On the Equivalence of Monge and Kantarovich Problems in Discrete Optimal Transport

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Abstract

Consider a discrete optimal transport problem that has at least two consumers. We show that the Monge and Kantorovich versions of such a discrete optimal transport problem are equivalent for all cost functions if and only if the supply from all the suppliers are equal, and the demand from every consumer is an integral multiple of the supply. When there is only one consumer, we show that the equivalence is trivially true irrespective of the supply and demand.

1 Discrete Optimal Transport

Consider an optimal transport problem in the discrete setting, where the set of suppliers and the set of consumers are denoted, respectively, as $X = \{x_1, \ldots, x_N\}$ and $Y = \{y_1, \ldots, y_M\}$. The supply from x_i and the demand from y_j are denoted, respectively, as μ_i and ν_j. The total supply must be equal to the total demand and therefore we have $\sum_{i=1}^{N} \mu_i = \sum_{j=1}^{M} \nu_j$. The cost of transporting a unit quantity from supplier x_i to consumer y_j is given by the cost function $c(x_i, y_j)$. See [2] and the references therein for more details on the problem.

The Monge version of the optimal transport problem in the discrete setting is given by

$$V_M = \inf_{\substack{T:\{1,\ldots,N\}\to\{1,\ldots,M\},\\ T:T\#\mu=\nu}} \sum_{i=1}^{N} \mu_i c(x_i, y_{T(i)}),$$

where $T\#\mu = \nu$ in the discrete setting refers to $\sum_{i:T(i)=j} \mu_i = \nu_j$. The relaxed problem by Kantorovich is given by

$$V_K = \min_{\{\gamma:\gamma\in\Pi(\mu,\nu)\}} \sum_{i=1}^{N} \sum_{j=1}^{M} \gamma_{ij} c(x_i, y_j),$$

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where $\Pi(\mu,\nu) = \{\gamma \in \mathbb{R}^{N\times M} : \gamma_{ij} \geq 0, \sum_{j=1}^{M} \gamma_{ij} = \mu_i, \sum_{i=1}^{N} \gamma_{ij} = \nu_j\}$. Observe that when we set

$$\gamma_{ij} = \begin{cases} \mu_i & \text{if } T(i) = j, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

for all $i = 1,\ldots,N$, we have $\sum_{j=1}^{M} \gamma_{ij} c(x_i, y_j) = \mu_i c(x_i, y_{T(i)})$. Summing over all i, we have $\sum_{i=1}^{N}\sum_{j=1}^{M} \gamma_{ij} c(x_i, y_j) = \sum_{i=1}^{N} \mu_i c(x_i, y_{T(i)})$. So for any function $T : \{1,\ldots,N\} \to \{1,\ldots,M\}$ there exists an equivalent $\gamma \in \Pi(\mu,\nu)$ such that the objective functions of the Monge and Kantorovich problems are equal. We thus have $V_K \leq V_M$ given that there could exist $\gamma \in \Pi(\mu,\nu)$ that are not of the form given in (1).

We now address the following question. Under what conditions do we have $V_K = V_M$? The two problems are not the same, in general, as can be readily seen from the following two examples.

Example 1: Consider $N = M = 2$, $\mu_1 = 1$, $\mu_2 = 3$, and $\nu_1 = \nu_2 = 2$. There are only four possibilities for T: (a) $T(i) = 1$, $i = 1, 2$, (b) $T(i) = 2$, $i = 1, 2$, (c) $T(i) = i$, $i = 1, 2$, and (d) $T(i) = 3 - i$, $i = 1, 2$. Note that $T\#\mu = \nu$ is not satisfied for any of these cases. Thus, the constraint set for the Monge problem is empty, and hence $V_M = \infty$. Now, observe that the set $\Pi(\mu,\nu)$ is given by

$$\Pi(\mu,\nu) = \{(\gamma \in \mathbb{R}^{2\times 2} : \gamma_{22} \in [1,2], (\gamma_{11}, \gamma_{12}, \gamma_{21}) = (\gamma_{22} - 1, 2 - \gamma_{22}, 3 - \gamma_{22})\}.$$

We thus have γ_{ij} bounded for all i, j, and so the solution always exists for the Kantorovich problem irrespective of the cost function. So, V_K is finite for any cost function, and thus $V_K < V_M$.

