# Generalizing single-level relaxations for bilevel linear programs

Mahmoud Ashraf[1], Jourdain Lamperski[1], and Oleg A. Prokopyev[2,*]

[1] 3700 O'Hara Street, Department of Industrial Engineering,
University of Pittsburgh, Pittsburgh, PA 15213, USA
[2] Plattenstrasse 14, Department of Business Administration,
University of Zurich, Zurich 8032, Switzerland
E-mails: {mam1371,lamperski}@pitt.edu, oleg.prokopyev@business.uzh.ch

**Abstract**

We consider a broad class of bilevel linear programs in which the follower's decisions are all continuous, while the leader's decisions may include integrality restrictions. Solving such problems to optimality is known to be $NP$-hard. A classical approach in bilevel optimization for constructing lower and upper bounds is based on a single-level relaxation, in which the follower's objective function is dropped. Equivalently, the leader is assumed to "fully control" all of the follower's variables, which reduces the original bilevel program to a single-level optimization problem. We propose a generalization of this approach in which the leader controls only a subset of the follower's variables. This simple generalization yields a hierarchy of lower and upper bounds, with an underlying trade-off between the quality of the bounds and the computational difficulty of obtaining them. Finally, we provide some preliminary computational observations.

**Keywords:** bilevel linear programming, mixed-integer programming, single-level relaxation

## 1 Introduction

We consider a class of bilevel optimization problems of the form:

$$
\begin{aligned}
\eta^* := \min_{x,y} \ & \sum_{i=1}^{p} \alpha_i x_i + \sum_{i=1}^{n} \beta_i y_i \\
\text{s.t.} \ & x \in \mathcal{X} := \left\{ Ax \geq b,\ x \in \mathbb{R}_+^{p_1} \times \mathbb{Z}_+^{p_2} \right\}, \\
& y := y(x) \in \arg\max_{\bar{y}} \Big\{ \sum_{i=1}^{n} \gamma_i \bar{y}_i :\ C\bar{y} \leq d - Bx,\ \bar{y} \in \mathbb{R}_+^n \Big\},
\end{aligned}
\tag{BP}
$$

where the lower-level problem is a linear program (LP), while the upper-level decisions may involve integrality restrictions. Let $p := p_1 + p_2$, $A \in \mathbb{Q}^{q\times p}$, $B \in \mathbb{Q}^{m\times p}$, $C \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^q$ and $d \in \mathbb{Q}^m$. Also, to simplify our further notation we assume that $\alpha \in \mathbb{Q}^p$, $\beta \in \mathbb{Q}^n$, and $\gamma \in \mathbb{Q}^n$ are all row vectors. We refer to $x$ and $y$ as the upper-level (leader's) variables and the lower-level (follower's) variables, respectively.

*Corresponding author.

In bilevel problems, the key underlying assumption is that the leader acts first by setting the values of $x$. Then, the follower determines the values of variables $y := y(x)$ in reaction to the leader's choice by solving the lower-level problem. Because the bilevel problem is considered from the leader's perspective, the leader's aim is to select the best solution while taking into account the follower's rational response.

Bilevel problems of the form given by (BP) model hierarchical decision-making processes in a variety of practically relevant application settings. Primary examples include network design [1], pricing [10], and defense [3]. We also refer the reader to recent detailed surveys in [2, 8].

Next, we formally define

$$\mathcal{S} = \{(x, y) \in \mathcal{X} \times \mathbb{R}^n_+ \ : \ Bx + Cy \leq d\},$$

which is the feasible region of the relaxation obtained by dropping the optimality conditions on the follower's variables at the lower level. Given $x \in \mathcal{X}$, we refer to

$$\mathcal{S}(x) = \{y \in \mathbb{R}^n_+ \ : \ (x, y) \in \mathcal{S}\}$$

as the follower's feasible region, which represents the feasible decisions available to the follower, given a fixed leader's solution $x$. Then, the follower's feasible reactions, i.e., the set of optimal solutions of the lower-level problem are given by

$$\mathcal{R}(x) = \{y \in \mathcal{S}(x) \ : \ \gamma y \geq \gamma \bar{y} \quad \forall \bar{y} \in \mathcal{S}(x)\},$$

which is referred to as the follower's (rational) reaction set. Note that the lower-level problem in (BP) can be then re-written as simply $y \in \mathcal{R}(x)$. We say that a pair of leader's and follower's decisions given by $(x, y)$ is *bilevel feasible* if $x \in \mathcal{X}$ and $y \in \mathcal{R}(x)$.

Throughout this note we make the following two assumptions:

**A1**: The leader's feasible set $\mathcal{X}$ is non-empty and bounded. Furthermore, for any $x \in \mathcal{X}$, the follower's set $\mathcal{S}(x)$ is also non-empty and bounded.

**A2**: We consider the *optimistic* version of bilevel problems. That is, for any $x \in \mathcal{X}$, if $\mathcal{R}(x)$ is not a singleton (i.e., the follower's problem has multiple optimal solutions), then the follower picks the one that is most favorable to the leader's objective function.

Assumption **A1** is relatively standard in bilevel optimization literature and simply ensures that (BP) has a finite optimal solution [4]. Assumption **A2** is directly incorporated in our model formulation (BP) as the follower's optimal solution $y$ appears under the minimization symbol at the upper level. Hence, if $(x, y)$ is optimal for (BP), then we must have $y \in \mathcal{R}(x)$ and $\beta y \leq \beta \bar{y}$ for all $\bar{y} \in \mathcal{R}(x)$.

Furthermore, we note that the *optimistic* model is, perhaps, the most commonly used approach in the bilevel optimization literature [8]. Admittedly, there also exists the pessimistic model, where the least favorable response by the follower is considered [16]. Finally, both the optimistic and pessimistic responses can be considered simultaneously, leading to what is known as the strong–weak model [9].

Bilevel problems given by (BP) reduce to bilevel LPs (BLPs) if there are no integrality restrictions at the upper level, i.e., $p_2 = 0$. In contrast to classical single-level LPs, BLPs are known to be computationally difficult, i.e., strongly $NP$-hard; see, e.g., [6, 7]. Moreover, finding even locally optimal solutions is difficult [11]. On the positive side, BLPs, as well as more general classes of bilevel problems with LPs at the lower level as in (BP), can be reformulated as single-level mixed integer linear programs (MILPs) via the LP optimality conditions; see our further discussion on this issue in Section 2.5.

If we relax the optimality requirement for the follower in the lower level, then the resulting *single-level relaxation* of (BP) is an MILP of the form:

$$\eta^{\mathrm{SLR}} := \min_{(x,y)\in\mathcal{S}} \{\alpha x + \beta y\}, \tag{SLR}$$

which provides a lower bound for (BP) because an optimal solution for (BP) is always feasible for (SLR). This problem is also known as the *high-point problem* in the related literature.

Furthermore, an optimal solution for (SLR) can also be used to find an upper bound for (BP). Specifically, assume $(x^0, y^0)$ is optimal for (SLR) and $\hat{y}^0 \in \mathcal{R}(x^0)$ is an optimal follower's decision for the lower-level problem corresponding to $x^0$. Denote by

$$\hat{\eta}^{\mathrm{SLR}}(x^0, \hat{y}^0) := \alpha x^0 + \beta \hat{y}^0$$

the leader's objective function value corresponding to $(x_0, \hat{y}_0)$. Then, we have that:

$$\eta^{\mathrm{SLR}} \leq \eta^* \leq \hat{\eta}^{\mathrm{SLR}}(x^0, \hat{y}^0), \tag{1}$$

which we refer to as the SLR-based bounds. Note that if the leader's decisions are all continuous in (BP), then both the upper and lower bounds in (1) are computable in polynomial time. The SLR-based bounds are commonly exploited for solving various classes of bilevel optimization problems, including those that involve integrality restrictions at the lower level; see, e.g., a recent example in [12] and the references therein.

Note that one way to interpret the SLR-based bounds is that the leader controls all lower-level variables. In this note, we propose a natural generalization of this approach, where the key idea is to assume that the leader controls only a subset of the follower's variables. Hence, the problem remains bilevel, where the lower-level problem becomes smaller as the follower controls only the remaining subset of the lower-level variables. This simple generalization yields a hierarchy of lower and upper bounds, with an underlying trade-off between the quality of the bounds and the computational difficulty of obtaining them.

The remainder of this note is organized as follows. In Section 2, we formally describe the proposed hierarchies of bounds and present a simple approach for computing them. We also discuss several basic properties of these bounds and provide illustrative examples. In Section 3, we report some preliminary numerical observations. Finally, Section 4 summarizes our results and outlines future research directions.

## 2 Hierarchy of bounds

First, in Section 2.1, we formally outline the idea of a partial relaxation of the follower's objective function, which leads to a bilevel problem with a lower-level problem of a reduced dimension. Sections 2.2 and 2.3 then describe the lower and upper bounds for the original problem obtained by solving this simpler bilevel problem. Section 2.4 provides some additional observations and examples. Finally, Section 2.5 presents a single-level MILP reformulation that can be used to compute the developed bounds.

### 2.1 Partial relaxation of the follower's objective function

Recall that the follower's problem has $n$ decision variables. Assume $\mathcal{F} \subseteq [n] := \{1, \ldots, n\}$ and let $f := |\mathcal{F}|$. Define $y_{\mathcal{F}}$ as a vector, which consists of the components of $y$ that are contained in $\mathcal{F}$. That is,

$$y_{\mathcal{F}} := (y_{i_1}, y_{i_2}, \ldots, y_{i_f})$$

where $\mathcal{F} := \{i_1, i_2, \ldots, i_f\}$. Similarly, we define $\mathcal{L} := [n] \setminus \mathcal{F}$, $\ell := |\mathcal{L}| = n - f$ and $y_{\mathcal{L}}$. We also introduce a vector of *auxiliary* variables $\tilde{y}_{\mathcal{F}}$, which has the same dimension as $y_{\mathcal{F}}$. Then, we assume that $(x, y_{\mathcal{L}}, \tilde{y}_{\mathcal{F}}) \in \mathbb{R}^{p_1}_+ \times \mathbb{Z}^{p_2}_+ \times \mathbb{R}^{\ell}_+ \times \mathbb{R}^{f}_+$ and $y_{\mathcal{F}} \in \mathbb{R}^{f}_+$ are leader's and the follower's variables, respectively.

Let $C_{\mathcal{F}}$ and $C_{\mathcal{L}}$ be sub-matrices of $C$ containing the columns that correspond to $\mathcal{F}$ and $\mathcal{L}$, respectively. Then, we define another bilevel problem with the leader's variables given by $(x, y_{\mathcal{L}}, \tilde{y}_{\mathcal{F}})$:

$$\begin{aligned}
\eta^*_{\mathcal{F}} := \min_{(x,y_{\mathcal{L}},\tilde{y}_{\mathcal{F}}),y_{\mathcal{F}}} \quad & \sum_{i=1}^{p} \alpha_i x_i + \sum_{i\in\mathcal{L}} \beta_i y_i + \sum_{i\in\mathcal{F}} \beta_i y_i \\
\text{s.t.} \quad & x \in \mathcal{X} := \big\{Ax \geq b,\ x \in \mathbb{R}^{p_1}_{+} \times \mathbb{Z}^{p_2}_{+}\big\}, \\
& (x, y_{\mathcal{L}}, \tilde{y}_{\mathcal{F}}) \in \mathcal{S}, \\
& y_{\mathcal{F}} \in \arg\max_{\bar{y}_{\mathcal{F}}} \Big\{ \sum_{i\in\mathcal{F}} \gamma_i \bar{y}_i \ :\ C_{\mathcal{F}}\bar{y}_{\mathcal{F}} \leq d - Bx - C_{\mathcal{L}} y_{\mathcal{L}},\ \bar{y}_{\mathcal{F}} \in \mathbb{R}^{f}_{+} \Big\},
\end{aligned} \tag{BP[$\mathcal{F}$]}$$

where in the constraint $(x, y_{\mathcal{L}}, \tilde{y}_{\mathcal{F}}) \in \mathcal{S}$, the vector of lower-level variables is understood as the vector whose $\mathcal{L}$-components are $y_{\mathcal{L}}$ and whose $\mathcal{F}$-components are $\tilde{y}_{\mathcal{F}}$. Recall from Section 1 that set $\mathcal{S}$ is defined for vectors $(x, y)$ formed by the leader's and follower's variables.

