# Aggregated Quadratic Formulations and Semidefinite Relaxations of the Stable Set Polytope

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**Abstract**

The stable set problem admits various binary linear and quadratic formulations. The Shor relaxation of a particular quadratic formulation is the well-known theta body. We consider aggregations of quadratic constraints of this formulation, yielding exact and inexact quadratic formulations of the stable set problem, and then establish conditions under which the aggregated quadratic formulation is exact. By applying the Shor relaxation to aggregated quadratic formulations, we obtain new families of semidefinite relaxations of the stable set polytope. Conditions are established for the well-known clique inequalities to remain valid for these relaxations. We also compare the SDP-based aggregation closures of such relaxations and establish the surprising result that exact aggregations do not necessarily yield stronger closures than their inexact counterparts. In particular, one of our aggregation closures equals the theta body, whereas the other one can be strictly weaker.

*Keywords.* Stable set polytope, constraint aggregation, Shor relaxation, aggregation closure

## 1 Introduction

For hard optimization problems with nonconvex feasible sets, one approach for building strong convex relaxations is to combine some of the constraints after scaling each of them. This procedure is commonly referred to as constraint aggregation. The aim is to derive a new set with fewer nonconvex constraints so that its convex hull, or some other convex relaxation, is easier to construct than that of the original set. An extreme case of constraint aggregation is surrogate duality, where all nonconvex constraints are scaled and combined into a single constraint (one-row); this was pioneered by [21, 20] and is still an active area of research [32]. Aggregations have been commonly used in integer programming to generate polyhedral relaxations through linear valid inequalities [7, 31, 10, 2] and recently for relaxations of nonconvex quadratics [11, 1, 12]. Many of these results are for one-row aggregations.

An aggregation is parametrized by the weights used to scale each constraint. Convexifying each aggregation by taking its convex hull and intersecting the convex hulls over a collection of aggregation weights yields the so-called *aggregation closure* of the original set. There are many studies [8, 13, 34, 1] on characterizing the conditions under which these closures yield the convex hull or admit finite representations.

Not every aggregation of the original problem formulation necessarily yields an exact formulation; some of them can have strictly larger feasible sets. The literature on aggregation closures does not address the characterization of exact and inexact aggregations, perhaps because it is NP-hard (as we show) in general to distinguish between the two. If the closure is taken over a family that contains at least one exact aggregation, then the closure is

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obviously equal to the convex hull of the original set. Hence, it becomes more interesting to ask what happens when the closures are built using a convexification operator other than computing the convex hull.

The focus of this paper is to investigate the role of exact versus inexact formulations in constructing strong convex relaxations through aggregation-based closures. To the best of our knowledge, our study is the first rigorous and detailed theoretical investigation of this kind. Hence, we restrict our attention to the stable set problem, which is a classical NP-hard combinatorial optimization problem. There are two standard formulations – the commonly used binary linear edge formulation, whose aggregation closure is a polytope [34], and the binary quadratic edge formulation. The convex hull operator is intractable in general, even for an aggregated set, and so we instead consider convex relaxations using a semidefinite programming (SDP) based operator applied to the quadratic edge formulation. The resulting closure is an intersection of a collection of projected spectrahedra and is called the *SDP aggregation closure*. It contains the theta body, which is a classical non-aggregated semidefinite relaxation that is known to be exact for perfect graphs [23]. We obtain many theoretical results on the strengths of our SDP aggregation closures, including some explicit characterizations, and also provide some preliminary numerical results on their effectiveness on some benchmark instances. A surprising conclusion from our analysis is that exactness of aggregations is not necessarily an important consideration when building SDP-based closures because we show that they can yield strictly weaker relaxations. Special types of one-row SDP aggregations have been proposed and tested computationally [27, 17], however, a theoretical analysis of their one-row closures has not been performed.

### 1.1 Our Contributions

The main results are Theorems 4.4, 5.4 and 6.3. The first one establishes that one-row SDP aggregation closure already gives the tightest possible SDP relaxation, which is the theta body in our case. The second one shows that the polytope defined by all clique inequalities for the stable set problem is equal to the one-row aggregation closure over finitely many intersections. The third result establishes that our SDP closure from one-row exact aggregations is equal to a relaxation of the theta body obtained by changing equalities to inequalities, and hence it is strictly weaker than the theta body in general, which is perhaps the most surprising deduction of our analysis. Since the emphasis of this paper is on versus inexact aggregations, we show in Corollary 3.7, using some conditions from Proposition 3.1, that it is NP-hard to distinguish this.

Section 2 has basic results about formulations and SDP relaxations of the stable set problem. Section 3 introduces our aggregations of the quadratic edge formulation, and provides algebraic and geometric conditions for an aggregation to be exact, and the complexity of deciding whether exactness holds. The SDP relaxation, through the Shor lifting technique, of an aggregation is discussed next in Section 4, which leads us to define our SDP aggregation closures and their relation to some other closures from the literature. The bulk of our theoretical analysis is in Sections 5 and 6, where one-row aggregations are addressed. First, we consider one-row aggregations with mixed-sign coefficients and then move to those with strictly positive coefficients, which are guaranteed to yield exact one-row aggregations. The proofs of our main theorems are in Sections 5.1, 5.2 and 6.2. Section 7 discusses some open questions. Appendices A - C give missing proofs of some technical details, and Appendix D gives some numerical evidence to justify that our aggregations can be useful in practice; an efficient algorithmic procedure is beyond the scope of this paper.

### 1.2 Notation

The space of $n \times n$ real symmetric matrices is denoted by $\mathbb{S}^n$, the spaces of reals and integers follow standard practice. We use Greek letters for scalars, with the exceptions of $m$ and $n$ reserved for the number of edges and vertices, respectively, and $i, j, k, l$ reserved for indices. Lowercase letters denote vectors, and uppercase letters denote matrices or graphs. Specifically, $\mathbf{0}$ denotes the all-zeros vector or all-zeros matrix, $e$ denotes the all-ones vector, and $e_j$ denotes the $j$th unit vector, $j = 1, \ldots, n$, with dimensions obvious from the context. We use serif fonts for sets, with the sole exception of using a calligraphic font for a lifted set. The trace of a square matrix is denoted by $\mathtt{tr}(\cdot)$. For $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times n}$, the trace inner product is denoted by $\langle A, B\rangle = \mathtt{tr}(A^\top B) = \sum_{i=1}^m \sum_{j=1}^n A_{ij} B_{ij}$. The diagonal matrix obtained from a vector $x \in \mathbb{R}^n$ is $\mathtt{Diag}(x) \in \mathbb{S}^n$ and the diagonal of a matrix $X \in \mathbb{R}^{n\times n}$ is $\mathtt{diag}(X) \in \mathbb{R}^n$. We extensively use matrices $B \in \mathbb{R}^{\kappa\times m}$ whose columns are indexed by the edge set $\mathsf{E}$ and whose $l^{th}$ row, denoted by $b^l \in \mathbb{R}^{1\times m}$, consists of the components $b^l_{ij}$ for $(i, j) \in \mathsf{E},\ l = 1, \ldots, k$. The complete graph and the cycle graph on $n$ vertices are denoted by $K_n$ and $C_n$, respectively. The convex hull of a set is denoted by $\mathtt{conv}(\cdot)$. For $X \in \mathbb{S}^n$, we use $X \succeq \mathbf{0}$ (resp., $X \succ \mathbf{0}$) to denote that $X$ is positive semidefinite (resp., positive definite).

## 2 Background & Preliminaries

Let $G = (\mathsf{V}, \mathsf{E})$ be a simple, undirected, connected graph, where $\mathsf{V} = \{1, \ldots, n\}$ and $|\mathsf{E}| = m$. Its complement is $\overline{G} = (\mathsf{V}, \overline{\mathsf{E}})$, and the subgraph induced by any $\mathsf{S} \subseteq \mathsf{V}$ is denoted by $G[\mathsf{S}] = (\mathsf{S}, \mathsf{E}[\mathsf{S}])$. A stable set is a subset $\mathsf{S} \subseteq \mathsf{V}$ such that $G[\mathsf{S}]$ is a graph with no edges. A clique in $G$ is a stable set in $\overline{G}$. The stable set polytope for a graph is the convex hull of the incidence vectors of stable sets, i.e.,

$$\mathsf{STAB}(G) = \mathtt{conv}(\mathsf{S}(G)), \quad \text{where } \mathsf{S}(G) \coloneqq \left\{\chi^{\mathsf{S}} \colon \mathsf{S} \subseteq \mathsf{V} \text{ is a stable set}\right\}, \tag{1}$$

where $\chi^{\mathsf{S}} \in \{0, 1\}^n$ is given by $\chi^{\mathsf{S}}_j = 1$ if $j \in \mathsf{S}$ and $\chi^{\mathsf{S}}_j = 0$ otherwise. The above combinatorial formulation for $\mathsf{S}(G)$ translates to many different algebraic formulations, which we study in this paper. In this section, we review some useful results on the relaxations of the stable set polytope.

### 2.1 Standard relaxations

$\mathsf{S}(G)$ has two standard formulations given by

$$\mathsf{S}(G) = \{x \in \{0, 1\}^n : x_i + x_j \le 1,\ (i, j) \in \mathsf{E}\} = \{x \in \{0, 1\}^n : x_i x_j = 0,\ (i, j) \in \mathsf{E}\}, \tag{2}$$

which are referred to as the binary linear edge formulation and the binary quadratic edge formulation, respectively. Each of these representations of $\mathsf{S}(G)$ can be used to obtain different convex relaxations of $\mathsf{STAB}(G)$. In particular, the linear programming (LP) relaxation of the binary linear edge formulation corresponds to the polytope $\mathsf{FRAC}(G)$. The binary quadratic edge formulation can be relaxed by using the Shor lifting and projecting onto the space of the original variables [30], which yields the well-known theta body of Grötschel et al. [23] given by

$$\mathsf{TH}(G) = \left\{x \in \mathbb{R}^n \colon \exists X \in \mathbb{S}^n \text{ s.t. } X_{ij} = 0,\ (i, j) \in \mathsf{E};\ \mathtt{diag}(X) = x;\ \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix} \succeq \mathbf{0}\right\}. \tag{3}$$

This makes the theta body a projection of a compact spectrahedron and hence a compact convex set in $[0, 1]^n$. It was originally introduced by Grötschel et al. [22] using alternate representations in connection with the theta function studied by Lovász [28], and admits different semidefinite formulations [23, 30, 14] that have various nonlinear transformations among them [24, 35, 17]. It has other characterizations, such as the *orthonormal representation* with infinitely many linear inequalities.

**Orthonormal system** A set of unit vectors $\{u^1, \dots, u^n\}$ is an *orthonormal representation of $G$* if $(u^i)^\top u^j = 0$ for each $(i, j) \notin \mathsf{E}$. Such a representation can be denoted by a matrix $U \in \mathbb{R}^{\gamma \times n}$ whose columns are the vectors in an orthonormal representation. Denote the set of such matrices by

$$\mathcal{O}_\gamma(G) := \left\{U = \left[u^1 \cdots u^n\right] \in \mathbb{R}^{\gamma\times n} \colon \|u^i\| = 1,\ i \in \mathsf{V};\ (u^i)^\top u^j = 0,\, (i,j) \notin \mathsf{E}\right\}.$$

The pairwise orthogonality requirement for non-edges and elementary linear algebra imply that $\mathcal{O}_\gamma(G)$ is nonempty only if $\gamma \geq \alpha(G)$. An orthonormal representation inequality is of the form $\sum_{i\in\mathsf{V}} \left(c^\top u^i\right)^2 x_i \leq 1$ for any $U \in \mathcal{O}_\gamma(G)$ and any unit vector $c \in \mathbb{R}^\gamma$. The theta body $\mathsf{TH}(G)$ admits the following alternative description by all orthonormal representation constraints [22, 23]:

$$\mathsf{TH}(G) = \left\{x \in \mathbb{R}^n_+ \colon \sum_{i\in\mathsf{V}} \left(c^\top u^i\right)^2 x_i \leq 1,\ U \in \mathcal{O}_\gamma(G),\, \forall\, \gamma \in [\alpha(G), n],\ \|c\| = 1\right\}. \tag{4}$$

$\mathsf{FRAC}(G)$ is equal to $\mathsf{S}(G)$ if and only if $G$ is bipartite [22, Proposition 1.3], and polyhedral relaxations of $\mathsf{S}(G)$ that are stronger than $\mathsf{FRAC}(G)$ are available through different families of linear valid inequalities. There is a rich body of literature on this topic, we refer the reader to the reviews in [29, 19, 26, 25]. *Rank inequalities* are a common class of valid inequalities and are of the form $\sum_{i\in\mathsf{S}} x_i \leq \alpha(G[\mathsf{S}])$ for any $\mathsf{S} \subseteq \mathsf{V}$. These include various families of inequalities from different subsets $\mathsf{S}$, such as cliques, odd holes (chordless cycles of length 5 or more), odd antiholes (complements of odd holes), webs and antiwebs. Clique inequalities yield the polytope

$$\mathsf{QSTAB}(G) := \left\{x \in \mathbb{R}^n_+ \colon \sum_{j\in\mathsf{C}} x_j \leq 1, \quad \mathsf{C} \text{ is a maximal clique}\right\}, \tag{5}$$

where non-maximal (with respect to set inclusion) cliques are discarded because they are dominated by maximal ones. This is a stronger relaxation of $\mathsf{STAB}(G)$ than $\mathsf{FRAC}(G)$ and also offers a formulation due to $\mathsf{QSTAB}(G) \cap \{0, 1\}^n = \mathsf{S}(G)$. However, it is NP-hard to separate over $\mathsf{QSTAB}(G)$ [23, Theorem 9.2.9]. Clique inequalities can be strengthened under some conditions [25]. Although it is not the emphasis of our work, we mention that linear valid inequalities in the $(x, X)$-space have been investigated for strengthening $\mathsf{TH}(G)$; see, e.g., [30, 24, 18, 19, 15, 16].

The importance of $\mathsf{TH}(G)$ and $\mathsf{QSTAB}(G)$ arose in the context of perfect graphs. A graph is perfect if it does not contain an induced subgraph that is either an odd hole or an odd antihole [4], and such graphs can be detected in polynomial time [5]. A classical result of Chvátal [6, Theorem 3.1] is that perfect graphs are exactly those graphs for which $\mathsf{QSTAB}(G)$ is exact, i.e., equal to the stable set polytope. This gives a polyhedral characterization of perfect graphs, and later, Grötschel et al. [22] established a characterization in terms of the theta body.

**Proposition 2.1** (cf. Grötschel et al. [23, §9.3])**.** *We have*

$$\left\{x \in \mathbb{R}^n_+ \colon y^\top x \leq 1,\ y \in \mathsf{QSTAB}(\overline{G})\right\} = \mathsf{STAB}(G) \subseteq \mathsf{TH}(G) \subseteq \mathsf{QSTAB}(G) \subseteq \mathsf{FRAC}(G).$$

*Furthermore, the following are equivalent:*

i) *$G$ is a perfect graph,*

ii) $\mathsf{TH}(G)$ *is a polytope,*

iii) $\mathsf{STAB}(G) = \mathsf{TH}(G) = \mathsf{QSTAB}(G)$.

*For imperfect graphs, every nontrivial inequality that defines a facet*[1] *of* $\mathsf{TH}(G)$ *is a positive multiple of a clique inequality.*

This underscores the importance of clique inequalities and hence our emphasis on them with regard to the aggregated relaxations analyzed in this paper. For imperfect graphs, the non-polyhedral structure of $\mathsf{TH}(G)$ means that its description also requires inequalities that define low-dimensional faces.

### 2.2 Binary quadratic edge inequality formulation

We finish these preliminaries with the properties of another quadratic formulation of $\mathsf{S}(G)$, where each $x_i x_j = 0$ in (2) is relaxed to $x_i x_j \leq 0$, given by

$$\mathsf{S}(G) = \{x \in \{0,1\}^n : x_i x_j \leq 0,\ (i,j) \in \mathsf{E}\}\,. \tag{6a}$$

We refer to (6a) as the binary quadratic edge inequality formulation. Note that replacing $x_i x_j = 0$ by $x_i x_j \geq 0$ for each $(i,j) \in \mathsf{E}$ does not give rise to a valid formulation of $\mathsf{S}(G)$ since the latter constraint is redundant for any $x \in \{0,1\}^n$. The projection of the Shor relaxation of (6a) is

$$\begin{aligned}\mathsf{TH}_{\leq}(G) \coloneqq \Big\{x \in \mathbb{R}^n \colon \exists\, X \in \mathbb{S}^n \text{ s.t. } & X_{ij} \leq 0,\ (i,j) \in \mathsf{E}, \\ & \mathtt{diag}(X) = x;\ \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix} \succeq \mathbf{0}\Big\}.\end{aligned} \tag{6b}$$

By construction, it is obvious that $\mathsf{TH}(G) \subseteq \mathsf{TH}_{\leq}(G)$. We do not know if $\mathsf{TH}_{\leq}(G)$ has an analogous counterpart to the orthonormal representation for $\mathsf{TH}(G)$. We establish that $\mathsf{TH}_{\leq}(G)$ is a stronger relaxation than $\mathsf{QSTAB}(G)$, which seems to have gone unnoticed in the literature.

**Proposition 2.2.** *We have* $\mathsf{TH}(G) \subseteq \mathsf{TH}_{\leq}(G) \subseteq \mathsf{QSTAB}(G)$.

The proof is in Appendix B and relies on the strong duality property of linear optimization problems over $\mathsf{TH}_{\leq}(G)$.

