

Classification of facial exposedness of completely positive cones over symmetric cones

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Abstract

We classify the facial exposedness of completely positive cones over symmetric cones in terms of the rank of the associated Euclidean Jordan algebras. The completely positive cones are facially exposed when the rank is at most 2, but are not facially exposed when the rank is at least 5. Facial exposedness is not completely determined by the rank when the rank is 3 or 4, but we provide a characterization in each case. The resulting classification of facial exposedness in fact agrees with the corresponding classification of spectrahedrality for completely positive cones.

Key words. completely positive cones, symmetric cones, facial structure, facial exposedness, spectrahedrality

1 Introduction

For a closed cone K in a finite-dimensional real inner product space, we define

$$(1.1) \quad \mathcal{CP}(K) := \left\{ \sum_{i=1}^k a_i \otimes a_i \mid k \geq 1 \text{ and } a_i \in K \text{ for all } i = 1, \dots, k \right\},$$

and refer to it as the *completely positive cone over K* . A typical choice for K in this paper is a *symmetric cone*, namely, a cone that is self-dual and homogeneous. Symmetric cones form a broad and fundamental class of convex cones, including nonnegative orthants, second-order cones, cones of real symmetric positive semidefinite matrices, and their finite direct products [10]. In the case where K is a nonnegative orthant, the associated completely positive cone is essentially the cone of classical completely positive matrices, and a vast literature exists on this cone [35]. Motivated by the ability of completely positive cones to convexify various NP-hard problems [1, 3, 6–8], researchers have studied the membership problem [11, 16, 27], approximations [25, 26], and the geometry [22–24] of completely positive cones and their duals, namely, copositive cones over symmetric cones.

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Table 1: The facial exposedness of completely positive cones over symmetric cones in terms of the rank of the associated EJAs. The symbol ✓ means that the completely positive cones are facially exposed without additional assumptions on the EJAs, and the symbol ✗ means that they are not.

Rank	Facial exposedness
≤ 2	✓ (Theorem 3.1)
3	iff an EJA is not simple (Theorem 4.6)
4	iff an EJA is Jordan-isomorphic to the Hadamard EJA $\mathbb{R}^4$ (Theorem 5.4)
≥ 5	✗ (Theorem 6.1)

This paper is devoted to investigating the *facial exposedness*, a geometric property, of completely positive cones over symmetric cones. A convex cone is called facially exposed if every face can be written as the intersection of the convex cone and its supporting hyperplane. From the viewpoint of mathematical optimization, the notion of facial exposedness arose in connection with facial reduction algorithms [4, 5]. Facial exposedness is also important because certain geometric properties such as niceness [29, 34] and spectrahedrality [31] require facial exposedness. Spectrahedrality of a set, a notion that will play a role later in this paper, refers to the property that the set can be represented by a positive semidefinite constraint without extra variables. See [19] and [23, section 1] for more detailed connections with other properties.

The facial exposedness of copositive cones over symmetric cones is completely settled. By [23, Theorem 3.3], we see that a copositive cone is facially exposed if and only if the dimension of the underlying symmetric cone is at most 1.

It is natural to ask whether the duals, completely positive cones over symmetric cones, are facially exposed. When the underlying symmetric cone is a nonnegative orthant, the associated completely positive cone is facially exposed if and only if the dimension of the nonnegative orthant is at most 4 [38]; see also [17, 18, 39] for the non-facial exposedness of completely positive cones over nonnegative orthants of dimension at least 5. When the underlying symmetric cones are a second-order cone and the direct product of the 1-dimensional nonnegative orthant and a second-order cone, the associated completely positive cones are spectrahedral by Theorems 1 and 3 of [36], respectively, and hence they are facially exposed. These results are independent of the dimension of the second-order cone, suggesting that one should look at a structural parameter of symmetric cones other than their dimension.

In this paper, we classify the facial exposedness of completely positive cones over symmetric cones. To do this, we realize symmetric cones as the cones of squares of *Euclidean Jordan algebras* (EJAs) and investigate the facial exposedness of the associated completely positive cones in terms of the *rank* of the EJAs. Here, the rank of an EJA is a structural parameter whose precise definition is given in Section 2.3. Table 1 summarizes the classification of the facial exposedness of completely positive cones over symmetric cones. When the rank is at most 2, the completely positive cones are facially exposed by existing results and simple

arguments. When the rank is at least 5, the completely positive cones are not facially exposed because of the non-facial exposedness of completely positive cones over nonnegative orthants of dimension at least 5. Challenging situations arise when the rank is 3 or 4. In these cases, additional case distinctions are essential to determine facial exposedness.

The classification of facial exposedness shown in Table 1 can also be read as that of spectrahedrality. As can be seen from the subsequent discussion, if a completely positive cone over a symmetric cone is facially exposed, it is actually spectrahedral. Note that [33, Theorem 4.1] provides an example of a closed convex cone that is facially exposed but not spectrahedral; these properties can be verified using the proof of [30, Proposition 5], or alternatively Proposition 2.3 and Corollary 3.5 of [19]. Thus, facial exposedness does not imply spectrahedrality in general. The classification of spectrahedrality of completely positive cones over symmetric cones is related to the question of whether the completely positive cones and their duals are *spectrahedral shadows*, namely projections of spectrahedra [22, Question 6.1]. This classification problem for spectrahedral shadows in the rank-3 and rank-4 cases remains open.

The organization of this paper is as follows. Section 2 collects the notation and technical results needed throughout the paper. Sections 3 to 6 discuss the facial exposedness of completely positive cones over symmetric cones when the ranks of the associated EJAs are at most 2, equal to 3, equal to 4, and at least 5, respectively.

2 Preliminaries

2.1 Basic notation

Let $\mathbb{R}^n$ denote the space of n -dimensional real vectors equipped with the standard inner product. Vectors in $\mathbb{R}^n$ are column vectors denoted by boldface lowercase letters, such as $\mathbf{a}$. Let $\mathbf{a} \in \mathbb{R}^n$. Vectors are indexed from 1, and a_i denotes the i th entry of $\mathbf{a}$. For $1 \leq i < j \leq n$, we write $\mathbf{a}_{i:j}$ for the vector obtained by extracting the i th to j th entries of $\mathbf{a}$. The transpose of $\mathbf{a}$ is denoted by $\mathbf{a}^\top$. We use $\mathbf{0}$ and $\mathbf{e}_i$ to denote the vector all of whose entries are 0 and the vector whose i th entry is 1 and the others are 0, respectively.

Matrices are denoted by boldface uppercase letters, such as $\mathbf{A}$. Rows and columns of matrices are indexed from 1, and A_{ij} denotes the (i, j) th entry of a matrix $\mathbf{A}$. We use $\mathbf{O}$, $\mathbf{I}_n$, and $\mathbf{J}_n$ to denote the matrix all of whose entries are 0, the identity matrix of order n , and the diagonal matrix of order n whose $(1, 1)$ th entry is 1 and the other diagonal entries are -1 , respectively. For a square matrix $\mathbf{A}$, we use $\text{Ker}(\mathbf{A})$, $\det(\mathbf{A})$, $\text{tr}(\mathbf{A})$, $\text{adj}(\mathbf{A})$, and $\{0\} \oplus \mathbf{A}$ to denote the kernel of $\mathbf{A}$, the determinant of $\mathbf{A}$, the trace of $\mathbf{A}$, the adjugate of $\mathbf{A}$, and the square matrix whose entries in the first row and column are 0 and whose principal submatrix obtained by deleting the first row and column is $\mathbf{A}$, respectively. We also use the symbol $\oplus$ to denote the direct sum of vector spaces. We use $\mathcal{S}^n$ and $\mathcal{S}_+^n$ to denote the space of real $n \times n$ symmetric matrices and the cone of positive semidefinite matrices in $\mathcal{S}^n$, respectively. For a real symmetric positive definite matrix $\mathbf{A}$, we define $\sqrt{\mathbf{A}}$ to be the unique real symmetric positive definite matrix satisfying $(\sqrt{\mathbf{A}})^2 = \mathbf{A}$ [15, page 440].

Let V be a finite-dimensional real vector space and S be a set in V . We use $\text{span } S$ and $\text{int } S$ to denote the smallest subspace containing S and the interior of S , respectively. In

addition, we define $\mathbb{R}S := \{ax \mid a \in \mathbb{R}, x \in S\}$ and $\mathbb{R}_+S := \{ax \mid a \geq 0, x \in S\}$. For $x \in V$, we write $\mathbb{R}\{x\}$ and $\mathbb{R}_+\{x\}$ as $\mathbb{R}x$ and $\mathbb{R}_+x$, respectively.

We further assume that V is equipped with an inner product denoted by $\bullet$. For a set S in V , we write $S^\perp$ for the space of $x \in V$ such that $x \bullet y = 0$ for all $y \in S$. For a set K in V , we call K a *cone* if $ax \in K$ for all $a \geq 0$ and $x \in K$, and we define K^* , the dual of K , as the cone of $x \in V$ such that $x \bullet y \geq 0$ for all $y \in K$. For two closed cones K_1 and K_2 , we say that K_1 is *linearly isomorphic* to K_2 if there exists a linear isomorphism $\phi: \text{span } K_1 \rightarrow \text{span } K_2$ such that $\phi(K_1) = K_2$. For $x \in V$, we write $x \otimes x$ for the tensor product of x with itself, which can be regarded as the linear transformation on V that maps y to $(x \bullet y)x$. In this paper, the notation $\mathbf{X} \otimes \mathbf{X}$ is used for a matrix $\mathbf{X}$, but this does not denote the Kronecker product. The space of self-adjoint linear transformations on V is denoted by $\mathcal{S}(V)$, and $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathcal{S}(V)$ induced by that on V . When $V = \mathbb{R}^n$, we identify $\mathcal{S}(\mathbb{R}^n)$ with $\mathcal{S}^n$, where the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{S}^n$ is given by $\langle \mathbf{X}, \mathbf{Y} \rangle = \sum_{i,j=1}^n X_{ij}Y_{ij}$ for $\mathbf{X}, \mathbf{Y} \in \mathcal{S}^n$. We define $H_m(V)$ as the space of homogeneous polynomials of degree m with real coefficients on V . We note that elements in $H_m(\mathbb{R}^n)$ are homogeneous polynomials in the usual sense. For $A \in \mathcal{S}(V)$, we define $q_A \in H_2(V)$ as

$$(2.1) \quad q_A(x) := x \bullet A(x)$$

for $x \in V$.

For functions $f_1, \dots, f_k$, we write $f_1 \circ \dots \circ f_k$ for the composite function defined recursively as $(f_1 \circ \dots \circ f_k)(x) := (f_1 \circ \dots \circ f_{k-1})(f_k(x))$. We also use the symbol $\circ$ to denote the product of EJAs defined in Section 2.3. For a linear mapping f from a finite-dimensional real inner product space to another, f^* denotes its adjoint.

2.2 Faces and their exposedness

Let K be a closed convex cone in a finite-dimensional real inner product space V . A *face* of K is a nonempty (necessarily closed) convex subcone F such that for any $a, b \in K$, we have $a, b \in F$ if $a + b \in F$. We say that a nonzero $x \in K$ generates an *extreme ray* of K if $\mathbb{R}_+x$ is a face of K . An *exposed face* of K is a set that can be written as the intersection of K and $\{d\}^\perp$ for some $d \in K^*$. An exposed face is necessarily a face as the name suggests. We call K *facially exposed* if all the faces of K are exposed.

One way to show that a closed convex cone is not facially exposed is to exhibit one of its faces that is not facially exposed as a cone. This argument is justified by the following theorem, which follows immediately from the discussion in [32, page 163].