Next, there are three important observations that need to be made here, which we outline below.

- First, the leader's variables $\tilde{y}_{\mathcal{F}}$ appear only in the constraints that require $(x, y_{\mathcal{L}}, \tilde{y}_{\mathcal{F}}) \in \mathcal{S}$ at the upper level. Indeed, these variables do not contribute to the leader's objective function, and do not appear in the follower's constraints; only variables $(x, y_{\mathcal{L}})$ appear in the follower's problem. We introduce these auxiliary variables to ensure that for any feasible leader's decision $(x, y_{\mathcal{L}}, \tilde{y}_{\mathcal{F}})$ in (BP[$\mathcal{F}$]), the corresponding follower's problem is feasible. That is, assumption **A1** also holds for (BP[$\mathcal{F}$]).

- Second, if $\mathcal{F} = [n]$, then $\mathcal{L} = \emptyset$, which implies that (BP[$\mathcal{F}$]) and (BP) coincide, i.e., (BP[$\mathcal{F}$]) reduces to the original bilevel problem. Indeed, auxiliary variables $\tilde{y}_{\mathcal{F}}$ can simply be dropped from the model.

- Finally, if $\mathcal{F} = \emptyset$, then $\mathcal{L} = [n]$, which implies that (BP[$\mathcal{F}$]) and (SLR) coincide, i.e., (BP[$\mathcal{F}$]) forms a single-level relaxation of the original bilevel problem.

Generally speaking, (BP[$\mathcal{F}$]) is a natural generalization of the single-level relaxation (SLR), where, instead of *all* follower's variables, the leader "controls" only a subset of the original follower's variables, denoted by set $\mathcal{L} \subseteq [n]$. Meanwhile, the follower retains the control of the variables in $\mathcal{F}$, while $[n] = \mathcal{L} \cup \mathcal{F}$.

### 2.2 Lower bounds

In the remainder of the note, we use $(x^{\mathcal{F}}, y^{\mathcal{F}}_{\mathcal{L}}, \tilde{y}^{\mathcal{F}}_{\mathcal{F}})$ and $y^{\mathcal{F}}_{\mathcal{F}}$ to denote the leader's and the follower's optimal solution for (BP[$\mathcal{F}$]). In this notation, superscripts indicate the relaxation set used to obtain a solution, while subscripts indicate the corresponding variable block.

We first show that (BP[$\mathcal{F}$]) provides a natural lower bound for (BP). Formally:

**Proposition 1.** *For any $\mathcal{F} \subseteq [n]$, $\eta^*_{\mathcal{F}}$ is a valid lower bound for $\eta^*$, i.e., $\eta^*_{\mathcal{F}} \leq \eta^*$. Furthermore, if $\mathcal{F}' \subset \mathcal{F}$, then $\eta^*_{\mathcal{F}'} \leq \eta^*_{\mathcal{F}}$.*

*Proof.* Let $(x^*, y^*)$ form an optimal solution for (BP), and $y^*_{\mathcal{L}}$ and $y^*_{\mathcal{F}}$ be the components of $y^*$ that correspond to $\mathcal{L}$ and $\mathcal{F}$, respectively. Let $\tilde{y}^*_{\mathcal{F}} := y^*_{\mathcal{F}}$. Then, $(x^*, y^*_{\mathcal{L}}, \tilde{y}^*_{\mathcal{F}}, y^*_{\mathcal{F}})$ is bilevel feasible for (BP[$\mathcal{F}$]), where $(x^*, y^*_{\mathcal{L}}, \tilde{y}^*_{\mathcal{F}})$ and $y^*_{\mathcal{F}}$ are the leader's and the follower's decisions, respectively. Hence, $\eta^*_{\mathcal{F}} \leq \eta^*$.

The proof of the second part of the statement proceeds in a similar manner. Recall that $(x^{\mathcal{F}}, y^{\mathcal{F}}_{\mathcal{L}}, \tilde{y}^{\mathcal{F}}_{\mathcal{F}}, y^{\mathcal{F}}_{\mathcal{F}})$ forms an optimal solution for (BP[$\mathcal{F}$]). Denote by $y^{\mathcal{F}}_{\mathcal{L}'}$ and $y^{\mathcal{F}}_{\mathcal{F}'}$ the components of $(y^{\mathcal{F}}_{\mathcal{L}}, y^{\mathcal{F}}_{\mathcal{F}})$ that correspond to $\mathcal{L}'$ and $\mathcal{F}'$, respectively. Also, let $\tilde{y}^{\mathcal{F}}_{\mathcal{F}'} := y^{\mathcal{F}}_{\mathcal{F}'}$. Then, $(x^{\mathcal{F}}, y^{\mathcal{F}}_{\mathcal{L}'}, \tilde{y}^{\mathcal{F}}_{\mathcal{F}'}, y^{\mathcal{F}}_{\mathcal{F}'})$ is bilevel feasible for (BP[$\mathcal{F}'$]) and the result follows. $\square$

In addition, these lower bounds improve progressively as the cardinality of $\mathcal{F}$ increases, or, equivalently the cardinality of $\mathcal{L}$ decreases. That is, we obtain the following hierarchy of lower bounds:

**Proposition 2.** *For any nested sequence $\emptyset = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = [n]$ such that $|\mathcal{F}_k| = k$ for all $k \in \{0\} \cup [n]$, we have that:*

$$\eta^{SLR} = \eta^*_{\mathcal{F}_0} \leq \eta^*_{\mathcal{F}_1} \leq \cdots \leq \eta^*_{\mathcal{F}_n} = \eta^*.$$

*Proof.* The facts that $\eta^{\text{SLR}} = \eta^*_{\mathcal{F}_0}$ for $\mathcal{F}_0 = \emptyset$ and $\eta^* = \eta^*_{\mathcal{F}_n}$ for $\mathcal{F}_n = [n]$ hold by the construction of (BP[$\mathcal{F}$]). The remainder of the proof follows directly from Proposition 1. ☐

### 2.3 Upper bounds

Similar to the construction of the SLR-based upper bound in Section 1, (BP[$\mathcal{F}$]) can be exploited to construct upper bounds for (BP). Formally, let $(x^{\mathcal{F}}, y^{\mathcal{F}}_{\mathcal{L}}, \tilde{y}^{\mathcal{F}}_{\mathcal{F}})$ be an optimal leader's solution for (BP[$\mathcal{F}$]) and $\hat{y}^{\mathcal{F}}$ be an optimal follower's decision to the lower-level problem in (BP), given that the leader's decision is $x^{\mathcal{F}}$, i.e., $\hat{y}^{\mathcal{F}} \in \mathcal{R}(x^{\mathcal{F}})$.

Next, define $\hat{\eta}_{\mathcal{F}}$ as the leader's objective function value in (BP) corresponding to $(x^{\mathcal{F}}, \hat{y}^{\mathcal{F}})$, i.e.,

$$\hat{\eta}_{\mathcal{F}}(x^{\mathcal{F}}, \hat{y}^{\mathcal{F}}) := \alpha x^{\mathcal{F}} + \beta \hat{y}^{\mathcal{F}}, \tag{2}$$

which provides a valid upper bound for (BP). That is:

**Proposition 3.** *For any $\mathcal{F} \subseteq [n]$, let $(x^{\mathcal{F}}, y^{\mathcal{F}}_{\mathcal{L}}, \tilde{y}^{\mathcal{F}}_{\mathcal{F}})$ be a leader's optimal solution for (BP[$\mathcal{F}$]) and $\hat{y}^{\mathcal{F}} \in \mathcal{R}(x^{\mathcal{F}})$. Then, $\hat{\eta}_{\mathcal{F}}(x^{\mathcal{F}}, \hat{y}^{\mathcal{F}})$ is a valid upper bound for $\eta^*$, i.e., $\eta^* \leq \hat{\eta}_{\mathcal{F}}(x^{\mathcal{F}}, \hat{y}^{\mathcal{F}})$.*

*Proof.* It follows directly from the definitions of $(x^{\mathcal{F}}, \hat{y}^{\mathcal{F}})$, where $(x^{\mathcal{F}}, \hat{y}^{\mathcal{F}})$ is clearly bilevel feasible for (BP) by its construction outlined above. ☐

**Corollary 1.** *If $\eta^*_{\mathcal{F}} = \hat{\eta}_{\mathcal{F}}(x^{\mathcal{F}}, \hat{y}^{\mathcal{F}})$, then $\eta^* = \eta^*_{\mathcal{F}}$ and $x^{\mathcal{F}}$ is a leader's optimal solution for (BP).*

It is important to point out that, for any $\mathcal{F} \subset [n]$, the value $\eta^*_{\mathcal{F}}$ is uniquely defined. However, the same does not hold for the upper bound. Indeed, for a given $\mathcal{F}$, a leader's optimal solution for (BP[$\mathcal{F}$]), i.e., $x^{\mathcal{F}}$, is not necessarily unique. Similarly, given $x^{\mathcal{F}}$, the set of the follower's optimal responses $\mathcal{R}(x^{\mathcal{F}})$ is not necessarily a singleton. Consequently, the value of the upper bound, $\hat{\eta}_{\mathcal{F}}$, is a function of both $x^{\mathcal{F}}$ and $\hat{y}^{\mathcal{F}} \in \mathcal{R}(x^{\mathcal{F}})$, which is reflected in (2). This observation also holds for the SLR-based upper bound in the right-hand side of (1), which corresponds to $\mathcal{F} = \emptyset$. Furthermore, in general, for $\mathcal{F} = [n]$, the value of $\hat{\eta}_{\mathcal{F}}$ can be strictly larger than $\eta^*$, since in (2) one is not required to pick $\hat{y}^{\mathcal{F}} \in \mathcal{R}(x^{\mathcal{F}})$ that is most favorable to the leader. However, if $\mathcal{R}(x^{\mathcal{F}})$ is a singleton, then $\hat{\eta}_{\mathcal{F}} = \eta^*$ for $\mathcal{F} = [n]$.

One implication of the preceding discussion is that, in contrast to the lower bound, the quality of the upper bound does not necessarily improve as the cardinality of $\mathcal{F}$ increases. That is, a result similar in spirit to Proposition 2 does not hold in general for $\hat{\eta}_{\mathcal{F}}$. We illustrate this issue below.

