Valid Inequalities for Potential-Based Network Design Including Compressors

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We study the steady-state expansion problem for potential-based flow networks. Constructing a cost-minimal network that admits a flow satisfying the underlying physical laws is a central problem in the design of gas, hydrogen, water, and electricity infrastructures. The physical behavior of such networks is governed by nonlinear relations between arc flows and the potential differences at their endpoints.

We derive a novel class of valid inequalities for network expansion problems that include compressor arcs, which may increase the potential in their prescribed direction and are essential for transporting flow over long distances.

For fixed terminal sets we further show that the corresponding separation problem can be solved in polynomial time by minimum s-t cut computations in an auxiliary graph.

1 Introduction

Börner et al. [3] introduce a cutting-plane approach for the *potential-based network flow expansion problem*. The underlying network is represented by a directed graph $G = (V, A)$, where the arcs in A correspond to possible connections that may be selected as part of the network design. Each arc $a \in A$ is associated with a resistance parameter $\beta_a > 0$ that captures its physical properties. A flow vector $f \in \mathbb{R}^A$ assigns a flow value to each arc. For an arc $a = (u, v)$, a positive value $f_a > 0$ indicates flow in the prescribed direction from u to v, whereas a negative value represents flow in the opposite direction. They consider the problem of designing a cost-minimal network that is capable of satisfying a prescribed balance vector $b \in \mathbb{R}^V$ with $\sum_{i\in V} b_i = 0$. The physical behavior of the network is described by a potential vector $\pi \in \mathbb{R}^V$ together with a nonlinear potential-flow relation. In particular, for every arc $a = (u, v) \in A$, the potentials and flows satisfy

$$\pi_u - \pi_v = \beta_a \operatorname{sign}(f_a)|f_a|^r.$$

The parameter $r > 0$ is referred to as the degree of the potential-based flow network. Given this network and the prescribed balance vector, the objective is to select a minimum-cost subset of arcs such that the resulting network admits a feasible flow satisfying both the nodal balance constraints and the nonlinear potential-flow equations.

The main result of Börner et al. [3] is a new class of valid inequalities for the *potential-based network flow expansion problem* described above. Moreover, they show that these inequalities can be separated in polynomial time. Their construction is based on a sequence of k nested node sets whose induced cuts separate two disjoint node sets X and Y. Any flow transmitted from X to Y must cross each of these, and each crossing requires a corresponding potential drop that depends on the conductance available across the cut. Since the total potential difference between the outermost cuts induced by S_1 and S_k is bounded by the prescribed potential limits, the cumulative potential drop required to transport the flow is bounded as well. This interplay between the required flow, the conductance of the outflow cutset, and the available potential range forms the basis of the resulting valid inequalities.

A major limitation of their approach is that the proposed cutting planes are restricted to purely passive networks. In practical network design, however, limits on the admissible potential range often prevent the transport of flow over long distances using passive elements alone. Active components, such as compressors, are therefore essential for many large-scale network systems and must be explicitly accounted for in the design problem. The main contribution of this work is therefore twofold:

1. We extend the approach of Börner et al. [3] to gas network design problems that include compressors. Building on the same underlying idea of sequences of graph cuts, we derive a new class of linear valid inequalities that explicitly accounts for the presence of compressors.

2. Furthermore, we show that, for fixed disjoint node sets X and Y the corresponding separation problem can be solved in polynomial time.

Börner et al. [3] additionally optimize over the terminal sets and preserve polynomial time, exploiting that the value of a minimum cut sequence is a submodular function of these sets. Our separation procedure does not retain this property.

The remainder of the paper is structured as follows: After discussing relevant prior works in Section 2, Section 3 introduces the necessary preliminaries and the underlying optimization model. In Section 4, we first derive a nonlinear valid inequality for this model. Building on this result, Section 4.2 develops a linearization that yields the proposed compressor-aware cutting plane. Finally, Section 5 shows that, for fixed terminal sets X and Y, the resulting separation problem can be solved in polynomial time.

2 Related Work

Network expansion is a classical and well-studied optimization problem in which network elements are selected at an investment cost such that prescribed demands can be routed through the resulting network [16]. In transmission networks such as gas, water, and electricity networks, the design problem contains an additional physical layer. Rather than being freely routed subject only to flow conservation and capacity constraints, arc flows are coupled to potentials at their incident nodes through physical flow equations. Among the early contributions to the study of potential-based flows, [2] investigated nonlinear network equations and established existence and uniqueness results for several boundary-value problems. For fixed network topologies, the gas transmission and nomination validation problem has since received considerable attention [9, 6, 17, 7, 8, 5, 15, 19]. A comprehensive review of optimization problems in natural gas transportation is given by [18].

