Robust Network Design for Potential-Based Flows with Controllable Elements

Kristina Rolsing and Johannes Thürauf

Abstract. We study adjustable robust network design for potential-based flows with controllable elements under load uncertainty. The resulting problem combines discrete here-and-now expansion decisions with wait-and-see operational decisions governed by nonconvex flow constraints. Moreover, controllable elements introduce adjustable integer decisions, which are algorithmically challenging. We equivalently characterize robust feasibility and robust optimality of a fixed network design using adversarial bilevel problems. For robust feasibility, we extend an existing characterization for potential-based networks without controllable elements to networks with controllable elements under the structural assumption that no controllable element is part of a cycle. This yields a characterization of robust feasibility consisting of polynomially many mixed-integer nonlinear bilevel problems. Since controllable elements make the objective depend on both here-and-now expansion cost and wait-and-see operating cost, we verify robust optimality of a fixed network design using an additional mixed-integer nonlinear bilevel problem. We then derive equivalent single-level reformulations of these bilevel problems under the stated structural assumption. Building on these reformulations, we present an exact adversarial solution approach for computing adjustable robust network designs with controllable elements and demonstrate its applicability to gas networks. More generally, the developed potential-based framework can be used to compute robust network designs for different types of utility networks with controllable elements, including hydrogen and water networks.

1. Introduction

Robust network design accounts for uncertain input data, such as supplies, demands, capacities, or component failures, in the planning of network systems. Neglecting such uncertainties may render the resulting network infeasible or inefficient. Typical applications include telecommunication [19], supply-chain [37], energy [28], and utility networks [35], where uncertain loads are often a major source of uncertainty.

In this paper, we study mixed-integer nonlinear adjustable robust network design under uncertainty in both supply and demand. We consider general potential-based flows with controllable elements, such as pumps in water networks and compressors in gas networks, which are essential for transporting flow over long distances in larger utility networks. This modeling framework makes our results applicable to water, gas, hydrogen, and lossless DC power flow networks.

The studied adjustable robust network design problems have the following structure. First, discrete here-and-now decisions for building new arcs or controllable elements have to be made before the uncertainty realizes. Then, a worst-case load scenario from a prescribed uncertainty set is revealed. Finally, this worst-case realization of supplies and demands has to be transported through the constructed network by choosing suitable flows and operations of the controllable elements. A corresponding adjustable

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robust network design therefore guarantees that all load scenarios in the uncertainty set, possibly infinitely many, can be transported through the network.

To model the physics governing transport in utility networks, we use general potential-based flows; see, e.g., [23] for a recent overview. Potential-based flows generalize classical capacitated linear flows by introducing nodal potentials and coupling these potentials to arc flows through potential functions that are nonlinear in general. One of the main advantages of potential-based flows is their broad applicability to different utility networks; see [11]. Moreover, we consider controllable elements that can increase or decrease potentials. In the following, we also refer to these elements as active elements; see, e.g., [11, 25]. Such elements are essential for the operation of larger utility networks but are mathematically challenging because modeling their operation typically requires integer decisions.

For nominal topology optimization with potential-based flows, i.e., without uncertainty, we refer to the recent survey [23], which also provides references on application-specific topology optimization, for instance in gas networks. For recent developments in mathematical optimization for water networks, we refer to [8, 22] and the references therein. Further recent work on topology optimization with general potential-based flows includes valid inequalities [4] and approximation approaches [16].

We now focus on adjustable robust network design with nonlinear potential-based flows. Compared with robust network design for linear flow models, see, e.g., [3, 7, 30], the literature on nonlinear potential-based flows is much more limited. One reason is that algorithmic approaches for linear flow models often exploit specific structural properties, such as strong duality or necessary and sufficient Karush-Kuhn-Tucker (KKT) conditions, which do not directly extend to the nonlinear and nonconvex potential-based flows considered here.

We now turn to robust potential-based network design under load uncertainty. In general, potential-based networks are not monotone with respect to load changes; see [11, 23]. Thus, feasibility of one load scenario does not imply feasibility of smaller loads, which makes certifying robust feasibility challenging.

For passive networks, i.e., without active elements, robust feasibility of a given network can be characterized by solving finitely many adversarial single-level optimization problems. If robust infeasibility is detected, their solutions yield worst-case scenarios that cannot be transported through the network. Embedding these scenarios in an adversarial approach then yields an exact solution method that computes a robust network design in finitely many steps; see [35]. For tree-shaped networks, a mixed-integer linear reformulation based on finitely many worst-case load scenarios is derived in [29] and used to compute robust hydrogen networks. Moreover, graph structures for which decreasing the load preserves feasibility are identified in [15].

The literature on robust potential-based network design with controllable elements is even sparser. Indeed, controllable elements add binary wait-and-see decisions to the already challenging nonlinear recourse problem. Such adjustable integer decisions are known to be particularly difficult in adjustable robust optimization [36]. Robust gas network design with controllable elements under interval uncertainty only in the demand is studied in [33]. Assuming unlimited capacities at every supply node, they show that robust feasibility can be verified by two worst-case demand scenarios, whereas this is no longer sufficient when supplies are bounded [35, Section 5]. Component failures instead of load uncertainty are considered in [24], where a nested Benders approach is developed to compute resilient networks protected against prescribed component failures. Moreover, robust potential-based flows with controllable elements have been studied in the related context of robust operation of potential-based networks [1, 2] and for the European gas market [25]. Since we focus on adjustable robust optimization, we refer to [23, Remarks 9 and 10] for complementary stochastic and probabilistic-robust approaches.

The reviewed literature shows a need for methods that compute adjustable robust network designs with controllable elements under load uncertainty affecting both supply and demand. In this paper, we provide a first stepping stone toward solving such challenging problems for general potential-based flows.

In a nutshell, the presented algorithm follows an adversarial scheme in which the uncertainty set U is replaced by a finite scenario set $S \subseteq U$ that is iteratively enlarged. For a fixed set S, the scenario-expanded design problem yields a candidate network design. This candidate is then checked against the full uncertainty set, first for robust feasibility and then for robust optimality with respect to the worst-case operating cost. If a violating scenario is found, it is added to S. Otherwise, if the operating cost is underestimated, an objective cut is added. The algorithm is exact and terminates once the current candidate is robust optimal or infeasibility of the robust problem is certified. Overall, our main contributions are as follows.

First, we extend the robust-feasibility characterization of [35] from passive networks to networks with controllable elements. In contrast to [35], the resulting characterization requires solving finitely many mixed-integer nonlinear bilevel problems due to the presence of controllable elements. These bilevel problems contain binary variables and nonconvex constraints in the lower-level problem and are therefore challenging to solve. To obtain this characterization of robust feasibility and to reformulate the resulting bilevel problems in an algorithmically tractable way, we impose the structural assumption that no active element lies on a cycle of the network. This assumption is also used, e.g., in [1, 2, 25]. Under this assumption, we exploit the algorithmic approach of [25] to reformulate the adversarial bilevel problems equivalently as bilevel problems with linear lower-level problems. Applying KKT-based reformulations of these bilevel problems enables us to verify robust feasibility of a given network design by solving polynomially many single-level optimization problems.

Second, because controllable elements make the objective depend on both here-and-now expansion cost and wait-and-see operating cost, we verify robust optimality of a fixed network design using an additional mixed-integer nonlinear bilevel problem. This allows us to integrate worst-case operating cost of the controllable elements into the network design problem. We then provide an algorithmically tractable reformulation of this bilevel problem.

Third, we use these results to extend the adversarial approach of [35] to networks with controllable elements. We derive KKT-based single-level reformulations of the resulting bilevel problems and prove valid primal and dual bounds for the corresponding big-M linearizations. This yields an exact adversarial algorithm that computes an optimal adjustable robust network design in finitely many steps under the stated structural assumption. Finally, we demonstrate the applicability of our approach on realistic gas networks.

The remainder of this paper is organized as follows. Section 2 introduces potential-based networks with controllable elements and the adjustable robust network design problem studied in this paper. In Section 3, we develop an adversarial solution approach based on novel characterizations of robust feasibility and robust optimality using finitely many mixed-integer nonlinear bilevel optimization problems. Moreover, we derive equivalent bilevel reformulations with linear lower-level problems under the stated structural assumption. In Section 4, we present single-level reformulations of the resulting bilevel problems and prove valid big-M values that improve their computational performance. Finally, in Section 5, we report computational results for realistic gas networks.

2. Problem Statement

We first introduce potential-based flows with controllable elements in Section 2.1. Building on this modeling framework, we then state the adjustable robust network design problem in Section 2.2.

2.1. Potential-based Networks with Controllable Elements.

Following [25, 35], we model a potential network as a directed multigraph $G = (V, A)$. The node set V is partitioned into injection nodes V_+, withdrawal nodes V_-, and inner nodes V_0. The arc set A is partitioned into passive flow arcs A^{arc}, which transport flow, and active elements A^{act}, which can modify incident node potentials. Moreover, the active elements A^{act} are partitioned into potential-increasing elements A^{cm}, e.g., compressors or pumps, and potential-decreasing elements A^{cv}, e.g., control valves. Each arc $a \in A$ is represented by a triple $(u, v, l) \in V \times V \times L$ with start node u, end node v, and label l. This notation allows parallel network elements. With a slight abuse of notation, we also write $(u, v, l) \in A$. A network is called passive if $A^{\mathrm{act}} = \emptyset$ and active otherwise.

We introduce flow variables $q \in \mathbb{R}^A$ and nodal potentials $\pi \in \mathbb{R}^V$. For a passive arc $a = (u, v, l) \in A^{\mathrm{arc}}$, the incident potentials π_u and π_v are coupled by a *potential function* $\Phi_a : \mathbb{R} \to \mathbb{R}$, which depends on the corresponding arc flow. This relation is explicitly given by

$$\pi_u - \pi_v = \Phi_a(q_a), \quad a = (u, v, l) \in A^{\mathrm{arc}}. \tag{1}$$

Each function Φ_a is assumed to be continuous, strictly increasing, and odd, i.e., $\Phi_a(-x) = -\Phi_a(x)$. These assumptions are natural in physical flow systems and potential-based networks; see, e.g., [23].

Depending on the choice of potential function, the model captures different types of utility networks. Examples include gas networks with $\Phi_a(q_a) = \Lambda_a |q_a| q_a$, water networks with $\Phi_a(q_a) = \Lambda_a \operatorname{sgn}(q_a)|q_a|^{1.852}$, and lossless DC power flow networks with $\Phi_a(q_a) = \Lambda_a q_a$; see [11]. Here, $\Lambda_a > 0$ is an arc-specific constant and the potentials represent squared pressures, hydraulic heads, or phase angles, respectively.

The flow and potential variables are restricted by lower and upper bounds

$$\pi_u^- \le \pi_u \le \pi_u^+, \quad u \in V, \qquad q_a^- \le q_a \le q_a^+, \quad a \in A. \tag{2}$$

We further consider a load scenario $d \in \mathbb{R}^V$. The value $d_v < 0$ represents an injection at node $v \in V_+$, whereas $d_v > 0$ represents a withdrawal at node $v \in V_-$. Moreover, $d_v = 0$ holds at every inner node $v \in V_0$. Mass flow conservation is modeled by

$$\sum_{a \in \delta^{\mathrm{in}}(u)} q_a - \sum_{a \in \delta^{\mathrm{out}}(u)} q_a = d_u, \quad u \in V. \tag{3}$$

Here, $\delta^{\mathrm{in}}(u)$ and $\delta^{\mathrm{out}}(u)$ denote the incoming and outgoing arcs of node u, respectively. Since we consider stationary flows, each load scenario d is balanced, i.e., $\sum_{v \in V} d_v = 0$. Otherwise, (3) cannot be satisfied.

It remains to model the active elements, which we also refer to as controllable elements. For modeling active elements in utility networks, a wide range of optimization models has been used in the literature. These range from idealized mixed-integer linear models [8, 22, 25] to detailed mixed-integer nonlinear formulations [17]. In our setting, we adopt the mixed-integer linear model for active elements proposed in [25] and follow the corresponding description.

We use control variables $\Delta \in \mathbb{R}^{A^{\mathrm{act}}}$ with bounds $0 \le \Delta \le \Delta^+$. For $a \in A^{\mathrm{act}}$, the constant $\Delta_a^+ \ge 0$ denotes the maximal potential increase or decrease. Due to technical requirements, an active element can operate only if a minimum flow $m_a \ge 0$ passes through the arc in the prescribed direction. Otherwise, it behaves as a short pipe. For $a = (u, v, l) \in A^{\text{act}}$, we impose

$$\pi_u - \pi_v = \begin{cases} -\Delta_a, & a \in A^{\text{cm}}, \\ \Delta_a, & a \in A^{\text{cv}}, \end{cases} \tag{4}$$

$$0 \le \Delta_a \le \Delta_a^+ \chi_a(q_a), \quad a \in A^{\text{act}}, \tag{5}$$

where the indicator function $\chi_a(q_a)$ is given by

$$\chi_a(q_a) = \begin{cases} 1, & \text{if } q_a > m_a, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

The indicator function $\chi_a(q_a)$ can be modeled using additional binary variables. Using the potential-based modeling framework introduced above, we now define feasible potential-based flows. For a given load scenario $d \in \mathbb{R}^V$, we call a tuple $(q, \pi, \Delta) \in \mathbb{R}^A \times \mathbb{R}^V \times \mathbb{R}^{A^{\text{act}}}$ a feasible potential-based flow if it satisfies (1)–(6).