Theorem 2.1. *Let K be a closed convex cone in V , let F_1 be a face of K , and let F_2 be a nonempty convex subcone of F_1 .*

- (i) *F_2 is a face of K if and only if F_2 is a face of F_1 .*
- (ii) *F_2 is a non-exposed face of K if F_2 is a non-exposed face of F_1 .*

In particular, K is not facially exposed if F_1 is not facially exposed as a cone.

In addition, we investigate the facial exposedness of a closed convex cone via another cone that is linearly isomorphic to the cone under consideration. It is well known that facial exposedness is invariant under linear isomorphism. We summarize this in the following theorem and provide its concise proof because it is crucial for the discussion in this paper.

Theorem 2.2. *For each $i \in \{1, 2\}$, let K_i be a closed convex cone in a finite-dimensional inner product space. Assume that K_1 is linearly isomorphic to K_2 , and let $\phi: \text{span } K_1 \rightarrow \text{span } K_2$ be a linear isomorphism satisfying $\phi(K_1) = K_2$. Let F be a nonempty convex subcone of K_1 .*

- (i) *F is a face of K_1 if and only if $\phi(F)$ is a face of K_2 .*
- (ii) *F is an exposed face of K_1 if and only if $\phi(F)$ is an exposed face of K_2 .*

In particular, K_1 is facially exposed if and only if so is K_2 .

Proof. We can verify (i) from the definition of faces. We can deduce (ii) from the fact that if $F = K_1 \cap \{d\}^\perp$ for some $d \in K_1^*$, then we have $(\phi^*)^{-1}(d) \in K_2^*$ and $\phi(F) = K_2 \cap \{(\phi^*)^{-1}(d)\}^\perp$. $\square$

Spectrahedrality of a closed convex cone is a sufficient condition for the facial exposedness of the cone [31, Corollary 1]. A closed convex cone K in V is *spectrahedral* if there exist $d \geq 1$ and an affine mapping $A: V \rightarrow \mathcal{S}^d$ such that $K = \{x \in V \mid A(x) \in \mathcal{S}_+^d\}$. For later use, we summarize this fact as the following theorem.

Theorem 2.3. *If K is a spectrahedral cone in V , then it is facially exposed.*

2.3 Euclidean Jordan algebras and symmetric cones

A *Jordan algebra* is a pair $(E, \circ)$ where E is a finite-dimensional real vector space and $\circ: E \times E \rightarrow E$ is a bilinear product that is commutative and satisfies $x \circ ((x \circ x) \circ y) = (x \circ x) \circ (x \circ y)$ for all $x, y \in E$. A Jordan algebra $(E, \circ)$ with a unit element is said to be *Euclidean* if it admits an inner product $\bullet: E \times E \rightarrow \mathbb{R}$ that is associative in the sense that $(x \circ y) \bullet z = x \bullet (y \circ z)$ for all $x, y, z \in E$. In this paper, we may write an EJA as E for simplicity or as $(E, \circ, \bullet)$ to specify an underlying associative inner product on E . For two EJAs $(E_1, \circ_1)$ and $(E_2, \circ_2)$, we say that E_1 is *Jordan-isomorphic* to E_2 if there exists a linear isomorphism $\phi: E_1 \rightarrow E_2$ such that $\phi(x \circ_1 y) = \phi(x) \circ_2 \phi(y)$ for all $x, y \in E_1$, and we call such a mapping ϕ a *Jordan isomorphism*.

Let $(E, \circ)$ be an EJA. An element $c \in E$ is called an *idempotent* if $c \circ c = c$. For $c_1, \dots, c_r \in E$, we call $\{c_1, \dots, c_r\}$ a *Jordan frame* of E if every c_i for $i \in \{1, \dots, r\}$ is a nonzero idempotent that cannot be written as the sum of two nonzero idempotents in E , the sum of $c_1, \dots, c_r$ is a unit element of E , and $i \neq j$ implies $c_i \circ c_j = 0$. The cardinality r of a Jordan frame does not depend on the choice of the Jordan frame; this integer is called the *rank* of E .

For every EJA E , the set $E_+ := \{x \circ x \mid x \in E\}$ forms a *symmetric cone*, namely, a self-dual and homogeneous cone [10, Theorem III.2.1]. Conversely, any symmetric cone can be represented as E_+ for some EJA E , and the EJA is unique up to Jordan isomorphism [10, Chapter III]. By the invariance of the rank of EJAs under Jordan isomorphism, we see that

for a given symmetric cone, the algebraically defined rank of a corresponding EJA is uniquely determined. Indeed, the rank is completely determined by the geometry of the symmetric cone, and is computed as the maximum number of mutually orthogonal extreme rays of the symmetric cone [22, section 2.2]. Therefore, it makes sense to classify the facial exposedness of completely positive cones over symmetric cones in terms of the rank of the associated EJAs.

Here we display some examples of EJAs. First, let $E := \mathbb{R}^n$ with $n \geq 1$ and define

$$\mathbf{x} \circ \mathbf{y} := (x_1 y_1, \dots, x_n y_n)^\top \quad \text{and} \quad \mathbf{x} \bullet \mathbf{y} := \mathbf{x}^\top \mathbf{y},$$

respectively, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then $(E, \circ, \bullet)$ forms an EJA, and we call it a *Hadamard EJA*. The rank of this EJA is n , and the associated symmetric cone is the nonnegative orthant $\mathbb{R}_+^n$, i.e., the cone of nonnegative vectors in $\mathbb{R}^n$. Second, let $E := \mathbb{R}^n$ with $n \geq 2$ and define

$$\mathbf{x} \circ \mathbf{y} := \begin{pmatrix} \mathbf{x}^\top \mathbf{y} \\ x_1 \mathbf{y}_{2:n} + y_1 \mathbf{x}_{2:n} \end{pmatrix} \quad \text{and} \quad \mathbf{x} \bullet \mathbf{y} := \mathbf{x}^\top \mathbf{y},$$

respectively, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. We call the EJA defined in this way a *Jordan spin algebra*, and it is denoted by L^n . The rank of this EJA is 2, and the associated symmetric cone is the second-order cone defined as

$$L_+^n := \left\{ \mathbf{x} \in \mathbb{R}^n \mid x_1 \geq \sqrt{x_2^2 + \dots + x_n^2} \right\}.$$

We note that L^n itself may be used to denote the second-order cone [2, 24], but in this paper, following [13, 28] for example, this symbol is used to represent a specific EJA associated with the cone, i.e., the Jordan spin algebra. Third, let $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ denote the sets of complex numbers, quaternions, and octonions, respectively. For $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ and $n \geq 1$, we define $\mathcal{H}^n(\mathbb{A})$ as the space of Hermitian $n \times n$ matrices with entries in $\mathbb{A}$. In addition, we define

$$\mathbf{X} \circ \mathbf{Y} := \frac{\mathbf{X}\mathbf{Y} + \mathbf{Y}\mathbf{X}}{2} \quad \text{and} \quad \mathbf{X} \bullet \mathbf{Y} := \text{tr}(\mathbf{X} \circ \mathbf{Y}),$$

respectively, for $\mathbf{X}, \mathbf{Y} \in \mathcal{H}^n(\mathbb{A})$. Then $(\mathcal{H}^n(\mathbb{A}), \circ, \bullet)$ forms an EJA for any n if $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ or for $n \leq 3$ if $\mathbb{A} = \mathbb{O}$, and this EJA has rank n [10, Chapter V]. In particular, $\mathcal{H}^n(\mathbb{R}) = \mathcal{S}^n$ and the associated symmetric cone is $\mathcal{S}_+^n$.

Again, let $(E, \circ)$ be an EJA. We refer to E' as a *subalgebra* of E if it is a subspace of E and closed under the product $\circ$. For every subalgebra E' of E , the orthogonal projection (with respect to any fixed associative inner product) of the unit element of E onto E' is a unit element of E' , and hence E' is an EJA. Two subalgebras E_1 and E_2 of E are called *orthogonal* if $x \circ y = 0$ for all $x \in E_1$ and $y \in E_2$. The EJA E is called *simple* if it is nonzero and cannot be written as the direct sum of two nonzero orthogonal subalgebras. By [10, Proposition III.4.4], every nonzero EJA E can be decomposed into the direct sum of simple subalgebras $E_1, \dots, E_k$ that are pairwise orthogonal. We note that the rank of E is the sum of the ranks of $E_1, \dots, E_k$ since the union of a Jordan frame of each E_i for $i \in \{1, \dots, k\}$ forms a Jordan frame of E . The classification of simple EJAs is well known [10, Chapter V]. The following theorem summarizes the relevant classification results for EJAs of ranks 1, 2, and 3 in a form convenient for our purposes.

Theorem 2.4. *Let E be a simple EJA.*

- (i) *If E has rank 1, then it is Jordan-isomorphic to the Hadamard EJA $\mathbb{R}$.*
- (ii) *If E has rank 2, then there exists $n \geq 3$ such that the algebra is Jordan-isomorphic to the Jordan spin algebra L^n .*
- (iii) *If E has rank 3, then it is Jordan-isomorphic to $\mathcal{H}^3(\mathbb{A})$ for some $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$.*

Proof. If E has rank 1, we have $E = \mathbb{R}e$, where e is the unit element of E . The linear mapping sending e to 1 is a Jordan isomorphism between E and the Hadamard EJA $\mathbb{R}$. Therefore, we obtain (i). (ii) is a consequence of [10, Corollary IV.1.5]. (iii) follows directly from [10, Theorem V.3.7]. $\square$

2.4 Completely positive and copositive cones

Let K be a closed cone in a finite-dimensional real inner product space V . Recall the definition of $\mathcal{CP}(K)$ provided in (1.1). Its dual is the *copositive cone over K* defined as

$$\mathcal{COP}(K) := \{A \in \mathcal{S}(V) \mid q_A(x) \geq 0 \text{ for all } x \in K\},$$

where q_A is defined in (2.1). See [36, section 2] for their duality.

First, we observe that a linear isomorphism between two underlying cones induces a linear isomorphism between the associated completely positive cones and preserves facial exposedness.

Lemma 2.5. *For each $i \in \{1, 2\}$, let K_i be a closed cone in a finite-dimensional real inner product space. We assume that K_1 is linearly isomorphic to K_2 . Then $\mathcal{CP}(K_1)$ is linearly isomorphic to $\mathcal{CP}(K_2)$. In particular, $\mathcal{CP}(K_1)$ is facially exposed if and only if so is $\mathcal{CP}(K_2)$.*

Proof. Let $\phi: \text{span } K_1 \rightarrow \text{span } K_2$ be a linear isomorphism satisfying $\phi(K_1) = K_2$. For any $A \in \mathcal{CP}(K_1)$, we have

$$(2.2) \quad \phi \circ A \circ \phi^* \in \mathcal{CP}(K_2)$$

since for any $a \in K_1$, it follows that $\phi \circ (a \otimes a) \circ \phi^* = \phi(a) \otimes \phi(a)$ and $\phi(a) \in K_2$.