Building on these physical flow models, a range of approaches has been proposed for the design and expansion of potential-driven networks. Borraz-Sánchez et al. [4] study steady-state gas network expansion with candidate pipelines and active components and develop a convex mixed-integer second-order cone relaxation of the expansion planning problem. Humpola et al. [12] fix the topology variables and derive Benders-like valid inequalities from Lagrangian dual information for a convex relaxation of the resulting transmission subproblem [12]. Building on this, Humpola and Serrano [11] give sufficient conditions under which the transmission subproblem for a fixed topology is infeasible. Li et al. [14] solve the gas network design problem by a nested-loop algorithm that combines a convex reformulation in the inner loop with an enumeration scheme in the outer loop [14]. Schweiger and Liers [20] consider network extension under multiple demand scenarios and propose a scenario decomposition in which each scenario gives rise to a mixed-integer nonlinear subproblem [20].

Closest related to our work, Börner et al. [3] presented a class of valid inequalities for the design of passive potential-based flow networks. Their construction considers nested sequences of graph cuts separating prescribed node sets. Since any transmitted flow must cross every cut in the sequence, the installed conductance across these cutsets determines the potential drop required to transport the flow. Comparing the accumulated potential drop with the available potential range yields valid inequalities in the network-design variables. Börner et al. further show that the resulting inequalities can be separated in polynomial time. Despite active components having been incorporated into gas-network expansion models, the potential-based cut construction of Börner et al. relies on passivity. Our work extends this cutting-plane approach to networks where compressors can already be contained and also be newly build.

3 Preliminaries

Let $G = (V, A)$ be a directed and weakly connected multi-graph without self-loops. Each arc has a tail node, a head node and an arc type. These are given by the maps

$$\text{tail} : A \to V, \qquad \text{head} : A \to V, \qquad \text{type} : A \to \mathcal{T},$$

where $\mathcal{T}$ is a finite set of arc types. For every ordered node pair $(i, j) \in V \times V$ and every type $\vartheta \in \mathcal{T}$, there is at most one arc $a \in A$ satisfying

$$\text{tail}(a) = i, \qquad \text{head}(a) = j, \qquad \text{type}(a) = \vartheta.$$

We divide the arc types into the passive pipe types $\mathcal{T}^p$ and the compressor types $\mathcal{T}^c$. We abbreviate the passive and compressor edges shortly with A^{p} and A^{c}. For a subset $U \subseteq V$ we define the arc set induced by a node set as $A[U] := \{a \in A : \text{tail}(a) \in U,\ \text{head}(a) \in U\}$. For a node we define the outgoing and incoming arc sets by $\delta^+(i) := \{a \in A : \text{tail}(a) = i\}$, $\delta^-(i) := \{a \in A : \text{head}(a) = i\}$, and $\delta(i) = \delta^+(i) \cup \delta^-(i)$. For a node set $S \subseteq V$ we write $\delta^+(S) := \{a \in A : \text{tail}(a) \in S, \text{head}(a) \notin S\}$, $\delta^-(S) := \{a \in A : \text{head}(a) \in S, \text{tail}(a) \notin S\}$, and $\delta(S) := \delta^+(S) \cup \delta^-(S)$.

A cut in G is a partition of V into two disjoint node sets S and $\bar{S} := V \setminus S$. The cut induced by a node set $S \subseteq V$ is $(S, \bar{S})$. We say that $(S, \bar{S})$ is an (X, Y)-cut if $X \subseteq S$ and $Y \subseteq \bar{S}$. Since the flow on a passive pipe may run against its reference orientation, whereas a compressor arc admits flow only in its prescribed direction, the arcs that are able to carry flow from S to $V \setminus S$ are those in

$$\delta^{\text{out}}(S) := \delta^+(S) \cup \big(\delta^-(S) \cap A^{\mathrm{p}}\big). \tag{1}$$

We call $\delta^{out}(S)$ the outflow cutset or short cutset of $(S, \bar{S})$. As G has no self-loops, $\delta^+(S) \cap \delta^-(S) = \emptyset$, so every arc of $\delta^{\text{out}}(S)$ crosses the induced cut of S in exactly one direction.