Example 2: Consider $N = M = 2$, $\mu_1 = \nu_1 = 1$, $\mu_2 = \nu_2 = 2$. Again, there are only four possibilities for T. Note that the constraint $T\#\mu = \nu$ is satisfied only when $T(i) = i$. Thus, the constraint set for the Monge problem is a singleton set, and hence the solution is $T(i) = i$ for any given cost function. So, if we consider $c(x_1, y_1) = c(x_2, y_2) = 100$ and $c(x_1, y_2) = c(x_2, y_1) = 50$, we have $V_M = 300$. However, the solution to the Kantorovich problem can be derived as $\gamma_{11} = 0$, $\gamma_{12} = \gamma_{21} = \gamma_{22} = 1$, and hence $V_K = 200$. This is an example where the constraint set in Monge problem is non-empty but we still have $V_K < V_M$.

The following theorem from [1] provides a sufficient condition on the values of N, M, μ_i's and ν_j's for which $V_M = V_K$ holds for all possible cost functions.

Theorem 1. *[1, Thm. 1.1] When $N = M$ and $\mu_i = \nu_j = 1$ for all $i, j \in \{1,\ldots,N\}$, we have $V_M = V_K$.*

In the following section, we provide a necessary and sufficient condition on μ_i's and ν_j's to have $V_M = V_K$ for all cost functions.

2 An Equivalence Result

Consider a discrete optimal transport problem with one consumer, that is, $M = 1$. Then the feasible set of the Monge version is a singleton set, with $T(i) = 1$

for all i. Similarly, the feasible set of the Kantarovich version is also a singleton set, with $\gamma_{i1} = \mu_i$ for all i. Thus, $V_M = \sum_{i=1}^{N} \mu_i c(x_i, y_1) = V_K$, and this holds for all cost functions irrespective of μ_i's and ν_1.

We now consider a discrete transport problem having more than one consumer, that is, $M \geq 2$. The following result shows that $V_M = V_K$ holds for all cost functions if and only if (i) the supply from each supplier is equal (say μ), and (ii) the demand for each consumer is an integral multiple of μ.

Theorem 2. *Let $M \geq 2$. Then $V_M = V_K$ holds for all cost functions if and only if (i) $\mu_i = \mu$ for all $i \in \{1, \dots, N\}$, and (ii) $\nu_j = k_j \mu$ for some $k_j \in \mathbb{Z}_+$ for all $j \in \{1, \dots, M\}$.*

Proof. We begin by proving the if part. Define P as the optimal transport problem with $\mu_i = \mu$ for all i and $\nu_j = k_j \mu$ for all j. Given that total supply must be equal to total demand, we have $\sum_{j=1}^{M} k_j = N$. We now consider an alternate optimal transport problem $\tilde{P}$, where

(i) a quantity of μ in the original problem P is considered as a unit quantity in $\tilde{P}$, with the cost of transportation being scaled up by a factor of μ, and

(ii) consumer y_j in the original problem P is considered as k_j different consumers $\tilde{y}_{j1}, \tilde{y}_{j2}, \dots, \tilde{y}_{jk_j}$ in $\tilde{P}$, each having a demand of unit quantity according to the modified metric.

So, $\tilde{P}$ is an optimal transport problem with

- N suppliers denoted by $\tilde{x}_i$, $i = 1, \dots, N$,
- N consumers denoted by $\tilde{y}_{jl}$, $j = 1, \dots, M$, $l = 1, \dots, k_j$,
- their quantities of supply and consumption given by $\tilde{\mu}_i = \tilde{\nu}_{jl} = 1$, and
- the unit cost of transport $\tilde{c}(\tilde{x}_i, \tilde{y}_{jl}) = \mu c(x_i, y_j)$.

Observe from Theorem 1 that the Monge and Kantorovich versions of $\tilde{P}$ are equivalent, that is, $\tilde{V}_M = \tilde{V}_K$. We now show that the optimal transport problems P and $\tilde{P}$ are equivalent. In other words, we show that $\tilde{V}_M = V_M$ and $\tilde{V}_K = V_K$.

