Convexlikeness and Supportedness in Quadratic Multiobjective Optimization

Gabriele Eichfelder[*], Christian Kugelmann[†], E. Alper Yıldırım[‡]

July 14, 2026

Abstract

This paper studies geometric and structural properties of quadratic multiobjective optimization problems. Thereby, a multiobjective optimization problem is called convexlike if the upper image, i.e., the image set plus the nonnegative orthant, is a convex set. Moreover, we say that a feasible point is supported in case it is a minimal solution of a weighted sum of the objective functions with nonnegative weights, and we are interested in the question if all efficient, i.e., optimal, solutions of the multiobjective optimization problem under consideration are supported. We investigate conditions under which weighted sum scalarizations admit an optimal solution and how this is related to the existence of positive (semi-)definite convex combinations of the matrices defining the quadratic terms. An equivalent dual characterization of the existence of such convex combinations is derived. In the convexlike unconstrained setting, we show that the absence of weakly efficient solutions is equivalent to the upper image set being the whole space. For the unconstrained biobjective case, we obtain simplified characterizations of the existence of positive (semi-)definite convex combinations. These results clarify conditions under which weighted sum scalarizations are appropriate for solving quadratic multiobjective optimization problems.

1 Introduction

Many decision-making problems require the simultaneous consideration of several conflicting criteria. Multiobjective optimization provides a mathematical framework for analyzing such compromises and has evolved into a wide field with important continuous, mixed-integer, and robust variants. Recent survey articles document both the methodological diversity of the field and the significant progress that has been made in continuous multiobjective optimization over the last decades; see, for instance, [13, 19].

In this paper, we focus on quadratic multiobjective problems. This class naturally arises when compromises are expressed in terms of variances, covariances, quadratic

[*]Institute of Mathematics, Technische Universität Ilmenau, Po 10 05 65, D-98684 Ilmenau, Germany, ORCID 0000-0002-1938-6316, gabriele.eichfelder@tu-ilmenau.de

[†]Institute of Mathematics, Technische Universität Ilmenau, Po 10 05 65, D-98684 Ilmenau, Germany, ORCID 0009-0002-2548-768X christian.kugelmann@tu-ilmenau.de

[‡]School of Mathematics and Maxwell Institute of Mathematical Sciences, The University of Edinburgh, James Clerk Maxwell Building, Peter Guthrie Tait Road, JCMB 5415, Edinburgh, EH9 3FD, United Kingdom, ORCID 0000-0003-4141-3189, E.A.Yildirim@ed.ac.uk

deviations, or quadratic loss terms. Portfolio selection according to the mean-variance principle is a classic example; see, for instance, [30, 37]. Further examples can be found in the planning of radiation therapies as in [8, 9], where quadratic dose target functions are commonly used, as well as in classification and credit rating models based on quadratic multiobjective programming; see, e.g., [12, 31]. More recent work deals with robust two-stage quadratic multiobjective models; see, for instance, [15]. Quadratic approximations have also been used to describe characteristic structures in low-dimensional multiobjective optimization problems as in [28].

Motivated by the considerations listed above, we study quadratic multiobjective optimization problems of the form

$$\begin{aligned} &\text{minimize} \qquad f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix} := \begin{pmatrix} x^\top \Gamma^1 x + 2(\beta^1)^\top x + \alpha_1 \\ \vdots \\ x^\top \Gamma^m x + 2(\beta^m)^\top x + \alpha_m \end{pmatrix} \\ &\text{subject to} \qquad x \in X, \end{aligned} \tag{QMOP}$$

with a nonempty feasible set $X \subseteq \mathbb{R}^n$ and a vector-valued objective function $f : X \to \mathbb{R}^m$ with quadratic components $f_i : X \to \mathbb{R}$, $f_i(x) = x^\top \Gamma^i x + 2(\beta^i)^\top x + \alpha_i$, $\Gamma^i \in \mathcal{S}^n$ real symmetric matrices, $\beta^i \in \mathbb{R}^n$ for all $i \in \{1, \ldots, m\}$, and $\alpha \in \mathbb{R}^m$. In particular, we consider two cases: $X = \mathbb{R}^n$ and $X = \{x \in \mathbb{R}^n \mid Ax = b\}$ with $A \in \mathbb{R}^{r \times n}, b \in \mathbb{R}^r$.

A widely used scalarization technique for solving multiobjective optimization problems is the weighted sum method; see, for instance, [3]. It reduces a multiobjective optimization problem to a single-objective one and is therefore easy to understand and implement. In convex settings, appropriate weights allow one to recover the relevant minimal boundary points of the image set. Outside convex settings, however, this is generally no longer true. This makes it important to understand under which conditions weighted sum scalarizations capture the relevant minimal points of the image set. The minimal solutions generated by weighted sum scalarizations play an important role in the literature, both from a theoretical and a computational point of view, since they often provide a first tractable approximation of the set of relevant minimal solutions.

In contrast to single-objective optimization (that is, for $m = 1$), where one typically seeks a single minimal value, for $m \geqq 2$, there is generally no unique minimal objective value of (QMOP). Instead, one aims to identify the minimal elements of the image set, called nondominated points. In the literature, the set of all nondominated points is also known as the Pareto front. The feasible points whose images are nondominated are called efficient solutions.

In multiobjective optimization, a relevant geometric object is the upper image set $f(X) + \mathbb{R}^m_{\geqq}$, where $\mathbb{R}^m_{\geqq}$ is the nonnegative orthant in $\mathbb{R}^m$. If the upper image set is convex, weighted sum scalarizations are sufficient to recover the relevant minimal boundary points of the image set; see, e.g., [18, Theorem 3.5, Theorem 3.11, Corollary 3.12, Theorem 3.13]. Recent sufficient conditions for such convexity properties in quadratic problems are given in [24, Theorem 3.1, Theorem 4.1]. Related convexity questions have also been studied from a hidden convexity perspective via associated epigraphical sets; see [23]. Thus, such convexity questions arise in several contexts, including scalarization-based vector optimization and hidden convexity analyses of quadratic and polynomial systems. Since the Minkowski sum of two convex sets is convex, a sufficient condition for the upper image to be convex is the convexity of the image set $f(X)$. Convexity and closedness properties of image sets of quadratic vector-valued functions have been studied extensively in the literature; see, for instance, [16, Theorem 1], [33, Theorem 2.1, Theorem 2.2, Proposition 3.7], [25, Theorem 4.1, Theorem 5.1], [10, Theorem 2.1]. For unconstrained quadratic multiobjective problems with quadratic and affine linear objectives, i.e., with $X = \mathbb{R}^n$, this property has been studied in [21]. For $m = 2$ and $X = \mathbb{R}^n$, it is known that $f(X) + \mathbb{R}^2_{\geqq}$ is convex although the image set $f(X)$ is in general not convex; see [20]. As shown in [24, Theorem 4.1], the convexity of the upper image also holds for $X = \{x \in \mathbb{R}^n \mid Ax = b\}$ in the biobjective case. We provide an alternative proof of this result. We also discuss counterexamples showing that the convexity of $f(X) + \mathbb{R}^m_{\geqq}$ may fail for linear inequality constraints in the biobjective case and for unconstrained problems with three or more objectives.

However, even if the set $f(X) + \mathbb{R}^m_{\geqq}$ is convex, it is not immediately clear that there are any nondominated points at all in (QMOP). For instance, $f(X) + \mathbb{R}^m_{\geqq}$ could be the whole space $\mathbb{R}^m$. Thus, it is of interest to understand conditions under which weighted sum scalarizations of (QMOP) actually admit minimal solutions. We show that this question is closely related to positive (semi-)definiteness properties of convex combinations of the matrices $\Gamma^1, \dots, \Gamma^m$. In particular, positive semidefinite convex combinations arise as necessary conditions, while positive definite convex combinations yield sufficient conditions.

The existence of positive (semi-)definite linear combinations of symmetric matrices has been studied extensively; see, e.g., [5, 17]. Related questions also arise in the context of matrix pencils; see [5, 41]. Moreover, such existence conditions play an important role in the study of convexity and closedness properties of image sets of quadratic vector-valued functions; see [33, Theorem 2.1, Theorem 2.2]. In this paper, we focus on the simplex-constrained subcase, that is, on convex combinations. In particular, positive definite convex combinations play an important role in sufficient conditions for exact convex relaxations of quadratically constrained quadratic programs (QCQP); see, e.g., [11, 43]. This provides further motivation for studying convex combinations in the present setting.

The remainder of this paper is organized as follows. In Section 2, we give some standard definitions and some basic properties of multiobjective optimization problems. We also introduce the classes of convex, hidden convex, and convexlike multiobjective optimization problems. In Section 3, we investigate the unconstrained quadratic multiobjective optimization problem and derive conditions under which the weighted sum scalarization of (QMOP) admits a minimal solution. In particular, we examine the existence of positive semidefinite convex combinations of the matrices $\Gamma^1, \dots, \Gamma^m$ as a necessary condition. We also show that in the convexlike setting, the absence of weakly efficient solutions is equivalent to the upper image set being the whole space. Section 4 focuses on the unconstrained biobjective case and provides simplified conditions for the existence of minimal solutions to the weighted sum scalarization. Section 5 deals with the constrained biobjective case and shows that convexlikeness is preserved under linear equality constraints, but may fail under linear inequality constraints. Finally, Section 6 concludes by summarizing the results, noting their application to test instances, and outlining a motivation for potential future work.

2 Preliminaries

For $y^1, y^2 \in \mathbb{R}^m$ with $m \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers, we write $y^1 \leqq y^2$ if $y^1_i \leqq y^2_i$ for all $i = 1, \dots, m$, and $y^1 \leq y^2$ for $y^1 \leqq y^2$, $y^1 \neq y^2$ as well as $y^1 < y^2$ if $y^1_i < y^2_i$ for all $i = 1, \dots, m$. The nonnegative orthant is denoted by $\mathbb{R}^m_{\geqq} := \{y \in \mathbb{R}^m \mid y \geqq 0\}$ and the nonnegative orthant without zero is denoted by $\mathbb{R}^m_{\geq} := \mathbb{R}^m_{\geqq} \setminus \{0\}$. The image set of X under f is denoted by $Y := f(X)$ and the set $f(X) + \mathbb{R}^m_{\geqq}$ is called the upper image set. For a positive integer $a \in \mathbb{N}$, we define $[a] := \{1, \dots, a\}$. The set of real symmetric $n \times n$ matrices is denoted by $\mathcal{S}^n$. The closed pointed convex cone of symmetric positive semidefinite matrices is denoted by $\mathcal{S}^n_+$ with interior $\mathcal{S}^n_{++} := \operatorname{int}(\mathcal{S}^n_+)$, the set of positive definite matrices. Thus, we have the disjoint decomposition $\mathcal{S}^n_+ = \mathcal{S}^n_{++} \cup \partial\mathcal{S}^n_+$, where $\partial\mathcal{S}^n_+$ is the boundary of $\mathcal{S}^n_+$. We will repeatedly use the trace inner product on $\mathcal{S}^n$, denoted by $\operatorname{tr}(\Gamma S)$ for $\Gamma, S \in \mathcal{S}^n$. With respect to this inner product, the cone $\mathcal{S}^n_+$ is selfdual. We collect the following elementary results for later reference.

Lemma 2.1. *Let $\Gamma, S \in \mathcal{S}^n_+$. Then, there exists a unique matrix, denoted by $S^{\frac{1}{2}} \in \mathcal{S}^n_+$, such that $S = S^{\frac{1}{2}}S^{\frac{1}{2}}$. Moreover, the following statements hold.*

(i) $\operatorname{tr}(\Gamma S) \geqq 0$.

(ii) If $\operatorname{tr}(\Gamma S) = 0$*, then* $\Gamma S = 0$ *and* $\operatorname{Im}(S) \subseteq \ker(\Gamma)$.

(iii) $\operatorname{Im}(S^{\frac{1}{2}}) = \operatorname{Im}(S)$.

Proof. For the properties of $S^{\frac{1}{2}}$, see [22, Theorem 7.2.6.]. Property *(i)* is a consequence of the cone $\mathcal{S}^n_+$ being selfdual, see also [22, Observation 7.1.8.(a), Corollary 7.1.5.]. For property *(ii)*, see [22, Corollary 7.1.5., Observation 7.1.6.]. Finally, for property *(iii)*, see [22, Theorem 7.2.6.(c)]. □

Lemma 2.2. *[45, Theorem 1.20] Let $A \in \mathcal{S}^p$, $C \in \mathcal{S}^q$, and $B \in \mathbb{R}^{p\times q}$. Further, let $A^\dagger \in \mathcal{S}^p$ and $C^\dagger \in \mathcal{S}^q$ be the Moore-Penrose generalized inverse of A and C, respectively. For $M := \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix} \in \mathcal{S}^{p+q}$, the following statements are equivalent.*

(i) $M \in \mathcal{S}^{p+q}_+$.

(ii) $A \in \mathcal{S}^p_+$, $\operatorname{Im}(B) \subseteq \operatorname{Im}(A)$, *and* $C - B^\top A^\dagger B \in \mathcal{S}^q_+$.

(iii) $C \in \mathcal{S}^q_+$, $\operatorname{Im}(B^\top) \subseteq \operatorname{Im}(C)$, *and* $A - BC^\dagger B^\top \in \mathcal{S}^p_+$.

We now turn to the multiobjective optimization framework and recall the basic concepts and relations that will be used throughout this work. A point $\bar{y} \in Y$ is called nondominated of (QMOP) if there exists no $y \in Y$ with $y \leq \bar{y}$. A point $\bar{x} \in X$ is called an efficient solution of (QMOP) if $f(\bar{x})$ is a nondominated point of (QMOP). We denote the sets of nondominated points of (QMOP) and efficient solutions of (QMOP) by Y_{nd} and X_e, respectively. The main goal in multiobjective optimization is to determine the sets X_e and Y_{nd}, where Y_{nd} is also known as the Pareto front in the literature. The set Y_{nd} consists of all image points for which no component can be improved without worsening at least one other component.

For theoretical studies, one often considers the so-called set of weakly nondominated points, which is a superset of Y_{nd}. A point $\bar{y} \in Y$ is called weakly nondominated of (QMOP) if there exists no $y \in Y$ such that $y < \bar{y}$. A point $\bar{x} \in X$ is called a weakly efficient solution of (QMOP) if $f(\bar{x})$ is a weakly nondominated point of (QMOP). The sets of weakly nondominated points of (QMOP) and of weakly efficient solutions of (QMOP) are denoted by Y_{wnd} and X_{we}, respectively. Note that for weakly nondominated points, an improvement in at least one objective may not necessarily involve a deterioration in another objective.

However, points in Y_{nd} may still involve unbounded trade-offs, which means that an improvement in one objective may require an arbitrarily large deterioration in others. In order to avoid such unbounded trade-offs, there is an even stronger notion, namely, that of properly nondominated points. Note that there are various definitions of this in the literature; for an overview, see, e.g., [18, Section 2.4]. We will use the concept in the sense of Geoffrion; see, e.g., [18, Definition 2.39]. A point $\bar{y} \in Y$ is called properly nondominated of (QMOP) if there exists $K > 0$ such that for all $y \in Y$ and all $i \in [m]$ with $y_i < \bar{y}_i$ there exists some $j \in [m]$ with $y_j > \bar{y}_j$ such that

$$\frac{\bar{y}_i - y_i}{y_j - \bar{y}_j} \leqq K.$$

A point $\bar{x} \in X$ is called a properly efficient solution of (QMOP) if $f(\bar{x})$ is a properly nondominated point of (QMOP). The sets of properly nondominated points of (QMOP) and properly efficient solutions of (QMOP) are denoted by Y_{pnd} and X_{pe}, respectively.

All these concepts of the various terms related to nondominance and efficiency can be found in [18, Chapter 2]. The above definitions result in the following inclusions:

$$Y_{pnd} \subseteq Y_{nd} \subseteq Y_{wnd}, \qquad X_{pe} \subseteq X_e \subseteq X_{we}.$$

Note that for $X = \mathbb{R}^n$ we write $\mathbb{R}^n_{pe}, \mathbb{R}^n_e$, and $\mathbb{R}^n_{we}$. The optimization problem

$$\min_{x \in X} \omega^\top f(x) = \sum_{i=1}^m \omega_i f_i(x) \tag{WS(ω)}$$

for some weight vector $\omega \in \mathbb{R}^m_{\geq}$ is referred to as a weighted sum scalarization of (QMOP). It is easy to see that the weight vectors can be further restricted to the standard simplex given by $\Delta := \{\omega \in \mathbb{R}^m_{\geq} \mid \sum_{i=1}^m \omega_i = 1\}$. Moreover, we define

$$\begin{aligned} \Delta_+ &:= \{\omega \in \Delta \mid \omega_i > 0 \text{ for all } i \in [m]\}, \\ X^{\text{supp}} &:= \bigcup_{\omega \in \Delta} \operatorname*{argmin}_{x \in X} \omega^\top f(x), \\ X^{\text{supp}}_+ &:= \bigcup_{\omega \in \Delta_+} \operatorname*{argmin}_{x \in X} \omega^\top f(x), \end{aligned}$$

where $\Delta_+ \subsetneq \Delta$ is the subset of the standard simplex with positive components. Note that for $X = \mathbb{R}^n$ we write $(\mathbb{R}^n)^{\text{supp}}$ and $(\mathbb{R}^n)^{\text{supp}}_+$. We refer to X^{supp} as the set of supported solutions and X^{supp}_+ as the set of positively supported solutions of (QMOP). Analogously, any element of the set X^{supp} is called a supported solution and any element of the set X^{supp}_+ is called a positively supported solution of (QMOP). This terminology is not uniformly used in the literature; see, for instance, [14, Chapter 8] and [27]. We use this terminology in order to reflect the corresponding classes of weight vectors directly: positively supported refers to positive weights, whereas supported refers to nonnegative weights. It is well known, see, for instance, [18, Proposition 3.9, Theorem 3.11], that the following inclusions hold:

$$X^{\text{supp}} \subseteq X_{we} \qquad \text{and} \qquad X^{\text{supp}}_+ \subseteq X_{pe}. \tag{1}$$

For the reverse implication, convexity plays an important role. We therefore distinguish between several notions of convexity as follows. We call (QMOP)

(a) convexlike if the upper image set $f(X) + \mathbb{R}^m_{\geqq}$ is convex.

Further, (QMOP) is called

(b) convex if X is a convex set and if for all $i \in [m]$ the functions f_i are convex.

If (QMOP) is convex, then $f(X) + \mathbb{R}^m_{\geqq}$ is convex. Hence, every convex (QMOP) is convexlike. Finally, we say that (QMOP) is

(c) hidden convex if it is convexlike but not convex.

The following standard result explains why weighted sum scalarizations are particularly useful in convexlike problems; see, e.g., [18, Theorem 3.5, Corollary 3.12, Theorem 3.13].

Proposition 2.3. *If* (QMOP) *is convexlike, then* $X_{we} = X^{\text{supp}}$ *and* $X_{pe} = X^{\text{supp}}_{+}$.

This means that, if (QMOP) is convexlike, the reverse inclusions of (1) also hold. Thus, it is of interest to find sufficient conditions for convexlikeness. This is in particular relevant in the quadratic biobjective setting: if $m = 2$ and either $X = \mathbb{R}^n$ or $X = \{x \in \mathbb{R}^n \mid Ax = b\}$ with $A \in \mathbb{R}^{r \times n}, b \in \mathbb{R}^r$, then (QMOP) is convexlike. We discuss these cases in Sections 4 and 5.

For more than two objectives, (QMOP) need not be convexlike. Indeed, [26, Example 2.1] provides an example with two quadratic and one linear objective on $X = \mathbb{R}^n$ for which the upper image is not convex. By adding further objective functions and projecting onto the first three components (since linear projections preserve convexity), one obtains corresponding examples for every $m \geqq 3$.

For (QMOP) with $m \geqq 3$, it is also possible that there are no supported solutions whereas the set of weakly efficient solutions is nonempty, i.e., $X^{\text{supp}} = \emptyset$ and $X_{we} \neq \emptyset$. This is illustrated in the forthcoming Example 3.15.

Even though convexlikeness has useful implications, it is still possible that the set of weakly efficient solutions X_{we} of (QMOP) is empty and that, for every $\omega \in \Delta$, the weighted sum scalarization (WS(ω)) has no minimal solution. The following example provides a convexlike (QMOP) with $X_{we} = \emptyset$.

Example 2.4. *[20, Following Remark 4.17.] Let*

$$\Gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \Gamma^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \beta^1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \beta^2 = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \text{ and } \alpha_1 = \alpha_2 = 0,$$

i.e., $f_1(x) = x_1^2 - x_2^2$, $f_2(x) = x_1$. *Then* $f(\mathbb{R}^2) + \mathbb{R}^2_{\geqq} = \mathbb{R}^2$, *and hence the upper image is convex, but the set of weakly efficient solutions* $\mathbb{R}^2_{we}$ *is empty. Also note that the image set* $f(\mathbb{R}^2) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \leqq y_2^2\} \subsetneq \mathbb{R}^2$ *is not convex. The image set* $f(\mathbb{R}^2)$ *is illustrated in blue in Figure 1.*

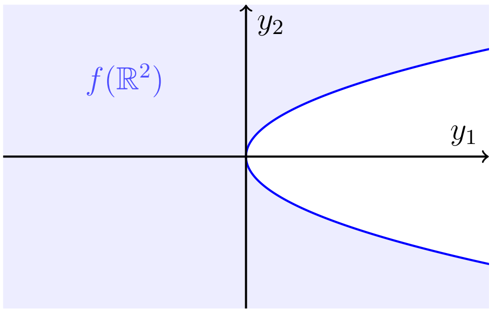

Figure 1: Image set $f(\mathbb{R}^2)$ of Example 2.4 in blue.

