}{4} \|\lambda_{p,e}(x - y)\|^2_{\mathbb{R}^{\deg(p)}}$$

is an inner product on $\mathbb{R}^n$ with associated norm $\|x\|_{p,e} = \|\lambda_{p,e}(x)\|_{\mathbb{R}^{\deg(p)}}$ and isometry group

$$\mathrm{Isom}(p, e) \coloneqq \left\{ A \in \mathrm{GL}_n(\mathbb{R}) \;\middle|\; \|Ax\|_{p,e} = \|x\|_{p,e} \text{ for all } x \in \mathbb{R}^n \right\}.$$

The BGLS inner product satisfies the *sharpened Cauchy-Schwarz inequality*,

$$\langle x, y \rangle_{p,e} \leq \langle \lambda_{p,e}(x), \lambda_{p,e}(y) \rangle_{\mathbb{R}^{\deg(p)}} \leq \|x\|_{p,e} \|y\|_{p,e}$$

that happens to be the sole axiom of a *semi-FTvN* system [16]. Automorphisms of (p, e) belong to $\mathrm{Isom}(p, e)$ almost by definition. When p has minimal degree, the same is true of $\mathrm{Aut}(\Lambda_{p,e})_e$ by Theorem 2.

Corollary 2. *If (p, e) is complete and if p has minimal degree, then*

$$\mathrm{Aut}(\Lambda_{p,e})_e = \mathrm{Aut}(p, e) \subseteq \mathrm{Aut}(\Lambda_{p,e}) \cap \mathrm{Isom}(p, e).$$

Whether or not we can reverse this inclusion depends on the point e. We borrow a lemma from Hofmann and Terp to show that there is at least one good choice [20], and then state our main result for hyperbolic pairs.

Lemma 2 (Hofmann and Terp, Lemmata 1.3–1.5)**.** *If K is a proper cone, then there exist maximal compact subgroups of $\mathrm{Aut}(K)$, and every such subgroup is of the form $\mathrm{Aut}(K)_e$ for some $e \in \mathrm{int}(K)$.*

Theorem 3. *If (p, e) is complete and if p has minimal degree, then there exists an $\tilde{e} \in \mathrm{int}(\Lambda_{p,e})$ such that $\Lambda_{p,e} = \Lambda_{p,\tilde{e}}$ and*

$$\mathrm{Aut}(p, \tilde{e}) = \mathrm{Aut}(\Lambda_{p,\tilde{e}})_{\tilde{e}} = \mathrm{Aut}(\Lambda_{p,\tilde{e}}) \cap \mathrm{Isom}(p, \tilde{e}).$$

Proof. Use Lemma 2 to find an $\tilde{e} \in \mathrm{int}(\Lambda_{p,e})$ such that $\mathrm{Aut}(\Lambda_{p,e})_{\tilde{e}}$ is a maximal compact subgroup of $\mathrm{Aut}(\Lambda_{p,e})$. From Lemma 1 we know that $(p, \tilde{e})$ is also hyperbolic, and that $\Lambda_{p,e} = \Lambda_{p,\tilde{e}}$. If two cones are equal, their automorphism groups are equal; thus $\mathrm{Aut}(\Lambda_{p,\tilde{e}})_{\tilde{e}}$ is a maximal compact subgroup of $\mathrm{Aut}(\Lambda_{p,\tilde{e}})$. Apply Corollary 2 to $(p, \tilde{e})$:

$$\mathrm{Aut}(\Lambda_{p,\tilde{e}})_{\tilde{e}} = \mathrm{Aut}(p, \tilde{e}) \subseteq \mathrm{Aut}(\Lambda_{p,\tilde{e}}) \cap \mathrm{Isom}(p, \tilde{e}).$$

Since $\mathrm{Aut}(\Lambda_{p,\tilde{e}}) \cap \mathrm{Isom}(p, \tilde{e})$ is a compact subgroup of $\mathrm{Aut}(\Lambda_{p,\tilde{e}})$, the maximality of $\mathrm{Aut}(\Lambda_{p,\tilde{e}})_{\tilde{e}}$ concludes the result. □

Though stated differently, the conditions on p and e in Theorem 3 are of a similar nature. A polynomial $\tilde{p}$ of minimal degree is guaranteed to exist, and switching from p to $\tilde{p}$ does not affect the cone. Thus: given any complete hyperbolic pair (p, e), there exists an "equivalent" pair $(\tilde{p}, \tilde{e})$ for which the desired identity holds. One particularly nice situation is when the cone is homogeneous, and the given point e is indistinguishable up to automorphism.

Definition 4. A proper cone K is *homogeneous* if for all $e, x \in \mathrm{int}(K)$, there exists an $A \in \mathrm{Aut}(K)$ such that $Ae = x$.

Güler showed that every homogeneous cone K can be viewed as a hyperbolicity cone [17]. Given $e \in \mathrm{int}(K)$, there exists a polynomial p such that (p, e) is hyperbolic and $K = \Lambda_{p,e}$. As the definition of a homogeneous cone includes being pointed [35], there is no loss of generality in supposing that (p, e) is complete.

Homogeneity obviates the need to find $\tilde{e}$ in Theorem 3. The next result can be traced back Proposition I.1.8 and the subsequent paragraph in Faraut and Korányi [8]. It was used in Theorem A.2 of Chua to similar ends [5]. We include a proof to make maximality explicit since Proposition I.1.8 does not claim or prove the existence of maximal compact subgroups. Some argument (in our case, Lemma 2) based on Zorn's lemma is needed.

Corollary 3. *If K is a homogeneous cone and if $e \in \operatorname{int}(K)$, then $\operatorname{Aut}(K)_e$ is a maximal compact subgroup of $\operatorname{Aut}(K)$.*

Proof. If $e \in \operatorname{int}(K)$, then Lemma 2 says that there exists an $x \in \operatorname{int}(K)$ such that $\operatorname{Aut}(K)_e \subseteq \operatorname{Aut}(K)_x$ where the latter is maximal as a compact subgroup of $\operatorname{Aut}(K)$. The homogeneity of K guarantees the existence of an $A \in \operatorname{Aut}(K)$ with $Ae = x$, and it follows that

$$\operatorname{Aut}(K)_x = A \operatorname{Aut}(K)_e A^{-1} \subseteq A \operatorname{Aut}(K)_x A^{-1}.$$

All three subgroups here are compact, so by the maximality of $\operatorname{Aut}(K)_x$, they are equal. Unconjugate the last two. ☐

We now specialize Theorem 3 to homogeneous cones. The proof is identical after using Corollary 3 to show that $\operatorname{Aut}(\Lambda_{p,e})_e$ is a maximal compact subgroup of $\operatorname{Aut}(\Lambda_{p,e})$.