**Example 1.** Given an instance of the knapsack interdiction problem:

$$\begin{aligned} \eta^* = \min_{x,y}\ & y_1 + 2y_2 + 2y_3 \\ \text{s.t.}\ & x_1 + 2x_2 + x_3 \leq 2, \quad x \in \{0,1\}^3, \\ & y \in \arg\max_{\bar{y}} \{\bar{y}_1 + 2\bar{y}_2 + 3\bar{y}_3 \ : \ \bar{y}_1 + \bar{y}_2 + \bar{y}_3 \leq 2,\ x_i + \bar{y}_i \leq 1 \ \ \forall i \in \{1,2,3\},\ \bar{y} \in \mathbb{R}^3_+\}, \end{aligned}$$

we consider $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3$, where $\mathcal{F}_0 = \emptyset$, $\mathcal{F}_1 = \{3\}$, $\mathcal{F}_2 = \{2,3\}$, and $\mathcal{F}_3 = \{1,2,3\}$. That is, $\mathcal{F}_0$ and $\mathcal{F}_3$ correspond to the single-level relaxation and the original bilevel problem respectively. Then:

• For $\mathcal{F}_0 = \emptyset$, the leader controls all follower's variables, which are set to 0. Then, $\eta^{\text{SLR}} = \eta^*_{\mathcal{F}_0} = 0$ and $x^{\mathcal{F}_0} = (0,0,0)^\top$ is one of the optimal leader's decisions for (BP[$\mathcal{F}$]). Thus, $\hat{y}^{\mathcal{F}_0} = (0,1,1)^\top$ and $\hat{\eta}_{\mathcal{F}_0} = 4$.

• For $\mathcal{F}_1 = \{3\}$, the leader controls $y_1$ and $y_2$, which are set to 0. Hence, $x^{\mathcal{F}_1} = (0,0,1)^\top$ is one of the leader's optimal decisions, with $\eta^*_{\mathcal{F}_1} = 0$. Then, $\hat{y}^{\mathcal{F}_1} = (1,1,0)^\top$ and $\hat{\eta}_{\mathcal{F}_1} = 3$.

• For $\mathcal{F}_2 = \{2,3\}$, the leader controls only $y_1$. However, observe that $x^{\mathcal{F}_2} = (0,0,1)^\top$ remains one of the leader's optimal decisions, with $\eta^*_{\mathcal{F}_2} = 2$. Then, $\hat{y}^{\mathcal{F}_2} = (1,1,0)^\top$ and $\hat{\eta}_{\mathcal{F}_2} = 3$.

• For $\mathcal{F}_3 = \{1,2,3\}$, the leader does not control any follower's variable. Hence, $x^{\mathcal{F}_3} = (1,0,1)^\top$ is a leader's optimal solution, with $\eta^*_{\mathcal{F}_3} = 2$. Then, $\hat{y}^{\mathcal{F}_3} = (0,1,0)^\top$ and $\hat{\eta}_{\mathcal{F}_3} = 2$.

In this example, we have a monotonically non-decreasing sequence of lower bounds, i.e., $0 \leq 0 \leq 2 \leq 2$, which is consistent with Proposition 2. Similarly, we have a monotonically non-increasing sequence of upper bounds, i.e., $4 \geq 3 \geq 3 \geq 2$. However, observe that for $\mathcal{F}_1 = \{3\}$, the leader has multiple optimal solutions. That is, $x^{\mathcal{F}_1} = (1,0,1)^\top$ is also optimal. Then, $\hat{y}^{\mathcal{F}_1} = (0,1,0)^\top$ and $\hat{\eta}_{\mathcal{F}_1} = 2$. Consequently, the corresponding sequence of upper bounds is not monotonically non-increasing, i.e., $4 \geq 2 \not\geq 3 \geq 2$. $\square$

To obtain a monotonically non-increasing sequence of upper bounds, one can slightly modify the definition in (2). Formally, consider a non-decreasing sequence $\emptyset = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = [n]$ such that $|\mathcal{F}_k| = k$ for all $k \in \{0\} \cup [n]$. Assume also that for each $\mathcal{F} \in \{\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_n\}$ we are given the corresponding optimal solution, $(x^{\mathcal{F}}, y^{\mathcal{F}}_{\mathcal{L}}, \tilde{y}^{\mathcal{F}}_{\mathcal{F}})$, for (BP[$\mathcal{F}$]) and $\hat{y}^{\mathcal{F}} \in \mathcal{R}(x^{\mathcal{F}})$. Then, we define:

$$\hat{\eta}'_{\mathcal{F}_k} := \min_{t=0,\ldots,k} \{\alpha x^{\mathcal{F}_t} + \beta \hat{y}^{\mathcal{F}_t}\}, \tag{3}$$

which allows us to provide the following hierarchy of upper bounds.

**Proposition 4.** *For any nested sequence $\emptyset = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = [n]$ such that $|\mathcal{F}_k| = k$ for all $k \in \{0\} \cup [n]$, we have that:*

$$\hat{\eta}^{SLR} = \hat{\eta}'_{\mathcal{F}_0} \geq \hat{\eta}'_{\mathcal{F}_1} \geq \ldots \geq \hat{\eta}'_{\mathcal{F}_n} \geq \eta^*.$$

*Proof.* Follows directly from Proposition 3 and the definition given by (3). $\square$

### 2.4 Additional observations

One may conjecture that, in order to obtain tighter relaxations, it is favorable to include into set $\mathcal{L}$ (i.e., by removing from $\mathcal{F}$) the indices corresponding to the follower's variables that take zero values in the bilevel optimal solution. Intuitively, this approach allows the leader to control those follower's variables that are zero (and hence, in a sense, not "used" by the follower) in the optimal bilevel solution. However, the following example illustrates that this strategy can be both advantageous and disadvantageous.

**Example 2.** Consider an instance of the knapsack interdiction problem given by:

$$\begin{aligned}
\eta^* = \min_{x,y}\ & y_1 + y_2 + My_3 \\
\text{s.t.}\ & x_1 + x_2 + x_3 \leq 1, \quad x \in \{0,1\}^3, \\
& y \in \arg\max_{\bar{y}} \big\{ \bar{y}_1 + 2\bar{y}_2 + 3\bar{y}_3 \ :\ \bar{y}_1 + \bar{y}_2 + \bar{y}_3 \leq 1,\ x_i + \bar{y}_i \leq 1 \ \ \forall i \in \{1,2,3\},\ \bar{y} \in \mathbb{R}^3_+ \big\},
\end{aligned}$$

where we assume $M$ is some constant parameter and $M > 1$. Naturally, the bilevel optimal solution is setting $x^* = (0,0,1)^\top$, which results in $\eta^* = 1$. Note that $y(x^*) = (0,1,0)^\top$. Given the intuition considered above, one can consider setting either $\mathcal{L} = \{1\}$, or $\mathcal{L} = \{3\}$.

First, if $\mathcal{L} = \{1\}$, i.e., $\mathcal{F} = \{2,3\}$, then the leader sets $x^{\mathcal{F}} = (0,0,1)^\top$. Indeed, the leader controls $y_1$; hence, the leader can set $y^{\mathcal{F}}_1 = 0$ in (BP[$\mathcal{F}$]). Consequently, $\eta^*_{\mathcal{F}} = 1$, and the corresponding $\hat{\eta}_{\mathcal{F}} = 1$, which implies that the upper and lower bounds coincide, and $x^{\mathcal{F}} = (0,0,1)^\top$ is optimal for (BP).

On the other hand, consider $\mathcal{L} = \{3\}$, i.e., $\mathcal{F} = \{1, 2\}$. Then, the leader may set $x^{\mathcal{F}} = (0, 1, 0)^{\top}$, which results in $\eta^*_{\mathcal{F}} = 1$. Consequently, $\hat{\eta}_{\mathcal{F}} = M$, which implies that the upper bound can be arbitrarily bad. □

Next, we define:

$$\Theta := \{\theta \in \mathbb{R}^m \ : \ \exists x \in \mathcal{X} \text{ such that } \theta = d - Bx\},$$

which is a set of all possible right-hand sides of the follower's LP in (BP). Then, we say that set $L \subseteq [n]$ is a follower's *null* set if for any $\theta \in \Theta$, there exists $y^0 \in \arg\max\{\gamma\bar{y} \ : \ C\bar{y} \leq \theta, \ \bar{y} \in \mathbb{R}^n_+\}$ such that:

$(i)$ $y^0_i = 0$ for all $i \in L$, and

$(ii)$ for all $y \in \arg\max\{\gamma\bar{y} \ : \ C\bar{y} \leq \theta, \ \bar{y} \in \mathbb{R}^n_+\}$, we have that $\beta y \geq \beta y^0$.

The latter condition is trivially satisfied if the follower has a unique optimal solution for every $\theta \in \Theta$. Otherwise, this condition ensures that $y^0$ is picked whenever the follower has multiple optimal solutions; recall Assumption **A2**.

**Proposition 5.** *Let $\mathcal{L} := [n] \setminus \mathcal{F}$ be a follower's null set. Let $(x^{\mathcal{F}}, y^{\mathcal{F}}_{\mathcal{L}}, \tilde{y}^{\mathcal{F}}_{\mathcal{F}})$ be the corresponding leader's optimal solution in (BP[$\mathcal{F}$]). If $y^{\mathcal{F}}_{\mathcal{L}} = 0$, then $\eta^*_{\mathcal{F}} = \eta^*$ and $x^{\mathcal{F}}$ is a leader's optimal solution for (BP).*

*Proof.* First, we note that $\eta^*_{\mathcal{F}} \leq \eta^*$; recall Proposition 1. By construction: $\eta^*_{\mathcal{F}} = \alpha x^{\mathcal{F}} + \sum_{i\in\mathcal{F}} \beta_i y^{\mathcal{F}}_i$, where vector $\{y^{\mathcal{F}}_i\}_{i\in\mathcal{F}}$ forms an optimal (optimistic) solution for the follower's LP in (BP[$\mathcal{F}$]).

Next, consider the follower's LP in (BP) for a given leader's decision $x^{\mathcal{F}}$. Let $\hat{y}^{\mathcal{F}} \in \mathcal{R}(x^{\mathcal{F}})$ be an optimal solution for the follower's LP, which is also most favorable to the leader among those in $\mathcal{R}(x^{\mathcal{F}})$. Since $\mathcal{L}$ is a null set, it implies that $\hat{y}^{\mathcal{F}}_i = 0$ for all $i \in \mathcal{L}$. Then, vector $y := (\{0\}_{i\in\mathcal{L}}, \{y^{\mathcal{F}}_i\}_{i\in\mathcal{F}}) \in \mathcal{R}(x^{\mathcal{F}})$. Furthermore, by construction, $\sum_{i\in\mathcal{F}} \beta_i y^{\mathcal{F}}_i \geq \sum_{i\in\mathcal{F}} \beta_i \hat{y}^{\mathcal{F}}_i$. Consequently, $\eta^*_{\mathcal{F}} = \hat{\eta}_{\mathcal{F}}(x^{\mathcal{F}}, \hat{y}^{\mathcal{F}})$ and the result follows from Corollary 1. □

A natural question is why verification of $y^{\mathcal{F}}_{\mathcal{L}} = 0$ is required. The underlying intuition is that the leader in (BP[$\mathcal{F}$]) may choose $y^{\mathcal{F}}_{\mathcal{L}} \neq 0$ to induce a favorable reaction from the follower in the lower-level problem.

As a final remark, we note that finding an appropriate null set, in general, should be nontrivial. Indeed, by its definition, it requires considering all possible right-hand sides in the follower's problem. In fact, such nontrivial sets may not even exist. Nevertheless, this analytical observation demonstrates that the proposed bounds can exhibit strong theoretical performance in certain settings.