For the practical applicability of weighted sum scalarizations, it is therefore crucial to establish the existence of supported or positively supported solutions of (QMOP). In convexlike problems, the set of supported solutions X^{supp} coincides with the set of weakly efficient solutions X_{we}, and the set of positively supported solutions X_+^{supp} coincides with the set of properly efficient solutions X_{pe}; see Proposition 2.3. We therefore focus on existence results for supported and positively supported solutions of (QMOP).

3 Unconstrained Quadratic Multiobjective Optimization

In this section, we consider the unconstrained quadratic multiobjective optimization problem

$$\begin{aligned} &\text{minimize} \qquad f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix} := \begin{pmatrix} x^\top \Gamma^1 x + 2(\beta^1)^\top x + \alpha_1 \\ \vdots \\ x^\top \Gamma^m x + 2(\beta^m)^\top x + \alpha_m \end{pmatrix} \qquad \text{(QMOP-UC)} \\ &\text{subject to} \qquad x \in \mathbb{R}^n, \end{aligned}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$, $\Gamma^i \in \mathcal{S}^n$, $\beta^i \in \mathbb{R}^n$ for all $i \in [m]$, and $\alpha \in \mathbb{R}^m$. Note that the constants α_i, $i \in [m]$, can be omitted, as they do not influence the sets of weakly and properly efficient solutions. Still, we keep them because the reformulation used in Section 5 naturally leads to the introduction of constant terms α_i, $i \in [m]$.

In the following, we apply a weighted sum scalarization to (QMOP-UC) and examine whether there exists a minimal solution of the corresponding single-objective optimization problem. Any such minimal solution yields a supported solution of (QMOP-UC). In the convexlike setting, this immediately translates into existence statements for weakly efficient and properly efficient solutions via Proposition 2.3. In contrast, without convexlikeness, supported solutions may not exist even though weakly efficient solutions exist, which is illustrated in the forthcoming Example 3.15.

3.1 Existence of Supported Solutions

Applying a weighted sum scalarization to (QMOP-UC) reduces the multiobjective optimization problem to an unconstrained single-objective quadratic optimization problem.

To this end, for $\omega \in \Delta$, we introduce the following notation:

$$\Gamma_\omega := \sum_{i=1}^{m} \omega_i \Gamma^i \in \mathcal{S}^n, \qquad \beta_\omega := \sum_{i=1}^{m} \omega_i \beta^i \in \mathbb{R}^n, \qquad \alpha_\omega := \sum_{i=1}^{m} \omega_i \alpha_i \in \mathbb{R} \tag{2}$$

as well as

$$\Psi_\omega : \mathbb{R}^n \to \mathbb{R}, \quad \Psi_\omega(x) := x^\top \Gamma_\omega x + 2(\beta_\omega)^\top x + \alpha_\omega. \tag{3}$$

Based on that, the weighted sum scalarization of (QMOP-UC) can be written as

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \;\; \Psi_\omega(x). \tag{WS-QMOP(ω)}$$

The following optimality conditions in convex single-objective quadratic optimization are well-known, see, for instance, [7, Example 4.5], [6, Section Quadratic Cost Functions], and [34, Theorem 27.1, Theorem 27.2].

Lemma 3.1. *Let $\Gamma \in \mathcal{S}^n, \beta \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$. For a quadratic function $\Psi : \mathbb{R}^n \to \mathbb{R}$ with $\Psi(x) = x^\top \Gamma x + 2\beta^\top x + \alpha$, the following properties are equivalent:*

(i) There exists $\mu \in \mathbb{R}$ such that $\Psi(x) \geqq \mu$ for all $x \in \mathbb{R}^n$.

(ii) $\Gamma \in \mathcal{S}^n_+$ and $\beta \in \mathrm{Im}(\Gamma)$.

(iii) The set $\operatorname{argmin}_{x \in \mathbb{R}^n} \Psi(x)$ is nonempty.

Furthermore, in case $\Gamma \in \mathcal{S}^n_{++}$ we have $\beta \in \mathrm{Im}(\Gamma)$ for any $\beta \in \mathbb{R}^n$ and $\operatorname{argmin}_{x \in \mathbb{R}^n} \Psi(x) = \{-\Gamma^{-1}\beta\}$ holds.

By applying Lemma 3.1 to the function Ψ_ω in (3), we obtain the following result.

Corollary 3.2. *Let $\omega \in \Delta$. Then,* (WS-QMOP(ω)) *admits a minimal solution if and only if $\Gamma_\omega \in \mathcal{S}^n_+$ and $\beta_\omega \in \mathrm{Im}(\Gamma_\omega)$.*

Since $\mathrm{Im}(\Gamma_\omega) = \ker(\Gamma_\omega)^\perp$ for $\omega \in \Delta$, the image condition $\beta_\omega \in \mathrm{Im}(\Gamma_\omega)$ is satisfied if and only if $\beta_\omega^\top x = 0$ for all $x \in \ker(\Gamma_\omega)$. This gives an equivalent formulation for the image condition. We are interested in conditions ensuring that the set of supported solutions of (QMOP-UC), namely $\bigcup_{\omega \in \Delta} \operatorname{argmin}_{x \in \mathbb{R}^n} \Psi_\omega(x)$, is nonempty. To this end, we present a sufficient condition for $\beta_\omega \in \mathrm{Im}(\Gamma_\omega)$ to hold for each $\omega \in \Delta$. The condition is easy to verify and only requires checking the feasibility of a single linear system.

Lemma 3.3. *If there exists a vector $\tilde{x} \in \mathbb{R}^n$ such that $\beta^i = \Gamma^i \tilde{x}$ for all $i \in [m]$, then $\beta_\omega \in \mathrm{Im}(\Gamma_\omega)$ for each $\omega \in \Delta$.*

Proof. Let $\omega \in \Delta$ and assume that there exists a vector $\tilde{x} \in \mathbb{R}^n$ such that $\beta^i = \Gamma^i \tilde{x}$ for all $i \in [m]$. Then $\Gamma_\omega \tilde{x} = \sum_{i=1}^m \omega_i \Gamma^i \tilde{x} = \sum_{i=1}^m \omega_i \beta^i = \beta_\omega \in \mathrm{Im}(\Gamma_\omega)$. □

The condition in Lemma 3.3 can be efficiently checked by solving the following linear system

$$\begin{pmatrix} \Gamma^1 \\ \vdots \\ \Gamma^m \end{pmatrix} x = \begin{pmatrix} \beta^1 \\ \vdots \\ \beta^m \end{pmatrix} \in \mathbb{R}^{mn}.$$

Note that the condition $\beta_\omega \in \mathrm{Im}(\Gamma_\omega)$ for all $\omega \in \Delta$ implies in particular that $\beta^i \in \mathrm{Im}(\Gamma^i)$ for each $i \in [m]$, by choosing ω as the i-th unit vector. Thus, the latter is a necessary condition. However, it is not sufficient in general, as the following example shows.

Example 3.4. *For $m = 2$, consider $\Gamma^1 = I$, $\Gamma^2 = -I$, $\beta^1 = (1, \dots, 1)^\top$, $\beta^2 = -2\beta^1$, where I denotes the identity matrix. Then $\beta^i \in \operatorname{Im}(\Gamma^i)$ holds for each $i \in [2]$ and with $\omega = (\frac{1}{2}, \frac{1}{2})^\top$ we have $\beta_\omega \notin \operatorname{Im}(\Gamma_\omega)$.*

Hence, the condition $\beta^i \in \operatorname{Im}(\Gamma^i)$, $i \in [m]$, is in general not sufficient to ensure that $\beta_\omega \in \operatorname{Im}(\Gamma_\omega)$ for all $\omega \in \Delta$. On the other hand, the sufficient condition from Lemma 3.3 is not necessary in general, as the following example shows.

Example 3.5. *For $m = 2$, consider $\Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\Gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\beta^1 = (1, 1)^\top$, $\beta^2 = (1, -2)^\top$. Then it is easy to see that there exists no vector $\tilde{x} \in \mathbb{R}^2$ such that $\beta^i = \Gamma^i \tilde{x}$ for all $i \in [2]$. However, for any $\omega = (1 - \lambda, \lambda)^\top \in \Delta$, where $\lambda \in [0, 1]$, consider $\Gamma_\omega = (1 - \lambda)\Gamma^1 + \lambda\Gamma^2$. Then $\det(\Gamma_\omega) = -\lambda^2 - (1 - \lambda)^2 < 0$ holds for each $\lambda \in [0, 1]$, which implies that $\beta_\omega \in \operatorname{Im}(\Gamma_\omega)$ for each $\omega \in \Delta$.*

By Corollary 3.2, a necessary condition for (WS-QMOP(ω)) to admit a minimal solution for some $\omega \in \Delta$ is the existence of a convex combination of the matrices $\Gamma^1, \dots, \Gamma^m$, which is contained in the cone of positive semidefinite matrices $\mathcal{S}^n_+$. Therefore, we define the sets

$$\Omega := \left\{ \omega \in \Delta \mid \operatorname*{argmin}_{x \in \mathbb{R}^n} \Psi_\omega(x) \neq \emptyset \right\}, \tag{4}$$

$$\Omega_+ := \left\{ \omega \in \Delta_+ \mid \operatorname*{argmin}_{x \in \mathbb{R}^n} \Psi_\omega(x) \neq \emptyset \right\}, \tag{5}$$

$$\Lambda := \left\{ \omega \in \Delta \mid \Gamma_\omega \in \mathcal{S}^n_+ \right\}, \tag{6}$$

$$\Lambda_+ := \left\{ \omega \in \Delta \mid \Gamma_\omega \in \mathcal{S}^n_{++} \right\}. \tag{7}$$

To analyze the geometry of the sets (4) - (7), we establish their convexity and provide their relations in the following lemma.

Lemma 3.6. *The following properties hold:*

(i) *The sets Ω, Ω_+, Λ, and Λ_+ are convex.*

(ii) $\Omega_+ \subseteq \Omega$ *and* $\Lambda_+ \subseteq \Lambda$.

(iii) $\Omega \subseteq \Lambda$.

(iv) $\Lambda_+ \subseteq \Omega$.

(v) $\Lambda_+ \cap \Delta_+ \subseteq \Omega_+$.

Proof. Property *(ii)* is obvious by definition. By Corollary 3.2 we obtain *(iii)* and additionally by Lemma 3.1 we obtain *(iv)* and *(v)*.

We now prove *(i)*. Suppose that $\Omega \neq \emptyset$. Let $\omega^1 \in \Omega$, $\omega^2 \in \Omega$, and $\lambda \in [0, 1]$. Further, set $\omega^\lambda := (1 - \lambda)\omega^1 + \lambda\omega^2$.

Since Δ is convex, we have $\omega^\lambda \in \Delta$. By (2), we have

$$\Gamma_{\omega^\lambda} = \sum_{i=1}^m \omega_i^\lambda \Gamma^i = (1-\lambda)\Gamma_{\omega^1} + \lambda\Gamma_{\omega^2}, \tag{8}$$

$$\beta_{\omega^\lambda} = \sum_{i=1}^m \omega_i^\lambda \beta^i = (1-\lambda)\beta_{\omega^1} + \lambda\beta_{\omega^2}, \tag{9}$$

$$\alpha_{\omega^\lambda} = \sum_{i=1}^m \omega_i^\lambda \alpha_i = (1-\lambda)\alpha_{\omega^1} + \lambda\alpha_{\omega^2}. \tag{10}$$

Since $\omega^1, \omega^2 \in \Omega$, it follows from Lemma 3.1 that Ψ_{ω^j} is bounded from below for all $j \in [2]$, i.e., there exist $\mu_1, \mu_2 \in \mathbb{R}$, such that for any $x \in \mathbb{R}^n$ we have

$$x^\top \Gamma_{\omega^1} x + 2(\beta_{\omega^1})^\top x + \alpha_{\omega^1} \geqq \mu_1, \qquad x^\top \Gamma_{\omega^2} x + 2(\beta_{\omega^2})^\top x + \alpha_{\omega^2} \geqq \mu_2.$$

Multiplying the first inequality by $1-\lambda \geqq 0$, the second by $\lambda \geqq 0$, and adding the two inequalities, it follows from (8), (9), and (10) that for any $x \in \mathbb{R}^n$ we have $\Psi_{\omega^\lambda}(x) \geqq (1-\lambda)\mu_1 + \lambda\mu_2$. By Lemma 3.1, we conclude that $\operatorname{argmin}_{x\in\mathbb{R}^n} \Psi_{\omega^\lambda}(x) \neq \emptyset$. Therefore, $\omega^\lambda \in \Omega$, which implies that Ω is a convex set. The proof of the convexity of Ω_+ follows from the same arguments and the convexity of Δ_+.

Now suppose that $\Lambda \neq \emptyset$. Let $\omega^1, \omega^2 \in \Lambda$, and $\lambda \in [0,1]$. Further, let $\omega^\lambda \in \Delta$ be defined as before. Since $\omega^1, \omega^2 \in \Lambda$, we have $\Gamma_{\omega^1}, \Gamma_{\omega^2} \in \mathcal{S}_+^n$ and as the cone of positive semidefinite matrices $\mathcal{S}_+^n$ is convex, we obtain $(1-\lambda)\Gamma_{\omega^1} + \lambda\Gamma_{\omega^2} \in \mathcal{S}_+^n$ and by (8), also $\Gamma_{\omega^\lambda} \in \mathcal{S}_+^n$ and thus $\omega^\lambda \in \Lambda$, which implies that Λ is convex. The proof of the convexity of Λ_+ follows from the same arguments and the convexity of $\mathcal{S}_{++}^n$. $\square$

To illustrate the sets Ω, Ω_+, Λ, and Λ_+, we give the following example. Note that for $m = 2$, we can identify Δ with the interval $[0,1]$ by writing every $\omega \in \Delta$ in the form $\omega = (\omega_1, \omega_2)^\top = (1-\lambda, \lambda)^\top$ for $\lambda \in [0,1]$. With this identification, we may view Ω, Ω_+, Λ, and Λ_+ as subsets of $[0,1]$.

Example 3.7. *Let* $\Gamma^1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \in \mathcal{S}_{++}^2$ *and* $\Gamma^2 = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$ *with eigenvalues* 1 *and* 3, *and* -2 *and* 2, *respectively. Then for* $\omega_2 \in [0,1]$ *we define* $\Gamma(\omega_2) := (1-\omega_2)\Gamma^1 + \omega_2\Gamma^2$, *which has the eigenvalues*

$$\lambda_{\min}(\omega_2) = 2(1-\omega_2) - \sqrt{5\omega_2^2 - 2\omega_2 + 1}, \quad \lambda_{\max}(\omega_2) = 2(1-\omega_2) + \sqrt{5\omega_2^2 - 2\omega_2 + 1}.$$

The eigenvalues are illustrated in Figure 2. The function $\lambda_{\min}(\omega_2)$ *changes its sign at* $\tilde{\omega}_2 := 2\sqrt{3} - 3 \approx 0.4641$ *and thus* $\Gamma(\omega_2) \in \mathcal{S}_+^2$ *if and only if* $\omega_2 \in [0, \tilde{\omega}_2] = \Lambda$.

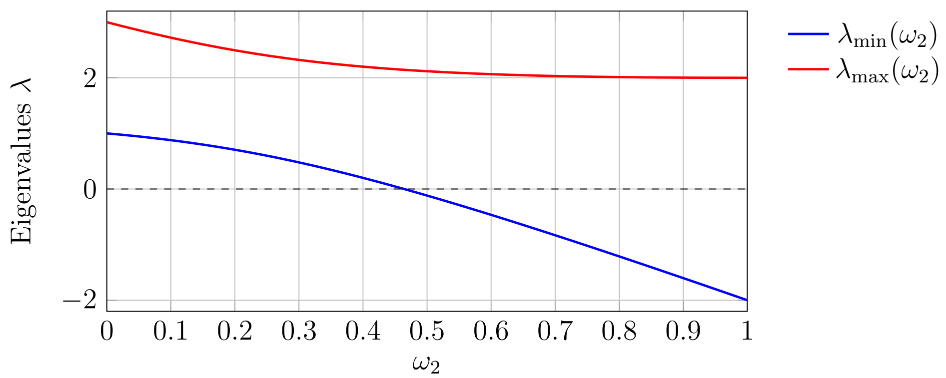

Figure 2: Spectrum of $\Gamma(\omega_2)$ of Example 3.7

In the following, we illustrate that the problem (WS-QMOP(ω)) *with* $\tilde{\omega} := (1-\tilde{\omega}_2, \tilde{\omega}_2)^\top$ *can have a different solution behavior depending on the further data of the problem.*

(a) *First, let* $\alpha = 0$ *and* $\beta^1 = \beta^2 = (1,0)^\top$*, i.e., assume an inhomogeneous case. Then we have* $\frac{1}{2}(1-\tilde{\omega}_2)^2 = 2 - 4\tilde{\omega}_2$*. This leads to the rank-*1 *representation*

$$\Gamma(\tilde{\omega}_2) = \tfrac{1}{2}\begin{pmatrix}1-\tilde{\omega}_2\\ 2\end{pmatrix}\begin{pmatrix}1-\tilde{\omega}_2 & 2\end{pmatrix} = \begin{pmatrix}2-4\tilde{\omega}_2 & 1-\tilde{\omega}_2\\ 1-\tilde{\omega}_2 & 2\end{pmatrix}$$

and thus, $\operatorname{Im}(\Gamma(\tilde{\omega}_2)) = \operatorname{span}\left\{(1-\tilde{\omega}_2, 2)^\top\right\}$*. We obtain* $\beta(\tilde{\omega}_2) := (1-\tilde{\omega}_2)\beta^1 + \tilde{\omega}_2\beta^2 = \beta^2 \notin \operatorname{Im}(\Gamma(\tilde{\omega}_2))$ *and hence,* (WS-QMOP(ω)) *has no minimal solution for the weight vector* $\tilde{\omega} := (1-\tilde{\omega}_2, \tilde{\omega}_2)^\top$*; see Corollary 3.2. Therefore, we obtain*

$$\Lambda_+ = \Omega = [0, \tilde{\omega}_2), \qquad \Omega_+ = (0, \tilde{\omega}_2) \subsetneq \Lambda_+,$$

which in particular shows that the inclusions in Lemma 3.6 (ii) - (iii) can be strict.

Note that in this case, by [33, Theorem 2.2], the image set $f(\mathbb{R}^2)$ *is closed and convex. In contrast to the homogeneous case in (b), it is neither a cone nor polyhedral; see Figure 3 (a).*

(b) *Second, let* $\alpha = 0$ *and* $\beta^1 = \beta^2 = 0$*. By [16, Theorem 1, Theorem 2], the image set* $f(\mathbb{R}^2)$ *is a closed convex cone. Moreover, we have* $\beta(\tilde{\omega}_2) = 0 \in \operatorname{Im}(\Gamma(\tilde{\omega}_2))$ *and by Corollary 3.2, the problem* (WS-QMOP(ω)) *has a minimal solution for the weight vector* $\tilde{\omega}$*. In this case, we have*

$$\Lambda = \Omega = [0, \tilde{\omega}_2], \qquad \Omega_+ = (0, \tilde{\omega}_2] \subsetneq \Lambda.$$

The weight vectors that lead to a minimal solution of (WS-QMOP(ω)) *therefore differ in both settings - depending on* β^1 *and* β^2 *- by only a single vector. This is illustrated in Figure 3. Note that in the homogeneous case (b), the line* $\{y \in \mathbb{R}^2 \mid \tilde{\omega}^\top y = \min_{z \in f(\mathbb{R}^2)} \tilde{\omega}^\top z\}$ *is a supporting hyperplane of the image set* $f(\mathbb{R}^2)$*.*

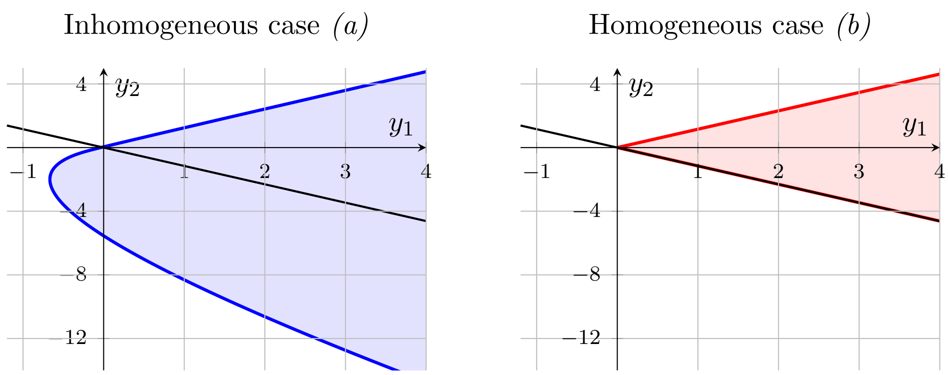

Figure 3: Image sets of Example 3.7 for the cases *(a)* (left) and *(b)* (right). In case *(b)*, the black line is the supporting hyperplane corresponding to the weight $\tilde{\omega}$. In case *(a)*, the scalarization with weight vector $\tilde{\omega}$ is unbounded from below; hence no supporting hyperplane parallel to the black line exists. The black line is included only for comparison with the homogeneous case.

We have thus seen that for $m = 2$ all four sets Ω, Ω_+, Λ, and Λ_+ can contain a nonempty interval. Now we give two more examples that show that these sets can also be empty or contain only one element.

Example 3.8. *(a) Let* $\Gamma^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\Gamma^2 = -\Gamma^1$, *and let* $\alpha = 0$. *Then, for* $\omega_2 \in [0,1]$, *we have*

$$\Gamma(\omega_2) := (1-\omega_2)\Gamma^1 + \omega_2\Gamma^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1-2\omega_2 \end{pmatrix},$$

which is positive semidefinite if and only if $\omega_2 \in \left[0, \frac{1}{2}\right] = \Lambda$. *In this case,* $\Gamma(\omega_2) \in \partial\mathcal{S}^2_+$ *and thus,* $\Lambda_+ = \emptyset$. *For* $\omega_2 \in \left[0, \frac{1}{2}\right)$ *we also have* $\mathrm{Im}(\Gamma(\omega_2)) = \mathrm{span}\left\{(0,1)^\top\right\}$ *and for* $\omega_2 = \frac{1}{2}$ *we obtain* $\mathrm{Im}\left(\Gamma\left(\frac{1}{2}\right)\right) = \{0\}$.