Theorem 4. *If $\Lambda_{p,e}$ is homogeneous and if p has minimal degree, then*

$$\operatorname{Aut}(p, e) = \operatorname{Aut}(\Lambda_{p,e})_e = \operatorname{Aut}(\Lambda_{p,e}) \cap \operatorname{Isom}(p, e).$$

When the cone is homogeneous, the difficulty lies solely in finding a p of minimal degree. As a surrogate we could also be told that the cone is *rank-one-generated*. Not all homogeneous cones have rank-one-generated realizations [14], but in every such $\Lambda_{p,e}$ the polynomial p has minimal degree [23].

Corollary 4. *If $\Lambda_{p,e}$ is homogeneous and rank-one-generated, then*

$$\operatorname{Aut}(p, e) = \operatorname{Aut}(\Lambda_{p,e})_e = \operatorname{Aut}(\Lambda_{p,e}) \cap \operatorname{Isom}(p, e).$$

3 Homogeneous cones

In the last section we developed a hierarchy of theorems for the symmetric $\subsetneq$ homogeneous $\subsetneq$ hyperbolicity hierarchy of cones. The remainder we devote to the homogeneous tier. All homogeneous cones can be viewed as hyperbolicity cones [17], but the polynomial used to do it (henceforth, the *Güler polynomial*) is of a generic nature and may not have the minimal degree required by Theorem 4.

The non-minimality of the Güler polynomial is common knowledge, but its degree has never been made explicit. In this section we have two goals, to find the degree of the Güler polynomial, and to clarify the relationship between two popular constructions of homogeneous cones. The former is something of an open problem, and the latter may be a boon to future research.

Remark 1. If you subscribe to Definition 4, then homogeneous cones are closed. Most of our sources disagree on this point, but without the boundary, homogeneous cones fail to be hyperbolicity cones. Fortunately the interior of the closure of an open convex set is itself [29], so we are able to switch from a homogeneous cone to its interior and back without loss of fidelity.

Several algebraic formalisms are used to construct and characterize homogeneous cones. In optimization, one popular choice is the T-algebra [5, 4, 6, 24, 13]. Theorem II.2.2, Proposition III.7.3, and Theorem III.9.4 in Vinberg [35] can be summarized as saying that, up to appropriate notions of isomorphism, the following are in one-to-one correspondence:

$$\text{homogeneous cones} \iff \text{unital clans} \iff \text{N-algebras} \iff \text{T-algebras.}$$

An *N-algebra* is the nilpotent part of a T-algebra. Clans we cover in Section 3.3.

Glossing over a slew of axioms, the elements of a T-algebra are square matrices whose entries come from various distinct algebras, rather than a single field like $\mathbb{R}$ or $\mathbb{C}$. A conjugate-transpose $x \mapsto x^*$ is defined on the T-algebra, and its diagonal components are isomorphic to $\mathbb{R}$. The length r of the diagonal (the size of the "matrix") is called the rank of the T-algebra. Denoting the T-algebra by $\mathcal{A}$, Vinberg shows that the set

$$K \coloneqq \{tt^* \mid t \in \mathcal{A} \text{ is upper-triangular with nonnegative diagonals}\}$$

is a homogeneous cone in the "Hermitian" subspace of $\mathcal{A}$, and that all homogeneous cones arise in this manner. If $x \in \operatorname{int}(K)$, the factorization $x = t(x)\, t(x)^*$ is unique and $t(x)$ has positive diagonal entries, analogous to the Cholesky decomposition of a symmetric positive-definite matrix. On the boundary of K, factorization remains unique subject to an additional condition [13].

Recall that the positive-semidefinite cone can be viewed as a hyperbolicity cone with respect to the hyperbolic pair $(p, e) \coloneqq (\det, I)$. In that setting, if $X = TT^*$ is the Cholesky decomposition of $X \in \operatorname{int}(\Lambda_{p,e})$, then

$$\det(X) = \det(T)^2 = \prod_{i=1}^{r} \lambda_i(T)^2 = \prod_{i=1}^{r} T_{ii}^2.$$

It is therefore tempting to consider the diagonal entries of $t(x)$ as a means of constructing a hyperbolic polynomial akin to the determinant in a T-algebra. This is essentially how Güler proceeds, though using an alternative construction of T-algebras due to Gindikin [17, 12, 11]. Define $\chi_i(x) \coloneqq t(x)_{ii}^2$ to be the square of the i^{th} diagonal entry of $t(x)$. If the diagonal is of length r, analogy compels us to try $p(x) \coloneqq \prod_{i=1}^{r} \chi_i(x)$, but p must be polynomial, and the χ_i are merely rational. Raising the χ_i to appropriate powers, however, produces a polynomial [12]. This is how Güler arrives at

$$p(x) \coloneqq \chi_1(x) \prod_{i=2}^{r} \chi_i(x)^{2^{i-2}} \qquad \text{(Güler polynomial)}$$

which, along with the unit element e of the T-algebra, forms a hyperbolic pair (p, e). A Güler polynomial exists for every homogeneous cone, but it need not have minimal degree. In a T-algebra of real matrices where K is the positive-semidefinite cone,

$$p(X) = t(X)_{11}^2 \, t(X)_{22}^2 \, t(X)_{33}^4 \cdots$$

cannot be minimal because it contains extra copies of $t(X)_{33}$ relative to the determinant. We are left wondering how to find the polynomial of minimal degree that we need to apply Theorem 4.

Gouveia, Ito, and Lourenço recently investigated this issue [14]. Using the *composite determinant* of Ishi [22], they showed that the minimal degree is at most 2^{r-1} in a T-algebra of rank r. Nakashima then provided an example where 2^{r-1} is required [25]. The authors remark that the degree of the Güler polynomial is not available for comparison; in the remainder of this section, we dispel the mystery. Most of the difficulty stems from the fact that the Güler polynomial is defined on an algebra arising from Siegel domains, rather than on a clan or (Vinberg style) T-algebra. The constructions must ultimately be equivalent, but it is not easy to see how everything is related, so it is not fair to simply conflate the two and call it a day.