We define $\Phi: \text{span } \mathcal{CP}(K_1) \rightarrow \text{span } \mathcal{CP}(K_2)$ as $\Phi(A) := \phi \circ A \circ \phi^*$ for every $A \in \text{span } \mathcal{CP}(K_1)$. Since every element in $\text{span } \mathcal{CP}(K_1)$ can be expressed as the difference of two elements in $\mathcal{CP}(K_1)$ [32, Theorem 2.7], it follows from (2.2) that Φ is well-defined. The inverse of Φ is given by $\Phi^{-1}(B) = \phi^{-1} \circ B \circ (\phi^*)^{-1}$ for $B \in \text{span } \mathcal{CP}(K_2)$, and thus Φ is a linear isomorphism. In addition, using (2.2), we obtain $\Phi(\mathcal{CP}(K_1)) = \mathcal{CP}(K_2)$. Therefore, $\mathcal{CP}(K_1)$ is linearly isomorphic to $\mathcal{CP}(K_2)$. Finally, by Theorem 2.2, we obtain the equivalence of facial exposedness for the two cones. $\square$

In particular, we see from Lemma 2.5 that for an EJA E , the facial exposedness of $\mathcal{CP}(E_+)$ is independent of the choice of an associative inner product on E . This means that it is not necessary to specify the associative inner product, and at the same time, we can choose it arbitrarily to investigate facial exposedness in practical computations. More generally, the following corollary states the invariance of the facial exposedness of $\mathcal{CP}(E_+)$ under Jordan isomorphism.

Corollary 2.6. *For each $i \in \{1, 2\}$, let $(E_i, \circ_i)$ be an EJA. We assume that E_1 is Jordan-isomorphic to E_2 . Then $\mathcal{CP}((E_1)_+)$ is linearly isomorphic to $\mathcal{CP}((E_2)_+)$. In particular, $\mathcal{CP}((E_1)_+)$ is facially exposed if and only if so is $\mathcal{CP}((E_2)_+)$.*

Proof. By assumption, the cone $(E_1)_+$ is linearly isomorphic to $(E_2)_+$. Therefore, from Lemma 2.5, we obtain the desired result. $\square$

Second, we summarize results on exposed faces of completely positive cones. The following result is quoted from [24, Proposition 5.4] for later use.

Theorem 2.7. *Let K be a closed cone in V and X be a subspace of V . Then $\mathcal{CP}(K \cap X)$ is an exposed face of $\mathcal{CP}(K)$.*

Theorem 2.7 yields a class of exposed faces of completely positive cones over symmetric cones as shown in the following lemma.

Lemma 2.8. *Let E be an EJA and E' be a subalgebra of E . Then $\mathcal{CP}(E'_+)$ is an exposed face of $\mathcal{CP}(E_+)$.*

Proof. We note that $E_+ \cap E' = E'_+$. We can derive this equality from [21, equation (9)], or alternatively prove it using spectral decomposition [10, Theorem III.1.2]. Since E'_+ can be written as the intersection of the cone E_+ and the subspace E' , by Theorem 2.7, we obtain the desired result. $\square$

The following lemma provides us with a way to describe exposed faces of completely positive cones in terms of zeros of quadratic forms. Its proof is omitted because it follows directly from the definitions. We note that Dickinson [9, page 393] pointed out the same result in the case where K is a nonnegative orthant.

Lemma 2.9. *Let K be a closed cone in V and $D \in \mathcal{COP}(K)$. We define*

$$(2.3) \quad Z_K(D) := \{x \in K \mid q_D(x) = 0\}.$$

Then we have $\mathcal{CP}(K) \cap \{D\}^\perp = \mathcal{CP}(Z_K(D))$.

For some specific symmetric cones, the facial exposedness (spectrahedrality) of the associated completely positive cones has been determined.

Theorem 2.10. *Let $n \geq 1$. Then $\mathcal{CP}(\mathbb{R}_+^n)$ is facially exposed (spectrahedral) if and only if $n \leq 4$.*

Proof. If $n \leq 4$, it follows from [20] that $\mathcal{CP}(\mathbb{R}_+^n)$ is equal to the intersection of $\mathcal{S}_+^n$ and the cone of entrywise nonnegative matrices in $\mathcal{S}^n$. This implies that $\mathcal{CP}(\mathbb{R}_+^n)$ is spectrahedral, and therefore facially exposed by Theorem 2.3. If $n \geq 5$, it follows from [38] that $\mathcal{CP}(\mathbb{R}_+^n)$ is not facially exposed and thus not spectrahedral either. $\square$

Theorem 2.11 ([36, Theorem 1]). *Let $n \geq 2$. Then we have*

$$\mathcal{CP}(L_+^n) = \{\mathbf{X} \in \mathcal{S}_+^n \mid \langle \mathbf{J}_n, \mathbf{X} \rangle \geq 0\}.$$

In particular, $\mathcal{CP}(L_+^n)$ is facially exposed.

Theorem 2.12 ([36, Theorem 3]). *Let $n \geq 2$. Then we have*

$$\mathcal{CP}(\mathbb{R}_+ \times L_+^n) = \{\mathbf{X} \in \mathcal{S}_+^{n+1} \mid \langle \{0\} \oplus \mathbf{J}_n, \mathbf{X} \rangle \geq 0, (\mathbf{X}\mathbf{e}_1)_{2:n+1} \in L_+^n\}.$$

In particular, $\mathcal{CP}(\mathbb{R}_+ \times L_+^n)$ is facially exposed.

Remark 2.13. Burer and Dong [8] defined

$$\mathcal{D} := \{\mathbf{X} \in \mathcal{S}_+^{n+1} \mid \mathbf{X}\mathbf{s} \in \mathbb{R}_+ \times L_+^n \text{ for all } \mathbf{s} \text{ generating extreme rays of } \mathbb{R}_+ \times L_+^n\}$$

and asserted that $\mathcal{D}$ is tractable (in the sense that the separation problem for $\mathcal{D}$ can be solved in polynomial time) and equal to $\mathcal{CP}(\mathbb{R}_+ \times L_+^n)$. In fact, $\mathcal{D}$ is *not* equal to $\mathcal{CP}(\mathbb{R}_+ \times L_+^n)$ for $n \geq 3$. To see this, let $\mathbf{X} := \{0\} \oplus \mathbf{I}_n$. Since the extreme rays of $\mathbb{R}_+ \times L_+^n$ are $\mathbb{R}_+\mathbf{e}_1$ and $\mathbb{R}_+(0, 1, \mathbf{v}^\top)^\top$ with $\mathbf{v} \in \mathbb{R}^{n-1}$ of norm 1, we can see that $\mathbf{X} \in \mathcal{D}$. On the other hand, since $\langle \{0\} \oplus \mathbf{J}_n, \mathbf{X} \rangle = -n + 2 < 0$, it follows from Theorem 2.12 that $\mathbf{X} \notin \mathcal{CP}(\mathbb{R}_+ \times L_+^n)$.

3 Case of rank at most 2

In this section, we show that $\mathcal{CP}(E_+)$ is facially exposed for any EJA E of rank at most 2.

Theorem 3.1. *Let E be an EJA of rank at most 2. Then $\mathcal{CP}(E_+)$ is facially exposed.*

Proof. If E has rank 0, i.e., $E = \{0\}$, then we have $\mathcal{CP}(E_+) = \{0\}$, which is facially exposed. If E has rank 1, then we see from (i) of Theorem 2.4 that E is Jordan-isomorphic to the Hadamard EJA $\mathbb{R}$. Since $\mathcal{CP}(\mathbb{R}_+)$ is facially exposed by Theorem 2.10, we see from Corollary 2.6 that $\mathcal{CP}(E_+)$ is facially exposed.

In what follows, we consider the case where E has rank 2. Let $E_1, \dots, E_k$ be simple subalgebras that are pairwise orthogonal and whose direct sum is E , and let $r_i \geq 1$ denote the rank of E_i for every $i \in \{1, \dots, k\}$. There are the following two possibilities for $(r_1, \dots, r_k)$ satisfying $\sum_{i=1}^k r_i = 2$:

Case 2.a $k = 2$ and $(r_1, r_2) = (1, 1)$,

Case 2.b $k = 1$ and $r_1 = 2$.

First, we consider Case 2.a. In this case, by (i) of Theorem 2.4, we see that E is Jordan-isomorphic to the Hadamard EJA $\mathbb{R}^2$. Since $\mathcal{CP}(\mathbb{R}_+^2)$ is facially exposed by Theorem 2.10, we see from Corollary 2.6 that $\mathcal{CP}(E_+)$ is facially exposed.

Second, we consider Case 2.b. By (ii) of Theorem 2.4, E is Jordan-isomorphic to a Jordan spin algebra L^n for some $n \geq 3$. Since $\mathcal{CP}(L_+^n)$ is facially exposed by Theorem 2.11, we see from Corollary 2.6 that $\mathcal{CP}(E_+)$ is facially exposed. $\square$

4 Case of rank 3

Let E be an EJA of rank 3. In addition, let $E_1, \dots, E_k$ be simple subalgebras that are pairwise orthogonal and whose direct sum is E , and let $r_i \geq 1$ denote the rank of E_i for every $i \in \{1, \dots, k\}$. Without loss of generality, we may assume that $r_1 \leq \dots \leq r_k$. There are the following three possibilities for $(r_1, \dots, r_k)$ satisfying $\sum_{i=1}^k r_i = 3$:

Case 3.a $k = 3$ and $(r_1, r_2, r_3) = (1, 1, 1)$,

Case 3.b $k = 2$ and $(r_1, r_2) = (1, 2)$,

Case 3.c $k = 1$ and $r_1 = 3$.

In Case 3.a, the facial exposedness can be shown in a way similar to that in the proof of Theorem 3.1, so we state the result and omit its proof.

Proposition 4.1. *Under the assumption of Case 3.a, $\mathcal{CP}(E_+)$ is facially exposed.*

In Case 3.b, $\mathcal{CP}(E_+)$ is facially exposed as shown in the following proposition.

Proposition 4.2. *Under the assumption of Case 3.b, $\mathcal{CP}(E_+)$ is facially exposed.*

Proof. By (i) and (ii) of Theorem 2.4, E is Jordan-isomorphic to the direct product of the Hadamard EJA $\mathbb{R}$ and a Jordan spin algebra L^n for some $n \geq 3$. Since $\mathcal{CP}(\mathbb{R}_+ \times L_+^n)$ is facially exposed by Theorem 2.12, we see from Corollary 2.6 that $\mathcal{CP}(E_+)$ is facially exposed. $\square$

In Case 3.c, i.e., the case where E is a simple EJA of rank 3, we will see that $\mathcal{CP}(E_+)$ is not facially exposed. We achieve this by finding a face of $\mathcal{CP}(E_+)$ that is linearly isomorphic to $\mathcal{CP}(\mathcal{S}_+^3)$ in Lemma 4.3, and then proving the non-facial exposedness of $\mathcal{CP}(\mathcal{S}_+^3)$ in Lemma 4.4.

Lemma 4.3. *Under the assumption of Case 3.c, $\mathcal{CP}(E_+)$ has an exposed face that is linearly isomorphic to $\mathcal{CP}(\mathcal{S}_+^3)$.*

Proof. By (iii) of Theorem 2.4, E is Jordan-isomorphic to $\mathcal{H}^3(\mathbb{A})$ for some $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, and we let $\phi: E \rightarrow \mathcal{H}^3(\mathbb{A})$ be a Jordan isomorphism. Since $\mathcal{S}^3$ is a subalgebra of $\mathcal{H}^3(\mathbb{A})$, we see that $\phi^{-1}(\mathcal{S}^3)$ is a subalgebra of E that is Jordan-isomorphic to $\mathcal{S}^3$. Therefore, by Corollary 2.6 and Lemma 2.8, $\mathcal{CP}(E_+)$ has the exposed face $\mathcal{CP}(\phi^{-1}(\mathcal{S}^3)_+)$ that is linearly isomorphic to $\mathcal{CP}(\mathcal{S}_+^3)$. $\square$

Lemma 4.4. *$\mathcal{CP}(\mathcal{S}_+^3)$ is not facially exposed.*

We defer the proof of Lemma 4.4 to Section 4.1 so as not to disturb the flow of the argument. Combining Lemmas 4.3 and 4.4, we obtain the following proposition.

