The subtle behavior of the facial distance

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Abstract

We give a simple example that disproves the following 2015 conjecture of Lacoste-Julien and Jaggi concerning the pyramidal width (aka facial distance):

> The pyramidal width of a set of vertices is non-increasing when another vertex is added (assuming that all previous points remain vertices).

In contrast to the recent example by Zhao (arXiv:2607.29555), our counterexample is readily verifiable via visual inspection. Furthermore, our counterexample shows that the relative increase in the pyramidal width can be arbitrarily large.

1 The counterexample

We give a simple example that disproves the following 2015 conjecture of Lacoste-Julien and Jaggi [1, Sec 3.1] concerning the pyramidal width:

> The pyramidal width of a set of vertices is non-increasing when another vertex is added (assuming that all previous points remain vertices).

In contrast to the recent counterexample provided by Zhao [3], our counterexample is readily verifiable via visual inspection. Furthermore, our counterexample shows that the relative increase in the pyramidal width can be arbitrarily large.

We rely on the equivalence between the pyramidal width and the facial distance established in [2, Thm 2]. Recall that the facial distance of a polytope $\mathcal{C}$ is defined as follows [2]:

$$\min_{\substack{F \in \mathrm{faces}(\mathcal{C}) \\ \emptyset \neq F \neq \mathcal{C}}} \mathrm{dist}(F, \mathrm{conv}(\mathrm{vert}(\mathcal{C}) \setminus F)).$$

Our counterexample is based on a distortion of the ℓ_1 unit ball. More precisely, suppose $0 < \epsilon \leq 1/4$ and consider the following six points in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \mathbf{e}_1, \quad \mathbf{v}_2 = -\mathbf{e}_1, \quad \mathbf{v}_3 = \mathbf{e}_2 - \epsilon\mathbf{e}_3, \quad \mathbf{v}_4 = -\mathbf{e}_2 - \epsilon\mathbf{e}_3, \quad \mathbf{v}_5 = \mathbf{e}_3, \quad \mathbf{v}_6 = -\mathbf{e}_3.$$

Let $\mathcal{C} := \mathrm{conv}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ and $\mathcal{C}' := \mathrm{conv}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$. These two polytopes are depicted in Figure 1.

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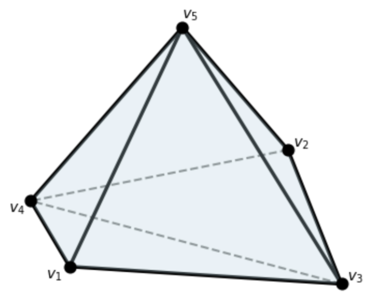
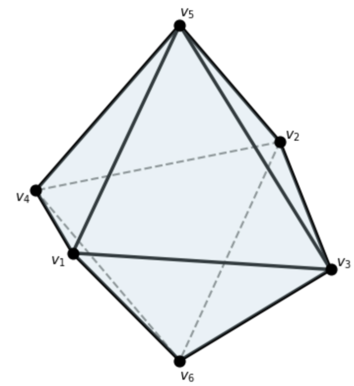


(a) $\mathcal{C}$



(b) $\mathcal{C}'$

Figure 1: Subtle behavior of the facial distance (pyramidal width)

The polytope $\mathcal{C}'$ is obtained by adding the new vertex $\mathbf{v}_6$ to $\mathcal{C}'$. The five old vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ of $\mathcal{C}$ remain vertices of $\mathcal{C}'$.

The polytope $\mathcal{C}'$ is a slight distortion of the ℓ_1 unit ball. A simple visual inspection shows that for $0 < \epsilon \ll 1/4$ its facial distance of $\mathcal{C}'$ is attained at one of its edges and is approximately the distance from the middle point of that edge to the origin, that is, $\approx 1/\sqrt{2}$.

A simple visual inspection also shows that the facial distance of $\mathcal{C}$ is attained at the edge connecting the vertices $\mathbf{v}_3$ and $\mathbf{v}_4$ and is equal to the distance between its middle point and the middle point of the edge connecting $\mathbf{v}_1$ and $\mathbf{v}_2$, that is, ϵ.

Thus for $0 < \epsilon \ll 1/4$ the ratio of the facial distance of the new polytope $\mathcal{C}'$ to that of the old polytope $\mathcal{C}$ is approximately

$$\frac{1}{\sqrt{2}\epsilon}$$

which can be arbitrarily large.

References

[1] Simon Lacoste-Julien and Martin Jaggi. On the global linear convergence of Frank-Wolfe optimization variants. In *Proceedings of Advances in Neural Information Processing Systems*, pages 496–504, 2015.

[2] Javier Pena and Daniel Rodriguez. Polytope conditioning and linear convergence of the Frank–Wolfe algorithm. *Mathematics of Operations Research*, 44(1):1–18, 2019.

[3] Jinze Zhao. Pyramidal width can increase under vertex insertion. *arXiv preprint arXiv:2607.29555*, 2026.