3.1 Siegel domain construction

To Vinberg's list we add an additional characterization. Up to isomorphism, the following are in one-to-one correspondence:

homogeneous cones $\Longleftrightarrow$ cones arising from homogeneous Siegel domains.

We present the Siegel domain construction [12, 11] following Gindikin[1] (§II.1). Gindikin works in $\mathbb{R}^n$, assuming in particular that every one-dimensional homogeneous cone is $\mathbb{R}_+$. We instead will remain abstract, choosing a basis only when required to do so. The works of Güler, Truong, Tunçel, Vandenberghe, and Rothaus contain further discussion of Siegel domains in relation to homogeneous cones [18, 33, 34, 30, 31].

Definition 5. Let U, V be finite-dimensional real vector spaces, and K be a pointed convex cone in V. If $F : (U \times U) \to V$ is symmetric and bilinear with $F(u, u) \in K$, and if $F(u, u) = 0$ implies that $u = 0$, then the *Siegel domain* in $V \oplus U$ corresponding to K and F is

$$S(K, F) \coloneqq \{x + u \in V \oplus U \mid x \in V,\ u \in U,\ x - F(u, u) \in \operatorname{int}(K)\}.$$

[1] Gindikin citations with chapter and section numbers refer to the book.

It is straightforward to extend the definition of homogeneity to non-conic convex domains using affine (as opposed to linear) automorphisms. Affine automorphisms of a pointed convex cone are linear by necessity, so the adjective can cause no confusion [12]. In this more general sense, homogeneous Siegel domains completely characterize line-free convex homogeneous domains.

Theorem 5 (Gindikin, Theorem II.1.1)**.** *Up to (affine) isomorphism, every line-free convex homogeneous domain is a homogeneous Siegel domain.*

We then have the following construction of a homogeneous cone from a homogeneous Siegel domain.

Lemma 3 (Gindikin, Lemma II.2.1)**.** *Suppose w generates the one-dimensional real vector space W. If $S(K,F)$ is a homogeneous Siegel domain in $V \oplus U$, then*

$$H(S(K,F)) \coloneqq \{x + u + \alpha w \in V \oplus U \oplus W \mid \alpha \geq 0,\ \alpha x - F(u,u) \in K\}$$

is a homogeneous cone in $V \oplus U \oplus W$.

Gindikin shows that, up to isomorphism, every homogeneous cone K arises in this manner: from a homogeneous Siegel domain $S(K', F)$ corresponding to a homogeneous cone K' in a subspace. We can make the same argument for K'... at each step the dimension is reduced, so eventually we reach the trivial cone in the trivial vector space.

With $\{0\}$ as a starting point, we now imagine this process taking place in reverse. Given a homogeneous cone K, we construct it step by step, starting from $\{0\}$, using Lemma 3. Doing so we obtain a sequence of homogeneous cones $K^{(j)}$ in spaces $V^{(j)} \coloneqq V^{(j-1)} \oplus U^{(j)} \oplus W^{(j)}$ corresponding to Siegel domains $D^{(j)} \coloneqq S\left(K^{(j-1)}, F^{(j)}\right)$. The process terminates at $K^{(r)} = K$ in $V^{(r)} = V$.

Step 1. At the outset, everything is trivial:

$$\begin{aligned} &K^{(0)} = V^{(0)} = U^{(1)} = \{0\}, \\ &F^{(1)} = (0,0) \mapsto 0, \text{ and} \\ &D^{(1)} = S\left(K^{(0)}, F^{(1)}\right) = \{0\}. \end{aligned}$$

The homogeneous cone associated with the Siegel domain $D^{(1)}$ is a (predetermined, as we are working backwards) ray

$$\begin{aligned} K^{(1)} &= H\left(D^{(1)}\right) \\ &\subseteq V^{(1)} = \{0\} \oplus \{0\} \oplus W^{(1)}, \end{aligned}$$

where $W^{(1)} = \operatorname{span}\left(\left\{w^{(1)}\right\}\right) \cong \mathbb{R}$ with $w^{(1)}$ determined up to a positive scalar by the requirement that $w^{(1)} \in K^{(1)}$.

Step 2. Keeping in mind that we are working backwards, we are "given" the bilinear form $F^{(2)} : \left(U^{(2)} \times U^{(2)}\right) \to V^{(1)}$ from which we construct the Siegel domain $D^{(2)} = S\left(K^{(1)}, F^{(2)}\right)$ and homogeneous cone

$$\begin{aligned} K^{(2)} &= H\left(D^{(2)}\right) \\ &\subseteq V^{(2)} = V^{(1)} \oplus U^{(2)} \oplus W^{(2)}. \end{aligned}$$

Again, $W^{(2)} = \operatorname{span}\left(\left\{w^{(2)}\right\}\right) \cong \mathbb{R}$ where canonically $w^{(2)} \in K^{(2)}$.

Step r. We reach the homogeneous cone

$$K = K^{(r)} = H\left(D^{(r)}\right)$$

in the space

$$V = V^{(r)} = V^{(r-1)} \oplus U^{(r)} \oplus W^{(r)}.$$

The number $r \in \mathbb{N}$ is the *rank* of the cone.

The form $F^{(j)}$ takes values in $V^{(j-1)}$, so if $i < j$, we may denote by $F_{ii}^{(j)}$ its component in $W^{(i)}$. If $U_i^{(j)}$ is the subspace of $U^{(j)}$ where $F_{ii}^{(j)}$ is positive-definite, then $U^{(j)} = \bigoplus_{i=1}^{j-1} U_i^{(j)}$, and (for $i < j < k$) we define $F_{ij}^{(k)}$ to be the component of $F^{(k)}$ in $U_i^{(j)}$. It is helpful to arrange these spaces as follows:

$$\begin{aligned}
V^{(1)} &= \begin{bmatrix} W^{(1)} \end{bmatrix} \\
V^{(2)} &= \begin{bmatrix} W^{(1)} & U_1^{(2)} \\ & W^{(2)} \end{bmatrix} &&= \begin{bmatrix} V^{(1)} & U_1^{(2)} \\ & W^{(2)} \end{bmatrix} \\
V^{(3)} &= \begin{bmatrix} W^{(1)} & U_1^{(2)} & U_1^{(3)} \\ & W^{(2)} & U_2^{(3)} \\ & & W^{(3)} \end{bmatrix} &&= \begin{bmatrix} V^{(2)} & U^{(3)} \\ & W^{(3)} \end{bmatrix} \\
&\vdots \\
V^{(r)} &= \begin{bmatrix} V^{(r-1)} & U^{(r)} \\ & W^{(r)} \end{bmatrix}.
\end{aligned}$$

In this representation, the bilinear forms $F^{(j)}$ are defined on the above-diagonal of the j^{th} column, and take values in $V^{(j-1)}$.