## 3 Aggregations

We consider aggregating the quadratic equality constraints $x_i x_j = 0$ for all $(i,j) \in \mathsf{E}$ in the binary quadratic edge formulation (2) to obtain a family of aggregated formulations that is parametrized by an aggregation matrix $B \in \mathbb{R}^{\kappa \times m}$ for $\kappa \in \mathbb{Z}_{++}$. For $l = 1, \ldots, \kappa$, the $l^{th}$ row of $B$ is denoted by the vector $b^l \in \mathbb{R}^m$ whose components, indexed by the edge set $\mathsf{E}$, are denoted by $b^l_{ij} \in \mathbb{R}$ for $(i,j) \in \mathsf{E}$,

$$B = \begin{bmatrix} (b^1)^\top \\ \vdots \\ (b^\kappa)^\top \end{bmatrix} \in \mathbb{R}^{\kappa \times m}. \tag{7}$$

[1] For closed convex sets, a facet is an exposed face of co-dimension 1.

The aggregated formulation associated with $B$ is given by

$$\mathsf{S}(G,B) \coloneqq \left\{x \in \{0,1\}^n : \sum_{(i,j)\in \mathsf{E}} b_{ij}^l\, x_i x_j = 0, \quad l = 1,\dots,\kappa\right\}, \quad B \in \mathbb{R}^{\kappa\times m}. \tag{8}$$

Observe that

$$\mathsf{S}(G) \subseteq \mathsf{S}(G,B), \quad B \in \mathbb{R}^{\kappa\times m}. \tag{9}$$

Our objective is to shed light on the set of matrices $B \in \mathbb{R}^{\kappa\times m}$ such that the relation (9) holds with equality. To that end, we formally define this set.

**Definition 1.** Let $\kappa \in \mathbb{Z}_{++}$. A matrix $B \in \mathbb{R}^{\kappa\times m}$ is a *feasibility-preserving aggregation matrix* if $\mathsf{S}(G,B) = \mathsf{S}(G)$. We denote the set of all such matrices by

$$\mathsf{FP}(G,\kappa) \coloneqq \left\{B \in \mathbb{R}^{\kappa\times m} \colon \mathsf{S}(G,B) = \mathsf{S}(G)\right\}, \quad \kappa \in \mathbb{Z}_{++}. \tag{10}$$

These sets are always nonempty since they contain componentwise positive matrices, as noted in Lemma 3.2.

### 3.1 Algebraic Description and Sufficient Conditions

In this section, we first give a complete characterization of $\mathsf{FP}(G,\kappa)$ in terms of nonempty induced subgraphs of $G$, and then present sufficient conditions for membership in $\mathsf{FP}(G,\kappa)$.

**Proposition 3.1.** *For any $\kappa \in \mathbb{Z}_{++}$ and $B \in \mathbb{R}^{\kappa\times m}$, the following statements are equivalent:*

i) *$B \in \mathsf{FP}(G,\kappa)$, where $\mathsf{FP}(G,\kappa)$ defined as in (10).*

ii) *For each $\mathsf{S} \subseteq \mathsf{V}$ with $\mathsf{E}[\mathsf{S}] \neq \emptyset$, $\exists\, l \in \{1,\dots,\kappa\}$ with $\sum\limits_{(i,j)\in \mathsf{E}[\mathsf{S}]} b_{ij}^l \neq 0$.*

iii) $\sum\limits_{l=1}^{\kappa} \left(\sum\limits_{(i,j)\in\mathsf{E}[\mathsf{S}]} b_{ij}^l\right)^2 > 0$ *for each $\mathsf{S} \subseteq \mathsf{V}$ with $\mathsf{E}[\mathsf{S}] \neq \emptyset$.*

*Proof.* The equivalence of (ii) and (iii) is trivial. Now let us argue (i) $\Rightarrow$ (ii). Each $\mathsf{S} \subseteq \mathsf{V}$ that induces a nonempty subgraph is not a stable set, and so $\chi^{\mathsf{S}} \in \{0,1\}^n \setminus \mathsf{S}(G)$. If (i) holds, we have $\mathsf{S}(G) = \mathsf{S}(G,B)$ and so each such $\chi^{\mathsf{S}} \in \{0,1\}^n \setminus \mathsf{S}(G,B)$. Since $\sum_{(i,j)\in\mathsf{E}} b_{ij}^l\, \chi_i^{\mathsf{S}} \chi_j^{\mathsf{S}} = \sum_{(i,j)\in\mathsf{E}[\mathsf{S}]} b_{ij}^l$ for all $l = 1,\dots,\kappa$, it follows from (8) that $\sum_{(i,j)\in\mathsf{E}[\mathsf{S}]} b_{ij}^l \neq 0$ for some $l$. Conversely, if (ii) holds, then we have $\{0,1\}^n \setminus \mathsf{S}(G) \subseteq \{0,1\}^n \setminus \mathsf{S}(G,B)$, which is equivalent to $\mathsf{S}(G) \supseteq \mathsf{S}(G,B)$. By (9), we conclude that $B \in \mathsf{FP}(G,\kappa)$. $\square$

Now we turn our attention to sufficient conditions for membership in $\mathsf{FP}(G,\kappa)$. Our next result provides a simple condition without using the combinatorial characterization in Proposition 3.1.

**Lemma 3.2.** *For any $\kappa \in \mathbb{Z}_{++}$ and $B \in \mathbb{R}^{\kappa\times m}$, if there exists $y \in \mathbb{R}^{\kappa}$ such that $B^\top y > \mathbf{0}$, we have $B \in \mathsf{FP}(G,\kappa)$. Consequently, every matrix having a positive row is feasibility-preserving, and hence $\mathsf{FP}(G,\kappa) \neq \emptyset$ for all $\kappa \in \mathbb{Z}_{++}$.*

*Proof.* Let $x \in \mathsf{S}(G,B)$. Multiplying each of the $\kappa$ equations by the corresponding components of $y \in \mathbb{R}^{\kappa}$ and adding them, we obtain $\sum_{(i,j)\in\mathsf{E}} \left(\sum_{l=1}^{\kappa} y_l\, b_{ij}^l\right) x_i x_j = 0$. Since $B^\top y > 0$ and $x \in \{0,1\}^n$, we obtain $x_i x_j = 0$ for each $(i,j) \in \mathsf{E}$. Therefore, $x \in \mathsf{S}(G)$. By (9), we conclude that $\mathsf{S}(G) = \mathsf{S}(G,B)$. Therefore, $B \in \mathsf{FP}(G,\kappa)$.

If $b^l > \mathbf{0}$ for some $l \in \{1,\dots,\kappa\}$, choosing $y = e_l \in \mathbb{R}^{\kappa}$ gives us $B^\top y = b^l > \mathbf{0}$ and so such a matrix is feasibility-preserving by the first part. $\square$

By positive homogeneity, this condition is equivalent to showing the existence of $y \in \mathbb{R}^\kappa$ such that $B^\top y \geq \mathbf{1}$, and hence it can be easily checked by solving a linear programming feasibility problem. However, as illustrated by our next result, this condition is, in general, not necessary for any $\kappa$.

**Lemma 3.3.** *For each $\kappa \in \mathbb{Z}_{++}$, there exists $B \in \mathsf{FP}(G, \kappa)$ that violates the sufficient condition of Lemma 3.2.*

*Proof.* First, suppose that $\kappa = 1$. Let $B = b^\top$, where $b \in \mathbb{R}^m$ satisfies $b_{ij} > m - 1$ for some $(i, j) \in \mathsf{E}$ and $b_{kl} \in [-1, 0)$ for all $(k, l) \in \mathsf{E} \setminus (i, j)$. We claim that $B \in \mathsf{FP}(G, \kappa)$. For any $x \in \mathsf{S}(G, B)$, we have $b_{ij} x_i x_j = -\sum_{(k,l)\in\mathsf{E}\setminus(i,j)} b_{kl} x_k x_l$. Since $x \in \{0, 1\}^n$, it follows that the right-hand side is bounded above by $m - 1$. Since $b_{ij} > m - 1$, we conclude that $x_i x_j = 0$. Since $b_{kl} \in [-1, 0)$ for all $(k, l) \in \mathsf{E} \setminus (i, j)$, we obtain $x_k x_l = 0$ for all $(k, l) \in \mathsf{E} \setminus (i, j)$. Therefore, $B \in \mathsf{FP}(G, 1)$. However, there does not exist $y \in \mathbb{R}$ such that $b\, y > \mathbf{0}$ since $b$ has positive and negative components.

Suppose now that $\kappa > 1$. By the Farkas' lemma, the system $B^\top y \geq \mathbf{1}$ is infeasible if and only if there exists a nonzero $v \in \mathbb{R}^m_+$ such that $B\, v = \mathbf{0}$. Consider $b \in \mathbb{R}^m$ given in the first part of the proof. Since $b\, y > \mathbf{0}$ is infeasible, there exists a nonzero $v \in \mathbb{R}^m_+$ such that $b^\top v = 0$. We construct $B \in \mathbb{R}^{\kappa\times m}$ given by (7) as follows. Define $b^1 = b$. For $l \in \{2, \ldots, \kappa\}$, choose $b^l \in \mathbb{R}^m$ such that $(b^l)^\top v = 0$. The Farkas dual system implies that the system $B^\top y \geq \mathbf{1}$ is infeasible, i.e., $B$ violates the sufficient condition of Lemma 3.2. On the other hand, $B \in \mathsf{FP}(G, \kappa)$ since $\mathsf{S}(G) \subseteq \mathsf{S}(G, B) \subseteq \{x \in \{0, 1\}^n : \sum_{(i,j)\in\mathsf{E}} b_{ij}\, x_i x_j = 0\} = \mathsf{S}(G)$. This completes the proof. □

On the other hand, if we restrict our aggregation matrix to be componentwise nonnegative, then our next result shows that the sufficient condition of Lemma 3.2 is also necessary and much simpler.

**Corollary 3.4.** *Let $B \in \mathbb{R}^{\kappa\times m}_+$. Then, $B \in \mathsf{FP}(G, \kappa)$ if and only if $B^\top e > \mathbf{0}$.*

*Proof.* Let $B \in \mathbb{R}^{\kappa\times m}_+$. If $B^\top e > \mathbf{0}$, then $B \in \mathsf{FP}(G, \kappa)$ by Lemma 3.2. Conversely, suppose that $B \in \mathsf{FP}(G, \kappa)$ and let $\mathsf{S} = \{i, j\} \subseteq \mathsf{V}$, where $(i, j) \in \mathsf{E}$. Since $\mathsf{E}[\mathsf{S}] = \{(i, j)\}$, Proposition 3.1 (ii) implies that there exists $l \in \{1, \ldots, \kappa\}$ such that $b^l_{ij} \neq 0$. Since $B$ is componentwise nonnegative, we conclude that $b^l_{ij} > 0$, which implies that $\sum_{l=1}^{\kappa} b^l_{ij} > 0$. Repeating the argument for each $(i, j) \in \mathsf{E}$ implies $B^\top e > \mathbf{0}$. □

### 3.2 Geometry of Feasibility-Preserving Aggregation Matrices

In this section, we focus on the geometry of the set of feasibility-preserving aggregation matrices. Our next result presents the geometric structure of $\mathsf{FP}(G, \kappa)$.

**Proposition 3.5.** *For each $\kappa \in \mathbb{Z}_{++}$, $\mathsf{FP}(G, \kappa)$ is given by the union of a finite number of open polyhedral cones.*

*Proof.* By Proposition 3.1 (ii),

$$\mathsf{FP}(G,\kappa) = \bigcap_{\substack{\mathsf{S}\subseteq\mathsf{V}:\\ \mathsf{E}[\mathsf{S}]\neq\emptyset}} \bigcup_{l=1}^{\kappa} \left\{ B \in \mathbb{R}^{\kappa\times m} : \sum_{(i,j)\in\mathsf{E}[\mathsf{S}]} b^l_{ij} \neq 0 \right\} = \bigcap_{\substack{\mathsf{S}\subseteq\mathsf{V}:\\ \mathsf{E}[\mathsf{S}]\neq\emptyset}} \bigcup_{l=1}^{\kappa} \left(\mathsf{C}^-(\mathsf{S}, l) \cup \mathsf{C}^+(\mathsf{S}, l)\right),$$

where $\mathsf{C}^-(\mathsf{S}, l) := \{B \in \mathbb{R}^{\kappa\times m} : \sum_{(i,j)\in\mathsf{E}[\mathsf{S}]} b^l_{ij} < 0\}$, and $\mathsf{C}^+(\mathsf{S}, l)$ is defined analogously by reversing the strict inequality. Therefore, for each $\mathsf{S} \subseteq \mathsf{V}$ such that $\mathsf{E}[\mathsf{S}] \neq \emptyset$, the corresponding inner set is given by the union of $2\kappa$ open halfspaces whose bounding hyperplanes go through the origin. Let $\mu = O(2^n)$ denote the number of sets $\mathsf{S} \subseteq \mathsf{V}$ such that $\mathsf{E}[\mathsf{S}] \neq \emptyset$. By using the distributive property, we conclude that $\mathsf{FP}(G, \kappa)$ is given by the union of at most $(2\kappa)^\mu$ open polyhedral cones. □

We next present an illustrative example.

*Example* 1. Let $G = K_3$, $\kappa = 1$, and $B \in \mathbb{R}^{1\times 3}$. By Proposition 3.5,

$$\begin{aligned}
\mathsf{FP}(K_3, 1) &= \left\{\{B \in \mathbb{R}^{1\times 3} \colon b_{12} + b_{23} + b_{13} > 0\} \cup \{B \in \mathbb{R}^{1\times 3} : b_{12} + b_{23} + b_{13} < 0\}\right\} \\
&\qquad \bigcap_{(i,j)\in \mathsf{E}} \left\{\{B \in \mathbb{R}^{1\times 3} : b_{ij} > 0\} \cup \{B \in \mathbb{R}^{1\times 3} : b_{ij} < 0\}\right\} \\
&= \bigcup \left\{B \in \mathbb{R}^{1\times 3} \colon b_{12} + b_{23} + b_{13} \,\square\, 0, \quad b_{12} \,\square\, 0, \quad b_{23} \,\square\, 0, \quad b_{13} \,\square\, 0\right\},
\end{aligned}$$

where $\square \in \{<, >\}$. Therefore, $\mathsf{FP}(K_3, 1)$ is given by the union of 16 open polyhedral cones, some of which may be the empty set. $\diamond$

We close this section with an interesting observation. By Proposition 3.5, the complement of $\mathsf{FP}(G, \kappa)$, denoted by $\overline{\mathsf{FP}(G, \kappa)}$, is given by

$$\overline{\mathsf{FP}(G, \kappa)} = \bigcup_{\substack{\mathsf{S}\subseteq \mathsf{V}:\\ \mathsf{E}[\mathsf{S}]\neq\emptyset}} \bigcap_{l=1}^{\kappa} \left\{B \in \mathbb{R}^{\kappa\times m} : \sum_{(i,j)\in \mathsf{E}[\mathsf{S}]} b^l_{ij} = 0\right\}.$$

For each $S \subseteq \mathsf{V}$ such that $\mathsf{E}[S] \neq \emptyset$, the corresponding inner set is given by a lower-dimensional subspace, which implies that $\overline{\mathsf{FP}(G, \kappa)}$ is given by the union of a finite number of subspaces. Therefore, it does not have an interior. In contrast, $\mathsf{FP}(G, \kappa)$ has a nonempty interior since it is given by the union of a finite number of full-dimensional polyhedral cones. Therefore, if the rows of $B$ are generated uniformly from the unit ball, it follows that $B \in \mathsf{FP}(G, \kappa)$ with probability one.

### 3.3 Checking the Feasibility-Preserving Property

We analyze the complexity of the question of deciding whether a given aggregation matrix is feasibility-preserving or not. Let us first address the easy cases. If $B \in \mathbb{R}^{\kappa\times m}$ is such that the linear program $B^\top y \geq \mathbf{1}$ is feasible in $y$, then Lemma 3.2 implies that $B \in \mathsf{FP}(G, \kappa)$. Corollary 3.4 establishes that the membership problem for nonnegative aggregation matrices can be solved by checking the positivity of the column sums of the matrix.

**Corollary 3.6.** *Any $B \in \mathbb{R}^{\kappa\times m}_+$ can be checked to belong to $\mathsf{FP}(G, \kappa)$ in $O(\kappa m)$ time. For $\kappa = 1$, $B$ is feasibility-preserving if and only if it is a positive row vector.*

Now we deduce two nontrivial results in Corollaries 3.7 and 3.8 whose proofs are given in Appendix A. For general matrices, i.e., those having positive and negative entries, the complementary question, which is to decide whether a given rational matrix is *not* feasibility-preserving, is NP-complete through a reduction from the subset sum problem to star graphs.

**Corollary 3.7.** *Membership in $\mathsf{FP}(G, \kappa)$ is co-NP-complete for any $\kappa \geq 1$.*

We do not know whether the membership problem is in NP or if it is NP-hard, but we note that this question can be solved through unconstrained quartic integer optimization. To state this result, define $\widetilde{b}_{ij} := \sum_{l=1}^{\kappa} (b^l_{ij})^2$ and $\widetilde{b}_{iji'j'} := \sum_{l=1}^{\kappa} b^l_{ij} b^l_{i'j'}$, and $\mathsf{E}' := \{((i,j),(i',j')) \in \mathsf{E}\times\mathsf{E} \colon (i,j) \prec (i',j')\}$ as all pairs of distinct edges ordered by the lexicographic order $\prec$ on $\mathbb{Z}^2$, and $\mathsf{E}'_- := \{((i,j),(i',j')) \in \mathsf{E}' \colon \widetilde{b}_{iji'j'} < 0\}$.