Consider a $\tilde{T}$ that is feasible for the Monge version of $\tilde{P}$, and define

$$T(i) = j \text{ if } \tilde{T}(i) = jl \text{ for some } l \in \{1, \dots, k_j\}. \tag{2}$$

We now show that T is feasible for the Monge version of P, and that the objective function of P under the transport mechanism T and that of $\tilde{P}$ under the transport mechanism $\tilde{T}$ are equal. We first argue that T is feasible. Note that $\tilde{T}$ feasible implies that there exists exactly one supplier i such that $\tilde{T}(i) = jl$ for any $l \in \{1, \dots, k_j\}$. So by the definition of T there exist exactly k_j suppliers who supply to consumer j in problem P. Thus, we have $\sum_{i:T(i)=j} \mu_i = k_j \mu = \nu_j$.

Therefore, T is feasible for the Monge version of P. Now, regarding the objective functions, we have

$$\sum_{i=1}^{N} \tilde{\mu}_i \tilde{c}(\tilde{x}_i, \tilde{y}_{\tilde{T}(i)}) = \sum_{i=1}^{N} \mu c(x_i, y_{T(i)}) = \sum_{i=1}^{N} \mu_i c(x_i, y_{T(i)}),$$

where the first equality follows from the definitions of $\tilde{c}$ and T.

We have thus shown that if T is constructed as in (2) from a $\tilde{T}$ that is feasible for $\tilde{P}$, then T is both feasible for P and gives the same value of the objective function as $\tilde{T}$. So, if $\tilde{T}^*$ is the solution of the Monge version of $\tilde{P}$, then there exists a feasible T^* such that $\sum_{i=1}^{N} \mu_i c(x_i, y_{T^*(i)}) = \tilde{V}_M$. We now show that T^* is the solution for P.

Consider a T that is feasible for P, and define

$$\tilde{T}(s_{jl}) = jl \text{ if } T(s_{j1}) = \cdots = T(s_{jk_j}) = j \text{ for some } s_{j1}, \ldots, s_{jk_j}. \tag{3}$$

A similar proof can be used to show that $\tilde{T}$ is feasible for $\tilde{P}$, and that it gives the same value of the objective function. So, if there exists a feasible T such that $\sum_{i=1}^{N} \mu_i c(x_i, y_{T^*(i)}) = V > \tilde{V}_M$, then there exists a feasible $\tilde{T}$ such that $\sum_{i=1}^{N} \tilde{\mu}_i \tilde{c}(\tilde{x}_i, \tilde{y}_{\tilde{T}^*(i)}) = V > \tilde{V}_M$, which is a contradiction. So T^* is the solution for P, and hence $V_M = \tilde{V}_M$.

We now proceed to prove $V_K = \tilde{V}_K$. Consider a $\tilde{\gamma}$ that is feasible for the Kantarovich version of $\tilde{P}$, and define

$$\gamma_{ij} = \sum_{l=1}^{k_j} \mu \tilde{\gamma}_{i(jl)} \tag{4}$$

We now show that γ is feasible for the Kantorovich version of P, and that the objective function of P under the transport mechanism γ and that of $\tilde{P}$ under the transport mechanism $\tilde{\gamma}$ are equal. We first argue that γ is feasible. We now have

$$\sum_{j=1}^{M} \gamma_{ij} = \sum_{j=1}^{M} \sum_{l=1}^{k_j} \mu \tilde{\gamma}_{i(jl)} = \mu, \quad \sum_{i=1}^{N} \gamma_{ij} = \sum_{i=1}^{N} \sum_{l=1}^{k_j} \mu \tilde{\gamma}_{i(jl)} = k_j \mu,$$

where the equalities follow from the definition of γ and the feasibility of $\tilde{\gamma}$. So γ is feasible for the Kantorovich version of P. Regarding the objective functions, we have

$$\sum_{i=1}^{N} \sum_{j=1}^{M} \sum_{l=1}^{k_j} \tilde{\gamma}_{i(jl)} \tilde{c}(\tilde{x}_i, \tilde{y}_{jl}) = \sum_{i=1}^{N} \sum_{j=1}^{M} \sum_{l=1}^{k_j} \tilde{\gamma}_{i(jl)} \mu c(x_i, y_j) = \sum_{i=1}^{N} \sum_{j=1}^{M} \gamma_{ij} c(x_i, y_j),$$

where the first and second equalities follows from the definitions of $\tilde{c}$ and γ, respectively.

We have thus shown that if γ is constructed as in (4) from a $\tilde{\gamma}$ that is feasible for $\tilde{P}$, then T is both feasible for P and gives the same value of the objective function as $\tilde{T}$. So, if $\tilde{\gamma}^*$ is the solution of the Kantarovich version of $\tilde{P}$, then there exists a feasible γ^* such that $\sum_{i=1}^{N}\sum_{j=1}^{M}\gamma_{ij}^* c(x_i, y_j) = \tilde{V}_K$. We now show that γ^* is the solution for P.