We introduce a coherent set of indices on all of $V = V^{(r)}$ by

$$V_{ij} := \begin{cases} U_i^{(j)} & \text{if } i < j \\ W^{(j)} & \text{if } i = j \end{cases}$$

Any $x \in V$ thus has components x_{ij} with $x_{ii} \in W^{(i)}$ and above-diagonal columns $x_j = \sum_{i<j} x_{ij} \in U^{(j)}$. That is,

$$x = \sum_{i \leq j \leq r} x_{ij} = \sum_{i=1}^{r} x_{ii} + \sum_{j=2}^{r} x_j = \sum_{i=1}^{r} x_{ii} + \sum_{j=2}^{r} \sum_{i=1}^{j-1} x_{ij}$$

where

$$x_{ii} \in W^{(i)}, \; x_j \in U^{(j)}, \text{ and } x_{ij} \in U_i^{(j)} \text{ for } j \neq i.$$

The diagonal components $W^{(j)} = \operatorname{span}\left(\left\{w^{(j)}\right\}\right)$ have a canonical basis, but in each $U_i^{(j)}$ we have a choice to make. All that Gindikin says about this is that we should choose in a manner that makes $F_{ii}^{(j)}$ a sum of squares, and that the resulting basis for $V = \bigoplus V_{ij}$ should be ordered lexicographically by (i, j) and arbitrarily within V_{ij}.

It is not obvious that such a choice is possible. "Sum of squares" in this context means that the $w^{(i)}$-coordinate of $F_{ii}^{(j)}(u, u) \in W^{(i)}$ should be a sum of the coordinates-squared of $u \in U^{(j)}$. Luckily it follows easily [11] that the support of $F_{ij}^{(k)}$ is $U_i^{(k)} \times U_j^{(k)}$ for all $i \leq j < k$. From this we conclude that $F_{ii}^{(j)}(u, v) = 0$ unless both $u, v \in U_i^{(j)}$, whereupon $F_{ii}^{(j)}$ is an inner product. A sum of squares can thus be achieved through orthogonalization and scaling.

3.2 Degree of the Güler polynomial

At this point we must switch to coordinates. Taking a basis of size $n := \dim(V)$ and with $n_{ij} := \dim(V_{ij})$, we shall assume for the remainder of the section that $V = \mathbb{R}^n$ and $V_{ij} = \mathbb{R}_{ij}^{n_{ij}}$. The components x_{ij} should be interpreted as real vectors of length n_{ij}. In particular, $x_{ii} \in \mathbb{R}$.

Gindikin argues that, relative to the point $e \in \operatorname{int}(K)$ such that $e_{ij} = \delta_{ij}$, a simply-transitive triangular subgroup can be used to map any $x \in \operatorname{int}(K)$ to an $\tilde{x} \in \operatorname{int}(K)$ whose off-diagonal components are zero. The diagonal components of $\tilde{x}$ are are then denoted by $\chi_i(x) \in \mathbb{R}$ for $i = 1, 2, \ldots, r$. Shortly thereafter, Gindikin shows us how to compute $\chi_i(x)$.

Define a sequence $x^{(k)} \in V^{(r-k)}$ componentwise, starting from $x^{(0)} := x$, by

$$x_{ij}^{(k)} := x_{ij}^{(k-1)} - \frac{F_{ij}^{(r-(k-1))}\left(x_{r-(k-1)}^{(k-1)}, x_{r-(k-1)}^{(k-1)}\right)}{x_{r-(k-1),r-(k-1)}^{(k-1)}}.$$

The diagonals $\chi_i(x) = \tilde{x}_{ii} \in \mathbb{R}$ are then given by

$$\chi_i(x) := x_{ii}^{(r-i)} = x_{ii}^{(r-i-1)} - \frac{F_{ii}^{(i+1)}\left(x_{i+1}^{(r-i-1)}, x_{i+1}^{(r-i-1)}\right)}{x_{i+1,i+1}^{(r-i-1)}},$$

with the base case being $\chi_r(x) = x_{rr}$ at $i = r$. Noticing that the denominators in these expressions are χ from previous iterations, we may rewrite them as

$$\begin{aligned} x_{ij}^{(k)} &= x_{ij}^{(k-1)} - \frac{F_{ij}^{(r-(k-1))}\left(x_{r-(k-1)}^{(k-1)},\, x_{r-(k-1)}^{(k-1)}\right)}{\chi_{r-(k-1)}\left(x\right)}, \\ \chi_i\left(x\right) &= x_{ii}^{(r-i-1)} - \frac{F_{ii}^{(i+1)}\left(x_{i+1}^{(r-i-1)},\, x_{i+1}^{(r-i-1)}\right)}{\chi_{i+1}\left(x\right)}. \end{aligned} \tag{1}$$

The base case $\chi_r\left(x\right) = x_{rr}$ is obviously a homogeneous polynomial of degree one in the coordinates of x. All other χ_i for $i < r$ are, a priori, rational functions. There is however a way to obtain homogeneous polynomials from them. Take χ_{r-1} as the first example. From Equation (1),

$$\chi_{r-1}\left(x\right)\chi_r\left(x\right) = x_{r-1,r-1}x_{rr} - F_{r-1,r-1}^{(r)}\left(x_r,\, x_r\right).$$

The bilinearity of $F_{r-1,r-1}^{(r)}$ makes it homogeneous of degree two in the coordinates of its argument, and the coordinates of x_r are coordinates of x, so the entire expression is homogeneous of degree two in the coordinates of x. We will go one step further. From Equation (1),

$$\chi_{r-2}\left(x\right)\chi_{r-1}\left(x\right) = x_{r-2,r-2}^{(1)}\chi_{r-1}\left(x\right) - F_{r-2,r-2}^{(r-1)}\left(x_{r-1}^{(1)},\, x_{r-1}^{(1)}\right).$$

We just saw that $\chi_{r-1}\left(x\right)$ will be homogeneous of degree two if multiplied by $\chi_r\left(x\right)$, and the same is true (for the same reason) of

$$x_{r-2,r-2}^{(1)} = x_{r-2,r-2} - \frac{F_{r-2,r-2}^{(r)}\left(x_r,\, x_r\right)}{\chi_r\left(x\right)}.$$