2.2. **Adjustable Robust Network Design.** We now present a model for adjustable robust potential-based network design under load uncertainty. To this end, we consider the following uncertainty set of balanced load scenarios

$$U := Z \cap \left\{ d \in \mathbb{R}^V : \sum_{u \in V_+} d_u + \sum_{u \in V_-} d_u = 0, \; d_u = 0 \text{ for all } u \in V_0 \right\}. \tag{7}$$

Here, the nonempty compact set $Z \subset \mathbb{R}^V$ encodes the sign restrictions on injections and withdrawals as well as any additional uncertainty description. Thus, we allow for general uncertainty sets, including convex, nonconvex, and discrete sets, that can capture uncertainty in both injections and withdrawals.

The goal is to compute a network design that admits a feasible transport for every load scenario $d \in U$ and minimizes construction cost together with worst-case operating cost. To state the corresponding adjustable robust optimization model, we partition passive and active elements into existing (A_{ex}) and candidate elements (A_{ca}) and define

$$A_{\text{ex}} := A_{\text{ex}}^{\text{arc}} \cup A_{\text{ex}}^{\text{cm}} \cup A_{\text{ex}}^{\text{cv}}, \qquad A_{\text{ca}} := A_{\text{ca}}^{\text{arc}} \cup A_{\text{ca}}^{\text{cm}} \cup A_{\text{ca}}^{\text{cv}},$$

where $A^{\text{arc}} = A_{\text{ex}}^{\text{arc}} \cup A_{\text{ca}}^{\text{arc}}$, $A^{\text{cm}} = A_{\text{ex}}^{\text{cm}} \cup A_{\text{ca}}^{\text{cm}}$, and $A^{\text{cv}} = A_{\text{ex}}^{\text{cv}} \cup A_{\text{ca}}^{\text{cv}}$ hold.

For each candidate element $a \in A_{\text{ca}}$, the binary variable $x_a \in \{0, 1\}$ indicates whether a is built, and $c_a \ge 0$ denotes the corresponding construction cost. For each active element $a \in A^{\text{act}}$, we consider a linear operating cost function parameterized by $w_a \ge 0$. Using the previously introduced potential-based flow model, we now state the adjustable robust network design model

$$\inf_{x \in X} \sup_{d \in U} \inf_{q, \pi, \Delta} \left\{ c^\top x + w^\top \Delta : (q, \pi, \Delta) \in Y(x, d) \right\}, \tag{8}$$

where, for fixed $x \in X$ and $d \in U$, the set of feasible wait-and-see decisions is defined by

$$Y(x, d) := \left\{ (q, \pi, \Delta) \in \mathbb{R}^A \times \mathbb{R}^V \times \mathbb{R}^{A^{\text{act}}} : q, \pi, \Delta \text{ satisfying (10)} \right\}. \tag{9}$$

Here, Constraints (10) describe the transport through the built network:

$$\sum_{a\in\delta^{\text{in}}(u)} q_a - \sum_{a\in\delta^{\text{out}}(u)} q_a = d_u, \quad u \in V, \tag{10a}$$

$$\pi_u - \pi_v = \Phi_a(q_a), \quad a = (u,v,l) \in A_{\text{ex}}^{\text{arc}}, \tag{10b}$$

$$\pi_u - \pi_v \le \Phi_a(q_a) + (1-x_a)M_a^+, \quad a = (u,v,l) \in A_{\text{ca}}^{\text{arc}}, \tag{10c}$$

$$\pi_u - \pi_v \ge \Phi_a(q_a) + (1-x_a)M_a^-, \quad a = (u,v,l) \in A_{\text{ca}}^{\text{arc}}, \tag{10d}$$

$$\pi_u - \pi_v = \begin{cases} -\Delta_a, & a = (u,v,l) \in A_{\text{ex}}^{\text{cm}}, \\ \Delta_a, & a = (u,v,l) \in A_{\text{ex}}^{\text{cv}}, \end{cases} \tag{10e}$$

$$\pi_u - \pi_v \le \begin{cases} -\Delta_a + (1-x_a)M_a^+, & a = (u,v,l) \in A_{\text{ca}}^{\text{cm}}, \\ \Delta_a + (1-x_a)M_a^+, & a = (u,v,l) \in A_{\text{ca}}^{\text{cv}}, \end{cases} \tag{10f}$$

$$\pi_u - \pi_v \ge \begin{cases} -\Delta_a + (1-x_a)M_a^-, & a = (u,v,l) \in A_{\text{ca}}^{\text{cm}}, \\ \Delta_a + (1-x_a)M_a^-, & a = (u,v,l) \in A_{\text{ca}}^{\text{cv}}, \end{cases} \tag{10g}$$

$$0 \le \Delta_a \le \Delta_a^+ \chi_a(q_a), \quad a \in A^{\text{act}}, \tag{10h}$$

$$q_a^- \le q_a \le q_a^+, \quad a \in A_{\text{ex}}, \tag{10i}$$

$$q_a^- x_a \le q_a \le q_a^+ x_a, \quad a \in A_{\text{ca}}, \tag{10j}$$

$$\pi_u^- \le \pi_u \le \pi_u^+, \quad u \in V. \tag{10k}$$

The expansion variables $x \in X \subseteq \{0,1\}^{A_{\text{ca}}}$ are so-called here-and-now decisions, whereas $(q,\pi,\Delta) \in Y(x,d)$ are wait-and-see decisions that adapt to the realized load scenario $d \in U$. The set X can contain additional restrictions on the expansion decisions.

For a given network design $x \in X$ and load $d \in U$, Constraints (10) model transport within the resulting potential-based network. If a candidate element $a \in A_{\text{ca}}$ is built, i.e., $x_a = 1$, or if an element already exists, then the corresponding constraints coincide with those of a feasible potential-based flow (1)–(6). If a candidate element $a \in A_{\text{ca}}$ is not built, i.e., $x_a = 0$, then Constraint (10j) enforces $q_a = 0$, while the corresponding potential-coupling constraints are made redundant by the big-M terms; see (10c)–(10g). In line with [32], we use $M_a^+ := \pi_u^+ - \pi_v^-$ and $M_a^- := \pi_u^- - \pi_v^+$ for $a = (u,v,l) \in A_{\text{ca}}$. For an unbuilt active candidate element, $q_a = 0$ further implies $\Delta_a = 0$ by (10h). Together with the potential bounds (10k), this makes the corresponding candidate inequalities (10f) and (10g) redundant.

A finite objective value of (8) implies that every load $d \in U$ can be transported through the built network. Conversely, if for a design $x \in X$ there exists a load scenario $d \in U$ with $Y(x,d) = \emptyset$, then the inner minimization problem evaluates to ∞, and the design is robust infeasible.

We use inf and sup in (8) because discrete wait-and-see decisions may prevent attainment in general; see, e.g., [13, 18]. However, we later show that, under Assumption 1, the infimum and supremum are attained.

From the perspective of robust optimization, Problem (8) belongs to the challenging class of adjustable robust mixed-integer nonlinear optimization problems, for which applicable methods are scarce. In particular, the wait-and-see decisions have to satisfy nonlinear potential-flow constraints and include adjustable discrete decisions induced by the operation of the active elements (10h).

3. Adversarial Approach

We follow the well-known adversarial approach in robust optimization [36] to solve Problem (8). The key idea is to replace the uncertainty set U by a finite scenario set $S \subseteq U$ that is iteratively enlarged by worst-case scenarios. For a fixed scenario set S, the corresponding scenario-expanded problem is a finite-dimensional MINLP whose solution yields a candidate expansion decision x.

The candidate x is then checked against the full uncertainty set U. If there exists a scenario $d \in U$ with $Y(x, d) = \emptyset$, then x is robust infeasible, and d is added to S. The restricted design problem is then resolved with the enlarged scenario set. This continues until no violating load scenario can be found, i.e., until x is robust feasible, or infeasibility of the original robust problem is verified.

In addition to robust feasibility, we have to verify robust optimality, particularly with respect to the wait-and-see operating cost. Once a robust feasible candidate x is found, we compute its true worst-case operating cost over the uncertainty set U. If the master problem already matches this worst-case operating cost, then x is robust optimal. Otherwise, we add an objective cut that enforces this worst-case cost whenever the same expansion decision is selected again. The algorithm terminates once the current candidate is robust optimal, or once robust infeasibility is certified.

For passive networks, the solution framework in [35] also follows an adversarial approach. Compared with that setting, Problem (8) includes controllable elements and worst-case operating cost. This prevents a direct transfer of the techniques developed in [35] as discussed in Remark 3 therein. To apply the sketched adversarial approach, we have to address the following challenges:

i. How can we verify whether a given expansion decision x is robust feasible?
ii. How can we verify whether a given expansion decision x is robust optimal?
iii. Does the adversarial approach terminate after finitely many iterations?

3.1. **Robust Feasibility.** We now characterize robust feasibility of a fixed expansion decision $x \in X$, thereby addressing Challenge i. In particular, we show that robust feasibility can be checked by solving polynomially many optimization problems.

The characterization extends the pipe-only result of [35] to networks with controllable elements. In contrast to the passive case, this extension requires solving mixed-integer nonlinear bilevel problems. We show, however, that these bilevel problems can be reformulated under the structural Assumption 1 as equivalent single-level problems by exploiting structural properties of potential-based flows and corresponding reformulation techniques of [25].

We now introduce three classes of optimization problems that characterize robust feasibility of a given network design $x \in X$. These problems allow us to verify whether, for every load $d \in U$, there exists a feasible transport $(q, \pi, \Delta) \in Y(x, d)$ through the built network. If this is not the case, at least one of these optimization problems produces a load $d \in U$ that cannot be transported through the network, i.e., $Y(x, d) = \emptyset$.

In the following adversarial problems, we consider the graph $G(x) := (V, A(x))$, where $A(x) := A_{\text{ex}} \cup \{a \in A_{\text{ca}} : x_a = 1\}$. This graph is the built network obtained by removing all candidate arcs $a \in A_{\text{ca}}$ with $x_a = 0$. We first focus on the case where $G(x)$ is weakly connected, since the characterization extends directly to multiple connected components. Whenever no ambiguity arises, we write G instead of $G(x)$ to simplify notation.

The first adversarial problem checks whether there exists a load that forces a violation of the potential bounds. This can be modeled as a max-min bilevel problem: the leader selects a load scenario $d \in U$, while the follower routes this load with minimum violation of the potential bounds. Violations of the lower and upper potential bounds are represented by auxiliary variables $y, z \in \mathbb{R}$. For fixed network design $x \in X$, this adversarial bilevel problem can be formulated as

$$
\begin{aligned}
\varphi_G(x) := \sup_{d\in U} \inf_{q,\pi,\Delta,y,z} \quad & y+z \\
\text{s.t.} \quad & \text{(10a)–(10h)}, \\
& \pi_u + y \geq \pi_u^-, \qquad u \in V, \\
& \pi_u - z \leq \pi_u^+, \qquad u \in V.
\end{aligned} \tag{11}
$$

If the objective value is positive, i.e., $\varphi_G(x) > 0$, then every feasible point with a positive objective value provides a load $d \in U$ that cannot be transported without violating the potential bounds.

Problem (11) was studied in [25] in the context of deciding feasibility of bookings in gas networks. In that setting, there are no network design decisions, and the uncertainty set is a specific interval uncertainty set. Moreover, we use sup in (11) because discrete wait-and-see decisions may prevent attainment in general; see, e.g., [13, 18]. However, we later show that, under Assumption 1, the supremum and infimum are attained.

We note that Problem (11) does not contain any arc flow bounds because they may cut off potential-bound violating scenarios. Flow bound violation is addressed by the next class of adversarial problems. To this end, we again use bilevel formulations to identify worst-case loads with respect to arc flow bounds, treating lower and upper bounds separately. For a fixed arc a, the leader selects a load $d \in U$ to minimize or maximize its flow, respectively, while the follower chooses (q, π, Δ) with the opposite objective.

$$
\underline{q}_a(x) := \inf_{d\in U} \sup_{q,\pi,\Delta} \; q_a \quad \text{s.t.} \quad \text{(10a)–(10h)}, \tag{12}
$$

and

$$
\bar{q}_a(x) := \sup_{d\in U} \inf_{q,\pi,\Delta} \; q_a \quad \text{s.t.} \quad \text{(10a)–(10h)}. \tag{13}
$$

If the optimal objective value violates the corresponding flow bound, i.e., $\underline{q}_a(x) < q_a^-$ or $\bar{q}_a(x) > q_a^+$, then any bilevel feasible point violating an arc flow bound provides a load that cannot be transported through the built network.