For every node a balance value $b \in \mathbb{R}$ is given. We say that the balance vector is balanced if $\sum_{i \in V} b_i = 0$. Moreover each node $v \in V$ has a potential value $\pi_v \in [\underline{\pi}_v, \overline{\pi}_v]$. Each arc $a \in A$ is associated with a resistance coefficient $\beta_a > 0$. For every passive arc $a = (v, w)$, the pressure-flow relation is described by the Weymouth equation

$$\pi_v - \pi_w = \beta_a \operatorname{sign}(f_a) |f_a|^r. \tag{2}$$

In addition to passive arcs, the network may contain compressor arcs $A^{\mathrm{c}} = (v, w)$, which can increase the potential in the prescribed arc direction by an amount $\alpha \in \mathbb{R}_{\geq 0}$. Compression is only possible when the flow is aligned with the compressor direction, i.e., from v to w.

The physical relation on a compressor arc is given by

$$\pi_v - \pi_w + \alpha = \beta_a \operatorname{sign}(f_a) f_a^r. \tag{3}$$

We assume that the pressure increase α can be chosen continuously within the admissible operating range of the compressor and is bounded above by a parameter $\gamma_a \in \mathbb{R}_{\geq 0}$, i.e., $0 \leq \alpha \leq \gamma_a$.

Each candidate arc a is associated with a construction cost k_a. To model the corresponding investment decisions, we introduce binary variables $x \in \{0,1\}^{A^{\mathrm{p}}}$ for candidate pipe arcs and $y \in \{0,1\}^{A^{\mathrm{c}}}$ for candidate compressor arcs, where a value of one indicates that the respective network element is constructed. For each passive pipe $a = (i,j) \in A^{\mathrm{p}}$, we assign an arbitrary reference orientation from i to j. The corresponding flow variable f_a is allowed to take both positive and negative values, where $f_a > 0$ denotes flow in the reference direction and $f_a < 0$ denotes flow in the opposite direction. Compressor arcs, in contrast, remain directed, and flow through a compressor is restricted to its prescribed orientation.

The resulting network expansion model is given by

$$
\begin{aligned}
\min \quad & \sum_{a \in A^{\mathrm{p}}} k_a x_a + \sum_{a \in A^{\mathrm{c}}} k_a y_a && && \text{(4a)}\\
\text{s.t.} \quad & \sum_{a \in \delta^+(i)} f_a - \sum_{a \in \delta^-(i)} f_a = b_i && \forall i \in V, && \text{(4b)}\\
& -M_a^{\mathrm{p}}(1 - x_a) \le \pi_i - \pi_j - \beta_a \operatorname{sign}(f_a) f_a^2 \le M_a^{\mathrm{p}}(1 - x_a) && \forall a = (i,j) \in A^{\mathrm{p}}, && \text{(4c)}\\
& -M_a^{\mathrm{c}}(1 - y_a) \le \pi_i - \pi_j - \beta_a f_a^2 + \alpha_a \le M_a^{\mathrm{c}}(1 - y_a) && \forall a = (i,j) \in A^{\mathrm{c}}, && \text{(4d)}\\
& -\overline{f}_a x_a \le f_a \le \overline{f}_a x_a && \forall a \in A^{\mathrm{p}}, && \text{(4e)}\\
& 0 \le f_a \le \overline{f}_a y_a && \forall a \in A^{\mathrm{c}}, && \text{(4f)}\\
& \underline{\pi}_i \le \pi_i \le \overline{\pi}_i && \forall i \in V, && \text{(4g)}\\
& 0 \le \alpha_a \le \gamma_a y_a && \forall a \in A^{\mathrm{c}}, && \text{(4h)}\\
& x_a \in \{0,1\} && \forall a \in A^{\mathrm{p}}, && \text{(4i)}\\
& y_a \in \{0,1\} && \forall a \in A^{\mathrm{c}}. && \text{(4j)}
\end{aligned}
$$

The objective minimizes the total investment cost of selected passive pipes and compressor arcs. Constraint (4b) imposes flow conservation at every vertex. The big-M constraints (4c) and (4d) enforce the corresponding potential-flow relations whenever the associated arc is built and relax them otherwise. Constraints (4e) and (4f) link the arc flows to the corresponding build decisions. For passive pipes, flow is permitted in both directions, whereas compressor arcs only allow flow in their prescribed direction. Constraint (4g) enforces the nodal potential bounds, while (4h) bounds the compressor boost by the installed compressor capacity. Finally, constraints (4i) and (4j) define the binary design variables.