Proposition 4.5. *Under the assumption of Case 3.c, $\mathcal{CP}(E_+)$ is not facially exposed.*

Proof. By Lemma 4.3, $\mathcal{CP}(E_+)$ has an exposed face that is linearly isomorphic to $\mathcal{CP}(\mathcal{S}_+^3)$. Since $\mathcal{CP}(\mathcal{S}_+^3)$ is not facially exposed by Lemma 4.4, it follows from Theorems 2.1 and 2.2 that $\mathcal{CP}(E_+)$ is not facially exposed. $\square$

Summarizing Propositions 4.1, 4.2 and 4.5, we obtain the following characterization of the facial exposedness of $\mathcal{CP}(E_+)$ when E has rank 3.

Theorem 4.6. *Let E be an EJA of rank 3. Then $\mathcal{CP}(E_+)$ is facially exposed if and only if E is not simple.*

4.1 Proof of Lemma 4.4

This subsection is devoted to the proof of Lemma 4.4. For clarity, we first outline the structure of the proof. In Section 4.1.1, we construct an exposed face of $\mathcal{CP}(\mathcal{S}_+^3)$ and then construct an exposed face of the former face. By (i) of Theorem 2.1, the second face is also a face of $\mathcal{CP}(\mathcal{S}_+^3)$. In Section 4.1.2, we show that the second face is not exposed in $\mathcal{CP}(\mathcal{S}_+^3)$, which implies that $\mathcal{CP}(\mathcal{S}_+^3)$ is not facially exposed.

We use adjugates of matrices in the subsequent discussion, so the following lemma summarizes formulas for adjugates for later use.

Lemma 4.7.

- (i) Let $\mathbf{X} \in \mathcal{S}^3$. If $\mathbf{X}$ has rank at most 1, then $\text{adj}(\mathbf{X}) = \mathbf{O}$.
- (ii) $\{(\text{adj}(\mathbf{X}))_{ij} \mid 1 \leq i \leq j \leq 3\}$ is linearly independent.
- (iii) Let $\mathbf{B} \in \mathcal{S}^3$. Then the gradient $\nabla \langle \mathbf{B}, \text{adj}(\mathbf{X}) \rangle \in \mathcal{S}^3$ satisfies

$$\begin{cases} (\nabla \langle \mathbf{B}, \text{adj}(\mathbf{X}) \rangle)_{11} = B_{33}X_{22} - 2B_{23}X_{23} + B_{22}X_{33}, \\ (\nabla \langle \mathbf{B}, \text{adj}(\mathbf{X}) \rangle)_{12} = -B_{33}X_{12} + B_{23}X_{13} + B_{13}X_{23} - B_{12}X_{33}, \\ (\nabla \langle \mathbf{B}, \text{adj}(\mathbf{X}) \rangle)_{13} = B_{23}X_{12} - B_{13}X_{22} - B_{22}X_{13} + B_{12}X_{23}. \end{cases}$$

Proof. For $\mathbf{X} \in \mathcal{S}^3$, its adjugate is calculated as

$$(4.1) \quad \text{adj}(\mathbf{X}) = \begin{pmatrix} X_{22}X_{33} - X_{23}^2 & X_{13}X_{23} - X_{12}X_{33} & X_{12}X_{23} - X_{22}X_{13} \\ X_{13}X_{23} - X_{12}X_{33} & X_{11}X_{33} - X_{13}^2 & X_{12}X_{13} - X_{11}X_{23} \\ X_{12}X_{23} - X_{22}X_{13} & X_{12}X_{13} - X_{11}X_{23} & X_{11}X_{22} - X_{12}^2 \end{pmatrix}$$

$$(4.2) \quad = \mathbf{X}^2 - \text{tr}(\mathbf{X})\mathbf{X} + \frac{1}{2}(\text{tr}(\mathbf{X})^2 - \text{tr}(\mathbf{X}^2))\mathbf{I}_3,$$

where the second equality follows from [37, Theorem 1]. (i) holds since every 2×2 minor of $\mathbf{X} \in \mathcal{S}^3$ of rank at most 1 is 0. (ii) holds since every $X_{ij}X_{kl}$ for $1 \leq i \leq j \leq 3$ and $1 \leq k \leq l \leq 3$ appears in the upper-triangular entries of $\text{adj}(\mathbf{X})$ at most once as shown in (4.1). (iii) can be derived from (4.2). $\square$

4.1.1 Construction of a candidate non-exposed face of $\mathcal{CP}(\mathcal{S}_+^3)$

Let

$$\mathbf{G} := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and let D_1 be the element in $\mathcal{S}(\mathcal{S}^3)$ such that the associated quadratic form satisfies

$$\begin{aligned} (4.3) \quad q_{D_1}(\mathbf{X}) &= \langle \mathbf{G}, \mathbf{X} \rangle^2 + 4\langle \mathbf{G}^{-1}, \text{adj}(\mathbf{X}) \rangle + 4\mathbf{e}_1^\top \text{adj}(\mathbf{X})\mathbf{e}_1 \\ &= (X_{11} + X_{22} + 2X_{13})^2 \\ &\quad + 4(X_{11}X_{33} - X_{13}^2 - X_{11}X_{22} + X_{12}^2 + 2X_{12}X_{23} - 2X_{22}X_{13}) \\ &\quad + 4(X_{22}X_{33} - X_{23}^2). \end{aligned}$$

Lemma 4.8. *For any $\mathbf{X} \in \text{int } \mathcal{S}_+^3$, we have*

$$(4.4) \quad \langle \mathbf{G}, \mathbf{X} \rangle^2 + 4\langle \mathbf{G}^{-1}, \text{adj}(\mathbf{X}) \rangle > 0,$$

$$(4.5) \quad \mathbf{e}_1^\top \text{adj}(\mathbf{X}) \mathbf{e}_1 > 0.$$

In particular, it follows that $D_1 \in \mathcal{COP}(\mathcal{S}_+^3)$.

Proof. Let $\mathbf{X} \in \text{int } \mathcal{S}_+^3$. Since $\mathbf{X}$ is positive definite, $\text{adj}(\mathbf{X})$ can be written as $\text{adj}(\mathbf{X}) = \det(\mathbf{X})\mathbf{X}^{-1}$ [15, equation (0.8.2.2)]. This implies that $\text{adj}(\mathbf{X})$ is also positive definite, and thus we have (4.5).

In what follows, we prove (4.4). The eigenvalues of $\mathbf{G}$ are $\frac{1 \pm \sqrt{5}}{2}$ and 1, and the inertia of $\mathbf{G}$ is equal to that of $\sqrt{\mathbf{X}}\mathbf{G}\sqrt{\mathbf{X}}$ by Sylvester's law of inertia [15, Theorem 4.5.8]. Let $l_1, l_2, -l_3$ denote the eigenvalues of $\sqrt{\mathbf{X}}\mathbf{G}\sqrt{\mathbf{X}}$, where $l_1, l_2, l_3 > 0$. Then each of the two terms that appear on the left-hand side of (4.4) is calculated as follows. First, we have

$$\langle \mathbf{G}, \mathbf{X} \rangle^2 = \text{tr}(\sqrt{\mathbf{X}}\mathbf{G}\sqrt{\mathbf{X}})^2 = (l_1 + l_2 - l_3)^2.$$

Second, we have

$$\begin{aligned} 4\langle \mathbf{G}^{-1}, \text{adj}(\mathbf{X}) \rangle &= 4\det(\mathbf{X}) \text{tr}((\sqrt{\mathbf{X}}\mathbf{G}\sqrt{\mathbf{X}})^{-1}) \\ &= -4\det(\sqrt{\mathbf{X}}\mathbf{G}\sqrt{\mathbf{X}}) \text{tr}((\sqrt{\mathbf{X}}\mathbf{G}\sqrt{\mathbf{X}})^{-1}) \\ &= 4(l_2l_3 + l_1l_3 - l_1l_2), \end{aligned}$$

where the first equality follows from $\text{adj}(\mathbf{X}) = \det(\mathbf{X})\mathbf{X}^{-1}$ and the second equality follows from $\det(\mathbf{G}) = -1$. Therefore, it follows that

$$\begin{aligned} \langle \mathbf{G}, \mathbf{X} \rangle^2 + 4\langle \mathbf{G}^{-1}, \text{adj}(\mathbf{X}) \rangle &= (l_1 + l_2 - l_3)^2 + 4(l_2l_3 + l_1l_3 - l_1l_2) \\ &= (l_1 - l_2)^2 + l_3^2 + 2(l_1 + l_2)l_3 \\ &> 0, \end{aligned}$$

so we have (4.4).

By (4.4) and (4.5), we see that $q_{D_1}(\mathbf{X}) > 0$ for all $\mathbf{X} \in \text{int } \mathcal{S}_+^3$. The nonnegativity of q_{D_1} on $\mathcal{S}_+^3$ follows from its continuity. Therefore, we obtain $D_1 \in \mathcal{COP}(\mathcal{S}_+^3)$. $\square$

Recall that $Z_{\mathcal{S}_+^3}(D_1)$, denoted by Z_1 for simplicity, is defined in (2.3). By Lemma 2.9, $\mathcal{CP}(Z_1)$ is an exposed face of $\mathcal{CP}(\mathcal{S}_+^3)$. The following lemma gives a more explicit description of Z_1 .

Lemma 4.9. *For $t \in \mathbb{R}$, we define*

$$(4.6) \quad \mathbf{X}(t) := \begin{pmatrix} 1 & t & -t^2 \\ t & 1 & -t \\ -t^2 & -t & t^2 \end{pmatrix} \in \mathcal{S}^3.$$

Then we have

$$(4.7) \quad Z_1 = \{\mathbf{x}\mathbf{x}^\top \mid q_{\mathbf{G}}(\mathbf{x}) = 0\} \cup \mathbb{R}_+\{\mathbf{X}(t) \mid t \in (-1, 1)\}.$$

Proof. We only prove that Z_1 is contained in the right-hand side of (4.7). The converse inclusion follows from the positive semidefiniteness of $\mathbf{X}(t)$ for every $t \in (-1, 1)$ and direct calculations. Let $\mathbf{X} \in Z_1$, i.e., $\mathbf{X} \in \mathcal{S}_+^3$ and $q_{D_1}(\mathbf{X}) = 0$. The rank of $\mathbf{X}$ must be at most 2; otherwise, $\mathbf{X}$ is positive definite, and Lemma 4.8 implies that $q_{D_1}(\mathbf{X}) > 0$, which contradicts $\mathbf{X} \in Z_1$.

First, suppose that $\mathbf{X}$ has rank at most 1. Then there exists $\mathbf{x} \in \mathbb{R}^3$ such that $\mathbf{X} = \mathbf{x}\mathbf{x}^\top$. It follows from (i) of Lemma 4.7 that $0 = q_{D_1}(\mathbf{X}) = q_{\mathbf{G}}(\mathbf{x})^2$, i.e., $q_{\mathbf{G}}(\mathbf{x}) = 0$. This means that $\mathbf{X}$ belongs to the right-hand side of (4.7).

Second, suppose that $\mathbf{X}$ has rank 2. Then the rank-nullity theorem implies that the dimension of $\text{Ker}(\mathbf{X})$ is 1, and we have $\text{Ker}(\mathbf{X}) = \mathbb{R}\mathbf{d}$ for some nonzero $\mathbf{d} \in \mathbb{R}^3$. It follows from the discussion in [15, section 0.8.2] that $\text{adj}(\mathbf{X}) = a\mathbf{d}\mathbf{d}^\top$ for some $a > 0$.