If $G(x)$ has multiple connected components, then Problems (11)–(13) can be solved componentwise. In addition, each component must be balanced under every load scenario. For a connected component $G^i = (V^i, A^i)$ of $G(x)$, we follow the pipe-only case of [35] and define the maximum load imbalance by

$$
\mu_{G^i}(x) \; := \; \max_{d\in U} \; |y| \quad \text{s.t.} \quad y = \sum_{u\in V^i\cap V_+} d_u \; + \sum_{u\in V^i\cap V_-} d_u. \tag{14}
$$

The objective value is positive, i.e., $\mu_{G^i}(x) > 0$, if and only if there exists a load scenario $d \in U$ for which component G^i has nonzero net load. Such a load cannot be transported through the network.

We now show that the three classes of adversarial problems together yield an equivalent characterization of robust feasibility for a fixed network design. As discussed above, the resulting adversarial problems include challenging bilevel problems. In particular, their lower-level problems contain both discrete decisions for controllable elements and nonconvex potential-flow constraints, leading to mixed-integer nonlinear bilevel formulations. Thus, to obtain algorithmically tractable single-level reformulations of these highly challenging bilevel problems and use them to characterize robust feasibility of a given network design, we impose the following structural assumption.

Assumption 1. *For every expansion decision $x \in X$, no active element is part of an undirected cycle of the expanded graph $G(x)$.*

We note that this assumption is also used in related work on gas networks; see, for example, [1, 2, 25]. In particular, it is shown in [25] that under Assumption 1, the flows and potential differences in $G(x)$ are uniquely determined for a given load scenario $d \in U$. In the present network design setting, this result can be stated as follows.

Theorem 1 ([25, Theorem 4.2])**.** *Suppose Assumption 1 holds and let a network design $x \in X$ be fixed. Then, for any load scenario $d \in U$, every feasible point (q, π) satisfying (10a)–(10d) admits the same unique flows q_a for all $a \in A(x)$ and the same unique potential differences $\pi_u - \pi_v$ for all $(u, v, l) \in A^{\text{arc}}(x)$.*

Although the proof in [25] is stated for gas networks, it extends directly to general potential-based networks, since the arguments rely on general graph properties and on uniqueness results of passive potential-based flows; see, e.g., Theorem 7.1 of [17].

Exploiting this additional structure of potential-based networks (Theorem 1), we now prove that the adversarial problems (11)–(14) provide an equivalent characterization of robust feasibility of a given network design x.

Theorem 2. *Suppose Assumption 1 holds and let a network design $x \in X$ be given. Moreover, let $G(x) = (V, A(x))$ denote the expanded graph and $\mathcal{G}'(x) = \{G^1, \dots, G^n\}$ its connected components.*

Then, the network design x is robust feasible, i.e., for every load $d \in U$, Constraints (10) are feasible, if and only if for every component $G^i = (V^i, A^i) \in \mathcal{G}'(x)$, the following conditions hold:

$$\mu_{G^i}(x) = 0, \tag{15a}$$

$$\varphi_{G^i}(x) \le 0, \tag{15b}$$

$$\underline{q}_a(x) \ge q_a^-, \qquad \text{for all } a \in A^i, \tag{15c}$$

$$\bar{q}_a(x) \le q_a^+, \qquad \text{for all } a \in A^i. \tag{15d}$$

Proof. Let $x \in X$ be a robust feasible network design, i.e., for all load scenarios $d \in U$, the Constraints (10) are feasible. We now show that Conditions (15) hold. Robust feasibility of the network directly implies $\mu_{G^i}(x) = 0$ for all $i \in \{1, \dots, n\}$ by the flow conservation constraints (10a).

We next consider a fixed component G^i. Robust feasibility implies that, for every load scenario $d \in U$, there exists a point (q, π, Δ) satisfying Constraints (10) w.r.t. G^i. Hence, there exist nonpositive slack variables $y, z \le 0$ such that (q, π, Δ, y, z) is feasible for the lower-level problem of (11). Moreover, the flows q are uniquely determined due to Theorem 1, and, thus, for fixed d, the lower-level problem attains an optimal solution with nonpositive objective value. Since this holds for every load $d \in U$, it also holds after taking the supremum over U, and the optimal objective value of Problem (11) is at most zero, i.e., $\varphi_{G^i}(x) \le 0$ holds.

The flow-bound conditions (15c) and (15d) follow analogously. For every load $d \in U$, robust feasibility provides a feasible potential-based flow (q, π, Δ) satisfying Constraints (10), including the flow bounds. Hence, this point is lower-level feasible for Problems (12) and (13), and its arc flow lies within the corresponding flow bounds. Therefore, the optimal values of the bilevel problems satisfy the bounds in Conditions (15c) and (15d).

We now prove the reverse implication and assume that Conditions (15) hold. For an arbitrary load $d \in U$, we now prove that there exists a feasible potential-based flow (q, π, Δ) satisfying Constraints (10).

Condition (15a) implies that d is balanced on every connected component G^i. Thus, the following arguments can be applied to each connected component.

For the fixed load d, there exist (q, π') satisfying Constraints (10a)–(10d), which follows from Theorem 7.1 of [17]. These flows q are unique due to Theorem 1. Thus, for fixed load d, there is an optimal lower-level solution (q, π, Δ, y, z) of Problem (11) restricted to G^i. It remains to show that the corresponding flow and potential bounds are satisfied.

We first verify the flow bounds. Since the arc flows q are unique, for every arc $a \in A^i$, the point (d, q, π, Δ) is bilevel feasible for Problems (12) and (13) with objective value q_a. Thus, $\underline{q}_a(x) \le q_a \le \bar{q}_a(x)$ holds. Together with Conditions (15c) and (15d), this yields

$$q_a^- \le \underline{q}_a(x) \le q_a \le \bar{q}_a(x) \le q_a^+.$$

Hence, the flow bounds (10i) and (10j) are satisfied within G^i.

We now verify the potential bounds. By Condition (15b), the optimal value of Problem (11) is nonpositive, i.e., $y + z \le 0$. If $y \le 0$ and $z \le 0$, then the potentials π directly satisfy the lower and upper potential bounds (10k). Thus, (d, q, π, Δ) satisfies Constraints (10).

If $y > 0$, then at least one lower potential bound is violated. We now shift the potentials $\tilde{\pi}_u := \pi_u + y$ for all $u \in V^i$ and set $\tilde{y} := 0$ as well as $\tilde{z} := y + z \le 0$. Thus, $\tilde{y} + \tilde{z} \le 0$. The potential shift does not change any potential difference nor the arc flows q or controls Δ. Thus, the corresponding flow and potential constraints remain valid. Moreover, the shifted potentials $\tilde{\pi}$ now satisfy the potential bounds (10k). Consequently, $(d, q, \tilde{\pi}, \Delta, \tilde{y}, \tilde{z})$ satisfies the Constraints (10). We note that the case $z > 0$ can be treated analogously.

Thus, for the arbitrary load $d \in U$, every component G^i admits feasible flows, potentials, and controls satisfying the potential and flow bounds. Combining the componentwise solutions and setting the flows on non-built candidate arcs to zero yields a point (q, π, Δ) satisfying Constraints (10). Since $d \in U$ was arbitrary, this shows robust feasibility of the considered network design. □

Theorem 2 enables us to verify robust feasibility of a given network design $x \in X$ by solving at most $2|\mathcal{G}'(x)| + 2|A(x)|$ optimization problems. If one of the Conditions (15) is violated, the corresponding adversarial problem yields a load scenario that certifies robust infeasibility. Thus, Theorem 2 resolves Challenge i.

However, Problems (11)–(13) are challenging mixed-integer nonlinear bilevel problems, for which standard bilevel optimization methods are not directly applicable. Even for networks without controllable elements, checking robust feasibility of a fixed network design under interval-based uncertainty is coNP-hard, which follows from [34].

Exploiting structural properties of potential-based flows, we now provide equivalent single-level reformulations of the presented bilevel problems.

Theorem 3. *Suppose that Assumption 1 holds and let $x \in X$ be a given network design. Then, for $a \in A(x)$, Problem (12) admits the same optimal value as the single-level problem*

$$\underline{q}_a(x) := \min_{d,q,\pi,\Delta} \; q_a \quad \text{s.t.} \quad (10a)–(10h), \; d \in U. \tag{16}$$

Proof. Let an arc $a \in A(x)$ be given. For a fixed load $d \in U$, Theorem 1 implies that all feasible lower-level points of Problem (12) admit the same flow value on arc a.

Thus, the inner maximization in Problem (12) returns this unique flow value. Consequently, Problem (12) is equivalent to optimizing q_a directly over all tuples (d, q, π, Δ) satisfying (10a)–(10h) and $d \in U$.

It remains to show that the resulting infimum is attained. By Definition 7, the uncertainty set U is compact. Moreover, passive potential-based flows are acyclic, see [12], which also holds in the present setting under Assumption 1. Hence, together with compactness of U, acyclicity yields finite bounds on the admissible arc flows in Problem (16). Moreover, the controls Δ do not actively alter the arc flows under Assumption 1 and can therefore be set to zero. Furthermore, by Theorem 1, for a given load, the potentials are unique up to constant shift. Thus, we can fix one reference potential in each connected component. Together with the previously derived bounds on the admissible flows, the strict monotonicity of the potential functions Φ_a, $a \in A$, then also yields bounds on the feasible potentials in Problem (16), without changing the objective value. Thus, Problem (16) can be equivalently considered over a bounded feasible set. Since the defining constraints are continuous, this feasible set is closed and therefore compact. Moreover, it is nonempty due to Theorem 7.1 of [17]. The objective function is continuous, and hence the minimum is attained by the Weierstrass theorem. □

An analogous argument shows that the adversarial problem (13) admits the same optimal value as the single-level optimization problem

$$\bar{q}_a(x) := \max_{d,q,\pi,\Delta} \; q_a \quad \text{s.t.} \quad (10a)-(10h), \; d \in U. \tag{17}$$

Thus, the adversarial flow-bound problems, although originally formulated as mixed-integer nonlinear bilevel problems, can equivalently be solved as single-level MINLPs under Assumption 1.

It remains to derive an algorithmically tractable reformulation of the potential-bound adversarial problem (11).

As noted above, Problem (11) was studied by [25] for a specific interval uncertainty set in a gas-specific setting without network design decisions. Under Assumption 1, they derive an equivalent single-level reformulation. Since the proof relies only on compactness of the uncertainty set and on the structural properties of the potential functions imposed here, it carries over directly to the present setting with arbitrary compact uncertainty sets and general potential functions. Consequently, their result yields a single-level reformulation of Problem (11). We now present this reformulation in our setting. To this end, we will use the notation of a reduced network taken from [25, 27].

Removing all active elements from the expanded graph $G(x)$ decomposes the expanded graph into passive connected components $\mathcal{G} := \{G_0, G_1, \ldots, G_l\}$.[1] For a component $G_j \in \mathcal{G}$, let $V(G_j)$ denote its node set. If an active element a connects a node in G_i with a node in G_j, we write $a = (G_i, G_j)$. By Assumption 1, every active element is a bridge between two passive components. Hence, contracting every passive component G_j to a single node yields a reduced graph $\tilde{G}(x) = (\mathcal{G}, A^{\text{act}}(x))$ that is a forest, and it is a tree whenever $G(x)$ is connected.

We now state an equivalent bilevel reformulation of Problem (11) based on [25, Theorem 4.3]. To this end, we define $M := \max_{d \in U} \sum_{u \in V_-} d_u$, which is a finite bound on the absolute arc flow through an arc.

[1] Although the passive connected components depend on x, that is, $\mathcal{G} = \mathcal{G}(x)$ and $G_j = G_j(x)$ for all $j \in \{0, \ldots, l\}$, we omit this dependence for notational simplicity.

Theorem 4. *Suppose Assumption 1 holds and let $x \in X$ be a given network design. Then, Problem (11) admits the same optimal value as the bilevel problem*

$$\varphi_G(x) = \max_{d,q,\pi,s} \; y + z \tag{18a}$$
$$\text{s.t.} \quad (10a)–(10d), \tag{18b}$$
$$d \in U, \tag{18c}$$
$$\pi_u = \pi_v, \quad (u,v) \in A^{\text{act}}(x), \tag{18d}$$
$$q_a \le m_a(1 - s_a) + M s_a, \quad a \in A^{\text{act}}(x), \tag{18e}$$
$$s_a \in \{0,1\}, \quad a \in A^{\text{act}}(x), \tag{18f}$$
$$(\Delta, \tau, y, z) \in \mathcal{S}(d, q, \pi, s). \tag{18g}$$

Here, $\mathcal{S}(d,q,\pi,s)$ is the set of optimal solutions of the (d,q,π,s)-parameterized lower-level problem

$$\min_{\Delta,\tau,y,z} \; y + z \tag{19a}$$
$$\text{s.t.} \quad \tau_i - \tau_j = \begin{cases} -\Delta_a, & a = (G_i, G_j) \in A^{\text{cm}}(x), \\ \;\;\,\Delta_a, & a = (G_i, G_j) \in A^{\text{cv}}(x), \end{cases} \tag{19b}$$
$$0 \le \Delta_a \le \Delta_a^+ s_a, \quad a \in A^{\text{act}}(x), \tag{19c}$$
$$\tau_j + y \ge \pi_u^- - \pi_u, \quad u \in V(G_j), \; G_j \in \mathcal{G}, \tag{19d}$$
$$\tau_j - z \le \pi_u^+ - \pi_u, \quad u \in V(G_j), \; G_j \in \mathcal{G}. \tag{19e}$$

In the bilevel problem (18), the leader chooses a load $d \in U$ and the corresponding flow and potential variables q and π of the passive network to maximize the potential-bound violation. Thus, the leader computes the flow and potential decisions that appear as lower-level decisions in the adversarial problem (11). This is valid because, by Theorem 1, the flows are uniquely determined and the potentials are unique up to an additive constant. Moreover, the binary variables s encode the operating status of the controllable elements: a controllable element a can operate only if $q_a > m_a$ and is inactive otherwise. The follower then chooses the settings of the controllable elements Δ and may shift the potentials in every connected component G_j by τ_j in order to minimize the resulting potential-bound violation.