### 2.5 Single-level reformulation

A standard approach for solving bilevel problems with an LP at the lower level is to reformulate them as single-level optimization problems using the LP complementary slackness (CS) conditions; see, e.g., [15] and the discussion therein. Such reformulations can then be tackled by standard off-the-shelf MIP solvers. Similarly, problem (BP[$\mathcal{F}$]) admits a single-level reformulation, which we provide next:

$$\begin{aligned}
\eta^*_{\mathcal{F}} = \min_{x,y} \ & \sum_{i=1}^{p} \alpha_i x_i + \sum_{i=1}^{n} \beta_i y_i \\
\text{s.t.} \ & x \in \mathcal{X} := \left\{Ax \geq b, \ x \in \mathbb{R}^{p_1}_+ \times \mathbb{Z}^{p_2}_+\right\}, \\
& C_{\mathcal{L}} y_{\mathcal{L}} + C_{\mathcal{F}} y_{\mathcal{F}} \leq d - Bx, \\
& C^{\top}_{\mathcal{F}} \lambda \geq \gamma^{\top}_{\mathcal{F}}, \\
& \lambda^{\top}(d - Bx - C_{\mathcal{L}} y_{\mathcal{L}} - C_{\mathcal{F}} y_{\mathcal{F}}) = 0, \\
& y^{\top}_{\mathcal{F}}(C^{\top}_{\mathcal{F}} \lambda - \gamma^{\top}_{\mathcal{F}}) = 0, \\
& y_{\mathcal{L}} \in \mathbb{R}^{n-f}_+, \ y_{\mathcal{F}} \in \mathbb{R}^{f}_+, \ \lambda \in \mathbb{R}^m_+,
\end{aligned} \tag{BP[$\mathcal{F}$]-CS}$$

where the dual variables for the follower's LP problem in (BP[$\mathcal{F}$]) are denoted by $\lambda$. Note that, in contrast to (BP[$\mathcal{F}$]), the single-level model (BP[$\mathcal{F}$]-CS) does not require the auxiliary variables $\tilde{y}_{\mathcal{F}}$, as the feasibility of the follower's LP is directly enforced. Indeed, one can simply apply the standard CS-based reformulation to (BP[$\mathcal{F}$]) and observe that the resulting model can be simplified by setting $\tilde{y}_{\mathcal{F}}$ to be equal to $y_{\mathcal{F}}$.

The *key insight* is that (BP[$\mathcal{F}$]-CS) contains $O(f+m)$ complementarity pairs, which can either be linearized using a big-$M$ method or addressed directly via the solver's capability to handle SOS1 constraints or bilinear terms. In either case, the model's complexity (and potentially, the solver's running time) decreases as $f := |\mathcal{F}|$, i.e., the number of variables controlled by the follower in (BP[$\mathcal{F}$]), decreases. On the other hand, the quality of the corresponding upper and lower bounds improves as $f$ increases; recall Propositions 2 and 4. Naturally, there is a clear trade-off, which we explore numerically in Section 3.

In fact, one can also point to another theoretical argument supporting the outlined approach. Bilevel linear programs (BLPs) with a fixed number of the follower's variables are known to be polynomially solvable [5, 7]. Hence, in view of (BP[$\mathcal{F}$]), when $p_2 = 0$, our approach provides polynomially computable upper and lower bounds for BLPs whenever the cardinality of the set $\mathcal{F}$, i.e., $f$, is fixed.

# 3 Proof-of-concept: preliminary computational observations

Next, we present the results of our preliminary computational experiments. Note that an extensive computational study is beyond the scope of this short note. Instead, our goal is to highlight potential benefits of the proposed bounding schemes and directions for future theoretical and computational work.

## 3.1 Experimental setup

Our test instances are available in a GitHub repository[1]. We consider two sets of test instances; see Sections 3.3 and 3.4, respectively. In the first set, all upper-level variables $x$ are integer-valued and upper bounded by 10. In the second set, half of the upper-level variables are integer-valued and the other half are continuous, with all upper-level variables bounded by 10. Recall also that all follower's decision variables are continuous in (BP). The remaining parameters are held the same for both test sets.

Specifically, the numbers of decision variables and constraints at each level are set equal, i.e., $p_1+p_2 = n$ and $q = m$. For the pure integer case (in the upper level), we consider $n \in \{10, 60, 110, 160, \ldots, 410, 460\}$ with $m = 0.4n$ and $m = 0.6n$. The instances sourced from BOBILib [13] (originally from [14]) cover $n \in \{10, 60, \ldots, 460\}$ with $m = 0.4n$. For $m = 0.6n$, we follow the approach introduced in [14]. For the mixed-integer case (in the upper level), we also consider larger instances, where $n \in \{600, 800, 1000\}$ with $m \in \{0.4n, 0.6n\}$. These instances are generated following the approach similar to that used in [14]. Specifically, all problem coefficients are integers generated randomly and independently from a discrete uniform distribution over specified intervals. The objective function coefficients, i.e., the elements of $\alpha$, $\beta$, and $\gamma$, are drawn from $[-50, 50]$. The coefficients of the constraint matrices $A$, $B$, and $C$ are generated within $[0, 10]$. The right-hand sides at the upper and lower levels, i.e., $b$ and $d$, are sampled from $[30, 130]$ and $[10, 110]$, respectively. For each pair $(n, m)$, we consider 10 instances and report average performance metrics.

For the performance metrics, we define $\delta_L$ and $\delta_U$ as the relative percentage deviation of the lower and upper bounds, $(\eta^*_{\mathcal{F}}, \hat{\eta}_{\mathcal{F}})$, from the true optimal objective function value $\eta^*$, computed as:

$$\delta_L := \frac{|\eta^*_{\mathcal{F}} - \eta^*|}{|\eta^*|} \times 100 \quad \text{and} \quad \delta_U := \frac{|\hat{\eta}_{\mathcal{F}} - \eta^*|}{|\eta^*|} \times 100,$$

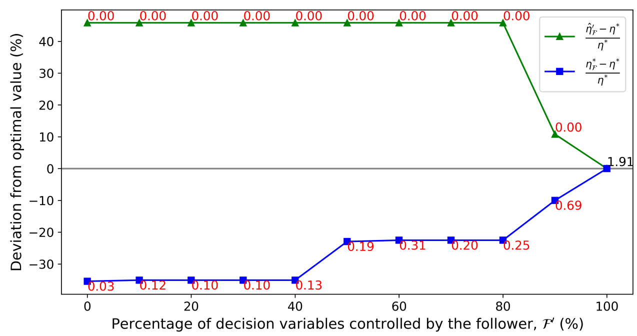
[1] https://github.com/mhmoud-ashraf/slr-generalization

respectively. Additionally, we define:

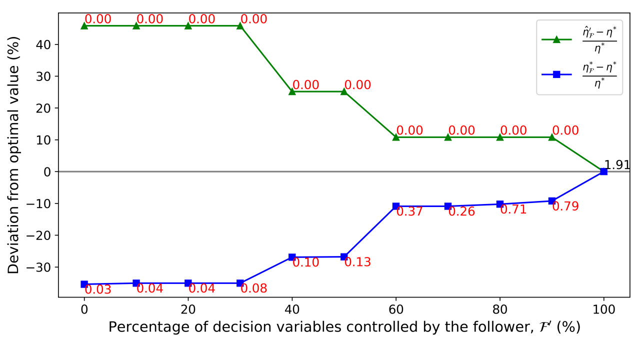
$$\delta := \frac{|\hat{\eta}_{\mathcal{F}} - \eta^*_{\mathcal{F}}|}{|\hat{\eta}_{\mathcal{F}}|} \times 100,$$

which is the percentage optimality gap between $\eta^*_{\mathcal{F}}$ and $\hat{\eta}_{\mathcal{F}}$.

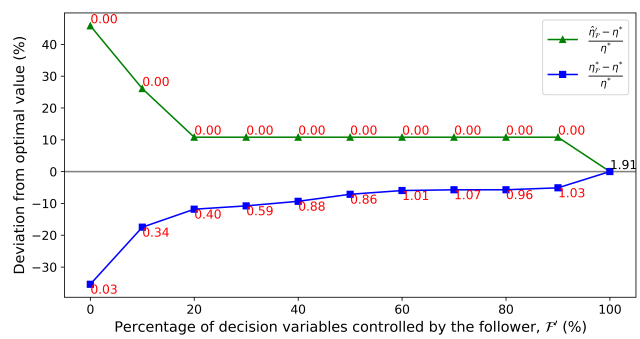
Finally, all computational experiments are performed using Gurobi 11.0.3. All experiments are coded in Python 3.10 and executed on a Windows PC with 3.2GHz Intel Core i5 processor and 28GB of RAM. We use Gurobi's native support for SOS1 constraints to handle the complementary slackness constraints in (BP[$\mathcal{F}$]-CS). In our computational experiments below, all running times are reported in seconds.




(a) Random relaxation ($H_R$)




(b) Non-increasing $\gamma$-based relaxation ($H_\gamma$)




(c) Non-increasing $\beta$-based relaxation ($H_\beta$)

Figure 1: Deviation percentage of upper (green)/lower (blue) bounds from the optimal value of (BP), i.e., $(\eta^*_{\mathcal{F}} - \eta^*)/\eta^*$ and $(\hat{\eta}'_{\mathcal{F}} - \eta^*)/\eta^*$ , for different relaxation approaches, where $n = 110$ and $m = 0.4n$. The runtimes (in seconds) for (BP[$\mathcal{F}$]) and (BP) are shown in red and black, respectively.

## 3.2 Constructing $\mathcal{F}$

In our study, we consider three simple approaches for constructing the set $\mathcal{F}$, hereafter referred to as $H_R$, $H_\gamma$, and $H_\beta$. Each approach first generates an ordering of the follower's variables. For a given cardinality $f = |\mathcal{F}|$, the set $\mathcal{F}$ is then defined as the first $f$ variables in this ordering.

The first approach, $H_R$, uses the `random.shuffle()` method in Python to generate a random ordering of the follower's variables. The second approach, $H_\gamma$, sorts the follower's variables in the *non-increasing order* of the $\gamma$ coefficients (i.e., the follower's objective function). That is, the follower controls the variables that are, in a sense, most important to the follower's objective function. Finally, the third approach, $H_\beta$, sorts the follower's variables in the *non-increasing order* of the $\beta$ coefficients. Thus, the follower controls the variables that are least favorable to the leader's minimization objective. In other words, these variables are potentially most detrimental for the leader's objective function.

### 3.3 Results for pure integer leader's decisions

Figure 1 illustrates the quality of the upper and lower bounds for an instance with $(n, m) = (110, 0.4n)$ at different relaxation levels, i.e., the cardinality of $\mathcal{F}$. The $x$-axis represents these relaxation levels in the percentage of decision variables controlled by the follower, i.e., $|\mathcal{F}|/n \times 100\%$. The $y$-axis shows the deviation percentage of the lower and upper bounds $(\eta^*_{\mathcal{F}}, \hat{\eta}'_{\mathcal{F}})$ from the true optimal objective function value $\eta^*$, depicted in blue and green, respectively. The runtime to compute each bound is labeled in red, while the black label represents the time required to solve (BP) to optimality.

Figures 1(a)-1(c) illustrate the progressive improvement in the obtained bounds as the follower controls more decision variables. The monotonic increase in the lower bounds and decrease in the upper bounds align with our theoretical observations in Propositions 2 and 4. Notably, the $\beta$-based relaxation $H_\beta$ seems to provide the tightest bounds. A possible explanation for this behavior is that the follower controls the variables that are, in a sense, most impactful for the leader's objective function.

Tables 1-3 summarize the average performance of each relaxation approach across the 10 instances for each size pair $(n, m)$. The columns $m$, $n$, $\text{Time}_{\text{BP}}$, $|\mathcal{F}| = 0\%$, and $\text{BP}_0$ are identical in all tables, as they are independent of the relaxation approach. The columns $m$ and $n$ report the number of constraints and decision variables at each level, respectively. The $\text{Time}_{\text{BP}}$ column reports the average runtime to solve (BP) to optimality using the solver. For each relaxation level, we report $\delta_L$, $\delta_U$, and $\delta$ as well as the runtime to compute these bounds (Time).