The vector $\mathbf{d}$ satisfies the following. First, it follows that $d_1 = 0$ since, by $q_{D_1}(\mathbf{X}) = 0$ and the discussion in Lemma 4.8, we have $0 = \mathbf{e}_1^\top (\text{adj}(\mathbf{X})) \mathbf{e}_1 = ad_1^2$. Second, it follows that $d_3 \neq 0$. Indeed, to derive a contradiction, we assume that $d_3 = 0$. Then $\mathbf{d} = d_2\mathbf{e}_2$. Note that $d_2 \neq 0$ by $\mathbf{d} \neq \mathbf{0}$. Since $\mathbf{X}$ is a positive semidefinite matrix of rank 2 and $\text{Ker}(\mathbf{X}) = \mathbb{R}\mathbf{d} = \mathbb{R}\mathbf{e}_2$, the second row and column of $\mathbf{X}$ are zero and $\begin{pmatrix} X_{11} & X_{13} \\ X_{13} & X_{33} \end{pmatrix}$ is positive definite; in particular, we have $X_{11}X_{33} - X_{13}^2 > 0$. Therefore, (4.3) leads to

$$q_{D_1}(\mathbf{X}) = (X_{11} + 2X_{13})^2 + 4(X_{11}X_{33} - X_{13}^2) > 0,$$

which contradicts $\mathbf{X} \in Z_1$. Thus, we conclude that $\mathbf{d}$ is a nonzero multiple of $(0, t, 1)^\top$ for some $t \in \mathbb{R}$. By

$$(4.8) \quad \text{Ker}(\mathbf{X}) = \mathbb{R}\mathbf{d} = \mathbb{R}(0, t, 1)^\top,$$

we see that

$$(4.9) \quad \mathbf{X} = \begin{pmatrix} X_{11} & X_{12} & -tX_{12} \\ X_{12} & X_{22} & -tX_{22} \\ -tX_{12} & -tX_{22} & t^2X_{22} \end{pmatrix}.$$

Substituting this into (4.3), we have

$$(4.10) \quad \begin{aligned} q_{D_1}(\mathbf{X}) &= (X_{11} + X_{22} - 2tX_{12})^2 - 4(X_{11}X_{22} - X_{12}^2)(1 - t^2) \end{aligned}$$

$$(4.11) \quad = \text{tr} \left(\begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix} \begin{pmatrix} 1 & -t \\ -t & 1 \end{pmatrix} \right)^2 - 4 \det \begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix} \det \begin{pmatrix} 1 & -t \\ -t & 1 \end{pmatrix}.$$

We note that since the positive semidefinite matrix $\mathbf{X}$ shown in (4.9) has rank 2 and is congruent to $\{0\} \oplus \begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix}$, the matrix $\begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix}$ is positive definite. In particular,

$$(4.12) \quad X_{11}X_{22} - X_{12}^2 > 0.$$

If $|t| > 1$, then (4.10) and (4.12) imply that $q_{D_1}(\mathbf{X}) > 0$, which contradicts $\mathbf{X} \in Z_1$. If $|t| = 1$, since

$$(X_{11} + X_{22})^2 - |2tX_{12}|^2 \geq 4(X_{11}X_{22} - X_{12}^2) > 0$$

holds by (4.12), we see from (4.10) that

$$q_{D_1}(\mathbf{X}) = (X_{11} + X_{22} - 2tX_{12})^2 > 0,$$

which is also a contradiction. Now, we assume that $|t| < 1$. Then $\begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix}$ and $\begin{pmatrix} 1 & -t \\ -t & 1 \end{pmatrix}$ are positive definite. In general, for $\mathbf{A}, \mathbf{B} \in \text{int } \mathcal{S}_+^2$, the AM-GM-type inequality $\text{tr}(\mathbf{AB}) \geq 2\sqrt{\det(\mathbf{A})\det(\mathbf{B})}$ holds with equality if and only if $\mathbf{A}$ is a positive multiple of $\mathbf{B}^{-1}$, which can be proved by applying the usual AM-GM inequality to the eigenvalues of $\sqrt{\mathbf{BA}}\sqrt{\mathbf{B}}$. By setting $\mathbf{A}$ and $\mathbf{B}$ to $\begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix}$ and $\begin{pmatrix} 1 & -t \\ -t & 1 \end{pmatrix}$ in that formula, respectively, we see from (4.11) that $q_{D_1}(\mathbf{X}) \geq 0$ with equality if and only if $\begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix}$ is a positive multiple of $\begin{pmatrix} 1 & -t \\ -t & 1 \end{pmatrix}^{-1} = \frac{1}{1-t^2} \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}$. By $q_{D_1}(\mathbf{X}) = 0$, there exists $b > 0$ such that $\begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix} = b \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}$. Substituting this into (4.9), we obtain $\mathbf{X} = b\mathbf{X}(t) \in \mathbb{R}_+\mathbf{X}(t)$. $\square$

For $t \in \mathbb{R}$, we define

$$\mathbf{y}(t) := (2, 2t, -1 - t^2)^\top.$$

Then the vectors generating the first component of the right-hand side of (4.7) can be described as

$$(4.13) \quad \{\mathbf{x} \in \mathbb{R}^3 \mid q_{\mathbf{G}}(\mathbf{x}) = 0\} = \mathbb{R}\{\mathbf{y}(t) \mid t \in \mathbb{R}\} \cup \{(0, 0, x_3)^\top \mid x_3 \in \mathbb{R}\}.$$

We only prove that the left-hand side of (4.13) is contained in the right-hand side because we can check the converse inclusion by direct calculations. Let $\mathbf{x} \in \mathbb{R}^3$ be such that

$$(4.14) \quad q_{\mathbf{G}}(\mathbf{x}) = x_1^2 + x_2^2 + 2x_1x_3 = 0.$$

First, we assume that $x_1 = 0$. Since $x_2 = 0$ by (4.14), we have $\mathbf{x} = (0, 0, x_3)^\top$, so it belongs to the right-hand side of (4.13). Second, we assume that $x_1 \neq 0$. Letting $t := x_2/x_1$, we have $\mathbf{x} = \frac{x_1}{2}\mathbf{y}(t)$, which also belongs to the right-hand side of (4.13).

Next, we construct an exposed face of $\mathcal{CP}(Z_1)$. Let

$$\mathbf{H} := \begin{pmatrix} 5 & 10 & 13 \\ 10 & -5 & 16 \\ 13 & 16 & 32 \end{pmatrix}$$

and let D_2 be the element in $\mathcal{S}(\mathcal{S}^3)$ such that the associated quadratic form satisfies

$$q_{D_2}(\mathbf{X}) = \langle \mathbf{H}, \mathbf{X} \rangle^2 - 108 \text{adj}(\mathbf{X})_{22}.$$

Lemma 4.10. *We have $D_2 \in \mathcal{COP}(Z_1)$. Furthermore, for $t \in \mathbb{R}$, we define $\mathbf{Y}(t) := \mathbf{y}(t)\mathbf{y}(t)^\top$. Then $Z_{Z_1}(D_2)$, denoted by Z_2 for simplicity, is given by*

$$(4.15) \quad Z_2 = \mathbb{R}_+\{\mathbf{Y}(-\tfrac{1}{2}), \mathbf{Y}(0), \mathbf{Y}(\tfrac{1}{2}), \mathbf{Y}(2), \mathbf{X}(0), \mathbf{X}(\tfrac{1}{2})\},$$

where $\mathbf{X}(\cdot)$ is defined in (4.6).

Proof. In the first step, we show that $D_2 \in \mathcal{COP}(Z_1)$. Let $\mathbf{X} \in Z_1$. It follows from Lemma 4.9 that $\mathbf{X}$ belongs to the first or second component on the right-hand side of (4.7). First, suppose that $\mathbf{X} = \mathbf{x}\mathbf{x}^\top$ for some $\mathbf{x} \in \mathbb{R}^3$ with $q_{\mathbf{G}}(\mathbf{x}) = 0$. By (i) of Lemma 4.7, we see that

$$(4.16) \quad q_{D_2}(\mathbf{X}) = q_{\mathbf{H}}(\mathbf{x})^2 \geq 0.$$

Second, suppose that $\mathbf{X} = a\mathbf{X}(t)$ for some $a \geq 0$ and $t \in (-1, 1)$. Then we have

$$(4.17) \quad q_{D_2}(\mathbf{X}) = 36a^2t^2(2t-1)^2 \geq 0.$$

Therefore, we obtain $D_2 \in \mathcal{COP}(Z_1)$.

In the second step, to prove (4.15), we only show that Z_2 is contained in the right-hand side of (4.15) because the converse inclusion follows from Lemma 4.9, (4.13), and direct calculations. First, if the inequality in (4.16) holds with equality, we have $q_{\mathbf{H}}(\mathbf{x}) = 0$. By $q_{\mathbf{G}}(\mathbf{x}) = 0$ and (4.13), $\mathbf{x}$ belongs to the first or second component on the right-hand side of (4.13). If $\mathbf{x} = (0, 0, x_3)^\top$, then since $q_{\mathbf{H}}(\mathbf{x}) = 32x_3^2$, the equality $q_{\mathbf{H}}(\mathbf{x}) = 0$ implies $\mathbf{x} = \mathbf{0}$. When $\mathbf{x} = \mathbf{0}$, we have $\mathbf{X} = \mathbf{O}$, which belongs to the right-hand side of (4.15). If $\mathbf{x} = a\mathbf{y}(t)$ for some $a \in \mathbb{R}$ and $t \in \mathbb{R}$, then since

$$(4.18) \quad q_{\mathbf{H}}(\mathbf{y}(t)) = 32 \left(t + \frac{1}{2} \right) t \left(t - \frac{1}{2} \right) (t-2),$$

the equality $q_{\mathbf{H}}(\mathbf{x}) = 0$ implies $a = 0$ or $t \in \{-\frac{1}{2}, 0, \frac{1}{2}, 2\}$; in either case, $\mathbf{X}$ belongs to the right-hand side of (4.15). Second, if the inequality in (4.17) holds with equality, we have $a = 0$ or $t \in \{0, \frac{1}{2}\}$; in either case, $\mathbf{X}$ belongs to the right-hand side of (4.15). Thus, we obtain (4.15). $\square$

We see from Lemma 2.9 that

$$(4.19) \quad \mathcal{CP}(Z_1) \cap \{D_2\}^\perp = \mathcal{CP}(Z_2).$$

Since $\mathcal{CP}(Z_1)$ is a face of $\mathcal{CP}(\mathcal{S}_+^3)$, by (i) of Theorem 2.1, $\mathcal{CP}(Z_2)$ is a face of $\mathcal{CP}(\mathcal{S}_+^3)$.

4.1.2 Non-exposedness of $\mathcal{CP}(Z_2)$

We begin by establishing the following lemma, which is crucial for proving that the face $\mathcal{CP}(Z_2)$ of $\mathcal{CP}(\mathcal{S}_+^3)$ is not exposed.