The reformulation moves the most challenging nonlinear constraints defining the potentials and flows, as well as the discrete activation decisions of the controllable elements, from the lower-level to the upper level. The resulting bilevel problem therefore has a linear lower-level problem. Consequently, Problem (18) can be reformulated as a single-level problem using the KKT conditions of the linear lower-level problem. The explicit reformulation is provided in Appendix A.

Overall, the presented reformulations, together with Theorem 2, enable us to verify robust feasibility of a given network design by solving polynomially many single-level optimization problems.

3.2. **Robust Optimality.** We now turn to the verification of robust optimality for a given robust feasible network design; see Challenge ii. This step is necessary because the objective function of the robust network design problem (8) explicitly depends on wait-and-see decisions, namely the operation of the controllable elements. This is in contrast to the pipe-only case [35], where the objective function depends only on here-and-now decisions. Hence, evaluating the objective value of a given robust network design with controllable elements requires computing its worst-case operating cost.

For a given robust feasible network design $x \in X$, we model the computation of the worst-case operating cost as the following bilevel problem

$$\psi(x) := \sup_{d \in U} \inf_{q,\pi,\Delta} \quad w^\top \Delta \qquad \text{s.t.} \quad (10a)-(10h), (10k). \tag{20}$$

The leader selects a load scenario $d \in U$ that maximizes the operating cost, while the follower reacts by choosing a cost-minimal feasible operation of the controllable elements. The flow bounds need not be included explicitly. Indeed, for each chosen load d, the flows are uniquely determined under Assumption 1. Since x is robust feasible, these unique flows satisfy the flow bounds. The potential bounds remain part of the lower-level problem, because the cost-minimal operation has to be feasible with respect to these bounds.

Analogously to the potential-bound adversarial problem (11), we can fix all quantities that are uniquely determined by the physics in the upper level. The lower-level then only chooses componentwise potential shifts and controls of the active elements. This yields the following result.

Theorem 5. *Let $x \in X$ be a given robust feasible network design. Under Assumption 1, Problem (20) has the same optimal value as the bilevel problem*

$$\psi(x) = \max_{d,q,\pi,s} \; w^\top \Delta \tag{21a}$$

$$\text{s.t.} \quad (18b)-(18f), \tag{21b}$$

$$(\Delta, \tau) \in \mathcal{R}(d, q, \pi, s). \tag{21c}$$

Here, $\mathcal{R}(d, q, \pi, s)$ is the set of optimal solutions of the (d, q, π, s)-parameterized lower-level problem

$$\min_{\Delta,\tau} \; w^\top \Delta \tag{22a}$$

$$\text{s.t.} \; \tau_i - \tau_j = \begin{cases} -\Delta_a, & a = (G_i, G_j) \in A^{\mathrm{cm}}(x), \\ \Delta_a, & a = (G_i, G_j) \in A^{\mathrm{cv}}(x), \end{cases} \tag{22b}$$

$$\tau_j \geq \pi_u^- - \pi_u, \quad u \in V(G_j), \; G_j \in \mathcal{G}, \tag{22c}$$

$$\tau_j \leq \pi_u^+ - \pi_u, \quad u \in V(G_j), \; G_j \in \mathcal{G}, \tag{22d}$$

$$\Delta_a \leq \Delta_a^+ s_a, \quad a \in A^{\mathrm{act}}(x), \tag{22e}$$

$$\Delta_a \geq 0, \quad a \in A^{\mathrm{act}}(x). \tag{22f}$$

Proof. The proof follows the one of Theorem 4.3 in [25]. The only difference is that, due to robust feasibility of the considered network design x, the slack variables y and z used in Theorem 4.3 of [25] are not needed. Instead, feasibility with respect to the potential bounds is imposed directly, and the lower-level objective is the operating cost. The remaining arguments are analogous to those in [25], which rely on general uniqueness properties of potential-based flows stated in Theorem 1. □

The bilevel problem (21) has a mixed-integer nonlinear upper level and a linear lower-level problem (22). Consequently, it admits a single-level reformulation using the KKT conditions of the linear lower-level problem. The resulting reformulation differs from the potential-bound adversarial reformulation of Problem (18), because the lower-level objective now minimizes the operating cost $w^\top \Delta$. We derive the corresponding KKT reformulation in Section 4, including valid big-M values for the dual variables of (22).

3.3. **Algorithm.** We now embed the developed characterization of robust feasibility and the verification of robust optimality into an adversarial algorithm for solving the adjustable robust network design problem (8) to global optimality. The resulting algorithm extends the adversarial framework of [35] from the pipe-only setting to networks with controllable elements and worst-case operating cost.

The proposed adversarial algorithm iteratively solves a master problem over a finite scenario set $S \subseteq U$. Its solution yields a candidate network design x, whose robust feasibility is verified by solving the finitely many optimization problems given by the robust characterization in Theorem 2. If a violating load scenario $d \in U$ is found, it is added to S, and the master problem is resolved. If no violating scenario exists, we compute the worst-case operating cost $\psi(x)$ of the robust feasible candidate network over the full uncertainty set by solving Problem (20). If the current master objective already accounts for $\psi(x)$, then x is robust optimal. Otherwise, we add an objective cut for x that incorporates its worst-case operating cost and resolve the master problem. The resulting adversarial procedure is summarized in Algorithm 1.

Before we prove correctness of Algorithm 1, we first discuss some technical details regarding the master problem. As introduced, a controllable element $a \in A^{\text{act}}$ can only be active if the corresponding arc flow exceeds a prescribed threshold value, i.e., $q_a > m_a$ holds. On the one hand, this modeling choice enables the bilevel reformulations in Theorems 4 and 5. On the other hand, the strict inequality $q_a > m_a$ cannot be represented directly in the single-level master problem.

To address this issue while still computing an optimal robust network design, we use the following modification. In the master problem, we model the activation of controllable elements by the relaxation $q_a \geq m_a$. This relaxation may have two effects.

First, for a fixed network design x, the relaxation may underestimate the operating cost, because it enlarges the feasible region of the master problem. We correct this underestimation by adding an x-parameterized objective cut that enforces the true worst-case operating cost for x; see Constraints (23c). Thus, these cuts correct the underestimation for robust feasible designs that have already been evaluated. Second, the relaxation may allow a robust infeasible network design x to remain feasible in the master problem, even after a violating load scenario has been added. To prevent this design from being selected again, we add a corresponding no-good cut whenever a network design has been verified as robust infeasible; see Constraints (23d).

These technical adaptations of the master problem enable us to correctly represent the operating range of the active elements and ensure that Algorithm 1 computes an optimal solution of the adjustable robust network design problem (8).

With the modifications outlined above, the master problem can be formulated as

$$\min_{x \in X, \eta} \quad \eta \tag{23a}$$

$$\text{s.t.} \quad \eta \geq c^\top x + w^\top \Delta^d, \quad d \in S, \tag{23b}$$

$$\eta \geq c^\top x + \psi(x') - \psi(x') \left(\sum_{i : x'_i = 1} (1 - x_i) + \sum_{i : x'_i = 0} x_i \right), \quad x' \in X', \tag{23c}$$

$$\sum_{i : x'_i = 1} (1 - x_i) + \sum_{i : x'_i = 0} x_i \geq 1, \quad x' \in X'' \tag{23d}$$

$$(q^d, \pi^d, \Delta^d, s^d) \text{ satisfies Constraints (10)}, \quad d \in S, \tag{23e}$$

where we explicitly model Constraints (10h) by using additional binary variables s and the discussed relaxation, i.e.,

$$0 \leq \Delta_a^d \leq \Delta_a^+ s_a^d, \quad q_a^d \geq m_a s_a^d + q_a^-(1 - s_a^d), \quad s_a^d \in \{0, 1\}, \quad a \in A^{\text{act}}. \tag{24}$$

The latter constraints imply that a controllable element can operate only if the corresponding flow reaches the threshold m_a. Otherwise, it is inactive and $s_a^d = 0 = \Delta_a^d$ holds.

The set X' contains all network designs for which the worst-case operating cost has already been computed in previous iterations by solving Problem (21). Analogously, the set X'' contains all network designs that have been previously identified as robust infeasible.

If several violating scenarios are found for a robust infeasible network design, we add the scenario with the largest violation of the corresponding bound.

We now prove that Algorithm 1 terminates and either computes a globally optimal solution to the adjustable robust network design problem (8) or verifies its infeasibility, thereby resolving Challenge iii.

Algorithm 1: Adversarial approach for solving the network design problem (8).

1 **Input:** Graph $G = (V, A_{\text{ex}} \cup A_{\text{ca}})$, uncertainty set U satisfying (7), construction cost c, and operating cost w.
2 **Output:** A globally optimal solution $x \in X$ to the adjustable robust network design problem (8) or verification of its infeasibility.
3 Determine a finite set of scenarios $S \subseteq U$ with $S \neq \emptyset$. Initialize $X' = \emptyset$, $X'' = \emptyset$.
4 Solve Problem (23) w.r.t. S, X', and X'' to obtain $(x, \eta, q, \pi, \Delta, s)$.
5 **if** *Problem* (23) *is infeasible* **then**
6   **return** infeasibility of Problem (8).
7 Determine the set $\mathcal{G}'(x)$ of connected components of the expanded graph $G(x) = (V, A(x))$.
8 **for** $G^i \in \mathcal{G}'(x)$ **do**
9   Solve Problem (14) to obtain d' with objective value $\mu_{G^i}(x)$.
10   **if** $\mu_{G^i}(x) > 0$ **then**
11   Set $S \leftarrow S \cup \{d'\}$ and $X'' \leftarrow X'' \cup \{x\}$. Then, go to Line 4.
12 Set $q^{\max} = 0$.
13 **for** $G^i \in \mathcal{G}'(x)$ **do**
14   **for** $a \in A^i$ **do**
15   Solve Problem (16) w.r.t. G^i to get $(d', q', \pi', \Delta', s')$ with value $\underline{q}_a(x)$.
16   **if** $\underline{q}_a(x) < q_a^-$ *and* $q_a^- - \underline{q}_a(x) > q^{\max}$ **then**
17   Set $q^{\max} \leftarrow q_a^- - \underline{q}_a(x)$ and $d^{\max} \leftarrow d'$.
18   Solve Problem (17) w.r.t. G^i to get $(d', q', \pi', \Delta', s')$ with value $\bar{q}_a(x)$.
19   **if** $\bar{q}_a(x) > q_a^+$ *and* $\bar{q}_a(x) - q_a^+ > q^{\max}$ **then**
20   Set $q^{\max} \leftarrow \bar{q}_a(x) - q_a^+$ and $d^{\max} \leftarrow d'$.
21 **if** $q^{\max} > 0$ **then**
22   Set $S \leftarrow S \cup \{d^{\max}\}$ and $X'' \leftarrow X'' \cup \{x\}$. Then, go to Line 4.
23 Set $\varphi^{\max} = 0$.
24 **for** $G^i \in \mathcal{G}'(x)$ **do**
25   Solve Problem (18) to obtain $(d', q', \pi', \Delta', s')$ with objective value $\varphi_{G^i}(x)$.
26   **if** $\varphi_{G^i}(x) > 0$ *and* $\varphi_{G^i}(x) > \varphi^{\max}$ **then**
27   Set $\varphi^{\max} \leftarrow \varphi_{G^i}(x)$ and $d^{\max} \leftarrow d'$.
28 **if** $\varphi^{\max} > 0$ **then**
29   Set $S \leftarrow S \cup \{d^{\max}\}$ and $X'' \leftarrow X'' \cup \{x\}$. Then, go to Line 4.
30 Solve Problem (21) to obtain the objective value $\psi(x)$.
31 **if** $\eta < c^\top x + \psi(x)$ **then**
32   Set $X' \leftarrow X' \cup \{x\}$ and go to Line 4.
33 **return** the optimal adjustable robust network design $x \in X$.

Theorem 6. *Suppose that Assumption 1 holds. Then, Algorithm 1 terminates after finitely many iterations and either returns an optimal solution of Problem* (8) *or verifies its infeasibility.*

Proof. We first show that the algorithm terminates after finitely many iterations. Since $X \subseteq \{0,1\}^{A_{\text{ca}}}$ is finite and robust infeasible network designs are iteratively excluded either by a violating scenario or a no-good cut (23d), cycling can only occur among robust feasible network designs. However, if a previously evaluated robust feasible network design is again optimal for the master problem in a later iteration, the algorithm terminates. Indeed, the previously added objective cut (23c) enforces the worst-case operating cost over the full uncertainty set for this design. Hence, the condition in Line 31 is not satisfied and the algorithm terminates.