4 Compressor-Aware Potential Cuts

The inequalities that we will develop in this section rest on the same observation as those of Börner et al. [3]: the amount of flow F that can be transported between two disjoint node sets X and Y is limited both by the arcs installed between them and by the potential difference available. While passive pipes consume potential through friction, compressors may add to the usable budget. This asymmetry is what makes the extension to compressors non-trivial: Once compressors are included into the model the available budget itself becomes a function of the extension decisions. In the purely passive case, every arc consumes part of the potential budget in order to carry its flow. Along a single path this is immediately clear. The potentials of the interior nodes cancel telescopically, so the total potential consumed is the difference of the potentials at the two ends of the path, which is bounded by the prescribed potential limits. Since the consumption of each arc grows with the flow it carries, this bounds the flow that the path can transport.

Enumerating all X-to-Y paths is not practical, as their number is in general exponential.

Instead, we place a sequence of cuts in the graph, chosen so that every X-to-Y path crosses each of them. Every crossing consumes a share of the potential budget that depends on the flow value and on the conductance installed on the corresponding cutset. Because a path must cross all cuts in turn, the individual consumptions compete for one and the same budget: the more potential is spent on one crossing, the less remains for the others. Their sum therefore cannot exceed the potential difference available between the two ends of the sequence, which yields a joint restriction on the conductances installed on the cutsets. Compressor arcs change this picture, since they do not consume potential but add to it: a path crossing a built compressor gains up to its maximal boost γ_a of additional potential budget. The restriction obtained from a cut sequence therefore depends on the expansion decision not only through the conductances installed on the cutsets, but also through the compressors built in the region enclosed by them, and the resulting inequalities must account for both.

We derive the cutting plane in two stages. In the first stage we decompose the flow between two sets X and Y in its individual used paths. Their individual flow capacity can be bounded by the potential budget $\varPi$ and the available compressor boost $H_{\mathcal{S}}(y)$. We introduce a nested sequence of cuts $\mathcal{S} = \{S_1, \dots, S_k\}$ with disjunct cutsets $\delta^{\text{out}}(S_i)$. These are aggregated into a bound on the total potential consumed by the flow across the entire cut sequence. The result of the first stage is the nonlinear condition

$$F^2 \sum_{i=1}^{k} \frac{1}{\varPhi_i(z)^2} \leq \varPi + H_{\mathcal{S}}(y),$$

In the second stage the two remaining nonlinearities in the design variables z and y are removed. Together these two steps yield a linear inequality in the design variables that is implied by the nonlinear condition, and hence valid for model (4).

Following the underlying construction of [3], the inequalities are based on sequences of k nested node sets $S_1, \dots, S_k$ with $k \in \mathbb{N}$.

Definition 4.1 (k-disjoint (X, Y)-cut sequence)**.** For two disjoint node sets $X, Y \subseteq V$, we call

$$\mathcal{S} = \{S_1, \dots, S_k\}$$

a *k-disjoint (X, Y)-cut sequence* if every cut induced by S_i is an (X, Y)-cut, the sets are nested, $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k$, and the corresponding cutsets are pairwise arc-disjoint,

$$\delta^{\text{out}}(S_i) \cap \delta^{\text{out}}(S_j) = \emptyset \qquad \forall\, i \neq j.$$

For a given cut sequence $\mathcal{S}$, we define the two sets of relevant arcs and contained compressors as

$$A(\mathcal{S}) := A \setminus (A[S_1] \cup A[V \setminus S_k]), \quad C(\mathcal{S}) := A^{\text{c}} \cap A(\mathcal{S}). \tag{5}$$

For notational convenience, we introduce a unified notation for the build variables and the potential relations of passive and compressor arcs. For each arc $a = (i, j) \in A$, let

$$z_a := \begin{cases} x_a, & a \in A^{\text{p}}, \\ y_a, & a \in A^{\text{c}}. \end{cases}$$

Given an incumbent expansion decision $\bar{z}$ the aggregate conductance weight associated with a node set S by

$$\varPhi(S, \bar{z}) := \sum_{a \in \delta^{\text{out}}(S)} \frac{\bar{z}_a}{\sqrt{\beta_a}}.$$

For a set $S_i \in \mathcal{S}$ we abbreviate $\varPhi_i = \varPhi(S_i, \bar{z})$.