This observation motivates us to consider $\chi_{r-2}\left(x\right)\chi_{r-1}\left(x\right)\chi_r\left(x\right)^2$ which, after distributing and regrouping, becomes

$$\left[\chi_{r-1}\left(x\right)\chi_r\left(x\right)\right]\left[x_{r-2,r-2}^{(1)}\chi_r\left(x\right)\right] - F_{r-2,r-2}^{(r-1)}\left(x_{r-1}^{(1)}\chi_r\left(x\right),\, x_{r-1}^{(1)}\chi_r\left(x\right)\right).$$

The minuend in this expression is homogeneous of degree four by choice. In the subtrahend, we have used the bilinearity of $F_{r-2,r-2}^{(r-1)}$ to move $\chi_r\left(x\right)^2$ inside. As before, $F_{r-2,r-2}^{(r-1)}$ is homogeneous of degree two in the coordinates of its argument, so it remains to show that those coordinates are themselves homogeneous of degree two in the coordinates of x. Recalling the support of $F_{r-2,r-2}^{(r-1)}$ from the end of Section 3.1, it suffices to confirm this for the coordinates of $x_{r-2,r-1}^{(1)}\chi_r\left(x\right)$. The m^{th} such coordinate is,

$$\left[x_{r-2,r-1}^{(1)}\chi_r\left(x\right)\right]_m = \left[x_{r-2,r-1}\right]_m x_{rr} - \left[F_{r-2,r-1}^{(r)}\left(x_r,\, x_r\right)\right]_m.$$

Both $\left[x_{r-2,r-1}\right]_m$ and x_{rr} are coordinates of x, so that term is homogeneous of degree two. And the orthogonal projection of $F_{r-2,r-1}^{(r)}$ onto the span of the m^{th} basis vector is real and bilinear, so it too will be homogeneous of degree two in the coordinates of x. We conclude that the entire expression $\chi_{r-2}(x)\,\chi_{r-1}(x)\,\chi_r(x)^2$ is homogeneous of degree four.

In the last step, re-group

$$\chi_{r-2}(x)\,\chi_{r-1}(x)\,\chi_r(x)^2 = \chi_{r-2}(x)\left[\chi_{r-1}(x)\,\chi_r(x)\right]\chi_r(x)$$

to emphasize that both $\chi_{r-1}(x)\,\chi_r(x)$ and $\chi_r(x)$ are homogeneous polynomials arising from previous steps—this pattern will continue to produce them. To simplify the notation, we shall from now on omit the argument x to the rational χ_k and the (forthcoming) polynomials q_k and π_k. Over $\mathbb{R}$, the resulting arithmetic is made legitimate by a ring isomorphism [2, 7].

Definition 6. Define the following expressions recursively for $k \in \{1, 2, \ldots, r\}$:

$$q_k \coloneqq \chi_k \prod_{j>k} q_j.$$

A priori each $q_k : \mathbb{R}^n \to \mathbb{R}$ is rational in the coordinates of x, because χ_k is. We will see however that the q_k are homogeneous polynomials of degree 2^{r-k}.

Lemma 4. *Let $\pi_k \coloneqq \prod_{j=r-(k-1)}^{r} q_j$ for $k \in \{0, 1, \ldots, r\}$. Then the coordinates of $\pi_k x^{(k)}$ are homogeneous polynomials of degree 2^k in the coordinates of x. In particular this holds for the diagonal coordinate $\pi_k x^{(k)}_{r-k,r-k} = \pi_k \chi_{r-k}$.*

Proof. We have already discussed the first few diagonal coordinates:

k	$r-(k-1)$	$q_{r-(k-1)}$	π_k	$\pi_k\chi_{r-k}$
0	$r+1$	undef	1	χ_r
1	r	χ_r	χ_r	$\chi_r\chi_{r-1}$
2	$r-1$	$\chi_{r-1}\chi_r$	$\chi_{r-1}\chi_r^2$	$\chi_{r-2}\chi_{r-1}\chi_r^2$

Every coordinate of $\pi_0 x^{(0)} = x$ is trivially homogeneous in the coordinates of x. Induction: suppose the result holds at $k-1$. In general we have

$$\pi_k x^{(k)} = q_{r-(k-1)}\pi_{k-1}x^{(k)} = \pi_{k-1}^2\chi_{r-(k-1)}x^{(k)}.$$

Choosing an arbitrary component $x^{(k)}_{ij}$ so that we may use Equation (1),

$$\pi_{k-1}^2\chi_{r-(k-1)}x^{(k)}_{ij} = \pi_{k-1}^2\left[\chi_{r-(k-1)}x^{(k-1)}_{ij} - F^{(r-(k-1))}_{ij}\left(x^{(k-1)}_{r-(k-1)},\, x^{(k-1)}_{r-(k-1)}\right)\right].$$

If we distribute the π_{k-1}^2 using bilinearity and regroup, we end up with

$$\left[\pi_{k-1}\chi_{r-(k-1)}\right]\left[\pi_{k-1}x^{(k-1)}_{ij}\right] - F^{(r-(k-1))}_{ij}\left(\pi_{k-1}x^{(k-1)}_{r-(k-1)},\, \pi_{k-1}x^{(k-1)}_{r-(k-1)}\right).$$

An arbitrary (say, m^{th}) coordinate of the first term is

$$\left(\left[\pi_{k-1}\chi_{r-(k-1)}\right]\left[\pi_{k-1}x_{ij}^{(k-1)}\right]\right)_m = \left[\pi_{k-1}\chi_{r-(k-1)}\right]\left[\pi_{k-1}x_{ij}^{(k-1)}\right]_m.$$

By supposition both bracketed expressions are homogeneous of degree 2^{k-1}, so their product is homogeneous of degree 2^k. Likewise,

$$\left[F_{ij}^{(r-(k-1))}\left(\pi_{k-1}x_{r-(k-1)}^{(k-1)}, \pi_{k-1}x_{r-(k-1)}^{(k-1)}\right)\right]_m$$

is the application of a real bilinear function, and is therefore homogeneous of degree two in the coordinates of its argument. By supposition, those coordinates are homogeneous of degree 2^{k-1} in the coordinates of x. It follows that the entire expression, an arbitrary coordinate of an arbitrary component of $\pi_k x^{(k)}$, is homogeneous of degree 2^k in the coordinates of x. □