(i) Considering $\beta^1 = \beta^2 = (1,0)^\top$ *and* $\beta(\omega_2) := (1-\omega_2)\beta^1 + \omega_2\beta^2$ *for* $\omega_2 \in \left[0, \frac{1}{2}\right] = \Lambda$, *this yields* $\beta(\omega_2) = \beta^2 \notin \mathrm{Im}(\Gamma(\omega_2))$. *Therefore,* (WS-QMOP($\omega$)) *has no minimal solution and* $\Omega = \emptyset$.

(ii) On the other hand, for $\beta^1 = (0,1)^\top$, $\beta^2 = (1,0)^\top$, *we have* $\beta(\omega_2) \in \mathrm{Im}(\Gamma(\omega_2))$ *if and only if* $\omega_2 = 0$. *Thus, in this case we obtain* $\Omega = \{0\}$, *which is a singleton and* $\Omega_+ = \emptyset$. *This in particular shows that the inclusion in Lemma 3.6 (iv) can be strict.*

(iii) Finally, for $\beta^1 = (0,1)^\top$ *and* $\beta^2 = -\beta^1$, *we obtain* $\beta(\omega_2) \in \mathrm{Im}(\Gamma(\omega_2))$ *for any* $\omega_2 \in \left[0, \frac{1}{2}\right]$. *Thus,* $\Lambda = \Omega = \left[0, \frac{1}{2}\right]$ *and* $\Omega_+ = \left(0, \frac{1}{2}\right]$, *where both sets are intervals. This in particular shows that the inclusion in Lemma 3.6 (v) can be strict.*

The image sets of the corresponding vector-valued functions of (i), (ii), and (iii) are given by $f(\mathbb{R}^2) = \{(y_1,y_2)^\top \in \mathbb{R}^2 \mid y_1 \geqq y_2\}$, $f(\mathbb{R}^2) = \{(y_1,y_2)^\top \in \mathbb{R}^2 \mid y_1 \geqq -1\}$, *and* $f(\mathbb{R}^2) = \{(t,-t)^\top \mid t \geqq -1\}$, *respectively. This is illustrated in Figure 4.*

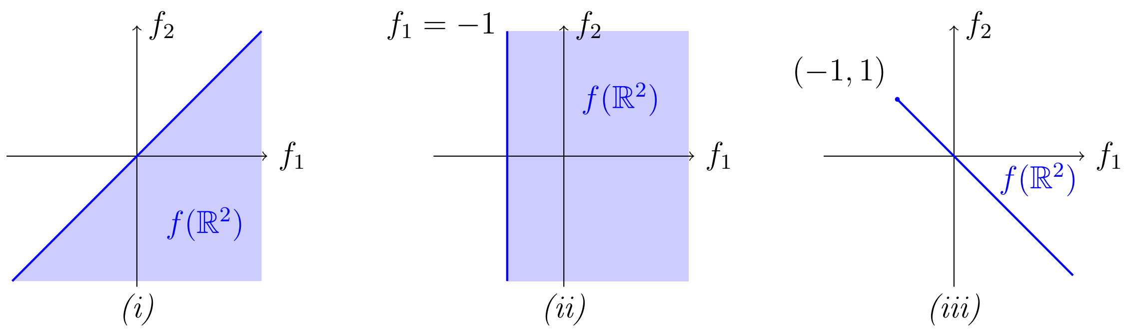

Figure 4: Image sets corresponding to the three choices of β^1 and β^2 in Example 3.8 *(a)* in blue.

(b) Let $\Gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\Gamma^2 = -\Gamma^1$. *Then, for* $\omega_2 \in [0,1]$, *we have*

$$\Gamma(\omega_2) := (1-\omega_2)\Gamma^1 + \omega_2\Gamma^2 = \begin{pmatrix} 1-2\omega_2 & 0 \\ 0 & -1+2\omega_2 \end{pmatrix},$$

which is positive semidefinite if and only if $\omega_2 = \frac{1}{2}$. With this we have $\Gamma\left(\frac{1}{2}\right) = 0$, $\operatorname{Im}\left(\Gamma\left(\frac{1}{2}\right)\right) = \{0\}$ and thus, $\Lambda_+ = \emptyset$. In addition, for $\beta(\omega_2) := (1-\omega_2)\beta^1 + \omega_2\beta^2$, we have $\beta\left(\frac{1}{2}\right) \in \operatorname{Im}\left(\Gamma\left(\frac{1}{2}\right)\right) = \{0\}$ if and only if $\beta^1 = -\beta^2$. Now assume $\alpha = 0$. Then for $\beta^1 = (1,0)^\top$, $\beta^2 = -\beta^1$, we obtain $\Lambda = \Omega = \Omega_+ = \left\{\frac{1}{2}\right\}$ and even Ω_+ is a singleton. The image set of the corresponding function is $f(\mathbb{R}^2) = \{(t,-t)^\top \mid t \in \mathbb{R}\}$.

Finally, the next example shows that $\Omega = \emptyset$ may already be forced by the quadratic part alone. More precisely, we state a case in which $\Lambda = \emptyset$ holds, so that $\Omega = \emptyset$ holds independently of the linear and constant terms.

Example 3.9. *Let $\Gamma^1 = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$, $\Gamma^2 = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$. For $\hat{x} = (1,-1)^\top$ we have $\hat{x}^\top\Gamma^1\hat{x} = -3 < 0$ and $\hat{x}^\top\Gamma^2\hat{x} = -3 < 0$. This implies $\Gamma_\omega \notin \mathcal{S}^2_+$ for all $\omega \in \Delta$ and thus $\Lambda = \emptyset$. Thus, regardless of the choices of β^1, β^2, α, Lemma 3.6 (iii) yields $\Omega = \emptyset$.*

In particular, these examples show that each of the inclusions in Lemma 3.6 *(ii)* - *(v)* can be strict.

3.2 Positive Semidefinite Convex Combinations of Matrices

Motivated by Corollary 3.2, we study conditions under which there exists a weight vector $\omega \in \Delta$ such that the convex combination $\Gamma_\omega = \sum_{i=1}^m \omega_i\Gamma^i$ is positive semidefinite, i.e., $\Lambda \neq \emptyset$. This property is necessary for the weighted sum scalarization (WS-QMOP(ω)) to admit a minimal solution for some $\omega \in \Delta$. Moreover, if Γ_ω is positive definite for some $\omega \in \Delta$, the scalarized problem has a unique minimal solution by Lemma 3.1 and therefore $\Omega \neq \emptyset$.

Linear combinations of matrices, i.e., $\sum_{i=1}^m \nu_i\Gamma^i$ with $\nu \in \mathbb{R}^m \setminus \{0\}$ and their definiteness properties have been studied extensively; see, e.g., [5, 41]. In particular, necessary and sufficient conditions for the existence of positive (semi-)definite linear combinations with real coefficients are given in [5, Theorem 2, Theorem 3] and [17, Theorem 1]. In contrast, we restrict the coefficients to the standard simplex Δ, i.e., we focus on positive (semi-)definite convex combinations. Within this setting, Theorem 3.12 provides an analogous characterization and complements the results of [5, 17].

A convenient way to identify whether a positive (semi-)definite convex combination of $\Gamma^1, \dots, \Gamma^m$ exists is to solve the semidefinite program (SDP)

$$\begin{aligned} \underset{\varepsilon,\ \omega}{\text{maximize}} \quad & \varepsilon \\ \text{subject to} \quad & \textstyle\sum_{i=1}^m \omega_i\Gamma^i - \varepsilon I \in \mathcal{S}^n_+, \\ & \varepsilon \in \mathbb{R},\ \omega \in \Delta. \end{aligned} \tag{P}$$

For (P), the following relations apply to the existence of positive (semi-)definite convex combinations of $\Gamma^1, \dots, \Gamma^m$.

Lemma 3.10. *Problem* (P) *admits an optimal solution $(\varepsilon^*, \omega^*)$ and*

$$\varepsilon^* = \lambda_{\min}\left(\sum_{i=1}^m \omega_i^*\Gamma^i\right) = \max_{\omega\in\Delta} \lambda_{\min}\left(\sum_{i=1}^m \omega_i\Gamma^i\right), \tag{11}$$

with $\lambda_{\min}(\cdot)$ denoting the minimal eigenvalue. Moreover, the following assertions are true.

(i) $\Lambda_+ \neq \emptyset$ *if and only if* $\varepsilon^* > 0$.

(ii) $\Lambda \neq \emptyset$ *and* $\Lambda_+ = \emptyset$ *if and only if* $\varepsilon^* = 0$.

(iii) $\Lambda = \emptyset$ *if and only if* $\varepsilon^* < 0$.

Proof. First, note that the constraint $\sum_{i=1}^m \omega_i \Gamma^i - \varepsilon I \in \mathcal{S}^n_+$ in (P) is equivalent to $\lambda_{\min}\left(\sum_{i=1}^m \omega_i \Gamma^i - \varepsilon I\right) \geqq 0$ and thus to $\lambda_{\min}\left(\sum_{i=1}^m \omega_i \Gamma^i\right) \geqq \varepsilon$. Hence, (P) is equivalent to $\max_{\omega \in \Delta} \lambda_{\min}\left(\sum_{i=1}^m \omega_i \Gamma^i\right)$. Since Δ is compact and the mapping $\omega \mapsto \lambda_{\min}\left(\sum_{i=1}^m \omega_i \Gamma^i\right)$ is continuous, Weierstrass' theorem implies that the maximum is attained. The assertions *(i)* - *(iii)* now follow immediately from the definitions of Λ and Λ_+. $\square$

Consequently, solving (P) provides an immediate certificate for the existence of positive (semi-)definite convex combinations. If $\varepsilon^* < 0$, then no such combination exists, and by Corollary 3.2, the weighted sum scalarization (WS-QMOP(ω)) has no minimal solution for any $\omega \in \Delta$. Note that for $m = 1$, we have $\varepsilon^* = \lambda_{\min}(\Gamma^1)$. Moreover, $\varepsilon^* = 0$ implies that every positive semidefinite convex combination lies on the boundary of $\mathcal{S}^n_+$. We next study the dual of (P) to obtain further structural insights:

$$\begin{aligned} \underset{\lambda,\, S}{\text{minimize}} \quad & \lambda \\ \text{subject to} \quad & \operatorname{tr}(S) = 1, \\ & \operatorname{tr}(\Gamma^i S) \leqq \lambda, \quad \forall\, i \in [m], \\ & \lambda \in \mathbb{R},\ S \in \mathcal{S}^n_+. \end{aligned} \tag{D}$$

We say that the Slater-Condition for (P) is satisfied if there exist $\bar{\omega} \in \Delta_+$ and $\bar{\varepsilon} \in \mathbb{R}$ such that $\sum_{i=1}^m \bar{\omega}_i \Gamma^i - \bar{\varepsilon} I \in \mathcal{S}^n_{++}$, i.e., (P) admits a strictly feasible point. Analogously, we say that the Slater-Condition for (D) is satisfied if there exist $\bar{\lambda} \in \mathbb{R}$ and $\bar{S} \in \mathcal{S}^n_{++}$ such that $\operatorname{tr}(\bar{S}) = 1$ and $\operatorname{tr}(\Gamma^i \bar{S}) < \bar{\lambda}$ for all $i \in [m]$, i.e., (D) admits a strictly feasible point. For more information on the Slater-Condition, see, e.g., [7, Section 5.2.3, Section 5.9], [4, Corollary 2.4.1], [42, Theorem 3.1]. Considering problems (P) and (D), we have the following relations.

Lemma 3.11. *Let* ε^* *be the optimal value of* (P)*. Then, the following properties are true.*

(i) Problem (D) *is the Lagrangian dual of* (P)*.*

(ii) The Slater-Condition for both problems (P) *and* (D) *is satisfied.*

(iii) Problem (D) *admits an optimal solution and its optimal value* λ^* *coincides with* ε^**.*

Proof. First, we prove *(i)*. An equivalent formulation of the maximization problem (P), up to a change in the sign of the optimal value, is

$$\begin{aligned} \underset{\varepsilon,\, \omega}{\text{minimize}} \quad & -\varepsilon \\ \text{subject to} \quad & \varepsilon I - \textstyle\sum_{i=1}^m \omega_i \Gamma^i \in -\mathcal{S}^n_+, \\ & \varepsilon \in \mathbb{R},\ -\omega \leqq 0,\ e^\top \omega - 1 = 0, \end{aligned} \tag{$\tilde{P}$}$$

where $e = (1, \dots, 1)^\top$ denotes the all ones vector. The Lagrange function of the minimization problem $(\tilde{P})$ reads as

$$\mathcal{L}(\varepsilon, \omega, S, \lambda, u) := -\varepsilon + \operatorname{tr}\left(\left(\varepsilon I - \sum_{i=1}^{m} \omega_i \Gamma^i\right) S\right) + \lambda(e^\top \omega - 1) - u^\top \omega,$$

where $S \in \mathcal{S}^n_+$, $\lambda \in \mathbb{R}$, and $u \in \mathbb{R}^m_{\geqq}$. Using the linearity of the trace and rearranging the terms yields

$$\mathcal{L}(\varepsilon, \omega, S, \lambda, u) = -\lambda + \varepsilon(\operatorname{tr}(S) - 1) + \sum_{i=1}^{m} \omega_i \left(\lambda - \operatorname{tr}(\Gamma^i S) - u_i\right).$$

For fixed (S, λ, u), the Lagrangian is affine in (ε, ω). Hence, $\inf_{\varepsilon \in \mathbb{R},\ \omega \in \mathbb{R}^m} \mathcal{L}(\varepsilon, \omega, S, \lambda, u)$ is finite if and only if the coefficients of ε and of each ω_i, $i \in [m]$, vanish, i.e.,

$$\operatorname{tr}(S) - 1 = 0 \quad \text{and} \quad \lambda - \operatorname{tr}(\Gamma^i S) - u_i = 0 \quad \text{for all} \quad i \in [m].$$

Hence, the dual problem of $(\tilde{P})$ and thus of (P) reads as

$$\begin{aligned} &\underset{S \in \mathcal{S}^n_+,\ \lambda \in \mathbb{R}}{\text{maximize}} && -\lambda \\ &\text{subject to} && \operatorname{tr}(S) = 1, \quad \operatorname{tr}(\Gamma^i S) \leqq \lambda, \qquad \forall i \in [m], \end{aligned}$$

which can be transformed to (D).

We now prove *(ii)*. Let $\bar{\omega} \in \Delta_+$ and let $\lambda_{\min}\left(\sum_{i=1}^m \bar{\omega}_i \Gamma^i\right)$ be the smallest eigenvalue of $\sum_{i=1}^m \bar{\omega}_i \Gamma^i$. Then we can choose $\bar{\varepsilon} < \lambda_{\min}\left(\sum_{i=1}^m \bar{\omega}_i \Gamma^i\right)$, which implies $\sum_{i=1}^m \bar{\omega}_i \Gamma^i - \bar{\varepsilon} I \in \mathcal{S}^n_{++}$. Thus, the Slater-Condition is satisfied for (P). Also, with the choices $\bar{S} := \frac{1}{n} I \in \mathcal{S}^n_{++}$ and $\bar{\lambda} > \max_{i \in [m]} \operatorname{tr}(\Gamma^i \bar{S}) = \frac{1}{n} \max_{i \in [m]} \operatorname{tr}(\Gamma^i)$, we have $\operatorname{tr}(\bar{S}) = 1$ and thus the Slater-Condition is satisfied for (D).

Finally, since both problems (P) and (D) satisfy the Slater-Condition, the duality gap is zero, and both problems have the same optimal value. This proves *(iii)*. □

Note that problem (D) can be formulated as the min-max problem

$$\min_{S \in \mathcal{S}^n_+,\, \operatorname{tr}(S) = 1} \ \max_{i \in [m]} \ \operatorname{tr}(\Gamma^i S).$$

By Lemma 3.11 and (11), we also have $\lambda^* = \varepsilon^* = \max_{\omega \in \Delta} \lambda_{\min}(\Gamma_\omega)$. Hence, (P) can also be seen as a max-min-eigenvalue problem; see also Example 3.7. Combining Lemma 3.11 and Lemma 3.10 with the dual problem (D) then gives us the following main result.

Theorem 3.12. *Let Λ and Λ_+ be defined as in* (6) *and* (7)*, respectively. Then the following properties are true.*

(i) $\Lambda_+ \neq \emptyset$ *if and only if*

$$\bigcap_{i \in [m]} \{S \in \mathcal{S}^n_+ \setminus \{0\} \mid \operatorname{tr}(\Gamma^i S) \leqq 0\} = \emptyset. \tag{12}$$

(ii) $\Lambda \neq \emptyset$ *and* $\Lambda_+ = \emptyset$ *if and only if*

$$\bigcap_{i\in[m]} \{S \in \mathcal{S}^n_+ \setminus \{0\} \mid \mathrm{tr}(\Gamma^i S) \leqq 0\} \neq \emptyset \qquad \wedge$$
$$\bigcap_{i\in[m]} \{S \in \mathcal{S}^n_+ \setminus \{0\} \mid \mathrm{tr}(\Gamma^i S) < 0\} = \emptyset.$$

(iii) $\Lambda = \emptyset$ *if and only if*

$$\bigcap_{i\in[m]} \{S \in \mathcal{S}^n_+ \setminus \{0\} \mid \mathrm{tr}(\Gamma^i S) < 0\} \neq \emptyset.$$

Proof. For the following proof, we consider (D) and its minimal value λ^*. Note that by Lemma 3.11 we have $\lambda^* = \varepsilon^*$ with ε^* the optimal value of (P). We make use of Lemma 3.10 in the following.

We now prove *(i)*. We have $\varepsilon^* = \lambda^* > 0$ if and only if there is no $(\lambda, S) \in \mathbb{R} \times \mathcal{S}^n_+$ with $\lambda \leqq 0$ feasible for (D). This is equivalent to

$$\bigcap_{i\in[m]} \{S \in \mathcal{S}^n_+ \mid \mathrm{tr}(S) = 1,\ \mathrm{tr}(\Gamma^i S) \leqq 0\} = \emptyset. \tag{13}$$

Note that for any $S \in \mathcal{S}^n_+ \setminus \{0\}$ we have $\mathrm{tr}(S) > 0$ and therefore $\hat{S} := \frac{1}{\mathrm{tr}(S)} S$ satisfies $\mathrm{tr}(\hat{S}) = 1$. Thus, (13) is in turn equivalent to (12). The statements *(ii)* - *(iii)* follow with the same arguments.

□

As $\mathcal{S}^n_+$ is the dual cone to $\mathcal{S}^n_+$, finding a matrix $S \in \mathcal{S}^n_+$ with $\mathrm{tr}(\Gamma^i S) < 0$ for some $i \in [m]$ is immediately a certificate for $\Gamma^i \notin \mathcal{S}^n_+$. Theorem 3.12, however, provides a simultaneous certificate: it requires one and the same matrix $S \in \mathcal{S}^n_+$ such that $\mathrm{tr}(\Gamma^i S) < 0$ holds for all $i \in [m]$. This distinction is essential. Even if each matrix Γ^i, $i \in [m]$, admits its own certificate of not being positive semidefinite, it may still hold that $\Lambda \neq \emptyset$; see also the forthcoming Example 3.27. In that example, such individual certificates exist, but no common matrix $S \in \mathcal{S}^n_+$ satisfying the above strict inequalities exists.

Remark 3.13. *Theorem 3.12 admits a geometric interpretation in terms of separation. Since*

$$\{\Gamma_\omega \mid \omega \in \Delta\} = \mathrm{conv}(\{\Gamma^1, \ldots, \Gamma^m\}),$$

the condition $\Lambda_+ = \emptyset$ *means that* $\mathrm{conv}(\{\Gamma^1, \ldots, \Gamma^m\})$ *does not intersect* $\mathcal{S}^n_{++}$*, whereas* $\Lambda = \emptyset$ *means that* $\mathrm{conv}(\{\Gamma^1, \ldots, \Gamma^m\})$ *does not intersect* $\mathcal{S}^n_+$*. Moreover, any matrix* $S \in \mathcal{S}^n_+ \setminus \{0\}$ *satisfying* $\mathrm{tr}(\Gamma^i S) \leqq 0$ *for all* $i \in [m]$ *yields by convexity*

$$\mathrm{tr}(BS) \leqq 0 \qquad \forall B \in \mathrm{conv}(\{\Gamma^1, \ldots, \Gamma^m\}),$$

and thus defines a half-space containing $\mathrm{conv}(\{\Gamma^1, \ldots, \Gamma^m\})$ *and excluding* $\mathcal{S}^n_{++}$*. Likewise, if* $\mathrm{tr}(\Gamma^i S) < 0$ *for all* $i \in [m]$*, then the induced half-space excludes* $\mathcal{S}^n_+$*.*

Under a simple trace condition, the identity matrix I is a feasible certificate in the intersections appearing in Theorem 3.12. For this, recall that $\mathrm{tr}(\Gamma^i)$ is the sum of all eigenvalues of Γ^i for $i \in [m]$.

Corollary 3.14. *Assume that the sum of all eigenvalues of* Γ^i*, or, equivalently,* $\operatorname{tr}(\Gamma^i)$*, is nonpositive for all* $i \in [m]$*. Then the set* Λ_+ *is empty. If, in addition,* $\operatorname{tr}(\Gamma^i) < 0$ *for all* $i \in [m]$*, then even the set* Λ *is empty.*

Considering again Example 3.9, we obtain $\operatorname{tr}(\Gamma^1) = \operatorname{tr}(\Gamma^2) = -1 < 0$ and thus, by Corollary 3.14 and Lemma 3.6 *(ii) - (iii)*, we have an alternative way to deduce $\Lambda = \Lambda_+ = \Omega = \Omega_+ = \emptyset$. We finally give an example showing that, for every $m \geqq 3$, weakly efficient solutions may exist although no supported solutions exist in (QMOP-UC). We provide the construction for $m = 3$; the example extends immediately to any $m > 3$ by copying one of the objective functions.