**Corollary 3.8.** *$B \in \mathbb{R}^{\kappa\times m}$ belongs to $\mathsf{FP}(G, \kappa)$ if and only if*

$$0 < \min_{x\in\mathbb{R}^n, s\in\mathbb{R}^n} \left\{q(x) + \rho\Big(\sum_{(i,j)\in \mathsf{E}} x_i x_j - s\Big)^2 \colon\ x \in \{0,1\}^n,\ s \in \mathbb{Z}_+,\ s \geq 1\right\},$$

*where* $q(x) := \sum_{(i,j)\in \mathsf{E}} \widetilde{b}_{ij}\, x_i x_j \;+\; \sum_{((i,j),(i',j'))\in \mathsf{E}'} 2\,\widetilde{b}_{iji'j'}\, x_i x_j x_{i'} x_{j'}$ *is a quartic, and*

$$\rho := \sum_{(i,j)\in \mathsf{E}[\mathsf{T}]} \widetilde{b}_{ij} + \sum_{(i,j),(i',j')\in \mathsf{E}'[\mathsf{T}]} 2\,\widetilde{b}_{iji'j'} - \sum_{((i,j),(i',j'))\in \mathsf{E}'_{-}} 2\,\widetilde{b}_{iji'j'},$$

*where* $\mathsf{T}\subseteq \mathsf{V}$ *is any subset such that* $\mathsf{E}[\mathsf{T}]\neq\emptyset$*, with* $\mathsf{E}'[\mathsf{T}]$ *denoting the subset of* $\mathsf{E}'$ *with edge pairs from* $\mathsf{T}$*, and* $\mathsf{E}'_{-}\subseteq \mathsf{E}'$ *is as defined just before the statement.*

We also establish membership through positivity of a mixed-binary linear programming problem, this is described later in Corollary A.2. The coefficient matrix for its constraints has entries in $\{0,\pm 1\}$ and integral right-hand side. Hence, if the constraints satisfy integrality properties such as total unimodularity or total dual integrality, it may be possible to obtain families of graphs for which the membership problem is polynomially solvable via linear optimization. This question is left for future research. Let us remark that, in practice, the optimization problem in Corollary 3.8 could potentially be solved faster than the MILP in Corollary A.2.

## 4 SDP Relaxations of Aggregated Formulations

In this section, given $\kappa\in\mathbb{Z}_{++}$ and an aggregation matrix $B\in\mathbb{R}^{\kappa\times m}$, we consider the Shor relaxation of the corresponding aggregated formulation $\mathsf{S}(G,B)$ given by (8). In Section 4.1, we introduce the Shor relaxation, its projection, and present some basic properties. In particular, we show that such relaxations give rise to a new family of semidefinite relaxations of $\mathsf{S}(G)$, which includes the theta body $\mathsf{TH}(G)$ given by (3) as a special case. We then consider aggregation closures by taking the intersection of the projections of Shor relaxations over all aggregation matrices in Section 4.3.

### 4.1 Basic Properties

Let $\kappa\in\mathbb{Z}_{++}$ and let $B\in\mathbb{R}^{\kappa\times m}$ be defined as in (7). Consider the corresponding aggregated formulation $\mathsf{S}(G,B)$ given by (8). The Shor lifting of $\mathsf{S}(G,B)$ is given by

$$\mathcal{R}(G,B) := \left\{ (x,X)\in\mathbb{R}^n\times\mathbb{S}^n\colon \begin{array}{l} \displaystyle\sum_{(i,j)\in\mathsf{E}} b^l_{ij}\, X_{ij} = 0,\ l=1,\dots,\kappa, \\ \mathsf{diag}(X) = x,\quad \begin{pmatrix} 1 & x^\top \\ x & X\end{pmatrix}\succeq \mathbf{0} \end{array} \right\}. \tag{11a}$$

The projection of $\mathcal{R}(G,B)$ onto the $x$-space is denoted by $\mathsf{R}(G,B)$:

$$\mathsf{R}(G,B) := \mathsf{Proj}_x\,\mathcal{R}(G,B) := \{x\in\mathbb{R}^n\colon \exists X\in\mathbb{S}^n \text{ s.t. } (x,X)\in\mathcal{R}(G,B)\}. \tag{11b}$$

We present several basic properties of $\mathsf{R}(G,B)$.

**Lemma 4.1.** *Let* $\kappa\in\mathbb{Z}_{++}$ *and* $B\in\mathbb{R}^{\kappa\times m}$*, and consider* $\mathcal{R}(G,B)$ *and* $\mathsf{R}(G,B)$ *defined as in* (11a) *and* (11b)*, respectively.*

i) *Each of* $\mathcal{R}(G,B)$ *and* $\mathsf{R}(G,B)$ *is a nonempty, compact, and convex set.*

ii) *It holds that* $\mathsf{STAB}(G)\subseteq\mathsf{TH}(G)\subseteq\mathsf{R}(G,B)$*.*

iii) *If* $B$ *has full column rank, then* $\mathsf{TH}(G)=\mathsf{R}(G,B)$*.*

iv) $\mathcal{R}(G,B)$ *has a strictly feasible (Slater) point.*

v) $\mathsf{R}(G,B)$ *is scale-invariant in aggregation:* $\mathsf{R}(G,B)=\mathsf{R}(G,\delta B)$ *for* $\delta\in\mathbb{R}\backslash\{0\}$*.*

*Proof.* (i) The set $\mathcal{R}(G,B)$ is nonempty since $(x,X) \in \mathcal{R}(G,B)$, where $x = \mathbf{0} \in \mathbb{R}^n$ and $X = \mathbf{0} \in \mathbb{S}^n$. It is closed and convex because it is a spectrahedron. It is also bounded since $\mathtt{diag}(X) = x$ and $X - xx^\top \succeq \mathbf{0}$ implies $\mathbf{0} \leq x \leq e$, which, in turn, implies that $\mathtt{tr}(X)$ is bounded. It follows that $\mathcal{R}(G,B)$ is bounded. Therefore, the projection $\mathsf{R}(G,B)$ is also a compact convex set in $[0,1]^n$.

(ii) The first inclusion follows from Proposition 2.1. For the second inclusion, let $x \in \mathsf{TH}(G)$. By (3), $\exists X \in \mathbb{S}^n$ such that $X_{ij} = 0$ for each $(i,j) \in \mathsf{E}$, $\mathtt{diag}(X) = x$, and $X - xx^\top \succeq \mathbf{0}$. Clearly, $(x,X) \in \mathcal{R}(G,B)$, which implies $x \in \mathsf{R}(G,B)$.

(iii) If $B$ has full column rank, then the only solution $z \in \mathbb{R}^m$ of the system $B\,z = \mathbf{0}$ is given by $z = \mathbf{0} \in \mathbb{R}^m$. Therefore, for any $(x,X) \in \mathcal{R}(G,B)$, it follows from (11a) that $X_{ij} = 0$ for each $(i,j) \in \mathsf{E}$. By (3) and (11b), we conclude that $\mathsf{R}(G,B) \subseteq \mathsf{TH}(G)$. The reverse inclusion follows from (ii).

(iv) Let $\tau \in (0, 1/n)$ and define $x = \tau\, e \in \mathbb{R}^n$ and $X = \tau\, I \in \mathbb{S}^n$. Clearly, $(x,X)$ satisfies all linear equality constraints $\mathcal{R}(G,B)$. Concerning the semidefinite constraint, note that $\begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix} \succ 0$ if and only if $X - xx^\top \succ 0$. Since $\tau \in (0,1/n)$, it is easy to verify that $X - xx^\top \succ 0$ since it is a strictly diagonally dominant matrix with positive diagonal entries. Therefore, $(x,X)$ is a strictly feasible point of $\mathcal{R}(G,B)$.

(v) This is trivial from the aggregation of equality constraints. □

For each $B \in \mathbb{R}^{\kappa \times m}$, Lemma 4.1 implies that $\mathsf{R}(G,B)$ is a convex relaxation of $\mathsf{STAB}(G)$, which, in general, is weaker than $\mathsf{TH}(G)$. However, for certain choices of $B$, the resulting convex relaxation agrees with $\mathsf{TH}(G)$. Therefore, $\mathsf{R}(G,B)$ can be viewed as a generalization of the theta body that gives rise to a new family of SDP relaxations of $\mathsf{STAB}(G)$.

### 4.2 Optimization over $\mathsf{R}(G,B)$

Recall that $\mathsf{R}(G,B)$ defined as in (11b) is the projection of the spectrahedron $\mathcal{R}(G,B)$ given by (11a) onto the $x$-space. In this section, we consider the linear optimization problem over $\mathsf{R}(G,B)$. To that end, for a given $w \in \mathbb{R}^n$, let us define

$$\nu(G,w,B) := \mathtt{max}\left\{ w^\top x : x \in \mathsf{R}(G,B) \right\}. \tag{12}$$

In general, note that $\mathsf{R}(G,B)$ may not admit an explicit representation in $\mathbb{R}^n$. However, it follows from (11b) that $\nu(G,w,B)$ can still be efficiently computed to any arbitrary accuracy by solving the following semidefinite programming problem:

$$\nu(G,w,B) = \mathtt{max}\left\{ w^\top x \colon x \in \mathcal{R}(G,B) \right\}. \tag{P(B,w)}$$

Before presenting the Lagrangian dual of (P(B,w)), let us define

$$\mathscr{A}_l(B) = \frac{1}{2} \sum_{(i,j)\in \mathsf{E}} b^l_{ij} \left( e_i e_j^\top + e_j e_i^\top \right) \in \mathbb{S}^n, \quad l = 1,\ldots,\kappa. \tag{13}$$

It follows from (11a) and (13) that

$$\mathcal{R}(G,B) = \left\{ (x,X) \in \mathbb{R}^n \times \mathbb{S}^n \colon \begin{array}{l} \langle \mathscr{A}_l(B), X \rangle = 0, \; l = 1,\ldots,\kappa, \\ \mathtt{diag}(X) = x, \quad \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix} \succeq \mathbf{0} \end{array} \right\}. \tag{14}$$

We introduce the Lagrange multipliers $z \in \mathbb{R}^\kappa$ and $y \in \mathbb{R}^n$ corresponding to the first and second set of equality constraints in (14), respectively, and the matrix variable $\begin{pmatrix} \delta & q^\top \\ q & Q \end{pmatrix} \in$

$\mathbb{S}^{n+1}$, with $\delta \in \mathbb{R}$, $q \in \mathbb{R}^n$, and $Q \in \mathbb{S}^n$, corresponding to the positive semidefinite constraint. After eliminating the dual variables $q$ and $Q$ using equality constraints, the Lagrangian dual of (P($B, w$)) is given by

$$\min_{(\delta,y,z)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^\kappa} \delta \quad \text{s.t.} \quad \begin{bmatrix} \delta & -\frac{1}{2}(w+y)^\top \\ -\frac{1}{2}(w+y) & \sum_{l=1}^{\kappa} z_l \mathscr{A}_l(B) + \mathsf{Diag}(y) \end{bmatrix} \succeq \mathbf{0}. \tag{D($B,w$)}$$

This primal-dual SDP pair is solvable and exhibits strong duality.

**Lemma 4.2.** *For any $w \in \mathbb{R}^n$ and any $B \in \mathbb{R}^{\kappa\times m}$, $\nu(G, w, B)$ defined as in (12) is finite. Furthermore, strong duality holds between (P($B, w$)) and (D($B, w$)), and the optimal value is attained in both problems.*

*Proof.* By Lemma 4.1(i), $\nu(G, w, B)$ is finite and the optimal value is attained in (12). Since (P($B, w$)) has a Slater point by Lemma 4.1(iv), it follows from the conic duality theory that strong duality holds between (P($B, w$)) and (D($B, w$)) and that the optimal value is attained in (D($B, w$)). $\square$

#### 4.2.1 Comparison to another SDP

The original SDP relaxation for the stability number of a graph is due to Lovász [28]. The feasible set of this so-called Lovász SDP is a nonlinear image of $\mathsf{TH}(G)$ (see [24, 35, 17] for various nonlinear transformations among them). Similar to $\mathsf{TH}(G)$ it also has equality constraints setting elements (corresponding to edges) of a matrix variable to zero. This is explained in Appendix C. It is reasonable to consider aggregating the edge equality constraints of the Lovász SDP to yield another aggregated SDP relaxation. We show in Proposition C.1, through a straightforward generalization of Galli and Letchford [17, Proposition 3], that for any aggregation of the Lovász SDP, there is a corresponding aggregation of the theta body such that the bound from the latter is at least as good as the former. Hence, we do not pursue aggregations of the Lovász SDP.

### 4.3 SDP Aggregation Closure

For $\kappa \in \mathbb{Z}_{++}$ and $\mathcal{B} \subseteq \mathbb{R}^{\kappa\times m}$, the SDP aggregation closure over matrices in $\mathcal{B}$ is defined as the intersection of aggregated relaxations $\mathsf{R}(G, B)$ from (11b) over all possible choices of $B \in \mathcal{B}$, i.e.,

$$\mathsf{R}_{\mathcal{B}}(G) := \bigcap_{B\in\mathcal{B}} \mathsf{R}(G, B). \tag{15a}$$

If the closure is taken over all $\kappa \times m$ matrices, we replace $\mathcal{B}$ with $\kappa$ in the notation and write this *full closure* of the $\kappa$-th order as

$$\mathsf{R}_\kappa(G) := \bigcap_{B\in\mathbb{R}^{\kappa\times m}} \mathsf{R}(G, B), \qquad \kappa \in \mathbb{Z}_{++}. \tag{15b}$$

The aggregation closure can be viewed as the limit of the strength of the relaxations arising from individual aggregations. It is the intersection of (possibly uncountably many) projected spectrahedra and its finite representability, which we define below, will help identify critical one-row aggregations.

**Definition 2.** We say that $\mathsf{R}_{\mathcal{B}}(G)$ *admits a finite representation* if there exist $B^1, B^2, \ldots, B^t \in \mathcal{B}$ for some finite $t$ such that $\mathsf{R}_{\mathcal{B}}(G) = \bigcap_{j=1}^t \mathsf{R}(G, B^j)$.

Following elementary properties serve as a buildup towards our main results.

**Lemma 4.3.** *For any $\mathcal{B} \subseteq \mathbb{R}^{\kappa\times m}$, we have $\mathsf{TH}(G) \subseteq \mathsf{R}_{\mathcal{B}}(G)$ and $\mathsf{R}_{\mathcal{B}}(G)$ is a nonempty, compact, and convex set. Furthermore, $\mathsf{R}_{\kappa+1}(G) \subseteq \mathsf{R}_{\kappa}(G)$ for all $\kappa \in \mathbb{Z}_{++}$.*

*Proof.* The first inclusion directly follows from Lemma 4.1(ii) and (15b), and this also makes the closure a nonempty set. It is compact and convex since $\mathsf{R}(G, B)$ is compact and convex by Lemma 4.1(i). For any $B \in \mathbb{R}^{\kappa\times m}$, define $B^+ \in \mathbb{R}^{(\kappa+1)\times m}$ as the matrix whose first $\kappa$ rows are given by $B$ and the last row is the zero vector. Then, $\mathsf{R}(G, B) = \mathsf{R}(G, B^+)$ since $\mathcal{R}(G, B) = \mathcal{R}(G, B^+)$, where $\mathcal{R}(G, B)$ is given by (11a). Therefore,

$$\mathsf{R}_{\kappa+1}(G) \subseteq \bigcap_{B^+\in\mathbb{R}^{(\kappa+1)\times m}} \mathsf{R}(G, B^+) = \bigcap_{B\in\mathbb{R}^{\kappa\times m}} \mathsf{R}(G, B) = \mathsf{R}_{\kappa}(G). \qquad \square$$

By Lemmas 4.1 and 4.3, we immediately obtain $\mathsf{TH}(G) = \mathsf{R}_{\kappa}(G)$ for all $\kappa \geq m$. Our main result is that all full closures are equal to the theta body.

**Theorem 4.4.** $\mathsf{TH}(G) = \mathsf{R}_{\kappa}(G)$ *for all* $\kappa \in \mathbb{Z}_{++}$.

We will prove Theorem 4.4 in Section 5.1 by analysing one-row aggregations. A direct, but important, consequence of this result is that the equivalence of the strength with the aggregation closure persists even if the theta body and aggregations are strengthened through the addition of valid inequalities in the $x$-space.

**Corollary 4.5.** *For any convex relaxation $\mathcal{C} \supset \mathsf{STAB}(G)$ and $\kappa \in \mathbb{Z}_{++}$,*

$$\mathsf{TH}(G) \cap \mathcal{C} = \bigcap_{B\in\mathbb{R}^{\kappa\times m}} \mathsf{Proj}_x\{(x, X) \in \mathcal{R}(G, B) \colon x \in \mathcal{C}\}.$$

*Proof.* Using the definition of projection, it is easy to verify that each projection on the right-hand side can be simplified to $\mathcal{C} \cap \mathsf{Proj}_x \mathcal{R}(G, B)$, and then the claim follows from applying Theorem 4.4 and using (11b) for $\mathsf{R}(G, B)$. $\square$

Since the projection and intersection operators do not commute in general, it is not clear whether the same holds after addition of valid inequalities in $(x, X)$-space.

## 4.4 Stronger closures

Let us elaborate on the construction of (15a). There are two points to discuss – first about the many steps in our procedure, and second about the differences with studies in the literature on aggregation closures.

There are four building blocks to (15a) – *aggregation* of constraints, SDP relaxation through *lifting*, *intersection* over multiple aggregations, *projection* onto the $x$-space. There are a total of 4! permutations for the sequence in which these operators are applied, but not all sequences are permissible – projection not before lifting, intersection not before aggregation. Moreover, the aggregation and lifting operators commute. Thus, there are essentially only two ways of taking closures. The first way, which is what we are doing in (15a), is to sequence the operators as follows (starting with leftmost) : aggregate $\sim$ lift-relax $\to$ project $\to$ intersect. The second closure is obtained by swapping the last two operations : aggregate $\sim$ lift-relax $\to$ intersect $\to$ project, and this is at least as strong as $\mathsf{R}_{\mathcal{B}}(G)$ due to non-commutativity properties of projection and intersection. In fact, this second closure is equal to $\mathsf{TH}(G)$ when $\mathcal{B}$ has matrices $B^1, B^2, \ldots, B^t$ for some finite $t$ such that the row-wise concatenated matrix $\tilde{B} := [B^1; B^2; \ldots; B^t]$ has full column rank (equal to $m$); this is due to $\tilde{B}$ serving as a new aggregation matrix and applying Lemma 4.1(iii). Indeed, in the lifted space, intersection of aggregations is equivalent to a new aggregation of higher order. Thus, unless $\mathcal{B}$ has very few elements, the nontrivial question is to study the weaker closure $\mathsf{R}_{\mathcal{B}}(G)$, which is what we focus on in this paper.