Consider a γ that is feasible for P, and define

$$\tilde{\gamma}_{i(jl)} = \frac{\gamma_{ij}}{k_j \mu} \text{ for all } l \in \{1, \ldots, k_j\}. \tag{5}$$

A similar proof can be used to show that $\tilde{\gamma}$ is feasible for $\tilde{P}$, and that it gives the same value of the objective function. So, if there exists a feasible γ such that $\sum_{i=1}^{N}\sum_{j=1}^{M}\gamma_{ij} c(x_i, y_j) = V > \tilde{V}_K$, then there exists a feasible $\tilde{\gamma}$ such that $\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{l=1}^{k_j}\tilde{\gamma}_{i(jl)}\tilde{c}(\tilde{x}_i, \tilde{y}_{jl}) = V > \tilde{V}_K$, which is a contradiction. So γ^* is the solution for P, and hence $V_K = \tilde{V}_K$. This completes the if part of the proof.

We now proceed to prove the only if part. We show that if the conditions in the statement of the theorem are not satisfied, then there exists a cost function for which $V_M > V_K$. Consider the case when $\mu_i = \mu$ for all $i = 1, \ldots, N$, but $\exists\, j$ such that $\frac{\nu_j}{\mu} \notin \mathbb{Z}_+$. Then $\sum_{i:T(i)=j}\mu_i$ will always be an integral multiple of μ, and thus the feasible set of the Monge problem is empty. So we have $V_M = \infty$. But the feasible set of the Kantorovich problem, $\Pi(\mu, \nu)$, is closed, bounded, and nonempty. It is bounded since $\gamma_{ij} \in [0, \mu_i]$, and is nonempty since $\gamma_{ij} = \frac{\mu_i \nu_j}{\sum_{j=1}^{M}\nu_j} \in \Pi(\mu, \nu)$. The objective function to be minimized is a linear function and thus is continuous. Since we minimize a continuous function on a compact set, the minimizer γ^* exists, and thus V_K is finite. Thus, we have $V_M > V_K$ for any cost function.

Consider the case when $\max_i \mu \neq \min_i \mu_i$. In case the supply and demand are such that the feasible set of the Monge problem is empty, then we again have $V_M = \infty > V_K$ for any cost function, and the theorem is proved. So, let us consider the case when the feasible set of the Monge problem is nonempty. Consider a feasible $\hat{T}$, and pick i and j such that $\mu_i \neq \mu_j$ and $\hat{T}(i) \neq \hat{T}(j)$. Observe that picking such a pair of suppliers is possible because of the following reason: Pick i and j such that $\mu_i \neq \mu_j$; If $\hat{T}(i) = \hat{T}(j)$, then pick k such that $\hat{T}(i) \neq \hat{T}(k)$; If $\mu_i = \mu_k$, then j and k can be picked; Otherwise, i and k can be picked.

Given that $\mu_i \neq \mu_j$ and $\hat{T}(i) \neq \hat{T}(j)$, we consider the following cost function:

$$c(x_k, y_{\hat{T}(k)}) = 1,\ c(x_k, y_l) = (N+2)\frac{\max_s \mu_s}{\mu_k},\ \forall l \neq \hat{T}(k), \forall k \neq i, k \neq j,$$
$$c(x_i, y_{\hat{T}(j)}) = 1,\ c(x_i, y_{\hat{T}(i)}) = 2,\ c(x_i, y_l) = (N+2)\frac{\max_s \mu_s}{\mu_i},\ \forall l \neq \hat{T}(i), l \neq \hat{T}(j),$$
$$c(x_j, y_{\hat{T}(i)}) = 1,\ c(x_j, y_{\hat{T}(j)}) = 2,\ c(x_j, y_l) = (N+2)\frac{\max_s \mu_s}{\mu_j},\ \forall l \neq \hat{T}(i), l \neq \hat{T}(j).$$