Example 3.15. *[1, Following Section 3] Let*

$$\Gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}, \Gamma^2 = \begin{pmatrix} -\frac{5}{4} & \frac{3\sqrt{3}}{4} \\ \frac{3\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}, \Gamma^3 = \begin{pmatrix} -\frac{5}{4} & -\frac{3\sqrt{3}}{4} \\ -\frac{3\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}, \beta^1 = \beta^2 = \beta^3 = 0, \alpha = 0,$$

i.e. $f_1(x) = x_1^2 - 2x_2^2$*,* $f_2(x) = -\frac{5}{4}x_1^2 + \frac{3\sqrt{3}}{2}x_1x_2 + \frac{1}{4}x_2^2$*, and* $f_3(x) = -\frac{5}{4}x_1^2 - \frac{3\sqrt{3}}{2}x_1x_2 + \frac{1}{4}x_2^2$*. We have* $\operatorname{tr}(\Gamma^i) = -1 < 0$ *for all* $i \in [3]$ *and by Corollary 3.14, we obtain* $(\mathbb{R}^2)^{\text{supp}} = \emptyset$*. Moreover, it is easy to see that for all* $x \in \mathbb{R}^2$ *there exists* $i \in [3]$ *such that* $f_i(x) \geqq 0$*. Since we have* $f_i(0) = 0$ *for all* $i \in [3]$*, this implies* $x = 0$ *is a weakly efficient solution of* (QMOP-UC)*.*

3.3 Existence Results of (Positively) Supported Solutions

After discussing the convexity properties and relations of the sets Ω, Ω_+, Λ, and Λ_+ in Section 3.1, and deriving conditions for the nonemptiness of Λ and Λ_+ in Section 3.2, we now turn to the specific question of when the sets $\mathbb{R}^n_{we}$ and $\mathbb{R}^n_{pe}$ are nonempty. More precisely, we provide further sufficient conditions ensuring $\Lambda_+ \neq \emptyset$, and hence that (QMOP-UC) admits positively supported solutions, i.e., $\Omega_+ \neq \emptyset$. In addition, we present conditions ensuring the image condition of Corollary 3.2 whenever $\Lambda_+ = \emptyset$ and $\Lambda \neq \emptyset$ and positively supported solutions and supported solutions exist, i.e., $\Omega_+ \neq \emptyset$ and $\Omega \neq \emptyset$, respectively.

By (1), every minimal solution of the weighted sum scalarization (WS-QMOP(ω)) yields a weakly efficient solution of (QMOP-UC), and every minimal solution corresponding to positive weights yields a properly efficient solution. Hence, whenever (WS-QMOP(ω)) admits a minimal solution for some $\omega \in \Delta$ ($\omega \in \Delta_+$), we obtain the existence of weakly (respectively, properly) efficient solutions of (QMOP-UC). We also want to emphasize at this point once again that in the unconstrained case the set of weakly efficient solutions is denoted by $\mathbb{R}^n_{we}$ and the set of properly efficient solutions by $\mathbb{R}^n_{pe}$, respectively.

We can initially state a result concerning the existence of weakly efficient solutions whenever the upper image set $f(\mathbb{R}^n) + \mathbb{R}^m_{\geqq}$ is closed.

Proposition 3.16. *For* (QMOP-UC) *suppose that* $f(\mathbb{R}^n) + \mathbb{R}^m_{\geqq}$ *is closed. Then,* $\mathbb{R}^n_{we} = \emptyset$ *if and only if* $f(\mathbb{R}^n) + \mathbb{R}^m_{\geqq} = \mathbb{R}^m$*.*

Proof. By definition, we have $\mathbb{R}^n_{we} = \emptyset$ if and only if $f(\mathbb{R}^n)_{wnd} = \emptyset$. We first show that $f(\mathbb{R}^n)_{wnd} = \emptyset$ holds if and only if $(f(\mathbb{R}^n) + \mathbb{R}^m_{\geqq})_{wnd} = \emptyset$. If $y \in (f(\mathbb{R}^n) + \mathbb{R}^m_{\geqq})_{wnd}$, then $y = z + r$ for some $z \in f(\mathbb{R}^n)$ and $r \in \mathbb{R}^m_{\geqq}$. If $z \notin f(\mathbb{R}^n)_{wnd}$, then there exists

$\tilde{z} \in f(\mathbb{R}^n)$ with $\tilde{z} < z$. Since $z \leqq y$, we have $\tilde{z} < y$, contradicting $y \in (f(\mathbb{R}^n) + \mathbb{R}^m_{\geqq})_{wnd}$. Thus, $z \in f(\mathbb{R}^n)_{wnd}$. The relation $f(\mathbb{R}^n)_{wnd} \subseteq (f(\mathbb{R}^n) + \mathbb{R}^m_{\geqq})_{wnd}$ is well-known. Since $f(\mathbb{R}^n) + \mathbb{R}^m_{\geqq}$ is closed by assumption, by [29, Proposition 1.40, Corollary 1.44] we obtain the desired result. $\square$

We also recall a basic distinction between the single- and multiobjective case. For $m = 1$, for a single quadratic function $f_1 : \mathbb{R}^n \to \mathbb{R}$, the image set $f_1(\mathbb{R}^n)$ is always a closed set; see, e.g., [7, Example 4.5]. In particular, if $\Gamma^1 \in \mathcal{S}^n$ is indefinite, then $f_1(\mathbb{R}^n) = \mathbb{R}$. For $m = 2$, closedness of the image set may fail, as the image set may be neither open nor closed. This affects the behavior of the existence of weakly efficient solutions, as the following example shows.

Example 3.17. *[36, Beispiel 3.19] Let* $\Gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \Gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \beta^1 = \beta^2 = 0, \alpha = 0$, *i.e.,* $f_1(x) = x_1^2,\ f_2(x) = 2x_1x_2$. *Then* $f(\mathbb{R}^2) = \left\{(y_1, y_2)^\top \in \mathbb{R}^2 \mid y_1 > 0\right\} \cup \{0\}$ *and* $f(\mathbb{R}^2) + \mathbb{R}^2_{\geqq} = \left\{(y_1, y_2)^\top \in \mathbb{R}^2 \mid y_1 > 0\right\} \cup \left\{(0, y_2)^\top \in \mathbb{R}^2 \mid y_2 \geqq 0\right\}$, *where both sets are neither open nor closed. The image set* $f(\mathbb{R}^2)$ *is illustrated in blue in Figure 5.*

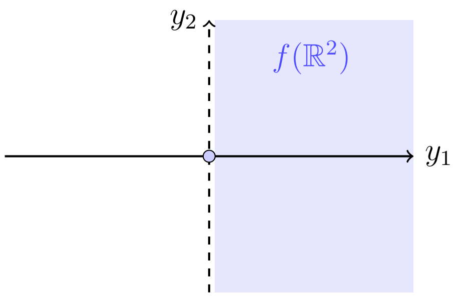

Figure 5: Image set $f(\mathbb{R}^2)$ of Example 3.17 in blue.

By Example 3.17, closedness of the upper image set is in general not given in (QMOP-UC). Hence, Proposition 3.16 does not apply to (QMOP-UC) in general. However, the following proposition provides a sufficient condition for the upper image set being closed.

Proposition 3.18. *Let* Λ_+ *be defined as in* (7). *If* $\Lambda_+ \neq \emptyset$, *then* $f(\mathbb{R}^n) + \mathbb{R}^m_{\geqq}$ *is closed.*

Proof. Let $\omega \in \Lambda_+$. Further, let $\lambda_{\min}(\Gamma_\omega) > 0$ be the minimal eigenvalue of Γ_ω. Then $x^\top \Gamma_\omega x \geqq \lambda_{\min}(\Gamma_\omega) \cdot ||x||_2^2$ for all $x \in \mathbb{R}^n$. Hence

$$\Psi_\omega(x) \geqq \lambda_{\min}(\Gamma_\omega) \cdot ||x||_2^2 - 2||\beta_\omega||_2||x||_2 + \alpha_\omega.$$

The right-hand side of the latter inequality tends to infinity as $||x||_2 \to \infty$. Thus, the function Ψ_ω is coercive.

Now, let $(y^k)_{k\in\mathbb{N}} \subseteq f(\mathbb{R}^n) + \mathbb{R}^m_{\geqq}$ be a sequence such that $y^k \to y \in \mathbb{R}^m$. Then, for every $k \in \mathbb{N}$, choose $x^k \in \mathbb{R}^n$ with $f(x^k) \leqq y^k$. Since $\omega \in \Delta$, we have

$$\Psi_\omega(x^k) = \sum_{i=1}^m \omega_i f_i(x^k) \leqq \sum_{i=1}^m \omega_i y_i^k.$$

Since $y^k \to y$, the sequence on the right-hand side of the latter inequality is bounded from above. Therefore, there exists $C \in \mathbb{R}$ such that $\Psi_\omega(x^k) \leqq C$ for all $k \in \mathbb{N}$. Hence,

$x^k \in \{x \in \mathbb{R}^n \mid \Psi_\omega(x) \leqq C\}$ for all $k \in \mathbb{N}$. Since Ψ_ω is continuous and coercive, it is lower semicontinuous and level-bounded in the sense of [35, Definition 1.8]. Hence, by the discussion following [35, Corollary 1.10], the sublevel set $\{x \in \mathbb{R}^n \mid \Psi_\omega(x) \leqq C\}$ is compact. Thus, the sequence $(x^k)_{k\in\mathbb{N}}$ admits a convergent subsequence $(x^{k_j})_{j\in\mathbb{N}}$ with limit $\bar{x} \in \mathbb{R}^n$, that is, $x^{k_j} \to \bar{x}$ as $j \to \infty$. Since also $y^{k_j} \to y$ as $j \to \infty$, continuity of f and $f(x^{k_j}) \leqq y^{k_j}$ imply $f(\bar{x}) \leqq y$. Hence, $y \in f(\mathbb{R}^n) + \mathbb{R}^m_{\geqq}$, which proves that $f(\mathbb{R}^n) + \mathbb{R}^m_{\geqq}$ is closed. $\square$

Note that closedness of the upper image set does not imply its convexity in general. Indeed, in [26, Example 2.1], Proposition 3.18 applies with $\omega = (0, 1, 0)^\top$, so that the upper image set is closed, while it is not convex. In the context of Proposition 3.18, we also recall the following observation from [36, Lemma 4.28].

Proposition 3.19. *Suppose that $f_i(x) = x^\top \Gamma^i x$ for all $i \in [m]$. Then $f(\mathbb{R}^n)$ is open if and only if $f(\mathbb{R}^n) = \mathbb{R}^m$.*

The proof of Proposition 3.19 uses that $f(\mathbb{R}^n)$ is a cone and that $0 \in f(\mathbb{R}^n)$. Thus, if $f(\mathbb{R}^n)$ is open, it contains a ball around the origin. Note that the same argument yields an analogous statement for the upper image set.

According to Example 3.17, Proposition 3.16 generally does not apply to (QMOP-UC). However, we can prove a similar result concerning the existence of weakly efficient solutions whenever the upper image set $f(\mathbb{R}^n) + \mathbb{R}^m_{\geqq}$ is convex.

Theorem 3.20. *Suppose that* (QMOP-UC) *is convexlike. Then $\mathbb{R}^n_{we} = \emptyset$ if and only if $f(\mathbb{R}^n) + \mathbb{R}^m_{\geqq} = \mathbb{R}^m$.*

Proof. First assume that $f(\mathbb{R}^n) + \mathbb{R}^m_{\geqq} = \mathbb{R}^m$. By noting that $f(\mathbb{R}^n_{we})$ is a subset of the boundary of the upper image set, we immediately obtain $\mathbb{R}^n_{we} = \emptyset$.

Now, assume that $f(\mathbb{R}^n) + \mathbb{R}^m_{\geqq} \neq \mathbb{R}^m$. Thus, there exists $y \notin f(\mathbb{R}^n) + \mathbb{R}^m_{\geqq}$. By assumption we have that $f(\mathbb{R}^n) + \mathbb{R}^m_{\geqq}$ is a convex set. By a separation theorem, see, for instance, [7, Section 2.5.1], there exists a vector $\tilde{\omega} \in \mathbb{R}^m \setminus \{0\}$ and a scalar $\tilde{\mu} \in \mathbb{R}$ such that

$$\tilde{\omega}^\top y \leqq \tilde{\mu} \leqq \tilde{\omega}^\top z \text{ for all } z \in f(\mathbb{R}^n) + \mathbb{R}^m_{\geqq}. \tag{14}$$

We now prove $\tilde{\omega} \in \mathbb{R}^m_{\geqq}$ by contradiction. Assume that $\tilde{\omega}_i < 0$ for some $i \in [m]$. For all $z \in f(\mathbb{R}^n) + \mathbb{R}^m_{\geqq}$ we have that $z + t \cdot e^i \in f(\mathbb{R}^n) + \mathbb{R}^m_{\geqq}$ for all $t \geqq 0$ with $e^i \in \mathbb{R}^m_{\geqq}$ the i-th unit vector. This implies $\lim_{t\to\infty} \tilde{\omega}^\top(z + t \cdot e^i) = -\infty$, which contradicts (14). Now set $\omega := \frac{1}{\|\tilde{\omega}\|_1}\tilde{\omega} \in \Delta$. This yields $\omega^\top y \leqq \omega^\top z$ for all $z \in f(\mathbb{R}^n) + \mathbb{R}^m_{\geqq}$. Due to $\omega \in \Delta$, we have $\min_{z\in\mathbb{R}^m_{\geqq}} \omega^\top z = 0$, and we get

$$\inf_{x\in\mathbb{R}^n} \omega^\top f(x) = \inf_{z\in f(\mathbb{R}^n)} \omega^\top z = \inf_{z\in f(\mathbb{R}^n)+\mathbb{R}^m_{\geqq}} \omega^\top z \geqq \omega^\top y > -\infty.$$

By Lemma 3.1, the quadratic problem $\inf_{x\in\mathbb{R}^n} \omega^\top f(x)$ has a minimal solution, which is a supported solution of (QMOP-UC) and thus also an element of $\mathbb{R}^n_{we}$. $\square$

In the forthcoming Theorem 4.1, see also [20, Theorem 4.19.], we recall that for $m = 2$, (QMOP-UC) is convexlike and therefore, Theorem 3.20 applies. However, note that, as stated in Section 2, the upper image set $f(\mathbb{R}^n) + \mathbb{R}^m_{\geqq}$ is in general not convex for $m \geqq 3$; see also [26, Example 2.1]. Moreover, it is possible that $f(\mathbb{R}^n) + \mathbb{R}^2_{\geqq} \subsetneq$

$\mathbb{R}^2$ and $\mathbb{R}^n_{pe} = \emptyset$ holds, see Example 3.8 *(a) (ii)*. Therefore, we cannot provide an analogous result in the manner of Theorem 3.20 for the existence of properly efficient solutions. Related geometric characterizations for $m = 2$ in terms of two homogeneous quadratic functions, i.e., $f_1(x) = x^\top \Gamma^1 x$ and $f_2(x) = x^\top \Gamma^2 x$, including conditions of the form $f(\mathbb{R}^n) = \mathbb{R}^2$, can be found in [20]. These results are of a different nature, however, since they concern the geometry of the homogeneous part, whereas Theorem 3.20 characterizes the absence of weakly efficient solutions under convexlikeness in the general quadratic setting.

In the following, regardless of the convexlikeness of (QMOP-UC) or closedness of the upper image set, we specify conditions under which the sets of supported solutions $(\mathbb{R}^n)^{\mathrm{supp}}$ and of positively supported solutions $(\mathbb{R}^n)^{\mathrm{supp}}_+$ of (QMOP-UC) are (non)empty, respectively. We split into three cases: whenever there exists a positive definite convex combination ($\Lambda_+ \neq \emptyset$), a positive semidefinite but not positive definite convex combination ($\Lambda \neq \emptyset$ and $\Lambda_+ = \emptyset$), and whenever there exists no positive semidefinite convex combination ($\Lambda = \emptyset$) of the matrices $\Gamma^1, \ldots, \Gamma^m$.

First, suppose that $m \geqq 2$ and $\Lambda_+ \neq \emptyset$. Since $\mathcal{S}^n_{++}$ is open and the mapping $\omega \mapsto \Gamma_\omega$ is continuous, Λ_+ is relatively open in Δ. By Lemma 3.6 *(iv)*, Ω therefore contains a nonempty relatively open subset of Δ and is uncountable. Moreover, since Δ_+ is dense in Δ, the set $\Lambda_+ \cap \Delta_+$ is nonempty and relatively open in Δ. By Lemma 3.6 *(v)*, it is contained in Ω_+, which is therefore also uncountable.

Corollary 3.21. *Let Λ_+ be defined as in* (7). *If $m \geqq 2$ and $\Lambda_+ \neq \emptyset$, then the set Ω_+ and thus Ω are nonempty and uncountable. In particular, $(\mathbb{R}^n)^{\mathrm{supp}}_+ \neq \emptyset$ and thus, $(\mathbb{R}^n)^{\mathrm{supp}} \neq \emptyset$.*

Remark 3.22. *Assume $\Lambda_+ \neq \emptyset$. Then for every $\omega \in \Lambda_+$, the function Ψ_ω as defined in* (3) *is strictly convex and thus admits a unique minimal solution $x(\omega) = -\Gamma_\omega^{-1}\beta_\omega$; see Lemma 3.1. Hence, the mapping $\omega \mapsto x(\omega)$ is well-defined and continuous on Λ_+. By Lemma 3.6, the set Λ_+ is convex. Thus, the set $\{x(\omega) \mid \omega \in \Lambda_+\}$ is either a singleton or uncountable. In particular, either all weight vectors $\omega \in \Lambda_+$ yield the same supported solution, or the set $(\mathbb{R}^n)^{\mathrm{supp}}$ is uncountable.*

Note that in Remark 3.22 the case that $\{x(\omega) \mid \omega \in \Lambda_+\}$ is a singleton can in fact appear as the next example shows.

Example 3.23. *Let $n = 1, m = 2$, and $f_1(x) = x^2$, $f_2(x) = 2x^2$. Then $\omega_1 f_1(x) + \omega_2 f_2(x) = (\omega_1 + 2\omega_2)x^2$, which is a strictly convex function with unique minimal solution $x = 0$ for all $\omega \in \Delta$. Hence $(\mathbb{R}^n)^{\mathrm{supp}} = \{0\}$ and $\Lambda_+ = \Delta$, where the latter is uncountable.*

The following lemma provides useful sufficient conditions for the case $\Lambda_+ \neq \emptyset$ as well as for the case $\Lambda \neq \emptyset$ and $\Lambda_+ = \emptyset$.

Lemma 3.24. *Let $\mathcal{I} \subseteq [m]$ be a nonempty index set such that $\Gamma^i \in \mathcal{S}^n_+$ for all $i \in \mathcal{I}$. Then the following properties are true.*

- *(i)* $\bigcap_{i \in \mathcal{I}} \ker(\Gamma^i) = \ker\left(\sum_{i \in \mathcal{I}} \Gamma^i\right)$.
- *(ii)* $\bigcap_{i \in \mathcal{I}} \ker(\Gamma^i) = \{0\}$ *if and only if* $\sum_{i \in \mathcal{I}} \Gamma^i \in \mathcal{S}^n_{++}$.

Proof. Assume that $x \in \ker\left(\sum_{i\in\mathcal{I}} \Gamma^i\right)$. This implies

$$0 = x^\top \left(\sum_{i\in\mathcal{I}} \Gamma^i\right) x = \sum_{i\in\mathcal{I}} x^\top \Gamma^i x.$$

Since $\Gamma^i \in \mathcal{S}^n_+$ for all $i \in \mathcal{I}$, this implies $x^\top \Gamma^i x = 0$ for all $i \in \mathcal{I}$. Therefore, by [22, Observation 7.1.6.], we have $\Gamma^i x = 0$ for all $i \in \mathcal{I}$. Thus, $x \in \bigcap_{i\in\mathcal{I}} \ker(\Gamma^i)$. On the other hand, assume that $x \in \bigcap_{i\in\mathcal{I}} \ker(\Gamma^i)$. Then we clearly have $x \in \ker\left(\sum_{i\in\mathcal{I}} \Gamma^i\right)$, which proves *(i)*. The remainder of the proof is straightforward. □

The relevance of Lemma 3.24 is that the condition $\bigcap_{i\in\mathcal{I}} \ker(\Gamma^i) = \{0\}$, which may be difficult to verify when several singular positive semidefinite matrices are involved, is equivalent to the positive definiteness of the single matrix $\sum_{i\in\mathcal{I}} \Gamma^i$. In order to obtain a sufficient condition for $\Lambda_+ \neq \emptyset$ based only on individually positive semidefinite matrices, it is not necessary to check all possible index sets. Instead, one may simply collect all positive semidefinite matrices.

Corollary 3.25. *Let Λ_+ be defined as in* (7) *and set $\mathcal{I}_+ := \{i \in [m] \mid \Gamma^i \in \mathcal{S}^n_+\}$. If $\mathcal{I}_+ \neq \emptyset$ and $\sum_{i\in\mathcal{I}_+} \Gamma^i \in \mathcal{S}^n_{++}$, then $\Lambda_+ \neq \emptyset$.*

If there exists some nonempty index set $\mathcal{I} \subseteq [m]$ with $\Gamma^i \in \mathcal{S}^n_+$ for all $i \in \mathcal{I}$ and $\sum_{i\in\mathcal{I}} \Gamma^i \in \mathcal{S}^n_{++}$, then the conditions in Corollary 3.25 are also satisfied, since adding positive semidefinite matrices preserves positive definiteness. However, although assigning equal weights to the matrices indexed by $\mathcal{I}_+$ yields an element of Λ_+, assigning equal weights to all matrices $\Gamma^1, \ldots, \Gamma^m$ need not even yield a positive semidefinite matrix. The following example illustrates this.