Table 1: The average performance of a random-based relaxation of the follower's objective $(H_R)$ for 10 instances of each size $(n, m)$. Superscripts in the $\text{BP}_0$ and $\text{BP}_{90}$ columns indicate the number of instances, where finite bounds are found within the specified time limit.

| $m$ | $n$ | $\text{Time}_{\text{BP}}$ | $\lvert\mathcal{F}\rvert = 0\%$ | | | | $\lvert\mathcal{F}\rvert = 50\%$ | | | | $\lvert\mathcal{F}\rvert = 90\%$ | | | | $\text{BP}_0$ | | | $\text{BP}_{90}$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $\delta_L$ | $\delta_U$ | $\delta$ | Time | $\delta_L$ | $\delta_U$ | $\delta$ | Time | $\delta_L$ | $\delta_U$ | $\delta$ | Time | $\delta_L$ | $\delta_U$ | $\delta$ | $\delta_L$ | $\delta_U$ | $\delta$ |
| $0.4n$ | 10 | 0.01 | 33.10 | 17.90 | 761.47 | 0.01 | 19.03 | 15.12 | 727.12 | 0.01 | 1.67 | 3.71 | 6.97 | 0.01 | 29.02$^{(9)}$ | 10.46$^{(9)}$ | 50.01$^{(9)}$ | 8.48$^{(10)}$ | 4.64$^{(10)}$ | 15.76$^{(10)}$ |
| | 60 | 0.16 | 33.94 | 54.78 | 608.72 | 0.02 | 20.17 | 37.40 | 219.73 | 0.05 | 8.15 | 26.98 | 75.36 | 0.10 | 272.65$^{(7)}$ | 27.47$^{(1)}$ | 115.08$^{(1)}$ | 6.59$^{(10)}$ | 4.20$^{(10)}$ | 12.23$^{(10)}$ |
| | 110 | 1.14 | 22.10 | 51.56 | 276.52 | 0.04 | 14.24 | 38.98 | 1180.29 | 0.15 | 7.52 | 16.82 | 48.46 | 0.37 | - | - | - | 7.82$^{(10)}$ | 9.27$^{(6)}$ | 17.84$^{(6)}$ |
| | 160 | 2.70 | 23.65 | 48.77 | 209.20 | 0.09 | 20.50 | 36.05 | 112.72 | 0.31 | 10.87 | 20.51 | 58.32 | 0.76 | - | - | - | 12.53$^{(10)}$ | 9.40$^{(7)}$ | 28.42$^{(7)}$ |
| | 210 | 6.74 | 25.75 | 56.94 | 1022.34 | 0.19 | 19.90 | 43.18 | 940.44 | 0.51 | 11.60 | 19.21 | 49.00 | 1.52 | 3466.96$^{(2)}$ | - | - | 13.65$^{(10)}$ | 12.70$^{(7)}$ | 34.75$^{(7)}$ |
| | 260 | 17.46 | 31.25 | 64.06 | 747.09 | 0.25 | 25.71 | 62.67 | 662.48 | 0.71 | 10.81 | 23.37 | 56.03 | 3.77 | - | - | - | 12.00$^{(10)}$ | 10.63$^{(9)}$ | 26.43$^{(9)}$ |
| | 310 | 34.12 | 32.18 | 52.00 | 393.09 | 0.39 | 26.39 | 44.29 | 218.24 | 1.23 | 15.07 | 24.52 | 70.01 | 3.54 | 2734.89$^{(2)}$ | 40.80$^{(1)}$ | 120.34$^{(1)}$ | 16.47$^{(10)}$ | 12.86$^{(10)}$ | 34.58$^{(10)}$ |
| | 360 | 125.31 | 28.87 | 72.93 | 697.36 | 0.63 | 24.16 | 56.65 | 566.76 | 2.17 | 14.12 | 32.28 | 108.72 | 7.37 | 2377.46$^{(5)}$ | 16.73$^{(1)}$ | 60.19$^{(1)}$ | 15.75$^{(10)}$ | 10.65$^{(8)}$ | 30.73$^{(8)}$ |
| | 410 | 285.95 | 33.62 | 65.39 | 431.89 | 0.66 | 27.52 | 53.82 | 234.83 | 2.49 | 16.71 | 29.07 | 86.47 | 11.51 | 2408.45$^{(3)}$ | - | - | 19.15$^{(10)}$ | 21.41$^{(7)}$ | 55.15$^{(7)}$ |
| | 460 | 552.38 | 41.14 | 81.98 | 465.23 | 1.11 | 31.89 | 79.36 | 327.34 | 3.88 | 13.89 | 40.38 | 120.10 | 54.52 | 5082.84$^{(7)}$ | 42.46$^{(2)}$ | 7829.06$^{(2)}$ | 18.81$^{(10)}$ | 15.08$^{(9)}$ | 48.36$^{(9)}$ |
| $0.6n$ | 10 | 0.01 | 82.26 | 85.78 | 265.82 | 0.01 | 26.72 | 31.66 | 158.55 | 0.01 | 13.18 | 12.23 | 61.71 | 0.01 | 0.38$^{(10)}$ | 4.71$^{(10)}$ | 9.61$^{(10)}$ | 12.13$^{(10)}$ | 1.91$^{(10)}$ | 16.21$^{(10)}$ |
| | 60 | 0.22 | 13.01 | 36.95 | 343.31 | 0.02 | 11.42 | 36.34 | 404.60 | 0.08 | 2.82 | 5.55 | 10.12 | 0.11 | 3.85$^{(10)}$ | 15.85$^{(10)}$ | 46.23$^{(10)}$ | 3.07$^{(10)}$ | 6.08$^{(10)}$ | 10.68$^{(10)}$ |
| | 110 | 0.99 | 22.77 | 37.48 | 156.25 | 0.06 | 17.31 | 32.67 | 120.47 | 0.20 | 7.20 | 14.99 | 31.27 | 0.43 | 14.02$^{(9)}$ | 0.05$^{(4)}$ | 8.51$^{(4)}$ | 10.34$^{(10)}$ | 0.31$^{(7)}$ | 7.97$^{(7)}$ |
| | 160 | 6.43 | 35.61 | 62.80 | 1431.79 | 0.13 | 26.52 | 57.42 | 290.02 | 0.46 | 12.21 | 33.98 | 298.03 | 1.31 | 24.38$^{(10)}$ | 36.92$^{(3)}$ | 156.42$^{(3)}$ | 14.03$^{(10)}$ | 12.58$^{(7)}$ | 32.99$^{(7)}$ |
| | 210 | 10.65 | 31.08 | 63.48 | 763.92 | 0.24 | 23.92 | 45.98 | 106.18 | 1.06 | 13.54 | 18.31 | 47.87 | 2.97 | 28.89$^{(10)}$ | 29.40$^{(4)}$ | 164.04$^{(4)}$ | 17.83$^{(10)}$ | 25.10$^{(8)}$ | 101.80$^{(8)}$ |
| | 260 | 22.04 | 35.37 | 85.97 | 987.29 | 0.42 | 28.83 | 78.99 | 976.72 | 1.40 | 15.84 | 30.29 | 260.09 | 4.59 | 554.20$^{(10)}$ | - | - | 19.17$^{(10)}$ | 8.78$^{(8)}$ | 27.00$^{(8)}$ |
| | 310 | 147.13 | 36.44 | 59.00 | 399.33 | 0.62 | 29.99 | 55.80 | 296.81 | 1.85 | 16.68 | 21.49 | 60.46 | 8.79 | 36.76$^{(10)}$ | 10.57$^{(3)}$ | 45.37$^{(3)}$ | 22.32$^{(10)}$ | 6.33$^{(6)}$ | 28.11$^{(6)}$ |
| | 360 | 289.85 | 32.42 | 70.15 | 1002.31 | 0.66 | 26.64 | 59.77 | 734.48 | 3.80 | 15.42 | 56.80 | 200.90 | 17.89 | 897.05$^{(10)}$ | 16.25$^{(1)}$ | 62.93$^{(1)}$ | 17.42$^{(10)}$ | 7.05$^{(7)}$ | 26.33$^{(7)}$ |
| | 410 | 1020.45 | 43.72 | 87.86 | 583.86 | 1.05 | 33.64 | 51.60 | 379.97 | 4.83 | 21.93 | 25.12 | 78.95 | 25.15 | 44.63$^{(10)}$ | - | - | 26.44$^{(10)}$ | 6.96$^{(4)}$ | 28.54$^{(4)}$ |
| | 460 | 6468.90 | 43.84 | 75.78 | 1415.02 | 1.29 | 38.61 | 84.55 | 1390.98 | 7.97 | 25.65 | 56.09 | 967.99 | 40.98 | 46.95$^{(10)}$ | 53.72$^{(1)}$ | 202.37$^{(1)}$ | 25.88$^{(10)}$ | 17.65$^{(10)}$ | 84.56$^{(10)}$ |

The $\text{BP}_0$ and $\text{BP}_{90}$ columns report $\delta_L$, $\delta_U$, and $\delta$ when solving (BP), with the solver limited to run for the same amount of time (using the `TimeLimit` parameter) needed to obtain the relaxation-based bounds at $|\mathcal{F}| = 0\%$ and $|\mathcal{F}| = 90\%$, respectively. We use the Gurobi attributes `ObjVal`, `ObjBound`, and `MIPGap`, which correspond to the incumbent objective function value, best lower bound, and the relative

Table 2: The average performance of a $\gamma$-based relaxation of the follower's objective ($H_\gamma$) for 10 instances of each size $(n, m)$. Superscripts in the $\text{BP}_0$ and $\text{BP}_{90}$ columns indicate the number of instances, where finite bounds are found within the specified time limit.