Corollary 5. *Each q_k is a homogeneous polynomial of degree 2^{r-k}.*

Proof. Observe that $q_k = \chi_k \prod_{j>k} q_j = \chi_k \pi_{r-k}$ and apply Lemma 4. □

Corollary 6. *The Güler polynomial is homogeneous of degree 2^{r-1}.*

Proof. The Güler polynomial is q_1, since

$$q_k = \chi_k \prod_{j=k+1}^{r} q_j = \chi_k \chi_{k+1} \prod_{j=k+2}^{r} q_j^2 = \chi_k \chi_{k+1} \chi_{k+2}^2 \prod_{j=k+3}^{r} q_j^4 = \cdots$$
$$\cdots = \chi_k \prod_{j=1}^{r-k} \chi_{k+j}^{2(j-1)}.$$
□

The proof of Corollary 6 reveals that the homogeneous polynomials q_k are compound power functions associated with certain K-integral vectors (Gindikin §II.2). Having remarked that the various characterizations of homogeneous cones are equivalent, it comes as no surprise that the q_k appear in other settings. Analogous polynomials are defined on a T-algebra by Vinberg (§III.3) and on a clan by Ishi [22]. If you are willing to take the equivalence on faith, the Güler polynomial is Ishi's *composite determinant* D_r. We make this relationship precise in the next section.

Corollary 6 confirms that the Güler polynomial can have minimal degree, but only in pathological[2] cases such as the one constructed by Nakashima [25]. This answers a question that arose in the context of rank-one-generated hyperbolicity cones [23], but in most cases, does not help us find a polynomial of minimal degree. In light of this, we heed the advice of Gouveia et al.

> It may be more advantageous to deal with a homogeneous cone... on its own terms... instead of seeing it as a hyperbolicity cone...

[2]In that their minimal degrees are maximal for a given rank.

In the next section, we elucidate the relationship between Siegel domains and clans. Certainly Vinberg and his contemporaries were aware of this connection—Rothaus implies that both are equivalent to his own construction [30, 31]—but it seems to have been lost to time. Depending on the application, clans may be preferable to T-algebras because the clan structure is on the space that the cone K lives in. The group $\operatorname{Aut}(K)_e$ acts on the clan, for example.

3.3 Clans

Recall from the introduction to Section 3 that, up to isomorphism on both sides, unital clans correspond to homogeneous cones and vice-versa. Vinberg (§II.1) describes in detail the construction of a compact left-symmetric algebra (a clan with unit e) from a homogeneous cone K and point $e \in \operatorname{int}(K)$. In the interest of brevity, and as we are following Vinberg in any case, we omit the construction and begin at the axioms [35].

Definition 7 (Vinberg §II.1)**.** A *clan* is an algebra $\mathcal{C} = (V, \Delta)$ on a finite-dimensional real vector space V with bilinear multiplication $x \Delta y$ and left-regular representation $L_x \coloneqq y \mapsto x \Delta y$ satisfying three properties:

1. $L_x L_y - L_y L_x = L_{(x\Delta y - y \Delta x)}$ (left-symmetry)
2. $(x, y) \mapsto \operatorname{trace}(L_{x\Delta y})$ is an inner product (compactness)
3. The eigenvalues of L_x are real for all $x \in V$

We will use the symbol $\mathcal{C} = (V, \Delta)$ to denote a clan, transparently referencing the underlying vector space and multiplication when the context is clear. Two clans are isomorphic if there is an invertible linear multiplication-preserving map (algebra isomorphism) between them. An important feature of clans is that they possess a triangular direct-sum decomposition [35].

Definition 8 (Vinberg §II.4)**.** A *normal decomposition* of a unital clan $\mathcal{C} = (V, \Delta)$ is a direct-sum decomposition

$$\mathcal{C} = \bigoplus_{i \leq j \leq r} \mathcal{C}_{ij}$$

of the underlying vector space V, where $r \in \mathbb{N}$ is the *rank* of the clan, and

1. The diagonal subspaces $\mathcal{C}_{ii}$ are one-dimensional and are generated by idempotents $e_i = e_i \Delta e_i$, and
2. The off-diagonal subspaces $\mathcal{C}_{ij}$ (with $i < j$) are characterized by

$$x \in \mathcal{C}_{ij} \implies \begin{cases} e_k \Delta x = \frac{1}{2} x & \text{if } k \in \{i, j\} \\ e_k \Delta x = 0 & \text{if } k \notin \{i, j\} \\ x \Delta e_k = x & \text{if } k = j \\ x \Delta e_k = 0 & \text{if } k \neq j \end{cases}$$

for all $k \in \{1, 2, \ldots, r\}$.

If the clan is constructed (cf. Vinberg) with respect to $e \in \operatorname{int}(K)$, then $\sum e_i = e$ becomes the unit element of the clan. The components of the normal decomposition are orthogonal under the inner product in Definition 7, and have the following multiplication table (Vinberg p. 376).

$$\mathcal{C}_{ij}\Delta\mathcal{C}_{k\ell} \subseteq \begin{cases} \{0\} & \text{if } j \notin \{k,\ell\} \\ \mathcal{C}_{i\ell} & \text{if } j = k \\ \mathcal{C}_{ik} \text{ or } \mathcal{C}_{ki} & \text{if } j = \ell \end{cases} \tag{2}$$

If the normal decomposition looks familiar, it is because the Siegel domain construction led also to a triangular decomposition of (and coordinates for) the ambient space. We will show, starting from a normal decomposition $\mathcal{C} = \bigoplus \mathcal{C}_{ij}$ and bases for the off-diagonal $\mathcal{C}_{ij}$, that there exists a Siegel domain construction in $V = \bigoplus V_{ij}$ whose coordinates are identical. To motivate this, we note the similarity between Theorem 5 and Lemma 3, and the following amalgamation of Vinberg's Theorem II.2.2 and Proposition II.1.3.

Theorem 6. *Up to isomorphism, homogeneous convex domains and clans are in one-to-one correspondence. Unital clans correspond to homogeneous cones, and every homogeneous cone arises from a homogeneous convex domain via adjunction of a unit element to the domain's clan.*

Start with a homogeneous convex cone. The cone corresponds to a unital clan, and the unital clan arises from a non-unital clan via adjunction of a unit element. The non-unital clan has a principal decomposition into two factors, the first of which is a unital subclan having the principal idempotent as its unit element (Vinberg §II.3). The unital subclan corresponds to homogeneous convex cone of lower dimension, and so on. Imagining this process in reverse, we "construct" the original cone.