Example 3.26. *For $\Gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \Gamma^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \Gamma^3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \Gamma^4 = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$, we have $\mathcal{I}_+ = \{1, 2, 3\}$. Moreover, $\Gamma^1 + \Gamma^2 \in \mathcal{S}^2_{++}$. Hence, also $\Gamma^1 + \Gamma^2 + \Gamma^3 \in \mathcal{S}^2_{++}$. By Corollary 3.25 this yields $\Lambda_+ \neq \emptyset$. However, this does not imply that the equally weighted convex combination of $\Gamma^1, \ldots, \Gamma^4$ is positive semidefinite: for $\bar\omega = (0.25, 0.25, 0.25, 0.25)^\top \in \Delta_+$, we obtain the indefinite matrix $\sum_{i=1}^4 \bar\omega_i \Gamma^i$. On the other hand, for instance, we have $\omega = (0.25, 0.5, 0.125, 0.125)^\top \in \Lambda_+$.*

Note that even if all matrices $\Gamma^1, \ldots, \Gamma^m$ are indefinite, a positive definite convex combination may still exist, as the next example shows.

Example 3.27. *For $\Gamma^1 = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \Gamma^2 = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$, we have $\frac{1}{2}\Gamma^1 + \frac{1}{2}\Gamma^2 = \frac{1}{2}I \in \mathcal{S}^2_{++}$ and therefore $(\frac{1}{2}, \frac{1}{2})^\top \in \Lambda_+$.*

Now we consider the case where there exists a positive semidefinite but no positive definite convex combination of the matrices $\Gamma^1, \ldots, \Gamma^m$, i.e., $\Lambda \neq \emptyset$ and $\Lambda_+ = \emptyset$. The following corollary provides a simple sufficient condition for this case in view of Lemma 3.24 *(ii)*. Indeed, if the matrices $\Gamma^1, \ldots, \Gamma^m$ are positive semidefinite and share a common nonzero kernel vector, then every convex combination is again positive semidefinite and singular.

Corollary 3.28. *Let Λ and Λ_+ be defined as in* (6) *and* (7)*, respectively. Further, let $\Gamma^1, \ldots, \Gamma^m \in \mathcal{S}^n_+$ and $\sum_{i\in[m]} \Gamma^i \in \partial\mathcal{S}^n_+$. Then $\Lambda = \Delta$ and $\Lambda_+ = \emptyset$.*

Corollary 3.28 is useful since it replaces the analysis of all convex combinations by a single verification of the given matrices and their sum, as the following example illustrates.

Example 3.29. *For* $\Gamma^1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, $\Gamma^2 = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix}$, $\Gamma^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, *we have* $\Gamma^1, \Gamma^2, \Gamma^3 \in \partial\mathcal{S}^3_+$ *and* $\Gamma^1 + \Gamma^2 + \Gamma^3 \in \partial\mathcal{S}^3_+$. *Then Corollary 3.28 yields* $\Lambda = \Delta$ *and* $\Lambda_+ = \emptyset$.

To derive necessary conditions for the existence of (positively) supported solutions of (QMOP-UC), we additionally require the image condition from Corollary 3.2. Practical sufficient conditions for this image condition were given in Lemma 3.3. We now consider a similar setting as in Corollary 3.28 and establish equivalent characterizations of the existence of (positively) supported solutions of (QMOP-UC).

Proposition 3.30. *Let* $\Gamma^1, \dots, \Gamma^m \in \mathcal{S}^n_+$ *and suppose that* $\sum_{i \in [m]} \Gamma^i \in \partial\mathcal{S}^n_+$. *Then the following properties are true.*

(i) For any $\omega \in \Delta$, *the kernel of* Γ_ω *is given by* $\ker(\Gamma_\omega) = \bigcap_{\{i \in [m] \mid \omega_i > 0\}} \ker(\Gamma^i)$.

(ii) The set of positively supported solutions $(\mathbb{R}^n)^{\text{supp}}_+$ *of* (QMOP-UC) *is nonempty, i.e.,* $\Omega_+ \neq \emptyset$ *if and only if there exists* $\omega \in \Delta_+$ *such that* $\beta_\omega \in \sum_{i \in [m]} \operatorname{Im}(\Gamma^i)$, *where* $\sum_{i \in [m]} \operatorname{Im}(\Gamma^i)$ *is the Minkowski sum of the image spaces.*

(iii) The set of supported solutions $(\mathbb{R}^n)^{\text{supp}}$ *of* (QMOP-UC) *is nonempty if and only if there exists* $\omega \in \Delta$ *such that* $\beta_\omega = \sum_{\{i \in [m] \mid \omega_i > 0\}} \omega_i \beta^i \in \sum_{\{i \in [m] \mid \omega_i > 0\}} \operatorname{Im}(\Gamma^i)$.

Proof. By Corollary 3.28, we have $\Lambda = \Delta$ and $\Lambda_+ = \emptyset$. Property *(i)* follows by applying Lemma 3.24 to the index set $\{i \in [m] \mid \omega_i > 0\}$, noting that multiplication by a positive scalar does not change the kernel. Now, let $\hat{\omega} \in \Delta_+$. Since $\Gamma_{\hat{\omega}}$ is symmetric, we have $\operatorname{Im}(\Gamma_{\hat{\omega}}) = \ker(\Gamma_{\hat{\omega}})^\perp$. Hence, $\beta_{\hat{\omega}} \in \operatorname{Im}(\Gamma_{\hat{\omega}})$ if and only if $\beta_{\hat{\omega}} \perp \ker(\Gamma_{\hat{\omega}})$. Moreover, since $\hat{\omega}_i > 0$ for all $i \in [m]$, Lemma 3.24 yields $\ker(\Gamma_{\hat{\omega}}) = \bigcap_{i \in [m]} \ker(\Gamma^i)$. Therefore, $\beta_{\hat{\omega}} \in \operatorname{Im}(\Gamma_{\hat{\omega}})$ if and only if $\beta_{\hat{\omega}} \in \left(\bigcap_{i \in [m]} \ker(\Gamma^i)\right)^\perp$. Since $\left(\bigcap_{i \in [m]} \ker(\Gamma^i)\right)^\perp = \sum_{i \in [m]} \ker(\Gamma^i)^\perp = \sum_{i \in [m]} \operatorname{Im}(\Gamma^i)$, we obtain property *(ii)* by Corollary 3.2. The proof of *(iii)* is analogous, using the index set $\{i \in [m] \mid \omega_i > 0\}$ instead of $[m]$. $\square$

For condition *(ii)* in Proposition 3.30, observe that $\sum_{i \in [m]} \operatorname{Im}(\Gamma^i) = \operatorname{Im}[\Gamma^1 \dots \Gamma^m]$, where $[\Gamma^1 \dots \Gamma^m] \in \mathbb{R}^{n \times (nm)}$. Moreover, $\beta_\omega = [\beta^1 \dots \beta^m]\omega$. Hence, *(ii)* is equivalent to the existence of $y \in \mathbb{R}^{nm}$ and $\omega \in \Delta_+$ such that $[\Gamma^1 \dots \Gamma^m]y = [\beta^1 \dots \beta^m]\omega$. This can be checked efficiently by solving the linear program

$$
\begin{aligned}
\underset{\tau, y, \omega}{\text{maximize}} \quad & \tau \\
\text{subject to} \quad & \textstyle\sum_{i=1}^m \Gamma^i y^i = \sum_{i=1}^m \omega_i \beta^i, \\
& \textstyle\sum_{i=1}^m \omega_i = 1, \quad \omega_i \geqq \tau \qquad \forall i \in [m], \\
& \tau \in \mathbb{R}_\geqq, \ y \in \mathbb{R}^{nm}, \ \omega \in \mathbb{R}^m.
\end{aligned} \tag{15}
$$

Condition *(ii)* in Proposition 3.30 holds if and only if the optimal value of the maximization problem (15) is positive. Condition *(iii)* can then be checked in a similar way by applying (15) to faces of the standard simplex. Alternatively, instead of checking the faces separately, one may consider a mixed-integer formulation in which binary variables encode the active index set.

Finally, note that if all matrices $\Gamma^1, \ldots, \Gamma^m$ are indefinite, it is possible that $\Lambda \neq \emptyset$ and $\Lambda_+ = \emptyset$ holds; see Example 3.8 *(b)*.

4 Unconstrained Quadratic Biobjective Optimization

We now focus on the case $m = 2$. The reason for treating this case separately is that the biobjective setting admits a significantly stronger geometric structure than the general case with $m \geqq 3$. In particular, while convexlikeness may fail in general quadratic multiobjective optimization, the forthcoming problem (BI-UC) is convexlike, which yields a much more precise description of the sets of weakly and properly efficient solutions. We therefore consider the optimization problem

$$\begin{aligned} &\text{minimize} \qquad f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} := \begin{pmatrix} x^\top \Gamma^1 x + 2(\beta^1)^\top x + \alpha_1 \\ x^\top \Gamma^2 x + 2(\beta^2)^\top x + \alpha_2 \end{pmatrix} \\ &\text{subject to} \qquad x \in \mathbb{R}^n \end{aligned} \tag{BI-UC}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$, $\Gamma^i \in \mathcal{S}^n$, $\beta^i \in \mathbb{R}^n$ for all $i \in [2]$, and $\alpha \in \mathbb{R}^2$. For $\beta^1 = \beta^2 = 0$ and $\alpha = 0$, the image set $f(\mathbb{R}^n)$ is convex; see [16, Theorem 1]. Moreover, in [39], the set of efficient solutions and the corresponding set of nondominated points are analyzed for convex quadratic biobjective optimization problems. In particular, the set of efficient solutions is shown to be a line segment when the Hessian matrices of the objectives are proportional, that is, when one is a positive scalar multiple of the other.

4.1 Convexlikeness and Supportedness of (BI-UC)

As we have seen in Example 2.4, convexity of $f(\mathbb{R}^n)$ is in general not present, not even for $m = 2$. However, the following result provides convexlikeness.

Theorem 4.1. *[20, Theorem 4.19.] The optimization problem* (BI-UC) *is convexlike.*

Proposition 2.3 immediately gives the following corollary, which is useful from a practical perspective in multiobjective optimization.

Corollary 4.2. *For* (BI-UC), $\mathbb{R}^n_{we} = (\mathbb{R}^n)^{\text{supp}}$ *and* $\mathbb{R}^n_{pe} = (\mathbb{R}^n)^{\text{supp}}_+$.

By Corollary 4.2, the set of positively supported solutions of (BI-UC) coincides with the set of properly efficient solutions, and the set of supported solutions of (BI-UC) coincides with the set of weakly efficient solutions. However, as seen in Example 2.4, it is possible that (BI-UC) has no weakly efficient solutions at all. Thus, Corollary 4.2 alone does not provide an existence result for weakly efficient solutions or properly efficient solutions. By combining Theorem 3.20 and Theorem 4.1, we can establish the following result.

Corollary 4.3. *For* (BI-UC), $\mathbb{R}^n_{we} = \emptyset$ *if and only if* $f(\mathbb{R}^n) + \mathbb{R}^2_{\geqq} = \mathbb{R}^2$.

4.2 Convex Combinations and the Homogeneous S-Lemma

We now specialize the existence results on (positively) supported solutions from the previous sections to the biobjective case. In this setting, Theorem 3.12 admits a simpler formulation, since the relevant objects can be represented by vectors instead of positive semidefinite matrices. Moreover, in the homogeneous case, i.e., when $f_i(x) = x^\top \Gamma^i x$ for all $i \in [2]$, the forthcoming Theorem 4.4 allows us to formulate an alternative proof of the classical S-Lemma of Yakubovich; see Proposition 4.5. This result is restated in [32, Theorem 2.2] based on its original formulation in [44].

Theorem 4.4. *Let Λ and Λ_+ be defined as in* (6) *and* (7)*, respectively. Further, let $m = 2$. Then the following properties are true.*

(i) $\Lambda_+ \neq \emptyset$ *if and only if* $\left\{x \in \mathbb{R}^n \setminus \{0\} \mid x^\top \Gamma^1 x \leqq 0 \ \wedge \ x^\top \Gamma^2 x \leqq 0\right\} = \emptyset$.

(ii) $\Lambda \neq \emptyset$ *and* $\Lambda_+ = \emptyset$ *if and only if*

$$\begin{aligned} &\left\{x \in \mathbb{R}^n \setminus \{0\} \mid x^\top \Gamma^1 x < 0 \ \wedge \ x^\top \Gamma^2 x < 0\right\} = \emptyset \qquad \wedge \\ &\left\{x \in \mathbb{R}^n \setminus \{0\} \mid x^\top \Gamma^1 x \leqq 0 \ \wedge \ x^\top \Gamma^2 x \leqq 0\right\} \neq \emptyset. \end{aligned}$$

(iii) $\Lambda = \emptyset$ *if and only if* $\left\{x \in \mathbb{R}^n \setminus \{0\} \mid x^\top \Gamma^1 x < 0 \ \wedge \ x^\top \Gamma^2 x < 0\right\} \neq \emptyset$.

Proof. We note two auxiliary observations. First, suppose that there exists $S \in \mathcal{S}^n_+ \setminus \{0\}$ such that $\operatorname{tr}(\Gamma^i S) < 0$ for all $i \in [2]$. Define $r_i := \operatorname{tr}(\Gamma^i S)$ for $i \in [2]$. Since $m = 2$, by [2, Theorem 2.2], there exists $\hat{S} \in \mathcal{S}^n_+$ of rank at most 1 such that $r_i = \operatorname{tr}(\Gamma^i \hat{S})$ for all $i \in [2]$. Since $(r_1, r_2) \neq (0, 0)$, we have $\hat{S} \neq 0$, and hence $\hat{S} = \hat{s}\hat{s}^\top$ for some $\hat{s} \in \mathbb{R}^n \setminus \{0\}$. Therefore, $\hat{s}^\top \Gamma^i \hat{s} < 0$ for all $i \in [2]$. Second, suppose that there exists $S \in \mathcal{S}^n_+ \setminus \{0\}$ such that $\operatorname{tr}(\Gamma^i S) \leqq 0$ for all $i \in [2]$. By [32, Lemma 2.4], there exist nonzero vectors $p^1, \ldots, p^r \in \mathbb{R}^n$, where $r = \operatorname{rank}(S) \geqq 1$, such that $S = \sum_{k=1}^r p^k (p^k)^\top$ and $(p^k)^\top \Gamma^1 p^k \leqq 0$ for all $k \in [r]$. Since $\sum_{k=1}^r (p^k)^\top \Gamma^2 p^k = \operatorname{tr}(\Gamma^2 S) \leqq 0$, there exists some $\ell \in [r]$ such that $(p^\ell)^\top \Gamma^2 p^\ell \leqq 0$. Thus, with $\hat{s} := p^\ell$, we obtain $\hat{s} \in \mathbb{R}^n \setminus \{0\}$ and $\hat{s}^\top \Gamma^i \hat{s} \leqq 0$ for all $i \in [2]$. The reverse implication in both cases is immediate by choosing $S = xx^\top$.

Property *(iii)* now follows directly from Theorem 3.12 *(iii)* and the first auxiliary observation. Property *(i)* follows directly from Theorem 3.12 *(i)* and the second auxiliary observation. Finally, property *(ii)* is a direct consequence of *(i)* and *(iii)*. $\square$

This yields the above-mentioned S-Lemma for the homogeneous setting. We provide a proof that differs from those presented in [32] and in [44].

Proposition 4.5 (Homogeneous case of the S-Lemma)**.** *Let $\Gamma^1, \Gamma^2 \in \mathcal{S}^n$ and let $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ be two homogeneous quadratic functions, i.e., $f_i(x) = x^\top \Gamma^i x$ for all $i \in [2]$. Suppose that there exists a vector $\bar{x} \in \mathbb{R}^n$ such that $f_2(\bar{x}) < 0$. Then the following two statements are equivalent.*

(i) There is no vector $x \in \mathbb{R}^n$ such that

$$f_1(x) < 0 \qquad \wedge \qquad f_2(x) \leqq 0.$$

(ii) There exists a nonnegative number $\lambda \geqq 0$ such that

$$f_1(x) + \lambda f_2(x) \geqq 0 \text{ for all } x \in \mathbb{R}^n.$$

Proof. We can equivalently reformulate the result as follows. Since the assumption that there exists $\bar{x} \in \mathbb{R}^n$ with $f_2(\bar{x}) < 0$ is equivalent to $\Gamma^2 \notin \mathcal{S}^n_+$, let $\Gamma^1 \in \mathcal{S}^n$ and let $\Gamma^2 \notin \mathcal{S}^n_+$. For $\omega \in \Delta$ with $\omega_1 > 0$, set $\lambda := \frac{\omega_2}{\omega_1} \geqq 0$. Then $\Gamma_\omega \in \mathcal{S}^n_+$ is equivalent to $x^\top \Gamma_\omega x = \omega_1 x^\top \Gamma^1 x + \omega_2 x^\top \Gamma^2 x \geqq 0$ for all $x \in \mathbb{R}^n$, which, in turn, is equivalent to $f_1(x) + \lambda f_2(x) \geqq 0$ for all $x \in \mathbb{R}^n$. Thus, the homogeneous S-Lemma is equivalent to the following statement: there exists a weight vector $\omega \in \{\bar{\omega} \in \Delta \mid \bar{\omega}_1 \neq 0\}$ such that $\Gamma_\omega \in \mathcal{S}^n_+$ if and only if

$$\left\{x \in \mathbb{R}^n \mid x^\top \Gamma^1 x < 0\right\} \cap \left\{x \in \mathbb{R}^n \mid x^\top \Gamma^2 x \leqq 0\right\} = \emptyset. \tag{16}$$

Now assume that (16) is satisfied. Then, in particular,

$$\left\{x \in \mathbb{R}^n \setminus \{0\} \mid x^\top \Gamma^1 x < 0 \ \wedge \ x^\top \Gamma^2 x < 0\right\} = \emptyset.$$

Hence, by Theorem 4.4 *(iii)*, we have $\Lambda \neq \emptyset$. Thus, there exists $\tilde{\omega} \in \Delta$ such that $\Gamma_{\tilde{\omega}} = \tilde{\omega}_1 \Gamma^1 + \tilde{\omega}_2 \Gamma^2 \in \mathcal{S}^n_+$. Since $\tilde{\omega}_1 = 0$ implies $\Gamma_{\tilde{\omega}} = \Gamma^2 \in \mathcal{S}^n_+$, contradicting $\Gamma^2 \notin \mathcal{S}^n_+$, we must have $\tilde{\omega}_1 \neq 0$. This proves the assertion.

Conversely, assume that there exists a weight vector $\hat{\omega} \in \{\bar{\omega} \in \Delta \mid \bar{\omega}_1 \neq 0\}$ such that $\Gamma_{\hat{\omega}} \in \mathcal{S}^n_+$. If there exists some $\hat{x} \in \mathbb{R}^n$ with $\hat{x}^\top \Gamma^1 \hat{x} < 0$ and $\hat{x}^\top \Gamma^2 \hat{x} \leqq 0$, then, since $\hat{\omega}_1 > 0$ and $\hat{\omega}_2 \geqq 0$, we obtain $\hat{x}^\top \Gamma_{\hat{\omega}} \hat{x} < 0$, contradicting $\Gamma_{\hat{\omega}} \in \mathcal{S}^n_+$. Hence, the intersection in (16) is empty. □

Note that the existence of a rank-1 matrix $\hat{S} \in \mathcal{S}^n_+$ as in the proof of Theorem 4.4 is generally not given for $m \geqq 3$; see [2, Theorem 2.2]. Hence, for $m \geqq 3$ the statement of Theorem 3.12 cannot be strengthened in the manner of Theorem 4.4.

4.3 Existence of Weakly and Properly Efficient Solutions of (BI-UC)

In this section, we specify conditions under which the sets of weakly efficient solutions $\mathbb{R}^n_{we}$ and of properly efficient solutions $\mathbb{R}^n_{pe}$ of (BI-UC) are nonempty, respectively. For a detailed overview, see Table 1 on page 31. At this point, we emphasize once again that, in the biobjective case, the set of positively supported solutions coincides with the set of properly efficient solutions, and the set of supported solutions coincides with the set of weakly efficient solutions.

In view of the preceding results, the question of the existence of weakly and properly efficient solutions can be separated into two parts. First, one has to determine the weights $\omega \in \Delta$ for which the convex combination Γ_ω is positive semidefinite or positive definite, respectively, where

$$\Gamma_\omega = (1 - \omega_j)\Gamma^i + \omega_j \Gamma^j, \qquad \omega_j \in [0,1],\ i, j \in [2],\ i \neq j. \tag{17}$$

This corresponds to the determination of the sets Λ and Λ_+. Second, for weights $\omega \in \Lambda$, one has to verify the additional image condition $\beta_\omega \in \mathrm{Im}(\Gamma_\omega)$, which is necessary and sufficient for (WS-QMOP(ω)) to have a minimal solution by Corollary 3.2.

We first distinguish the possible definiteness properties of one of the two matrices. If $\Gamma^i \in \mathcal{S}^n_{++}$ for some $i \in [2]$, then $e^i \in \Lambda_+$, where e^i is the i-th unit vector. Hence, $\Lambda_+ \neq \emptyset$, and Corollary 3.21 yields $\Omega_+ \neq \emptyset$ and, in particular, $\mathbb{R}^n_{pe} \neq \emptyset$.

If $\Gamma^i \in \partial \mathcal{S}^n_+$ for some $i \in [2]$, then the situation is more delicate and will be analyzed by considering the restriction of Γ^j to $\ker(\Gamma^i)$, where $j \in [2]$ with $i \neq j$. Finally, if neither of the two matrices belongs to $\mathcal{S}^n_+$, then the preceding boundary analysis cannot be applied directly; this case will be discussed afterwards.