For the second point, we note that aggregation closures in the literature take the convex hull of each aggregated set before intersecting a collection of these convex hulls, whereas we are applying the SDP relaxation operator instead of the convex hull operator. If we are to consider closures from convexification, obviously $\mathsf{STAB}(G) = \cap_{B\in\mathcal{B}} \mathfrak{conv}\, \mathsf{S}(G,B)$ whenever $\mathcal{B} \cap \mathsf{FP}(G,\kappa) \neq \emptyset$, because $\mathsf{STAB}(G) = \mathfrak{conv}\, \mathsf{S}(G,B)$ for any $B \in \mathsf{FP}(G,\kappa)$. However, it is not clear whether there is some collection $\mathcal{B}$ with $\mathcal{B} \cap \mathsf{FP}(G,\kappa) = \emptyset$ for which this closure is exact; we leave this for future research.

# 5 General One-Row Aggregations

We consider general one-row aggregations obtained by aggregating all quadratic edge equalities in (2),

$$\mathsf{S}(G,B) = \Big\{x \in \{0,1\}^n : \sum_{(i,j)\in\mathsf{E}} b_{ij}\, x_i x_j = 0\Big\}, \qquad B \in \mathbb{R}^{1\times m}. \tag{16}$$

This is a special case of the general aggregation $\mathsf{S}(G,B)$ defined in (8), by restricting $\kappa = 1$. By adapting the duplication argument used in the proof of Lemma 4.3, we conclude that this one-row $\mathsf{S}(G,B)$ is the weakest member within the family of general $\mathsf{S}(G,B)$s, in the sense that any $\kappa$-row $\mathsf{S}(G,B)$ is at least as strong as each of the $\kappa$ one-row $\mathsf{S}(G,B)$s. Similarly, the projection of the corresponding Shor relaxation associated with a one-row $\mathsf{S}(G,B)$, which we denote as a one-row $\mathsf{R}(G,B)$, is the weakest within the family (11b).

Our main goal in this section is to shed light on the strength of such general one-row relaxations $\mathsf{R}(G,B)$. In addition, our objective is to give a complete description of $\mathsf{R}_1(G)$ which is defined as

$$\mathsf{R}_1(G) \coloneqq \bigcap_{B\in\mathbb{R}^{1\times m}} \mathsf{R}(G,B) = \bigcap_{B\in\mathbb{R}^{1\times m}: B^\top e = 1} \mathsf{R}(G,B), \tag{17}$$

where the second equality is due to scale-invariance of $\mathsf{R}(G,B)$ in Lemma 4.1(v).

For our subsequent analysis, we recall the standard terminology that a linear inequality is said to be *valid* to a set, or a *valid inequality* for that set, if it is satisfied by all points in that set.

## 5.1 Theta Body and Aggregation Closure

Our main goal in this section is to show that the single-row aggregation closure $\mathsf{R}_1(G)$ from (17) is equal to the theta body. By Lemma 4.3, we have $\mathsf{TH}(G) \subseteq \mathsf{R}_1(G)$. To establish the reverse inclusion, we rely on the orthonormal representation (4) for the theta body. To that end, we next investigate the validity of orthonormal representation constraints for $\mathsf{R}(G,B)$.

**Lemma 5.1.** *Let $U = [u^1,\dots,u^n] \in \mathcal{O}_\gamma(G)$ and $c \in \mathbb{R}^\gamma$ be a unit vector. Then, the corresponding orthonormal representation constraint $\sum_{i\in\mathsf{V}}(c^\top u^i)^2 x_i \le 1$ is valid for $\mathsf{R}(G,B)$, where $B \in \mathbb{R}^{1\times m}$ is given by $b_{ij} = \left(c^\top u^i\right)\left(c^\top u^j\right)\left((u^i)^\top u^j\right)$ for $(i,j) \in \mathsf{E}$.*

*Proof.* Let $\{u^1,\dots,u^n\} \subset \mathbb{R}^\gamma$ be an orthonormal representation of $G$, let $c \in \mathbb{R}^\gamma$ be a unit vector, and let $B$ be defined as in the hypothesis. Let $w = \sum_{j\in\mathsf{V}}(c^\top u^j)^2 e_j$. By (12), the orthonormal representation constraint $\sum_{i\in\mathsf{V}}(c^\top u^i)^2 x_i \le 1$ is valid for $\mathsf{R}(G,B)$ if and only if $\nu(G,w,B) \le 1$. By weak duality between the primal-dual SDP pair given by (P$(B,w)$) and (D$(B,w)$) for $l = 1$, this inequality is valid if there exists $(\hat{y},\hat{z}) \in \mathbb{R}^n \times \mathbb{R}$ such that $(1,\hat{y},\hat{z})$ is dual feasible.

To that end, let us first define $M := W W^\top \in \mathbb{S}^{n+1}$ where $W \in \mathbb{R}^{(n+1)\times n}$ has its first row, indexed by 0, equal to $c^\top$ and for $i = 1,\dots,n$, the $i^{th}$ row is equal to $-(c^\top u^i)(u^i)^\top$.

Observe that $M \succeq \mathbf{0}$, $M_{00} = 1$, $M_{0j} = -(c^\top u^j)^2$, $j = 1, \ldots, n$, $M_{ii} = (c^\top u^i)^2$, $i = 1, \ldots, n$, $M_{ij} = b_{ij}$, $(i,j) \in \mathsf{E}$, and $M_{ij} = 0$, $(i,j) \notin \mathsf{E}$. Now define $\hat{y} = w$ and $\hat{z} = 2$. It is easy to verify that (the dual positive semidefinite constraint is) $(1, \hat{y}, \hat{z})$ is feasible for ($\mathrm{D}(B, w)$) since

$$\begin{bmatrix} \hat{\delta} & -\frac{1}{2}(w + \hat{y})^\top \\ -\frac{1}{2}(w + \hat{y}) & \hat{z}\mathscr{A}(B) + \mathsf{Diag}(\hat{y}) \end{bmatrix} = \begin{bmatrix} 1 & -w^\top \\ -w & 2\mathscr{A}(B) + \mathsf{Diag}(w) \end{bmatrix} = M \succeq \mathbf{0},$$

where $\mathscr{A}(B)$ is a special case of (13) and is given by

$$\mathscr{A}(B) = \frac{1}{2} \sum_{(i,j) \in \mathsf{E}} b_{ij} \left( e_i e_j^\top + e_j e_i^\top \right) \in \mathbb{S}^n. \tag{18}$$

It follows that $(1, w, 2)$ is a feasible solution of ($\mathrm{D}(B, w)$), which gives the validity of the orthonormal representation constraint for $\mathsf{R}(G, B)$. $\square$

This enables us to establish our first main result on aggregation closures.

***Proof of Theorem 4.4.*** By Lemma 4.3(i) and (iii), it suffices to argue $\mathsf{TH}(G) \supseteq \mathsf{R}_1(G)$. By Lemma 5.1, for each orthonormal representation constraint in (4), there exists some $B \in \mathbb{R}^{1 \times m}$ such that the constraint is valid for the corresponding one-row relaxation $\mathsf{R}(G, B)$. Therefore, $\mathsf{R}_1(G) = \bigcap_{B \in \mathbb{R}^{1 \times m}} \mathsf{R}(G, B) \subseteq \mathsf{TH}(G)$. $\square$

Despite the fact that one-row relaxations give rise to the weakest relaxations of $\mathsf{S}(G)$, Theorem 4.4 reveals that their aggregation closure closes the possible relaxation gap with $\mathsf{TH}(G)$ (cf. Lemma 4.1(ii)).

In the next section, we consider the question of finite representability of $\mathsf{R}_\kappa(G)$.

## 5.2 Aggregations over Clique Edges

We consider the specific family of aggregations over edges in cliques and then establish their important role in the finite representability of $\mathsf{R}_1(G)$. We first show that each clique inequality is valid to the aggregated relaxation $\mathsf{R}(G, B)$, where the aggregation is taken over the clique edges. To do this, we use the well-known observation that every clique inequality (5) can be expressed as an orthonormal representation inequality (4).

**Lemma 5.2.** *For any clique $\mathsf{C} \subseteq \mathsf{V}$, the clique inequality $\sum_{j \in \mathsf{C}} x_j \leq 1$ is valid for $\mathsf{R}(G, B(\mathsf{C}))$ where $B(\mathsf{C}) \in \mathbb{R}^{1 \times m}$ is given by*

$$b_{ij}(\mathsf{C}) = 1, \quad \text{if } (i,j) \in \mathsf{E}[\mathsf{C}], \qquad b_{ij}(\mathsf{C}) = 0, \; (i,j) \in \mathsf{E} \setminus \mathsf{E}[\mathsf{C}]. \tag{19}$$

*Proof.* Without loss of generality, let $\mathsf{C} = \{1, \ldots, \eta\}$. Choose $c = e_1 \in \mathbb{R}^n$. Define $u^i = c$, $i \in \mathsf{C}$. For each $j \notin \mathsf{C}$, let $u^j = e_j$. It is easy to verify that this is an orthonormal representation of $G$. Therefore, we obtain $\sum_{i \in \mathsf{V}} (c^\top u^i)^2 x_i = \sum_{i \in \mathsf{C}} x_i \leq 1$. By Lemma 5.1, this orthonormal representation constraint is valid for $\mathsf{R}(G, B)$, where $B \in \mathbb{R}^{1 \times m}$ is given by $b_{ij} = \left(c^\top u^i\right)\left(c^\top u^j\right)\left((u^i)^\top u^j\right)$ for $(i,j) \in \mathsf{E}$. It is easy to verify that $B = B(\mathsf{C})$, where $B(\mathsf{C})$ is defined as in (19). $\square$

Lemma 5.2 highlights that the intersection of the projections of one-row aggregations over all clique edges is at least as strong as $\mathsf{QSTAB}(G)$. We utilize this in our next main result Theorem 5.4 along with the following technical lemma.

**Lemma 5.3** (Bomze et al. [3, Lemma 4.4])**.** *For any $x \in \mathbb{R}^n$, we have $\mathsf{Diag}(x) - xx^\top \succeq \mathbf{0}$ if and only if $x \in \mathbb{R}^n_+$ and $e^\top x \leq 1$.*

**Theorem 5.4.** *Let $\mathsf{C}'$ denote the collection of all maximal cliques in $G$. Then,*

$$\mathsf{TH}(G) = \mathsf{R}_1(G) \subseteq \bigcap_{\mathsf{C}\in\mathsf{C}'} \mathsf{R}(G, B(\mathsf{C})) = \mathsf{QSTAB}(G), \tag{20}$$

*where $B(\mathsf{C})$ is defined as in (19). For a perfect graph, equality holds throughout in (20) and hence $\mathsf{R}_1(G)$ admits a finite representation.*

*Proof.* The first equality and first inclusion are from Theorem 4.4 and (17), so it remains to prove the last equality. The forward inclusion $\bigcap_{\mathsf{C}\in\mathsf{C}'} \mathsf{R}(G, B(\mathsf{C})) \subseteq \mathsf{QSTAB}(G)$ is by Lemma 5.2 and (5). Showing the reverse inclusion is equivalent to arguing $\mathsf{QSTAB}(G) \subseteq \mathsf{R}(G, B(\mathsf{C}))$ for all $\mathsf{C} \in \mathsf{C}'$. Take any $\hat{x} \in \mathsf{QSTAB}(G)$ and any maximal clique $\mathsf{C}$. Without loss of generality, assume that $\mathsf{C} = \{1, \ldots, \eta\}$, and let $\mathsf{D} := \mathsf{V} \setminus \mathsf{C} = \{\eta+1, \ldots, n\}$. Partition $\hat{x}$ into subvectors indexed by $\mathsf{C}$ and $\mathsf{D}$ as $\hat{x} = (\hat{x}_{\mathsf{C}},\ \hat{x}_{\mathsf{D}}) \in \mathbb{R}^{\eta} \times \mathbb{R}^{n-\eta}$. We argue that by defining the matrix $\hat{X}$ as

$$\hat{X} := \begin{bmatrix} \mathtt{Diag}(\hat{x}_{\mathsf{C}}) & \hat{x}_{\mathsf{C}}(\hat{x}_{\mathsf{D}})^\top \\ \hat{x}_{\mathsf{D}}(\hat{x}_{\mathsf{C}})^\top & \hat{x}_{\mathsf{D}}(\hat{x}_{\mathsf{D}})^\top + \mathtt{Diag}(\hat{x}_{\mathsf{D}} - \hat{x}_{\mathsf{D}} \circ \hat{x}_{\mathsf{D}}) \end{bmatrix},$$

where $\hat{x}_{\mathsf{D}} \circ \hat{x}_{\mathsf{D}}$ denotes the componentwise product, the point $(\hat{x}, \hat{X})$ satisfies all the constraints in the SDP formulation of $\mathsf{R}(G, B(\mathsf{C}))$, which implies that $\hat{x} \in \mathsf{R}(G, B(\mathsf{C}))$ and thus completes our proof.

Since $\hat{X}_{ij} = 0$ for all $i, j \in \mathsf{C}$ due to $\hat{X}$ having a diagonal block in the top-left, we obtain $\langle \mathscr{A}(B(\mathsf{C})), \hat{X}\rangle = 0$, where $\mathscr{A}(B(\mathsf{C}))$ follows from (18) and (19). It is easy to check that $\mathtt{diag}(\hat{X}) = \hat{x}$. Since $\hat{x} \in \mathsf{QSTAB}(G) \subseteq [0,1]^n$, we have $\mathtt{Diag}(\hat{x}_{\mathsf{D}} - \hat{x}_{\mathsf{D}} \circ \hat{x}_{\mathsf{D}}) \succeq \mathbf{0}$. Also, $\hat{x}_{\mathsf{C}} \in \mathbb{R}^{\eta}_{+}$ and $\sum_{j\in\mathsf{C}} \hat{x}_j = e^\top \hat{x}_{\mathsf{C}} \leq 1$, and then Lemma 5.3 implies $\mathtt{Diag}(\hat{x}_{\mathsf{C}}) - \hat{x}_{\mathsf{C}}(\hat{x}_{\mathsf{C}})^\top \succeq \mathbf{0}$. The construction of $\hat{X}$ gives us

$$\hat{X} - \hat{x}\,\hat{x}^\top = \begin{bmatrix} \mathtt{Diag}(\hat{x}_{\mathsf{C}}) - \hat{x}_{\mathsf{C}}\,(\hat{x}_{\mathsf{C}})^\top & 0 \\ 0 & \mathtt{Diag}(\hat{x}_{\mathsf{D}} - \hat{x}_{\mathsf{D}} \circ \hat{x}_{\mathsf{D}}) \end{bmatrix},$$

and this block-diagonal matrix is positive semidefinite because we have already argued the diagonal blocks to be positive semidefinite. Thus, we have argued all the feasibility constraints to conclude $\hat{x} \in \mathsf{R}(G, B(\mathsf{C}))$, which yields the last equality in (20).

For a perfect graph, the equalities follow directly from Proposition 2.1(iii) and (20), and finite representability of $\mathsf{R}_1(G)$ is due to having finitely many cliques. $\square$

**??** 6.3 notes later that there are imperfect graphs for which the inclusion in the above theorem is strict.

## 6 Positive One-Row Aggregations

In this section, we focus on feasibility-preserving one-row aggregations of $\mathsf{S}(G, B)$. By Corollary 3.6, a sufficient condition is that $B \in \mathbb{R}^{1\times m}_{++}$ is a positive row vector, and hence

$$\mathsf{S}(G, B) = \Big\{x \in \{0,1\}^n : \sum_{(i,j)\in\mathsf{E}} b_{ij}\, x_i x_j = 0\Big\}, \qquad B \in \mathbb{R}^{1\times m}_{++}. \tag{21}$$

We remark that any such $B$ is a feasibility-preserving aggregation vector by Lemma 3.2 (see Definition 1). Therefore, our main objective is to investigate the strength of the relaxations $\mathsf{R}(G, B)$ arising from $\mathsf{S}(G, B)$ by (21), i.e., $\mathsf{R}(G, B)$ defined as in (11b), where $B \in \mathbb{R}^{1\times m}_{++}$, and their *positive aggregation closure* given by

$$\mathsf{R}_1^{>}(G) := \bigcap_{B\in\mathbb{R}^{1\times m}_{++}} \mathsf{R}(G, B) = \bigcap_{B\in\mathbb{R}^{1\times m}_{++}:\, B^\top e=1} \mathsf{R}(G, B), \tag{22}$$

which is the general closure (15a) with $\mathcal{B} = \mathbb{R}^{1\times m}_{++}$, and the second equality is due to the scale invariance of $\mathsf{R}(G, B)$ from Lemma 4.1(v). This contains the one-row aggregation closure $\mathsf{R}_1(G)$ from (17),

$$\mathsf{R}_1(G) \coloneqq \bigcap_{B\in\mathbb{R}^{1\times m}} \mathsf{R}(G, B) \subseteq \mathsf{R}^{>}_1(G). \tag{23}$$

For any $B \in \mathbb{R}^{1\times m}_{++}$, we have $\mathsf{R}^{>}_1(G) \subseteq \mathsf{R}(G, B)$. Thus, the positive aggregation closure $\mathsf{R}^{>}_1(G)$ can be viewed as the limit of the strength of the relaxations $\mathsf{R}(G, B)$ arising from each feasibility-preserving aggregation $\mathsf{S}(G, B)$ given by (21). Our aim in this section is to obtain a complete description of $\mathsf{R}^{>}_1(G)$. Furthermore, in view of (23), such a description allows the comparison of $\mathsf{R}_1(G)$ and $\mathsf{R}^{>}_1(G)$.

