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FRANK E. CURTIS, LINGJUN GUO, AND DANIEL P. ROBINSON

Department of Industrial and Systems Engineering, Lehigh University

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FRANK E. CURTIS^{*1}, LINGJUN GUO^{†1}, AND DANIEL P. ROBINSON^{‡1}

¹Department of Industrial and Systems Engineering, Lehigh University

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Abstract

For solving nonconvex equality-constrained optimization problems, a recent Gradient-Eigenstep Algorithm by Goyens et al. is an iteration-efficient approach, based on minimizing Fletcher’s augmented Lagrangian function, for finding an approximate second-order stationary point from an arbitrary starting point. In this paper, the analysis of this algorithm is extended, offering a two-fold contribution. First, it is shown that a local-linear rate of convergence can be obtained by this method if it is initiated sufficiently close to a strong second-order stationary point and employs a sufficiently small step-size parameter and sufficiently large penalty parameter. In this case, the algorithm reduces to a gradient descent algorithm applied to minimize Fletcher’s augmented Lagrangian. Second, as a particularly useful application of the first result, it is shown that the Gradient-Eigenstep algorithm can be used as an iteration-efficient subproblem solver in the context of a progressive sampling strategy for solving equality-constrained optimization problems when the objective and constraint functions are defined by large sample averages, ultimately offering an algorithm with an improved worst-case sample complexity when compared to an approach that solves a full-sample problem directly.

1 Introduction

This work studies the theoretical convergence-rate behavior of an algorithm for solving nonlinear equality-constrained continuous optimization problems. Problems of this type, and algorithms for solving them, have been the subject of research for decades. Many of the most successful algorithms can be characterized as being penalty methods or primal-dual methods; see, e.g., [12, Chapter 17–18]. These are local-search algorithms that seek a point satisfying either first- or second-order stationary conditions for optimality.

Our focus in this paper is on a particular penalty method. Penalty methods solve a constrained problem by introducing a measure of constraint violation and adding it to the objective function, weighted by a penalty parameter. The aim is to solve the original constrained problem through unconstrained optimization techniques, where the penalty parameter is adjusted in a principled manner to ensure that the constraints are eventually satisfied, at least in the limit. The penalty function used in a penalty method can be characterized as being either *inexact* or *exact*. An inexact penalty function is one for which the penalty parameter might need to be driven to infinity to ensure that constraint satisfaction is achieved in an asymptotic sense. By contrast, an exact penalty function is one for which, for a given problem, there exists a threshold for the penalty parameter, above which there is a correspondence between stationary points for the original constrained problem and stationary points for the penalty function. This feature of exact penalty functions is attractive, but an issue with many exact penalty functions is that they are nonsmooth.

^{*}E-mail: frank.e.curtis@lehigh.edu

[†]E-mail: lig423@lehigh.edu

[‡]E-mail: daniel.p.robinson@lehigh.edu

Fletcher [6] provides an exact penalty function—which came to be known as Fletcher’s augmented Lagrangian function—that is both exact and smooth. After it was introduced and shown to be exact, several articles [2, 3, 4] explored the potential use of it in practical penalty methods for solving constrained problems, focusing in particular on its computational costs related to the computation of least-squares Lagrange multipliers and their derivatives. Recently, the article [5] proposed an algorithm based on Fletcher’s augmented Lagrangian that only requires a linear system solve in each iteration. Even more recently, [9] extended the work in [3] by proposing an algorithm that exploits potential negative curvature directions to obtain an approximate second-order stationary point with a complexity guarantee.

Our contributions in this paper are twofold. First, we show that if an initial point is given that is sufficiently close to a strong second-order stationary point satisfying a regularity condition, then if the penalty parameter is sufficiently large and the step sizes are chosen sufficiently small, then an application of gradient descent to minimize Fletcher’s augmented Lagrangian function can be used to obtain a linear rate of decrease in the norm of Fletcher’s augmented Lagrangian, ultimately yielding an even more accurate approximate strong second-order stationary point. In this pursuit, we show that the Gradient-Eigenstep method in [9] reduces to gradient descent with a local-linear rate of convergence in the vicinity of certain strong second-order stationary points. Second, as a particularly useful application of these first results, we prove strong worst-case complexity properties for a recently proposed progressive sampling method for solving certain equality-constrained problems; see [1, Algorithm 1]. In particular, the progressive sampling method is designed for solving problems where the objective and constraint functions are formed by large-scale sample average approximations of functions defined by expectations. The idea of progressive sampling is to solve a subsampled problem to modest accuracy, and then use the solution as a starting point for solving a subsequently subsampled problem to higher accuracy, progressively increasing the sample size until the original sample average approximation problem is solved to a desired accuracy. If the original problem satisfies a so-called strong-Morse property and the subsampled problems are solved through an application of gradient descent on Fletcher’s augmented Lagrangian function with appropriate parameter choices, we show that the progressive sampling strategy yields an improved worst-case sample complexity property compared to a method that solves the full sample average problem directly.

1.1 Notation

We use $\mathbb{R}$ to denote the set of real numbers, $\mathbb{R}_{\geq r}$ (resp., $\mathbb{R}_{> r}$) to denote the set of real numbers greater than or equal to (resp., greater than) $r \in \mathbb{R}$, $\mathbb{R}^n$ to denote the set of n -dimensional real vectors, and $\mathbb{R}^{m \times n}$ to denote the set of m -by- n -dimensional real matrices. We define $\mathbb{N} := \{0, 1, 2, \dots\}$, and, for any integer $N \geq 1$, we use $[N]$ to denote $\{1, \dots, N\}$. For any finite set $\mathcal{S}$, we use $|\mathcal{S}|$ to denote its cardinality. For vectors, we define $\|\cdot\| := \|\cdot\|_2$, or else specify the norm explicitly. We use $\|\cdot\|$ to denote the spectral norm of any matrix.

For any $A \in \mathbb{R}^{m \times n}$ we use $\sigma_i(A)$ for each $i \in [\min\{m, n\}]$ to denote its i th largest singular value, and for any symmetric $A \in \mathbb{R}^{n \times n}$ we use $\lambda_i(A)$ for each $i \in [n]$ to denote its i th largest eigenvalue. Given any such A , we use $\text{Null}(A)$ to denote its null space, i.e., $\{d \in \mathbb{R}^n : Ad = 0\}$. Assuming that $B \in \mathbb{R}^{n \times m}$ has full-column rank, its Moore-Penrose pseudoinverse is $B^\dagger := (B^T B)^{-1} B^T$. Note that in this case of B having full-column rank, the pseudoinverse $B^\dagger$ is a left inverse of B in the sense that $B^\dagger B = I$. Given $B \in \mathbb{R}^{n \times m}$ with full column rank, we use $\mathcal{R}(B) := BB^\dagger$ and $\mathcal{N}(B^T) = I - \mathcal{R}(B)$ to denote orthogonal projection matrices onto the span of the columns of B and the null space of B^T , respectively. That such $\mathcal{R}(B)$ and $\mathcal{N}(B^T)$ are orthogonal projection matrices follows since each of them are both symmetric and idempotent.

1.2 Organization

In §2, we show that the Gradient-Eigenstep method ([9, Algorithm 1]) can reduce to a gradient-descent method that converges linearly in the neighborhood of certain second-order stationary points of an equality constrained optimization problem. We apply this result to prove strong worst-case complexity properties of a progressive sampling method for equality constrained optimization in §3. A summary and concluding remarks are given in §4.

2 Local-Linearly Convergent Algorithm

In this section, we prove that the Gradient-Eigenstep method, namely, [9, Algorithm 1]—if the step size is sufficiently small and the penalty parameter is sufficiently large—converges linearly when initiated in the vicinity of certain second-order stationary points of the optimization problem

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to (s.t.) } c(x) = 0 \quad (1)$$

when, among other properties, the functions f and c satisfy the following.

Assumption 2.1. *The objective function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and constraint function $c : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are twice-continuously differentiable.*

We begin this section with some additional assumptions and definitions, then proceed with our theoretical convergence-rate analysis.

As in [9], we assume the following about the constraint function.

Assumption 2.2. *There exists a positive real number $R \in \mathbb{R}_{>0}$ such that the set $\mathcal{C}_R := \{x \in \mathbb{R}^n : \|c(x)\| \leq R\}$ is compact.*

Under Assumption 2.1, let us denote the objective gradient function as $\nabla f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, the objective Hessian function as $\nabla^2 f : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$, the constraint derivative function as $\nabla c : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$, and for each $j \in [m]$ the j th constraint Hessian function as $\nabla^2 [c]_j : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$. Here and throughout, for a function c , we use $[c]_j$ to denote its j th component. Similarly as in [9], we make the following assumption about the constraint Jacobian function, i.e., ∇c^T . Here, the margin δ appears in our analysis as a tolerance to ensure that trial steps computed by an algorithm remain within the set $\hat{\mathcal{C}}_R$.

Assumption 2.3. *There exists $\delta \in \mathbb{R}_{>0}$, a bounded open set $\hat{\mathcal{C}}_R$ containing $\mathcal{C}_R + \{v \in \mathbb{R}^n : \|v\| \leq \delta\}$ and a positive real number $\sigma_{\min} \in \mathbb{R}_{>0}$ such that, for any $x \in \hat{\mathcal{C}}_R$, the constraint Jacobian has $\sigma_m(\nabla c(x)^T) \geq \sigma_{\min}$.*

The Lagrangian of (1) is $L : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ defined by

$$L(x, \hat{y}) = f(x) + c(x)^T \hat{y}, \quad (2)$$

where $\hat{y} \in \mathbb{R}^m$ is known as the Lagrange multiplier. Defined with respect to L , an approximate second-order stationary point of (1) is the following.

Definition 2.1. *For a pair of real numbers $(\epsilon, \zeta) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$, a point $x \in \mathbb{R}^n$ is (ϵ, ζ) -stationary for (1) if there exists $\hat{y} \in \mathbb{R}^m$ such that*

$$\begin{aligned} \|\nabla L(x, \hat{y})\| &\leq \epsilon \\ \text{and } d^T \nabla_{xx}^2 L(x, \hat{y}) d &\geq -\zeta \|d\|^2 \text{ for all } d \in \text{Null}(\nabla c(x)^T). \end{aligned}$$

Due to the first of the second-order stationarity conditions in Definition 2.1, for any $x \in \mathbb{R}^n$ a particular Lagrange multiplier of interest is any that minimizes a norm of the gradient of the Lagrangian. Such a Lagrange multiplier is also of interest since it is used within the algorithm that we analyze. Let us define an ℓ_2 -norm least-squares Lagrange multiplier function $y : \hat{\mathcal{C}}_R \rightarrow \mathbb{R}^m$ by

$$y(x) = \arg \min_{\hat{y} \in \mathbb{R}^m} \|\nabla f(x) + \nabla c(x) \hat{y}\|^2, \quad (3)$$

and note that, under Assumption 2.3, for any $x \in \hat{\mathcal{C}}_R$ the unique solution is

$$y(x) = -(\nabla c(x)^T \nabla c(x))^{-1} \nabla c(x)^T \nabla f(x) = -\nabla c(x)^\dagger \nabla f(x). \quad (4)$$

We now introduce Fletcher's augmented Lagrangian function.

Definition 2.2. Using y defined by (4), Fletcher's augmented Lagrangian with penalty parameter $\rho \in \mathbb{R}_{>0}$ is the function $F_\rho : \hat{\mathcal{C}}_R \rightarrow \mathbb{R}$ defined by

$$F_\rho(x) = f(x) + c(x)^T y(x) + \rho \|c(x)\|^2.$$

We also include the following additional assumption about the Lagrangian with respect to the least-squares Lagrange multiplier function and the constraint violation measure $\|c\|^2$, i.e., the two component functions defining Fletcher's augmented Lagrangian for any penalty parameter.

Assumption 2.4. Along with Assumptions 2.1, 2.2, and 2.3, the function y defined by (4) is twice-continuously differentiable over $\hat{\mathcal{C}}_R$, and over $\hat{\mathcal{C}}_R$ the functions $f + c^T y$ and $\|c\|^2$ have Hessian functions that are Lipschitz continuous in the sense that there exists $(M_f, M_c) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ such that, for all $(x, \bar{x}) \in \hat{\mathcal{C}}_R \times \hat{\mathcal{C}}_R$, one has that

$$\begin{aligned} \|\nabla^2(f + c^T y)(x) - \nabla^2(f + c^T y)(\bar{x})\| &\leq M_f \|x - \bar{x}\| \\ \text{and } \|\nabla^2(\|c\|^2)(x) - \nabla^2(\|c\|^2)(\bar{x})\| &\leq M_c \|x - \bar{x}\|. \end{aligned}$$

Under these conditions, for any $\rho \in \mathbb{R}_{>0}$, define $M_\rho := M_f + \rho M_c$.

We now commence our analysis. For our first lemma, we introduce a uniform bound for the norms of various quantities and state useful properties of F_ρ at approximate second-order stationary points.

Lemma 2.1. Under Assumptions 2.1, 2.2, 2.3, and 2.4 there is $\kappa \in \mathbb{R}_{>0}$ with

$$\max_{x \in \hat{\mathcal{C}}_R} \left\{ \|\nabla f(x)\|, \|y(x)\|, \|\nabla y(x)\|, \|\nabla c(x)\|, \|\nabla^2 f(x)\|, \max_{j \in [m]} \{ \|\nabla^2[y]_j(x)\|, \|\nabla^2[c]_j(x)\| \} \right\} \leq \kappa, \quad (5)$$

and along with any $\rho \in \mathbb{R}_{>0}$ one has

$$\lambda_{R,\rho} := \sup_{x \in \hat{\mathcal{C}}_R} \{ \lambda_1(\nabla^2 F_\rho(x)) \} < \infty. \quad (6)$$

Moreover, for any $\epsilon \in (0, R]$ and $\beta \in \mathbb{R}_{>0}$, if $x \in \mathbb{R}^n$ has

$$\begin{aligned} \|\nabla L(x, y(x))\| &\leq \epsilon \\ \text{and } d^T \nabla_{xx}^2 L(x, y(x)) d &\geq \beta \|d\|^2 \text{ for all } d \in \text{Null}(\nabla c(x)^T), \end{aligned} \quad (7)$$

then for any $\rho \in \mathbb{R}_{>0}$ and with positive real numbers

$$\begin{aligned} \omega_\rho &:= 1 + \kappa + 2\rho\kappa \\ \text{and } \eta_\rho &:= \frac{2\sqrt{m}\kappa}{\sigma_{\min}} + m\kappa + 2m\rho\kappa \end{aligned} \quad (8)$$

one finds that

$$\begin{aligned} \|c(x)\| &\leq R, \\ \|\nabla F_\rho(x)\| &\leq \omega_\rho \epsilon, \\ \text{and } \lambda_n(\nabla^2 F_\rho(x)) &\geq \min\{\beta - \eta_\rho \epsilon, 2\rho\sigma_{\min}^2 - (\kappa + m\kappa^2) - \eta_\rho \epsilon\}. \end{aligned} \quad (9)$$

Proof. Consider (5). Continuity of $\nabla f(x)$, $\nabla c(x)$, $\nabla^2 f(x)$, and $\nabla^2[c]_j(x)$ for all $j \in [m]$ follow by Assumption 2.1. Continuity of $y(x)$, $\nabla y(x)$, and $\nabla^2[y]_j(x)$ for all $j \in [m]$ follow by Assumptions 2.1, 2.3, and 2.4. Since $\mathcal{C}_R$ is compact by Assumption 2.2, the existence of $\kappa \in \mathbb{R}_{>0}$ follows from continuity of these functions, continuity of norms, the fact that compositions of continuous functions are continuous, and the Weierstrauss Extreme Value Theorem.

Consider next (6). The fact that F_ρ is twice-continuously differentiable follows under Assumption 2.4, and by continuity of eigenvalues of $\nabla^2 F_\rho$ and compactness of $\mathcal{C}_R$ it follows that the supremum in (6) is finite.