The relationship of this to the Siegel domain construction goes beyond mere analogy. Given a homogeneous cone and a normal decomposition of its unital clan, we will show that the bilinear forms $F^{(k)}$ in the Siegel domain construction are determined by the clan product, and that the coordinates in both settings can be made to agree.

The inspiration for this is Vinberg's proof (Proposition II.4.8) of the normal decomposition. Beginning with a clan and assuming that it can be done in a clan of lesser rank, Vinberg peels off an idempotent, performs a principal decomposition (§II.3), computes a normal decomposition of its unital subalgebra (induction), and then brings back the idempotent. If we place the idempotent at the bottom-right of the subalgebra's normal decomposition, we can place the column from the principal decomposition above it on the superdiagonal. The result is a normal decomposition of the original clan, and the steps we've taken correspond to those in the Siegel domain construction—all that remains is to work out the details.

Theorem 7. *Let $K = K^{(r)}$ be a homogeneous cone of rank r in $V = V^{(r)}$, and suppose $\mathcal{C} = (V, \Delta)$ is a clan corresponding to $e \in \operatorname{int}(K)$. If the normal* *decomposition of* $\mathcal{C}$ *is*

$$\mathcal{C} = \begin{bmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \cdots & \mathcal{C}_{1r} \\ & \mathcal{C}_{22} & \cdots & \mathcal{C}_{2r} \\ & & \ddots & \vdots \\ & & & \mathcal{C}_{rr} \end{bmatrix},$$

then there exist spaces $V^{(j-1)}$*,* $U^{(j)}$ *and Siegel domains* $D^{(j)} = S\left(K^{(j-1)}, F^{(j)}\right)$ *for* $j \in \{1, 2, \ldots, r\}$ *such that* $K^{(r)} = H\left(D^{(r)}\right)$ *is a homogeneous cone in*

$$V^{(r)} = \begin{bmatrix} W^{(1)} & U_1^{(1)} & \cdots & U_1^{(r)} \\ & W^{(2)} & \cdots & U_2^{(r)} \\ & & \ddots & \vdots \\ & & & W^{(r)} \end{bmatrix},$$

where $W^{(j)} = \operatorname{span}(\{e_j\}) = \mathcal{C}_{jj}$*, and the basis of* $U_i^{(j)}$ *is the same as that of* $\mathcal{C}_{ij}$*. Moreover, each* $2F^{(j)}$ *is the restriction of the clan product to its domain.*

Proof (induction on rank). When $r = 1$, the normal decomposition of $\mathcal{C}$ is $\mathcal{C}_{11} = \operatorname{span}(\{e_1\})$ for some idempotent e_1. The cone is $K = K^{(1)} = \mathbb{R}_+ e_1$, and trivially arises from the Siegel domain construction with $K^{(0)} = V^{(0)} = U^{(1)} = \{0\}$, $W^{(1)} = \operatorname{span}(\{e_1\}) = \mathcal{C}_{11}$, and $F^{(1)} \equiv 0$. On $U^{(1)} \times U^{(1)} = \{(0,0)\}$, the clan product clearly agrees with $2F^{(1)}$.

Suppose that the result holds for rank $r-1$. Let

$$U^{(r)} := \bigoplus_{i=1}^{r-1} \mathcal{C}_{ir} \quad \text{and} \quad V^{(r-1)} := \bigoplus_{i,j=1}^{r-1} \mathcal{C}_{ij}.$$

It is easy to see from the multiplication table (2) that both $V^{(r-1)}$ and $V^{(r-1)} \oplus U^{(r)}$ are subalgebras of $\mathcal{C}$, and from Definition 7 it is clear that every subalgebra of a clan is a clan. Moreover if $\mathcal{C}_{rr} = \operatorname{span}(\{e_r\})$, then $e - e_r$ is the unit element of $V^{(r-1)}$. It follows that there is a homogeneous cone $K^{(r-1)}$ in $V^{(r-1)}$, and that $V^{(r-1)} \oplus U^{(r)}$ is a principal decomposition (of itself) with respect to $e - e_r$. Citing Equation (24) in Vinberg §II.3,

$$x \Delta y = y \Delta x \text{ for all } x, y \in U^{(r)}.$$

A few pages later, Proposition II.3.7 shows that

$$x \Delta x \in K^{(r-1)} \text{ for all } x \in U^{(r)}.$$

Finally, since $\langle x, x \rangle := \operatorname{trace}(L_{x \Delta x})$ is the square of a norm, we have that $x \Delta x = 0$ if and only if $x = 0$. Proposition II.3.5 thus confirms that the homogeneous convex domain associated with the clan $V^{(r-1)} \oplus U^{(r)}$ is precisely the

Siegel domain $D^{(r)} = S\left(K^{(r-1)}, F^{(r)}\right)$, where $2F^{(r)}$ is the restriction of the clan product to $U^{(r)}$. Put $w^{(r)} = e_r$ and form the cone associated with the Siegel domain to recover $K^{(r)}$.

It remains to specify the coordinates on $U^{(r)}$ in a manner that makes $F_{ii}^{(r)}$ a sum of squares. For this it is instructive to introduce the inner product $\langle x, y\rangle_I := \frac{1}{2}\operatorname{tr}(x\Delta y)$, where

$$\operatorname{tr}(x) = \operatorname{tr}\left(\sum_{i=1}^{r} \alpha_i e_i + \sum_{i<j\leq r} x_{ij}\right) := \sum_{i=1}^{r} \alpha_i$$

is the sum of the diagonal coordinates with respect to the idempotents of the normal decomposition. This inner product is due to Ishi, and it preserves the orthogonality of the normal decomposition [21, 22].