## 6.1 Validity of Clique Inequalities

For each clique $\mathsf{C} \subseteq \mathsf{V}$ in $G = (\mathsf{V}, \mathsf{E})$, recall from Proposition 2.1 that the clique inequality $\sum_{j\in\mathsf{C}} x_j \leq 1$ is valid for $\mathsf{STAB}(G)$. Lemma 5.2 gives a recipe for choosing $B \in \mathbb{R}^{1\times m}$, which ensures the validity of the clique inequality for $\mathsf{R}(G, B)$. In this section, we address the same question under the additional assumption that $B \in \mathbb{R}^{1\times m}_{++}$.

For complete graphs, we first observe that a single positive aggregation vector is sufficient to obtain an exact relaxation.

**Observation 6.1.** *If $G = K_n$, the unique maximal clique inequality $\sum_{j=1}^{n} x_j \leq 1$ is valid for $\mathsf{R}(K_n, \frac{1}{n}e^\top)$, and $\mathsf{STAB}(K_n) = \mathsf{R}^{>}_1(K_n) = \mathsf{R}(K_n, \frac{1}{n}e^\top)$.*

*Proof.* The first claim is directly from Lemma 5.2 since $B(\{1, \ldots, n\}) = e^\top$ and the scale invariance of $\mathsf{R}(G, B)$ due to Lemma 4.1(v). Since $K_n$ is a perfect graph, it follows from Proposition 2.1(iii) that $\mathsf{STAB}(K_n) = \mathsf{TH}(K_n) = \mathsf{QSTAB}(K_n) = \{x \in \mathbb{R}^n_+ : \sum_{j=1}^{n} x_j \leq 1\}$. Combining with Lemma 4.1(ii), (22), and the validity of the maximal clique inequality for $\mathsf{R}(K_n, \frac{1}{n}e^\top)$, we obtain the equality. □

We next consider non-complete graphs, and so note that any clique is now a proper subset of $\mathsf{V}$. As such, the recipe of Lemma 5.2 cannot be used to ensure the validity of the clique inequality corresponding to such a clique $\mathsf{C}$ since $B(\mathsf{C}) \notin \mathbb{R}^{1\times m}_{++}$.

Our next result gives a complete characterization of the set of cliques for which there is a $B \in \mathbb{R}^{1\times m}_{++}$ such that the corresponding clique inequality is valid for $\mathsf{R}(G, B)$.

**Proposition 6.1.** *Let $G \neq K_n$ and $\mathsf{C} \subset \mathsf{V}$ be a clique. Then, there exists a $B \in \mathbb{R}^{1\times m}_{++}$ such that the clique inequality $\sum_{j\in\mathsf{C}} x_j \leq 1$ is valid for $\mathsf{R}(G, B)$ if and only if $\mathsf{C} \cup \{k\}$ is a clique for each $k \in \mathsf{V} \setminus \mathsf{C}$.*

*Proof.* Without loss of generality, suppose that $\mathsf{C} = \{1, \ldots, \eta\}$, and let $\mathsf{D} = \mathsf{V} \setminus \mathsf{C} = \{\eta + 1, \ldots, n\}$. Note that $\mathsf{D} \neq \emptyset$ since $G \neq K_n$. Let $w = \chi^{\mathsf{C}} = \sum_{j=1}^{\eta} e_j$. By (12), the clique inequality $\sum_{j\in\mathsf{C}} x_j \leq 1$ is valid for $\mathsf{R}(G, B)$ if and only if $\nu(G, w, B) \leq 1$.

( $\Longrightarrow$ ) Suppose that $\nu(G, w, B) \leq 1$ for some $B \in \mathbb{R}^{1\times m}_{++}$. By strong duality of the primal–dual SDP pair (P(B, w)) and (D(B, w)) from Lemma 4.2, there exist $(\hat{y}, \hat{z}) \in \mathbb{R}^n \times \mathbb{R}$ such that

$$M \coloneqq \begin{bmatrix} 1 & -\frac{1}{2}(w + \hat{y})^\top \\ -\frac{1}{2}(w + \hat{y}) & \hat{z}\,\mathscr{A}(B) + \mathsf{Diag}(\hat{y}) \end{bmatrix} \succeq \mathbf{0},$$

where $\mathscr{A}(B)$ is given by (18). We can partition $M$ as follows:

$$M = \begin{bmatrix} 1 & -\frac{1}{2}(e + \hat{y}_{\mathsf{C}})^\top & -\frac{1}{2}(\hat{y}_{\mathsf{D}})^\top \\ -\frac{1}{2}(e + \hat{y}_{\mathsf{C}}) & \mathsf{Diag}(\hat{y}_{\mathsf{C}}) + \hat{z}\,[\mathscr{A}(B)]_{\mathsf{CC}} & \hat{z}\,[\mathscr{A}(B)]_{\mathsf{CD}} \\ -\frac{1}{2}\hat{y}_{\mathsf{D}} & \hat{z}\,[\mathscr{A}(B)]_{\mathsf{CD}}^\top & \mathsf{Diag}(\hat{y}_{\mathsf{D}}) + \hat{z}\,[\mathscr{A}(B)]_{\mathsf{DD}} \end{bmatrix} \succeq \mathbf{0}, \tag{24}$$

where $e \in \mathbb{R}^\eta$, $\hat{y}_{\mathsf{C}} \in \mathbb{R}^\eta$, and $\hat{y}_{\mathsf{D}} \in \mathbb{R}^{n-\eta}$ denote the subvectors of $\hat{y}$ indexed by $\mathsf{C}$ and $\mathsf{D}$, respectively, and $[\mathscr{A}(B)]_{\mathsf{CC}} \in \mathbb{S}^\eta$, $[\mathscr{A}(B)]_{\mathsf{CD}} \in \mathbb{R}^{\eta \times (n-\eta)}$, and $[\mathscr{A}(B)]_{\mathsf{DD}} \in \mathbb{S}^{n-\eta}$ denote the submatrices of $\mathscr{A}(B) \in \mathbb{S}^n$ corresponding to rows and columns indexed by $\mathsf{C} \times \mathsf{C}$, $\mathsf{C} \times \mathsf{D}$ and $\mathsf{D} \times \mathsf{D}$, respectively. We index the first row and first column of $M$ by 0.

Since $M \succeq \mathbf{0}$, each $2 \times 2$ principal submatrix of $M$ corresponding to the indices 0 and $j \in \mathsf{C}$ is positive semidefinite, i.e.,

$$\begin{bmatrix} 1 & -\frac{1}{2}(\hat{y}_j + 1) \\ -\frac{1}{2}(\hat{y}_j + 1) & \hat{y}_j \end{bmatrix} \succeq \mathbf{0}, \quad j \in \mathsf{C},$$

which holds if and only $\hat{y}_j \geq 0$ and

$$0 \leq \hat{y}_j - \tfrac{1}{4}\left(\hat{y}_j + 1\right)^2 = -\tfrac{1}{4}\left(\hat{y}_j - 1\right)^2, \quad j \in \mathsf{C}.$$

We therefore conclude that

$$\hat{y}_j = 1, \quad j \in \mathsf{C}, \tag{25}$$

which implies that $\hat{y}_{\mathsf{C}} = e \in \mathbb{R}^\eta$. Combining this with the Schur complement on the top left $2 \times 2$ block of $M$, we obtain $I + \hat{z}\,[\mathscr{A}(B)]_{\mathsf{CC}} - ee^\top \succeq \mathbf{0}$. Since $\mathtt{Diag}([\mathscr{A}(B)]_{\mathsf{CC}}) = \mathbf{0}$, we conclude

$$\mathtt{Diag}\left(I + \hat{z}\ [\mathscr{A}(B)]_{\mathsf{CC}} - ee^\top\right) = \mathbf{0}.$$

Since $I + \hat{z}\ [\mathscr{A}(B)]_{\mathsf{CC}} - ee^\top \succeq \mathbf{0}$, it follows that $I + \hat{z}\ [\mathscr{A}(B)]_{\mathsf{CC}} - ee^\top = 0$, or equivalently that $\hat{z}\ [\mathscr{A}(B)]_{\mathsf{CC}} = ee^\top - I$. By (18), we obtain

$$\tfrac{1}{2}\,\hat{z}\,b_{ij}\,A_{ij} = \tfrac{1}{2}\,\hat{z}\,b_{ij} = 1, \quad i, j \in \mathsf{C},\ i \neq j, \tag{26}$$

where $A = \sum\limits_{(i,j) \in \mathsf{E}} \left(e_i e_j^\top + e_j e_i^\top\right) \in \mathbb{S}^n$ denotes the adjacency matrix of $G$. Since $B \in \mathbb{R}^{1 \times m}_{++}$, we obtain

$$\hat{z} > 0. \tag{27}$$

Substituting $\hat{y}_{\mathsf{C}} = e \in \mathbb{R}^\eta$ and $\hat{z}\ [\mathscr{A}(B)]_{\mathsf{CC}} = ee^\top - I$ in (24), and using the Schur complement of the entire matrix $M$, we arrive at

$$\begin{bmatrix} 0 & \hat{z}\,[\mathscr{A}(B)]_{\mathsf{CD}} - \frac{1}{2}\,e\,(\hat{y}_{\mathsf{D}})^\top \\ \hat{z}\ [\mathscr{A}(B)]_{\mathsf{CD}}^\top - \frac{1}{2}\,\hat{y}_{\mathsf{D}}\,e^\top & \mathtt{Diag}(\hat{y}_{\mathsf{D}}) + \hat{z}\ [\mathscr{A}(B)]_{\mathsf{DD}} - \frac{1}{4}\,\hat{y}_{\mathsf{D}}\,(\hat{y}_{\mathsf{D}})^\top \end{bmatrix} \succeq \mathbf{0}, \tag{28}$$

which implies that

$$\hat{z}\ [\mathscr{A}(B)]_{\mathsf{CD}} = \tfrac{1}{2}\,e\,(\hat{y}_{\mathsf{D}})^\top. \tag{29}$$

Since $\mathtt{Diag}([\mathscr{A}(B)]_{\mathsf{DD}}) = \mathbf{0}$, we have $\hat{y}_{\mathsf{D}} \geq \mathbf{0}$ by the lower right block of the left-hand side of (28). Next, we argue that $\hat{y}_{\mathsf{D}} > \mathbf{0}$. Suppose, for a contradiction, that there exists $k \in \mathsf{D}$ such that $\hat{y}_k = 0$. By (18), (27) and (29), we obtain

$$[[\mathscr{A}(B)]_{\mathsf{CD}}]_{jk} = \tfrac{1}{2}\,b_{jk}\,A_{jk} = 0, \quad j \in \mathsf{C}. \tag{30}$$

Using the $(3,3)$-block of the matrix $M$, we have $\mathtt{Diag}(\hat{y}_{\mathsf{D}}) + \hat{z}\ [\mathscr{A}(B)]_{\mathsf{DD}} \succeq \mathbf{0}$. Since $\hat{y}_k = 0$ for some $k \in \mathsf{D}$, and $\mathtt{Diag}([\mathscr{A}(B)]_{\mathsf{DD}}) = \mathbf{0}$, we conclude that

$$[[\mathscr{A}(B)]_{\mathsf{DD}}]_{lk} = \tfrac{1}{2}\,b_{lk}\,A_{lk} = 0, \quad l \in \mathsf{D}. \tag{31}$$

Since $B \in \mathbb{R}^{1 \times m}_{++}$, it follows from (30) and (31) that $A_{ik} = 0$ for each $i \in \mathsf{V}$, i.e., $k \in \mathsf{D}$ is an isolated vertex of $G$, which contradicts our hypothesis that $G$ is connected. Therefore, we conclude that

$$\hat{y}_k > 0, \quad k \in \mathsf{D}. \tag{32}$$

Next, it follows from (27), (29) and (32) that

$$[[\mathscr{A}(B)]_{\mathsf{CD}}]_{jk} = \tfrac{1}{2}\, b_{jk} A_{jk} = \tfrac{\hat{y}_k}{2\,\hat{z}} > 0, \quad j \in \mathsf{C}, k \in \mathsf{D}, \tag{33}$$

where we used (27) and (32) to derive the last inequality. We therefore conclude that $A_{jk} = 1$, i.e., $(j,k) \in \mathsf{E}$ for each $j \in \mathsf{C}$ and $k \in \mathsf{D} = \mathsf{V} \setminus \mathsf{C}$. Since $\mathsf{C}$ is a clique, it follows that $\mathsf{C} \cup \{k\}$ is a clique for each $k \in \mathsf{V} \setminus \mathsf{C}$. This establishes the forward implication.

($\Longleftarrow$) Suppose $\mathsf{C} = \{1, \ldots, \eta\}$ is a clique and $\mathsf{C} \cup \{k\}$ is a clique for each $k \in \mathsf{V} \setminus \mathsf{C}$. Let $B \in \mathbb{R}^{1\times m}_{++}$ be given by

$$b_{ij} = 1, \quad i,j \in \mathsf{C},\ i \neq j, \qquad b_{jk} = \frac{1}{2(n-\eta)}, \quad j \in \mathsf{C},\ k \in \mathsf{D}, \tag{34a}$$

$$b_{kl} = \frac{1}{4(n-\eta)^2}, \quad k,l \in \mathsf{D} \text{ such that } (k,l) \in \mathsf{E}. \tag{34b}$$

We argue that $\nu(G,w,B) \leq 1$. By the hypothesis, we have $(i,j) \in \mathsf{E}$ and $(j,k) \in \mathsf{E}$ for each distinct $i,j \in \mathsf{C}$ and each $k \in \mathsf{D}$. We next construct $(\hat{y},\hat{z}) \in \mathbb{R}^n \times \mathbb{R}$ such that the matrix $M \in \mathbb{S}^{n+1}$ from (24) is positive semidefinite, which would imply that $(1,\hat{y},\hat{z})$ is a feasible solution of ($\mathrm{D}(B,w)$) and complete our proof.

Let $\hat{z} = 2$ and $\hat{y} \in \mathbb{R}^n$ be given by

$$\hat{y}_j = 1, \quad j \in \mathsf{C}, \qquad \hat{y}_k = \frac{1}{n-\eta}, \quad k \in \mathsf{D}. \tag{35}$$

Consider the semidefiniteness constraint for $M$. By (28), it suffices to show that

$$\mathtt{Diag}(\hat{y}_{\mathsf{D}}) + \hat{z}\,[\mathscr{A}(B)]_{\mathsf{DD}} - \frac{1}{4}\,\hat{y}_{\mathsf{D}}\,(\hat{y}_{\mathsf{D}})^\top \succeq \mathbf{0}.$$

(18), (34) and (35) and $\hat{z} = 2$ yield

$$\mathtt{Diag}(\hat{y}_{\mathsf{D}}) + \hat{z}\,[\mathscr{A}(B)]_{\mathsf{DD}} - \frac{1}{4}\,\hat{y}_{\mathsf{D}}\,\hat{y}_{\mathsf{D}}^\top = \left(\frac{1}{n-\eta}\right) I + \left(\frac{1}{4(n-\eta)^2}\right) A_{\mathsf{DD}} - \left(\frac{1}{4(n-\eta)^2}\right) ee^\top,$$

where $A_{\mathsf{DD}} \in \mathbb{S}^{n-\eta}$ is the adjacency matrix of the induced subgraph $G[\mathsf{D}]$. We claim that the matrix on the right-hand side is diagonally dominant, which implies it is positive semidefinite as required.

Indeed, for each $k \in \mathsf{D}$, we have

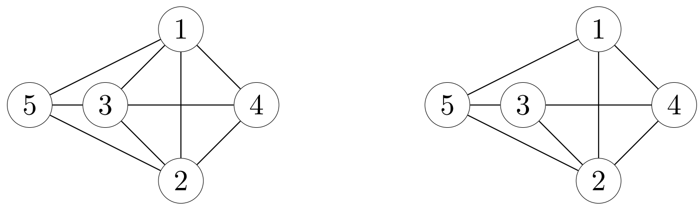
$$\frac{1}{n-\eta} - \frac{1}{4(n-\eta)^2} \sum_{l \in \mathsf{D}} |A_{kl} - 1| \geq \frac{1}{n-\eta} - \frac{n-\eta}{4(n-\eta)^2} = \frac{3}{4(n-\eta)} \geq 0,$$

where we used $|A_{kl} - 1| \leq 1$ for each $l \in \mathsf{D}$ to derive the first inequality, and $n - \eta > 0$ to derive the second one. Therefore, the positive semidefiniteness constraint is satisfied by (28). We conclude that $(1,\hat{y},\hat{z})$ is a feasible solution of ($\mathrm{D}(B,w)$). Therefore, $\nu(G,w,B) \leq 1$, which implies $\sum_{j \in \mathsf{C}} x_j \leq 1$ is valid for $\mathsf{R}(G,B)$, and we are done. □

Lemma 5.2 establishes that for every clique one can construct a $B \in \mathbb{R}^{1\times m}$ such that the corresponding clique inequality is valid for $\mathsf{R}(G,B)$. In contrast, under the additional restriction that $B \in \mathbb{R}^{1\times m}_{++}$, Proposition 6.1 provides a complete description of the subset of cliques for which the corresponding clique inequality is valid for $\mathsf{R}(G,B)$ and gives a recipe for constructing such a $B \in \mathbb{R}^{1\times m}_{++}$. The following example illustrates such a clique and the construction procedure. We also illustrate another graph for which a clique inequality is not valid for $\mathsf{R}(G,B)$ for any choice of $B \in \mathbb{R}^{1\times m}_{++}$.