| $m$ | $n$ | $\text{Time}_{\text{BP}}$ | $\lvert\mathcal{F}\rvert = 0\%$ | | | | $\lvert\mathcal{F}\rvert = 50\%$ | | | | $\lvert\mathcal{F}\rvert = 90\%$ | | | | $\text{BP}_0$ | | | $\text{BP}_{90}$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $\delta_L$ | $\delta_U$ | $\delta$ | Time | $\delta_L$ | $\delta_U$ | $\delta$ | Time | $\delta_L$ | $\delta_U$ | $\delta$ | Time | $\delta_L$ | $\delta_U$ | $\delta$ | $\delta_L$ | $\delta_U$ | $\delta$ |
| | 10 | 0.01 | 33.10 | 17.90 | 761.47 | 0.01 | 9.34 | 4.79 | 17.11 | 0.01 | 7.32 | 3.83 | 14.00 | 0.01 | $29.02^{(9)}$ | $10.46^{(9)}$ | $50.01^{(9)}$ | $23.17^{(10)}$ | $24.86^{(10)}$ | $477.27^{(10)}$ |
| | 60 | 0.16 | 33.94 | 54.78 | 608.72 | 0.02 | 21.17 | 46.21 | 407.71 | 0.02 | 8.20 | 22.20 | 196.45 | 0.10 | $272.65^{(7)}$ | $27.47^{(1)}$ | $115.08^{(1)}$ | $6.32^{(10)}$ | $7.75^{(10)}$ | $17.39^{(10)}$ |
| | 110 | 1.14 | 22.10 | 51.56 | 276.52 | 0.04 | 13.89 | 36.75 | 1354.30 | 0.08 | 5.98 | 19.12 | 48.19 | 0.32 | - | - | - | $9.29^{(10)}$ | $19.24^{(5)}$ | $41.49^{(5)}$ |
| | 160 | 2.70 | 23.65 | 48.77 | 209.20 | 0.09 | 16.77 | 25.40 | 89.33 | 0.15 | 10.86 | 23.31 | 82.41 | 0.59 | - | - | - | $14.77^{(10)}$ | $11.65^{(7)}$ | $35.03^{(7)}$ |
| $0.4n$ | 210 | 6.74 | 25.75 | 56.94 | 1022.34 | 0.19 | 16.10 | 35.10 | 863.87 | 0.30 | 9.80 | 17.04 | 52.72 | 1.34 | $3466.96^{(2)}$ | - | - | $11.38^{(10)}$ | $9.48^{(7)}$ | $24.89^{(7)}$ |
| | 260 | 17.46 | 31.25 | 64.06 | 747.09 | 0.25 | 19.85 | 45.18 | 215.73 | 0.43 | 11.12 | 29.20 | 65.86 | 2.25 | - | - | - | $14.83^{(10)}$ | $11.95^{(10)}$ | $33.68^{(10)}$ |
| | 310 | 34.12 | 32.18 | 52.00 | 393.09 | 0.39 | 20.88 | 50.42 | 234.84 | 0.59 | 13.51 | 20.00 | 53.40 | 3.13 | $2734.89^{(2)}$ | $40.80^{(1)}$ | $120.34^{(1)}$ | $18.07^{(10)}$ | $17.24^{(9)}$ | $50.07^{(9)}$ |
| | 360 | 125.31 | 28.87 | 72.93 | 697.36 | 0.63 | 21.58 | 57.00 | 295.79 | 0.92 | 12.60 | 38.55 | 146.26 | 6.03 | $2377.46^{(5)}$ | $16.73^{(1)}$ | $60.19^{(1)}$ | $17.42^{(10)}$ | $9.76^{(9)}$ | $31.04^{(9)}$ |
| | 410 | 285.95 | 33.62 | 65.39 | 431.89 | 0.66 | 25.08 | 37.63 | 172.16 | 1.06 | 17.23 | 29.42 | 70.60 | 9.56 | $2408.45^{(3)}$ | - | - | $20.77^{(10)}$ | $27.61^{(8)}$ | $90.07^{(8)}$ |
| | 460 | 552.38 | 41.14 | 81.98 | 465.23 | 1.11 | 29.08 | 60.71 | 255.82 | 1.38 | 18.45 | 49.90 | 175.28 | 16.14 | $5082.84^{(7)}$ | $42.46^{(2)}$ | $7829.06^{(2)}$ | $23.74^{(10)}$ | $18.88^{(9)}$ | $60.91^{(9)}$ |
| | 10 | 0.01 | 82.26 | 85.78 | 265.82 | 0.01 | 51.63 | 41.42 | 196.49 | 0.01 | 23.02 | 14.87 | 124.10 | 0.01 | $0.38^{(10)}$ | $4.71^{(10)}$ | $9.61^{(10)}$ | $21.15^{(10)}$ | $4.45^{(10)}$ | $42.78^{(10)}$ |
| | 60 | 0.22 | 13.01 | 36.95 | 343.31 | 0.02 | 7.95 | 22.88 | 67.17 | 0.03 | 3.58 | 11.78 | 23.74 | 0.10 | $3.85^{(10)}$ | $15.85^{(10)}$ | $46.23^{(10)}$ | $4.70^{(10)}$ | $14.04^{(9)}$ | $32.17^{(9)}$ |
| | 110 | 0.99 | 22.77 | 37.48 | 156.25 | 0.06 | 12.90 | 33.18 | 143.32 | 0.10 | 8.73 | 23.08 | 59.97 | 0.39 | $14.02^{(9)}$ | $0.05^{(4)}$ | $8.51^{(4)}$ | $12.76^{(10)}$ | $0.36^{(6)}$ | $9.79^{(6)}$ |
| | 160 | 6.43 | 35.61 | 62.80 | 1431.79 | 0.13 | 18.40 | 24.51 | 62.60 | 0.26 | 11.09 | 18.65 | 46.33 | 1.11 | $24.38^{(10)}$ | $36.92^{(3)}$ | $156.42^{(3)}$ | $14.93^{(10)}$ | $18.68^{(8)}$ | $65.98^{(8)}$ |
| $0.6n$ | 210 | 10.65 | 31.08 | 63.48 | 763.92 | 0.24 | 23.25 | 58.90 | 198.27 | 0.37 | 15.27 | 15.92 | 39.17 | 2.06 | $28.89^{(10)}$ | $29.40^{(4)}$ | $164.04^{(4)}$ | $19.87^{(10)}$ | $20.77^{(8)}$ | $88.46^{(8)}$ |
| | 260 | 22.04 | 35.37 | 85.97 | 987.29 | 0.42 | 24.75 | 68.43 | 1038.95 | 0.55 | 15.49 | 25.87 | 120.31 | 3.86 | $554.20^{(10)}$ | - | - | $20.39^{(10)}$ | $4.51^{(6)}$ | $19.49^{(6)}$ |
| | 310 | 147.13 | 36.44 | 59.00 | 399.33 | 0.62 | 29.23 | 50.64 | 224.75 | 0.88 | 16.30 | 33.22 | 92.74 | 7.58 | $36.76^{(10)}$ | $10.57^{(3)}$ | $45.37^{(3)}$ | $22.64^{(10)}$ | $7.93^{(7)}$ | $31.42^{(7)}$ |
| | 360 | 289.85 | 32.42 | 70.15 | 1002.31 | 0.66 | 24.48 | 56.87 | 234.20 | 1.26 | 14.86 | 34.25 | 107.83 | 14.35 | $897.05^{(10)}$ | $16.25^{(1)}$ | $62.93^{(1)}$ | $16.86^{(10)}$ | $8.15^{(7)}$ | $27.67^{(7)}$ |
| | 410 | 1020.45 | 43.72 | 87.86 | 583.86 | 1.05 | 31.07 | 60.65 | 400.94 | 1.59 | 20.58 | 42.63 | 211.81 | 19.04 | $44.63^{(10)}$ | - | - | $26.07^{(10)}$ | $5.57^{(5)}$ | $28.52^{(5)}$ |
| | 460 | 6468.90 | 43.84 | 75.78 | 1415.02 | 1.29 | 33.25 | 75.78 | 1317.30 | 2.01 | 22.64 | 65.53 | 964.59 | 55.37 | $46.95^{(10)}$ | $53.72^{(1)}$ | $202.37^{(1)}$ | $25.54^{(10)}$ | $15.08^{(10)}$ | $73.78^{(10)}$ |

optimality gap, respectively. Superscripts in the $\text{BP}_0$ and $\text{BP}_{90}$ columns indicate the number of instances for which finite bounds were found within the specified time limit.

Table 3: The average performance of a $\beta$-based relaxation of the follower's objective ($H_\beta$) for 10 instances of each size $(n, m)$. Superscripts in the $\text{BP}_0$ and $\text{BP}_{90}$ columns indicate the number of instances, where finite bounds are found within the specified time limit.

| $m$ | $n$ | $\text{Time}_{\text{BP}}$ | $\lvert\mathcal{F}\rvert = 0\%$ | | | | $\lvert\mathcal{F}\rvert = 50\%$ | | | | $\lvert\mathcal{F}\rvert = 90\%$ | | | | $\text{BP}_0$ | | | $\text{BP}_{90}$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $\delta_L$ | $\delta_U$ | $\delta$ | Time | $\delta_L$ | $\delta_U$ | $\delta$ | Time | $\delta_L$ | $\delta_U$ | $\delta$ | Time | $\delta_L$ | $\delta_U$ | $\delta$ | $\delta_L$ | $\delta_U$ | $\delta$ |
| | 10 | 0.01 | 33.10 | 17.90 | 761.47 | 0.01 | 6.55 | 4.01 | 12.97 | 0.01 | 0.09 | 0.63 | 0.77 | 0.01 | $29.02^{(9)}$ | $10.46^{(9)}$ | $50.01^{(9)}$ | $4.38^{(10)}$ | $1.90^{(10)}$ | $6.74^{(10)}$ |
| | 60 | 0.16 | 33.94 | 54.78 | 608.72 | 0.02 | 6.56 | 21.61 | 189.35 | 0.11 | 1.72 | 15.14 | 52.71 | 0.15 | $272.65^{(7)}$ | $27.47^{(1)}$ | $115.08^{(1)}$ | $4.01^{(10)}$ | $1.87^{(10)}$ | $6.23^{(10)}$ |
| | 110 | 1.14 | 22.10 | 51.56 | 276.52 | 0.04 | 5.08 | 10.73 | 20.59 | 0.38 | 1.03 | 1.80 | 3.07 | 0.64 | - | - | - | $4.04^{(10)}$ | $1.53^{(8)}$ | $4.76^{(8)}$ |
| | 160 | 2.70 | 23.65 | 48.77 | 209.20 | 0.09 | 9.06 | 17.35 | 64.35 | 0.85 | 0.98 | 5.44 | 7.88 | 2.30 | - | - | - | $2.17^{(10)}$ | $0.95^{(10)}$ | $3.33^{(10)}$ |
| $0.4n$ | 210 | 6.74 | 25.75 | 56.94 | 1022.34 | 0.19 | 8.51 | 21.38 | 51.11 | 1.47 | 2.67 | 1.15 | 3.98 | 3.38 | $3466.96^{(2)}$ | - | - | $6.13^{(10)}$ | $4.03^{(9)}$ | $9.91^{(9)}$ |
| | 260 | 17.46 | 31.25 | 64.06 | 747.09 | 0.25 | 8.07 | 30.98 | 70.50 | 5.61 | 1.39 | 8.35 | 17.92 | 12.66 | - | - | - | $3.47^{(10)}$ | $2.65^{(10)}$ | $6.55^{(10)}$ |
| | 310 | 34.12 | 32.18 | 52.00 | 393.09 | 0.39 | 10.17 | 11.06 | 25.32 | 6.48 | 2.89 | 8.88 | 13.72 | 12.45 | $2734.89^{(2)}$ | $40.80^{(1)}$ | $120.34^{(1)}$ | $7.07^{(10)}$ | $7.01^{(10)}$ | $16.02^{(10)}$ |
| | 360 | 125.31 | 28.87 | 72.93 | 697.36 | 0.63 | 7.32 | 23.14 | 84.48 | 19.46 | 1.34 | 2.39 | 4.06 | 43.49 | $2377.46^{(5)}$ | $16.73^{(1)}$ | $60.19^{(1)}$ | $6.11^{(10)}$ | $6.43^{(10)}$ | $14.34^{(10)}$ |
| | 410 | 285.95 | 33.62 | 65.39 | 431.89 | 0.66 | 11.14 | 22.62 | 46.16 | 41.43 | 3.09 | 1.65 | 5.65 | 157.05 | $2408.45^{(3)}$ | - | - | $6.06^{(10)}$ | $7.80^{(10)}$ | $15.61^{(10)}$ |
| | 460 | 552.38 | 41.14 | 81.98 | 465.23 | 1.11 | 9.67 | 25.88 | 60.25 | 174.88 | 2.04 | 4.00 | 7.33 | 390.27 | $5082.84^{(7)}$ | $42.46^{(2)}$ | $7829.06^{(2)}$ | $4.66^{(10)}$ | $5.42^{(10)}$ | $11.24^{(10)}$ |
| | 10 | 0.01 | 82.26 | 85.78 | 265.82 | 0.01 | 48.40 | 48.43 | 239.06 | 0.01 | 4.70 | 7.61 | 20.85 | 0.01 | $0.38^{(10)}$ | $4.71^{(10)}$ | $9.61^{(10)}$ | $10.63^{(10)}$ | $0.00^{(9)}$ | $0.00^{(9)}$ |
| | 60 | 0.22 | 13.01 | 36.95 | 343.31 | 0.02 | 2.99 | 11.16 | 22.23 | 0.14 | 1.96 | 7.12 | 15.19 | 0.19 | $3.85^{(10)}$ | $15.85^{(10)}$ | $46.23^{(10)}$ | $0.64^{(10)}$ | $3.04^{(10)}$ | $4.46^{(10)}$ |
| | 110 | 0.99 | 22.77 | 37.48 | 156.25 | 0.06 | 6.61 | 20.88 | 68.98 | 0.50 | 1.63 | 0.31 | 1.96 | 0.72 | $14.02^{(9)}$ | $0.05^{(4)}$ | $8.51^{(4)}$ | $3.64^{(10)}$ | $2.70^{(10)}$ | $7.08^{(10)}$ |
| | 160 | 6.43 | 35.61 | 62.80 | 1431.79 | 0.13 | 9.75 | 11.99 | 29.78 | 1.71 | 3.15 | 8.37 | 16.91 | 3.25 | $24.38^{(10)}$ | $36.92^{(3)}$ | $156.42^{(3)}$ | $7.06^{(10)}$ | $2.98^{(10)}$ | $10.92^{(10)}$ |
| $0.6n$ | 210 | 10.65 | 31.08 | 63.48 | 763.92 | 0.24 | 9.78 | 12.05 | 27.78 | 4.19 | 2.01 | 6.11 | 9.67 | 6.31 | $28.89^{(10)}$ | $29.40^{(4)}$ | $164.04^{(4)}$ | $7.06^{(10)}$ | $15.43^{(10)}$ | $57.16^{(10)}$ |
| | 260 | 22.04 | 35.37 | 85.97 | 987.29 | 0.42 | 12.42 | 18.02 | 258.19 | 6.73 | 2.58 | 4.77 | 8.21 | 12.05 | $554.20^{(10)}$ | - | - | $9.40^{(10)}$ | $5.64^{(8)}$ | $13.36^{(8)}$ |
| | 310 | 147.13 | 36.44 | 59.00 | 399.33 | 0.62 | 10.08 | 15.88 | 44.36 | 20.29 | 2.91 | 2.85 | 6.60 | 51.05 | $36.76^{(10)}$ | $10.57^{(3)}$ | $45.37^{(3)}$ | $9.12^{(10)}$ | $7.45^{(9)}$ | $18.00^{(9)}$ |
| | 360 | 289.85 | 32.42 | 70.15 | 1002.31 | 0.66 | 7.66 | 16.65 | 59.32 | 70.05 | 2.20 | 0.01 | 2.21 | 137.31 | $897.05^{(10)}$ | $16.25^{(1)}$ | $62.93^{(1)}$ | $3.77^{(10)}$ | $6.41^{(10)}$ | $11.33^{(10)}$ |
| | 410 | 1020.45 | 43.72 | 87.86 | 583.86 | 1.05 | 13.62 | 19.98 | 47.97 | 172.02 | 1.22 | 1.98 | 3.47 | 599.01 | $44.63^{(10)}$ | - | - | $6.75^{(10)}$ | $6.37^{(10)}$ | $14.65^{(10)}$ |
| | 460 | 6468.90 | 43.84 | 75.78 | 1415.02 | 1.29 | 13.86 | 24.27 | 98.21 | 591.86 | 8.98 | 7.06 | 9.71 | 1018.29 | $46.95^{(10)}$ | $53.72^{(1)}$ | $202.37^{(1)}$ | $8.77^{(10)}$ | $7.67^{(10)}$ | $20.50^{(10)}$ |