Consequently, the algorithm either returns infeasibility or a network design $x \in X$. If the master problem (23) is infeasible, then the adjustable robust network design problem (8) is infeasible as well, since all cuts added to the master problem are valid for Problem (8). The objective cuts (23c) only enforce the true worst-case operating cost of previously evaluated robust feasible designs. The no-good cuts (23d) only exclude robust infeasible network designs.

Thus, we consider the case that a network design $x \in X$ is returned. By Theorem 2, this design is robust feasible. Moreover, the condition in Line 31 ensures that the operating cost for x represented in the master problem coincides with the worst-case operating cost over the full uncertainty set. It remains to note that the master problem (23) is a relaxation of Problem (8): it only contains finitely many scenarios and valid cuts. Therefore, its optimal value is a lower bound on the optimal value of Problem (8). Since the returned robust feasible design $x \in X$ attains this lower bound, it is globally optimal. □

4. KKT-Based Single-Level Reformulation and Big-M Linearization

The reformulated adversarial bilevel problems (18) and (21) both have a linear lower-level problem. Hence, a single-level reformulation can be obtained by replacing the lower-level problem with its necessary and sufficient KKT conditions. However, this introduces additional nonlinearities through complementarity constraints.

These nonlinear complementarity constraints can be modeled by so-called SOS1 constraints or by big-M linearizations using valid bounds on the corresponding dual variables and primal constraint functions; see, e.g., [9]. In practice, big-M linearizations are typically computationally more efficient, as we also observe in our computational study, but they require computationally effective big-M values.

For the adversarial problem (18), we can directly use the KKT-based single-level reformulation of [25], including the valid big-M values proven therein. However, for the adversarial problem (21), which computes the worst-case operating cost, these big-M values do not carry over directly, since the objective function differs and the slack variables of (18) are removed as well. Thus, we now present a KKT-based single-level reformulation of Problem (21) in Section 4.1 and then prove valid bounds on the dual variables and primal constraints in Section 4.2.

4.1. **KKT-Based Single-Level Reformulation.** For the remainder of this section, we consider a weakly connected component of a robust feasible network design x. Hence, for every fixed upper-level decision (d, q, π, s), the linear lower-level problem of (21) is feasible and has a finite optimal value. Thus, it has an optimal solution. We now present the dual problem associated with (22), using the dual variables α for Constraints (22b), δ^- and δ^+ for Constraints (22c)–(22d), and β for the upper bounds (22e).

$$\max_{\alpha,\beta,\delta^+,\delta^-} \quad -\sum_{a \in A^{\text{act}}(x)} \Delta_a^+ s_a \beta_a + \sum_{u \in V} \left((\pi_u^- - \pi_u)\delta_u^- - (\pi_u^+ - \pi_u)\delta_u^+ \right) \tag{25a}$$

$$\text{s.t.} \quad \sum_{a \in \delta^{\text{out}}(G_j)} \alpha_a - \sum_{a \in \delta^{\text{in}}(G_j)} \alpha_a = \sum_{u \in V(G_j)} (\delta_u^+ - \delta_u^-), \quad G_j \in \mathcal{G}, \tag{25b}$$

$$\alpha_a \le w_a + \beta_a, \quad a \in A^{\text{cm}}(x), \tag{25c}$$

$$-\alpha_a \le w_a + \beta_a, \quad a \in A^{\text{cv}}(x), \tag{25d}$$
$$\beta_a \ge 0, \quad a \in A^{\text{act}}(x), \tag{25e}$$
$$\delta_u^+ \ge 0,\ \delta_u^- \ge 0, \quad u \in V. \tag{25f}$$

Here, the graphs G_j again denote the connected components obtained after removing the active elements. If each passive component G_j is contracted to a single node, we obtain the reduced graph $\tilde{G}(x) = (\mathcal{G}, A^{\text{act}}(x))$, which is a tree in the considered setting.

Problem (25) can be interpreted as a flow problem on this reduced network $\tilde{G}(x)$ with variable supply and demand. The variables α represent dual flows on active arcs, while arc capacities are given by $w_a + \beta_a$. Constraints (25b) enforce dual flow conservation on $\tilde{G}(x)$ and the right-hand side represents the net load of a passive component $G_j \in \mathcal{G}$.

Combining upper-level feasibility (21b), lower-level primal feasibility (22b)–(22f), lower-level dual feasibility (25b)–(25f), and the complementarity conditions, we obtain the following single-level KKT reformulation of Problem (21).

$$\psi(x) = \max_{\xi}\ w^\top \Delta \tag{26a}$$
$$\text{s.t. (21b)}, \tag{26b}$$
$$\text{(22b)–(22f)}, \quad \text{(25b)–(25f)}, \tag{26c}$$
$$\delta_u^- \left(\tau_j + \pi_u - \pi_u^-\right) = 0, \quad u \in V(G_j),\ G_j \in \mathcal{G}, \tag{26d}$$
$$\delta_u^+ \left(\pi_u^+ - \pi_u - \tau_j\right) = 0, \quad u \in V(G_j),\ G_j \in \mathcal{G}, \tag{26e}$$
$$\beta_a \left(\Delta_a^+ s_a - \Delta_a\right) = 0, \quad a \in A^{\text{act}}(x), \tag{26f}$$
$$(w_a - \alpha_a + \beta_a)\Delta_a = 0, \quad a \in A^{\text{cm}}(x), \tag{26g}$$
$$(w_a + \alpha_a + \beta_a)\Delta_a = 0, \quad a \in A^{\text{cv}}(x), \tag{26h}$$

where $\xi = (d, q, \pi, s, \Delta, \tau, \alpha, \beta, \delta^+, \delta^-)$ is the vector of upper-level, lower-level primal, and lower-level dual variables.

4.2. **Big-M Linearization.** The KKT-based single-level reformulation (26) involves nonlinear complementarity conditions (26d)–(26h). We tackle these challenging constraints by using the well-known big-M linearization; see [9]. The latter replaces a complementarity constraint of the form $\lambda g(x) = 0$, where $\lambda \ge 0$ is a dual variable and $g(x) \ge 0$ is the corresponding primal constraint, by

$$\lambda \le M_\lambda u, \quad g(x) \le M_g(1-u), \quad u \in \{0,1\},$$

where $u \in \{0,1\}$ is an additional binary variable and M_λ and M_g are valid upper bounds on λ and $g(x)$, respectively. Although the existence of valid big-M values is theoretically guaranteed in general, see [6], the resulting values are typically too large for practical computations. It is therefore often necessary and beneficial to derive valid and tight bounds for the dual variables and primal constraint functions in order to solve the corresponding single-level reformulations efficiently. However, even verifying the correctness of given big-M values is NP-hard; see [14].

In the following, we derive valid big-M values for (26) by exploiting structural properties of potential-based flows and the underlying graph.

We start with valid bounds for the dual variables. The dual problem (25) can be interpreted as a flow problem on the reduced network $\tilde{G}(x)$, which is a tree. Thus, once the net load $\sum_{u \in V(G_j)} (\delta_u^+ - \delta_u^-)$ is fixed for every passive component $G_j \in \mathcal{G}$, the dual flow α is uniquely determined by (25b). Moreover, redistributing the dual supply δ^+ and demand δ^- within a passive component does not change the arc flows α.

The next lemma shows that the redistribution can be chosen such that each passive component contains at most one node with positive supply or demand.

Lemma 7. *Let $(d, q, \pi, s, \Delta, \tau)$ be bilevel feasible for Problem (21). Then, there exists an optimal solution $(\alpha, \beta, \delta^+, \delta^-)$ of the lower-level dual problem (25) such that, for every passive component $G_j \in \mathcal{G}$, there is at most one node $u \in V(G_j)$ with $\delta_u^+ > 0$ or $\delta_u^- > 0$ and such that $\delta_w^+ \delta_w^- = 0$ holds for all $w \in V(G_j)$, i.e.,*

$$\left|\left\{u \in V(G_j) : \delta_u^+ > 0 \text{ or } \delta_u^- > 0\right\}\right| \leq 1, \qquad G_j \in \mathcal{G}, \tag{27a}$$

$$\delta_u^+ \delta_u^- = 0, \qquad u \in V(G_j),\ G_j \in \mathcal{G}. \tag{27b}$$

Proof. Let $(\alpha, \beta, \delta^+, \delta^-)$ be an optimal solution of (25). Fix a passive component $G_j \in \mathcal{G}$. The dual flow-balance constraint (25b) depends on δ^+ and δ^- only through the net load $\sum_{u \in V(G_j)} (\delta_u^+ - \delta_u^-)$. Thus, any modification of the supply δ^+ and demand δ^- does not change the dual flow α if the aggregated net load remains unchanged. In particular, such modifications preserve dual feasibility and can affect only the objective function.

We now use this observation to eliminate pairs of positive supply and demand inside G_j. For nodes $v, w \in V(G_j)$ with $\delta_v^+ > 0$ and $\delta_w^- > 0$, we reduce both by $\varepsilon := \min\{\delta_v^+, \delta_w^-\} > 0$. This modification still preserves dual feasibility, but changes the objective function by

$$-\varepsilon \left((\pi_v - \pi_v^+) + (\pi_w^- - \pi_w) \right).$$

However, by Constraints (22c) and (22d), we obtain

$$\pi_w^- - \pi_w \leq \tau_j \leq \pi_v^+ - \pi_v \quad \implies \quad (\pi_v - \pi_v^+) + (\pi_w^- - \pi_w) \leq 0.$$

Thus, the modifications do not decrease the dual objective function value. Because the original point is optimal, the dual objective value cannot increase either. Hence, the modified point is again optimal. Repeating this operation finitely many times yields a dual solution in which positive entries on $V(G_j)$ occur either only in δ^+ or only in δ^- and hence $\delta_u^+ \delta_u^- = 0$ for all $u \in V(G_j)$.

Suppose now that only δ^+ has multiple positive entries w.r.t. $V(G_j)$. Redistributing all positive entries of δ^+ in G_j to a node $u \in V(G_j)$ with $\pi_u - \pi_u^+ = \max_{l \in V(G_j)} \pi_l - \pi_l^+$ preserves the net load and does not decrease the dual objective. The same argument applies if only δ^- has multiple positive entries w.r.t. $V(G_j)$, by moving all positive entries of δ^- to a node $u \in V(G_j)$ with $\pi_u^- - \pi_u = \max_{l \in V(G_j)} \pi_l^- - \pi_l$. Applying this argument to every passive component yields a dual solution with the claimed property. □

Using this additional structure on the supply and demand at the nodes, we next derive an explicit upper bound on the supplies δ^+.

Lemma 8. *Let the point $(d, q, \pi, s, \Delta, \tau)$ be bilevel feasible for Problem (21). Then, there exists a corresponding KKT point $(\Delta, \tau, \alpha, \beta, \delta^+, \delta^-)$ satisfying (26c)–(26h), Condition (27), and*

$$\delta_v^+ \leq |\mathcal{G}| \sum_{a \in A^{\text{act}}(x)} w_a, \qquad v \in V. \tag{28}$$

Proof. Let $(d, q, \pi, s, \Delta, \tau)$ be bilevel feasible for Problem (21).

The case of a single passive component, i.e., $|\mathcal{G}| = 1$, is straightforward. Due to Lemma 7, there exists a KKT point $(\Delta, \tau, \alpha, \beta, \delta^+, \delta^-)$ satisfying Condition (27). Moreover, Constraint (25b) implies $\sum_{v \in V} (\delta_v^+ - \delta_v^-) = 0$. Together with $\delta^+ \geq 0$, $\delta^- \geq 0$, and Condition (27), this implies $\delta^+ = \delta^- = 0$, which proves the claim.

We now assume that $|\mathcal{G}| > 1$ holds. Let $(\Delta, \tau, \alpha, \beta, \delta^+, \delta^-)$ be a KKT point satisfying (26c)–(26h). For the fixed primal point Δ and τ, we further choose the multipliers $(\alpha, \beta, \delta^+, \delta^-)$ such that $\sum_{v \in V} \delta_v^+$ is minimized. Such a choice exists since for fixed (d, q, π, s) and fixed (Δ, τ), all primal slacks in (26d)–(26h) are fixed. Hence, the complementarity conditions are linear in the multipliers $(\alpha, \beta, \delta^+, \delta^-)$. Together with dual feasibility, this shows that choosing dual multipliers that minimize $\sum_{v \in V} \delta_v^+$ corresponds to solving a feasible linear problem. Moreover, since $\delta^+ \geq 0$, its objective value is bounded from below. Hence, this linear problem admits an optimal point. Consequently, we can choose dual multipliers $(\alpha, \beta, \delta^+, \delta^-)$ among all KKT multipliers associated with fixed (Δ, τ) such that $\sum_{v \in V} \delta_v^+$ is minimal.