Consider now (9). Consider arbitrary $\epsilon \in (0, R]$, $\beta \in \mathbb{R}_{>0}$, $x \in \mathbb{R}^n$ satisfying (7), and $\rho \in \mathbb{R}_{>0}$. Therefore,

$$\max\{\|\nabla f(x) + \nabla c(x)y(x)\|, \|c(x)\|\} \leq \|\nabla L(x, y(x))\| \leq \epsilon, \quad (10)$$

which with $\epsilon \leq R$ gives the first inequality in (9). Thus, $x \in \mathcal{C}_R$, so the triangle inequality and submultiplicity of the matrix 2-norm give

$$\begin{aligned} \|\nabla F_\rho(x)\| &\leq \|\nabla f(x) + \nabla c(x)y(x)\| + (\|\nabla y(x)\| + 2\rho\|\nabla c(x)\|)\|c(x)\| \\ &\leq \|\nabla f(x) + \nabla c(x)y(x)\| + (\kappa + 2\rho\kappa)\|c(x)\|. \end{aligned} \quad (11)$$

Combining (11) with (10), one obtains the second inequality in (9). Our final aim is to prove the third inequality in (9). Toward this end, note that

$$\begin{aligned} &\nabla^2 F_\rho(x) \\ &= \nabla_{xx}^2 L(x, y(x)) + \nabla y(x) \nabla c(x)^T + \nabla c(x) \nabla y(x)^T \\ &\quad + 2\rho \nabla c(x) \nabla c(x)^T + \sum_{j \in [m]} (\nabla^2 [y]_j(x) + 2\rho \nabla^2 [c]_j(x)) [c]_j(x). \end{aligned} \quad (12)$$

Let us now express this Hessian in an equivalent form involving a decomposition into orthogonal spaces defined by the constraint Jacobian at x . Let us first derive an expression for $\nabla y(x) \nabla c(x)^T$ by differentiating the linear system that defines $y(x)$ through (4). Specifically, by (4), one finds that

$$\nabla c(x)^T \nabla c(x) y(x) = -\nabla c(x)^T \nabla f(x). \quad (13)$$

Abbreviating notation and differentiating the left-hand side yields

$$\begin{aligned} &\nabla(\nabla c^T \nabla c y)|_x \\ &= \nabla(\nabla c^T|_x \nabla c|_x y)|_x + \nabla(\nabla c^T \nabla c|_x y|_x)|_x + \nabla(\nabla c^T|_x \nabla c y|_x)|_x \\ &= \nabla y|_x \nabla c^T|_x \nabla c|_x + [\nabla^2 [c]_1|_x \nabla c|_x y|_x \quad \cdots \quad \nabla^2 [c]_m|_x \nabla c|_x y|_x] \\ &\quad + \left(\sum_{j \in [m]} \nabla^2 [c]_j|_x [y]_j|_x \right) \nabla c|_x, \end{aligned}$$

while at the same time differentiating the right-hand side yields

$$\begin{aligned} \nabla(-\nabla c^T \nabla f)|_x &= -\nabla(\nabla c^T|_x \nabla f)|_x - \nabla(\nabla c^T \nabla f|_x)|_x \\ &= -\nabla^2 f|_x \nabla c|_x - [\nabla^2 [c]_1|_x \nabla f|_x \quad \cdots \quad \nabla^2 [c]_m|_x \nabla f|_x]. \end{aligned}$$

Combining these derivations and rearranging yields

$$\begin{aligned} &\nabla y(x) \nabla c(x)^T \nabla c(x) \\ &= -\nabla_{xx}^2 L(x, y(x)) \nabla c(x) \\ &\quad - \underbrace{[\nabla^2 [c]_1(x) \nabla_x L(x, y(x)) \quad \cdots \quad \nabla^2 [c]_m(x) \nabla_x L(x, y(x))]}_{=: \mathcal{E}(x)}. \end{aligned} \quad (14)$$

Multiplying (14) on the right by $(\nabla c(x)^T \nabla c(x))^{-1} \nabla c(x)^T$ yields

$$\begin{aligned} &\nabla y(x) \nabla c(x)^T \\ &= -(\nabla_{xx}^2 L(x, y(x)) \nabla c(x) + \mathcal{E}(x)) (\nabla c(x)^T \nabla c(x))^{-1} \nabla c(x)^T \end{aligned}$$

$$= -\nabla_{xx}^2 L(x, y(x)) \mathcal{R}(\nabla c(x)) - \mathcal{E}(x) \nabla c(x)^\dagger. \quad (15)$$

On the other hand, denoting $\mathcal{R}(x) := \mathcal{R}(\nabla c(x))$, $\mathcal{N}(x) := \mathcal{N}(\nabla c(x)^T)$, and $H_{xx} := \nabla_{xx}^2 L(x, y(x))$ and using $\mathcal{R}(x) + \mathcal{N}(x) = I$, one finds that

$$\begin{aligned} H_{xx} - H_{xx} \mathcal{R}(x) - \mathcal{R}(x) H_{xx} &= H_{xx} \mathcal{N}(x) - \mathcal{R}(x) H_{xx} \\ &= (\mathcal{N}(x) + \mathcal{R}(x)) H_{xx} \mathcal{N}(x) - \mathcal{R}(x) H_{xx} \\ &= \mathcal{N}(x) H_{xx} \mathcal{N}(x) - \mathcal{R}(x) H_{xx} \mathcal{R}(x). \end{aligned} \quad (16)$$

Combining (12), (15), and (16) now yields the expression

$$\begin{aligned} \nabla^2 F_\rho(x) &= \mathcal{N}(x) H_{xx} \mathcal{N}(x) - \mathcal{R}(x) H_{xx} \mathcal{R}(x) + 2\rho \nabla c(x) \nabla c(x)^T \\ &\quad - \mathcal{E}(x) \nabla c(x)^\dagger - (\nabla c(x)^\dagger)^T \mathcal{E}(x)^T \\ &\quad + \sum_{j \in [m]} (\nabla^2 [y]_j(x) + 2\rho \nabla^2 [c]_j(x)) [c]_j(x). \end{aligned} \quad (17)$$

Let us now observe that from the definition of $\mathcal{E}(x)$, norm inequalities, submultiplicity of the matrix 2-norm, $x \in \mathcal{C}_R$, (5), and (10) that

$$\begin{aligned} \|\mathcal{E}(x)\| &= \|\mathcal{E}(x)^T\| \\ &= \max_{d \in \mathbb{R}^n \text{ s.t. } \|d\|=1} \|\mathcal{E}(x)^T d\| \\ &= \max_{d \in \mathbb{R}^n \text{ s.t. } \|d\|=1} \left\| \begin{bmatrix} \nabla_x L(x, y(x))^T \nabla^2 [c]_1(x)^T d \\ \vdots \\ \nabla_x L(x, y(x))^T \nabla^2 [c]_m(x)^T d \end{bmatrix} \right\| \\ &= \max_{d \in \mathbb{R}^n \text{ s.t. } \|d\|=1} \sqrt{\sum_{j \in [m]} (\nabla_x L(x, y(x))^T \nabla^2 [c]_j(x)^T d)^2} \\ &\leq \sqrt{m} \max_{d \in \mathbb{R}^n \text{ s.t. } \|d\|=1} \left(\max_{j \in [m]} |\nabla_x L(x, y(x))^T \nabla^2 [c]_j(x)^T d| \right) \\ &\leq \sqrt{m} \max_{d \in \mathbb{R}^n \text{ s.t. } \|d\|=1} \left(\max_{j \in [m]} \|\nabla_x L(x, y(x))\| \|\nabla^2 [c]_j(x)\| \|d\| \right) \\ &\leq \sqrt{m} \max_{d \in \mathbb{R}^n \text{ s.t. } \|d\|=1} (\epsilon \kappa \|d\|) = \sqrt{m} \kappa \epsilon. \end{aligned} \quad (18)$$

Now consider arbitrary $d \in \mathbb{R}^n \setminus \{0\}$ decomposed as $d \equiv d_{\mathcal{N}} + d_{\mathcal{R}}$, where $d_{\mathcal{N}} \in \text{Null}(\nabla c(x)^T)$ and $d_{\mathcal{R}} \in \text{Range}(\nabla c(x))$. One has from (17) that

$$\begin{aligned} &d^T \nabla^2 F_\rho(x) d \\ &= d^T \mathcal{N}(x) \nabla_{xx}^2 L(x, y(x)) \mathcal{N}(x) d - d^T \mathcal{R}(x) \nabla_{xx}^2 L(x, y(x)) \mathcal{R}(x) d \\ &\quad + 2\rho d^T \nabla c(x) \nabla c(x)^T d - 2d^T \mathcal{E}(x) \nabla c(x)^\dagger d \\ &\quad + d^T \left(\sum_{j \in [m]} (\nabla^2 [y]_j(x) + 2\rho \nabla^2 [c]_j(x)) [c]_j(x) \right) d \\ &= d_{\mathcal{N}}^T \nabla_{xx}^2 L(x, y(x)) d_{\mathcal{N}} - d_{\mathcal{R}}^T \nabla_{xx}^2 L(x, y(x)) d_{\mathcal{R}} \\ &\quad + 2\rho \|\nabla c(x)^T d_{\mathcal{R}}\|^2 - 2d^T \mathcal{E}(x) \nabla c(x)^\dagger d \\ &\quad + d^T \left(\sum_{j \in [m]} (\nabla^2 [y]_j(x) + 2\rho \nabla^2 [c]_j(x)) [c]_j(x) \right) d, \end{aligned} \quad (19)$$

where for the latter two terms one has from (5), (10), (18), and the fact that the spectral norm of the pseudoinverse of a matrix is at most the reciprocal of the matrix's smallest singular value [8, Chapter 21] that

$$\begin{aligned}
& -2d^T \mathcal{E}(x) \nabla c(x)^\dagger d + d^T \left(\sum_{j \in [m]} (\nabla^2[y]_j(x) + 2\rho \nabla^2[c]_j(x)) [c]_j(x) \right) d \\
& \geq -2\|\mathcal{E}(x)\| \|\nabla c(x)^\dagger\| \|d\|^2 - \left\| \sum_{j \in [m]} (\nabla^2[y]_j(x) + 2\rho \nabla^2[c]_j(x)) [c]_j(x) \right\| \|d\|^2 \\
& \geq -\frac{2\sqrt{m}\kappa}{\sigma_{\min}} \epsilon \|d\|^2 - \sum_{j \in [m]} (\|\nabla^2[y]_j(x)\| + 2\rho \|\nabla^2[c]_j(x)\|) \|c\|_j(x) \|d\|^2 \\
& \geq -\frac{2\sqrt{m}\kappa}{\sigma_{\min}} \epsilon \|d\|^2 - \sum_{j \in [m]} (\|\nabla^2[y]_j(x)\| + 2\rho \|\nabla^2[c]_j(x)\|) \|c(x)\| \|d\|^2 \\
& \geq -\frac{2\sqrt{m}\kappa}{\sigma_{\min}} \epsilon \|d\|^2 - (m\kappa + 2m\rho\kappa) \epsilon \|d\|^2.
\end{aligned} \tag{20}$$

Since $d_{\mathcal{N}} \in \text{Null}(\nabla c(x)^T)$, Definition 2.1 gives $d_{\mathcal{N}}^T \nabla_{xx}^2 L(x, y(x)) d_{\mathcal{N}} \geq \beta \|d_{\mathcal{N}}\|^2$. With (5), the triangle inequality, and submultiplicity of the matrix 2-norm,

$$\begin{aligned}
\|\nabla_{xx}^2 L(x, y(x))\| &= \left\| \nabla^2 f(x) + \sum_{j \in [m]} \nabla^2[c]_j(x) [y]_j(x) \right\| \\
&\leq \|\nabla^2 f(x)\| + \sum_{j \in [m]} \|\nabla^2[c]_j(x)\| \|y(x)\| \leq \kappa + m\kappa^2.
\end{aligned} \tag{21}$$

At the same time, let us derive a lower bound for $\|\nabla c(x)^T d_{\mathcal{R}}\|$. Let $\nabla c(x)$ have the singular value decomposition $\sum_{i \in [m]} u_i \sigma_i v_i^T$ where $(u_i, \sigma_i, v_i) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m$ for all $i \in [m]$ with $\{u_i\}_{i \in [m]}$ and $\{v_i\}_{i \in [m]}$ being sets of orthonormal vectors. Then, by Assumption 2.3 and the fact that $d_{\mathcal{R}} \in \text{Range}(\nabla c(x))$,

$$\begin{aligned}
\|\nabla c(x)^T d_{\mathcal{R}}\|^2 &= \left\| \sum_{i \in [m]} v_i \sigma_i u_i^T d_{\mathcal{R}} \right\|^2 = \sum_{i \in [m]} \|v_i \sigma_i u_i^T d_{\mathcal{R}}\|^2 = \sum_{i \in [m]} \sigma_i^2 (u_i^T d_{\mathcal{R}})^2 \\
&\geq \sigma_{\min}^2 \sum_{i=1}^m (u_i^T d_{\mathcal{R}})^2 = \sigma_{\min}^2 \|d_{\mathcal{R}}\|^2.
\end{aligned} \tag{22}$$

Combining (19), (20), (21), (22), and the decomposition $d = d_{\mathcal{N}} + d_{\mathcal{R}}$,

$$\begin{aligned}
& d^T \nabla^2 F_\rho(x) d \\
& \geq \beta \|d_{\mathcal{N}}\|^2 + (2\rho\sigma_{\min}^2 - (\kappa + m\kappa^2)) \|d_{\mathcal{R}}\|^2 \\
& \quad - \left(\frac{2\sqrt{m}\kappa}{\sigma_{\min}} + m\kappa + 2m\rho\kappa \right) \epsilon \|d\|^2 \\
& = \left(\beta - \left(\frac{2\sqrt{m}\kappa}{\sigma_{\min}} + m\kappa + 2m\rho\kappa \right) \epsilon \right) \|d_{\mathcal{N}}\|^2 \\
& \quad + \left(2\rho\sigma_{\min}^2 - (\kappa + m\kappa^2) - \left(\frac{2\sqrt{m}\kappa}{\sigma_{\min}} + m\kappa + 2m\rho\kappa \right) \epsilon \right) \|d_{\mathcal{R}}\|^2.
\end{aligned} \tag{23}$$

Combined with the definition of η_ρ in (8), the proof is complete. $\square$ $\square$

We make a few observations regarding Lemma 2.1. First, the bounds on the norms of $c(x)$ and $\nabla F_\rho(x)$ in (9) follow from the fact that x is assumed to satisfy (7) with $\epsilon \leq R$; the factor on the right-hand side of the second inequality in (9) merely accounts for the difference between the gradients of the Lagrangian and Fletcher's augmented Lagrangian. Second, we note that our lower bound on the smallest eigenvalue of the Hessian of Fletcher's augmented Lagrangian in (9) can be related to [5, Theorem 6], which gives a lower bound for ρ to ensure positive semidefiniteness of $\nabla^2 F_\rho(x)$, namely,

$$\rho \geq \max\{0, \lambda_1(\nabla c(x)^\dagger \nabla_{xx}^2 L(x) (\nabla c(x)^\dagger)^T)\}. \quad (24)$$

In particular, (23) with $\epsilon = 0$ yields

$$d^T \nabla^2 F_\rho(x) d \geq \beta \|d_{\mathcal{N}}\|^2 + (2\rho\sigma_{\min}^2 - (\kappa + m\kappa^2)) \|d_{\mathcal{R}}\|^2,$$

so positive semidefiniteness of $\nabla^2 F_\rho(x)$ follows from $\rho \geq \frac{\kappa + m\kappa^2}{2\sigma_{\min}^2}$. This is of the same form as (24) since, as mentioned in the proof of the previous lemma, $\|\nabla c(x)^\dagger\| \leq \frac{1}{\sigma_{\min}}$ and from the definition of κ one has $\|\nabla_{xx}^2 L(x)\| \leq \kappa + m\kappa^2$. Finally, let us make two further observations related to (23). That is, for directions in $\text{Null}(\nabla c(x)^T)$, a small ϵ relative to β is sufficient to ensure positive curvature with respect to $\nabla^2 F_\rho(x)$. However, for directions in $\text{Range}(\nabla c(x))$, the combination of a large ρ and a small ϵ are needed to ensure positive curvature with respect to $\nabla^2 F_\rho(x)$. This occurs since (7) corresponds to sufficiently positive curvature of $\nabla_{xx}^2 L(x, y(x))$ in $\text{Null}(\nabla c(x)^T)$, but to ensure positive curvature of $\nabla^2 F_\rho(x)$ in $\text{Range}(\nabla c(x))$ one needs a large penalty parameter ρ to exploit the curvature of the Hessian of $\|c(x)\|^2$.

We now prove our main theorem of this section.