The results in Tables 1-3 are consistent with the illustrative example in Figure 1. Our observations can be summarized as follows.

- The relaxed models are solved to optimality significantly faster than the original problem (BP). This observation holds even for the least relaxed cases, $|\mathcal{F}| = 90\%$, although the effect is less pronounced.
- There is a consistent improvement in $\delta_L$ and $\delta_U$ as the relaxation level decreases, or equivalently, $|\mathcal{F}|$ increases as the follower gains control over more decision variables. This improvement is consistent with our theoretical findings in Propositions 2-4.
- The $\beta$-based relaxation ($H_\beta$) generally outperforms the other two relaxation methods in terms of the quality of the obtained bounds. However, obtaining these tighter bounds comes at the cost of significantly increased runtimes.
- By comparing the columns that correspond to $|\mathcal{F}| = 0\%$ and $\mathrm{BP}_0$, we observe that early termination severely restricts the solver's ability to achieve high-quality bounds. A similar observation holds, albeit to a lesser extent, for $\mathrm{BP}_{90}$.
- By comparing the columns that correspond to $|\mathcal{F}| = 90\%$ and $\mathrm{BP}_{90}$, we note that the solver-based bounds still tend to perform worse than the relaxation-based bounds generated using $H_\beta$; see Table 3. The solver is more competitive against $H_R$ and $H_\gamma$; see Tables 1 and 2, respectively. However, $H_R$ and $H_\gamma$ are typically faster than $H_\beta$, and thus, the solver often fails to provide a finite optimality gap within the time limit.

## 3.4 Results for mixed-integer leader's decision

Next, we relax the integrality constraints for the second half of the leader's decision variables, allowing them to take fractional values, while keeping the constraints that all upper-level variables are bounded above by 10. We note that this modification significantly reduces the computational difficulty of the problem. As shown in the $\mathrm{Time}_{\mathrm{BP}}$ columns of Tables 4-6, the runtime required to solve (BP) to optimality is drastically reduced compared to the pure integer setting. Consequently, for this test set we consider significantly larger instances, with $n$ ranging from 360 up to 1000.

We assess the quality of the bounds obtained under $H_R$, $H_\gamma$, and $H_\beta$ using the same metrics defined previously. The results are summarized in Tables 4-6. While the results are relatively consistent with those from Section 3.3, two additional observations are worth noting. First, for larger instances, computing bounds under $H_\beta$ for $|\mathcal{F}| = 90\%$ may require more time than actually solving the model exactly; see the rows with $n = 1000$ in Table 6. On the other hand, the performance of the solver-based bounds computed by $\mathrm{BP}_{90}$ is not as good as in the instances from Section 3.3.

Table 4: The average performance of a random-based relaxation of the follower's objective ($H_R$) for 10 instances of each size $(n, m)$. Superscripts in the $\mathrm{BP}_0$ and $\mathrm{BP}_{90}$ columns indicate the number of instances, where finite bounds are found within the specified time limit.

| $m$ | $n$ | $\mathrm{Time}_{\mathrm{BP}}$ | $\lvert\mathcal{F}\rvert = 0\%$ | | | | $\lvert\mathcal{F}\rvert = 50\%$ | | | | $\lvert\mathcal{F}\rvert = 90\%$ | | | | $\mathrm{BP}_0$ | | | $\mathrm{BP}_{90}$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $\delta_L$ | $\delta_U$ | $\delta$ | Time | $\delta_L$ | $\delta_U$ | $\delta$ | Time | $\delta_L$ | $\delta_U$ | $\delta$ | Time | $\delta_L$ | $\delta_U$ | $\delta$ | $\delta_L$ | $\delta_U$ | $\delta$ |
| $0.4n$ | 360 | 3.94 | 8.66 | 56.93 | 225.79 | 0.47 | 5.93 | 46.80 | 129.18 | 1.59 | 3.04 | 16.89 | 25.80 | 3.12 | $5731.93^{(5)}$ | - | - | $3.26^{(10)}$ | $4.11^{(9)}$ | $7.70^{(9)}$ |
| | 410 | 6.16 | 7.74 | 56.61 | 239.25 | 0.53 | 5.12 | 39.67 | 221.49 | 2.52 | 0.80 | 16.58 | 22.75 | 5.00 | $5906.39^{(1)}$ | - | - | $1.54^{(10)}$ | $2.26^{(9)}$ | $3.61^{(9)}$ |
| | 460 | 9.52 | 10.46 | 67.08 | 454.62 | 0.61 | 6.41 | 49.38 | 136.54 | 3.57 | 1.16 | 23.66 | 36.34 | 6.51 | - | - | - | $4.09^{(10)}$ | $12.47^{(10)}$ | $20.19^{(10)}$ |
| | 600 | 16.34 | 8.32 | 59.58 | 914.11 | 1.18 | 6.81 | 53.50 | 376.52 | 5.05 | 2.13 | 30.13 | 71.35 | 9.12 | $6894.90^{(5)}$ | - | - | $3.62^{(10)}$ | $37.97^{(4)}$ | $66.00^{(4)}$ |
| | 800 | 42.46 | 9.45 | 64.88 | 370.45 | 1.94 | 7.03 | 45.79 | 121.35 | 10.31 | 1.79 | 25.46 | 41.22 | 19.20 | $7637.14^{(3)}$ | - | - | $2.36^{(10)}$ | $37.25^{(5)}$ | $58.76^{(5)}$ |
| | 1000 | 125.22 | 7.31 | 64.06 | 368.78 | 3.13 | 5.92 | 55.55 | 912.12 | 18.13 | 1.09 | 28.05 | 46.67 | 59.54 | $15609.10^{(7)}$ | - | - | $3.58^{(10)}$ | $42.65^{(5)}$ | $100.73^{(5)}$ |
| $0.6n$ | 360 | 9.93 | 7.38 | 56.56 | 304.34 | 0.57 | 4.58 | 45.60 | 110.47 | 3.30 | 0.96 | 18.25 | 26.51 | 6.73 | - | - | - | $0.71^{(10)}$ | $1.60^{(8)}$ | $2.07^{(8)}$ |
| | 410 | 11.96 | 8.58 | 50.90 | 164.76 | 0.83 | 3.82 | 43.09 | 113.45 | 4.38 | 1.38 | 21.44 | 36.47 | 8.11 | $6764.33^{(2)}$ | - | - | $1.07^{(10)}$ | $3.03^{(10)}$ | $4.44^{(10)}$ |
| | 460 | 24.05 | 9.85 | 64.15 | 1077.02 | 1.01 | 7.93 | 55.28 | 519.97 | 6.17 | 3.03 | 39.47 | 113.44 | 13.96 | $7885.77^{(6)}$ | - | - | $2.01^{(10)}$ | $6.78^{(10)}$ | $11.73^{(10)}$ |
| | 600 | 32.46 | 8.07 | 72.94 | 1540.22 | 1.65 | 5.25 | 55.07 | 223.43 | 9.96 | 1.90 | 24.30 | 45.53 | 18.78 | $9468.92^{(4)}$ | - | - | $5.72^{(10)}$ | $37.31^{(2)}$ | $52.53^{(2)}$ |
| | 800 | 93.28 | 7.22 | 59.92 | 240.19 | 3.25 | 3.50 | 48.47 | 337.50 | 29.48 | 0.92 | 25.18 | 40.14 | 61.92 | $4311.30^{(9)}$ | $79.28^{(1)}$ | $410.54^{(1)}$ | $4.67^{(10)}$ | $46.41^{(4)}$ | $88.09^{(4)}$ |
| | 1000 | 222.35 | 7.43 | 71.22 | 621.61 | 4.89 | 4.99 | 56.50 | 231.42 | 41.39 | 1.74 | 33.63 | 62.51 | 153.21 | $9122.93^{(7)}$ | $78.44^{(1)}$ | $407.20^{(1)}$ | $2.60^{(10)}$ | $56.74^{(7)}$ | $148.80^{(7)}$ |

Table 5: The average performance of a $\gamma$-based relaxation of the follower's objective ($H_\gamma$) for 10 instances of each size $(n, m)$. Superscripts in the $\mathrm{BP}_0$ and $\mathrm{BP}_{90}$ columns indicate the number of instances, where finite bounds are found within the specified time limit.