Lemma 4.11. *It follows that*

$$\mathcal{COP}(\mathcal{S}_+^3) \cap \mathcal{CP}(Z_2)^\perp \subseteq \{\mathbf{x}\mathbf{x}^\top \otimes \mathbf{x}\mathbf{x}^\top \mid \mathbf{x} \in \mathbb{R}^3 \text{ satisfies } q_{\mathbf{G}}(\mathbf{x}) = 0\}^\perp.$$

Proof. Let $A \in \mathcal{COP}(\mathcal{S}_+^3) \cap \mathcal{CP}(Z_2)^\perp$. By $A \in \mathcal{CP}(Z_2)^\perp$ and (4.15), we have

$$(4.20) \quad q_A(\mathbf{X}) = 0 \text{ for all } \mathbf{X} \in \{\mathbf{Y}(-\frac{1}{2}), \mathbf{Y}(0), \mathbf{Y}(\frac{1}{2}), \mathbf{Y}(2), \mathbf{X}(0), \mathbf{X}(\frac{1}{2})\}.$$

Let $p(\mathbf{x}) := q_A(\mathbf{x}\mathbf{x}^\top)$. Since p belongs to $H_4(\mathbb{R}^3)$ and $p(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^3$ by $A \in \mathcal{COP}(\mathcal{S}_+^3)$, it follows from [14] that p is a sum of squares; there exist $p_1, \dots, p_k \in H_2(\mathbb{R}^3)$ such that

$$(4.21) \quad p(\mathbf{x}) = \sum_{i=1}^k p_i(\mathbf{x})^2.$$

Every p_i for $i \in \{1, \dots, k\}$ belongs to $\text{span}\{q_G, q_H\}$. It is sufficient to show this for $i = 1$. By $q_A(\mathbf{Y}(t)) = p(\mathbf{y}(t))$, (4.20), and (4.21), we have $(p_1 \circ \mathbf{y})(t) = 0$ for all $t \in \{-\frac{1}{2}, 0, \frac{1}{2}, 2\}$. By the factor theorem, there exists $\lambda \in \mathbb{R}$ such that

$$(p_1 \circ \mathbf{y})(t) = \lambda \left(t + \frac{1}{2}\right) t \left(t - \frac{1}{2}\right) (t - 2) = \frac{\lambda}{32} (q_H \circ \mathbf{y})(t),$$

where we use (4.18) to derive the second equality. Let $\mathbf{C} \in \mathcal{S}^3$ be such that $q_{\mathbf{C}} = p_1 - \frac{\lambda}{32} q_H$. Then we see that $(q_{\mathbf{C}} \circ \mathbf{y})(t)$ is the zero polynomial. Calculating $(q_{\mathbf{C}} \circ \mathbf{y})(t)$ yields the following identity:

$$C_{33}t^4 - 4C_{23}t^3 + 2(2C_{22} + C_{33} - 2C_{13})t^2 + 4(2C_{12} - C_{23})t + 4C_{11} + C_{33} - 4C_{13} = 0.$$

Comparing the coefficients yields $C_{11} = C_{22} = C_{13}$, denoted by $c \in \mathbb{R}$, and $C_{12} = C_{23} = C_{33} = 0$. Therefore, we obtain $q_{\mathbf{C}}(\mathbf{x}) = c(x_1^2 + x_2^2 + 2x_1x_3) = cq_G(\mathbf{x})$, and $p_1 = cq_G + \frac{\lambda}{32} q_H \in \text{span}\{q_G, q_H\}$.

For each $i \in \{1, \dots, k\}$, there exist $a_i, b_i \in \mathbb{R}$ such that $p_i = a_i q_G + b_i q_H$. We define

$$(4.22) \quad \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} := \sum_{i=1}^k \begin{pmatrix} a_i \\ b_i \end{pmatrix} \begin{pmatrix} a_i \\ b_i \end{pmatrix}^\top \in \mathcal{S}_+^2.$$

Then (4.21) leads to

$$(4.23) \quad p(\mathbf{x}) = \alpha q_G(\mathbf{x})^2 + 2\beta q_G(\mathbf{x})q_H(\mathbf{x}) + \gamma q_H(\mathbf{x})^2.$$

Using α, β , and γ defined in (4.22), we can represent q_A as

$$(4.24) \quad q_A(\mathbf{X}) = \alpha \langle \mathbf{G}, \mathbf{X} \rangle^2 + 2\beta \langle \mathbf{G}, \mathbf{X} \rangle \langle \mathbf{H}, \mathbf{X} \rangle + \gamma \langle \mathbf{H}, \mathbf{X} \rangle^2 + \langle \mathbf{B}, \text{adj}(\mathbf{X}) \rangle$$

for some $\mathbf{B} \in \mathcal{S}^3$. Indeed, let $f: H_2(\mathcal{S}^3) \rightarrow H_4(\mathbb{R}^3)$ be the linear mapping sending a polynomial $q(\mathbf{X})$ to the polynomial $q(\mathbf{x}\mathbf{x}^\top)$ in $\mathbf{x}$. We see that f is surjective, since every monomial $x_i x_j x_k x_l$ for $1 \leq i \leq j \leq k \leq l \leq 3$ is realized by $f(X_{ij}X_{kl})$. In addition, the dimensions of $H_2(\mathcal{S}^3)$ and $H_4(\mathbb{R}^3)$ are 21 and 15, respectively. The rank-nullity theorem implies that the dimension of $\text{Ker}(f)$, the kernel of f , is 6. Since the set $\{(\text{adj}(\mathbf{X}))_{ij} \mid 1 \leq i \leq j \leq 3\}$ consists of six elements in $\text{Ker}(f)$ by (i) of Lemma 4.7 and is linearly independent by (ii) of Lemma 4.7, it is a basis for $\text{Ker}(f)$. In other words, we obtain the following equation:

$$(4.25) \quad \text{Ker}(f) = \{\langle \mathbf{B}, \text{adj}(\mathbf{X}) \rangle \mid \mathbf{B} \in \mathcal{S}^3\}.$$

Now, we define

$$q(\mathbf{X}) := q_A(\mathbf{X}) - (\alpha \langle \mathbf{G}, \mathbf{X} \rangle^2 + 2\beta \langle \mathbf{G}, \mathbf{X} \rangle \langle \mathbf{H}, \mathbf{X} \rangle + \gamma \langle \mathbf{H}, \mathbf{X} \rangle^2).$$

It follows from the definition of p and (4.23) that

$$q(\mathbf{x}\mathbf{x}^\top) = p(\mathbf{x}) - (\alpha q_{\mathbf{G}}(\mathbf{x})^2 + 2\beta q_{\mathbf{G}}(\mathbf{x})q_{\mathbf{H}}(\mathbf{x}) + \gamma q_{\mathbf{H}}(\mathbf{x})^2) = 0,$$

i.e., $q \in \text{Ker}(f)$. Combining this with (4.25), we obtain (4.24).

Both β and γ defined in (4.22) are in fact 0. To see this, fix $t \in \{0, \frac{1}{2}\}$ arbitrarily. Since $q_A(\mathbf{X})$ is nonnegative on $\mathcal{S}_+^3$ and it attains 0 at $\mathbf{X} = \mathbf{X}(t)$ by (4.20), the gradient of q_A at $\mathbf{X}(t)$, i.e., $\nabla q_A(\mathbf{X}(t))$, satisfies

$$\begin{aligned} (4.26) \quad -\nabla q_A(\mathbf{X}(t)) &\stackrel{(a)}{\in} \{\mathbf{Y} \in \mathcal{S}^3 \mid \langle \mathbf{Y}, \mathbf{X} - \mathbf{X}(t) \rangle \leq 0 \text{ for all } \mathbf{X} \in \mathcal{S}_+^3\} \\ &\stackrel{(b)}{=} -\mathcal{S}_+^3 \cap \{\mathbf{X}(t)\}^\perp \\ &\stackrel{(c)}{=} -\{\mathbf{Y} \in \mathcal{S}_+^3 \mid \mathbf{Y}\mathbf{X}(t) = \mathbf{X}(t)\mathbf{Y} = \mathbf{O}\}, \end{aligned}$$

where we use the first-order optimality condition [32, Theorem 27.4] to derive (a), use the formula for the normal cone of $\mathcal{S}_+^3$ [12, Theorem 2.1] to derive (b), and use $\mathbf{X}(t) \in \mathcal{S}_+^3$ to derive (c). It follows from (4.26) and $\text{Ker}(\mathbf{X}(t)) = \mathbb{R}(0, t, 1)^\top$ (see (4.8)) that $\nabla q_A(\mathbf{X}(t))$ is a (nonnegative) multiple of $\begin{pmatrix} 0 & 0 & 0 \\ 0 & t^2 & t \\ 0 & t & 1 \end{pmatrix}$. In particular, it follows that

$$(4.27) \quad (\nabla q_A(\mathbf{X}(t)))_{1j} = 0 \text{ for all } j \in \{1, 2, 3\}.$$

From (4.24), we have

$$\begin{aligned} (4.28) \quad \nabla q_A(\mathbf{X}) &= 2(\alpha \langle \mathbf{G}, \mathbf{X} \rangle + \beta \langle \mathbf{H}, \mathbf{X} \rangle) \mathbf{G} + 2(\beta \langle \mathbf{G}, \mathbf{X} \rangle + \gamma \langle \mathbf{H}, \mathbf{X} \rangle) \mathbf{H} + \nabla \langle \mathbf{B}, \text{adj}(\mathbf{X}) \rangle. \end{aligned}$$

We substitute $\mathbf{X}(0)$ for $\mathbf{X}$ in (4.28). It follows from Eq. (4.27) and (iii) of Lemma 4.7 that $0 = (\nabla q_A(\mathbf{X}(0)))_{12} = 40\beta$, which implies that $\beta = 0$. With this, by substituting $\mathbf{X}(\frac{1}{2})$ for $\mathbf{X}$ in (4.28), we see again from (4.27) and (iii) of Lemma 4.7 that

$$0 = -(\nabla q_A(\mathbf{X}(\frac{1}{2})))_{11} - 2(\nabla q_A(\mathbf{X}(\frac{1}{2})))_{12} + (\nabla q_A(\mathbf{X}(\frac{1}{2})))_{13} = 108\gamma,$$

which implies that $\gamma = 0$.

It follows from (4.23) and $\beta = \gamma = 0$ that $p(\mathbf{x}) = \alpha q_{\mathbf{G}}(\mathbf{x})^2$. Using this equality, for any $\mathbf{x} \in \mathbb{R}^3$ with $q_{\mathbf{G}}(\mathbf{x}) = 0$, we have

$$\langle A, \mathbf{x}\mathbf{x}^\top \otimes \mathbf{x}\mathbf{x}^\top \rangle = q_A(\mathbf{x}\mathbf{x}^\top) = p(\mathbf{x}) = 0.$$

Therefore, we obtain the desired result. $\square$

We are ready to show the non-exposedness of $\mathcal{CP}(Z_2)$. To derive a contradiction, we assume that $\mathcal{CP}(Z_2)$ is an exposed face of $\mathcal{CP}(\mathcal{S}_+^3)$. Then there exists $D \in \mathcal{COP}(\mathcal{S}_+^3)$ such that $\mathcal{CP}(Z_2) = \mathcal{CP}(\mathcal{S}_+^3) \cap \{D\}^\perp$. Since $D \in \mathcal{COP}(\mathcal{S}_+^3) \cap \mathcal{CP}(Z_2)^\perp$, it follows from Lemma 4.11 that

$$(4.29) \quad \{\mathbf{x}\mathbf{x}^\top \otimes \mathbf{x}\mathbf{x}^\top \mid \mathbf{x} \in \mathbb{R}^3 \text{ satisfies } q_{\mathbf{G}}(\mathbf{x}) = 0\} \subseteq \mathcal{CP}(\mathcal{S}_+^3) \cap \{D\}^\perp = \mathcal{CP}(Z_2).$$

Let $\mathbf{x}^* := (2, 2, -2)^\top$ and $\mathbf{X}^* := \mathbf{x}^*(\mathbf{x}^*)^\top$. It follows from $q_G(\mathbf{x}^*) = 0$ and (4.29) that $\mathbf{X}^* \otimes \mathbf{X}^* \in \mathcal{CP}(Z_2)$. However, $\mathbf{X}^*$ satisfies

$$\langle D_2, \mathbf{X}^* \otimes \mathbf{X}^* \rangle = q_{D_2}(\mathbf{X}^*) = ((\mathbf{x}^*)^\top \mathbf{H} \mathbf{x}^*)^2 = 576 > 0,$$

where we use (i) of Lemma 4.7 to derive the second equality. By (4.19), we have $\mathbf{X}^* \otimes \mathbf{X}^* \notin \mathcal{CP}(Z_2)$, which is a contradiction. Thus, $\mathcal{CP}(Z_2)$ is a non-exposed face of $\mathcal{CP}(\mathcal{S}_+^3)$, and this completes the proof of Lemma 4.4.