Theorem 2.1. *Suppose that Assumptions 2.1, 2.2, 2.3, and 2.4 hold and let $\kappa \in \mathbb{R}_{>0}$ satisfy (5). In addition, for some $\beta \in \mathbb{R}_{>0}$, suppose that*

$$\rho > \rho_\beta := \max\left\{\frac{\max\{1, \kappa\}}{\sigma_{\min}}, \frac{\beta + \kappa + m\kappa^2}{2\sigma_{\min}^2}\right\}, \quad (25)$$

and that $x \in \mathbb{R}^n$ satisfies (7), where with $M_\rho \in \mathbb{R}_{>0}$ defined in Assumption 2.4, $(\omega_\rho, \eta_\rho) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ defined in (8), and $\lambda_\rho \in \mathbb{R}_{>0}$ satisfying

$$\lambda_\rho \geq \beta + \lambda_{R,\rho} \quad (\text{where } \lambda_{R,\rho} \text{ is defined in (6)})$$

the tolerance $\epsilon \in \mathbb{R}_{>0}$ satisfies (with δ defined in Assumption 2.3)

$$\epsilon \leq \epsilon_\rho := \min\left\{\frac{\beta}{2\left(\eta_\rho + \frac{4M_\rho\omega_\rho}{\beta} + \left(1 + \frac{2}{\sigma_{\min}}\right)\frac{\kappa}{\sqrt{5}}\right)}, \frac{R}{\omega_\rho}, \frac{\beta\lambda_\rho}{2M_\rho\omega_\rho}, \frac{\lambda_\rho\delta}{2\omega_\rho}\right\}. \quad (26)$$

Furthermore, suppose that $t \in \mathbb{R}_{>0}$ satisfies

$$0 < t \leq t_\rho := \min\left\{\frac{1}{\lambda_\rho}, \frac{1}{\beta - \eta_\rho\epsilon - \frac{4M_\rho\omega_\rho}{\beta}\epsilon}\right\}. \quad (27)$$

Then, for any $\epsilon' \in (0, \epsilon)$ and with the parameters

$$\underbrace{(\epsilon_1, \alpha_{01}, c_1)}_{(\text{notation in [9]})} \equiv \left(\frac{\epsilon'}{\sqrt{5}}, t, \frac{1}{2}\right), \quad (28)$$

the method [9, Algorithm 1] reduces to applying gradient descent with the constant step size t to minimize F_ρ with initial point x . Thus, in at most

$$T = \left\lceil \log_{\frac{4}{4-\beta t}} \frac{\sqrt{5}\omega_\rho\epsilon}{\epsilon'} \right\rceil \text{ iterations (where } 4 - \beta t > 0) \quad (29)$$

the method gives a point satisfying (7) with (ϵ, β) replaced by (ϵ', β') , where

$$\beta' := \beta - \eta_\rho \epsilon - \frac{4M_\rho \omega_\rho}{\beta} \epsilon - \left(1 + \frac{2}{\sigma_{\min}}\right) \frac{\kappa}{\sqrt{5}} \epsilon' \geq \frac{1}{2} \beta. \quad (30)$$

Proof. Let us begin by showing that (26) implies (30) as well as positivity of other quantities that are used throughout the proof, including the upper bound for the step size in (27). By (26) and $\epsilon' \in (0, \epsilon)$, one finds

$$\begin{aligned} \beta' &= \beta - \eta_\rho \epsilon - \frac{4M_\rho \omega_\rho}{\beta} \epsilon - \left(1 + \frac{2}{\sigma_{\min}}\right) \frac{\kappa}{\sqrt{5}} \epsilon' \\ &\geq \beta - \left(\eta_\rho + \frac{4M_\rho \omega_\rho}{\beta} + \left(1 + \frac{2}{\sigma_{\min}}\right) \frac{\kappa}{\sqrt{5}}\right) \epsilon \geq \frac{1}{2} \beta, \end{aligned}$$

which shows that (30) holds. From these inequalities, one also finds that

$$\beta_{\rho, \epsilon} := \beta - \eta_\rho \epsilon \geq \frac{1}{2} \beta \quad \text{and} \quad \tilde{\beta}_{\rho, \epsilon} := \beta - \eta_\rho \epsilon - \frac{4M_\rho \omega_\rho}{\beta} \epsilon \geq \frac{1}{2} \beta, \quad (31)$$

which shows that the upper bound in (27) is positive.

Let us now consider gradient descent applied to minimize F_ρ from x with step size t , where $\rho \in \mathbb{R}_{>0}$ satisfies (25) and $t \in \mathbb{R}_{>0}$ satisfies (27), which by combining (26) and (31) can be seen to satisfy

$$0 < t \leq \frac{1}{\lambda_\rho} \leq \frac{\beta}{2M_\rho \omega_\rho \epsilon} \leq \frac{\tilde{\beta}_{\rho, \epsilon}}{M_\rho \omega_\rho \epsilon} \quad \text{and} \quad 0 < t \leq \frac{1}{\tilde{\beta}_{\rho, \epsilon}}. \quad (32)$$

Once this is completed, we confirm that [9, Algorithm 1] with the parameters in (28) indeed reduces to such an application of gradient descent. Let $x^0 \leftarrow x$ and consider the iterative method defined for all $i = 0, 1, 2, \dots$ by

$$x^{i+1} \leftarrow x^i - t \nabla F_\rho(x^i) \quad \text{until} \quad \|\nabla F_\rho(x^{i+1})\| \leq \frac{\epsilon'}{\sqrt{5}}. \quad (33)$$

Our first aim is to show that $x^0 \in \mathcal{C}_R$, $\|\nabla F_\rho(x^0)\| \leq \omega_\rho \epsilon$, and $\lambda_n(\nabla^2 F_\rho(x^0)) \geq \beta_{\rho, \epsilon}$, and that, for any $i \in \mathbb{N}$ with $i \geq 1$, one finds that

$$\begin{aligned} x^i &\in \mathcal{C}_R, \\ \|\nabla F_\rho(x^i)\| &\leq \left(1 - \frac{1}{2} \tilde{\beta}_{\rho, \epsilon} t\right)^i \|\nabla F_\rho(x^0)\| \\ \text{and } \lambda_n(\nabla^2 F_\rho(x^i)) &\geq \beta_{\rho, \epsilon} - M_\rho t \|\nabla F_\rho(x^0)\| \sum_{l=0}^{i-1} \left(1 - \frac{1}{2} \tilde{\beta}_{\rho, \epsilon} t\right)^l \geq \tilde{\beta}_{\rho, \epsilon}, \end{aligned} \quad (34)$$

where $1 - \frac{1}{2} \tilde{\beta}_{\rho, \epsilon} t > 0$ by (32). First, for $i = 0$, by (25) one has that $\rho \geq \frac{\beta + \kappa + m\kappa^2}{2\sigma_{\min}^2}$, so with (31) one has

$$2\sigma_{\min}^2 \rho - (\kappa + m\kappa^2) - \eta_\rho \epsilon \geq \beta - \eta_\rho \epsilon = \beta_{\rho, \epsilon}. \quad (35)$$

Since $x^0 = x$ satisfies (7), it follows from (26) and $\omega_\rho \geq 1$ that $\|c(x^0)\| \leq R$, which means $x^0 \in \mathcal{C}_R$, and it follows from Lemma 2.1, (26), and (35) that

$$\begin{aligned} \|\nabla F_\rho(x^0)\| &\leq \omega_\rho \epsilon \\ \text{and } \lambda_n(\nabla^2 F_\rho(x^0)) &\geq \min\{\beta - \eta_\rho \epsilon, 2\sigma_{\min}^2 \rho - (\kappa + m\kappa^2) - \eta_\rho \epsilon\} \\ &\geq \beta_{\rho, \epsilon}, \end{aligned} \quad (36)$$

as claimed. Now consider arbitrary $i \in \mathbb{N}$ with $i \geq 1$ such that (34) holds. To complete the induction, our aim is to prove that all of (34) holds with i replaced by $i + 1$ as well. Denote $G_i := \nabla F_\rho(x^i)$ for all $i \geq 0$. Furthermore, note that, by (26) and the fact that $t \leq 1/\lambda_\rho$, one has

$$\|x^{i+1} - x^i\| = t\|G_i\| \leq t\|G_0\| \leq t\omega_\rho\epsilon \leq \frac{\omega_\rho\epsilon}{\lambda_\rho} \leq \frac{\delta}{2},$$

meaning that the line segment $[x^i, x^{i+1}]$ lies in $\hat{\mathcal{C}}_R$ and Assumption 2.4 applies along the segment. Next, one has from Assumptions 2.1 and 2.4 that

$$\begin{aligned} \|G_{i+1}\| &\leq \|G_i + \nabla^2 F_\rho(x^i)(x^{i+1} - x^i)\| + \frac{1}{2}M_\rho\|x^{i+1} - x^i\|^2 \\ &= \|G_i - t\nabla^2 F_\rho(x^i)G_i\| + \frac{1}{2}M_\rho t^2\|G_i\|^2 \\ &\leq \|I - t\nabla^2 F_\rho(x^i)\|\|G_i\| + \frac{1}{2}M_\rho t^2\|G_i\|^2. \end{aligned} \quad (37)$$

By $x^i \in \mathcal{C}_R$ from (34), (6), $\lambda_\rho \geq \lambda_{R,\rho}$, and (32), one has $t\lambda_1(\nabla^2 F_\rho(x^i)) \leq t\lambda_\rho \leq 1$, which with (34) gives

$$\begin{aligned} &\|I - t\nabla^2 F_\rho(x^i)\| \\ &= \max_{v \in \mathbb{R}^n \text{ s.t. } \|v\|=1} |v^T v - tv^T \nabla^2 F_\rho(x^i)v| \\ &= \max \{ |1 - t\lambda_n(\nabla^2 F_\rho(x^i))|, |1 - t\lambda_1(\nabla^2 F_\rho(x^i))| \} \\ &= 1 - t\lambda_n(\nabla^2 F_\rho(x^i)) \\ &\leq 1 - t\tilde{\beta}_{\rho,\epsilon}. \end{aligned} \quad (38)$$

At the same time, combining (32), (34), and (36) yields

$$M_\rho t\|G_i\| \leq M_\rho t\|G_0\| \leq M_\rho t\omega_\rho\epsilon \leq \tilde{\beta}_{\rho,\epsilon}. \quad (39)$$

Now combining (32), (37), (38), and (39) one obtains that

$$\begin{aligned} \|G_{i+1}\| &\leq \left(1 - t\tilde{\beta}_{\rho,\epsilon} + \frac{1}{2}M_\rho t^2\|G_i\|\right) \|G_i\| \\ &\leq \left(1 - \frac{1}{2}t\tilde{\beta}_{\rho,\epsilon}\right) \|G_i\|. \end{aligned} \quad (40)$$

This gives the first inequality in (34), and in fact shows that $\|G_{i+1}\| \leq \|G_i\|$ for all $i \in \mathbb{N}$. Also, from [13, Theorem 6.6] (or see [15, Eq. (3)]), one finds

$$\begin{aligned} &|\lambda_n(\nabla^2 F_\rho(x^{i+1})) - \lambda_n(\nabla^2 F_\rho(x^i))| \\ &\leq \|\nabla^2 F_\rho(x^{i+1}) - \nabla^2 F_\rho(x^i)\| \leq M_\rho\|x^{i+1} - x^i\| = M_\rho t\|G_i\|. \end{aligned}$$

Combining (26), (31), (32), (34), and (40), one now obtains that

$$\begin{aligned} &\lambda_n(\nabla^2 F_\rho(x^{i+1})) \\ &\geq \lambda_n(\nabla^2 F_\rho(x^i)) - |\lambda_n(\nabla^2 F_\rho(x^{i+1})) - \lambda_n(\nabla^2 F_\rho(x^i))| \\ &\geq \beta_{\rho,\epsilon} - M_\rho t\|G_0\| \sum_{l=0}^{i-1} \left(1 - \frac{1}{2}\tilde{\beta}_{\rho,\epsilon}t\right)^l - M_\rho t\|G_i\| \\ &\geq \beta_{\rho,\epsilon} - M_\rho t\|G_0\| \sum_{l=0}^i \left(1 - \frac{1}{2}\tilde{\beta}_{\rho,\epsilon}t\right)^l \end{aligned}$$

$$\begin{aligned}
&\geq \beta_{\rho,\epsilon} - M_{\rho}t \|G_0\| \sum_{l=0}^{\infty} \left(1 - \frac{1}{2}\tilde{\beta}_{\rho,\epsilon}t\right)^l \\
&= \beta_{\rho,\epsilon} - M_{\rho}t \|G_0\| \frac{2}{\tilde{\beta}_{\rho,\epsilon}t} \\
&\geq \beta_{\rho,\epsilon} - \frac{4M_{\rho}\omega_{\rho}\epsilon}{\beta} = \tilde{\beta}_{\rho,\epsilon}.
\end{aligned} \tag{41}$$

All that remains to complete the induction is to show that $x^{i+1} \in \mathcal{C}_R$. Toward this end, let us employ [9, Corollary 2.7]. Employing this result requires that the singular values of ∇c are bounded away from zero and that $\mathcal{C}_R$ is compact, which hold here by Assumptions 2.2 and 2.3. It also requires that

$$\rho > \max_{x \in \mathcal{C}_R} \frac{\max\{1, \|\nabla y(x)\|\}}{\sigma_{\min}(\nabla c(x))}. \tag{42}$$

This also holds here since for any $x \in \mathcal{C}_R$ one has $\sigma_{\min}(\nabla c(x)) \geq \sigma_{\min}$ by Assumption 2.3 and $\|\nabla y(x)\| \leq \kappa$ by (5), which when combined with (25) shows that (42) holds. Now since (40) and (41) give

$$\|G_{i+1}\| \leq \tilde{\epsilon}_{i+1} := \left(1 - \frac{1}{2}\tilde{\beta}_{\rho,\epsilon}t\right)^{i+1} \|G_0\| \quad \text{and} \quad \nabla^2 F_{\rho}(x^{i+1}) \succeq \tilde{\beta}_{\rho,\epsilon}I, \tag{43}$$

by [9, Corollary 2.7] one has x^{i+1} is an $(\tilde{\epsilon}_{i+1}, 2\tilde{\epsilon}_{i+1}, -\tilde{\beta}_{\rho,\epsilon} + (1 + \frac{2}{\sigma_{\min}})\kappa\tilde{\epsilon}_{i+1})$ stationary point of (1) in the sense (defined in [9]) that

$$\begin{aligned}
&\|c(x^{i+1})\| \leq \tilde{\epsilon}_{i+1}, \quad \|\nabla_x L(x^{i+1}, y(x^{i+1}))\| \leq 2\tilde{\epsilon}_{i+1}, \\
&\text{and } d^T \nabla_{xx}^2 L(x^{i+1}, y(x^{i+1}))d \geq \left(\tilde{\beta}_{\rho,\epsilon} - \left(1 + \frac{2}{\sigma_{\min}}\right)\kappa\tilde{\epsilon}_{i+1}\right) \|d\|^2 \\
&\text{for all } d \in \text{Null}(\nabla c(x^{i+1})^T).
\end{aligned} \tag{44}$$

Noting that $1 - \frac{1}{2}\tilde{\beta}_{\rho,\epsilon}t < 1$, one has $\tilde{\epsilon}_{i+1} \leq \|G_0\|$. Combined with the fact that $\|G_0\| \leq \omega_{\rho}\epsilon \leq R$ by (26), one finds that $\|c(x^{i+1})\| \leq \tilde{\epsilon}_{i+1} \leq R$. Thus, $x^{i+1} \in \mathcal{C}_R$ has been proved, and the desired induction to prove (34) is complete.

Let us now give an upper bound for the number of iterations until the gradient descent method defined by (33) terminates. Let $T' \in \mathbb{N}$ be the smallest positive integer such that $\tilde{\epsilon}_{T'} \leq \frac{\epsilon'}{\sqrt{5}}$. Then,

$$T' \leq \left\lceil \log_{1 - \frac{1}{2}\tilde{\beta}_{\rho,\epsilon}t} \frac{\epsilon'}{\sqrt{5}\|G_0\|} \right\rceil, \tag{45}$$

and since $\|G_{T'}\| \leq \tilde{\epsilon}_{T'}$ one has $T \leq T'$. Since the definition of λ_{ρ} , (31), (32), and (34) imply that

$$4 - \beta t \geq 4 - \frac{\beta}{\lambda_{\rho}} \geq 4 - \frac{\beta}{\lambda_n(\nabla^2 F_{\rho}(x^T))} \geq 4 - \frac{\beta}{\tilde{\beta}_{\rho,\epsilon}} \geq 2,$$

one finds with (31) that

$$\frac{4}{4 - \beta t} = \frac{1}{1 - \frac{1}{4}\beta t} \leq \frac{1}{1 - \frac{1}{2}\tilde{\beta}_{\rho,\epsilon}t},$$

so by considering (45) and $\|G_0\| \leq \omega_{\rho}\epsilon$ one finds that

$$\begin{aligned}
\log_{1 - \frac{1}{2}\tilde{\beta}_{\rho,\epsilon}t} \frac{\epsilon'}{\sqrt{5}\|G_0\|} &= \log_{\frac{1}{1 - \frac{1}{2}\tilde{\beta}_{\rho,\epsilon}t}} \frac{\sqrt{5}\|G_0\|}{\epsilon'} \\
&\leq \log_{\frac{1}{1 - \frac{1}{4}\beta t}} \frac{\sqrt{5}\|G_0\|}{\epsilon'} \leq \log_{\frac{4}{4 - \beta t}} \frac{\sqrt{5}\omega_{\rho}\epsilon}{\epsilon'},
\end{aligned}$$

which gives the upper bound for $T \leq T'$ stated in the theorem.