For the considered KKT point $(\Delta, \tau, \alpha, \beta, \delta^+, \delta^-)$, whose multipliers are chosen to minimize $\sum_{v \in V} \delta_v^+$, Condition (27) of Lemma 7 is satisfied. Indeed, if two nodes $u, v \in V(G_j)$ in the same passive component $G_j \in \mathcal{G}$ satisfy $\delta_u^+ > 0$ and $\delta_v^- > 0$, then reducing both values by $\varepsilon := \min\{\delta_u^+, \delta_v^-\}$ preserves dual feasibility, optimality, and all complementarity conditions, but contradicts the minimality of $\sum_{v \in V} \delta_v^+$. The case of multiple positive entries of δ^+ w.r.t. $V(G_j)$ can be handled similarly because the primal constraints corresponding to positive entries of δ^+ are tight. This allows us to concentrate all positive load of δ^+ w.r.t. $V(G_j)$ at one of these positive entries in $V(G_j)$, which preserves dual feasibility, dual optimality, and complementarity. The same argument applies to δ^-. Consequently, Condition (27) can be imposed for the considered KKT point $(\Delta, \tau, \alpha, \beta, \delta^+, \delta^-)$, whose multipliers minimize $\sum_{v \in V} \delta_v^+$.

We now prove that the considered KKT point satisfies the bounds (28) by deriving a contradiction to the minimality of $\sum_{v \in V} \delta_v^+$. To this end, suppose that there exists a node $u \in V$ with

$$\delta_u^+ > |\mathcal{G}| \sum_{a \in A^{\mathrm{act}}(x)} w_a.$$

We note that the operating-cost coefficients are nonnegative, i.e., $w_a \geq 0$ for all $a \in A^{\mathrm{act}}(x)$. By Condition (27), each passive component $G_j \in \mathcal{G}$ contains at most one node $u \in V(G_j)$ with $\delta_u^+ + \delta_u^- > 0$. Moreover, Constraints (25b) imply $\sum_{v \in V} \delta_v^+ = \sum_{v \in V} \delta_v^-$. Thus, in an acyclic flow decomposition of the dual flow on the reduced tree-shaped network, more than $\sum_{a \in A^{\mathrm{act}}(x)} w_a$ units of supply from node u have to be routed to some demand node w contained in a passive component different from that of u. We choose such a node w, which satisfies

$$\delta_w^- > \sum_{a \in A^{\mathrm{act}}(x)} w_a.$$

Let now $P(u, w)$ denote the unique path between u and w in the underlying undirected graph of the reduced network. We further denote by $A(P(u, w)) \subseteq A^{\mathrm{act}}(x)$ the set of directed arcs corresponding to this path. Because the reduced network is a tree, the dual flows α are uniquely determined by δ^+ and δ^-. Thus, every active element $a \in A(P(u, w))$ carries a dual flow satisfying

$$|\alpha_a| > \sum_{a' \in A^{\mathrm{act}}(x)} w_{a'}.$$

Note that for every arc $a \in A(P(u, w))$ that is oriented from u to w, the corresponding arc flow is positive and satisfies $\alpha_a > \sum_{a' \in A^{\mathrm{act}}(x)} w_{a'} \geq 0$. Otherwise, it is negative and satisfies $\alpha_a < -\sum_{a' \in A^{\mathrm{act}}(x)} w_{a'} \leq 0$.

For $a \in A^{\mathrm{cm}}(x) \cap A(P(u, w))$ with orientation from u to w, Constraint (25c) implies $\sum_{a' \in A^{\mathrm{act}}(x)} w_{a'} < \alpha_a \leq w_a + \beta_a$, and hence $\beta_a > 0$. Moreover, we can w.l.o.g. assume that β_a is chosen so that Constraint (25c) is tight, since decreasing β_a to this value preserves KKT feasibility and does not change δ^+. For $a \in A^{\mathrm{cv}}(x) \cap A(P(u, w))$

with orientation from u to w, Constraint (25d) implies $-\alpha_a < -\sum_{a' \in A^{\text{act}}(x)} w_{a'} \leq w_a + \beta_a$. It follows that w.l.o.g. $\beta_a = 0$ holds and Constraint (25d) is not tight. Analogously, if the corresponding arc a is directed from w to u, we obtain $\beta_a > 0$ for $a \in A^{\text{cv}}(x) \cap A(P(u,w))$ and $\beta_a = 0$ for $a \in A^{\text{cm}}(x) \cap A(P(u,w))$. Consequently, for all $a \in A(P(u,w))$ with $\beta_a = 0$, it holds $\Delta_a = 0$ due to the complementarity constraints (26g) and (26h). Moreover, for all arcs $a \in A(P(u,w))$ with $\beta_a > 0$, the corresponding constraints in (25c) and (25d) are tight.

For fixed primal variables Δ and τ, we now construct a new KKT point. Choosing a sufficiently small $\varepsilon > 0$, we reduce the supply at node u and demand at node w such that $\bar{\delta}^+_u = \delta^+_u - \varepsilon \geq 0$ and $\bar{\delta}^-_w = \delta^-_w - \varepsilon \geq 0$ holds, while leaving all other components of δ^+ and δ^- unchanged. The modified supplies $\bar{\delta}^+$ and demands $\bar{\delta}^-$ are still balanced.

Since the reduced network is a tree, the dual arc flows only change along the path $A(P(u,w))$. More precisely, each positive dual arc flow on this path is reduced by ε, whereas each negative dual arc flow is increased by ε. We denote the resulting dual arc flows by $\bar{\alpha}$. The modified variables $\bar{\alpha}$, $\bar{\delta}^+$, and $\bar{\delta}^-$ satisfy the dual flow-balance constraints (25b) and the nonnegativity constraints (25f).

In addition, we reduce the corresponding values of β. For all $a \in A(P(u,w))$, we set $\bar{\beta}_a := \max\{\beta_a - \varepsilon, 0\}$, while all remaining entries of β are left unchanged. These modified values $\bar{\beta}$, together with $\bar{\alpha}$, $\bar{\delta}^+$, and $\bar{\delta}^-$, also satisfy the Constraints (25c)–(25e). Moreover, we can choose $\varepsilon > 0$ sufficiently small so that for each arc a, $\beta_a > 0$ holds if and only if $\bar{\beta}_a > 0$ holds.

This directly ensures that the complementarity Constraints (26f) are satisfied. Moreover, as previously shown, for all $a \in A(P(u,w))$ with $\beta_a = 0$, it holds $\Delta_a = 0$, which directly renders the corresponding complementarity constraints feasible for the newly constructed point $(\Delta, \tau, \bar{\alpha}, \bar{\beta}, \bar{\delta}^+, \bar{\delta}^-)$. Additionally, for all arcs $a \in A(P(u,w))$ with $\beta_a > 0$, respectively $\bar{\beta}_a > 0$, the corresponding Constraints (25c) and (25d) are tight for the original KKT point which also carries over to the newly constructed one because the corresponding arc flow $\bar{\alpha}_a$ and $\bar{\beta}_a$ are both exactly reduced by ε. Consequently, all complementarity constraints (26d)–(26h) are satisfied by the newly constructed point $(\Delta, \tau, \bar{\alpha}, \bar{\beta}, \bar{\delta}^+, \bar{\delta}^-)$. Together with unchanged primal feasibility and the dual feasibility shown above, this proves that the latter is, indeed, a feasible KKT point.

However, the newly constructed KKT point satisfies

$$\sum_{v \in V} \bar{\delta}^+_v = \sum_{v \in V} \delta^+_v - \varepsilon < \sum_{v \in V} \delta^+_v,$$

which contradicts the minimal choice of the initial KKT point w.r.t. $\sum_{v \in V} \delta^+_v$ and concludes the proof. □

Since we can bound the supply at every node of the reduced network by Lemma 8, we can now derive a corresponding bound on the demand δ^-.

Lemma 9. *Let the point $(d, q, \pi, s, \Delta, \tau)$ be bilevel feasible for Problem (21). Then, there exists a KKT point $(\Delta, \tau, \alpha, \beta, \delta^+, \delta^-)$ satisfying (26c)–(26h), (27), (28), and*

$$\delta^-_v \leq (|\mathcal{G}| - 1)|\mathcal{G}| \sum_{a \in A^{\text{act}}(x)} w_a, \qquad v \in V. \tag{29}$$

Proof. Lemma 8 implies the existence of a KKT point satisfying (26c)–(26h), (27), and (28). From condition (27) of Lemma 7, it follows that each passive component contains at most one node v with positive dual load and either $\delta^+_v > 0$ or $\delta^-_v > 0$ holds. Due to flow conservation constraints (25b), it holds $\sum_{v \in V} \delta^-_v = \sum_{v \in V} \delta^+_v$. Hence, there are at most $|\mathcal{G}|-1$ nodes with $\delta_v^+ > 0$. Together with (28), it follows

$$\sum_{v\in V} \delta_v^+ \leq (|\mathcal{G}|-1)|\mathcal{G}| \sum_{a\in A^{\text{act}}(x)} w_a.$$

Using again $\sum_{v\in V} \delta_v^- = \sum_{v\in V} \delta_v^+$ and $\delta^- \geq 0$, for every $v \in V$, we obtain

$$\delta_v^- \leq \sum_{u\in V} \delta_u^- = \sum_{u\in V} \delta_u^+ \leq (|\mathcal{G}|-1)|\mathcal{G}| \sum_{a\in A^{\text{act}}(x)} w_a,$$

which proves the claim. □

Exploiting the previously obtained bounds on the supply and demand, we now bound the corresponding dual arc flows α.

Lemma 10. *Let the point $(d, q, \pi, s, \Delta, \tau)$ be bilevel feasible for Problem (21). Then, there exists a KKT point $(\Delta, \tau, \alpha, \beta, \delta^+, \delta^-)$ satisfying (26c)–(26h), (27)–(29), and, for every arc $a \in A^{\text{act}}(x)$,*

$$|\alpha_a| \leq (|\mathcal{G}|-1)|\mathcal{G}| \sum_{a'\in A^{\text{act}}(x)} w_{a'}. \tag{30}$$

Proof. Due to Lemma 9, there exists a KKT point $(\Delta, \tau, \alpha, \beta, \delta^+, \delta^-)$ satisfying (26c)–(26h) and (27)–(29). For fixed supply δ^+ and demand δ^-, the variables α represent the corresponding arc flows in the reduced network $\tilde{G}(x)$. Since the reduced graph is a tree under Assumption 1, these arc flows are uniquely determined and acyclic. Thus, the absolute arc flow can directly be bounded by the total amount of supply, i.e., by $\sum_{v\in V} \delta_v^+$. By (27) of Lemma 7 and flow conservation (25b), it holds $\sum_{v\in V} \delta_v^+ \leq (|\mathcal{G}|-1)|\mathcal{G}| \sum_{a\in A^{\text{act}}(x)} w_a$, which concludes the proof. □

Lastly, we prove bounds on the dual variables β.

Lemma 11. *Let the point $(d, q, \pi, s, \Delta, \tau)$ be bilevel feasible for Problem (21). Then, there exists a KKT point $(\Delta, \tau, \alpha, \beta, \delta^+, \delta^-)$ satisfying (26c)–(26h), (27)–(30), and for every $a \in A^{\text{act}}(x)$,*

$$0 \leq \beta_a \leq (|\mathcal{G}|-1)|\mathcal{G}| \sum_{a'\in A^{\text{act}}(x)} w_{a'}$$

holds.

Proof. Due to Lemma 10, there exists a KKT point $(\Delta, \tau, \alpha, \beta, \delta^+, \delta^-)$ satisfying (26c)–(26h) and (27)–(30). The nonnegative variables β enter the dual objective in (25) with nonpositive coefficients and only occur in Constraints (25c)–(25e). Thus, while maintaining feasibility of Constraints (25c)–(25e), we can always choose β componentwise minimal by setting

$$\beta_a = \max\{0, \alpha_a - w_a\},\ a \in A^{\text{cm}}(x), \quad \beta_a = \max\{0, -\alpha_a - w_a\},\ a \in A^{\text{cv}}(x).$$

This componentwise minimal choice preserves all complementarity conditions and therefore feasibility of the corresponding KKT point. Consequently, the claim follows from $w \geq 0$ and the absolute flow bounds (30). □

Finally, we turn to bounds on the primal variables. Since the operational variables Δ_a, $a \in A^{\text{act}}(x)$, are already bounded due to (22e) and (22f), we now state valid bounds for the potentials π and the shift variables τ.

Lemma 12. *For two nodes $u, v \in V$, let $P(u,v)$ denote a shortest path between u and v in the underlying undirected graph of $G(x)$. We denote by $A(P(u,v)) \subseteq A$ the set of directed arcs corresponding to this path.*

For every optimal solution $(d, q, \pi, s, \Delta, \tau)$ of Problem (21), there exists an optimal solution $(d, q, \pi', s, \Delta, \tau')$ of Problem (21) such that

$$0 \leq \pi'_v \leq \max_{u \in V} \sum_{a \in A(P(u,v)) \cap A^{\mathrm{arc}}} \Phi_a\big(\max\{|q_a^-|, |q_a^+|\}\big), \qquad v \in V, \tag{31}$$

and for every passive component $G_j \in \mathcal{G}$, the shift variable τ'_j satisfies

$$\max_{k \in V(G_j)} \left(\pi_k^- - \max_{r \in V} \sum_{a \in A(P(r,k)) \cap A^{\mathrm{arc}}} \Phi_a\big(\max\{|q_a^-|, |q_a^+|\}\big) \right) \leq \tau'_j \leq \min_{k \in V(G_j)} \pi_k^+.$$

Since the proof relies on standard arguments for potential-based flows, we defer it to Appendix B.