We now consider the case where one matrix belongs to the boundary of the positive semidefinite cone. Thus, suppose that $\Gamma^i \in \partial\mathcal{S}^n_+$ for some $i \in [2]$, and let $j \in [2]$ with $i \neq j$. Then the behavior of the convex combinations in (17) is closely related to the restriction of Γ^j to $\ker(\Gamma^i)$. More precisely, if $Z \in \mathbb{R}^{n\times \dim(\ker(\Gamma^i))}$ is a matrix whose columns form a basis of $\ker(\Gamma^i)$, then the definiteness properties of $Z^\top\Gamma^j Z$ provide a unified way to distinguish whether positive definite convex combinations exist, whether no nontrivial positive semidefinite convex combination exists, or whether the positive semidefinite convex combinations form an interval. The following results first describe the possible structures of Λ and Λ_+ in terms of this restriction. Afterwards, we combine this matrix characterization with the image condition above in order to obtain criteria for the nonemptiness of $\mathbb{R}^n_{we}$ and $\mathbb{R}^n_{pe}$. We use the notation

$$\sigma_+ := \max\left\{\omega_j \in [0,1] \mid (1-\omega_j)\Gamma^i + \omega_j\Gamma^j \in \mathcal{S}^n_+\right\}. \tag{18}$$

Since $\Gamma^i \in \partial\mathcal{S}^n_+$, the choice $\omega_j = 0$ satisfies $(1-\omega_j)\Gamma^i + \omega_j\Gamma^j \in \mathcal{S}^n_+$. Hence, the set of feasible weights in (18) is nonempty. In addition, for

$$h : [0,1] \to \mathcal{S}^n, \qquad h(\omega_j) := (1-\omega_j)\Gamma^i + \omega_j\Gamma^j,$$

we have $\left\{\omega_j \in [0,1] \mid h(\omega_j) \in \mathcal{S}^n_+\right\} = h^{-1}(\mathcal{S}^n_+)$. Since $\mathcal{S}^n_+$ is closed and h is continuous, the set $h^{-1}(\mathcal{S}^n_+)$ is closed and hence compact. Thus, the supremum is attained and σ_+ in (18) is indeed a maximum. Moreover, by Lemma 3.6, the set $h^{-1}(\mathcal{S}^n_+)$ is convex. Since it is also closed, it is in fact a closed interval in $[0,1]$. We first obtain the following characterization of the possible structures of Λ and Λ_+.

Proposition 4.6. *Let $\Gamma^i \in \partial\mathcal{S}^n_+$ for some $i \in [2]$, and let $j \in [2]$ with $i \neq j$. Further, let $Z \in \mathbb{R}^{n\times\dim(\ker(\Gamma^i))}$ be such that its columns form a basis of $\ker(\Gamma^i)$, and let e^k be the k-th unit vector for $k \in [2]$. Then the following properties are true.*

(i) If $Z^\top\Gamma^j Z \in \mathcal{S}^{\dim(\ker(\Gamma^i))}_{++}$, then $\Lambda_+ \neq \emptyset$.

(ii) If $Z^\top\Gamma^j Z \notin \mathcal{S}^{\dim(\ker(\Gamma^i))}_{+}$, then $\Lambda = \{e^i\}$ and $\Lambda_+ = \emptyset$.

(iii) If $Z^\top\Gamma^j Z \in \partial\mathcal{S}^{\dim(\ker(\Gamma^i))}_{+}$, then $\Lambda_+ = \emptyset$ and

$$\Lambda = \left\{(1-\omega_j)e^i + \omega_j e^j \mid \omega_j \in [0,\sigma_+]\right\}.$$

Moreover, $\sigma_+ = 0$ if and only if there exists additionally a vector $\hat{u} \in \ker(Z^\top\Gamma^j Z)\setminus\{0\}$ such that $\Gamma^i\Gamma^j Z\hat{u} \neq 0$. In particular, if no such vector exists, then $\sigma_+ > 0$. If no such vector exists and, in addition, $\Gamma^j \notin \mathcal{S}^n_+$, then $0 < \sigma_+ < 1$.

Proof. Since $\Gamma^i \in \partial\mathcal{S}^n_+$, every $x \in \mathbb{R}^n\setminus\{0\}$ with $x^\top\Gamma^i x \leqq 0$ belongs to $\ker(\Gamma^i)$; see, e.g., [22, Observation 7.1.6.]. Hence, there exists $u \in \mathbb{R}^{\dim(\ker(\Gamma^i))}\setminus\{0\}$ such that $x = Zu$. If $Z^\top\Gamma^j Z \in \mathcal{S}^{\dim(\ker(\Gamma^i))}_{++}$, then $x^\top\Gamma^j x = u^\top Z^\top\Gamma^j Zu > 0$. Thus, no vector $x \in \mathbb{R}^n\setminus\{0\}$ can satisfy $x^\top\Gamma^i x \leqq 0$ and $x^\top\Gamma^j x \leqq 0$ simultaneously. Therefore, Theorem 4.4 *(i)* implies $\Lambda_+ \neq \emptyset$, which proves property *(i)*.

We now prove property *(ii)*. Let $Z^\top\Gamma^j Z \notin \mathcal{S}^{\dim(\ker(\Gamma^i))}_{+}$. Then there exists a vector $u \in \mathbb{R}^{\dim(\ker(\Gamma^i))}\setminus\{0\}$ such that $u^\top Z^\top\Gamma^j Zu < 0$. Setting $x := Zu$, we have $x \in \ker(\Gamma^i)\setminus\{0\}$. Hence, for every $\omega_j > 0$, we have

$$x^\top\Gamma_\omega x = (1-\omega_j)x^\top\Gamma^i x + \omega_j x^\top\Gamma^j x = \omega_j u^\top Z^\top\Gamma^j Zu < 0.$$

Thus, $\Gamma_\omega \notin \mathcal{S}_+^n$ for every $\omega_j > 0$. Since $\Gamma^i \in \partial\mathcal{S}_+^n$, the only positive semidefinite convex combination is obtained for $\omega = e^i$. Therefore, $\Lambda = \{e^i\}$. Moreover, since $\Gamma^i \in \partial\mathcal{S}_+^n$, it follows that $e^i \notin \Lambda_+$, and hence $\Lambda_+ = \emptyset$.

It remains to prove property *(iii)*. Let $Z^\top \Gamma^j Z \in \partial\mathcal{S}_+^{\dim(\ker(\Gamma^i))}$. Then there exists a vector $u \in \mathbb{R}^{\dim(\ker(\Gamma^i))} \setminus \{0\}$ such that $u^\top Z^\top \Gamma^j Z u = 0$. Setting $x := Zu$, we obtain $x \in \ker(\Gamma^i) \setminus \{0\}$ and therefore $x^\top \Gamma^i x = 0$ and $x^\top \Gamma^j x = 0$. Thus,

$$\left\{x \in \mathbb{R}^n \setminus \{0\} \mid x^\top \Gamma^i x \leqq 0 \ \wedge \ x^\top \Gamma^j x \leqq 0\right\} \neq \emptyset.$$

By Theorem 4.4 *(i)*, it follows that $\Lambda_+ = \emptyset$. By the definition of σ_+ and the discussion preceding this proposition, the set of feasible weights $\omega_j \in [0,1]$ satisfying $(1-\omega_j)\Gamma^i + \omega_j \Gamma^j \in \mathcal{S}_+^n$ is the closed interval $[0, \sigma_+]$. Hence, $\Lambda = \{(1-\omega_j)e^i + \omega_j e^j \mid \omega_j \in [0,\sigma_+]\}$.

It remains to characterize the case $\sigma_+ = 0$. First, we consider a Schur decomposition of Γ^i. Therefore, let $Q = [U_{\mathrm{Im}} \mid U_{\ker}] \in \mathbb{R}^{n\times n}$ be an orthogonal matrix, where the columns of U_{Im} form an orthonormal basis of $\mathrm{Im}(\Gamma^i)$ and the columns of $U_{\ker}$ form an orthonormal basis of $\ker(\Gamma^i)$. Then

$$Q^\top \Gamma^i Q = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad D_1 \in \mathcal{S}_{++}^{n-\dim(\ker(\Gamma^i))}.$$

Writing

$$Q^\top \Gamma^j Q = \begin{pmatrix} E_1 & E_2 \\ E_2^\top & G \end{pmatrix},$$

we have $G = U_{\ker}^\top \Gamma^j U_{\ker} \in \partial\mathcal{S}_+^{\dim(\ker(\Gamma^i))}$, since G and $Z^\top \Gamma^j Z$ have the same inertia by Sylvester's law of inertia. Since the columns of Z and of $U_{\ker}$ both form a basis of $\ker(\Gamma^i)$, there exists a regular matrix $T \in \mathbb{R}^{\dim(\ker(\Gamma^i))\times\dim(\ker(\Gamma^i))}$ such that $Z = U_{\ker}T$. For $\hat{u} \in \mathbb{R}^{\dim(\ker(\Gamma^i))}$, set $y := T\hat{u}$. Then $\hat{u} \in \ker(Z^\top \Gamma^j Z) \setminus \{0\}$ if and only if $y \in \ker(G) \setminus \{0\}$. Moreover,

$$Q^\top \Gamma^j Z \hat{u} = Q^\top \Gamma^j U_{\ker} y = Q^\top \Gamma^j Q \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} E_2 y \\ G y \end{pmatrix}.$$

Thus, for $y \in \ker(G)$, we have

$$Q^\top \Gamma^i \Gamma^j Z \hat{u} = Q^\top \Gamma^i Q Q^\top \Gamma^j Z \hat{u} = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} E_2 y \\ G y \end{pmatrix} = \begin{pmatrix} D_1 E_2 y \\ 0 \end{pmatrix}.$$

Since D_1 is regular, it follows that $\Gamma^i \Gamma^j Z \hat{u} \neq 0$ if and only if $E_2 y \neq 0$. Hence, the existence of a vector $\hat{u} \in \ker(Z^\top \Gamma^j Z) \setminus \{0\}$ with $\Gamma^i \Gamma^j Z \hat{u} \neq 0$ is equivalent to $\ker(G) \not\subseteq \ker(E_2)$. Since G is symmetric, the latter is equivalent to $\mathrm{Im}(E_2^\top) \not\subseteq \mathrm{Im}(G)$. For $\omega_j > 0$, we obtain

$$Q^\top \Gamma_\omega Q = \begin{pmatrix} H_\omega & \omega_j E_2 \\ \omega_j E_2^\top & \omega_j G \end{pmatrix}, \qquad H_\omega := (1-\omega_j) D_1 + \omega_j E_1.$$

Note that by Sylvester's law of inertia, the matrices $Q^\top \Gamma_\omega Q$ and Γ_ω have the same inertia. Therefore, it suffices to examine the definiteness properties of $Q^\top \Gamma_\omega Q$. By Lemma 2.2, we have $Q^\top \Gamma_\omega Q \in \mathcal{S}_+^n$ if and only if $\omega_j G \in \mathcal{S}_+^{\dim(\ker(\Gamma^i))}$, $\mathrm{Im}(\omega_j E_2^\top) \subseteq \mathrm{Im}(\omega_j G)$, and $H_\omega - (\omega_j E_2)(\omega_j G)^\dagger (\omega_j E_2)^\top \in \mathcal{S}_+^{n-\dim(\ker(\Gamma^i))}$. Since $\omega_j > 0$ and $(\omega_j G)^\dagger = \frac{1}{\omega_j} G^\dagger$ (see the definition of the Moore-Penrose generalized inverse [45, (1.1.35)]), the latter is equivalent to $G \in \mathcal{S}_+^{\dim(\ker(\Gamma^i))}$, $\mathrm{Im}(E_2^\top) \subseteq \mathrm{Im}(G)$, and $H_\omega - \omega_j E_2 G^\dagger E_2^\top \in \mathcal{S}_+^{n-\dim(\ker(\Gamma^i))}$. Hence, the image condition $\mathrm{Im}(E_2^\top) \subseteq \mathrm{Im}(G)$ is necessary for the existence of any $\omega_j > 0$ with $Q^\top \Gamma_\omega Q \in \mathcal{S}_+^n$. Thus, if there exists $\hat{u} \in \ker(Z^\top \Gamma^j Z) \setminus \{0\}$ with $\Gamma^i \Gamma^j Z \hat{u} \neq 0$, then $\mathrm{Im}(E_2^\top) \not\subseteq \mathrm{Im}(G)$ and therefore $\Gamma_\omega \notin \mathcal{S}_+^n$ for all $\omega_j > 0$. Consequently, $\sigma_+ = 0$.

Conversely, assume that no such vector $\hat{u}$ exists. Then $\mathrm{Im}(E_2^\top) \subseteq \mathrm{Im}(G)$. Moreover,

$$H_\omega - \omega_j E_2 G^\dagger E_2^\top = D_1 + \omega_j (E_1 - D_1 - E_2 G^\dagger E_2^\top).$$

For $\omega_j = 0$, this matrix equals $D_1 \in \mathcal{S}_{++}^{n-\dim(\ker(\Gamma^i))}$. Hence, by continuity, $H_\omega - \omega_j E_2 G^\dagger E_2^\top$ is positive definite for all sufficiently small $\omega_j > 0$. Thus, by Lemma 2.2, there exists $\omega_j > 0$ such that $Q^\top \Gamma_\omega Q \in \mathcal{S}_+^n$. This proves $\sigma_+ > 0$.

Finally, if no such vector $\hat{u}$ exists and $\Gamma^j \notin \mathcal{S}_+^n$, then the choice $\omega_j = 1$ is not feasible in (18). Since the feasible set is a closed interval in $[0, 1]$, it follows that $\sigma_+ < 1$. Hence, $0 < \sigma_+ < 1$. This proves property *(iii)* and ends the proof. □

From a practical point of view, the vector $\hat{u} \in \ker(Z^\top \Gamma^j Z) \setminus \{0\}$ in Proposition 4.6 *(iii)* can be obtained from the eigenvalue decomposition of $Z^\top \Gamma^j Z$. Since $\lambda_{\min}(Z^\top \Gamma^j Z) = 0$, the eigenspace corresponding to the eigenvalue 0 is nontrivial. Let $\{\hat{u}^1, \ldots, \hat{u}^r\}$ be a basis of this eigenspace. Then it suffices to check whether $\Gamma^i \Gamma^j Z \hat{u}^\ell \neq 0$ holds for at least one $\ell \in [r]$. Indeed, if $\Gamma^i \Gamma^j Z \hat{u}^\ell = 0$ for all $\ell \in [r]$, then by linearity $\Gamma^i \Gamma^j Z \hat{u} = 0$ for every $\hat{u} \in \ker(Z^\top \Gamma^j Z)$.

Remark 4.7. *The assumption in Proposition 4.6 does not depend on the particular choice of the basis matrix Z of* $\ker(\Gamma^i)$*. Indeed, let $Z, Z' \in \mathbb{R}^{n \times \dim(\ker(\Gamma^i))}$ be such that the columns of both matrices form a basis of* $\ker(\Gamma^i)$*. Then, there exists a regular matrix $T \in \mathbb{R}^{\dim(\ker(\Gamma^i)) \times \dim(\ker(\Gamma^i))}$ such that $Z = Z'T$. Hence, we obtain $Z^\top \Gamma^j Z = T^\top (Z'^\top \Gamma^j Z') T$. By Sylvester's law of inertia, both matrices have the same inertia. In particular, it follows that $Z^\top \Gamma^j Z \in \mathcal{S}_{++}^{\dim(\ker(\Gamma^i))}$ if and only if $Z'^\top \Gamma^j Z' \in \mathcal{S}_{++}^{\dim(\ker(\Gamma^i))}$.*

In the following, we consider the case where $\ker(\Gamma^i) \subseteq \ker(\Gamma^j)$. Under this assumption, we can describe the structure of the positive semidefinite convex combinations on the interval $[0, \sigma_+]$ more precisely.

Lemma 4.8. *Let $\Gamma^i \in \partial \mathcal{S}_+^n$ and $\Gamma^j \notin \mathcal{S}_+^n$ for $i, j \in [2]$, $i \neq j$, and assume that* $\ker(\Gamma^i) \subseteq \ker(\Gamma^j)$*. Further, let σ_+ be defined as in* (18)*. Then, $0 < \sigma_+ < 1$ and* $\ker(\Gamma_\omega) = \ker(\Gamma^i)$ *for all $\omega_j \in [0, \sigma_+)$.*

Proof. Since the columns of Z form a basis of $\ker(\Gamma^i)$ and $\ker(\Gamma^i) \subseteq \ker(\Gamma^j)$, we have $\Gamma^j Z = 0$. Hence, $Z^\top \Gamma^j Z = 0 \in \partial \mathcal{S}_+^{\dim(\ker(\Gamma^i))}$ and $\Gamma^i \Gamma^j Z \hat{u} = 0$ for all $\hat{u} \in \mathbb{R}^{\dim(\ker(\Gamma^i))}$. Since $\Gamma^j \notin \mathcal{S}_+^n$ by assumption, Proposition 4.6 *(iii)* yields $0 < \sigma_+ < 1$.

It remains to prove $\ker(\Gamma_\omega) = \ker(\Gamma^i)$ for all $\omega_j \in [0, \sigma_+)$. First, the inclusion $\ker(\Gamma^i) \subseteq \ker(\Gamma_\omega)$ follows directly from $\ker(\Gamma^i) \subseteq \ker(\Gamma^j)$.

Conversely, let $\omega_j \in [0, \sigma_+)$ and $x \in \ker(\Gamma_\omega)$. If $\omega_j = 0$, then $\Gamma_\omega = \Gamma^i$ and hence $x \in \ker(\Gamma^i)$. Thus, let $\omega_j > 0$. Then, we have in particular $x^\top \Gamma_\omega x = 0$. Suppose, to the contrary, that $x \notin \ker(\Gamma^i)$. Since $\Gamma^i \in \mathcal{S}_+^n$, we obtain $x^\top \Gamma^i x > 0$. From

$$0 = x^\top \Gamma_\omega x = (1 - \omega_j) x^\top \Gamma^i x + \omega_j x^\top \Gamma^j x$$

it follows that $x^\top \Gamma^j x < 0$. Hence, increasing ω_j slightly yields $x^\top \big((1 - \omega_j - \varepsilon)\Gamma^i + (\omega_j + \varepsilon)\Gamma^j\big) x < 0$ for sufficiently small $\varepsilon > 0$ with $\omega_j + \varepsilon < \sigma_+$. This contradicts the definition of σ_+. Thus, $x \in \ker(\Gamma^i)$. Therefore, $\ker(\Gamma_\omega) = \ker(\Gamma^i)$ for all $\omega_j \in [0, \sigma_+)$, which ends the proof. □

We now combine the preceding matrix characterization with the image condition from Corollary 3.2. This yields the following criteria for the nonemptiness of $\mathbb{R}^n_{we}$ and $\mathbb{R}^n_{pe}$.

Proposition 4.9. *Let $\Gamma^i \in \partial\mathcal{S}^n_+$ for some $i \in [2]$, and let $j \in [2]$ with $i \neq j$. Further, let σ_+ be defined as in (18). Then the following properties are true.*

(i) If $\Lambda_+ \neq \emptyset$, then $\mathbb{R}^n_{pe} \neq \emptyset$.

(ii) If $\sigma_+ = 0$, then $\mathbb{R}^n_{pe} = \emptyset$. Moreover, $\mathbb{R}^n_{we} \neq \emptyset$ if and only if $\beta^i \in \mathrm{Im}(\Gamma^i)$. In the latter case, we have in particular $\mathbb{R}^n_{we} = \mathrm{argmin}_{x\in\mathbb{R}^n} f_i(x) \neq \emptyset$.

(iii) If $\sigma_+ > 0$ and $\Lambda_+ = \emptyset$, then $\mathbb{R}^n_{we} \neq \emptyset$ if and only if there exists $\omega_j \in [0, \sigma_+]$ such that $(1-\omega_j)\beta^i + \omega_j\beta^j \in \mathrm{Im}((1-\omega_j)\Gamma^i + \omega_j\Gamma^j)$.

Moreover, $\mathbb{R}^n_{pe} \neq \emptyset$ if and only if there exists $\omega_j \in [0, \sigma_+] \cap (0, 1)$ such that $(1-\omega_j)\beta^i + \omega_j\beta^j \in \mathrm{Im}((1-\omega_j)\Gamma^i + \omega_j\Gamma^j)$.

(iv) Suppose, in addition, that $\Gamma^j \notin \mathcal{S}^n_+$ and $\ker(\Gamma^i) \subseteq \ker(\Gamma^j)$. Then $\mathbb{R}^n_{pe} \neq \emptyset$ if and only if either $(1-\sigma_+)\beta^i + \sigma_+\beta^j \in \mathrm{Im}((1-\sigma_+)\Gamma^i + \sigma_+\Gamma^j)$ or there exists $\omega_j \in (0, \sigma_+)$ such that $(1-\omega_j)\beta^i + \omega_j\beta^j \in \mathrm{Im}(\Gamma^i)$.

Proof. Property *(i)* follows directly by Corollary 3.21 and Corollary 4.2.

We now prove property *(ii)*. If $\sigma_+ = 0$, then we have $\Lambda = \{e^i\}$. Since $e^i \notin \Delta_+$, it follows from Corollary 4.2 that $\mathbb{R}^n_{pe} = \emptyset$. Moreover, by Corollary 3.2, the weighted sum scalarization corresponding to $\omega = e^i$ has a minimal solution if and only if $\beta^i \in \mathrm{Im}(\Gamma^i)$. Since this is the only positive semidefinite convex combination, we obtain $\mathbb{R}^n_{we} \neq \emptyset$ if and only if $\beta^i \in \mathrm{Im}(\Gamma^i)$. In this case, the set of weakly efficient solutions is given by $\mathbb{R}^n_{we} = \mathrm{argmin}_{x\in\mathbb{R}^n} f_i(x) \neq \emptyset$.