| $m$ | $n$ | $\mathrm{Time}_{\mathrm{BP}}$ | $\lvert\mathcal{F}\rvert = 0\%$ | | | | $\lvert\mathcal{F}\rvert = 50\%$ | | | | $\lvert\mathcal{F}\rvert = 90\%$ | | | | $\mathrm{BP}_0$ | | | $\mathrm{BP}_{90}$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $\delta_L$ | $\delta_U$ | $\delta$ | Time | $\delta_L$ | $\delta_U$ | $\delta$ | Time | $\delta_L$ | $\delta_U$ | $\delta$ | Time | $\delta_L$ | $\delta_U$ | $\delta$ | $\delta_L$ | $\delta_U$ | $\delta$ |
| $0.4n$ | 360 | 3.94 | 8.66 | 56.93 | 225.79 | 0.47 | 4.59 | 44.52 | 111.15 | 0.71 | 2.33 | 25.29 | 43.89 | 2.64 | $5731.93^{(5)}$ | - | - | $3.91^{(10)}$ | $6.67^{(10)}$ | $12.78^{(10)}$ |
| | 410 | 6.16 | 7.74 | 56.61 | 239.25 | 0.53 | 3.35 | 29.83 | 65.74 | 0.86 | 1.25 | 15.14 | 20.55 | 3.81 | $5906.39^{(1)}$ | - | - | $2.21^{(10)}$ | $13.81^{(8)}$ | $27.28^{(8)}$ |
| | 460 | 9.52 | 10.46 | 67.08 | 454.62 | 0.61 | 5.24 | 46.74 | 121.80 | 1.00 | 1.56 | 31.95 | 56.94 | 5.78 | - | - | - | $4.68^{(10)}$ | $10.88^{(9)}$ | $18.67^{(9)}$ |
| | 600 | 16.34 | 8.32 | 59.58 | 914.11 | 1.18 | 5.97 | 49.71 | 838.52 | 1.98 | 1.43 | 29.54 | 55.79 | 8.95 | $6894.90^{(5)}$ | - | - | $2.77^{(10)}$ | $38.22^{(4)}$ | $71.78^{(4)}$ |
| | 800 | 42.46 | 9.45 | 64.88 | 370.45 | 1.94 | 2.76 | 40.62 | 101.74 | 3.00 | 1.02 | 19.04 | 32.03 | 22.66 | $7637.14^{(3)}$ | - | - | $3.21^{(10)}$ | $34.12^{(6)}$ | $51.25^{(6)}$ |
| | 1000 | 125.22 | 7.31 | 64.06 | 368.78 | 3.13 | 3.40 | 43.07 | 95.83 | 4.49 | 1.10 | 28.87 | 60.89 | 69.11 | $15609.10^{(7)}$ | - | - | $3.72^{(10)}$ | $41.57^{(6)}$ | $91.83^{(6)}$ |
| $0.6n$ | 360 | 9.93 | 7.38 | 56.56 | 304.34 | 0.57 | 3.04 | 32.37 | 60.43 | 0.96 | 1.35 | 20.66 | 34.34 | 5.33 | - | - | - | $0.86^{(10)}$ | $1.78^{(8)}$ | $2.34^{(8)}$ |
| | 410 | 11.96 | 8.58 | 50.90 | 164.76 | 0.83 | 3.13 | 39.35 | 89.86 | 1.40 | 1.19 | 21.15 | 34.01 | 8.39 | $6764.33^{(2)}$ | - | - | $1.80^{(10)}$ | $4.65^{(9)}$ | $6.81^{(9)}$ |
| | 460 | 24.05 | 9.85 | 64.15 | 1077.02 | 1.01 | 4.81 | 47.61 | 219.74 | 1.67 | 2.64 | 34.56 | 110.37 | 11.23 | $7885.77^{(6)}$ | - | - | $2.06^{(10)}$ | $7.43^{(10)}$ | $11.77^{(10)}$ |
| | 600 | 32.46 | 8.07 | 72.94 | 1540.22 | 1.65 | 3.29 | 38.29 | 108.06 | 2.42 | 0.93 | 22.03 | 37.27 | 19.66 | $9468.92^{(4)}$ | - | - | $4.32^{(10)}$ | $47.12^{(2)}$ | $83.55^{(2)}$ |
| | 800 | 93.28 | 7.22 | 59.92 | 240.19 | 3.25 | 2.79 | 50.61 | 391.75 | 7.17 | 0.85 | 20.86 | 47.96 | 81.76 | $4311.30^{(9)}$ | $79.28^{(1)}$ | $410.54^{(1)}$ | $5.54^{(10)}$ | $42.11^{(3)}$ | $74.84^{(3)}$ |
| | 1000 | 222.35 | 7.43 | 71.22 | 621.61 | 4.89 | 3.15 | 49.91 | 125.22 | 10.03 | 0.72 | 24.11 | 36.98 | 166.28 | $9122.93^{(7)}$ | $78.44^{(1)}$ | $407.20^{(1)}$ | $2.66^{(10)}$ | $57.89^{(6)}$ | $155.79^{(6)}$ |

Table 6: The average performance of a $\beta$-based relaxation of the follower's objective ($H_\beta$) for 10 instances of each size $(n, m)$. Superscripts in the $\mathrm{BP}_0$ and $\mathrm{BP}_{90}$ columns indicate the number of instances, where finite bounds are found within the specified time limit.

| $m$ | $n$ | $\mathrm{Time}_{\mathrm{BP}}$ | $\lvert\mathcal{F}\rvert = 0\%$ | | | | $\lvert\mathcal{F}\rvert = 50\%$ | | | | $\lvert\mathcal{F}\rvert = 90\%$ | | | | $\mathrm{BP}_0$ | | | $\mathrm{BP}_{90}$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $\delta_L$ | $\delta_U$ | $\delta$ | Time | $\delta_L$ | $\delta_U$ | $\delta$ | Time | $\delta_L$ | $\delta_U$ | $\delta$ | Time | $\delta_L$ | $\delta_U$ | $\delta$ | $\delta_L$ | $\delta_U$ | $\delta$ |
| $0.4n$ | 360 | 3.94 | 8.66 | 56.93 | 225.79 | 0.47 | 0.69 | 14.45 | 19.55 | 4.02 | 0.04 | 4.01 | 5.10 | 3.82 | $5731.93^{(5)}$ | - | - | $1.56^{(10)}$ | $2.97^{(10)}$ | $5.18^{(10)}$ |
| | 410 | 6.16 | 7.74 | 56.61 | 239.25 | 0.53 | 0.79 | 11.32 | 14.58 | 5.75 | 0.03 | 3.51 | 4.43 | 6.26 | $5906.39^{(1)}$ | - | - | $1.28^{(10)}$ | $2.00^{(9)}$ | $3.08^{(9)}$ |
| | 460 | 9.52 | 10.46 | 67.08 | 454.62 | 0.61 | 0.52 | 16.87 | 25.39 | 9.02 | 0.24 | 4.41 | 7.15 | 8.41 | - | - | - | $3.58^{(10)}$ | $8.18^{(10)}$ | $13.81^{(10)}$ |
| | 600 | 16.34 | 8.32 | 59.58 | 914.11 | 1.18 | 0.19 | 12.91 | 22.02 | 12.34 | 0.04 | 2.44 | 2.77 | 10.46 | $6894.90^{(5)}$ | - | - | $4.64^{(10)}$ | $36.88^{(5)}$ | $62.91^{(5)}$ |
| | 800 | 42.46 | 9.45 | 64.88 | 370.45 | 1.94 | 0.34 | 8.97 | 10.77 | 25.53 | 0.03 | 1.12 | 1.24 | 24.23 | $7637.14^{(3)}$ | - | - | $4.51^{(10)}$ | $33.82^{(6)}$ | $43.85^{(6)}$ |
| | 1000 | 125.22 | 7.31 | 64.06 | 368.78 | 3.13 | 0.30 | 21.00 | 33.59 | 127.54 | 0.04 | 7.43 | 16.22 | 144.48 | $15609.10^{(7)}$ | - | - | $5.56^{(10)}$ | $39.45^{(6)}$ | $83.68^{(6)}$ |
| $0.6n$ | 360 | 9.93 | 7.38 | 56.56 | 304.34 | 0.57 | 0.50 | 14.78 | 23.24 | 6.71 | 0.24 | 5.10 | 7.67 | 8.74 | - | - | - | $0.55^{(10)}$ | $2.43^{(9)}$ | $3.21^{(9)}$ |
| | 410 | 11.96 | 8.58 | 50.90 | 164.76 | 0.83 | 0.81 | 19.18 | 30.81 | 8.35 | 0.00 | 0.00 | 0.00 | 11.57 | $6764.33^{(2)}$ | - | - | $1.06^{(10)}$ | $2.59^{(10)}$ | $4.14^{(10)}$ |
| | 460 | 24.05 | 9.85 | 64.15 | 1077.02 | 1.01 | 0.84 | 18.74 | 27.30 | 39.30 | 0.00 | 0.71 | 0.77 | 21.05 | $7885.77^{(6)}$ | - | - | $0.39^{(10)}$ | $1.72^{(10)}$ | $2.26^{(10)}$ |
| | 600 | 32.46 | 8.07 | 72.94 | 1540.22 | 1.65 | 0.53 | 16.92 | 27.82 | 21.58 | 0.18 | 8.67 | 13.66 | 26.36 | $9468.92^{(4)}$ | - | - | $6.80^{(10)}$ | $32.63^{(3)}$ | $42.21^{(3)}$ |
| | 800 | 93.28 | 7.22 | 59.92 | 240.19 | 3.25 | 0.25 | 14.26 | 26.13 | 70.61 | 0.07 | 1.99 | 2.28 | 82.43 | $4311.30^{(9)}$ | $79.28^{(1)}$ | $410.54^{(1)}$ | $6.23^{(10)}$ | $46.41^{(4)}$ | $87.03^{(4)}$ |
| | 1000 | 222.35 | 7.43 | 71.22 | 621.61 | 4.89 | 0.17 | 12.21 | 15.85 | 203.31 | 0.01 | 1.43 | 1.67 | 258.65 | $9122.93^{(7)}$ | $78.44^{(1)}$ | $407.20^{(1)}$ | $3.24^{(10)}$ | $53.70^{(7)}$ | $136.50^{(7)}$ |

# 4 Concluding remarks

A classical approach in bilevel optimization for constructing lower and upper bounds is based on the concept of a single-level relaxation. In this note, we propose a generalization of this concept by allowing the leader to control only a subset of the follower's variables. This simple generalization yields a hierarchy of lower and upper bounds, with an underlying trade-off between the quality of the bounds and the computational difficulty of obtaining them. Our preliminary computational experiments clearly demonstrate this trade-off.

In conclusion, the following three points are worth noting:

- First, our bounds, as well as the hierarchy of the lower and upper bounds (i.e., the results in Sections 2.2 and 2.3), also apply to bilevel programs with integrality restrictions at the lower level. We focus on the case with an LP lower level because these bounds are simpler to compute for this class of bilevel problems; recall Section 2.5. Indeed, when integer variables remain at the lower level, the resulting bilevel problem remains relatively difficult, although it becomes easier as the number of lower-level variables is reduced. Thus, extending our bounds in this direction is a natural avenue for future research.

- Second, we consider bilevel problems without coupling constraints. That is, the follower's optimal decisions do not appear in the leader's constraints. We expect that the results in Section 2.2 should hold for these problems as well. However, generalizing the upper bounds is likely to be less straightforward.

- Finally, our experiments show that the structure of $\mathcal{F}$ (i.e., the subset of variables that remain under the follower's control) affects both the running time and the quality of the resulting bounds. Hence, it may be of interest to derive more sophisticated methods for constructing this set, possibly tailored to different types of lower-level problems.

We believe that these directions provide promising starting points for further developing the proposed bounding framework.

**Declaration of generative AI use.** During the preparation of this work, the authors used AI for language refinement and clarity. After using this tool, the authors reviewed and edited the content as needed and take full responsibility for the content of the published article.

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