Fix $i \in \{1, 2, \ldots, r\}$. If $u, v \in \mathcal{C}_{ir}$, then from the multiplication table (2),

$$\langle u, v\rangle_I = 0 \iff u\Delta v = 0.$$

If $\{u_1, u_2, \ldots, u_{n_{ir}}\}$ is an orthonormal basis for $\mathcal{C}_{ir} = U_i^{(r)}$ with respect to $\langle\cdot,\cdot\rangle_I$, then for any $x \in U^{(r)}$ it follows that

$$(x\Delta x)_{ii} = x_{ir}\Delta x_{ir} = \left(\sum_{k=1}^{n_{ir}} \alpha_k u_k\right)\Delta\left(\sum_{k=1}^{n_{ir}} \alpha_k u_k\right) = \sum_{k=1}^{n_{ir}} \alpha_k^2 \left(u_k \Delta u_k\right).$$

We know from (2) that each $u_k\Delta u_k$ is $\beta_k e_i$ for some $\beta_k \in \mathbb{R}$, but

$$\left[1 = \|u_k\|_I^2 = \frac{1}{2}\operatorname{tr}\left(u_k\Delta u_k\right) = \frac{1}{2}\beta_k\right] \implies \beta_k = 2$$

for all k. So, ultimately,

$$F_{ii}^{(r)}(x, x) = \frac{1}{2}(x\Delta x)_{ii} = \frac{1}{2}\sum_{k=1}^{n_{ir}} \alpha_k^2 \beta_k e_i = \left(\sum_{k=1}^{n_{ir}} \alpha_k^2\right) e_i$$

will be a sum of squares. ☐

Recall the polynomials q_k, homogeneous of degree 2^{r-k}, from the Siegel domain construction (Definition 6 and Corollary 5). At the end of Section 3.2 we promised to clarify the relationship between the q_k and the *determinant-type polynomials* D_k, homogeneous of degree 2^{k-1}, defined on a clan by Ishi [22]. Though it is clear that Ishi considers them interchangeable, Theorem 7 makes the connection explicit.

Corollary 7. *The polynomials q_k from Definition 6 and the determinant-type polynomials $D_{r-(k-1)}$ of Ishi are equivalent. In particular, the Güler polynomial q_1 is the composite determinant D_r.*

To an extent this obsoletes Section 3.2, because the degree of the composite determinant is known. In any case, it is reassuring that the answers agree.

4 Conclusions and future work

Finding a polynomial of minimal degree for a homogeneous cone remains of course an open problem. But having clarified the relationship between the formalisms in Section 3.3, we are freed to work in whichever is most convenient. One example: for a linear map, preserving the extreme rays of a proper cone is equivalent to membership in its automorphism group [18, 32]. Truong and Tunçel characterized the extreme rays of a homogeneous cone using Siegel domains [33], but it is desirable to know the extreme rays in other settings [13]. Translating this (and similar results) to clans now poses no difficulty.

In Section 2, for want of algebra automorphisms, we began investigating the spectral preservers $\text{Aut}\,(p, e)$ of a hyperbolic pair. When our cone is homogeneous, we do however have a set of algebra automorphisms—those that preserve the clan product—so it is natural to wonder if $\text{Aut}\,(\mathcal{C}) = \text{Aut}\,(K)_e$ in the clan $\mathcal{C}$ associated with the homogeneous cone K and $e \in \text{int}\,(K)$. Unfortunately the answer is negative.

Example 1. Begin with the space of 2×2 real matrices which by design satisfies the axioms of a T-algebra. The homogeneous cone associated with this T-algebra is the 2×2 real PSD cone. Vinberg (§III.2) explains how to obtain a clan from a T-algebra, keeping the homogeneous cone intact [35]. Consider first the subspace $\mathcal{S}^2$ of symmetric matrices in which the positive-semidefinite cone $K = \mathcal{S}^2_+$ lives. On $\mathcal{S}^2$ we define the multiplication,

$$x\Delta y \coloneqq \hat{x}y + y\underset{\sim}{x},$$

where

$$\hat{x} \coloneqq \sum_{i<j} x_{ij} + \frac{1}{2}\sum_i x_{ii}$$
$$\underset{\sim}{x} \coloneqq \sum_{i>j} x_{ij} + \frac{1}{2}\sum_i x_{ii}$$

so that $x = \hat{x} + \underset{\sim}{x}$. Along with the inner product

$$\langle x, y\rangle \coloneqq \text{trace}\,(L_{x\Delta y}) = \text{tr}\,(x\Delta y) = \text{trace}\,(xy)$$

this forms a clan having the identity matrix as its unit element. The cone $\mathcal{S}^2_+$ is symmetric, so we know that the spectral preservers on $\mathcal{S}^2$ are

$$\text{Aut}\left(\mathcal{S}^2_+\right)_e = \text{Aut}\left(\mathcal{S}^2_+\right) \cap \text{Isom}\left(\mathcal{S}^2\right)$$

with respect to the trace inner product (cf. Theorem 1). One of these maps however is not a clan automorphism. Define,

$$P \coloneqq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{aligned} &\phi : \mathcal{S}^2 \to \mathcal{S}^2 \\ &\phi \coloneqq X \mapsto PXP^T. \end{aligned}$$

It is known that ϕ is a PSD cone automorphism (it preserves eigenvalues) that both fixes the identity and happens to be a trace isometry [27]. Yet it fails to preserve the clan product. To see this, note that P itself belongs to $\mathcal{S}^2$, so we may let $Y \coloneqq \operatorname{diag}(1, 2)$ and then simply compute $\phi(P \Delta Y) \neq \phi(P) \Delta \phi(Y)$.

We close with a related problem. If $\mathcal{C}$ is the clan associated with K and $e \in \operatorname{int}(K)$, then there are at least two inner products on $\mathcal{C}$ such that $\operatorname{Aut}(K)_e = \operatorname{Isom}(\mathcal{C})$. The first we encountered in Theorem 4, and the second follows from the compactness of $\operatorname{Aut}(K)_e$ and a Haar measure argument (Theorem II.1.7 of Bröcker and tom Dieck [3], for example). Both of these are implicit however, and there is no guarantee that they differ.

On the other hand, we know two inner products on $\mathcal{C}$ explicitly—the one from Definition 7, and the one we used to normalize the basis in Theorem 7:

$$\langle x, y \rangle \coloneqq \begin{cases} \operatorname{trace}(L_{x \Delta y}) & \text{(Vinberg)} \\ \frac{1}{2} \operatorname{tr}(x \Delta y) & \text{(Ishi)} \end{cases}$$

These are structurally equivalent to inner products that satisfy the assumptions of Theorem 1 in a Euclidean Jordan algebra [26]. Does $\operatorname{Aut}(K)_e = \operatorname{Isom}(\mathcal{C})$ hold for either of them? (We suspect the answer is "no," but a counterexample is not readily available.) Failing that, is there an explicit inner product for which $\operatorname{Aut}(K)_e = \operatorname{Isom}(\mathcal{C})$ does hold?

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