Next, let us show that x^T satisfies (7) with (ϵ, β) replaced by (ϵ', β') . Note that $\|G_T\| \leq \frac{\epsilon'}{\sqrt{5}}$ by the termination condition in (33) and $\lambda_n(\nabla^2 F_\rho(x^T)) \geq \tilde{\beta}_{\rho, \epsilon}$ by (41). Therefore, [9, Corollary 2.7] applied at x^T gives

$$\begin{aligned} \|c(x^T)\| &\leq \|G_T\|, \quad \|\nabla_x L(x^T, y(x^T))\| \leq 2\|G_T\|, \\ \text{and } d^T \nabla_{xx}^2 L(x^T, y(x^T))d &\geq \left(\tilde{\beta}_{\rho, \epsilon} - \left(1 + \frac{2}{\sigma_{\min}}\right) \kappa \|G_T\| \right) \|d\|^2 \\ \text{for all } d &\in \text{Null}(\nabla c(x^T)^T). \end{aligned} \tag{46}$$

From this, the termination condition in (33), and the properties (34),

$$\begin{aligned} &\|\nabla L(x^T, y(x^T))\| \\ &= \sqrt{\|\nabla_x L(x^T, y(x^T))\|^2 + \|\nabla_{\hat{y}} L(x^T, y(x^T))\|^2} \\ &\leq \sqrt{5}\|G_T\| \leq \sqrt{5} \left(\frac{\epsilon'}{\sqrt{5}} \right) = \epsilon'. \end{aligned}$$

At the same time, combining (31), (46), and $\|G_T\| \leq \frac{\epsilon'}{\sqrt{5}}$ one has

$$\begin{aligned} &\tilde{\beta}_{\rho, \epsilon} - \left(1 + \frac{2}{\sigma_{\min}}\right) \kappa \|G_T\| \\ &\geq \tilde{\beta}_{\rho, \epsilon} - \left(1 + \frac{2}{\sigma_{\min}}\right) \frac{\kappa}{\sqrt{5}} \epsilon' \\ &= \beta - \eta_\rho \epsilon - \frac{4M_\rho \omega_\rho}{\beta} \epsilon - \left(1 + \frac{2}{\sigma_{\min}}\right) \frac{\kappa}{\sqrt{5}} \epsilon' \geq \frac{1}{2}\beta. \end{aligned}$$

which gives (30). Overall, x^T satisfies (7) with (ϵ', β') , as claimed.

Lastly, let us explain that [9, Algorithm 1] with the parameter choice in (28) reduces to the gradient descent method (33). In contrast to (33) itself, [9, Algorithm 1] first moves along negative gradient directions until $\|\nabla F_\rho\| \leq \epsilon_1$. Then, if the curvature is sufficiently negative, it starts a second-order phase by moving along a direction computed by using second-order information. The stepsizes for both directions are determined by using a line-search strategy. From (31) and (41), one can observe $\lambda_n(\nabla^2 F_\rho(x^{i+1})) \geq \frac{1}{2}\beta > 0$ for all i , and hence the second-order phase will never be triggered. Moreover, let us show that $\lambda_1(\nabla^2 F_\rho(x)) \leq \lambda_\rho$ for all x on the line segment between x^i and x^{i+1} . For any $s \in [0, 1]$, let $x(s) = x^i + s(x^{i+1} - x^i)$. Then, by [13, Theorem 6.6], Assumption 2.4, $x^i \in \mathcal{C}_R$ from (34) along with the definition of $\lambda_{R, \rho}$ in (6), the fact that $\|x^{i+1} - x^i\| = t\|G_i\|$, (39), the fact that $\tilde{\beta}_{\rho, \epsilon} \leq \beta$ by (31), and $\lambda_\rho \geq \beta + \lambda_{R, \rho}$, one finds that

$$\begin{aligned} \lambda_1(\nabla^2 F_\rho(x(s))) &\leq \lambda_1(\nabla^2 F_\rho(x^i)) + \|\nabla^2 F_\rho(x(s)) - \nabla^2 F_\rho(x^i)\| \\ &\leq \lambda_{R, \rho} + M_\rho s \|x^{i+1} - x^i\| \\ &\leq \lambda_{R, \rho} + M_\rho t \|G_i\| \\ &\leq \lambda_{R, \rho} + \tilde{\beta}_{\rho, \epsilon} \leq \lambda_{R, \rho} + \beta \leq \lambda_\rho. \end{aligned}$$

Therefore, from this bound, Taylor's theorem, and t in (32), one has

$$\begin{aligned} F_\rho(x^{i+1}) &\leq F_\rho(x^i) + \nabla F_\rho(x^i)^T (x^{i+1} - x^i) + \frac{1}{2} \lambda_\rho \|x^{i+1} - x^i\|^2 \\ &= F_\rho(x^i) - t \|\nabla F_\rho(x^i)\|^2 + \frac{1}{2} \lambda_\rho t^2 \|\nabla F_\rho(x^i)\|^2 \\ &\leq F_\rho(x^i) - \frac{1}{2} t \|\nabla F_\rho(x^i)\|^2 \end{aligned}$$

$$= F_\rho(x^i) - c_1 \alpha_{01} \|\nabla F_\rho(x^i)\|^2. \quad (47)$$

Hence, the line-search termination condition in iteration i of [9, Algorithm 2] is satisfied at x^{i+1} , so [9, Algorithm 1] reduces to gradient descent with a fixed step size t . Moreover, the choice ϵ_1 in (28) ensures the output of [9, Algorithm 1], say $\tilde{x}^T$, gives $\|\nabla F_\rho(\tilde{x}^T)\| \leq \epsilon_1 = \frac{\epsilon'}{\sqrt{5}}$, which is exactly the termination condition in (33). $\square$ $\square$

3 Application: Progressive Sampling

The purpose of this section is to show that Theorem 2.1 can be leveraged to prove a strong complexity guarantee in a specific setting of interest. In particular, we consider the setting of progressive sampling as proposed in [1] to solve instances of problem (1) when the objective and constraint functions are defined through expectations, which in turn are approximated by a large-scale sample average approximation (SAA) strategy. We show that when [9, Algorithm 1] is employed as the subproblem solver, an improved sample complexity can be obtained by a progressive sampling strategy to solve the SAA problem as compared to an approach that solves the full SAA problem directly.

The SAA problem type that we consider in this section can be written as

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \equiv \frac{1}{N} \sum_{i \in [N]} f_i(x) \\ \text{s.t.} \quad & c(x) \equiv \frac{1}{N} \sum_{i \in [N]} c_i(x) = 0, \end{aligned} \quad (48)$$

where, among other properties, the objective function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and constraint function $c : \mathbb{R}^n \rightarrow \mathbb{R}^m$ satisfy Assumption 2.1. Generally speaking, the objective and constraint functions could be defined with respect to different numbers of terms, but to simplify our notation we assume that f and c are each composed of N terms. A main challenge of solving instances of problem (48) is that N can be quite large, meaning that each evaluation of f , c , or either of their first-order derivatives requires taking a sum over N terms.

The progressive sampling strategy proposed in [1] has been designed for such a context. The main hope is to exploit the fact that many of the objective and constraint function terms may be relatively similar to each other, say, in settings when the objective and constraints are averages over functions defined by independent and identically distributed draws of a random variable. The main idea of the approach is to solve (48) in a progressive manner by first selecting a subset of terms (e.g., at random), solving the corresponding sampled problem to moderate accuracy, then increasing the sample size and tightening the accuracy tolerance in an iterative manner, where in each case the solve is warm-started by the approximate solution obtained for the previous sample. Denoting $\mathcal{S}_k \subseteq [N]$, a corresponding approximate objective function as $f_{\mathcal{S}_k} : \mathbb{R}^n \rightarrow \mathbb{R}$, and a corresponding approximate constraint function as $c_{\mathcal{S}_k} : \mathbb{R}^n \rightarrow \mathbb{R}^m$, one can write a subproblem of progressive sampling as

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f_{\mathcal{S}_k}(x) \quad \text{s.t.} \quad c_{\mathcal{S}_k}(x) = 0, \\ \text{where} \quad & f_{\mathcal{S}_k}(x) = \frac{1}{|\mathcal{S}_k|} \sum_{i \in \mathcal{S}_k} f_i(x) \quad \text{and} \quad c_{\mathcal{S}_k}(x) = \frac{1}{|\mathcal{S}_k|} \sum_{i \in \mathcal{S}_k} c_i(x). \end{aligned} \quad (49)$$

A statement of the progressive sampling method from [1] (assuming equal sample sizes for the objective and constraint functions) is given as Algorithm 1. Also, a simplified statement of [9, Algorithm 1] is given as Algorithm 2. (The step-size selection mechanisms that may be employed by the algorithm are specified in our subsequent analysis.) Analogously to Definition 2.2, for any $\mathcal{S} \subseteq [N]$ the algorithm defines $F_{\mathcal{S},\rho} : \hat{\mathcal{C}}_{\mathcal{S},R} \rightarrow \mathbb{R}$ (see Assumption 3.4 below for the definition of $\hat{\mathcal{C}}_{\mathcal{S},R}$), which for all $x \in \hat{\mathcal{C}}_{\mathcal{S},R}$ is given by

$$F_{\mathcal{S},\rho}(x) = f_{\mathcal{S}}(x) + c_{\mathcal{S}}(x)^T y_{\mathcal{S}}(x) + \rho \|c_{\mathcal{S}}(x)\|^2,$$

where analogously to (4), for any $x \in \hat{\mathcal{C}}_{S,R}$, the multiplier is given by

$$y_S(x) = -\nabla c_S(x)^\dagger \nabla f_S(x). \quad (50)$$

Let us also introduce for our analysis, as in (2), the Lagrangian

$$L_S(x, \hat{y}) = f_S(x) + c_S(x)^T \hat{y}.$$

For these quantities, we have $F \equiv F_{[N]}$, $y \equiv y_{[N]}$, and $L \equiv L_{[N]}$.

Algorithm 1 Progressive Sampling Algorithm for Solving (48)

Require: Initial sample size $p_1 \in [N]$, sample increase factor $\theta \in (1, \infty)$, initial point $x_0 \in \mathbb{R}^n$, iteration limit $K = \lceil \log_\theta \frac{N}{p_1} \rceil + 1$, and tolerances $\{(\epsilon_k, \zeta_k)\}_{k=1}^K \subset \mathbb{R}_{>0} \times \mathbb{R}_{>0}$

- 1: set $\mathcal{S}_0 \leftarrow \emptyset$
- 2: **for** $k \in [K]$ **do**
- 3: choose $\mathcal{S}_k \supseteq \mathcal{S}_{k-1}$ with $|\mathcal{S}_k| = p_k$
- 4: choose $\rho_k \in \mathbb{R}_{>0}$
- 5: using x_{k-1} , $\mathcal{S}_k$, ρ_k , and (ϵ_k, ζ_k) , call Algorithm 2 to obtain x_k
- 6: set $p_{k+1} \leftarrow \min\{\theta p_k, N\}$
- 7: **end for**
- 8: **return** $(x_K, y(x_K))$

Algorithm 2 Adapted [9, Algorithm 1] for Solving (49)

Require: Initial point $x_{k-1} \in \mathbb{R}^n$, sample set $\mathcal{S}_k \subseteq [N]$, penalty parameter $\rho_k \in \mathbb{R}_{>0}$, and tolerances $(\epsilon_k, \zeta_k) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ from Algorithm 1

- 1: set $i \leftarrow 0$ and $x_{k-1}^0 \leftarrow x_{k-1}$.
- 2: **for** $i \in \mathbb{N}$ **do**
- 3: **if** $\|\nabla F_{\mathcal{S}_k, \rho_k}(x_{k-1}^i)\| > \epsilon_k$ **then**
- 4: set $x_{k-1}^{i+1} \leftarrow x_{k-1}^i - t_g \nabla F_{\mathcal{S}_k, \rho_k}(x_{k-1}^i)$ for some $t_g \in \mathbb{R}_{>0}$
- 5: **else if** $\lambda_n(\nabla^2 F_{\mathcal{S}_k, \rho_k}(x_{k-1}^i)) < -\zeta_k$ **then**
- 6: set $d \in \mathbb{R}^n$ with $\|d\| = 1$, $d^T \nabla^2 F_{\mathcal{S}_k, \rho_k}(x_{k-1}^i) d < -\zeta_k$, and $d^T \nabla F_{\mathcal{S}_k, \rho_k}(x_{k-1}^i) \leq 0$
- 7: set $x_{k-1}^{i+1} \leftarrow x_{k-1}^i + t_H d$ for some $t_H \in \mathbb{R}_{>0}$
- 8: **else**
- 9: set $x_k \leftarrow x_{k-1}^i$
- 10: **return** x_k
- 11: **end if**
- 12: **end for**

For our analysis of Algorithm 1, we make the following assumptions. (These assumptions are consistent with those in [1], so making them allows us to employ the results up through Theorem 3.8 in that paper.) First, extending Assumption 2.1, we assume the following.

Assumption 3.1. *The objective function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and constraint function $c : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are twice-continuously differentiable. Moreover, there exists $(\kappa_{\nabla f}, \kappa_{\nabla c}, \kappa_{\nabla^2 f}, \kappa_{\nabla^2 c}) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ such that, for all $x \in \mathbb{R}^n$ and $j \in [m]$, one has $\|\nabla f(x)\| \leq \kappa_{\nabla f}$, $\|\nabla c(x)\| \leq \kappa_{\nabla c}$, $\|\nabla^2 f(x)\| \leq \kappa_{\nabla^2 f}$, and $\|\nabla^2 [c]_j(x)\| \leq \kappa_{\nabla^2 c}$, where recall that $[c]_j$ is the j th component of c .*

To connect Assumption 3.1 to properties of the approximation functions, we make the following loose assumption about the component functions. The main consequence of this assumption can be seen in [1, Lemma 3.2], which shows that by increasing the sample size one obtains approximation function and derivative values that more closely approximate those of f and c .

Assumption 3.2. *There exists $(\gamma_f, \gamma_c, \gamma_{\nabla f}, \gamma_{\nabla c}, \gamma_{\nabla^2 f}, \gamma_{\nabla^2 c}) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ such that, for all $(x, j) \in \mathbb{R}^n \times [m]$, one has*

$$\begin{aligned} \frac{1}{N} \sum_{i \in [N]} |f_i(x) - f(x)|^2 &\leq \gamma_f, \\ \frac{1}{N} \sum_{i \in [N]} \|c_i(x) - c(x)\|^2 &\leq \gamma_c, \\ \frac{1}{N} \sum_{i \in [N]} \|\nabla f_i(x) - \nabla f(x)\|^2 &\leq \gamma_{\nabla f}, \\ \frac{1}{N} \sum_{i \in [N]} \|\nabla c_i(x) - \nabla c(x)\|^2 &\leq \gamma_{\nabla c} \|\nabla c(x)\|^2, \\ \frac{1}{N} \sum_{i \in [N]} \|\nabla^2 f_i(x) - \nabla^2 f(x)\|^2 &\leq \gamma_{\nabla^2 f} \\ \text{and } \frac{1}{N} \sum_{i \in [N]} \|\nabla^2 [c_i]_j(x) - \nabla^2 [c]_j(x)\|^2 &\leq \gamma_{\nabla^2 c}. \end{aligned}$$

Next, extending Assumptions 2.2, 2.3, and 2.4 for any sample problem encountered by the algorithm, we make the following assumptions.

Assumption 3.3. *There exists $R \in \mathbb{R}_{>0}$ such that, for any $\mathcal{S} \subseteq [N]$ with $|\mathcal{S}| \geq p_1$, the set $\mathcal{C}_{\mathcal{S}, R} := \{x \in \mathbb{R}^n : \|c_{\mathcal{S}}(x)\| \leq R\}$ is compact.*

Assumption 3.4. *There exists $\delta \in \mathbb{R}_{>0}$ (uniform over $\mathcal{S} \subseteq [N]$ with $|\mathcal{S}| \geq p_1$), a bounded open set $\hat{\mathcal{C}}_{\mathcal{S}, R}$ containing $\mathcal{C}_{\mathcal{S}, R} + \{v \in \mathbb{R}^n : \|v\| \leq \delta\}$, and a positive real number $\sigma_{\min} \in \mathbb{R}_{>0}$ such that, for any $\mathcal{S} \subseteq [N]$ with $|\mathcal{S}| \geq p_1$ and $x \in \hat{\mathcal{C}}_{\mathcal{S}, R}$, the constraint Jacobian has $\sigma_m(\nabla c_{\mathcal{S}}(x)^T) \geq \sigma_{\min}$.*

Assumption 3.5. *Along with Assumptions 3.1, 3.2, 3.3, and 3.4, for any $\mathcal{S} \subseteq [N]$ with $|\mathcal{S}| \geq p_1$ the function $y_{\mathcal{S}}$ defined by (50) is twice-continuously differentiable over $\hat{\mathcal{C}}_{\mathcal{S}, R}$, and over $\hat{\mathcal{C}}_{\mathcal{S}, R}$ the functions $f_{\mathcal{S}} + c_{\mathcal{S}}^T y_{\mathcal{S}}$ and $\|c_{\mathcal{S}}\|^2$ have Hessian functions that are Lipschitz continuous in the sense that there exists $(M_f, M_c) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ (independent of $\mathcal{S}$) such that, for all $(x, \bar{x}) \in \hat{\mathcal{C}}_{\mathcal{S}, R} \times \hat{\mathcal{C}}_{\mathcal{S}, R}$, one has that*

$$\begin{aligned} \|\nabla^2(f_{\mathcal{S}} + c_{\mathcal{S}}^T y_{\mathcal{S}})(x) - \nabla^2(f_{\mathcal{S}} + c_{\mathcal{S}}^T y_{\mathcal{S}})(\bar{x})\| &\leq M_f \|x - \bar{x}\| \\ \text{and } \|\nabla^2(\|c_{\mathcal{S}}\|^2)(x) - \nabla^2(\|c_{\mathcal{S}}\|^2)(\bar{x})\| &\leq M_c \|x - \bar{x}\|. \end{aligned}$$

Under these conditions, for any $\rho \in \mathbb{R}_{>0}$, define $M_{\rho} := M_f + \rho M_c$.