5 Case of rank 4

Let E be an EJA of rank 4. We classify the facial exposedness of $\mathcal{CP}(E_+)$ depending on whether E is Jordan-isomorphic to the Hadamard EJA $\mathbb{R}^4$.

First, we discuss the case where E is not Jordan-isomorphic to $\mathbb{R}^4$.

Lemma 5.1. *Let $(E, \circ)$ be an EJA of rank 4 that is not Jordan-isomorphic to the Hadamard EJA $\mathbb{R}^4$. Then $\mathcal{CP}(E_+)$ has an exposed face that is linearly isomorphic to $\mathcal{CP}(\mathbb{R}_+^2 \times L_+^n)$ for some $n \geq 3$.*

Proof. Fix a Jordan frame $\{c_1, c_2, c_3, c_4\}$ of E . By [10, Theorem IV.2.1], E can be decomposed into the following direct sum of subspaces:

$$E = \mathbb{R}c_1 \oplus \mathbb{R}c_2 \oplus \mathbb{R}c_3 \oplus \mathbb{R}c_4 \oplus E_{12} \oplus E_{13} \oplus E_{23} \oplus E_{14} \oplus E_{24} \oplus E_{34},$$

where $E_{ij} := \{x \in E \mid c_i \circ x = c_j \circ x = \frac{1}{2}x\}$ for every $1 \leq i < j \leq 4$. By the assumption that E is not Jordan-isomorphic to the Hadamard EJA $\mathbb{R}^4$, there exist $1 \leq i < j \leq 4$ such that $E_{ij} \neq \{0\}$. We may assume that $E_{34} \neq \{0\}$ without loss of generality. We define $E_{3:4} := \mathbb{R}c_3 \oplus \mathbb{R}c_4 \oplus E_{34}$, which is a subalgebra of E by [13, Lemma 20.a] and [10, Proposition IV.1.1]. Since $E_{3:4}$ is a simple EJA of rank 2 by $E_{34} \neq \{0\}$ and [21, page 168], (ii) of Theorem 2.4 implies that $E_{3:4}$ is Jordan-isomorphic to a Jordan spin algebra L^n for some $n \geq 3$. Therefore, $E' := \mathbb{R}c_1 \oplus \mathbb{R}c_2 \oplus E_{3:4}$ is a subalgebra of E and E' is Jordan-isomorphic to the direct product of the Hadamard EJA $\mathbb{R}^2$ and L^n . Since the cone of squares in $\mathbb{R}^2 \times L^n$ is $\mathbb{R}_+^2 \times L_+^n$, it follows from Corollary 2.6 and Lemma 2.8 that $\mathcal{CP}(E_+)$ has the exposed face $\mathcal{CP}(E'_+)$ that is linearly isomorphic to $\mathcal{CP}(\mathbb{R}_+^2 \times L_+^n)$. $\square$

Lemma 5.2. *$\mathcal{CP}(\mathbb{R}_+^2 \times L_+^n)$ is not facially exposed for any $n \geq 3$.*

We defer the proof of Lemma 5.2 to Section 5.1 so as not to disturb the flow of the argument. Lemmas 5.1 and 5.2 lead to the non-facial exposedness of $\mathcal{CP}(E_+)$ in the case where E is not Jordan-isomorphic to $\mathbb{R}^4$, as stated in the following proposition.

Proposition 5.3. *Let E be an EJA of rank 4 that is not Jordan-isomorphic to the Hadamard EJA $\mathbb{R}^4$. Then $\mathcal{CP}(E_+)$ is not facially exposed.*

Proof. By Lemma 5.1, $\mathcal{CP}(E_+)$ has an exposed face that is linearly isomorphic to $\mathcal{CP}(\mathbb{R}_+^2 \times L_+^n)$ for some $n \geq 3$. Since $\mathcal{CP}(\mathbb{R}_+^2 \times L_+^n)$ is not facially exposed by Lemma 5.2, it follows from Theorems 2.1 and 2.2 that $\mathcal{CP}(E_+)$ is not facially exposed. $\square$

Combining Proposition 5.3 with the case where E is Jordan-isomorphic to $\mathbb{R}^4$, we obtain the following characterization of the facial exposedness of $\mathcal{CP}(E_+)$ when E has rank 4.

Theorem 5.4. *Let E be an EJA of rank 4. Then $\mathcal{CP}(E_+)$ is facially exposed if and only if E is Jordan-isomorphic to the Hadamard EJA $\mathbb{R}^4$.*

Proof. If E is not Jordan-isomorphic to the Hadamard EJA $\mathbb{R}^4$, the non-facial exposedness of $\mathcal{CP}(E_+)$ follows from Proposition 5.3. If E is Jordan-isomorphic to $\mathbb{R}^4$, the facial exposedness of $\mathcal{CP}(E_+)$ can be shown by an argument similar to the proof of Theorem 3.1. $\square$

5.1 Proof of Lemma 5.2

This subsection is devoted to the proof of Lemma 5.2. The outline of the proof is similar to that of Lemma 4.4 presented in Section 4.1. In Section 5.1.1, we observe that $\mathcal{CP}(\mathbb{R}_+^2 \times L_+^n)$ has a face that is linearly isomorphic to $\mathcal{CP}(\mathbb{R}_+^2 \times L_+^3)$ and construct a face of $\mathcal{CP}(\mathbb{R}_+^2 \times L_+^3)$ by constructing a chain of faces. In Section 5.1.2, we show that the final face is not exposed in $\mathcal{CP}(\mathbb{R}_+^2 \times L_+^3)$, which implies that $\mathcal{CP}(\mathbb{R}_+^2 \times L_+^n)$ is not facially exposed.

5.1.1 Construction of a candidate non-exposed face of $\mathcal{CP}(\mathbb{R}_+^2 \times L_+^3)$

In the first step, we show that $\mathcal{CP}(\mathbb{R}_+^2 \times L_+^n)$ has an exposed face that is linearly isomorphic to $\mathcal{CP}(\mathbb{R}_+^2 \times L_+^3)$. We define $X := \text{span}\{\mathbf{e}_1, \dots, \mathbf{e}_5\} \subseteq \mathbb{R}^{n+2}$. It follows from Theorem 2.7 that $\mathcal{CP}((\mathbb{R}_+^2 \times L_+^n) \cap X)$ is an exposed face of $\mathcal{CP}(\mathbb{R}_+^2 \times L_+^n)$. Since

$$(\mathbb{R}_+^2 \times L_+^n) \cap X = \{(x_1, \dots, x_5, 0, \dots, 0)^\top \in \mathbb{R}^{n+2} \mid (x_1, \dots, x_5)^\top \in \mathbb{R}_+^2 \times L_+^3\}$$

and it is linearly isomorphic to $\mathbb{R}_+^2 \times L_+^3$, it follows from Lemma 2.5 that $\mathcal{CP}((\mathbb{R}_+^2 \times L_+^n) \cap X)$ is linearly isomorphic to $\mathcal{CP}(\mathbb{R}_+^2 \times L_+^3)$.

In the second step, we construct an exposed face of $\mathcal{CP}(\mathbb{R}_+^2 \times L_+^3)$. Let $\mathbf{D}_1 \in \mathcal{S}^5$ be such that the associated quadratic form satisfies

$$(5.1) \quad q_{\mathbf{D}_1}(\mathbf{x}) = \mathbf{x}_{3:5}^\top \mathbf{J}_3 \mathbf{x}_{3:5} + (x_1 - x_2 + x_3 + x_4 - x_5)^2 + 2x_1(x_3 - x_5).$$

We see that $\mathbf{D}_1 \in \mathcal{COP}(\mathbb{R}_+^2 \times L_+^3)$ since each of the three terms on the right-hand side of (5.1) is nonnegative for any $\mathbf{x} \in \mathbb{R}_+^2 \times L_+^3$. Recall that $Z_{\mathbb{R}_+^2 \times L_+^3}(\mathbf{D}_1)$, denoted by Z_1 for simplicity, is defined in (2.3). It follows from Lemma 2.9 that $\mathcal{CP}(Z_1)$ is an exposed face of $\mathcal{CP}(\mathbb{R}_+^2 \times L_+^3)$. The following lemma gives a more explicit description of Z_1 .

Lemma 5.5. *Let*

$$\begin{aligned} \mathbf{x}(s) &:= \left(0, 1-s, \frac{1+s^2}{2}, \frac{1-s^2}{2}, s\right)^\top \quad (s \in \mathbb{R}), \\ \mathbf{x}^* &:= (0, 0, 1, -1, 0)^\top \end{aligned}$$

and define

$$C := \{\alpha(\mathbf{e}_1 + \mathbf{e}_2) + \beta\mathbf{x}(1) \mid \alpha, \beta \geq 0\} = \{(\alpha, \alpha, \beta, 0, \beta)^\top \mid \alpha, \beta \geq 0\}.$$

Then we have

$$(5.2) \quad Z_1 = \mathbb{R}_+ \{\mathbf{x}(s) \mid s \leq 1\} \cup \mathbb{R}_+ \mathbf{x}^* \cup C.$$

Proof. We only prove that Z_1 is contained in the right-hand side of (5.2) because the converse inclusion can be verified through direct calculations. Let $\mathbf{x} \in Z_1$, i.e., $\mathbf{x} \in \mathbb{R}_+^2 \times L_+^3$ and $q_{\mathbf{D}_1}(\mathbf{x}) = 0$. Since $q_{\mathbf{D}_1}(\mathbf{x}) = 0$, each of the three terms on the right-hand side of (5.1) is also 0, and we have

$$(5.3) \quad (x_3 + x_4)(x_3 - x_4) = x_5^2,$$

$$(5.4) \quad x_1 - x_2 + x_3 + x_4 - x_5 = 0,$$

$$(5.5) \quad x_1(x_3 - x_5) = 0.$$

First, we consider the case where $\beta := x_3 + x_4$ is not 0. We note that $\beta > 0$ by $\mathbf{x}_{3:5} \in L_+^3$. Let $s_\beta := x_5/\beta$. Then (5.3) implies that $x_3 - x_4 = \beta s_\beta^2$. Combining this with $x_3 + x_4 = \beta$, we have

$$\mathbf{x}_{3:5} = \beta \left(\frac{1 + s_\beta^2}{2}, \frac{1 - s_\beta^2}{2}, s_\beta \right)^\top.$$

We further distinguish two cases depending on whether $s_\beta = 1$ or not. If $s_\beta = 1$, then we have $\mathbf{x}_{3:5} = \beta(1, 0, 1)^\top$. Substituting this into (5.4), we have $x_1 = x_2$, and this common value is nonnegative by $\mathbf{x}_{1:2} \in \mathbb{R}_+^2$. Then we have

$$\mathbf{x} = (x_1, x_1, \beta, 0, \beta)^\top \in C.$$

If $s_\beta \neq 1$, then since $x_3 - x_5 = \frac{\beta}{2}(1 - s_\beta)^2 \neq 0$, it follows from (5.5) that $x_1 = 0$. Consequently, (5.4) implies that $x_2 = \beta(1 - s_\beta)$. Since $\mathbf{x}_{1:2} \in \mathbb{R}_+^2$, we have $s_\beta \leq 1$. Therefore, we obtain

$$\mathbf{x} = \beta \mathbf{x}(s_\beta) \in \mathbb{R}_+ \{ \mathbf{x}(s) \mid s \leq 1 \}.$$