*Example* 2. For the graph of order 5 given in Figure 1a, consider the clique $\mathsf{C} = \{1, 2\}$. Note that $\mathsf{C} \cup \{k\}$ is a clique for each $k \in \mathsf{V} \setminus \mathsf{C} = \{3, 4, 5\}$. By Proposition 6.1, there exists a $B \in \mathbb{R}^{1\times 9}_{++}$ such that the clique inequality $\sum_{j\in\mathsf{C}} x_j = x_1 + x_2 \leq 1$ is valid for $\mathsf{R}(G, B)$. The construction in (34a) gives us $b_{12} = 1$, $b_{13} = b_{14} = b_{15} = b_{23} = b_{24} = b_{25} = 1/6$, and the construction in (34b) gives us $b_{34} = b_{35} = 1/36$. Our numerical experiments confirm the validity of $x_1 + x_2 \leq 1$ for $\mathsf{R}(G, B)$.

For the graph in Figure 1b, obtained by removing the edge $(1, 3)$, consider the clique $\mathsf{C} = \{1, 2\}$ and note that $\mathsf{C} \cup \{3\}$ is not a clique. By Proposition 6.1, there does not exist a $B \in \mathbb{R}^{1\times 8}_{++}$ such that $x_1 + x_2 \leq 1$ is valid for $\mathsf{R}(G, B)$. $\diamond$




(a) $x_1 + x_2 \leq 1$ is valid (b) $x_1 + x_2 \leq 1$ is not valid

Figure 1: An edge inequality may or may not be valid to a positive one-row aggregation

The above example illustrates that even an edge inequality may not be valid for $\mathsf{R}(G, B)$ for any choice of $B \in \mathbb{R}^{1\times m}_{++}$. Concerning the validity of stronger maximal clique inequalities for $\mathsf{R}(G, B)$, the following result highlights a significant weakness of the additional restriction of $B \in \mathbb{R}^{1\times m}_{++}$ in comparison with $B \in \mathbb{R}^{1\times m}$.

**Corollary 6.2.** *Let $G \neq K_n$ and $\mathsf{C} \subseteq \mathsf{V}$ be a maximal clique. Then, for any $B \in \mathbb{R}^{1\times m}_{++}$, the clique inequality $\sum_{j\in\mathsf{C}} x_j \leq 1$ is not valid for $\mathsf{R}(G, B)$, and hence $\mathsf{STAB}(G) \subsetneq \mathsf{R}(G, B)$ and $\mathsf{R}(G, B) \setminus \mathsf{QSTAB}(G) \neq \emptyset$.*

*Proof.* Since $\mathsf{C}$ is not contained in a larger clique, the first assertion follows from Proposition 6.1. By (5), we therefore have $\mathsf{R}(G, B) \setminus \mathsf{QSTAB}(G) \neq \emptyset$. Since $\mathsf{STAB}(G) \subseteq \mathsf{QSTAB}(G)$ by Proposition 2.1 and $\mathsf{STAB}(G) \subseteq \mathsf{R}(G, B)$ by Lemma 4.1(ii), the strict inclusion follows. $\square$

Unless $G = K_n$, Corollary 6.2 reveals that the additional restriction of strict positivity on $B$ always gives rise to strictly weaker relaxations $\mathsf{R}(G, B)$ of $\mathsf{STAB}(G)$. Moreover, for a triangle-free graph, these relaxations are not a subset of $\mathsf{FRAC}(G)$, since every maximal clique is an edge. Although this may indicate that these aggregated relaxations are weaker than the trivial LP, we note that there is no dominance relationship between the two for odd cycles, which are one of the simplest classes of triangle-free graphs.

**Observation 6.2.** *For odd $n \leq 21$, $\mathsf{R}(C_n, e^\top)$ and $\mathsf{FRAC}(C_n)$ are not subsets of each other.*

*Proof.* By Corollary 6.2, each edge inequality $x_i + x_j \leq 1$ is not valid for $\mathsf{R}(C_n, e^\top)$, which implies $\mathsf{R}(C_n, e^\top) \nsubseteq \mathsf{FRAC}(C_n)$. To argue $\mathsf{R}(C_n, e^\top) \nsupseteq \mathsf{FRAC}(C_n)$, we numerically solved with a SDP solver the problem of maximizing $e^\top x$ over $\mathsf{R}(C_n, e^\top)$ for odd $n \leq 21$. All these values were strictly less than $n/2 = \max\{e^\top x \colon x \in \mathsf{FRAC}(C_n)\}$. $\square$

We conjecture the above observation is true for all odd $n$ and leave it as a future research question.

In the subsequent sections, we consider the impact of this additional restriction of positivity on the aggregation closure $\mathsf{R}^{>}_1(G)$ given by (22) and also relate this closure to the Shor relaxation of an alternative binary quadratic formulation of $\mathsf{S}(G)$.

## 6.2 Positive Aggregation Closure

We present a description of the positive aggregation closure $\mathsf{R}_1^>(G)$ defined in (22). A similar argument as in the proof of Lemma 4.1(i) reveals that $\mathsf{R}_1^>(G)$ is a nonempty, compact, and convex set. Furthermore, it follows from Theorem 4.4 and (23) that $\mathsf{TH}(G) = \mathsf{R}_1(G) \subseteq \mathsf{R}_1^>(G)$. Our main goal in this section is to establish $\mathsf{R}_1^>(G) = \mathsf{TH}_\leq(G)$, where we recall $\mathsf{TH}_\leq(G)$ defined in (6b), and both these sets are contained in $\mathsf{QSTAB}(G)$.

**Theorem 6.3.** $\mathsf{R}_1^>(G) = \mathsf{TH}_\leq(G) \subseteq \mathsf{QSTAB}(G)$.

For the first equality in Theorem 6.3, our proof strategy is to show that each set is contained in the other one. We start with the first inclusion.

**Lemma 6.4.** $\mathsf{R}_1^>(G) \subseteq \mathsf{TH}_\leq(G)$.

*Proof.* We will prove by contraposition. For any $\hat{x} \notin \mathsf{TH}_\leq(G)$, we want to show that $\hat{x} \notin \mathsf{R}_1^>(G)$. Towards this end, we construct a $B \in \mathbb{R}^{1\times m}_{++}$ such that $\hat{x} \notin \mathsf{R}(G,B)$; this implies $\hat{x} \notin \mathsf{R}_1^>(G)$ because $\mathsf{R}_1^>(G) \subseteq \mathsf{R}(G,B)$ from the definition (22).

Since $\mathsf{TH}_\leq(G)$ is a closed convex set and $\hat{x} \notin \mathsf{TH}_\leq(G)$, there exists $w \in \mathbb{R}^n$ such that $w^\top \hat{x} > \vartheta_\leq(G,w) := \max\{w^\top x : x \in \mathsf{TH}_\leq(G)\}$. The dual problem is solvable and has strong duality; see Lemma B.1. Let $(\delta^*, y^*, s^*) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$ denote a dual optimal solution. Note that $s^* \in \mathbb{R}^m_+$ and $\delta^* = \vartheta_\leq(G,w)$. Let us consider two cases for the set $\mathsf{E}_0 := \{(i,j) \in \mathsf{E} : s^*_{ij} = 0\}$.

*Case 1*: $\mathsf{E}_0 = \emptyset$. Note that we have $s^* \in \mathbb{R}^m_{++}$. We claim that $\hat{x} \notin \mathsf{R}(G,B)$ for $B = (s^*)^\top \in \mathbb{R}^{1\times m}_{++}$. We will prove this claim by showing that $w^\top \hat{x} > \nu(G,w,B) = \max\{w^\top x : x \in \mathsf{R}(G,B)\}$. To establish this inequality, we will construct an appropriate feasible solution $(\hat{\delta}, \hat{y}, \hat{z})$ of (D($B,w$)) relying on the dual optimal solution $(\delta^*, y^*, s^*)$. Let us define $\hat{\delta} = \delta^*$, $\hat{y} = y^*$, and $\hat{z} = 1$. Then,

$$M^0 := \begin{bmatrix} \hat{\delta} & -\frac{1}{2}(w+\hat{y})^\top \\ -\frac{1}{2}(w+\hat{y}) & \hat{z}\mathscr{A}(B) + \mathsf{Diag}(\hat{y}) \end{bmatrix} = \begin{bmatrix} \delta^* & -\frac{1}{2}(w+y^*)^\top \\ -\frac{1}{2}(w+y^*) & \mathscr{A}((s^*)^\top) + \mathsf{Diag}(y^*) \end{bmatrix} \succeq \mathbf{0},$$

where the last relation is due to the dual feasibility of $(\delta^*, y^*, s^*)$. It follows that $(\hat{\delta}, \hat{y}, \hat{z})$ is a feasible solution of (D($B,w$)). Since $\hat{\delta} = \delta^* = \vartheta_\leq(G,w)$, it follows from weak duality between (P($B,w$)) and (D($B,w$)) that

$$\nu(G,w,B) = \max\{w^\top x : x \in \mathsf{R}(G,B)\} \leq \hat{\delta} = \delta^* = \vartheta_\leq(G,w) < w^\top \hat{x}.$$

Therefore, we conclude that $\hat{x} \notin \mathsf{R}(G,B)$, which implies that $\hat{x} \notin \mathsf{R}_1^>(G)$ by (22).

*Case 2*: $\mathsf{E}_0 \neq \emptyset$. In this case, $s^* \in \mathbb{R}^m_+ \backslash \mathbb{R}^m_{++}$. Therefore, we will use a different procedure to construct a $B \in \mathbb{R}^m_{++}$ such that $\hat{x} \notin \mathsf{R}(G,B)$.

Since $(\delta^*, y^*, s^*) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$ is a dual optimal solution of (39), we have

$$M^0 = \begin{bmatrix} \delta^* & -\frac{1}{2}(w+y^*)^\top \\ -\frac{1}{2}(w+y^*) & \mathscr{A}(s^*) + \mathsf{Diag}(y^*) \end{bmatrix} \succeq \mathbf{0}.$$

Our procedure is based on a careful perturbation of $M^0$ in such a way that the resulting matrix yields a feasible solution of (D($B,w$)) for some $B \in \mathbb{R}^{1\times m}_{++}$. To that end, let us define $M^{(i,j)} \in \mathbb{S}^{n+1}$ by

$$M^{(i,j)} := \left(\tfrac{1}{2}e_0 - e_i - e_j\right)\left(\tfrac{1}{2}e_0 - e_i - e_j\right)^\top \succeq \mathbf{0}, \quad (i,j) \in \mathsf{E}_0.$$

We index the first row and first column of $M^{(i,j)}$ by 0. Note that $M^{(i,j)}_{00} = \frac{1}{4}$; $M^{(i,j)}_{0i} = M^{(i,j)}_{i0} = M^{(i,j)}_{0j} = M^{(i,j)}_{j0} = -\frac{1}{2}$; $M^{(i,j)}_{ii} = M^{(i,j)}_{jj} = M^{(i,j)}_{ij} = M^{(i,j)}_{ji} = 1$; and all remaining entries of $M^{(i,j)}$ are equal to zero.

Let $\epsilon = w^\top \hat{x} - \vartheta_{\leq}(G, w) = w^\top \hat{x} - \delta^* > 0$. We next define

$$M := M^0 + \frac{2\epsilon}{|\mathsf{E}_0|} \sum_{(i,j)\in \mathsf{E}_0} M^{(i,j)},$$

and define $(\hat{\delta}, \hat{y}, \hat{z}) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ and $B \in \mathbb{R}^{1\times m}_{++}$ as follows:

$$\hat{\delta} = \delta^* + \frac{\epsilon}{2}, \quad \hat{z} = 1, \quad \hat{y}_k = y_k^* + \sum_{\substack{l \in \mathsf{V}\backslash\{k\}:\\ (k,l)\in \mathsf{E}_0}} \frac{2\epsilon}{|\mathsf{E}_0|}, \; k \in \mathsf{V}, \quad b_{ij} = \begin{cases} \frac{2\epsilon}{|\mathsf{E}_0|}, & \text{if } (i,j) \in \mathsf{E}_0, \\ s_{ij}^*, & \text{if } (i,j) \in \mathsf{E}\backslash \mathsf{E}_0. \end{cases}$$

Then, it is easy to verify that

$$M = M^0 + \frac{2\epsilon}{|\mathsf{E}_0|} \sum_{(i,j)\in \mathsf{E}_0} M^{(i,j)} = \begin{bmatrix} \hat{\delta} & -\frac{1}{2}(w + \hat{y})^\top \\ -\frac{1}{2}(w+\hat{y}) & \hat{z}\mathscr{A}(B) + \mathsf{Diag}(\hat{y}) \end{bmatrix}.$$

Furthermore, $M \succeq \mathbf{0}$ since $M^0 \succeq \mathbf{0}$, $\epsilon > 0$, and $M^{(i,j)} \succeq \mathbf{0}$ for each $(i,j) \in \mathsf{E}_0$. Therefore, we conclude that $(\hat{\delta}, \hat{y}, \hat{z}) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ is a feasible solution of (D($B, w$)). From the weak duality relation between (P($B, w$)) and (D($B, w$)), we arrive at

$$\nu(G, w, B) = \max\{w^\top x : x \in \mathsf{R}(G, B)\} \leq \hat{\delta} = \delta^* + \tfrac{\epsilon}{2} < \delta^* + \epsilon = w^\top \hat{x}.$$

Therefore, we conclude that $\hat{x} \notin \mathsf{R}(G, B)$, which implies that $\hat{x} \notin \mathsf{R}_1^{>}(G)$ by (22). □

We next establish the reverse inclusion.

**Lemma 6.5.** $\mathsf{TH}_{\leq}(G) \subseteq \mathsf{R}_1^{>}(G)$.

*Proof.* We establish $\mathsf{TH}_{\leq}(G) \subseteq \mathsf{R}(G, B)$ for all $B \in \mathbb{R}^m_{++}$, which immediately leads to our assertion due to the definition of $\mathsf{R}_1^{>}(G)$ in (22).

Let $\hat{x} \in \mathsf{TH}_{\leq}(G)$. By (6b), there exists $\hat{X} \in \mathbb{S}^n$ such that $\hat{X}_{ij} \leq 0$ for all $(i,j) \in \mathsf{E}$, $\mathsf{diag}(\hat{X}) = \hat{x}$ and $\hat{X} - \hat{x}\,\hat{x}^\top \succeq \mathbf{0}$.

First, suppose that $\hat{X}_{ij} = 0$ for all $(i,j) \in \mathsf{E}$. Then, it follows from (11a) that $(\hat{x}, \hat{X}) \in \mathcal{R}(G, B)$ for any $B \in \mathbb{R}^m_{++}$, which, in turn, implies that $\hat{x} \in \mathsf{R}(G, B)$ for any $B \in \mathbb{R}^m_{++}$ by (11b).

Suppose now that $\hat{X}_{ij} < 0$ for some $(i,j) \in \mathsf{E}$. Since $\mathbf{0} \leq \hat{x} \leq e$, its components can be partitioned into the following sets:

$$\mathsf{V}_0 = \{i \in \mathsf{V} : \hat{x}_i = 0\}, \quad \mathsf{V}_1 = \{i \in \mathsf{V} : \hat{x}_i = 1\}, \quad \mathsf{V}_f = \{i \in \mathsf{V} : 0 < \hat{x}_i < 1\}.$$

Let $\hat{Y} := \hat{X} - \hat{x}\,\hat{x}^\top \in \mathbb{S}^n$. Since $\mathsf{diag}(\hat{X}) = \hat{x}$, we observe that $\hat{Y}_{ii} = 0$ for each $i \in \mathsf{V}_0 \cup \mathsf{V}_1$. Since $\hat{Y} \succeq \mathbf{0}$, we conclude that $\hat{Y}_{ij} = \hat{Y}_{ji} = 0$ for each $i \in \mathsf{V}_0 \cup \mathsf{V}_1$ and each $j \in \mathsf{V}$. Therefore, $\hat{X}_{ij} = \hat{X}_{ji} = \hat{x}_i \hat{x}_j \geq 0$ for each $i \in \mathsf{V}_0 \cup \mathsf{V}_1$ and each $j \in \mathsf{V}$. It follows that, for any $(i,j) \in \mathsf{E}$ such that $\hat{X}_{ij} < 0$, we have $i \in \mathsf{V}_f$ and $j \in \mathsf{V}_f$. Therefore, we conclude that $|\mathsf{V}_f| \geq 2$ and there exists an $(i,j) \in \mathsf{E}$ such that $i \in \mathsf{V}_f$, $j \in \mathsf{V}_f$, and $\hat{X}_{ij} < 0$.

Let $B \in \mathbb{R}^m_{++}$ be arbitrary. Since $\hat{X}_{ij} \leq 0$ for each $(i,j) \in \mathsf{E}$, $\hat{X}_{ij} < 0$ for at least one $(i,j) \in \mathsf{E}$ and $B \in \mathbb{R}^m_{++}$, we have $\langle \mathscr{A}(B), \hat{X}\rangle = \sum_{(i,j)\in\mathsf{E}} b_{ij}\hat{X}_{ij} < 0$. Now consider the symmetric matrix $\overline{X} := \hat{x}\,\hat{x}^\top + \mathsf{Diag}(d)$, where $d \in \mathbb{R}^n$ is given by $d_i = 0$ for each $i \in \mathsf{V}_0 \cup \mathsf{V}_1$ and $d_i = \hat{x}_i - \hat{x}_i^2 > 0$ for each $i \in \mathsf{V}_f$. Clearly, $\overline{X} \succeq \mathbf{0}$ and $\mathsf{diag}(\overline{X}) = \hat{x}$. Recall that there exists an $(i,j) \in \mathsf{E}$ such that $i \in \mathsf{V}_f$ and $j \in \mathsf{V}_f$. Since $B \in \mathbb{R}^m_{++}$, $\overline{X}_{ij} \geq 0$ for each $i = 1, \ldots, n$; $j = 1, \ldots, n$, and $\overline{X}_{ij} > 0$ for each $i \in \mathsf{V}_f$ and $j \in \mathsf{V}_f$, we obtain $\langle \mathscr{A}(B), \overline{X}\rangle = \sum_{(i,j)\in\mathsf{E}} b_{ij}\hat{X}_{ij} > 0$.