Our last assumption is the one that allows us to leverage the local-linear convergence rate of Algorithm 2 in the neighborhood of certain stationary points (as shown in the previous section) in order to prove an overall improved sample complexity bound for a progressive sampling strategy. It builds on the concept of a Morse function [11] to that of a Morse program [7]. As discussed in [1, Remark 3.2], the assumption is relatively mild in the context of constrained optimization problems with isolated local minima [10].

Assumption 3.6. *For some pair of positive real numbers $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$, problem (48) is (α, β) -strongly Morse in the sense that, for any $x \in \mathbb{R}^n$, if the pair $(x, y(x))$ satisfies $\|\nabla L(x, y(x))\| \leq \alpha$, then*

$$|d^T \nabla_{xx}^2 L(x, y(x)) d| \geq \beta \|d\|^2 \quad \text{for all } d \in \text{Null}(\nabla c(x)^T).$$

For our analysis, we also introduce the useful quantity

$$\xi_{\mathcal{S}} := \sqrt{\frac{N(N - |\mathcal{S}|)}{|\mathcal{S}|^2}} \in [0, \sqrt{N(N - 1)}] \quad \text{for all nonempty } \mathcal{S} \subseteq [N]. \quad (51)$$

The lemma below shows conditions under which we can carry over the theoretical guarantees from [9] for our employment of Algorithm 2 within the context of Algorithm 1. In the lemma, we use $\tau_{\mathcal{S}_1} \in \mathbb{R}_{>0}$ to denote the positive real number “ C ” introduced in [9, Corollary 2.7], which depends on second-order derivatives of $\{[c_{\mathcal{S}_1}]_j\}_{j \in [m]}$ and $\{[y_{\mathcal{S}_1}]_j\}_{j \in [m]}$ over $\mathcal{C}_{\mathcal{S}_1, R}$. We use this notation to indicate the dependence of this quantity on the sample set $\mathcal{S}_1$.

Lemma 3.1. *Suppose that Assumptions 3.1, 3.2, 3.3, 3.4 and 3.5 hold, a sample set $\mathcal{S}_1 \subseteq [N]$ is given with $|\mathcal{S}_1| \geq p_1$, and an initial point $x_0 \in \mathcal{C}_{\mathcal{S}_1, R}$ is given. Then, there exists $\hat{\rho}_1 \in \mathbb{R}_{>0}$ such that the requirements of [9, Theorem 3.5] hold for all $\rho_1 \geq \hat{\rho}_1$. Consequently, for any such ρ_1 there exists $(u_{\mathcal{S}_1, 1}, u_{\mathcal{S}_1, 2}) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ such that for any $(\bar{\epsilon}, \bar{\zeta}) \in (0, \frac{\sqrt{5}}{2}R] \times (0, 1]$ one has that Algorithm 2 with inputs x_0 , $\mathcal{S}_1$, ρ_1 , and $(\epsilon_1, \zeta_1) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ satisfying*

$$\epsilon_1 \leq \bar{\epsilon}_1(\bar{\epsilon}, \bar{\zeta}) := \min \left\{ \frac{\sqrt{5}\bar{\epsilon}}{5}, \frac{\bar{\zeta}}{4\tau_{\mathcal{S}_1}} \right\} \quad (52)$$

$$\text{and } \zeta_1 \leq \bar{\zeta}_1(\bar{\epsilon}, \bar{\zeta}) := \frac{\bar{\zeta}}{2} + \tau_{\mathcal{S}_1} \bar{\epsilon}_1(\bar{\epsilon}, \bar{\zeta})$$

(using either the line-search methods in [9, Algorithms 2–3] or setting the step sizes according to the lower bounds in [9, Lemmas 3.2 and 3.4]) terminates in a number of iterations that is at most

$$T_{\mathcal{S}_1} = \max\{u_{\mathcal{S}_1, 1}\epsilon_1^{-2}, u_{\mathcal{S}_1, 2}\zeta_1^{-3}\}. \quad (53)$$

Moreover, the final iterate is $(\bar{\epsilon}, \bar{\zeta})$ -stationary (see Definition 2.1).

Proof. Consider arbitrary $\mathcal{S}_1$ and x_0 as stated. Our first aim is to prove that the requirements of [9, Theorem 3.5] hold for all ρ_1 above a threshold $\hat{\rho}_1 \in \mathbb{R}_{>0}$. This requires showing that Assumptions A1–A5 in [9] hold. Assumption A1 requires the existence of $\hat{R} \in \mathbb{R}_{>0}$ and $\hat{\sigma} \in \mathbb{R}_{>0}$ such that for all $x \in \mathbb{R}^n$ with $\|c_{\mathcal{S}_1}(x)\| \leq \hat{R}$ one has $\sigma_{\min}(\nabla c_{\mathcal{S}_1}(x)^T) \geq \hat{\sigma}$. This holds in our present setting with Assumption 3.4. Assumption A2 requires the existence of R as stated in Assumption 3.3. Assumption A3 requires the existence of $\eta_{\mathcal{S}_1} \in \mathbb{R}_{>0}$ such that

$$\|c_{\mathcal{S}_1}(x+d) - c_{\mathcal{S}_1}(x) - \nabla c_{\mathcal{S}_1}(x)^T d\| \leq \eta_{\mathcal{S}_1} \|d\|^2 \quad \text{for all } (x, d) \in \mathcal{C}_{\mathcal{S}_1, R} \times \mathbb{R}^n. \quad (54)$$

To show that this holds here, observe that for all $(x, j) \in \mathbb{R}^n \times [m]$ it follows from Assumption 3.2, [1, (3.3f)], and the triangle inequality that

$$\begin{aligned} \|\nabla^2[c_{\mathcal{S}_1}]_j(x)\| &= \|\nabla^2[c]_j(x) + \nabla^2[c_{\mathcal{S}_1}]_j(x) - \nabla^2[c]_j(x)\| \\ &\leq \|\nabla^2[c]_j(x)\| + \|\nabla^2[c_{\mathcal{S}_1}]_j(x) - \nabla^2[c]_j(x)\| \\ &\leq \|\nabla^2[c]_j(x)\| + \xi_{\mathcal{S}_1} \sqrt{\gamma \nabla^2 c} \\ &\leq \kappa_{\nabla^2 c} + \sqrt{\frac{N(N-p_1)\gamma \nabla^2 c}{p_1^2}} =: \kappa_{p_1}. \end{aligned} \quad (55)$$

Now consider arbitrary $(x, d) \in \mathcal{C}_{\mathcal{S}_1, R} \times \mathbb{R}^n$ and observe that, by Taylor’s theorem and (55), it follows that for all $j \in [m]$ there exists $\tilde{x}_j \in \mathbb{R}^n$ with

$$\begin{aligned} &|c_{\mathcal{S}_1}]_j(x+d) - [c_{\mathcal{S}_1}]_j(x) - \nabla[c_{\mathcal{S}_1}]_j(x)^T d| \\ &= \frac{1}{2} |d^T \nabla^2[c_{\mathcal{S}_1}]_j(\tilde{x}_j) d| \leq \frac{1}{2} \|\nabla^2[c_{\mathcal{S}_1}]_j(\tilde{x}_j)\| \|d\|^2 \leq \frac{1}{2} \kappa_{p_1} \|d\|^2. \end{aligned} \quad (56)$$

Consequently, one finds that

$$\|c_{\mathcal{S}_1}(x+d) - c_{\mathcal{S}_1}(x) - \nabla c_{\mathcal{S}_1}(x)^T d\| \leq \sqrt{\sum_{j \in [m]} \frac{1}{4} \kappa_{p_1}^2 \|d\|^4} = \frac{1}{2} \sqrt{m} \kappa_{p_1} \|d\|^2,$$

from which it follows that (54) holds for $\eta_{\mathcal{S}_1} = \frac{1}{2}\sqrt{m}\kappa_{p_1}$. Assumption A4 requires that $x_0 \in \mathcal{C}_{\mathcal{S}_1, R}$, as is stated for this lemma. Finally, Assumption A5 requires that the penalty parameter is chosen greater than

$$\max_{x \in \mathcal{C}_{\mathcal{S}_1, R}} \max \left\{ \frac{\sigma_1(\nabla c_{\mathcal{S}_1}(x))\sigma_1(\nabla y_{\mathcal{S}_1}(x))}{2\sigma_m(\nabla c_{\mathcal{S}_1}(x))^2}, \frac{\sigma_1(\nabla y_{\mathcal{S}_1}(x))}{\sigma_m(\nabla c_{\mathcal{S}_1}(x))}, \frac{1}{\sigma_m(\nabla c_{\mathcal{S}_1}(x))} \right\}.$$

Since $\mathcal{C}_{\mathcal{S}_1, R}$ is compact, it follows under Assumption 3.4 and the extreme value theorem that there exists $\hat{\rho}_1$ as stated in the lemma. All together, one can conclude that the requirements of [9, Theorem 3.5] hold in our present setting.

Next we prove the claimed worst-case complexity bound and the claimed $(\bar{\epsilon}, \bar{\zeta})$ -stationarity of the final iterate produced by Algorithm 2. Toward these ends, one first has from [9, Theorem 3.5] that there exists $(\bar{u}_{\mathcal{S}_1, 1}, \bar{u}_{\mathcal{S}_1, 2}, \bar{u}_{\mathcal{S}_1, 3}) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ such that in a number of iterations that is at most

$$\bar{T}_{\mathcal{S}_1} := \max \{ \bar{u}_{\mathcal{S}_1, 1}\epsilon_1^{-2}, \bar{u}_{\mathcal{S}_1, 2}\zeta_1^{-1}, \bar{u}_{\mathcal{S}_1, 3}\zeta_1^{-3} \} \quad (57)$$

the algorithm produces a point $\bar{x} \in \mathbb{R}^n$ satisfying

$$\|\nabla F_{\mathcal{S}_1}(\bar{x})\| \leq \epsilon_1 \quad \text{and} \quad d^T \nabla^2 F_{\mathcal{S}_1}(\bar{x})d \geq -\zeta_1 \|d\|^2 \quad \text{for all } d \in \mathbb{R}^n, \quad (58)$$

meaning that Algorithm 2 terminates with $\bar{x}$ as the final iterate. (Note that to apply [9, Theorem 3.5] it has been observed that $\epsilon_1 \leq \frac{\sqrt{5}}{5}\bar{\epsilon} \leq \frac{1}{2}R$.) To show that this is achieved in a number of iterations that is at most $T_{\mathcal{S}_1}$ in (53), one only needs to observe that $\bar{\zeta} \in (0, 1]$ implies

$$\zeta_1 \leq \frac{\bar{\zeta}}{2} + \tau_{\mathcal{S}_1} \left(\frac{\bar{\zeta}}{4\tau_{\mathcal{S}_1}} \right) \leq \bar{\zeta} \leq 1 \implies \zeta_1^{-1} \leq \zeta_1^{-3},$$

so from (57) the bound in (53) holds with $u_{\mathcal{S}_1, 1} := \bar{u}_{\mathcal{S}_1, 1}$ and $u_{\mathcal{S}_1, 2} := \max\{\bar{u}_{\mathcal{S}_1, 2}, \bar{u}_{\mathcal{S}_1, 3}\}$. Finally, [9, Theorem 3.5] also shows that $\bar{x}$ satisfies

$$\begin{aligned} \|\nabla_x L(\bar{x}, y(\bar{x}))\| &\leq 2\epsilon_1, \quad \|\nabla_y L(\bar{x}, y(\bar{x}))\| \leq \epsilon_1, \\ \text{and } d^T \nabla_{xx}^2 L(\bar{x}, y(\bar{x}))d &\geq -(\zeta_1 + \tau_{\mathcal{S}_1}\epsilon_1)\|d\|^2 \quad \text{for all } d \in \text{Null}(\nabla c_{\mathcal{S}_1}(\bar{x})^T). \end{aligned}$$

Observing that this means $\|\nabla L(\bar{x}, y(\bar{x}))\| \leq \sqrt{5}\epsilon_1 \leq \bar{\epsilon}$, and observing that $\zeta_1 + \tau_{\mathcal{S}_1}\epsilon_1 \leq \frac{\bar{\zeta}}{2} + 2\tau_{\mathcal{S}_1}(\frac{\bar{\zeta}}{4\tau_{\mathcal{S}_1}}) \leq \bar{\zeta}$, it follows that $\bar{x}$ is $(\bar{\epsilon}, \bar{\zeta})$ -stationary. $\square$ $\square$

We now state a useful result, namely, [1, Lemma 3.6], which shows that the strong Morse property extends from the full-sample problem to any sampled problem as long as the sample size is sufficiently large.

Lemma 3.2. ([1, Lemma 3.6]) Suppose that Assumptions 3.1, 3.2, 3.3, 3.4, 3.5, and 3.6 hold, and define the tuple $(\kappa_1, \kappa_2, \kappa_3) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ by

$$\begin{aligned} \kappa_1 &:= \frac{\sqrt{\gamma \nabla c \kappa \nabla c}}{\sigma_{\min}}, \quad \kappa_2 := \sqrt{(2\sqrt{\kappa \nabla f} + \kappa_1 \kappa \nabla f)^2 + \gamma_c}, \quad \text{and} \\ \kappa_3 &:= \left(3\kappa \nabla^2 f \kappa_1 + \sqrt{\kappa \nabla^2 f} + \frac{\sqrt{m}(3\kappa \nabla^2 c \kappa_1 + \sqrt{\gamma \nabla^2 c})(5\kappa \nabla f \kappa_1 + \sqrt{\gamma \nabla f})}{2\sigma_{\min} \kappa_1} \right). \end{aligned}$$

Then, with $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ from Assumption 3.6, if $\mathcal{S}_{k+1} \subseteq [N]$ has

$$\xi_{\mathcal{S}_{k+1}} \leq \min \left\{ \frac{1}{3\kappa_1}, \frac{\alpha}{2\kappa_2}, \frac{7\beta}{18\kappa_3} \right\}, \quad (59)$$

then (49) with k replaced by $k+1$ is $(\alpha_{\mathcal{S}_{k+1}}, \beta_{\mathcal{S}_{k+1}})$ -strongly Morse with

$$\begin{aligned} \alpha_{\mathcal{S}_{k+1}} &:= \alpha - \kappa_2 \xi_{\mathcal{S}_{k+1}} \geq \frac{1}{2}\alpha \\ \text{and } \beta_{\mathcal{S}_{k+1}} &:= \left(1 - \frac{1}{3}\kappa_1 \xi_{\mathcal{S}_{k+1}} \right) \beta - \kappa_3 \xi_{\mathcal{S}_{k+1}} \geq \frac{1}{2}\beta. \end{aligned}$$

Our next lemma shows that if the achieved termination tolerances for a sample problem are set in an appropriate manner, then the initial point for the solve of the next sample problem (if $K > 1$) satisfies certain critical bounds. Here, we first see the significance of Assumption 3.6.