Second, we consider the case where $x_3 + x_4 = 0$. Since (5.3) implies that $x_5 = 0$, we see that $\mathbf{x}_{3:5} = x_3(1, -1, 0)$. Substituting this into (5.4) and (5.5), we have $x_1 = x_2$ and $x_1 x_3 = 0$, respectively. We note that $x_3 \geq 0$ by $\mathbf{x}_{3:5} \in L_+^3$. If $x_3 > 0$, then $x_1 = 0$, and we have

$$\mathbf{x} = x_3(0, 0, 1, -1, 0)^\top \in \mathbb{R}_+ \mathbf{x}^*.$$

If $x_3 = 0$, then we have

$$\mathbf{x} = (x_1, x_1, 0, 0, 0)^\top \in C.$$

Therefore, we obtain the desired result. $\square$

In the third step, we construct an exposed face of $\mathcal{CP}(Z_1)$. Let $\mathbf{D}_2 \in \mathcal{S}^5$ be such that the associated quadratic form satisfies

$$q_{\mathbf{D}_2}(\mathbf{x}) = (x_3 - x_4)(x_3 + x_4 - x_5).$$

Using Lemma 5.5, we can easily check that $\mathbf{D}_2 \in \mathcal{COP}(Z_1)$. For simplicity, we let $Z_2 := Z_{Z_1}(\mathbf{D}_2)$. By Lemma 2.9, we have

$$(5.6) \quad \mathcal{CP}(Z_1) \cap \{\mathbf{D}_2\}^\perp = \mathcal{CP}(Z_2).$$

Since $\mathcal{CP}(Z_1)$ is a face of $\mathcal{CP}(\mathbb{R}_+^2 \times L_+^3)$, by (i) of Theorem 2.1, $\mathcal{CP}(Z_2)$ is a face of $\mathcal{CP}(\mathbb{R}_+^2 \times L_+^3)$.

We can represent Z_2 more explicitly as

$$(5.7) \quad Z_2 = \mathbb{R}_+ \{\mathbf{x}(0), \mathbf{x}^*\} \cup C.$$

We only show that Z_2 is contained in the right-hand side of (5.7) because the converse inclusion follows from Lemma 5.5 and direct calculations. Let $\mathbf{x} \in Z_2$, i.e., $\mathbf{x} \in Z_1$ and $q_{\mathbf{D}_2}(\mathbf{x}) = 0$. By Lemma 5.5, we see that $\mathbf{x}$ belongs to one of the three components on the right-hand side of (5.2). First, if $\mathbf{x} \in \mathbb{R}_+ \mathbf{x}^*$ or $\mathbf{x} \in C$, then $\mathbf{x}$ belongs to the right-hand side of (5.7). Next, we assume that there exist $\beta \geq 0$ and $s \leq 1$ such that $\mathbf{x} = \beta \mathbf{x}(s)$. Since $0 = q_{\mathbf{D}_2}(\mathbf{x}) = \beta^2 s^2 (1-s)$, we have $\beta = 0$, $s = 0$, or $s = 1$. If $\beta = 0$, then $\mathbf{x} = \mathbf{0}$, which belongs to the right-hand side of (5.7). If $s = 0$, then $\mathbf{x} = \beta \mathbf{x}(0) \in \mathbb{R}_+ \mathbf{x}(0)$. If $s = 1$, then $\mathbf{x} = \beta \mathbf{x}(1) \in C$. Therefore, we obtain (5.7).

5.1.2 Non-exposedness of $\mathcal{CP}(Z_2)$

To show that $\mathcal{CP}(Z_2)$ is a non-exposed face of $\mathcal{CP}(\mathbb{R}_+^2 \times L_+^3)$ by contradiction, we assume that $\mathcal{CP}(Z_2)$ is an exposed face of $\mathcal{CP}(\mathbb{R}_+^2 \times L_+^3)$. Then there exists $\mathbf{D} \in \mathcal{COP}(\mathbb{R}_+^2 \times L_+^3)$ such that

$$(5.8) \quad \mathcal{CP}(Z_2) = \mathcal{CP}(\mathbb{R}_+^2 \times L_+^3) \cap \{\mathbf{D}\}^\perp.$$

We define $p(s) := q_{\mathbf{D}}(\mathbf{x}(s))$, which is a polynomial in s .

Lemma 5.6. *p is the zero polynomial.*

Proof. We observe that s^2 divides $p(s)$, i.e., $p(0) = p'(0) = 0$. Indeed, $p(0) = q_{\mathbf{D}}(\mathbf{x}(0)) = 0$ by $\mathbf{x}(0) \in Z_2$ (see (5.7)) and (5.8). We see that $p(s) \geq 0$ for any $s \leq 1$ since $\mathbf{x}(s) \in \mathbb{R}_+^2 \times L_+^3$ (Lemma 5.5) and $\mathbf{D} \in \mathcal{COP}(\mathbb{R}_+^2 \times L_+^3)$. Therefore, $s = 0$ is a local minimizer of $p(s)$, and we have $p'(0) = 0$.

Moreover, $(s-1)^2$ divides $p(s)$, i.e., $p(1) = p'(1) = 0$. Indeed, $p(1) = q_{\mathbf{D}}(\mathbf{x}(1)) = 0$ by $\mathbf{x}(1) \in C \subseteq Z_2$ (see (5.7)) and (5.8). Since $\mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{x}(1)$ belong to the convex cone C , which is contained in Z_2 , it follows from (5.8) that

$$(5.9) \quad 0 = \frac{1}{2} q_{\mathbf{D}}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{x}(1)) = \mathbf{e}_1^\top \mathbf{D} \mathbf{x}(1) + \mathbf{e}_2^\top \mathbf{D} \mathbf{x}(1).$$

Let $i \in \{1, 2\}$ and $t > 0$ be arbitrary. By $t\mathbf{e}_i + \mathbf{x}(1) \in \mathbb{R}_+^2 \times L_+^3$ and $\mathbf{D} \in \mathcal{COP}(\mathbb{R}_+^2 \times L_+^3)$, we have

$$0 \leq q_{\mathbf{D}}(t\mathbf{e}_i + \mathbf{x}(1)) = t^2 q_{\mathbf{D}}(\mathbf{e}_i) + 2t\mathbf{e}_i^\top \mathbf{D} \mathbf{x}(1).$$

Dividing this by $2t$ and letting $t \downarrow 0$, we have $\mathbf{e}_i^\top \mathbf{D} \mathbf{x}(1) \geq 0$. Combining this with (5.9), since $i \in \{1, 2\}$ is arbitrary, we obtain

$$(5.10) \quad \mathbf{e}_1^\top \mathbf{D} \mathbf{x}(1) = \mathbf{e}_2^\top \mathbf{D} \mathbf{x}(1) = 0.$$

For $s \in \mathbb{R}$, let

$$\bar{\mathbf{x}}(s) := \left(0, 0, \frac{1+s^2}{2}, \frac{1-s^2}{2}, s\right)^\top \in \mathbb{R}_+^2 \times L_+^3$$

and $\bar{p}(s) := q_D(\bar{\mathbf{x}}(s))$. The polynomial $\bar{p}$ satisfies $\bar{p}(s) \geq 0$ for all $s \in \mathbb{R}$ by $\mathbf{D} \in \mathcal{COP}(\mathbb{R}_+^2 \times L_+^3)$ and $\bar{p}(1) = 0$ by $\bar{\mathbf{x}}(1) = \mathbf{x}(1) \in Z_2$ and (5.8). This implies that

$$0 = \bar{p}'(1) = 2\bar{\mathbf{x}}'(1)^\top \mathbf{D}\bar{\mathbf{x}}(1) = 2(\mathbf{x}'(1) + \mathbf{e}_2)^\top \mathbf{D}\mathbf{x}(1) = p'(1) + 2\mathbf{e}_2^\top \mathbf{D}\mathbf{x}(1) = p'(1),$$

where the first equality follows from the optimality of $\bar{p}$ at $s = 1$, the third equality follows from $\bar{\mathbf{x}}(s) = \mathbf{x}(s) - (1-s)\mathbf{e}_2$, and the fifth equality follows from (5.10). Therefore, $s^2(s-1)^2$ divides $p(s)$.

The degree of p is at most 3 since $\mathbf{x}(s) = \frac{s^2}{2}\mathbf{x}^* + (0, 1-s, \frac{1}{2}, \frac{1}{2}, s)^\top$ and $q_D(\mathbf{x}^*) = 0$ by $\mathbf{x}^* \in Z_2$ (see (5.7)). Since $s^2(s-1)^2$ divides the polynomial $p(s)$ of degree at most 3, we conclude that p is the zero polynomial. $\square$

It follows from Lemma 5.6 that $0 = p(-1) = q_D(\mathbf{x}(-1))$. Combining this with $\mathbf{x}(-1) \in \mathbb{R}_+^2 \times L_+^3$ and (5.8) yields $\mathbf{x}(-1)\mathbf{x}(-1)^\top \in \mathcal{CP}(Z_2)$. However, since $q_{D_2}(\mathbf{x}(-1)) = 2 \neq 0$, (5.6) implies that $\mathbf{x}(-1)\mathbf{x}(-1)^\top \notin \mathcal{CP}(Z_2)$, which is a contradiction. Therefore, $\mathcal{CP}(Z_2)$ is a non-exposed face of $\mathcal{CP}(\mathbb{R}_+^2 \times L_+^3)$, and thus $\mathcal{CP}(\mathbb{R}_+^2 \times L_+^3)$ is not facially exposed. Since $\mathcal{CP}(\mathbb{R}_+^2 \times L_+^3)$ has an exposed face that is linearly isomorphic to $\mathcal{CP}(\mathbb{R}_+^2 \times L_+^3)$, by Theorems 2.1 and 2.2, we conclude that $\mathcal{CP}(\mathbb{R}_+^2 \times L_+^3)$ is not facially exposed.

6 Case of rank at least 5

In this section, we show that $\mathcal{CP}(E_+)$ is not facially exposed for any EJA E of rank at least 5. We achieve this by constructing one of its faces that is linearly isomorphic to a completely positive cone over a nonnegative orthant.

Theorem 6.1. *Let E be an EJA of rank at least 5. Then $\mathcal{CP}(E_+)$ is not facially exposed.*

Proof. Let $r \geq 5$ denote the rank of E and $\{c_1, \dots, c_r\}$ be a Jordan frame of E . Then $E' := \text{span}\{c_1, \dots, c_r\}$ is a subalgebra of E that is Jordan-isomorphic to the Hadamard EJA $\mathbb{R}^r$, where a Jordan isomorphism is the linear mapping sending $c_i \in E'$ to $\mathbf{e}_i \in \mathbb{R}^r$ for every $i \in \{1, \dots, r\}$. It follows from Lemma 2.8 that $\mathcal{CP}(E'_+)$ is an exposed face of $\mathcal{CP}(E_+)$. Since $\mathcal{CP}(\mathbb{R}_+^r)$ is not facially exposed by $r \geq 5$ and Theorem 2.10, it follows from Corollary 2.6 and Theorem 2.1 that $\mathcal{CP}(E_+)$ is not facially exposed. $\square$

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