Since the inner products of $\mathscr{A}(B)$ with $\hat{X}$ and $\overline{X}$ have opposite signs, it follows that a convex combination of the two inner products can be made equal to zero, i.e., there exists a $\theta \in (0,1)$ such that

$$0 = \theta \langle \mathscr{A}(B), \overline{X} \rangle + (1-\theta) \langle \mathscr{A}(B), \hat{X} \rangle = \langle \mathscr{A}(B), \theta \overline{X} + (1-\theta) \hat{X} \rangle = \langle \mathscr{A}(B), \widetilde{X} \rangle,$$

where $\widetilde{X} := \theta \overline{X} + (1-\theta) \hat{X} \in \mathbb{S}^n$. Since $\mathtt{diag}(\hat{X}) = \mathtt{diag}(\overline{X}) = \hat{x}$, we obtain $\mathtt{diag}(\widetilde{X}) = \hat{x}$. Furthermore, since $\hat{X} - \hat{x}\hat{x}^\top \succeq \mathbf{0}$ and $\overline{X} - \hat{x}\hat{x}^\top = \mathtt{Diag}(d) \succeq \mathbf{0}$,

$$\theta \left(\hat{X} - \hat{x}\hat{x}^\top\right) + (1-\theta) \left(\overline{X} - \hat{x}\hat{x}^\top\right) = \widetilde{X} - \hat{x}\hat{x}^\top \succeq \mathbf{0}.$$

It follows from (11a) that $(\hat{x}, \widetilde{X}) \in \mathcal{R}(G, B)$. Therefore, $\hat{x} \in \mathsf{R}(G, B)$ by (11b). ☐

***Proof of Theorem 6.3.*** The equality in our main result follows directly from Lemmas 6.4 and 6.5, and the inclusion is from Proposition 2.2. ☐

Similar to Corollary 4.5, this equivalence of strength persists after addition of valid inequalities in the $x$-space; we omit the details.

By Theorems 4.4 and 6.3, it follows that the positive aggregation closure $\mathsf{R}_1^>(G)$ can be weaker than the aggregation closure $\mathsf{R}_1(G)$ arising from general aggregations. Recall that each $B \in \mathbb{R}^{1\times m}_{++}$ is a feasibility-preserving aggregation by Lemma 3.2. We have therefore obtained a surprising conclusion that the SDP aggregation closure from feasibility-preserving aggregations can yield a strictly weaker relaxation than the theta body (which is the closure from all aggregations). Thus, feasibility-preservation of an aggregation is not necessarily a desirable or crucial property when constructing SDP relaxations. However, we caution that our result only applies to a subclass of one-row feasibility-preserving aggregations characterized by strictly positive weights, and it is possible that there are other classes of one-row feasibility-preserving aggregations, with a mixture of positive and negative weights, whose SDP closures are exact. In the next section, we shed more light on the differences between $\mathsf{R}_1^>(G)$ and $\mathsf{R}_1(G)$.

### 6.3 A Comparison of Different Relaxations

In this section, we give a comparison of the strengths of the different convex relaxations $\mathsf{TH}(G)$, $\mathsf{QSTAB}(G)$, $\mathsf{R}_1(G)$, $\mathsf{R}_1^>(G)$, and $\mathsf{TH}_\le(G)$ given by (3), (5), (6b), (17) and (22), respectively, and $\bigcap_{\mathsf{C}\in\mathsf{C}'} \mathsf{R}(G, B(\mathsf{C}))$ described as in (20). We start by combining all the relations established thus far and their consequences for perfect and imperfect graphs.

**Corollary 6.6.** *Let $\mathsf{C}'$ denote the collection of all maximal cliques in $G$ and recall the vector $B(\mathsf{C})$ constructed in (19). We have*

$$\mathsf{STAB}(G) \subseteq \mathsf{TH}(G) = \mathsf{R}_1(G) \subseteq \mathsf{R}_1^>(G) = \mathsf{TH}_\le(G) \subseteq \bigcap_{\mathsf{C}\in\mathsf{C}'} \mathsf{R}(G, B(\mathsf{C})) = \mathsf{QSTAB}(G).$$

*Equality holds throughout for a perfect graph, and we have $\mathsf{STAB}(G) \subsetneq \mathsf{R}_1(G)$ for any imperfect graph. Furthermore, each of the second and third inclusions can be strict.*

*Proof.* The chain of inclusions and equalities is from Theorems 5.4 and 6.3. For any imperfect graph, we have $\mathsf{STAB}(G) \subsetneq \mathsf{TH}(G)$ from Proposition 2.1, and then the third claim is due to $\mathsf{TH}(G) = \mathsf{R}_1(G)$ for all graphs from Theorem 5.4. The last assertion follows from the examples in **??** 6.3. ☐

Next, we consider the second and third inclusions in the above result for imperfect graphs. By fixing $w = e \in \mathbb{R}^n$, a numerical approach was adopted to find the smallest

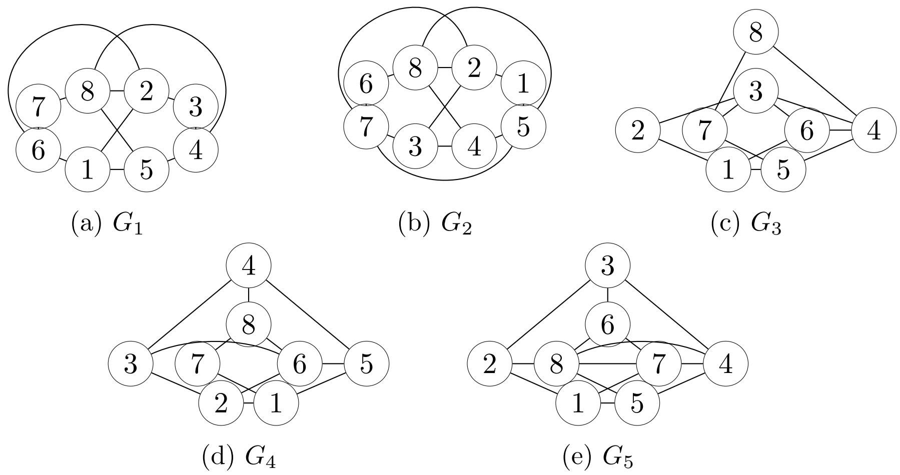



Figure 2: Five imperfect graphs of order 8

number of vertices $n \geq 5$ for which both inclusions are strict[2]. Thus, letting $\mathcal{G}_n$ denote the set of all imperfect graphs with $n$ vertices for $n \geq 5$, our goal is to compute (or bound) the number

$$\overline{n} := \min \left\{ n \colon \exists G \in \mathcal{G}_n \text{ s.t. } \max_{x \in \mathsf{TH}(G)} e^\top x < \max_{x \in \mathsf{TH}_{\leq}(G)} e^\top x < \max_{x \in \mathsf{QSTAB}(G)} e^\top x \right\}.$$

We enumerated the graphs in $\mathcal{G}_n$ using Nauty [9], our findings are presented next.

**Observation 6.3.** *For $w = e$, we have $\overline{n} = 8$ and the five graphs depicted in Figure 2, each of which has exactly two $C_5$ cycles as induced subgraphs, are the only ones of order 8 that achieve strict inequalities between the three maximizations. By Corollary 6.6, we conclude that each of these five graphs satisfies* $\mathsf{STAB}(G) \subsetneq \mathsf{R}_1(G) \subsetneq \mathsf{R}_1^{>}(G) \subsetneq \mathsf{QSTAB}(G)$.

## 7 Concluding Remarks and Future Directions

In this paper, we consider aggregations of binary quadratic formulations of the stable set problem. We give a characterization of the set of aggregations that yields an exact reformulation. The Shor relaxation of each of these aggregated formulations gives rise to a family of semidefinite programming relaxations for the stable set problem that subsumes the theta body. We describe the aggregation closures of one-row aggregations and establish the surprising result that feasibility-preserving relaxations can yield an aggregation closure that is strictly weaker than that of arbitrary aggregations.

Some interesting questions are still open for future research. Most are structural questions, whereas the last one is of an algorithmic nature.

i) What is the smallest number of aggregations $\mathsf{R}(G, B)$ required to represent $\mathsf{R}_1(G)$ for perfect graphs? Note that this is bounded above by the number of maximal cliques.

ii) Is there a finite representation of $\mathsf{R}_1(G)$ using $\mathsf{R}(G, B)$ with unrestricted aggregation coefficients for imperfect graphs? Theorem 5.4 establishes that this is true for perfect graphs.

iii) Is there a finite representation of $\mathsf{R}_1^{>}(G)$ for perfect graphs?

[2]Recall that any imperfect graph has at least 5 vertices.

iv) What is the smallest $\kappa$ such that $\mathsf{TH}(G) = \mathsf{R}^{\geq}_{\kappa}(G)$ for imperfect graphs? Corollary 6.6 reveals that that $\kappa = 1$ for perfect graphs.

v) Given a $w \in \mathbb{R}^n_+$, is it possible to efficiently construct a vector $B \in \mathbb{R}^{1\times m}$ such that the optimum of $w^\top x$ over the one-row aggregation $\mathsf{R}(G, B)$ is equal to that over $\mathsf{TH}(G)$? It is easy to argue the existence of such a vector by setting $B = (z^*)^\top$, where $(\delta^*, y^*, z^*) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ is an optimal solution of the dual $(\mathrm{D}(I, w))$ for $\mathsf{TH}(G)$. However, the question is to construct $B$ without needing to solve the dual problem for $\mathsf{TH}(G)$, which would defeat the whole purpose of using aggregations.

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# A Complexity Analysis

***Proof of Corollary 3.7.*** Proposition 3.1 (ii) gives us the following characterization of aggregation matrices that do not preserve feasibility,

$$B \notin \mathsf{FP}(G,\kappa) \iff \exists\, \mathsf{S} \subseteq \mathsf{V} \text{ with } \mathsf{E}[\mathsf{S}] \neq \emptyset \text{ s.t.} \quad \sum_{(i,j)\in \mathsf{E}[\mathsf{S}]} b_{ij}^l = 0,\ l = 1,\dots,\kappa. \tag{36}$$

The encoding size of any nonempty induced subgraph is obviously polynomial in the input length. The condition $\sum_{(i,j)\in \mathsf{E}[\mathsf{S}]} b_{ij}^l = 0$ for every $l$ can be checked in $O(\kappa m)$ time. Therefore, membership in $\mathsf{FP}(G,\kappa)$ is in co-NP. For co-NP-hardness, there is a simple reduction from the subset sum problem to the case $\kappa = 1$. Take any subset sum instance, which is given by positive integers $a_0, a_1, \dots, a_n$ and asks for existence of a nonempty subset $\mathsf{T} \subset \{1,\dots,n\}$ such that $\sum_{i\in\mathsf{T}} a_i = a_0$. Construct a weighted star graph $S_{n+2}$ whose vertices are $0,\dots,n+1$ and edges are $(0,i)$ for $i = 1,\dots,n+1$, with weights $a_1, a_2, \dots, a_n, -a_0$, respectively. We claim that the subset sum instance is YES if and only if $B := (a_1, a_2, \dots, a_n, -a_0) \notin \mathsf{FP}(S_{n+2}, 1)$, which implies membership in $\mathsf{FP}(S_{n+2}, 1)$ is co-NP-hard. Let $\mathsf{T}$ be such that $\sum_{i\in\mathsf{T}} a_i = a_0$. Then the nonempty induced subgraph $\mathsf{S} := \{0, n+1\} \cup \mathsf{T}$ in $S_{n+2}$ satisfies the right-hand side in (36), implying $(a_1, a_2, \dots, a_n, -a_0) \notin \mathsf{FP}(S_{n+2}, 1)$. For the reverse direction, we know from (36) that there exists some nonempty induced subgraph $\mathsf{S} \subset \{0,\dots,n+1\}$ satisfying $\sum_{(i,j)\in\mathsf{E}[\mathsf{S}]} b_{ij} = 0$. All edges in a star graph are incident to the vertex 0, and so we must have $0 \in \mathsf{S}$. Since $b_{0,i} = a_i > 0$ for $1 \leq i \leq n$ and $b_{0,n+1} = -a_0 < 0$, to satisfy $\sum_{(i,j)\in\mathsf{E}[\mathsf{S}]} b_{ij} = 0$ we must have $n+1 \in \mathsf{S}$ and $\mathsf{S} \cap \{1,\dots,n\} \neq \emptyset$. Then, $\mathsf{T} = \{i \in \mathsf{S} \colon 1 \leq i \leq n\}$ becomes a YES certificate for the subset sum instance. □

For our next proof, we need the following technicality about converting a constrained boolean optimization problem to a simplified version using a penalty function approach.

**Lemma A.1.** *For $b \in \mathbb{Z}_+$ and any functions $f : \{0,1\}^p \to \mathbb{R}$ and $g : \{0,1\}^p \to \mathbb{Z}_+$ of $p \geq 2$ binary variables, we have*

$$\min_{x\in\{0,1\}^p} \{f(x) \colon g(x) \geq b\} = \min_{x\in\{0,1\}^p, s\in\mathbb{Z}_+} f(x) + \rho\,(g(x) - s)^2 \quad \text{s.t.} \quad s \geq b$$

*where $\rho = f(\tilde{x}) - l$ for $\tilde{x} \in \{0,1\}^p$ with $g(\tilde{x}) \geq b$ and $l \leq \min\{f(x) : x \in \{0,1\}^p\}$.*

*Proof.* If $x$ is feasible for the left-hand side, then $g(x) \in \mathbb{Z}_+$ allows us to set $s = g(x)$ and obtain $(x, g(x))$ to be feasible to rhs, thus implying the $\geq$ inequality for any $\rho$. For the reverse inequality, we argue that the objective function value of any feasible point for the right-hand side is bounded below by the optimal value $z^*$ of the left-hand side. Take any $(x, s)$ feasible for the right-hand side. If $g(x) = s$, then the objective function value is $f(x)$ and $x$ is feasible for the left-hand side because $s = g(x) \geq b$, and so $f(x) \geq z^*$. Now let $g(x) \neq s$. This means $(g(x) - s)^2$ is equal to some positive integer $\sigma$. The objective function value is equal to $f(x) + \rho\,\sigma$, with $\sigma \geq 1$ and $\rho \geq 0$, which is bounded below by $l + \rho = f(\tilde{x})$. The feasibility of $\tilde{x}$ for the left-hand side ensures that $f(\tilde{x}) \geq z^*$, establishing the reverse inequality. □

*Proof of Corollary 3.8.* Use $x \in \{0,1\}^n$ to denote the indicator vector of any $\mathsf{S} \subseteq \mathsf{V}$ so that $x_i = 1$ if and only if $i \in \mathsf{S}$. Then, a nonempty induced subgraph corresponds to at least one $x_i x_j$ term, over $(i,j) \in \mathsf{E}$, being one. Thus, the set of all nonempty induced subgraphs in $G$ is in bijection with the set $\{x \in \{0,1\}^n \colon \sum_{(i,j)\in\mathsf{E}} x_i x_j \geq 1\}$. Furthermore, $\sum_{(i,j)\in\mathsf{E}[\mathsf{S}]} b^l_{ij} = \sum_{(i,j)\in\mathsf{E}} b^l_{ij} x_i x_j$. Using the characterization from Proposition 3.1 (iii) and squaring and expanding the summation in terms of $x$ requires us to certify positivity of the quartic

$$q(x) \;:=\; \sum_{(i,j)\in\mathsf{E}} \widetilde{b}_{ij}\, x_i x_j \;+\; \sum_{((i,j),(i',j'))\in\mathsf{E}'} 2\,\widetilde{b}_{iji'j'}\, x_i x_j x_{i'} x_{j'}$$

the set $\{x \in \{0,1\}^n \colon \sum_{(i,j)\in\mathsf{E}} x_i x_j \geq 1\}$, where we define $\widetilde{b}_{ij} := \sum_{l=1}^{\kappa} (b^l_{ij})^2$ and $\widetilde{b}_{iji'j'} := \sum_{l=1}^{\kappa} b^l_{ij} b^l_{i'j'}$. Applying Lemma A.1 with $f(x) = q(x)$, $g(x) = \sum_{(i,j)\in\mathsf{E}} x_i x_j$, and $b = 1$ gives us the simplified quartic integer optimization of checking positivity of minimising $q(x) + \rho(\sum_{(i,j)\in\mathsf{E}} x_i x_j - s)^2$ over $x \in \{0,1\}^n$ and $s \in \{1,2,\dots\}$, where the penalty parameter $\rho$ is given by

$$\rho = \sum_{(i,j)\in\mathsf{E}[\mathsf{T}]} \widetilde{b}_{ij} + \sum_{(i,j),(i',j')\in\mathsf{E}'[\mathsf{T}]} 2\,\widetilde{b}_{iji'j'} - \sum_{((i,j),(i',j'))\in\mathsf{E}'_-} 2\,\widetilde{b}_{iji'j'},$$

where $\mathsf{T} \subseteq \mathsf{V}$ is any subset such that $\mathsf{E}[\mathsf{T}] \neq \emptyset$, with $\mathsf{E}'[\mathsf{T}]$ denoting the subset of $\mathsf{E}'$ with edge pairs from $\mathsf{T}$, and $\mathsf{E}'_- \subseteq \mathsf{E}'$ is as defined just before the statement of Corollary 3.8. The last term in $\rho$ follows from the observation that $\widetilde{b}_{ij} \geq 0$ for each $(i,j) \in E$, implying a trivial lower bound on $q$ over $\{0,1\}^n$. □

Now we characterize membership in terms of mixed-binary linear optimization that is not simply a linearization of the simplified problem in Corollary 3.8, using the same edge pair sets $\mathsf{E}'$ and $\mathsf{E}'_-$ that were defined earlier.