Lemma 3.3. *Suppose that Assumptions 3.1, 3.2, 3.3, 3.4, 3.5, and 3.6 hold and $K > 1$. Define $(\kappa_1, \kappa_2, \kappa_3) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ as in Lemma 3.2 and define the pair of positive real numbers $(\tau_1, \tau_2) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ by*

$$\tau_1 := \kappa_2 \quad \text{and} \quad \tau_2 := \left(\kappa_{\nabla^2 f} + \frac{\sqrt{m} \kappa_{\nabla f} \kappa_{\nabla^2 c}}{\sigma_{\min}} \right) \kappa_1. \quad (60)$$

Suppose that $p_1 \in [N]$ is sufficiently large such that, for all $k \in [K]$,

$$\xi_{S_k} \leq \xi_{S_1} \leq \min \left\{ \frac{1}{3\kappa_1}, \frac{\alpha}{4\kappa_2}, \frac{2\beta}{9\kappa_3} \right\}, \quad (61)$$

and that, for all $k \in [K-1]$, a pair $(\hat{\epsilon}_k, \hat{\zeta}_k) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ satisfies

$$\hat{\epsilon}_k \leq \tau_1 \xi_{S_1} \quad \text{and} \quad \hat{\zeta}_k \leq \tau_2 \xi_{S_k}. \quad (62)$$

Then, for all $k \in [K-1]$, if $x_k \in \mathbb{R}^n$ is $(\hat{\epsilon}_k, \hat{\zeta}_k)$ -stationary with multiplier $y_{S_k}(x_k)$ respect to (49), then for (49) with k replaced by $k+1$ one has

$$\begin{aligned} \|\nabla L_{S_{k+1}}(x_k, y_{S_{k+1}}(x_k))\| &\leq 3\tau_1 \xi_{S_1} \leq \alpha_{S_{k+1}} \quad \text{and} \\ d^T \nabla_{xx}^2 L_{S_{k+1}}(x_k, y_{S_{k+1}}(x_k)) d &\geq \beta_{S_{k+1}} \|d\|^2 \quad \text{for all } d \in \text{Null}(\nabla c_{S_{k+1}}(x_k)^T), \end{aligned} \quad (63)$$

where one defines $(\alpha_{S_{k+1}}, \beta_{S_{k+1}})$ as in Lemma 3.2.

Proof. Suppose that $K > 1$ and $x_k \in \mathbb{R}^n$ is $(\hat{\epsilon}_k, \hat{\zeta}_k)$ -stationary with respect to (49) for $\mathcal{S} = S_k$ with $(\hat{\epsilon}_k, \hat{\zeta}_k)$ satisfying (62). Our aim is to prove (63). First let us bound the norm of the difference between the gradients of L_{S_k} and $L_{S_{k+1}}$ with respect to the point x_k . Toward this end, using (50), define $y_k := y_{S_k}(x_k)$ and $y_{k+1} := y_{S_{k+1}}(x_k)$. Then, by the triangle inequality,

$$\begin{aligned} &\|\nabla_x L_{S_{k+1}}(x_k, y_{k+1}) - \nabla_x L_{S_k}(x_k, y_k)\| \\ &\leq \|\nabla_x L_{S_{k+1}}(x_k, y_{k+1}) - \nabla_x L(x_k, y_k)\| \\ &\quad + \|\nabla_x L(x_k, y_k) - \nabla_x L_{S_k}(x_k, y_k)\|. \end{aligned} \quad (64)$$

Then, by similar arguments as led to [1, (29)], one finds

$$\begin{aligned} \|\nabla_x L_{S_k}(x_k, y_k) - \nabla_x L(x_k, y_k)\| &\leq \xi_{S_k} (2\sqrt{\kappa_{\nabla f}} + \kappa_1 \kappa_{\nabla f}) \quad \text{and} \\ \|\nabla_x L_{S_{k+1}}(x_k, y_{k+1}) - \nabla_x L(x_k, y_k)\| &\leq \xi_{S_{k+1}} (2\sqrt{\kappa_{\nabla f}} + \kappa_1 \kappa_{\nabla f}) \\ &\leq \xi_{S_k} (2\sqrt{\kappa_{\nabla f}} + \kappa_1 \kappa_{\nabla f}), \end{aligned}$$

where the last inequality follows from (51) and $|S_{k+1}| > |S_k|$. Combining these bounds with (64), one obtains that

$$\|\nabla_x L_{S_{k+1}}(x_k, y_{k+1}) - \nabla_x L_{S_k}(x_k, y_k)\| \leq 2\xi_{S_k} (2\sqrt{\kappa_{\nabla f}} + \kappa_1 \kappa_{\nabla f}). \quad (65)$$

On the other hand, from the triangle inequality, [1, (10b)], and $\xi_{S_{k+1}} \leq \xi_{S_k}$,

$$\begin{aligned} &\|\nabla_{\hat{y}} L_{S_{k+1}}(x_k, y_{k+1}) - \nabla_{\hat{y}} L_{S_k}(x_k, y_k)\| \\ &= \|c_{S_{k+1}}(x_k) - c_{S_k}(x_k)\| \\ &\leq \|c_{S_{k+1}}(x_k) - c(x_k)\| + \|c(x_k) - c_{S_k}(x_k)\| \\ &\leq \sqrt{\gamma_c} \xi_{S_{k+1}} + \sqrt{\gamma_c} \xi_{S_k} \leq 2\sqrt{\gamma_c} \xi_{S_k}. \end{aligned} \quad (66)$$

Combining (65) and (66), one finds that

$$\begin{aligned} & \|\nabla L_{\mathcal{S}_{k+1}}(x_k, y_{k+1}) - \nabla L_{\mathcal{S}_k}(x_k, y_k)\|^2 \\ & \leq 4(\gamma_c + (2\sqrt{\kappa_{\nabla f}} + \kappa_1 \kappa_{\nabla f})^2) \xi_{\mathcal{S}_k}^2 = 4\kappa_2^2 \xi_{\mathcal{S}_k}^2, \end{aligned} \quad (67)$$

which further gives under the conditions of the lemma that

$$\begin{aligned} & \|\nabla L_{\mathcal{S}_{k+1}}(x_k, y_{k+1})\| \\ & \leq \|\nabla L_{\mathcal{S}_{k+1}}(x_k, y_{k+1}) - \nabla L_{\mathcal{S}_k}(x_k, y_k)\| + \|\nabla L_{\mathcal{S}_k}(x_k, y_k)\| \\ & \leq 2\kappa_2 \xi_{\mathcal{S}_k} + \hat{\epsilon}_k \leq 2\tau_1 \xi_{\mathcal{S}_1} + \tau_1 \xi_{\mathcal{S}_1} \leq 3\tau_1 \xi_{\mathcal{S}_1} = 3\kappa_2 \xi_{\mathcal{S}_1} \leq \frac{3}{4}\alpha. \end{aligned} \quad (68)$$

At the same time, by $\xi_{\mathcal{S}_{k+1}} \leq \xi_{\mathcal{S}_k}$ and $\kappa_2 \xi_{\mathcal{S}_k} \leq \frac{1}{4}\alpha$, one has that

$$\alpha_{\mathcal{S}_{k+1}} = \alpha - \kappa_2 \xi_{\mathcal{S}_{k+1}} \geq \alpha - \kappa_2 \xi_{\mathcal{S}_k} \geq \alpha - \frac{1}{4}\alpha = \frac{3}{4}\alpha. \quad (69)$$

Combining (68) and (69), one obtains the first conclusion in (63) that

$$\|\nabla L_{\mathcal{S}_{k+1}}(x_k, y_{k+1})\| \leq 3\tau_1 \xi_{\mathcal{S}_1} \leq \alpha_{\mathcal{S}_{k+1}}. \quad (70)$$

Our next goal is to prove the second inequality in (63). Toward this end, first note that Lemma 3.2 applies here since the right-hand side of (61) is less than or equal to the right-hand side of (59). Hence, by Lemma 3.2, it follows that (49) with k replaced by $k+1$ is $(\alpha_{\mathcal{S}_{k+1}}, \beta_{\mathcal{S}_{k+1}})$ -strongly Morse. Combined with (70), this means that one has for all $d_{\mathcal{S}_{k+1}} \in \text{Null}(\nabla c_{\mathcal{S}_{k+1}}(x_k)^T)$ that

$$|d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_k, y_{k+1}) d_{\mathcal{S}_{k+1}}| \geq \beta_{\mathcal{S}_{k+1}} \|d_{\mathcal{S}_{k+1}}\|^2. \quad (71)$$

To prove the second inequality in (63), it is necessary to show that this inequality holds without the absolute value on the left-hand side. Toward showing this, let us define $\bar{d}_{\mathcal{S}_{k+1}} := \mathcal{N}(\nabla c_{\mathcal{S}_k}(x_k)^T) d_{\mathcal{S}_{k+1}}$. Then, one finds

$$\begin{aligned} & d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_k, y_{k+1}) d_{\mathcal{S}_{k+1}} \\ & = \bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_k}(x_k, y_k) \bar{d}_{\mathcal{S}_{k+1}} \\ & \quad + (d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_k, y_{k+1}) d_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_k}(x_k, y_k) \bar{d}_{\mathcal{S}_{k+1}}) \\ & \geq \bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_k}(x_k, y_k) \bar{d}_{\mathcal{S}_{k+1}} \\ & \quad - |d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_k, y_{k+1}) d_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_k}(x_k, y_k) \bar{d}_{\mathcal{S}_{k+1}}|. \end{aligned} \quad (72)$$

Since $\mathcal{N}(\nabla c_{\mathcal{S}_k}(x_k)^T)$ is a projection matrix, $\|\mathcal{N}(\nabla c_{\mathcal{S}_k}(x_k)^T)\| \leq 1$ and

$$\begin{aligned} \|\bar{d}_{\mathcal{S}_{k+1}}\|^2 & = \|\mathcal{N}(\nabla c_{\mathcal{S}_k}(x_k)^T) d_{\mathcal{S}_{k+1}}\|^2 \\ & \leq \|\mathcal{N}(\nabla c_{\mathcal{S}_k}(x_k)^T)\|^2 \|d_{\mathcal{S}_{k+1}}\|^2 \leq \|d_{\mathcal{S}_{k+1}}\|^2. \end{aligned}$$

Along with the fact that x_k is $(\hat{\epsilon}_k, \hat{\zeta}_k)$ -stationary, this means that the first-term on the right-hand side of (72) satisfies the inequalities

$$\bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_k}(x_k, y_k) \bar{d}_{\mathcal{S}_{k+1}} \geq -\hat{\zeta}_k \|\bar{d}_{\mathcal{S}_{k+1}}\|^2 \geq -\hat{\zeta}_k \|d_{\mathcal{S}_{k+1}}\|^2. \quad (73)$$

On the other hand, with respect to the second term on the right-hand side of (72), observe that with the triangle inequality one finds

$$|d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_k, y_{k+1}) d_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_k}(x_k, y_k) \bar{d}_{\mathcal{S}_{k+1}}|$$

$$\begin{aligned}
&\leq |d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_k, y_{k+1}) d_{\mathcal{S}_{k+1}} - d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L(x_k, y_k) d_{\mathcal{S}_{k+1}}| \\
&\quad + |d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L(x_k, y_k) d_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L(x_k, y_k) \bar{d}_{\mathcal{S}_{k+1}}| \\
&\quad + |\bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L(x_k, y_k) \bar{d}_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_k}(x_k, y_k) \bar{d}_{\mathcal{S}_{k+1}}|. \tag{74}
\end{aligned}$$

Next, one can observe that the first and third terms on the right-hand side of (74) can be bounded as in [1, (41)] with respect to $\mathcal{S} = \mathcal{S}_k$ and $\mathcal{S} = \mathcal{S}_{k+1}$, respectively. To see this, note that a bound of the form in [1, (36)] holds since [1, (36)] followed using general matrix-norm inequalities. Furthermore, a bound of the form in [1, (37)] that relies on [1, (36)] and [1, (10f)] holds, and bounds of the form in [1, (38)–(39)] hold since these rely on $\xi_{\mathcal{S}_k} \leq \frac{1}{3\kappa_1}$ and $\xi_{\mathcal{S}_{k+1}} \leq \frac{1}{3\kappa_1}$, which hold here. Thus, since [1, (41)] follows from a combination of [1, (36)–(39)], one obtains that with

$$\lambda := \sqrt{\kappa_{\nabla^2 f}} + \frac{3\sqrt{m}\kappa_{\nabla^2 c}\kappa_1(3\kappa_{\nabla f}\kappa_1 + \sqrt{\gamma_{\nabla f}}) + \sqrt{m}\sqrt{\gamma_{\nabla^2 c}}(5\kappa_{\nabla f}\kappa_1 + \sqrt{\gamma_{\nabla f}})}{2\sigma_{\min}\kappa_1},$$

the first and third terms on the right-hand side of (74) respectively satisfy

$$\begin{aligned}
&|d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_k, y_{k+1}) d_{\mathcal{S}_{k+1}} - d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L(x_k, y_k) d_{\mathcal{S}_{k+1}}| \\
&\leq \lambda \xi_{\mathcal{S}_{k+1}} \|d_{\mathcal{S}_{k+1}}\|^2 \leq \lambda \xi_{\mathcal{S}_k} \|d_{\mathcal{S}_{k+1}}\|^2 \tag{75}
\end{aligned}$$

$$\begin{aligned}
&\text{and } |\bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L(x_k, y_k) \bar{d}_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_k}(x_k, y_k) \bar{d}_{\mathcal{S}_{k+1}}| \\
&\leq \lambda \xi_{\mathcal{S}_k} \|\bar{d}_{\mathcal{S}_{k+1}}\|^2 \leq \lambda \xi_{\mathcal{S}_k} \|d_{\mathcal{S}_{k+1}}\|^2. \tag{76}
\end{aligned}$$

As for the second term on the right of (74), submultiplicity of the matrix 2-norm, the triangle inequality, $\|\bar{d}_{\mathcal{S}_{k+1}}\| \leq \|d_{\mathcal{S}_{k+1}}\|$, and [1, (8c)] give

$$\begin{aligned}
&|d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L(x_k, y_k) d_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L(x_k, y_k) \bar{d}_{\mathcal{S}_{k+1}}| \\
&= |(d_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}})^T \nabla_{xx}^2 L(x_k, y_k) (d_{\mathcal{S}_{k+1}} + \bar{d}_{\mathcal{S}_{k+1}})| \\
&\leq \|\nabla_{xx}^2 L(x_k, y_k)\| \|d_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}\| \|d_{\mathcal{S}_{k+1}} + \bar{d}_{\mathcal{S}_{k+1}}\| \\
&\leq 2 \left(\kappa_{\nabla^2 f} + \frac{\sqrt{m}\kappa_{\nabla f}\kappa_{\nabla^2 c}}{\sigma_{\min}} \right) \|d_{\mathcal{S}_{k+1}}\| \|d_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}\| \\
&= 2 \frac{\tau_2}{\kappa_1} \|d_{\mathcal{S}_{k+1}}\| \|d_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}\|. \tag{77}
\end{aligned}$$

By Assumption 3.4 the matrices $\nabla c(x_k)$ and $\nabla c_{\mathcal{S}_k}(x_k)$ have full row rank, i.e., the same rank, which by [14, Theorems 2.3–2.4] means that

$$\|(\mathcal{R}(\nabla c(x_k)) - \mathcal{R}(\nabla c_{\mathcal{S}_k}(x_k)))\| \leq \kappa_1 \xi_{\mathcal{S}_k}.$$

Hence, with respect to $\|d_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}\|$, one finds from the triangle inequality, submultiplicity of the matrix 2-norm, and [1, Lemma 3.5] that

$$\begin{aligned}
&\|d_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}\| \\
&= \|\mathcal{R}(\nabla c_{\mathcal{S}_k}(x_k)) d_{\mathcal{S}_{k+1}}\| \\
&\leq \|\mathcal{R}(\nabla c(x_k)) d_{\mathcal{S}_{k+1}}\| + \|(\mathcal{R}(\nabla c(x_k)) - \mathcal{R}(\nabla c_{\mathcal{S}_k}(x_k)))\| \|d_{\mathcal{S}_{k+1}}\| \\
&\leq \kappa_1 \xi_{\mathcal{S}_{k+1}} \|d_{\mathcal{S}_{k+1}}\| + \kappa_1 \xi_{\mathcal{S}_k} \|d_{\mathcal{S}_{k+1}}\| \leq 2\kappa_1 \xi_{\mathcal{S}_k} \|d_{\mathcal{S}_{k+1}}\|. \tag{78}
\end{aligned}$$

Combining (72)–(78) with [1, (25)], (60), and $\hat{\zeta}_k \leq \tau_2 \xi_{\mathcal{S}_k}$, one finds

$$\begin{aligned}
&d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_k, y_{k+1}) d_{\mathcal{S}_{k+1}} \\
&\geq -\hat{\zeta}_k \|d_{\mathcal{S}_{k+1}}\|^2 - 2\lambda \xi_{\mathcal{S}_k} \|d_{\mathcal{S}_{k+1}}\|^2 - 4\tau_2 \xi_{\mathcal{S}_k} \|d_{\mathcal{S}_{k+1}}\|^2
\end{aligned}$$

$$= (-5\tau_2 - 2\lambda) \xi_{\mathcal{S}_k} \|d_{\mathcal{S}_{k+1}}\|^2.$$

On the other hand, from the definitions of τ_2 , λ , and κ_3 one finds

$$\begin{aligned} & (-5\tau_2 - 2\lambda) \\ & \geq -2 \left(3\kappa_{\nabla^2 f} \kappa_1 + \sqrt{\kappa_{\nabla^2 f}} + \frac{\sqrt{m}(3\kappa_{\nabla^2 c} \kappa_1 + \sqrt{\gamma_{\nabla^2 c}})(5\kappa_{\nabla f} \kappa_1 + \sqrt{\gamma_{\nabla f}})}{2\sigma_{\min} \kappa_1} \right) \\ & = -2\kappa_3. \end{aligned}$$