Combining Lemmas 8–12 yields valid bounds for all dual variables and for the primal and dual constraint functions appearing in the complementarity conditions. Hence, the complementarity conditions in (26) can be replaced by standard big-M linearizations; see [9]. We use this reformulation in the following computational study, where it leads to improved computational performance.

5. Computational Study

We apply Algorithm 1 to gas network instances with controllable elements to compute robust network designs under load uncertainty. In Section 5.1, we describe the considered gas network instances and uncertainty sets. In Section 5.2, we discuss the implementation of the algorithm and additional algorithmic enhancements for solving the resulting MINLPs more efficiently. Finally, in Section 5.3, we present and discuss the computational results.

5.1. Gas Network Instances and Uncertainty Modeling.

For our computational study, we use gas network instances derived from the publicly available GasLib library [31]. Specifically, we consider the GasLib-40 and GasLib-134 instances, where GasLib-134 roughly represents the Greek gas transmission network. In addition, we construct an instance, which we call GasLib-37, from GasLib-135 by removing six arcs and retaining the connected component with the largest number of sources.[2] The resulting instance has 37 nodes, including 3 sources, and 44 arcs.

To satisfy Assumption 1, we replace active elements on cycles by short pipes, i.e., pipes of zero length without pressure loss. After this modification, GasLib-40 contains five of the original six compressors, GasLib-134 remains unchanged with one compressor and one control valve, and GasLib-37 contains four compressors.

For each base network $G = (V, A)$, we keep the node set V fixed and construct three network design variants as in [35]. The *unchanged* setting corresponds to robustifying the existing network, i.e., all arcs of the original network are existing arcs and $A_{\mathrm{ex}} = A$. In the *spanning-tree* setting, the existing arcs $A_{\mathrm{ex}} \subseteq A$ form a spanning tree of the original network. In the *greenfield* setting, no arc exists, i.e., $A_{\mathrm{ex}} = \emptyset$, and the network has to be built from scratch. We note that no spanning-tree variant is considered for GasLib-134 because this network is already a tree.

For these design variants, we define the set A_{ca} of candidate elements that can be built as follows. All active elements that are not existing arcs, i.e., $A^{\mathrm{act}} \cap (A \setminus A_{\mathrm{ex}})$, are included in A_{ca}. For each such candidate compressor, we additionally add a parallel

[2]The removed arcs are pipe_93, pipe_46, pipe_96, pipe_106, pipe_103, and pipe_104.

short pipe. For each pipe in A, we add multiple parallel candidate pipes whose diameters are scaled versions of the original diameter in the corresponding GasLib instance. For GasLib-37 and GasLib-40 in the *unchanged* and *spanning-tree* settings, we use the scaling factors $\{0.3, 0.7, 1.0, 1.3\}$. For the substantially more challenging *greenfield* setting, we use the coarser set $\{0.5, 1.0, 1.5\}$. For GasLib-134, which is computationally easier due to its tree structure, we use the finer set $\{0.3, 0.5, 0.7, 0.9, 1.0, 1.1, 1.3\}$ for both design variants.

We next specify construction and operating cost. For each compressor $a \in A^{\mathrm{cm}}$, we allow a maximum operating range of 10 bar, as in [25], and use an annualized construction cost of 0.053 million €/year. The linear operating-cost coefficient is $w_a = 1.26 \cdot 10^{-6} q_a^{\max}$, where $q_a^{\max}$ denotes the maximum flow through compressor a over the given uncertainty set. For each active element $a \in A^{\mathrm{act}}$, the minimum arc-flow threshold m_a is set to zero. Pipe construction cost follow [26]: building a pipe a with diameter D_a costs $278.24 \exp(1.6 D_a)$ €/m. We annualize this cost using a lifetime of 40 years and annual operation and maintenance cost of 5 €/m. Short pipes and control valves have zero construction cost.

For gas networks, the potential function of each pipe a is given by $\Phi_a(q_a) = \Lambda_a q_a |q_a|$; see [11]. Here, the pressure loss coefficient $\Lambda_a > 0$ is computed as described in [10, 35]. In general, we obtain a nonlinear and nonconvex potential-based flow model.

Uncertainty Sets: We use the same four polyhedral uncertainty sets as in [35]. For a given base load scenario d^{base}, these sets contain balanced loads that deviate from the base scenario.

The box uncertainty set U_{box} contains all balanced loads for which the withdrawals $u \in V_-$ vary in $[0.6 d_u^{\mathrm{base}}, 1.4 d_u^{\mathrm{base}}]$ and the injections $u \in V_+$ vary in $[0.7 d_u^{\mathrm{base}}, 1.3 d_u^{\mathrm{base}}]$. The set U_{sum} refines U_{box} by imposing lower and upper bounds on the aggregate withdrawal $\sum_{u \in V_-} d_u$. The set U_{corr} additionally models correlations between randomly selected withdrawals by bounding differences between their relative deviations from the base values, which models consumers with similar demand patterns. Finally, $U_{\mathrm{all}} := U_{\mathrm{box}} \cap U_{\mathrm{sum}} \cap U_{\mathrm{corr}}$ combines all restrictions. For a detailed description of the uncertainty sets and their parameter values, we refer to Section 6.2 of [35].

For GasLib-40 and GasLib-134, we use the base scenarios provided by [31]. For GasLib-134, this is the load scenario corresponding to October 24, 2014. For GasLib-37, each withdrawal has base load 156, and each source has base injection 1144, both measured in $1000\,\mathrm{m}^3\,\mathrm{h}^{-1}$.

5.2. **Algorithmic and Computational Setup.** We now briefly describe the implementation of Algorithm 1, including the enhancements used to solve the challenging master problem (23) and adversarial subproblems more efficiently. Both problem classes are MINLPs. Since most algorithmic enhancements used to accelerate the solution process are adapted from [35], we keep the following description concise and refer to that paper for the technical details.

For modeling the nonconvex potential-drop constraints (1), we use a well-established equivalent formulation based on additional binary flow-direction variables, which indicate the flow direction between incident nodes. Such formulations are often computationally more effective and have been used frequently in the gas network optimization literature; see, e.g., [5, 21, 35]. A detailed description of this reformulation for general potential-based flows is given in [35, Appendix A].

We also add the valid inequalities of [12] to all potential-based optimization problems. These inequalities exclude cyclic flows and are valid for passive potential-based flows, which are acyclic; see [12]. By Assumption 1, active elements are not part of any cycle, so the inequalities apply directly in our setting.

The considered gas networks typically do not impose binding bounds on the arc flows, and the available large flow bounds in the GasLib instances are redundant in our setting. Consequently, we do not explicitly solve the adversarial flow-bound problems (16) and (17). However, we note that the flows are implicitly bounded by the potential bounds at the incident nodes through the coupling of potentials and arc flows. Nevertheless, tighter flow bounds are useful for improving the computational performance of the resulting MINLPs. We therefore use the presolve procedure of [35, Section 4.3] to compute tightened lower and upper flow bounds.

Since the scenario set S in the master problem only grows during Algorithm 1, the optimal objective value of a previous iteration remains a valid lower bound in the current iteration. We enforce this bound by adding an additional objective cut.

Moreover, we impose that at most one element can be selected from each set of parallel candidate elements. This does not restrict the model, since multiple new parallel pipes can be represented by a single equivalent pipe; see [20].

Combining the enhancements described above yields our baseline approach, denoted by MINLP_base in the following.

As observed in [35], this baseline approach is often not sufficient for the most challenging *greenfield* instances. We therefore also consider an adapted enhanced variant, denoted by MINLP_enhanced. This variant is identical to the enhanced algorithmic approach of [35], except that we additionally include valid inequalities from classical network optimization.

In each iteration of Algorithm 1, MINLP_enhanced solves the master problem in up to three stages. It starts with solving a reduced master problem containing only the most recently added adversarial scenario. If the resulting solution is feasible for the full master problem, the algorithm proceeds. Otherwise, it solves a mixed-integer second-order cone relaxation of the full master problem. If this again does not yield a feasible master solution, the full MINLP master problem is solved. A complete description of this enhanced approach is given in [35, Section 6.3], where it is referred to as Reduced Relaxation.

Although MINLP_enhanced outperforms the baseline approach in our computational results, its default version from [35] does not solve all *greenfield* instances. We therefore strengthen it by adding two classes of valid inequalities from classical network design and from gas network design [5].

First, for every supply or demand node with nonzero load and no incident existing arc, we add a linear cut requiring at least one incident network element to be built. In addition, for inner nodes V_0 with no incident existing arc, we enforce that either no incident candidate or at least two incident candidates are built. This excludes inner nodes as leaf nodes of the expanded network, which would only incur additional construction cost.

Second, we add the cuts of [5, Section 4.3], which only involve the binary flow-direction variables. These cuts ensure that, at every supply node with nonzero load, at least one incident arc allows outgoing flow and, analogously, that at every withdrawal node, at least one incident arc allows incoming flow. In addition, for inner nodes with exactly two incident elements, they enforce one incoming and one outgoing flow direction, thereby preserving mass flow balance.

All models are implemented in Python 3.11.13 with Pyomo 6.10.0 and solved with Gurobi 13.0.1. The computations are carried out on a single server node equipped with Intel Xeon E3-1240 v5 CPUs. We impose a memory limit of 64 GB, a total time limit of 24 h, and use at most 4 threads.

5.3. **Computational Results.** We apply Algorithm 1 to the gas network instances and uncertainty sets described in Sections 5.1 and 5.2. The goal of this section is to demonstrate the applicability of the developed adversarial approach and to assess its computational performance.

In the following, an instance is counted as solved if Algorithm 1 either terminates with a globally optimal robust network design or proves infeasibility within the time limit. Moreover, we refer to adversarial scenarios generated by verifying robust feasibility (Theorem 2) as *pressure scenarios* (*pressure scn* in the tables). Adversarial scenarios generated by computing the worst-case operating cost (Theorem 5) are called *operational scenarios* (*operational scn* in the tables). For the considered instances, the adversarial problem for load balancedness in connected components (Problem (14)) did not generate any adversarial scenarios.

Tree-shaped networks: Table 1 reports the results for the two considered variants of Algorithm 1.

Table 1. Runtimes and number of adversarial scenarios for GasLib-134 solved with the approach MINLP_enhanced (top) and with MINLP_base (bottom).

	Unchanged #Solved 4 of 4			Greenfield #Solved 4 of 4		
	Min	Median	Max	Min	Median	Max
#Pressure scn	1	1	1	3	4	4
#Operational scn	0	0	0	0	0	0
Runtime (s)	76.13	90.97	112.27	194.82	265.86	350.14

	Unchanged #Solved 4 of 4			Greenfield #Solved 4 of 4		
	Min	Median	Max	Min	Median	Max
#Pressure scn	1	1	1	4	4	4
#Operational scn	0	0	0	0	0	1
Runtime (s)	63.93	104.18	106.96	124.80	236.28	353.32

For GasLib-134, both algorithmic variants solve all considered instances and the runtimes of both approaches are comparable. In particular, the runtimes are low and even in the greenfield setting, the maximum runtime of MINLP_enhanced is 350.14 s. However, this can be explained by the tree-shaped structure of the network, which makes the corresponding network design instances significantly easier to solve than cyclic networks.

The number of generated adversarial scenarios is very moderate relative to the size of the network. For the enhanced formulation, the unchanged instances require only one pressure scenario, whereas the greenfield instances require between 3 and 4 pressure scenarios. In addition, only one operational scenario is generated, namely in the *greenfield* setting with MINLP_base. The number of generated adversarial scenarios may differ between variants of Algorithm 1, since neither robust designs nor adversarial scenarios need to be unique. In Remarks 13 and 14, we discuss the observed small number of adversarial scenarios in detail.

Cyclic networks: We now turn to the cyclic networks GasLib-37 and GasLib-40, which are computationally much more demanding than tree-shaped networks. Tables 2 and 3 report the corresponding computational results for both variants of Algorithm 1.

A detailed comparison of the runtimes shows that MINLP_enhanced clearly outperforms MINLP_base on the cyclic instances. It solves all cyclic instances to global optimality, including the challenging *greenfield* instances, whereas MINLP_base solves only 19 of the 24 instances.

Across all tree-shaped and cyclic instances that could be solved by both approaches, MINLP_enhanced is faster than MINLP_base in 77.78 % of these instances.

Table 2. Runtimes and number of adversarial scenarios for GasLib-37 solved with the approach MINLP_enhanced (top) and with MINLP_base (bottom).

	Unchanged #Solved 4 of 4			Spanning tree #Solved 4 of 4			Greenfield #Solved 4 of 4		
	Min	Median	Max	Min	Median	Max	Min	Median	Max
#Pressure scn	1	1	2	1	2	2	2	2	4
#Operational scn	0	0	0	0	0	0	0	0	0
Runtime (s)	407.01	9 240.37	81 259.85	120.93	313.86	447.14	2 056.27	6 215.65	60 415.56

	Unchanged #Solved 4 of 4			Spanning tree #Solved 4 of 4			Greenfield #Solved 2 of 4		
	Min	Median	Max	Min	Median	Max	Min	Median	Max
#Pressure scn	1	1	2	2	2	2	2	2	2
#Operational scn	0	0	0	0	0	0	0	0	0
Runtime (s)	798.27	12 813.18	41 258.57	2 439.48	3 426.94	5 468.21	20 306.97	25 427.24	30 547.50

Table 3. Runtimes and number of adversarial scenarios for GasLib-40 solved with the approach MINLP_enhanced (top) and with MINLP_base (bottom).