We now prove property *(iii)*. Let $\sigma_+ > 0$ and $\Lambda_+ = \emptyset$. By the definition of σ_+ and Proposition 4.6, the positive semidefinite convex combinations are precisely those corresponding to weights $\omega = (1-\omega_j)e^i + \omega_j e^j$ with $\omega_j \in [0, \sigma_+]$. By Corollary 3.2, such a weighted sum scalarization has a minimal solution if and only if

$$(1-\omega_j)\beta^i + \omega_j\beta^j \in \mathrm{Im}\big((1-\omega_j)\Gamma^i + \omega_j\Gamma^j\big).$$

Since supported solutions coincide with weakly efficient solutions in the biobjective case, this proves the assertion for $\mathbb{R}^n_{we}$. For properly efficient solutions, we additionally have to require that the corresponding weight vector belongs to Δ_+. In the biobjective case, this is equivalent to $\omega_j \in (0, 1)$. Combining this with $\omega_j \in [0, \sigma_+]$, we obtain the desired result.

It remains to prove property *(iv)*. Suppose that $\Gamma^j \notin \mathcal{S}^n_+$ and $\ker(\Gamma^i) \subseteq \ker(\Gamma^j)$. By Lemma 4.8, we have $0 < \sigma_+ < 1$ and $\ker(\Gamma_\omega) = \ker(\Gamma^i)$ for all $\omega_j \in [0, \sigma_+)$. Hence,

$$\mathrm{Im}(\Gamma_\omega) = \ker(\Gamma_\omega)^\perp = \ker(\Gamma^i)^\perp = \mathrm{Im}(\Gamma^i)$$

for all $\omega_j \in [0, \sigma_+)$. Since $0 < \sigma_+ < 1$, a weight vector $\omega = (1-\omega_j)e^i + \omega_j e^j$ belongs to Δ_+ and satisfies $\Gamma_\omega \in \mathcal{S}^n_+$ if and only if either $\omega_j \in (0, \sigma_+)$ or $\omega_j = \sigma_+$. Therefore, by property *(iii)*, the set $\mathbb{R}^n_{pe}$ is nonempty if and only if either $(1-\sigma_+)\beta^i + \sigma_+\beta^j \in \mathrm{Im}((1-\sigma_+)\Gamma^i + \sigma_+\Gamma^j)$ or there exists $\omega_j \in (0, \sigma_+)$ such that $(1-\omega_j)\beta^i + \omega_j\beta^j \in \mathrm{Im}(\Gamma^i)$. This proves property *(iv)* and ends the proof. $\square$

The conditions in Proposition 4.9 can be checked by linear algebraic computations. First, the condition $\ker(\Gamma^i) \subseteq \ker(\Gamma^j)$ is equivalent to $\operatorname{Im}(\Gamma^j) \subseteq \operatorname{Im}(\Gamma^i)$ and hence to the solvability of the matrix equation $\Gamma^i V = \Gamma^j$ for some $V \in \mathbb{R}^{n\times n}$. Indeed, writing $V = (v_1, \dots, v_n)$ and $\Gamma^j = (g_1, \dots, g_n)$, this means precisely that $\Gamma^i v_k = g_k$ for all $k \in [n]$, i.e., that each column of Γ^j belongs to $\operatorname{Im}(\Gamma^i)$. This criterion is also stated in [38, Chapter 5].

If the kernel condition holds, then the determination of σ_+ reduces to the positive semidefinite interval of the reduced pencil $D_1 + tE_1$, $t \in \mathbb{R}$, similar to the proof of Proposition 4.6. By [38, Theorem 2.6, Theorem 4.3, Theorem 3.3], the endpoints of the reduced pencil are determined by the extreme eigenvalues of a matrix M satisfying $D_1 M = E_1$ and $\operatorname{Im}(M) \subseteq \operatorname{Im}(D_1)$. Since D_1 is nonsingular, one can use $M = D_1^{-1}E_1$. Hence, σ_+ can be computed by solving the eigenvalue problem for $D_1^{-1}E_1$ and transforming the corresponding right endpoint $\bar{t}$ of the interval for $D_1 + tE_1$ via

$$\sigma_+ = \frac{\bar{t}}{1+\bar{t}}. \tag{19}$$

Moreover, the existence of a weight $\omega_j \in (0, \sigma_+)$ satisfying $(1-\omega_j)\beta^i + \omega_j \beta^j \in \operatorname{Im}(\Gamma^i)$ is equivalent to the feasibility of the linear system $\Gamma^i x + (\beta^i - \beta^j)\omega_j = \beta^i$ with constraint $0 < \omega_j < \sigma_+$. Thus, in this case, the question of the existence of weakly and properly efficient solutions is reduced to linear feasibility checks and, in the case of σ_+, to an eigenvalue computation for the reduced matrix pencil.

We illustrate the preceding cases by the following examples. The first example illustrates the case where the restriction of Γ^j to $\ker(\Gamma^i)$ is not positive semidefinite. In this case, Proposition 4.6 *(ii)* yields $\Lambda = \{e^i\}$ and $\Lambda_+ = \emptyset$.

Example 4.10. *For* $\Gamma^1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \in \partial\mathcal{S}^2_+$ *and* $\Gamma^2 = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \in -\mathcal{S}^2_+$ *we set* $i = 1$ *and* $j = 2$. *We have* $\ker(\Gamma^1) = \operatorname{span}\left\{(1,1)^\top\right\}$. *For* $Z = (1,1)^\top$, *it follows that* $Z^\top\Gamma^2 Z = -4 < 0$. *Hence, by Proposition 4.6 (ii), we obtain* $\Lambda = \{e^1\}$ *and* $\Lambda_+ = \emptyset$. *In particular,* $\sigma_+ = 0$. *By Proposition 4.9 (ii), it follows that* $\mathbb{R}^2_{pe} = \emptyset$ *independently of the choices of* β^1, β^2, α. *Moreover,* $\mathbb{R}^2_{we} \neq \emptyset$ *if and only if* $\beta^1 \in \operatorname{Im}(\Gamma^1)$. *Since* $\operatorname{Im}(\Gamma^1) = \operatorname{span}\left\{(1,-1)^\top\right\}$, *we obtain with the choice* $\beta^1 = (1,-1)^\top$, *i.e., for*

$$f_1(x) = (x_1 - x_2)^2 + 2(x_1 - x_2) + \alpha_1 = (x_1 - x_2 + 1)^2 + \alpha_1 - 1,$$

that $\mathbb{R}^2_{we} = \operatorname{argmin}_{x\in\mathbb{R}^2} f_1(x) = \left\{x \in \mathbb{R}^2 \mid x_1 - x_2 = -1\right\}$. *In particular, this characterization is independent of the choices of* β^2 *and* α. *In addition, for any* $\beta^1 \notin \operatorname{span}\left\{(1,-1)^\top\right\}$, *we obtain* $\mathbb{R}^2_{we} = \emptyset$. *By Corollary 4.3, the latter is equivalent to* $f(\mathbb{R}^2) + \mathbb{R}^2_{\geqq} = \mathbb{R}^2$.

The next example illustrates the degenerate singular case in Proposition 4.6 *(iii)*, where $Z^\top\Gamma^j Z \in \partial\mathcal{S}^n_+$ but $\sigma_+ = 0$.

Example 4.11. *We consider again Example 3.17. Then we have* $i = 1$, $j = 2$, $Z = (0,1)^\top$, *and* $Z^\top\Gamma^2 Z = 0$. *Thus,* $Z^\top\Gamma^2 Z \in \partial\mathcal{S}^1_+$. *Moreover,*

$$\ker(\Gamma^1) = \operatorname{span}\left\{(0,1)^\top\right\} \not\subseteq \ker(\Gamma^2) = \{0\}.$$

With the choice $\hat{u} := 1$, *we obtain* $\Gamma^1\Gamma^2 Z\hat{u} = (1,0)^\top \neq 0$. *Hence, Proposition 4.6 (iii) yields* $\sigma_+ = 0$, *and therefore* $\Lambda = \{e^1\}$ *and* $\Lambda_+ = \emptyset$. *By Proposition 4.9 (ii), it follows that* $\mathbb{R}^2_{pe} = \emptyset$. *Moreover, since* $\beta^1 = 0 \in \operatorname{Im}(\Gamma^1)$, *we obtain*

$$\mathbb{R}^2_{we} = \operatorname*{argmin}_{x\in\mathbb{R}^2} f_1(x) = \operatorname*{argmin}_{x\in\mathbb{R}^2} x_1^2 = \left\{(0,x_2)^\top \mid x_2 \in \mathbb{R}\right\}.$$

In particular, we have $Y_{wnd} = \{(0,0)^\top\}$.

The final example illustrates the case covered by Lemma 4.8, where $\ker(\Gamma^i) \subseteq \ker(\Gamma^j)$, $\Gamma^j \notin \mathcal{S}^n_+$, and therefore $0 < \sigma_+ < 1$.

Example 4.12. *For* $\Gamma^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \partial\mathcal{S}^3_+$ *and* $\Gamma^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in -\mathcal{S}^3_+$ *we set* $i = 1$ *and* $j = 2$. *We have*

$$\ker(\Gamma^1) = \operatorname{span}\left\{(0,0,1)^\top\right\} \subseteq \operatorname{span}\left\{(0,1,0)^\top, (0,0,1)^\top\right\} = \ker(\Gamma^2).$$

Thus, Lemma 4.8 applies. Moreover, for the reduced matrices we have

$$D_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad E_1 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, $\lambda_{\min}(D_1^{-1}E_1) = -1$. *By [38, Theorem 2.6, Theorem 4.4, Theorem 3.3], we obtain* $\bar{t} = 1$, *and therefore, by* (19), $\sigma_+ = \frac{1}{2}$. *In addition,* $\operatorname{Im}\big((1-\sigma_+)\Gamma^1 + \sigma_+\Gamma^2\big) = \operatorname{span}\left\{(0,1,0)^\top\right\}$. *For* $\beta^1 = (1,1,0)^\top$ *and* $\beta^2 = (-1,0,0)^\top$, *we have*

$$(1-\sigma_+)\beta^1 + \sigma_+\beta^2 = \left(0, \tfrac{1}{2}, 0\right)^\top \in \operatorname{span}\left\{(0,1,0)^\top\right\}.$$

Thus, by Proposition 4.9 (iv), we obtain $\mathbb{R}^3_{pe} \neq \emptyset$ *for* (BI-UC).

Finally, we consider the cases where neither matrix is positive semidefinite. If both matrices Γ^1 and Γ^2 are indefinite, all three situations may occur: one may have $\Lambda_+ \neq \emptyset$, as in Example 3.27; $\Lambda \neq \emptyset$ and $\Lambda_+ = \emptyset$ with possibly $\Omega_+ \neq \emptyset$, as in Example 3.8 *(b)*; or $\Lambda = \emptyset$, as in Example 3.9. If both matrices satisfy $\Gamma^1, \Gamma^2 \in -\mathcal{S}^n_+ \setminus \{0\}$, then $\Lambda = \emptyset$, since $-\mathcal{S}^n_+$ is a convex cone and every convex combination Γ_ω belongs to $-\mathcal{S}^n_+ \setminus \{0\}$. If $\Gamma^i \in -\mathcal{S}^n_+ \setminus \{0\}$ and Γ^j is indefinite, then also $\Lambda = \emptyset$, since for $\omega_j = 0$ we have $\Gamma_\omega = \Gamma^i \notin \mathcal{S}^n_+$, while for every $\omega_j > 0$ there exists $x \in \mathbb{R}^n \setminus \{0\}$ such that $x^\top\Gamma^j x < 0$, and hence $x^\top\Gamma_\omega x < 0$.

We now provide a summary of the results of Section 4 in Table 1 below. According to Lemma 3.10, classes 1, 2, and 3 are pairwise disjoint for given matrices Γ^1 and Γ^2. The symbol $\dot\vee$ should thus be understood to mean that exactly one of the identities 1, 2, or 3 is fulfilled. This means, for example, that the combination $\Gamma^1, \Gamma^2 \in \partial\mathcal{S}^n_+ \setminus \{0\}$ can end up either in class 1 or 2, depending on the characteristics of the matrices Γ^1 and Γ^2. Note that in case 2, the existence of weakly efficient solutions does not immediately follow, since the image condition from Corollary 3.2 must also be satisfied.

1: $\Lambda_+ \neq \emptyset$ ($\Rightarrow \mathbb{R}^n_{pe} \neq \emptyset$) 2: $\Lambda_+ = \emptyset$ and $\Lambda \neq \emptyset$ 3: $\Lambda = \emptyset$ ($\Rightarrow \mathbb{R}^n_{we} = \emptyset$)

Γ^2 \ Γ^1	0	$\in \mathcal{S}^n_{++}$	$\in \partial\mathcal{S}^n_+ \setminus \{0\}$	indef.	$\in -\partial\mathcal{S}^n_+ \setminus \{0\}$	$\in -\mathcal{S}^n_{++}$
0	2	1	2	2	2	2
$\in \mathcal{S}^n_{++}$	1	1	1	1	1	1
$\in \partial\mathcal{S}^n_+ \setminus \{0\}$	2	1	1 $\dot\vee$ 2	1 $\dot\vee$ 2	2	2
indef.	2	1	1 $\dot\vee$ 2	1 $\dot\vee$ 2 $\dot\vee$ 3	3	3
$\in -\partial\mathcal{S}^n_+ \setminus \{0\}$	2	1	2	3	3	3
$\in -\mathcal{S}^n_{++}$	2	1	2	3	3	3

Table 1: Summarizing overview

We close this section by observing how the results in this section can be extended to unconstrained quadratic multiobjective optimization problems with more than two objective functions. We first state the following lemma.

Lemma 4.13. *Consider the unconstrained quadratic multiobjective optimization problem* (QMOP-UC)*. Let* $\{i, j\} \subseteq [m]$ *be such that* $i \neq j$*.*

(i) If there exists $\sigma \in [0, 1]$ *such that* $(1-\sigma)\Gamma^i + \sigma\Gamma^j \in \mathcal{S}^n_+$ *and* $(1-\sigma)\beta^i + \sigma\beta^j \in \mathrm{Im}((1-\sigma)\Gamma^i + \sigma\Gamma^j)$*, then* $\mathbb{R}^n_{we} \neq \emptyset$*.*

(ii) If there exists $\sigma \in [0, 1]$ *such that* $(1-\sigma)\Gamma^i + \sigma\Gamma^j \in \mathcal{S}^n_{++}$*, then* $\mathbb{R}^n_{pe} \neq \emptyset$*.*

Proof. Let $\omega = (1-\sigma)e^i + \sigma e^j \in \Delta$, where $e^i \in \mathbb{R}^m$ denotes the i-th unit vector. Then, the assertion *(i)* follows from Corollary 3.2 and (1). Concerning the assertion *(ii)*, there exists an $\varepsilon > 0$ such that $\Gamma_{\omega'} \in \mathcal{S}^n_{++}$ for all $\omega' \in \Delta_+$ such that $\|\omega' - \omega\|_2 < \varepsilon$. The assertion follows from (1). □

For unconstrained quadratic multiobjective optimization problems with more than two objective functions, Lemma 4.13 reveals that our results in this section can be used to establish sufficient conditions for the existence of weakly and properly efficient solutions whenever any pair of objective functions satisfies the corresponding assumptions of Lemma 4.13.

5 Constrained Biobjective Optimization Problems

The previous sections show that convexlikeness in quadratic multiobjective optimization is very sensitive to the number of objectives. For $m \geqq 3$, it can already fail in the unconstrained setting. In order to distinguish the effects of constraints from the effects of multiple objectives, we restrict this section to the case with two objectives, i.e., $m = 2$. We then examine how linear constraints influence convexlikeness. While linear equality constraints preserve convexlikeness, we show that this property may be violated by linear inequality constraints.

5.1 Biobjective Optimization with Linear Equality Constraints

We now consider the quadratic biobjective optimization problem with linear equality constraints of the form

$$\begin{aligned} &\text{minimize} \qquad g(u) = \begin{pmatrix} g_1(u) \\ g_2(u) \end{pmatrix} := \begin{pmatrix} u^\top Q^1 u + 2(c^1)^\top u \\ u^\top Q^2 u + 2(c^2)^\top u \end{pmatrix} \\ &\text{subject to} \qquad Au = b, \quad u \in \mathbb{R}^d, \end{aligned} \tag{BI-LIN}$$

where $g_i : \mathbb{R}^d \to \mathbb{R}$ for all $i \in [2]$, $Q^1, Q^2 \in \mathcal{S}^d$ are real symmetric matrices, $c^1, c^2 \in \mathbb{R}^d$, $A \in \mathbb{R}^{r\times d}$, $b \in \mathbb{R}^r$, $r, d \in \mathbb{N}$, and $1 \leqq r = \mathrm{rank}(A) < d$. The feasible set of (BI-LIN) is $\tilde{X} := \{u \in \mathbb{R}^d \mid Au = b\}$. Since A has full row rank $r \geqq 1$, it follows that $\mathrm{Im}(A) = \mathbb{R}^r$, so the set $\tilde{X}$ is nonempty. The feasible region $\tilde{X}$ is, in fact, an affine subspace. Adding constants to the objective components does not change the set of weakly (respectively, properly) efficient solutions. For this reason, we omit constants in this section.

To show that (BI-LIN) is also convexlike, we reformulate it into an unconstrained biobjective problem of the form (BI-UC). By using the above results for the unconstrained case, we can then draw conclusions about the constrained problem (BI-LIN). For this purpose, we need the following result.

Lemma 5.1. *Let $u_p \in \mathbb{R}^d$ be a particular solution of the system $Au = b$ and let the columns of $Z \in \mathbb{R}^{d\times(d-r)}$ be a basis of the kernel of A. Then, the mapping*

$$\Phi : \mathbb{R}^{d-r} \to \tilde{X}, \quad x \mapsto u_p + Zx$$

is a bijection.

Proof. For any $x \in \mathbb{R}^{d-r}$, we have $u := u_p + Zx \in \mathbb{R}^d$. Since $A(u_p + Zx) = Au_p + AZx = b$, we have that $u \in \tilde{X}$. Thus, $\Phi(x) \in \tilde{X}$ and Φ is well-defined.

Now let $u \in \tilde{X}$, which implies $Au = b$. Set $\tilde{u} := u - u_p$, then $A\tilde{u} = Au - Au_p = 0$, hence $\tilde{u} \in \ker(A)$. Since the columns of Z form a basis of the kernel of A, there exists a vector $x \in \mathbb{R}^{d-r}$ such that $\tilde{u} = Zx$. Hence, $u = u_p + \tilde{u} = u_p + Zx = \Phi(x)$, and therefore Φ is surjective.

Next, let $x^1, x^2 \in \mathbb{R}^{d-r}$ and $\Phi(x^1) = \Phi(x^2)$, which is equivalent to $u_p + Zx^1 = u_p + Zx^2$. This implies $Z(x^1 - x^2) = 0$. Since the columns of Z form a basis of the kernel of A, the latter is only possible if $x^1 = x^2$. Thus, Φ is injective. □

We now substitute any feasible solution $u \in \tilde{X}$ with the term $u_p + Zx$ for $x \in \mathbb{R}^{d-r}$ in the objectives for all $i \in [2]$ for some fixed particular solution $u_p \in \tilde{X}$, which results in

$$g_i(u_p + Zx) = x^\top (Z^\top Q^i Z)x + 2(Q^i u_p + c^i)^\top Zx + 2(c^i)^\top u_p + (u_p)^\top Q^i u_p.$$

We further define

$$\begin{aligned}
\Gamma^i &:= Z^\top Q^i Z \in \mathcal{S}^{d-r}, \\
\beta^i &:= Z^\top (Q^i u_p + c^i) \in \mathbb{R}^{d-r}, \\
\alpha_i &:= 2(c^i)^\top u_p + (u_p)^\top Q^i u_p \in \mathbb{R}
\end{aligned}$$

for all $i \in [2]$, which brings us to the reformulated unconstrained optimization problem

$$\underset{x\in\mathbb{R}^{d-r}}{\text{minimize}} \qquad f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} := \begin{pmatrix} x^\top \Gamma^1 x + 2(\beta^1)^\top x + \alpha_1 \\ x^\top \Gamma^2 x + 2(\beta^2)^\top x + \alpha_2 \end{pmatrix}, \qquad \text{(BI-UC-REF)}$$

which is of the form (BI-UC) for $n := d - r$. Note here again that the inertia of Γ^i for $i \in [2]$ is always constant, regardless of the choice of matrix Z. The following result provides identity of the image sets of (BI-LIN) and (BI-UC-REF).

Lemma 5.2. *The sets of objective values of* (BI-LIN) *and* (BI-UC-REF) *coincide, that is, $g(\tilde{X}) = f(\mathbb{R}^{d-r})$. More precisely, for every feasible solution $u \in \tilde{X}$ there exists a unique vector $x \in \mathbb{R}^{d-r}$ such that $u = u_p + Zx$, and for this vector we have $g(u) = f(x)$. Conversely, for every $x \in \mathbb{R}^{d-r}$ there exists a unique feasible vector $u = u_p + Zx \in \tilde{X}$ such that $g(u) = f(x)$.*

Proof. According to Lemma 5.1, Φ is a bijection. Thus,

$$g(\tilde{X}) = \{g(u) \mid u \in \tilde{X}\} = \{g(\Phi(x)) \mid x \in \mathbb{R}^{d-r}\} = \{f(x) \mid x \in \mathbb{R}^{d-r}\} = f(\mathbb{R}^{d-r}).$$

□

The following example illustrates how, by applying the above reformulation to an optimization problem of the form (BI-LIN), one obtains an optimization problem of the form (BI-UC-REF). In particular, the reformulated problem satisfies $\alpha_i \neq 0$ for $i \in [2]$.