Hence, since (61) requires $\xi_{\mathcal{S}_k} \leq \frac{2\beta}{9\kappa_3}$, it follows that

$$d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_k, y_{k+1}) d_{\mathcal{S}_{k+1}} \geq -2\kappa_3 \xi_{\mathcal{S}_k} \|d_{\mathcal{S}_{k+1}}\|^2 \geq -\frac{4}{9}\beta \|d_{\mathcal{S}_{k+1}}\|^2. \quad (79)$$

On the other hand, recall $\beta_{\mathcal{S}_{k+1}} = (1 - \frac{1}{3}\kappa_1 \xi_{\mathcal{S}_{k+1}}) \beta - \kappa_3 \xi_{\mathcal{S}_{k+1}}$, which along with $\xi_{\mathcal{S}_{k+1}} \leq \xi_{\mathcal{S}_k}$ and (61) (specifically, with $\xi_{\mathcal{S}_k} \leq \min\{\frac{1}{3\kappa_1}, \frac{2\beta}{9\kappa_3}\}$) implies

$$\beta_{\mathcal{S}_{k+1}} \geq \left(1 - \frac{1}{3}\kappa_1 \xi_{\mathcal{S}_k}\right) \beta - \kappa_3 \xi_{\mathcal{S}_k} \geq \beta - \frac{1}{9}\beta - \frac{2}{9}\beta = \frac{2}{3}\beta.$$

Since (49) with k replaced by $k+1$ is $(\alpha_{k+1}, \beta_{k+1})$ -strongly Morse (which follows as a consequence of Lemma 3.2), one has that

$$|d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_k, y_{k+1}) d_{\mathcal{S}_{k+1}}| \geq \beta_{\mathcal{S}_{k+1}} \|d_{\mathcal{S}_{k+1}}\|^2 \geq \frac{2}{3}\beta \|d_{\mathcal{S}_{k+1}}\|^2.$$

That said, since (79) holds and $-\frac{4}{9} > -\frac{2}{3}$, it must hold that

$$d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_k, y_{k+1}) d_{\mathcal{S}_{k+1}} \geq \beta_{\mathcal{S}_{k+1}} \|d_{\mathcal{S}_{k+1}}\|^2,$$

which completes the proof. $\square$

We are almost prepared to prove our main theorem about progressive sampling. We remark upfront that, in order for our theoretical guarantee to hold for $p_1 < N$, the algorithmic parameters need to be chosen within prescribed intervals. For notational simplicity, we shall define these intervals with respect to a single prescribed parameter $\mu \in (0, 1)$. Choosing the parameters in such intervals ensures that the computational effort is balanced appropriately for all $k \in [K]$, so that, for example, the tolerance for solving one sampled problem is not arbitrarily tighter than the tolerances for solving the other sampled problems in the sequence. Such requirements can be seen to be natural for worst-case complexity guarantees of the type proved in the theorem. We emphasize, however, that for good practical performance, we do not expect these parameter requirements to be absolutely necessary. Indeed, the experiments in [1] show good performance without such stringent parameter choices.

To define our parameter intervals, we refer to the user-defined $\mu \in (0, 1)$ along with (α, β) from Assumption 3.6. First, for each $\mathcal{S} \subseteq [N]$ with $|\mathcal{S}| \geq p_1$, let $\kappa_{\mathcal{S}}$ be defined in the same manner as κ is defined in Lemma 2.1, in this case with respect to problem (49) with $\mathcal{S}$ in place of $\mathcal{S}_k$, then let

$$\bar{\kappa} := \max\{\kappa_{\mathcal{S}} : \mathcal{S} \subseteq [N] \text{ and } |\mathcal{S}| \geq p_1\}.$$

The fact that this value is finite follows under the present assumptions from this section using the same arguments as in Lemma 2.1. Second, let $\hat{\rho}_1$ be defined as in Lemma 3.1. Third, as in (25), but with κ replaced by the uniform bound $\bar{\kappa}$ defined above and $\hat{\rho}_1$ included, let

$$\rho_{\beta} := \max \left\{ \frac{\max\{1, \bar{\kappa}\}}{\sigma_{\min}}, \frac{\beta + \bar{\kappa} + m\bar{\kappa}^2}{2\sigma_{\min}^2}, \hat{\rho}_1 \right\} \quad \text{and} \quad \rho_{\max} := \mu^{-1} \rho_{\beta}.$$

Fourth, following the notation in Theorem 2.1, let

$$\begin{aligned}\omega &:= 1 + \bar{\kappa} + 2\rho_{\max}\bar{\kappa}, \\ \eta &:= \frac{2\sqrt{m\bar{\kappa}}}{\sigma_{\min}} + m\bar{\kappa} + 2m\rho_{\max}\bar{\kappa}, \\ M &:= M_f + \rho_{\max}M_c,\end{aligned}$$

and let $\bar{\lambda} \in \mathbb{R}_{>0}$ be any real number satisfying

$$\begin{aligned}\bar{\lambda} &\geq \beta + \sup_{(\rho, \mathcal{S})} \max \left\{ 0, \sup_{x \in \mathcal{C}_{\mathcal{S}, R}} \lambda_1(\nabla^2 F_{\mathcal{S}, \rho}(x)) \right\} \\ \text{s.t. } &\rho \in (\rho_{\beta}, \rho_{\max}] \\ &\mathcal{S} \subseteq [N] \text{ with } |\mathcal{S}| \geq p_1.\end{aligned}\tag{80}$$

Finally, define the values

$$\epsilon_* := \min \left\{ \frac{\beta}{4 \left(\eta + \frac{8M\omega}{\beta} + \left(1 + \frac{2}{\sigma_{\min}} \right) \frac{\bar{\kappa}}{\sqrt{5}} \right)}, \frac{R}{\omega}, \frac{\beta\bar{\lambda}}{4M\omega}, \frac{\bar{\lambda}\delta}{2\omega} \right\}\tag{81}$$

$$\epsilon_c := \tau_1 \xi_{\mathcal{S}_1}\tag{82}$$

$$\text{and } c := \min \left\{ \frac{1}{3\kappa_1}, \frac{\alpha}{4\kappa_2}, \frac{2\beta}{9\kappa_3}, \frac{\epsilon_*}{3\tau_1}, \frac{\sqrt{5}R}{2\tau_1}, \frac{1}{\tau_2} \right\}.\tag{83}$$

It is useful to state the following technical lemma.

Lemma 3.4. *For any $b \in \mathbb{R}_{>0}$, $\kappa \in \mathbb{R}_{>0}$, and $\rho \in \mathbb{R}_{>0}$, let $\epsilon_{\rho}(b, \kappa)$ denote the right-hand side of (26) with β replaced by b , $(\omega_{\rho}, \eta_{\rho})$ defined as in (8) for the given κ , $M_{\rho} = M_f + \rho M_c$, and λ_{ρ} replaced by $\bar{\lambda}$ defined in (80), i.e.,*

$$\epsilon_{\rho}(b, \kappa) = \min \left\{ \frac{b}{2 \left(\eta_{\rho} + \frac{4M_{\rho}\omega_{\rho}}{b} + \left(1 + \frac{2}{\sigma_{\min}} \right) \frac{\kappa}{\sqrt{5}} \right)}, \frac{R}{\omega_{\rho}}, \frac{b\bar{\lambda}}{2M_{\rho}\omega_{\rho}}, \frac{\bar{\lambda}\delta}{2\omega_{\rho}} \right\}.\tag{84}$$

Then, $\epsilon_{\rho}(b, \kappa)$ is nondecreasing in b and nonincreasing in each of κ and ρ . Consequently, for every $\rho \in (\rho_{\beta}, \rho_{\max}]$, $\kappa \in (0, \bar{\kappa}]$, and $b \in [\beta/2, \beta]$ one has that $\epsilon_ \leq \epsilon_{\rho}(b, \kappa)$, while at the same time for any $\epsilon \in (0, \epsilon_{\rho}(b, \kappa)]$ and with*

$$t_{\rho}(b, \kappa) := \min \left\{ \frac{1}{\bar{\lambda}}, \frac{1}{b - \eta_{\rho}\epsilon - \frac{4M_{\rho}\omega_{\rho}}{b}\epsilon} \right\}\tag{85}$$

(see (27)) one has that $1/\bar{\lambda} = t_{\rho}(b, \kappa)$, so $[\mu/\bar{\lambda}, 1/\bar{\lambda}] \subset (0, t_{\rho}(b, \kappa)]$.

Proof. For the sake of brevity, let $C(\kappa) := \left(1 + \frac{2}{\sigma_{\min}} \right) \frac{\kappa}{\sqrt{5}}$. Consider first monotonicity with respect to b . The first term in (84) is

$$\frac{b}{2 \left(\eta_{\rho} + \frac{4M_{\rho}\omega_{\rho}}{b} + C(\kappa) \right)} = \frac{b^2}{2 \left((\eta_{\rho} + C(\kappa))b + 4M_{\rho}\omega_{\rho} \right)},$$

which is strictly increasing in b . On the other hand, the second and fourth terms in (84) are independent of b while the third term is increasing in b . Hence, overall, $\epsilon_{\rho}(b, \kappa)$ is nondecreasing in b , as claimed. Consider next monotonicity with respect to κ . By (8), both ω_{ρ} and η_{ρ} are increasing in κ . At the same time, $C(\kappa)$ is increasing in κ while M_{ρ} is independent of κ . Consequently, each of the four terms in (84) is nonincreasing

in κ , and therefore so is $\epsilon_\rho(b, \kappa)$. Consider finally monotonicity with respect to ρ . Each of ω_ρ , η_ρ , and M_ρ is increasing in ρ while $\bar{\lambda}$ is independent of ρ , so by similar reasoning each of the four terms in (84) is nonincreasing in ρ , and therefore so is $\epsilon_\rho(b, \kappa)$.

Now consider arbitrary $\rho \in (\rho_\beta, \rho_{\max}]$, $\kappa \in (0, \bar{\kappa}]$, and $b \in [\beta/2, \beta]$. Combining the three monotonicity properties established above, one finds that

$$\epsilon_\rho(b, \kappa) \geq \epsilon_{\rho_{\max}}(\tfrac{1}{2}\beta, \bar{\kappa}). \quad (86)$$

On the other hand, since $(\omega_{\rho_{\max}}, \eta_{\rho_{\max}}, M_{\rho_{\max}}) = (\omega, \eta, M)$ by the definitions of ω , η , and M , evaluating (84) at $(b, \kappa, \rho) = (\beta/2, \bar{\kappa}, \rho_{\max})$ yields

$$\epsilon_{\rho_{\max}}(\beta/2, \bar{\kappa}) = \min \left\{ \frac{\beta}{4 \left(\eta + \frac{8M\omega}{\beta} + \left(1 + \frac{2}{\sigma_{\min}} \right) \frac{\bar{\kappa}}{\sqrt{5}} \right)}, \frac{R}{\omega}, \frac{\beta\bar{\lambda}}{4M\omega} \right\} = \epsilon^*,$$

which with (86) shows that $\epsilon^* \leq \epsilon_\rho(b, \kappa)$, as desired.

Lastly, for the arbitrary $\rho \in (\rho_\beta, \rho_{\max}]$, $\kappa \in (0, \bar{\kappa}]$, and $b \in [\beta/2, \beta]$, consider arbitrary $\epsilon \in (0, \epsilon_\rho(b, \kappa)]$. By (80) one has $\bar{\lambda} \geq \beta$, which with $b \leq \beta$ gives

$$b - \eta_\rho \epsilon - \frac{4M_\rho \omega_\rho}{b} \epsilon \leq b \leq \beta \leq \bar{\lambda}.$$

Consequently, the first term in the minimum defining $t_\rho(b, \kappa)$ is the smaller one and $t_\rho(b, \kappa) = 1/\bar{\lambda}$. Thus, indeed, $1/\bar{\lambda} \leq t_\rho(b, \kappa)$, and since $\mu \in (0, 1)$ it follows that $[\mu/\bar{\lambda}, 1/\bar{\lambda}] \subseteq (0, t_\rho(b, \kappa)]$, which completes the proof. $\square$

We now prove our main theorem for progressive sampling. In the theorem, for a reasonable basis of comparison, our primary measures of performance of an algorithm are the number of individual objective gradients and individual constraint Jacobians that need to be computed prior to termination with an approximate second-order stationary point of a desired user-defined accuracy. In other words, we are concerned with the number of times that $\nabla f_i(x)$ is evaluated for some $i \in [N]$ and some $x \in \mathbb{R}^n$, and the number of times that $\nabla c_i(x)$ is evaluated for some $i \in [N]$ and some $x \in \mathbb{R}^n$. One might also be interested in the numbers of individual objective function evaluations, individual constraint function evaluations, or evaluations of a least square multiplier. That said, if the methods are implemented with fixed step-size strategies, then the algorithmic complexity with respect to these quantities would be the same as for the derivative computations that we consider.

Theorem 3.1. *Suppose that Assumptions 3.1, 3.2, 3.3, 3.4, 3.5, and 3.6 hold. Then, the following hold for Algorithm 1, where it is assumed that calls to Algorithm 2 for $k = 1$ use either the line-search methods in [9, Algorithms 2–3] or set the step sizes according to the lower bounds in [9, Lemmas 3.2 and 3.4]. (In the following parts, we refer to $\bar{\epsilon}_1(\cdot, \cdot)$ and $\bar{\zeta}_1(\cdot, \cdot)$ defined in Lemma 3.1.)*

- (a) *Suppose that $p_1 = N$, $\mathcal{S}_1 = [N]$, $x_0 \in \mathcal{C}_{[N], R}$, and $\rho_1 \geq \hat{\rho}_1$, where $\hat{\rho}_1$ is defined as in Lemma 3.1. Then, for any $(\epsilon, \zeta) \in (0, \frac{\sqrt{5}}{2}R] \times (0, 1]$, Algorithm 1 with initial point x_0 and tolerances (ϵ_1, ζ_1) satisfying*

$$\epsilon_1 \in [\mu \bar{\epsilon}_1(\epsilon, \zeta), \bar{\epsilon}_1(\epsilon, \zeta)] \quad \text{and} \quad \zeta_1 \in [\mu \bar{\zeta}_1(\epsilon, \zeta), \bar{\zeta}_1(\epsilon, \zeta)]$$

for some $\mu \in (0, 1)$ requires

$$\mathcal{O}(N \lceil \max\{\epsilon^{-2}, \zeta^{-3}\} \rceil) \quad (87)$$

individual objective gradients and individual constraint Jacobians until it terminates, yielding x_1 that is (ϵ, ζ) -stationary for (48).

- (b) *Suppose that $p_1 < N$, $\mathcal{S}_1 \subset [N]$ with $|\mathcal{S}_1| = p_1$ satisfies $\xi_{\mathcal{S}_1} \leq c$ (where c is defined in (83)), $x_0 \in \mathcal{C}_{\mathcal{S}_1, R}$, and the ultimate desired tolerances satisfy $(\epsilon, \zeta) \in (0, 3\epsilon_c) \times \mathbb{R}_{>0}$ (where ϵ_c is defined in (82)). Suppose further that the parameters of Algorithm 1 and the calls to Algorithm 2 have:*

- $\rho_k \in (\rho_\beta, \rho_{\max}]$ for all $k \in [K]$;
- for each $k \in \{2, \dots, K\}$, line 4 of Algorithm 2 employs a fixed step size chosen to satisfy $t_g \in [\mu/\bar{\lambda}, 1/\bar{\lambda}]$;
- for all $k \in [K-1]$, target accuracies $(\hat{\epsilon}_k, \hat{\zeta}_k) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ are chosen such that $\hat{\epsilon}_k \in [\mu\epsilon_c, \epsilon_c]$ and $\hat{\zeta}_k \in [\mu\tau_2\xi_{S_k}, \tau_2\xi_{S_k}]$;
- the tolerances input to Algorithm 2 satisfy

$$\begin{aligned}\epsilon_1 &\in [\mu\bar{\epsilon}_1(\hat{\epsilon}_1, \hat{\zeta}_1), \bar{\epsilon}_1(\hat{\epsilon}_1, \hat{\zeta}_1)], \\ \zeta_1 &\in [\mu\bar{\zeta}_1(\hat{\epsilon}_1, \hat{\zeta}_1), \bar{\zeta}_1(\hat{\epsilon}_1, \hat{\zeta}_1)], \\ (\epsilon_k, \zeta_k) &= (\hat{\epsilon}_k/\sqrt{5}, \hat{\zeta}_k) \text{ for all } k \in \{2, \dots, K-1\}, \\ \text{and } (\epsilon_K, \zeta_K) &= (\epsilon/\sqrt{5}, \zeta).\end{aligned}$$

Then, with

$$B := \frac{4}{4 - \frac{\mu\beta}{2\lambda}} > 1, \quad \bar{T} := \left\lceil \log_B \left(\frac{3\sqrt{5}\omega}{\mu} \right) \right\rceil, \quad (88)$$

$(u_{S_1,1}, u_{S_1,2})$ defined as in Lemma 3.1, and

$$C_1 := \max \left\{ \frac{u_{S_1,1}}{\mu^4 \xi_{S_1}^2} \max \left\{ \frac{5}{\tau_1^2}, \frac{16\tau_{S_1}^2}{\tau_2^2} \right\}, \frac{8u_{S_1,2}}{\mu^6 (\tau_2 \xi_{S_1})^3} \right\}, \quad (89)$$

Algorithm 1 terminates with x_K satisfying

$$\begin{aligned}\|\nabla L(x_K, y(x_K))\| &\leq \epsilon \\ \text{and } d^T \nabla_{xx}^2 L(x_K, y(x_K)) d &\geq \frac{1}{2} \beta \|d\|^2 \text{ for all } d \in \text{Null}(\nabla c(x_K)^T),\end{aligned}$$

so that x_K is (ϵ, ζ') -stationary for (48) for every $\zeta' \in \mathbb{R}_{\geq 0}$. Moreover, the total number of individual objective gradients and individual constraint Jacobians computed before termination is at most

$$p_1 C_1 + \bar{T} \frac{\theta(N - p_1)}{\theta - 1} + N \left\lceil \log_B \left(\frac{3\sqrt{5}\omega\epsilon_c}{\epsilon} \right) \right\rceil, \quad (90)$$

where the first two terms are independent of ϵ and ζ .