	Unchanged #Solved 4 of 4			Spanning tree #Solved 4 of 4			Greenfield #Solved 4 of 4		
	Min	Median	Max	Min	Median	Max	Min	Median	Max
#Pressure scn	1	1	1	1	1	1	1	2	2
#Operational scn	0	0	0	0	0	1	0	0	1
Runtime (s)	91.80	121.52	147.79	219.31	261.50	375.22	20 172.52	25 255.70	30 411.01

	Unchanged #Solved 4 of 4			Spanning tree #Solved 4 of 4			Greenfield #Solved 1 of 4		
	Min	Median	Max	Min	Median	Max	Min	Median	Max
#Pressure scn	1	1	1	1	1	1	1	1	1
#Operational scn	0	0	0	0	0	1	1	1	1
Runtime (s)	235.43	296.15	691.72	498.01	1 763.12	15 312.76	52 453.26	52 453.26	52 453.26

Moreover, the median total runtime decreases from 798 s for MINLP_base to 322 s for MINLP_enhanced. The adversarial problems contribute differently to the overall runtime. For MINLP_base, adversarial problems account for a median share of 16.89 %, whereas for MINLP_enhanced, adversarial problems account for a median share of 41.52 % of the total runtime. This indicates that the algorithmic improvements in MINLP_enhanced significantly reduce the runtimes of the master problems, which are the main computational bottleneck in the baseline approach.

For MINLP_enhanced, the robust-feasibility adversarial problems alone account for a median share of 32.76 %, whereas the robust-optimality adversarial problems account for only 4.63 %. Thus, certifying robust feasibility is the dominant adversarial task, whereas verifying worst-case operating cost is comparatively inexpensive for the considered instances.

Instead of the big-M linearizations used in the KKT-based single-level reformulations, we also tested SOS1-based reformulations of the corresponding complementarity conditions. We implemented this variant for both MINLP_enhanced and MINLP_base. The SOS1-based version of MINLP_enhanced solves only 27 of the 32 instances. In particular, it fails on one *unchanged* and one *spanning-tree* GasLib-37 instance, as well as on two *unchanged* and one *spanning-tree* GasLib-40 instance. The SOS1-based version of MINLP_base performs even worse and solves only 20 of the 32 instances. This comparison shows that the valid big-M bounds derived in Section 4, together with the resulting linearizations of the complementarity conditions, improve the computational performance.

Finally, we discuss the observed moderate number of adversarial scenarios needed to ensure robust feasibility and robust optimality of the considered network designs.

Remark 13 (Number of Pressure Scenarios)**.** *The observed small number of adversarial scenarios needed to guarantee robust feasibility is in line with the results for passive robust network design in [35]. As discussed in Section 5 of [35], networks with only a few sources that can supply several withdrawal nodes at maximum demand typically require only a small number of pressure scenarios. This structure is present in the realistic networks considered here, and our computational results indicate that, also with controllable elements, only a few pressure scenarios are often needed to guarantee robust feasibility.*

Remark 14 (Number of Operational Scenarios)**.** *The number of operational scenarios generated by the adversarial Problem* (20)*, which computes the worst-case operating cost, is even smaller than the number of pressure scenarios. One possible explanation is that the pressure adversarial problem* (18) *computes load scenarios that maximize violations of the pressure bounds, and hence induce large pressure drops in the network. Such scenarios are often also natural candidates for high operating cost, because large pressure drops typically have to be compensated by increased operation of the controllable elements. Consequently, worst-case pressure scenarios may already induce high operating cost for the current design and are included in the master problem before the worst-case operating cost is computed. However, there is no guarantee that a pressure scenario is also worst-case with respect to the operating cost. Hence, the adversarial problem* (20)*, which computes the worst-case operating cost, cannot be omitted. Indeed, certain GasLib-40 instances in the spanning-tree and greenfield settings require an additional operational scenario, which shows that robust feasibility alone does not suffice to certify robust optimality.*

Overall, the computational results demonstrate the applicability of the developed adversarial approach to realistic gas network design instances with controllable elements. Across all considered instances, a moderate number of worst-case pressure and operational scenarios are needed to guarantee robust feasibility and robust optimality. Moreover, the approach is effective for robustifying existing networks and, when combined with the enhanced solution techniques, also for the challenging greenfield setting in which a robust network under load uncertainty has to be built from scratch.

6. Conclusion

We have developed an exact adversarial framework for adjustable robust network design with controllable elements under load uncertainty. Under the structural assumption that no controllable element is part of a cycle, we characterize robust feasibility and robust optimality of a fixed network design by finitely many mixed-integer nonlinear bilevel problems. We reformulate these bilevel problems as equivalent single-level optimization problems and embed the resulting reformulations in an exact adversarial approach for computing optimal robust network designs with uncertain injections and withdrawals.

Since the framework is developed for general potential-based networks, it covers utility networks such as gas, hydrogen, and water systems. Our computational results on realistic gas network instances show that optimal robust designs with controllable elements can be computed using a moderate number of worst-case feasibility and operating-cost scenarios.

The results provide a first stepping stone toward solving robust network design problems with more general controllable elements. Relaxing the structural assumption that no controllable element is contained in a cycle is an important but highly challenging direction for future research. Such an extension may require new techniques from mixed-integer nonlinear robust or bilevel optimization, since structural properties of potential-based flows, such as uniqueness of flows, may fail in the presence of controllable elements on cycles.

Finally, decision-dependent uncertainty sets are a promising direction for future research and, to the best of our knowledge, have not yet been studied for robust potential-based network design. In the present setting, all suppliers and consumers represented in the uncertainty set have to be included in the robust network design. Decision-dependent uncertainty sets instead allow one to decide which potential suppliers or consumers are connected to the network, possibly at additional cost.

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Appendix A.

The KKT reformulation of Problem (18) is given by

$$
\begin{aligned}
\max_{\xi}\ & y+z \\
\text{s.t. }\ & (18b)–(18f), (19b)–(19e), (25b) \\
& \alpha_a \le \beta_a,\ \beta_a \ge 0, \quad a \in A^{\text{cm}}(x), \\
& -\alpha_a \le \beta_a,\ \beta_a \ge 0, \quad a \in A^{\text{cv}}(x), \\
& \sum_{u\in V} \delta_u^+ = 1,\ \sum_{u\in V} \delta_u^- = 1, \\
& \delta_u^+ \ge 0,\ \delta_u^- \ge 0, \quad u \in V, \\
& \delta_u^- \left(\tau_j + y + \pi_u - \pi_u^-\right) = 0, \quad u \in V(G_j),\ G_j \in \mathcal{G}, \\
& \delta_u^+ \left(\tau_j - z + \pi_u - \pi_u^+\right) = 0, \quad u \in V(G_j),\ G_j \in \mathcal{G}, \\
& \beta_a\left(\Delta_a^+ s_a - \Delta_a\right) = 0, \quad a \in A^{\text{act}}(x), \\
& (-\alpha_a + \beta_a)\Delta_a = 0, \quad a \in A^{\text{cm}}(x), \\
& (\alpha_a + \beta_a)\Delta_a = 0, \quad a \in A^{\text{cv}}(x),
\end{aligned}
$$

where $\xi = (d, q, \pi, s, \Delta, \tau, y, z, \alpha, \beta, \delta^+, \delta^-)$ is the vector of upper-level, lower-level primal, and lower-level dual variables. Moreover, the following valid primal and dual bounds have been proved in [25]:

- $\alpha_a \in [-1, 1],\ \beta_a \in [0, 1]$, for all $a \in A^{\text{act}}(x)$,
- $\delta_u^+, \delta_u^- \in [0, 1]$, for all $u \in V$,
- $0 \le \pi_u \le \sum_{a\in A(x)} \Lambda_a M^2$, for all $u \in V$, with $M = \max_{d\in U} \sum_{u\in V_-} d_u$, and
- $0 \le \tau_j \le \sum_{a\in A^{\text{act}}(x)} \Delta_a^+$, for all $G_j \in \mathcal{G}$.

Appendix B.

We now present the proof of Lemma 12.

Proof. Let $(d, q, \pi, s, \Delta, \tau)$ be a bilevel feasible point of (21). For fixed $\varepsilon \in \mathbb{R}$, we now consider the point $(d, q, \pi + \varepsilon, s, \Delta, \tau - \varepsilon)$ and show that it is still bilevel feasible.

Because the upper-level variables (d, q, s) stay unchanged and only the potential vector π is shifted by ε, upper-level feasibility is preserved due to Theorem 7.1 of [17]. Also the lower-level Constraints (22e) and (22f) remain satisfied since the variables Δ are unchanged. Moreover, Constraints (22b) are valid due to

$$
\begin{aligned}
(\tau_i - \varepsilon) - (\tau_j - \varepsilon) &= \tau_i - \tau_j = -\Delta_a, && a \in A^{\text{cm}}(x), \\
(\tau_i - \varepsilon) - (\tau_j - \varepsilon) &= \tau_i - \tau_j = \Delta_a, && a \in A^{\text{cv}}(x).
\end{aligned}
$$

In addition, Constraints (22c) and (22d) remain satisfied because, for all $v \in V(G_j)$ and $G_j \in \mathcal{G}$,

$$
\begin{aligned}
\tau_j - \varepsilon \ge \pi_v^- - (\pi_v + \varepsilon) \quad &\Longleftrightarrow \quad \tau_j \ge \pi_v^- - \pi_v, \\
\tau_j - \varepsilon \le \pi_v^+ - (\pi_v + \varepsilon) \quad &\Longleftrightarrow \quad \tau_j \le \pi_v^+ - \pi_v,
\end{aligned}
$$

holds. Overall, $(d, q, \pi + \varepsilon, s, \Delta, \tau - \varepsilon)$ is a bilevel feasible point of (21). Therefore, shifting π and τ by ε preserves feasibility and optimality for the lower-level problem.

Let $u \in V$ satisfy $\pi_u = \min_{v\in V} \pi_v$. We now explicitly choose $\varepsilon := -\min_{v\in V} \pi_v$. Then, $\pi_u + \varepsilon = \min_{v\in V}\{\pi_v + \varepsilon\} = 0$ holds. Consequently, there exists an optimal solution, denoted again by $(d, q, \pi, s, \Delta, \tau)$, with $\pi_u = 0$ and $\pi_v \ge 0$ for each $v \in V$.

For an arbitrary node $v \in V$, set $A(P(u,v)) \subseteq A$ contains the directed arcs of the corresponding shortest path $P(u,v)$. For each arc a on $A(P(u,v))$, let $\eta_{a,P} = 1$ if a is traversed in its orientation, and let $\eta_{a,P} = -1$ otherwise.

Summing up the upper-level constraints (10b)–(10d) and (18d) over the passive path arcs in $A(P(u,v))$ leads to

$$\pi_v = \pi_u - \sum_{a \in A(P(u,v)) \cap A^{\mathrm{arc}}} \eta_{a,P}\, \Phi_a(q_a) \leq \sum_{a \in A(P(u,v)) \cap A^{\mathrm{arc}}} |\Phi_a(q_a)|.$$

The robust feasibility of x implies that the flows q satisfy $q_a^- \leq q_a \leq q_a^+$ for all $a \in A^{\mathrm{arc}}(x)$. Together with the property that the potential function Φ_a is odd and strictly increasing, it follows

$$0 \leq \pi_v \leq \sum_{a \in A(P(u,v)) \cap A^{\mathrm{arc}}} \Phi_a(\max\{|q_a^-|, |q_a^+|\}).$$

Finally, maximizing over all possible reference nodes $u \in V$ yields (31).

We now derive the bounds on τ and consider a fixed passive component $G_j \in \mathcal{G}$. By Constraint (22c) and the derived potential bounds (31), we obtain

$$\tau_j \geq \pi_v^- - \pi_v \geq \pi_v^- - \max_{r \in V} \sum_{a \in A(P(r,v)) \cap A^{\mathrm{arc}}} \Phi_a(\max\{|q_a^-|, |q_a^+|\}), \quad v \in V(G_j).$$

Taking the maximum over all $v \in V(G_j)$ yields the lower bound. For the upper bound on τ, we combine Constraint (22d) with $\pi_v \geq 0$ and obtain

$$\tau_j \leq \pi_v^+ - \pi_v \leq \pi_v^+, \quad v \in V(G_j).$$

Taking the minimum over all $v \in V(G_j)$ yields the upper bound. □

(K. Rolsing, J. Thürauf) University of Technology Nuremberg (UTN), Department Liberal Arts and Social Sciences, Discrete Optimization, Dr.-Luise-Herzberg-Str. 4, 90461 Nuremberg, Germany

Email address: kristina.rolsing@utn.de

Email address: johannes.thuerauf@utn.de