Example 5.3. *Consider* (BI-LIN) *with* $Q^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $Q^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $c^1 = (0,0)^\top$, $c^2 = (1,0)^\top$, $A = (1,-1)$, *and* $b = 1$. *Then* $g_1(u) = u_1^2$, $g_2(u) = u_2^2 + 2u_1$ *and* $\ker(A) = \operatorname{span}\{(1,1)^\top\}$. *For* $u_p = (1,0)^\top$ *and* $Z = (1,1)^\top$, *the reformulated problem* (BI-UC-REF) *satisfies* $\Gamma^1 = \Gamma^2 = 1$, $\beta^1 = \beta^2 = 1$, *and* $\alpha_1 = 1$, $\alpha_2 = 2$. *Thus,* $f_1(x) = x^2 + 2x + 1$, $f_2(x) = x^2 + 2x + 2$ *and, in particular,* $\alpha_i \neq 0$ *for all* $i \in [2]$.

The efficiency properties are retained as follows.

Lemma 5.4. *Let* $u_p \in \mathbb{R}^d$ *be a particular solution of the system* $Au = b$, *and let the columns of* $Z \in \mathbb{R}^{d\times(d-r)}$ *be a basis of the kernel of* A. *Furthermore, let* $\bar{x} \in \mathbb{R}^{d-r}$ *be arbitrary and define* $\bar{u} := u_p + Z\bar{x} \in \tilde{X}$. *Then the following properties are true:*

(i) $\bar{x} \in \mathbb{R}^{d-r}_e$ *for* (BI-UC-REF) *if and only if* $\bar{u} \in \tilde{X}_e$ *for* (BI-LIN)*;*

(ii) $\bar{x} \in \mathbb{R}^{d-r}_{pe}$ *for* (BI-UC-REF) *if and only if* $\bar{u} \in \tilde{X}_{pe}$ *for* (BI-LIN)*;*

(iii) $\bar{x} \in \mathbb{R}^{d-r}_{we}$ *for* (BI-UC-REF) *if and only if* $\bar{u} \in \tilde{X}_{we}$ *for* (BI-LIN)*.*

Proof. Let $\bar{u} = u_p + Z\bar{x} \in \tilde{X}$ be an efficient solution of (BI-LIN). Assume that $\bar{x} \in \mathbb{R}^{d-r}$ is not efficient. Then there exists a vector $x \in \mathbb{R}^{d-r}$ with $f_i(x) \leq f_i(\bar{x})$ for all $i \in [2]$ and $f_j(x) < f_j(\bar{x})$ for at least one $j \in [2]$. By Lemma 5.2, $u = u_p + Zx \in \tilde{X}$, and therefore $g_i(u) = f_i(x) \leq f_i(\bar{x}) = g_i(\bar{u})$ for all $i \in [2]$ and $g_j(u) = f_j(x) < f_j(\bar{x}) = g_j(\bar{u})$ for at least one $j \in [2]$, which contradicts the efficiency of $\bar{u} \in \tilde{X}$. Thus, $\bar{x}$ must be efficient for (BI-UC-REF). The proof of the converse and the weakly efficient property is analogous, which proves *(i)* and *(iii)*.

Now, let $\bar{u} = u_p + Z\bar{x} \in \mathbb{R}^d$ be properly efficient for (BI-LIN), that is, there exists $K > 0$ such that for all $g(u)$, $u \in \tilde{X}$, and all $i \in [2]$ with $g_i(u) < g_i(\bar{u})$ there exists $j \in [2]$ with $g_j(u) > g_j(\bar{u})$ such that

$$\frac{g_i(\bar{u}) - g_i(u)}{g_j(u) - g_j(\bar{u})} \leqq K.$$

According to Lemma 5.2, we have $g(\tilde{X}) = f(\mathbb{R}^{d-r})$ and any vector $x \in \mathbb{R}^{d-r}$ corresponds one by one to a vector $u = u_p + Zx \in \tilde{X}$ such that $g_i(u) = f_i(x)$ for all $i \in [2]$ and thus

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} = \frac{g_i(\bar{u}) - g_i(u)}{g_j(u) - g_j(\bar{u})} \leqq K,$$

which implies that $\bar{x}$ is properly efficient for (BI-UC-REF). The reverse direction is again analogous, which completes the proof. $\square$

We now state the convexlikeness result for (BI-LIN). Since the feasible set of (BI-LIN) is affine, this result is naturally related to existing convexity results for pairs of quadratic functions on affine sets; see, for instance, [40, Corollary 10] and the recent discussion in [24, Theorem 4.1]. Here, we indicate explicitly how the unconstrained convexlikeness result stated in Theorem 4.1 can be transferred to the linear equality constrained setting: by Lemma 5.2, the reformulation introduced above has the same image set as (BI-LIN).

Proposition 5.5. *The optimization problem* (BI-LIN) *is convexlike.*

As an immediate consequence of Lemma 5.2 and Lemma 5.4, the image sets of efficient, properly efficient, and weakly efficient solutions are preserved under the reformulation from (BI-LIN) to (BI-UC-REF).

Corollary 5.6. *Let $u_p \in \mathbb{R}^d$ be a particular solution of the system $Au = b$, and let the columns of $Z \in \mathbb{R}^{d\times(d-r)}$ form a basis of the kernel of A. Then $g(\tilde{X}_e) = f(\mathbb{R}^{d-r}_e)$, $g(\tilde{X}_{pe}) = f(\mathbb{R}^{d-r}_{pe})$, and $g(\tilde{X}_{we}) = f(\mathbb{R}^{d-r}_{we})$.*

Since (BI-LIN) is convexlike by Proposition 5.5, the general results established in Section 4 for unconstrained biobjective convexlike problems can be applied to the present setting via the reformulation (BI-UC-REF). We illustrate this by translating Theorem 4.4 *(i)* to the constrained setting here.

Corollary 5.7. *Let $u_p \in \mathbb{R}^d$ be a particular solution of the system $Au = b$, and let the columns of $Z \in \mathbb{R}^{d\times(d-r)}$ form a basis of the kernel of A. Define*

$$\tilde{\Lambda}_+ := \{\omega \in \Delta \mid \omega_1 Z^\top Q^1 Z + \omega_2 Z^\top Q^2 Z \in \mathcal{S}^{d-r}_{++}\}.$$

Then, $\tilde{\Lambda}_+ \neq \emptyset$ if and only if $\{v \in \ker(A) \setminus \{0\} \mid v^\top Q^1 v \leqq 0 \ \wedge \ v^\top Q^2 v \leqq 0\} = \emptyset$.

The following example illustrates this.

Example 5.8. *We consider again Example 5.3. For $\omega \in \Delta$ we have $\omega_1 Z^\top Q^1 Z + \omega_2 Z^\top Q^2 Z = \omega_1 + \omega_2 = 1 > 0$ and therefore $\tilde{\Lambda}_+ = \Delta$. On the other hand, any vector $v \in \ker(A) \setminus \{0\}$ is of the form $v_t = t\ (1,1)^\top$ with $t \neq 0$. This yields $v_t^\top Q^i v_t = t^2 > 0$ for all $i \in [2]$. In particular, $\{v \in \ker(A) \setminus \{0\} \mid v^\top Q^1 v \leqq 0 \ \wedge \ v^\top Q^2 v \leqq 0\} = \emptyset$. Moreover, since $f_1(x) = (x+1)^2$ and $f_2(x) = (x+1)^2 + 1$, the image set is given by $g(\tilde{X}) = f(\mathbb{R}) = \{(y_1, y_2)^\top \in \mathbb{R}^2 \mid y_2 = y_1 + 1, y_1 \geqq 0\}$. Furthermore, for every $\omega \in \Delta$, the weighted sum scalarization of* (BI-UC-REF) *is given by $\omega_1 f_1(x) + \omega_2 f_2(x) = (x+1)^2 + \omega_2$ with unique minimal solution $\bar{x} = -1$. Hence, $\bar{x}$ is a positively supported solution and therefore properly efficient for* (BI-UC-REF)*. Since $f(\bar{x}) = (0,1)^\top$ is the unique weakly nondominated point of $f(\mathbb{R})$, we obtain $\mathbb{R}_{pe} = \mathbb{R}_e = \mathbb{R}_{we} = \{\bar{x}\}$ for* (BI-UC-REF)*, $\tilde{X}_{pe} = \tilde{X}_e = \tilde{X}_{we} = \{(0,-1)^\top\}$ for* (BI-LIN)*, and $g(\tilde{X}_{pe}) = g(\tilde{X}_e) = g(\tilde{X}_{we}) = f(\mathbb{R}_{pe}) = f(\mathbb{R}_e) = f(\mathbb{R}_{we}) = \{f(\bar{x})\} = \{(0,1)^\top\}$. This is illustrated in Figure 6.*

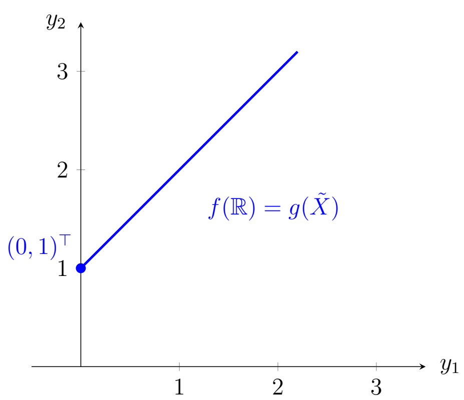

Figure 6: Image set $f(\mathbb{R}) = g(\tilde{X})$ of Example 5.8.

5.2 Biobjective Optimization with Linear Inequality Constraints

For $m = 2$, convexlikeness persists under linear equality constraints but can already fail under linear inequality constraints. To illustrate this, we consider the quadratic biobjective optimization problem

$$\begin{aligned} &\text{minimize} \quad && g(u) = \begin{pmatrix} g_1(u) \\ g_2(u) \end{pmatrix} := \begin{pmatrix} u^\top Q^1 u + 2(c^1)^\top u \\ u^\top Q^2 u + 2(c^2)^\top u \end{pmatrix} \\ &\text{subject to} \quad && Au \leqq b, \quad u \in \mathbb{R}^d, \end{aligned} \tag{BI-LIN-INEQ}$$

where $g_i : \mathbb{R}^d \to \mathbb{R}$ for all $i \in [2]$, $Q^1, Q^2 \in \mathcal{S}^d$ are real symmetric matrices, $c^1, c^2 \in \mathbb{R}^d$, $A \in \mathbb{R}^{r \times d}$, $b \in \mathbb{R}^r$, $r, d \in \mathbb{N}$. The feasible set is denoted by $\bar{X} = \{u \in \mathbb{R}^d \mid Au \leqq b\}$. The following example illustrates that (BI-LIN-INEQ), in general, does not enjoy the convexlikeness property.

Example 5.9. *Let $d = m = 2$ and the objective functions $g_1, g_2 \colon \mathbb{R}^2 \to \mathbb{R}$ be given by*

$$g_1(u) = -u_1^2 - u_2^2 - u_1 u_2, \qquad g_2(u) = -3u_1^2 - u_1 u_2 - 4u_2.$$

We assume the feasible set is a polytope defined by the following three constraints

$$u_1 + u_2 \leqq 1, \qquad -2u_1 - u_2 \leqq 0, \qquad u_1 \leqq 80.$$

The polytope has the vertices $v_1 := (-1, 2)^\top, v_2 := (80, -79)^\top$, and $v_3 := (80, -160)^\top$. Hence, in particular, every feasible point satisfies $-1 \leqq u_1 \leqq 80$ and $-160 \leqq u_2 \leqq 2$. The feasible set $\bar{X}$ and its image $g(\bar{X})$ are illustrated in Figure 7, with $\bar{X}$ shown on the left and $g(\bar{X})$ shown on the right. One can easily see that the upper image $g(\bar{X}) + \mathbb{R}^2_{\geqq}$ is not convex.

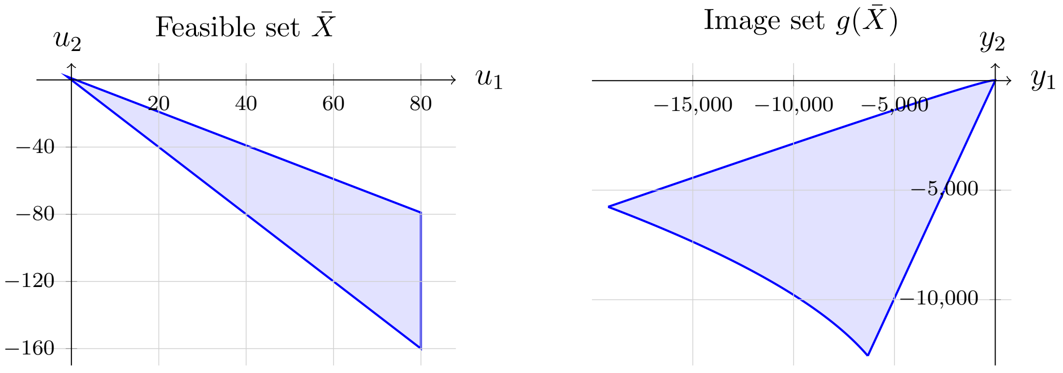

Figure 7: Feasible set $\bar{X}$ and image set $g(\bar{X})$ for the constrained biobjective problem in Example 5.9. The feasible set is shown on the left and the corresponding image set on the right.

6 Conclusion and Outlook

In this paper, we studied structural properties of quadratic multiobjective optimization problems. We characterized the solvability of weighted sum scalarizations in terms of positive semidefinite convex combinations of the matrices defining the quadratic terms.

We also analyzed convexlikeness, in particular for the biobjective unconstrained case and for problems with linear equality constraints. In convexlike settings, we related the absence of weakly efficient solutions to the upper image set being the whole space. Moreover, we showed that convexlikeness does not generally hold in the biobjective case with linear inequality constraints. These results clarify when weighted sum scalarizations are structurally justified and when their limitations should be expected.

Besides their theoretical relevance, our results can be used to construct test instances with prescribed scalarization behavior. In particular, one can generate problems for which all, only some, or none of the weights lead to scalarized problems with minimizers. This may be useful for testing scalarization-based methods and for illustrating structural phenomena in nonconvex quadratic multiobjective optimization.

Further research may address extensions to feasible sets described by quadratic constraints, including trust-region type constraints, and the derivation of further sufficient conditions for convexlikeness, especially beyond the biobjective setting.

Acknowledgments

The authors express their gratitude to Oliver Stein and Felix Neussel for fruitful discussions. The work of the first and the second author is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - Project-ID 528525668.

References

[1] M. Anitescu. Degenerate nonlinear programming with a quadratic growth condition. *SIAM Journal on Optimization*, 10(4):1116–1135, 2000.

[2] A. I. Barvinok. Problems of distance geometry and convex properties of quadratic maps. *Discrete & Computational Geometry*, 13(2):189–202, 1995.

[3] C. Bazgan, S. Ruzika, C. Thielen, and D. Vanderpooten. The power of the weighted sum scalarization for approximating multiobjective optimization problems. *Theory of Computing Systems*, 66(1):395–415, 2022.

[4] A. Ben-Tal and A. Nemirovski. *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*. MPS/SIAM Series on Optimization. SIAM, 2001.

[5] A. Berman and A. Ben-Israel. A note on pencils of Hermitian or symmetric matrices. *SIAM Journal on Applied Mathematics*, 21(1):51–54, 1971.

[6] D. P. Bertsekas. *Nonlinear Programming.* Athena Scientific, Belmont, MA, 2nd edition, 1999.

[7] S. Boyd and L. Vandenberghe. *Convex Optimization.* Cambridge University Press, Cambridge, 2004.

[8] S. Breedveld, D. Craft, R. van Haveren, and B. Heijmen. Multi-criteria optimization and decision-making in radiotherapy. *European Journal of Operational Research*, 277(1):1–19, 2019.

[9] S. Breedveld, P. Storchi, M. H. de Keijzer, and B. Heijmen. Fast, multiple optimizations of quadratic dose objective functions in IMRT. *Physics in Medicine and Biology*, 51(14):3569–3579, 2006.

[10] L. Brickman. On the field of values of a matrix. *Proceedings of the American Mathematical Society*, 12(1):61–66, 1961.

[11] S. Burer and Y. Ye. Exact semidefinite formulations for a class of (random and non-random) nonconvex quadratic programs. *Mathematical Programming*, 181:1–17, 2020.

[12] X. Chao and Y. Peng. A cost-sensitive multi-criteria quadratic programming model for imbalanced data. *Journal of the Operational Research Society*, 69(4):500–516, 2018.

[13] A. Chinchuluun and P. M. Pardalos. A survey of recent developments in multiobjective optimization. *Annals of Operations Research*, 154(1):29–50, 2007.

[14] F. Chlumsky-Harttmann. *Robust Multi-Objective Optimization: Analysis and Algorithmic Approaches*. Dissertation, Rheinland-Pfälzische Technische Universität Kaiserslautern-Landau, 2025.

[15] T. D. Chuong, X. Yu, C. Liu, A. Eberhard, and C. Li. Solving two-stage quadratic multiobjective problems via optimality and relaxations. *Journal of Optimization Theory and Applications*, 203(1):676–713, 2024.

[16] L. L. Dines. On the mapping of quadratic forms. *Bulletin of the American Mathematical Society*, 47:494–498, 1941.

[17] L. L. Dines. On linear combinations of quadratic forms. *Bulletin of the American Mathematical Society*, 49:388–393, 1943.

[18] M. Ehrgott. *Multicriteria Optimization*. Springer, Berlin, Heidelberg, 2nd edition, 2005.

[19] G. Eichfelder. Twenty years of continuous multiobjective optimization in the twenty-first century. *EURO Journal on Computational Optimization*, 9, 2021.

[20] F. Flores-Bazán and F. Opazo. Characterizing the convexity of joint-range for a pair of inhomogeneous quadratic functions and strong duality. *Minimax Theory and its Applications*, 1(2):257–290, 2016.

[21] F. Flores-Bazán and F. Opazo. Characterizing convexity of images for quadratic-linear mappings with applications in nonconvex quadratic optimization. *SIAM Journal on Optimization*, 31(3):1774–1796, 2021.

[22] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, 2nd edition, 2013.

[23] Y. Huang, N. D. Dizon, V. Jeyakumar, and G. Li. Hidden convexity of separable polynomial systems: exact semi-definite programs for a class of moment-ambiguity distributionally robust optimization problems. *Optimization*, 0(0):1–32, 2025.

[24] N. Q. Huy, N. H. Hung, T. V. Nghi, H. N. Tuan, and N. V. Tuyen. Hidden convexity of quadratic systems and its application to quadratic programming, 2026. arXiv preprint:2601.13511.

[25] V. Jeyakumar, G. M. Lee, and G. Y. Li. Alternative theorems for quadratic inequality systems and global quadratic optimization. *SIAM Journal on Optimization*, 20(2):983–1001, 2009.

[26] V. Jeyakumar and G. Y. Li. Trust-region problems with linear inequality constraints: exact SDP relaxation, global optimality and robust optimization. *Mathematical Programming*, 147(1):171–206, 2014.

[27] D. Könen and M. Stiglmayr. On supportedness in multi-objective integer linear programming. *Journal of Multi-Criteria Decision Analysis*, 32(3), 2025.

[28] A. Kulkarni, M. Kohns, M. Bortz, K.-H. Küfer, and H. Hasse. Regularities of Pareto sets in low-dimensional practical multi-criteria optimisation problems: Analysis, explanation, and exploitation. *Optimization and Engineering*, 24(3):1611–1632, 2023.

[29] A. Löhne. *Vector Optimization with Infimum and Supremum*. Springer, Berlin, Heidelberg, 2011.

[30] H. Markowitz. Portfolio selection. *The Journal of Finance*, 7(1):77–91, 1952.

[31] Y. Peng, G. Kou, Y. Shi, and Z. Chen. A multi-criteria convex quadratic programming model for credit data analysis. *Decision Support Systems*, 44(4):1016–1030, 2008.

[32] I. Pólik and T. Terlaky. A survey of the S-lemma. *SIAM Review*, 49(3):371–418, 2007.

[33] B. T. Polyak. Convexity of quadratic transformations and its use in control and optimization. *Journal of Optimization Theory and Applications*, 99(3):553–583, 1998.

[34] R. T. Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, NJ, 1970.

[35] R. T. Rockafellar and R. J.-B. Wets. *Variational Analysis*. Springer, Berlin, Heidelberg, 2009.

[36] J. Schelinski. Beiträge zum Verständnis quadratischer multikriterieller Optimierungsprobleme. Master's thesis, Technische Universität Ilmenau, 2024.

[37] A. Shkolnik, A. Kercheval, H. Gurdogan, L. R. Goldberg, and H. Bar. Portfolio selection revisited. *Annals of Operations Research*, 346(1):137–155, 2025.

[38] H. Song. Positive semidefinite intervals for matrix pencils. Master's thesis, University of Windsor, Windsor, Ontario, Canada, 2004.

[39] C. Touré, A. Auger, D. Brockhoff, and N. Hansen. On bi-objective convex-quadratic problems. In *EMO 2019 – 10th International Conference on Evolutionary Multi-Criterion Optimization*, East Lansing, Michigan, United States, 2019.

[40] H. Tuy and H. D. Tuan. Generalized S-Lemma and strong duality in nonconvex quadratic programming. *Journal of Global Optimization*, 56(3):1045–1072, 2013.

[41] F. Uhlig. Definite and semidefinite matrices in a real symmetric matrix pencil. *Pacific Journal of Mathematics*, 49(2):561–568, 1973.

[42] L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Review*, 38(1):49–95, 1996.

[43] A. L. Wang and F. Kılınç-Karzan. On the tightness of SDP relaxations of QCQPs. *Mathematical Programming*, 193:33–73, 2022.

[44] V. A. Yakubovich. S-Procedure in nonlinear control theory. *Vestnik Leningrad University*, 4:73–93, 1977. English translation.

[45] F. Zhang. *The Schur Complement and its Applications*, volume 4 of *Numerical Methods and Algorithms*. Springer, New York, 2005.