Proof. Consider part (a). By Lemma 3.1, there exists a pair $(u_{[N],1}, u_{[N],2}) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ such that, for any $(\epsilon, \zeta) \in (0, \frac{\sqrt{5}}{2}R] \times (0, 1]$ and tolerances (ϵ_1, ζ_1) satisfying the conditions of part (a), Algorithm 1 requires at most

$$T_{S_1} = \max\{u_{[N],1}\epsilon_1^{-2}, u_{[N],2}\zeta_1^{-3}\}$$

iterations until x_1 is produced, which is (ϵ, ζ) -stationary, as desired. Further, since $\epsilon_1 \geq \mu\bar{\epsilon}_1(\epsilon, \zeta) = \mu \min\{\frac{\sqrt{5}\epsilon}{5}, \frac{\zeta}{4\tau_{[N]}}\}$ and $\zeta_1 \geq \mu\bar{\zeta}_1(\epsilon, \zeta) \geq \mu\frac{\zeta}{2}$, one finds

$$\begin{aligned}T_{S_1} &\leq \max\{5\mu^{-2}u_{[N],1}\epsilon^{-2}, 16\tau_{[N]}^2\mu^{-2}u_{[N],1}\zeta^{-2}, 8\mu^{-3}u_{[N],2}\zeta^{-3}\} \\ &\leq \max\{5\mu^{-2}u_{[N],1}\epsilon^{-2}, 16\tau_{[N]}^2\mu^{-2}u_{[N],1}\zeta^{-3}, 8\mu^{-3}u_{[N],2}\zeta^{-3}\}.\end{aligned}$$

Thus, the claim follows since each iteration the call to Algorithm 2 requires N individual objective gradients and N individual constraint Jacobians.

Now consider part (b). Throughout the proof of this part, for each $k \in [K]$ let $\kappa_{S_k} \leq \bar{\kappa}$ denote a value satisfying (5) with respect to (49) and define the pair $(\alpha_{S_k}, \beta_{S_k})$ in the manner of Lemma 3.2. Since $p_{k+1} = \min\{\theta p_k, N\}$ for all $k \in [K]$ and $K = \lceil \log_\theta(N/p_1) \rceil + 1$, one finds that $p_k = \theta^{k-1}p_1 < N$ for all

$k \in [K-1]$ and $p_K = N$. Consequently, $\xi_{S_k} > 0$ for all $k \in [K-1]$ while $\xi_{S_K} = 0$ so $\alpha_{S_K} = \alpha$ and $\beta_{S_K} = \beta$. In addition, since $\xi_{S_1} \leq c$ and, by (51) and $|\mathcal{S}_k| \geq p_1$, one has $\xi_{S_k} \leq \xi_{S_1}$ for all $k \in [K]$, it follows from the definition of c that (61) holds. Hence, by Lemma 3.2, $\beta_{S_k} \in [\beta/2, \beta]$ for all $k \in [K]$. Finally, observe from (82) and (83) and the conditions of part (b) that

$$3\epsilon_c = 3\tau_1\xi_{S_1} \leq 3\tau_1c \leq \epsilon^* \leq \frac{R}{\omega} \leq R. \quad (91)$$

We now claim that, for any $k \in \{2, \dots, K\}$ and any $\epsilon' \in (0, 3\epsilon_c)$, if

$$\begin{aligned} \|\nabla L_{S_k}(x_{k-1}, y_{S_k}(x_{k-1}))\| &\leq 3\epsilon_c \\ \text{and } d^T \nabla_{xx}^2 L_{S_k}(x_{k-1}, y_{S_k}(x_{k-1}))d &\geq \beta_{S_k} \|d\|^2 \\ \text{for all } d &\in \text{Null}(\nabla c_{S_k}(x_{k-1})^T), \end{aligned} \quad (92)$$

then Algorithm 2, called with inputs x_{k-1} , $\mathcal{S}_k$, ρ_k , and (ϵ_k, ζ_k) replaced by tolerances $(\epsilon'/\sqrt{5}, \zeta_k)$ and with the fixed step size t_g , reduces to gradient descent applied to minimize F_{S_k, ρ_k} , terminates in at most $\lceil \log_B(3\sqrt{5}\omega/\epsilon')\epsilon_c \rceil$ —more precisely, at most $\lceil \log_B(3\sqrt{5}\omega\epsilon_c/\epsilon') \rceil$ —iterations, and returns a point x_k satisfying (7) with respect to (49) with (ϵ, β) replaced by $(\epsilon', \beta_{S_k}/2)$.

To prove this claim, we verify the conditions of Theorem 2.1 for (49) with $\rho = \rho_k$, $\beta = \beta_{S_k}$, $\kappa = \kappa_{S_k}$, $\epsilon = 3\epsilon_c$, $\lambda_\rho = \bar{\lambda}$, and $t = t_g$. First, the threshold in (25) is increasing in each of β and κ , so with $\beta_{S_k} \leq \beta$ and $\kappa_{S_k} \leq \bar{\kappa}$ its value is at most ρ_β , meaning that (25) holds since $\rho_k > \rho_\beta$. Second, by (80) one has $\bar{\lambda} \geq \beta + \sup_{x \in \mathcal{C}_{S_k, R}} \lambda_1(\nabla^2 F_{S_k, \rho_k}(x)) \geq \beta_{S_k} + \lambda_{R, \rho_k}$, so $\bar{\lambda}$ is valid in place of λ_ρ . Third, by Lemma 3.4 with $(b, \kappa, \rho) = (\beta_{S_k}, \kappa_{S_k}, \rho_k)$ —for which $b \in [\beta/2, \beta]$, $\kappa \leq \bar{\kappa}$, and $\rho \in (\rho_\beta, \rho_{\max}]$ hold—and by (91), one finds $3\epsilon_c \leq \epsilon^* \leq \epsilon_{\rho_k}(\beta_{S_k}, \kappa_{S_k})$, so (26) holds. In addition, $3\epsilon_c \in (0, R)$ by (91), as required in Lemma 2.1. Fourth, by Lemma 3.4 one has $t_g \leq 1/\bar{\lambda} \leq t_{\rho_k}$, so (27) holds. Finally, (92) states that x_{k-1} satisfies (7) with $\epsilon = 3\epsilon_c$ and $\beta = \beta_{S_k}$, and $\epsilon' \in (0, 3\epsilon_c)$. Consequently, Theorem 2.1 applies. By the parameter correspondence (28), the tolerance $\epsilon'/\sqrt{5}$ and step size t_g input to Algorithm 2 are exactly those in (28), so Algorithm 2 reduces to the gradient-descent iteration (33); indeed, by (31) and (41) one has $\lambda_n(\nabla^2 F_{S_k, \rho_k}(x_{k-1}^i)) \geq \beta_{S_k}/2 > 0$ for all i , so the condition in line 5 of Algorithm 2 never holds. Moreover, by (29), the number of iterations performed is at most

$$\left\lceil \log_{\frac{4}{4-\beta_{S_k}t_g}} \left(\frac{3\sqrt{5}\omega\rho_k\epsilon_c}{\epsilon'} \right) \right\rceil \leq \left\lceil \log_B \left(\frac{3\sqrt{5}\omega\epsilon_c}{\epsilon'} \right) \right\rceil,$$

where the inequality follows since $\omega_{\rho_k} \leq \omega$ (as $\rho_k \leq \rho_{\max}$ and $\kappa_{S_k} \leq \bar{\kappa}$), since $3\sqrt{5}\omega\epsilon_c/\epsilon' > 1$, and since the function $4/(4-a)$ is increasing over $a \in [0, 4)$, which with $\beta_{S_k}t_g \geq \beta\mu/(2\bar{\lambda})$ shows that $4/(4-\beta_{S_k}t_g) \geq B$. Lastly, by (30), the returned point x_k satisfies (7) with (ϵ, β) replaced by (ϵ', β') where $\beta' \geq \beta_{S_k}/2 \geq \beta/4 > 0$, which proves the claim.

Now consider $k = 1$. By (83) one has $\hat{\epsilon}_1 \leq \epsilon_c = \tau_1\xi_{S_1} \leq \tau_1c \leq \frac{\sqrt{5}}{2}R$ and $\hat{\zeta}_1 \leq \tau_2\xi_{S_1} \leq \tau_2c \leq 1$, so $(\hat{\epsilon}_1, \hat{\zeta}_1) \in (0, \frac{\sqrt{5}}{2}R] \times (0, 1]$ satisfies the conditions of $(\bar{\epsilon}, \bar{\zeta})$ in Lemma 3.1. In addition, $x_0 \in \mathcal{C}_{S_1, R}$ and $\rho_1 > \rho_\beta \geq \hat{\rho}_1$. Hence, by Lemma 3.1, the call to Algorithm 2 with inputs x_0 , $\mathcal{S}_1$, ρ_1 , and (ϵ_1, ζ_1) terminates in at most $T_{S_1} = \max\{u_{S_1,1}\epsilon_1^{-2}, u_{S_1,2}\zeta_1^{-3}\}$ iterations with x_1 being $(\hat{\epsilon}_1, \hat{\zeta}_1)$ -stationary with respect to (49) with $k = 1$. Furthermore,

$$\begin{aligned} \epsilon_1 &\geq \mu\bar{\epsilon}_1(\hat{\epsilon}_1, \hat{\zeta}_1) = \mu \min\{\sqrt{5}\hat{\epsilon}_1/5, \hat{\zeta}_1/(4\tau_{S_1})\} \\ &\geq \mu^2\xi_{S_1} \min\{\sqrt{5}\tau_1/5, \tau_2/(4\tau_{S_1})\} \\ \text{and } \zeta_1 &\geq \mu\bar{\zeta}_1(\hat{\epsilon}_1, \hat{\zeta}_1) \geq \mu\hat{\zeta}_1/2 \geq \mu^2\tau_2\xi_{S_1}/2, \end{aligned}$$

so one finds from (89) that $T_{S_1} \leq C_1$. Since each iteration of Algorithm 2 in this case requires p_1 individual objective gradients and p_1 individual constraint Jacobians, $k = 1$ requires at most p_1C_1 of each, the first term in (90).

We now prove by induction that, for each $k \in \{2, \dots, K\}$, the point x_{k-1} satisfies (92) and, if $k \leq K-1$, that Algorithm 2 returns x_k that is $(\hat{\epsilon}_k, \hat{\zeta}_k)$ -stationary with respect to (49) in at most $\bar{T}$ iterations. For the base case, recall from the previous paragraph that x_1 is $(\hat{\epsilon}_1, \hat{\zeta}_1)$ -stationary with respect to (49) with $k=1$, where $\hat{\epsilon}_1 \leq \epsilon_c = \tau_1 \xi_{S_1}$ and $\hat{\zeta}_1 \leq \tau_2 \xi_{S_1}$, so that (62) holds for $k=1$. Since (61) holds by previous arguments, Lemma 3.3 with $k=1$ yields (63), which is exactly (92) for $k=2$ since $3\tau_1 \xi_{S_1} = 3\epsilon_c$. Next, for the inductive step, consider $k \in \{2, \dots, K-1\}$ and suppose that x_{k-1} satisfies (92). Since $\epsilon_k = \hat{\epsilon}_k / \sqrt{5}$ with $\hat{\epsilon}_k \in [\mu \epsilon_c, \epsilon_c]$, one has $\hat{\epsilon}_k \in (0, 3\epsilon_c)$, so Theorem 2.1 applies with $\epsilon' = \hat{\epsilon}_k$. Hence Algorithm 2 returns x_k in at most

$$\left\lceil \log_B \left(\frac{3\sqrt{5}\omega\epsilon_c}{\hat{\epsilon}_k} \right) \right\rceil \leq \left\lceil \log_B \left(\frac{3\sqrt{5}\omega}{\mu} \right) \right\rceil = \bar{T}$$

iterations, and x_k satisfies (7) with respect to (49) with (ϵ, β) replaced by $(\hat{\epsilon}_k, \frac{1}{2}\beta_{S_k})$. In particular, $\|\nabla L_{S_k}(x_k, y_{S_k}(x_k))\| \leq \hat{\epsilon}_k$ and, since one trivially finds $\beta_{S_k}/2 > 0 \geq -\hat{\zeta}_k$, it follows that x_k is $(\hat{\epsilon}_k, \hat{\zeta}_k)$ -stationary with respect to (49). Now since $\hat{\epsilon}_k \leq \epsilon_c = \tau_1 \xi_{S_1}$ and $\hat{\zeta}_k \leq \tau_2 \xi_{S_k}$, the tolerances satisfy (62), so Lemma 3.3 yields (63), which is (92) with k replaced by $k+1$. This completes the induction. Overall, since each iteration of Algorithm 2 in iteration k requires p_k individual objective gradients and p_k individual constraint Jacobians, and since $p_k = \theta^{k-1} p_1$ for all $k \in [K-1]$, the total of these evaluations required over iterations $k \in \{2, \dots, K-1\}$ is at most

$$\bar{T} \sum_{k=2}^{K-1} p_k = \bar{T} p_1 \sum_{j=1}^{K-2} \theta^j = \bar{T} \frac{\theta(p_{K-1} - p_1)}{\theta - 1} \leq \bar{T} \frac{\theta(N - p_1)}{\theta - 1},$$

which is the second term in (90). (We remark specifically that if $K=2$, then this sum is empty and the bound holds trivially.)

Finally, consider $k=K$. By the previous induction, x_{K-1} satisfies (92) with $k=K$, where $\beta_{S_K} = \beta$ and $L_{S_K} = L$ by previous arguments. Since $\epsilon_K = \epsilon / \sqrt{5}$ with $\epsilon \in (0, 3\epsilon_c)$, Theorem 2.1 applies with $\epsilon' = \epsilon$, so Algorithm 2 returns x_K in at most $\lceil \log_B(3\sqrt{5}\omega\epsilon_c/\epsilon) \rceil$ iterations, and x_K satisfies (7) with respect to (48) with (ϵ, β) replaced by $(\epsilon, \beta/2)$, i.e., $\|\nabla L(x_K, y(x_K))\| \leq \epsilon$ and $d^T \nabla_{xx}^2 L(x_K, y(x_K)) d \geq \frac{1}{2}\beta \|d\|^2$ for all $d \in \text{Null}(\nabla c(x_K)^T)$. Since $\frac{1}{2}\beta > 0 \geq -\zeta'$ for any $\zeta' \in \mathbb{R}_{\geq 0}$, it follows that x_K is (ϵ, ζ') -stationary for (48) for every $\zeta' \in \mathbb{R}_{\geq 0}$, as claimed. Since each iteration in this case requires N individual objective gradients and N individual constraint Jacobians, iteration $k=K$ requires at most $N \lceil \log_B(3\sqrt{5}\omega\epsilon_c/\epsilon) \rceil$ of each, which is the third term in (90). Summing the bounds from the previous arguments completes the proof. $\square \quad \square$

4 Conclusion

We have proved that a recently proposed Gradient-Eigenstep algorithm for solving equality constrained continuous optimization problems, which is based on minimizing Fletcher's augmented Lagrangian function, reduces to a local-linearly convergent gradient-descent method in the vicinity of certain approximate second-order stationary points when the step size is sufficiently small and the penalty parameter is sufficiently large. We have also leveraged this result to show that a progressive sampling strategy for solving sample-average-type problems can obtain an improved worst-case sample complexity bound as compared to an approach that solves a full-sample problem directly.

Data Availability

We do not analyze or generate any datasets because our work proceeds within a theoretical and mathematical approach.

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