Observe that $\sum_{s=1}^{N}\mu_s c(x_s, y_{\hat{T}(s)}) = \mu_i + \mu_j + \sum_{s=1}^{N}\mu_s < (N+2)\max_s \mu_s$. We now show that $\hat{T}$ is the solution of the Monge problem for the given cost function. If $T(k) \neq \hat{T}(k)$ for $k \neq i$, $k \neq j$, then a cost of $(N+2)\max_s \mu_s$ is incurred, and thus T cannot be the minimizer. A similar argument holds if

$T(i) \notin \{\hat{T}(i), \hat{T}(j)\}$. Furthermore, $T(i) = \hat{T}(j)$ and $T(j) = \hat{T}(i)$ is not feasible since $\mu_i \neq \mu_j$. Thus, $\hat{T}$ is the solution of the Monge problem for the given cost function, and thus $V_M = \mu_j + \mu_i + \sum_{s=1}^{N} \mu_s$. Assuming without loss of generality that $\mu_i < \mu_j$, we consider the following values for γ:

$$\gamma_{k(\hat{T}(k))} = \mu_k,\ \gamma_{kl} = 0,\ \forall l \neq \hat{T}(k), \forall k \neq i, k \neq j,$$
$$\gamma_{i(\hat{T}(j))} = \mu_i,\ \gamma_{il} = 0,\ \forall l \neq \hat{T}(j),$$
$$\gamma_{j(\hat{T}(i))} = \mu_i,\ \gamma_{j(\hat{T}(j))} = \mu_j - \mu_i,\ \gamma_{jl} = 0,\ \forall l \neq \hat{T}(i), l \neq \hat{T}(j).$$

We now show that γ is feasible. Note that $\sum_{t=1}^{M} \gamma_{st} = \mu_s$ is true for all $s = 1, \ldots, N$. Now, to show that $\sum_{s=1}^{N} \gamma_{st} = \nu_t$ for all t, we first observe that the transport mechanism under $\hat{T}$ and γ differs only with suppliers i and j, and consumers $\hat{T}(i)$ and $\hat{T}(j)$. Every other supplier k transports all his supplies to the consumer $\hat{T}(k)$ under both the transport mechanisms $\hat{T}$ and γ. So, for all $t \neq \hat{T}(i), t \neq \hat{T}(j)$, we have

$$\sum_{s=1}^{N} \gamma_{st} = \sum_{s:\hat{T}(s)=t} \mu_s = \nu_t,$$

where the final equality follows from the feasibility of $\hat{T}$. When $t \in \{\hat{T}(i), \hat{T}(j)\}$, we have

$$\sum_{s=1}^{N} \gamma_{st} = \gamma_{it} + \gamma_{jt} + \sum_{\substack{s:\hat{T}(s)=t,\\ s\neq i, s\neq j}} \mu_s = \sum_{s:\hat{T}(s)=t} \mu_s = \nu_t,$$

where the second equality follows since $\gamma_{i(\hat{T}(i))} + \gamma_{j(\hat{T}(i))} = \mu_i$ and $\gamma_{i(\hat{T}(j))} + \gamma_{j(\hat{T}(j))} = \mu_j$. This proves the feasibility of γ.

Now we have

$$V_K \leq \sum_{i=1}^{N} \sum_{j=1}^{M} \gamma_{ij} c(x_i, y_j) = \mu_j - \mu_i + \sum_{s=1}^{N} \mu_s < \mu_j + \mu_i + \sum_{s=1}^{N} \mu_s = V_M.$$

We have thus constructed a cost function for which $V_K < V_M$. This completes the proof of the theorem. □

Our results can thus be summarized as follows.

- The objective functions of the Monge and Kantorovich problems are equal for all cost functions when $M = 1$.
- When $M \geq 2$, the necessary and sufficient condition for the objective functions to be equal for all cost functions is (i) $\mu_i = \mu$ for all i and (ii) $\nu_j = k_j \mu$ for some $k_j \in \mathbb{Z}_+$ for all j. In other words, the objective functions are equal for all cost functions when the supply from every supplier is the same and the demand of every consumer is an integral multiple of the supply; when this condition is not satisfied, then there exists a cost function for which the objective functions are unequal.

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References

[1] Brezis, H. "Remarks on the Monge-Kantorovich problem in the discrete setting", *Comptes Rendus Mathématique Academies des Sciences*, Vol. 356 (2018) no. 2, pp. 207–213.

[2] Francesco Maggi, "Optimal Mass Transport on Euclidean Spaces", *Cambridge Studies on Advanced Mathematics*, Cambridge University Press, 2023.