**Corollary A.2.** *Any $B \in \mathbb{R}^{\kappa\times m}$ belongs to $\mathsf{FP}(G,\kappa)$ if and only if*

$$\begin{aligned}
0 \;<\; \min_{x\in\mathbb{R}^n, w\in\mathbb{R}^m, v\in\mathbb{R}^{|\mathsf{E}'|}} \quad & \sum_{(i,j)\in\mathsf{E}} \widetilde{b}_{ij} w_{ij} \;+\; \sum_{((i,j),(i',j'))\in\mathsf{E}'} 2\,\widetilde{b}_{iji'j'} v_{iji'j'} \\
\text{s.t.} \quad & \sum_{(i,j)\in\mathsf{E}} w_{ij} \geq 1, \qquad w_{ij} \leq x_i,\; w_{ij} \leq x_j, \quad (i,j) \in \mathsf{E} \\
& w_{ij} \geq x_i + x_j - 1, \quad (i,j) \in \mathsf{E} \textit{ s.t. } b^l_{ij} \neq 0 \textit{ some } l \\
& v_{iji'j'} \leq x_i,\; v_{iji'j'} \leq x_j,\; v_{iji'j'} \leq x_{i'},\; v_{iji'j'} \leq x_{j'}, \quad (i,j),(i',j') \in \mathsf{E}'_- \\
& v_{iji'j'} \geq x_i + x_j + x_{i'} + x_{j'} - 3, \quad (i,j),(i',j') \in \mathsf{E}' \colon \widetilde{b}_{iji'j'} > 0 \\
& x \in \{0,1\}^n,\; w \in \mathbb{R}^m_+,\; v \in \mathbb{R}^{|\mathsf{E}'|}_+
\end{aligned}$$

*Proof.* By Lemma A.1, the optimization problem in Corollary 3.8 has the same optimal value as the following optimization problem:

$$z^* := \min_{x\in\mathbb{R}^n} q(x) \quad \text{s.t.} \sum_{(i,j)\in\mathsf{E}} x_i x_j \geq 1,\; x \in \{0,1\}^n . \tag{37}$$

Since all nonlinear terms are products of binary variables, this problem easily transforms to a mixed-binary linear program as follows. Each $x_i x_j$ term in the $\geq$-constraint with a coefficient $+1$ can be replaced with its concave envelope $\min\{x_i, x_j\}$, and its appearance in the minimization objective can be replaced with its convex envelope $\max\{0, x_i + x_j - 1\}$ whenever its coefficient $\sum_l (b^l_{ij})^2$ is positive, i.e., some $b^l_{ij} \neq 0$. Therefore, each $x_i x_j$ for $(i,j) \in \mathsf{E}$ is linearized with a new variable $w_{ij} \geq 0$, and the first and the second line of constraints in the proposed formulation. For the quartic term for two distinct edges, whenever its coefficient $\widetilde{b}_{iji'j'} = \sum_l b^l_{ij} b^l_{i'j'}$ is negative (resp., positive), it can be linearized with its concave (resp., convex) envelope using a new variable $v_{iji'j'} \geq 0$, which is what is done in the third and the fourth line of constraints, respectively. This completes our derivation. $\square$

This mixed-binary LP can be solved as a binary LP because at an optimal solution, each $w$ and $v$ variable will be equal to the corresponding concave or convex envelope and will therefore take a binary value.

# B SDP from quadratic inequalities

This section gives missing proofs for the SDP relaxation $\mathsf{TH}_{\leq}(G)$ from (6b). First, we show the strong duality property. To describe this, denote

$$\mathscr{A} : s \in \mathbb{R}^m \mapsto \tfrac{1}{2} \sum_{(i,j)\in \mathsf{E}} s_{ij} \left( e_i e_j^\top + e_j e_i^\top \right). \tag{38}$$

**Lemma B.1.** *The linear optimization problem over* $\mathsf{TH}_{\leq}(G)$ *has a strong dual, i.e.,*

$$\max_{x \in \mathsf{TH}_{\leq}(G)} \; w^\top x \;=\; \min_{(\delta,y,s)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m_+} \; \delta \quad \text{s.t.} \quad \begin{bmatrix} \delta & -\frac{1}{2}(w+y)^\top \\ -\frac{1}{2}(w+y) & \mathscr{A}(s) + \mathtt{Diag}(y) \end{bmatrix} \succeq \mathbf{0}, \tag{39}$$

*for any* $w \in \mathbb{R}^n$*, and both primal and dual problems are solvable.*

*Proof.* The SDP dual is derived as follows. For constraints $X_{ij} \leq 0$ for all $(i,j) \in \mathsf{E}$, we define the Lagrangian multiplier $s \in \mathbb{R}^m_+$, whose components are indexed by the edge set $\mathsf{E}$ and are denoted by $s_{ij} \in \mathbb{R}_+$, $(i,j) \in E$. We introduce the Lagrangian multipliers $y \in \mathbb{R}^n$ corresponding to the constraints $\mathtt{diag}(X) = x$, and the matrix variable $\begin{pmatrix} \delta & q^\top \\ q & Q \end{pmatrix} \in \mathbb{S}^{n+1}$, with $\delta \in \mathbb{R}$, $q \in \mathbb{R}^n$, and $Q \in \mathbb{S}^n$, corresponding to the positive semidefinite constraint. The dual in (39) follows after eliminating $q$ and $Q$.

The primal optimal solution is attained since the primal feasible region is a compact set using a similar argument as in the proof of Lemma 4.1(i). For strong duality and dual solvability, we show that the following solution is a primal Slater point:

$$\hat{x} = \left(\frac{1}{n+1}\right) e \in \mathbb{R}^n, \qquad \hat{X} = \left(\frac{1}{n+1}\right) I + \frac{1}{2n(n+1)^2}\left(I - ee^\top\right) \in \mathbb{S}^n.$$

Note that $\hat{X}_{ii} = \hat{x}_i = 1/(n+1)$ for each $i$. Every off-diagonal entry of $\hat{X}$ is negative because it is equal to $-1/(2n(n+1)^2)$. The matrix $\hat{Y} := \hat{X} - \hat{x}\,\hat{x}^\top \in \mathbb{S}^n$ is strictly diagonally dominant, and hence is positive definite, since for each $i = 1, \dots, n$,

$$\begin{aligned} \hat{Y}_{ii} - \sum_{j\in\{1,\dots,n\}\setminus\{i\}} |\hat{Y}_{ij}| \;&=\; \frac{1}{n+1} - \left(\frac{1}{n+1}\right)^2 - \frac{n-1}{2n(n+1)^2} - \frac{n-1}{(n+1)^2} \\ &=\; \left(\frac{1}{n+1}\right)^2 \left(n+1-1-\frac{n-1}{2n} - n + 1\right) \;=\; \frac{1}{2n(n+1)} > 0, \end{aligned}$$

Therefore, $(\hat{x}, \hat{X}) \in \mathcal{R}_{\leq}(G)$ is a Slater point. $\square$

This allows us to show $\mathsf{TH}_{\leq}(G)$ is a stronger relaxation than $\mathsf{QSTAB}(G)$.

***Proof of Proposition 2.2.*** The first inclusion is obvious, so we have to show that each maximal clique $\mathsf{C} \subseteq \mathsf{V}$ induces a valid inequality for $\mathsf{TH}_{\leq}(G)$. Suppose, without loss of generality, that $\mathsf{C} = \{1, \dots, \eta\}$ and let $w = \chi^{\mathsf{C}} = \sum_{j=1}^{\eta} e_j$. Lemma B.1 implies that $\sum_{j \in \mathsf{C}} x_j \leq 1$ is valid for $\mathsf{TH}_{\leq}(G)$ if and only if there exists $(\hat{y}, \hat{s}) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $(1, \hat{y}, \hat{s})$ is a feasible solution to the dual SDP problem in (39) with $w = \chi^{\mathsf{C}}$. We claim that the following choice works: take $\hat{y} = w$, $\hat{s}_{ij} = 2$ for all distinct $i, j \in \mathsf{C}$, and $\hat{s}_{ij} = 0$ for all $(i,j) \in \mathsf{E}$ such that $i \notin \mathsf{C}$ or $j \notin \mathsf{C}$. Note that $\hat{s} \in \mathbb{R}^m_+$, and the positive semidefinite constraint holds because, using $\mathscr{A}(\hat{s})$ given by (38),

$$\begin{bmatrix} \hat{\delta} & -\frac{1}{2}(w + \hat{y})^\top \\ -\frac{1}{2}(w + \hat{y}) & \mathscr{A}(\hat{s}) + \mathsf{Diag}(\hat{y}) \end{bmatrix} = \begin{bmatrix} 1 & -w^\top \\ -w & w\,w^\top \end{bmatrix} = \begin{bmatrix} 1 \\ -w \end{bmatrix} \begin{bmatrix} 1 \\ -w \end{bmatrix}^\top \succeq \mathbf{0}. \qquad \square$$

# C Aggregation of another SDP for $\vartheta$-function

Lovász [28, Theorem 4] proposed an upper bound on the stability number of a graph in terms of the $\vartheta$-function $\vartheta(G, e) := \max_{Z \succeq \mathbf{0}} \{\langle e\,e^\top, Z\rangle \colon Z_{ij} = 0,\ (i,j) \in \mathsf{E};\ \mathsf{tr}(Z) = 1\}$. Grötschel et al. [23, Theorem 9.3.12] generalized this upper bound to the weighted stability number $\alpha(G, w) := \max\{w^\top x \colon x \in \mathsf{STAB}(G)\}$, where $w \in \mathbb{R}^n_+$, with objective $\langle \sqrt{w}\sqrt{w}^\top, Z\rangle$ for $\sqrt{w} := (\sqrt{w_1}, \dots, \sqrt{w_n})^\top \in \mathbb{R}^n$, to get

$$\vartheta(G, w) = \max_{Z \succeq \mathbf{0}} \left\{\langle \sqrt{w}\sqrt{w}^\top, Z\rangle \colon Z_{ij} = 0,\ (i,j) \in \mathsf{E};\ \mathsf{tr}(Z) = 1\right\} = \max_{x \in \mathsf{TH}(G)} w^\top x.$$

Galli and Letchford [17, Proposition 3] showed that, if $w = e \in \mathbb{R}^n$, using $B \in \mathbb{R}^{1 \times m}$ to aggregate the constraints of $\mathsf{TH}(G)$ yields a tighter bound than using the same $B$ to aggregate the above SDP in the $Z$-space. We adopt their arguments and generalize this dominance to any $w \in \mathbb{R}^n_+$. Recall $\nu(G, w, B)$ defined in (12).

**Proposition C.1.** *For any $w \in \mathbb{R}^n_+$ and $B \in \mathbb{R}^{1 \times m}$, let $\hat{B} \in \mathbb{R}^{1 \times m}$ be given by $\hat{b}_{ij} = \sqrt{w_i}\sqrt{w_j}\, b_{ij}$ for all $(i,j) \in \mathsf{E}$. Then,*

$$\nu(G, w, \hat{B}) \ \leq\ \max_{Z \succeq \mathbf{0}} \langle \sqrt{w}\sqrt{w}^\top, Z\rangle \ \text{s.t.} \sum_{(i,j) \in \mathsf{E}} b_{ij} Z_{ij} = 0,\ \mathsf{tr}(Z) = 1. \tag{40}$$

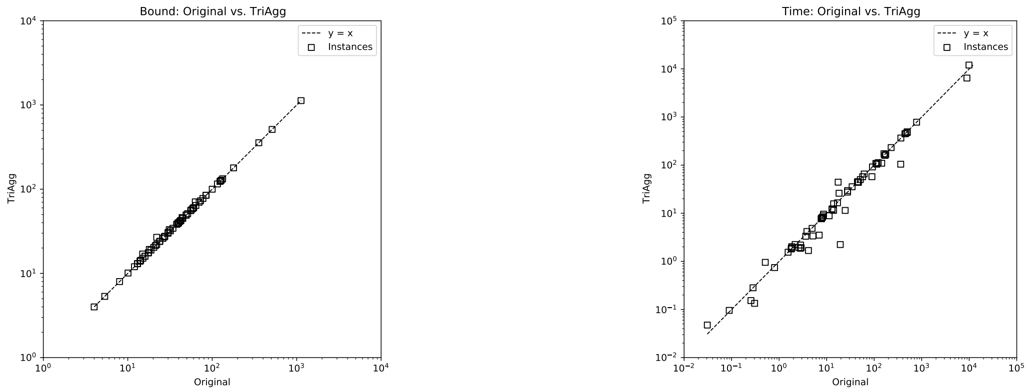
*Proof.* If $w = \mathbf{0}$, then (40) holds trivially. Suppose that $w \in \mathbb{R}^n_+ \backslash \{\mathbf{0}\}$, which implies that $\nu(G, w, \hat{B}) > 0$. Our first step is to argue that the following mapping

$$(x, X) \mapsto Z(x, X) = Z := \frac{\mathsf{Diag}(\sqrt{w}) X\, \mathsf{Diag}(\sqrt{w})}{w^\top x}.$$

maps every feasible solution $(x, X)$ of $\nu(G, w, \hat{B})$ such that $w^\top x > 0$ to a feasible solution of the aggregated SDP on the right-hand side of (40). It is clear that $Z \in \mathbb{S}^n$. We have $\sum_{(i,j) \in \mathsf{E}} b_{ij} Z_{ij} = \sum_{(i,j) \in \mathsf{E}} b_{ij} \sqrt{w_i}\sqrt{w_j} X_{ij} = \sum_{(i,j) \in \mathsf{E}} \hat{b}_{ij} X_{ij} = 0$. Since $\mathsf{diag}(X) = x$, the trace of $\mathsf{Diag}(\sqrt{w}) X\, \mathsf{Diag}(\sqrt{w})$ equals $w^\top x$, and hence $\mathsf{tr}(Z) = 1$. The numerator in $Z$ is positive semidefinite because $X \succeq \mathbf{0}$ and $\mathsf{Diag}(\sqrt{w})$ is a diagonal matrix, and then the denominator being positive implies $Z \succeq \mathbf{0}$. Thus, $Z$ is feasible.

Now we show that the mapping does not improve the objective value. Applying this to an optimal solution of $\nu(G, w, \hat{B})$ finishes our proof. Let $(x, X)$ be any feasible solution of $\nu(G, w, \hat{B})$ such that $w^\top x > 0$. Since $X - xx^\top \succeq \mathbf{0}$, the matrix $\mathsf{Diag}(\sqrt{w})\,(X - xx^\top)\, \mathsf{Diag}(\sqrt{w})$ is also positive semidefinite, and hence the trace inner product of the latter with the positive semidefinite matrix $\sqrt{w}\sqrt{w}^\top$ yields a nonnegative value, leading to

$$\begin{aligned} w^\top x\, \langle \sqrt{w}\sqrt{w}^\top, Z(x, X)\rangle &= \langle \sqrt{w}\sqrt{w}^\top, \mathsf{Diag}(\sqrt{w})\, X\, \mathsf{Diag}(\sqrt{w})\rangle \\ &\geq \langle \sqrt{w}\sqrt{w}^\top, \mathsf{Diag}(\sqrt{w})\, xx^\top\, \mathsf{Diag}(\sqrt{w})\rangle \\ &= \sqrt{w}^\top \mathsf{Diag}(\sqrt{w})\, xx^\top\, \mathsf{Diag}(\sqrt{w}) \sqrt{w} \ = \ (w^\top x)^2. \qquad \square \end{aligned}$$




(a) SDP bounds

(b) Wall-clock time

Figure 3: Comparison of the original SDP and the triangle-aggregated SDP on DIMACS instances. The dashed line indicates equality between the two SDPs.

# D Preliminary Numerical Results

We report some numerical results to illustrate the utility of the aggregation framework analysed in this paper. Galli and Letchford [17] did computational experiments with the aggregated SDP with all ones weight vector on Erdős–Rényi random graphs using `CSDP` version 6.1.1. They observed substantial reduction in computation time, albeit at the cost of weaker upper bounds. We revisit their experiments using the state-of-art operator splitting solver `SCS` version 3.2.11 [33]. Our preliminary results indicate that this simple aggregation does not consistently outperform the original SDP (the theta body) in computation time. Motivated by this observation, we consider a more structured aggregation based on an edge partition derived from a deterministic vertex cover using cliques of sizes 3 (triangles) and 2. This aggregation yields an exact quadratic formulation whose Shor relaxation, referred to as the *triangle-aggregated* SDP, has the potential to reduce computation time while maintaining good-quality upper bounds.

A heuristic algorithm is used to aggregate edges over all the triangles in a graph and then compare the bound quality and computation time for this triangle-aggregated SDP versus the theta body. Our algorithm computes an edge partition $\mathcal{E}$ of a graph by deterministically extracting triangles. At each iteration, it selects the lexicographically smallest triangle, adds its three edges as a block in $\mathcal{E}$, removes the corresponding vertices and edges, and assigns any incident edges to singleton edge blocks. Once no triangles remain, all remaining edges are added to $\mathcal{E}$ as singleton edges. The resulting edge partition is a collection of triangles and individual edges. The triangle aggregation matrix $B(\mathcal{E})$ has $|\mathcal{E}|$ rows and $m$ columns with entries in row $l$ corresponding to edges in the $l^{th}$ element of $\mathcal{E}$. Thus, the nonzeros in this matrix are $b^l_{ij} = 1$ if and only if $(i,j) \in \mathcal{E}_l$ for $l = 1, \dots, |\mathcal{E}|$. Since $B(\mathcal{E})^\top e > \mathbf{0}$, Lemma 3.2 implies that $B(\mathcal{E})$ is feasibility-preserving.

Figure 3 illustrates the potential advantage of our triangle-aggregated SDP on DIMACS instances. The left plot shows that the bounds produced by the triangle-aggregated SDP closely match those of the original SDP (the theta body), indicating tiny loss in bound quality. The right plot compares computation times and shows that the triangle-aggregated SDP outperforms the original SDP on more than half of the instances (those below the reference line). For example, on the instance `keller6`, the triangle-aggregated SDP reduces the computation time by about 2500 seconds compared with the theta body (from 8935 seconds), while sacrificing only a small amount in the upper bound.