The reference orientation of a passive pipe is an arbitrary modeling choice, whereas the direction of a compressor arc is a physical restriction. This is captured by the following notion of a traversal, which records an arc together with the direction in which it is used and allows both arc types to be treated uniformly along a path.

Definition 4.2 (Traversals)**.** For an arc $a \in A$ we define the set of *admissible orientations*

$$O(a) \coloneqq \begin{cases} \{+1, -1\}, & a \in A^{\mathrm{p}}, \\ \{+1\}, & a \in A^{\mathrm{c}}, \end{cases}$$

and call $\bar{A} \coloneqq \{(a, \sigma) : a \in A,\ \sigma \in O(a)\}$ the set of *traversals* of G. For a traversal $(a, \sigma) \in \bar{A}$ we write

$$t(a, \sigma) \coloneqq \begin{cases} \mathrm{tail}(a), & \sigma = +1, \\ \mathrm{head}(a), & \sigma = -1, \end{cases} \qquad h(a, \sigma) \coloneqq \begin{cases} \mathrm{head}(a), & \sigma = +1, \\ \mathrm{tail}(a), & \sigma = -1, \end{cases}$$

for its initial and terminal node, and define the *traversal potential drop*

$$\eta_{(a,\sigma)}(\pi, \alpha) \coloneqq \pi_{t(a,\sigma)} - \pi_{h(a,\sigma)} + \mathbb{1}\{a \in A^{\mathrm{c}}\}\alpha_a.$$

For a node set $S \subseteq V$ we write

$$\bar{\delta}^{\mathrm{out}}(S) \coloneqq \big\{(a, \sigma) \in \bar{A} : t(a, \sigma) \in S,\ h(a, \sigma) \notin S\big\}$$

for the set of traversals leaving S and $\mathrm{proj} : \bar{A} \to A$, $(a, \sigma) \mapsto a$, for the projection of a traversal to its underlying arc. Since G has no self-loops, the restriction of proj to $\bar{\delta}^{\mathrm{out}}(S)$ is a bijection onto $\delta^{\mathrm{out}}(S)$.

Informally, one may picture each passive pipe as having one additional copy pointing in the opposite direction, while the compressor has none, which ensures its physical limitation. Therefore, each passive pipe can be traversed with positive flow in the opposite direction through its copy. Note that this does not change model (4) because the traversals are only used to analyze paths within the network. The following observation is the formal counterpart of the discussion above. It shows that, for a traversal aligned with the flow, the potential drop is nonnegative and equals the friction loss, uniformly for both arc types.

Observation 4.3 (Orientation consistency)**.** Let (x, y, f, π, α) be feasible for model (4), let $a \in A$ with $z_a = 1$, and let $\sigma \in O(a)$ with $\sigma f_a > 0$. Then

$$\eta_{(a,\sigma)}(\pi, \alpha) = \beta_a f_a^2 \geq 0.$$

Proof. Let $a = (u, v)$. If $a \in A^{\mathrm{p}}$ and $\sigma = +1$, then $f_a > 0$ and (4c) gives $\eta_{(a,+1)} = \pi_u - \pi_v = \beta_a \operatorname{sign}(f_a) f_a^2 = \beta_a f_a^2$. If $a \in A^{\mathrm{p}}$ and $\sigma = -1$, then $f_a < 0$ and $\eta_{(a,-1)} = \pi_v - \pi_u = -\beta_a \operatorname{sign}(f_a) f_a^2 = \beta_a f_a^2$. If $a \in A^{\mathrm{c}}$, then $O(a) = \{+1\}$, so $\sigma = +1$, and (4d) gives $\eta_{(a,+1)} = \pi_u - \pi_v + \alpha_a = \beta_a f_a^2$. In all cases the value is nonnegative since $\beta_a > 0$. □

4.1 A nonlinear necessary condition

As a first step we derive a nonlinear condition that every feasible expansion must satisfy, which is then linearized in Section 4.2.

Definition 4.4 (Flow-consistent path)**.** Let $f \in \mathbb{R}^A$. A traversal $(a, \sigma) \in \bar{A}$ is f*-consistent* if $\sigma f_a > 0$. An f*-consistent path* is a sequence

$$p = \big(v_0, (a_1, \sigma_1), v_1, \ldots, (a_m, \sigma_m), v_m\big)$$

of pairwise distinct nodes $v_0, \ldots, v_m$ and f-consistent traversals with $t(a_r, \sigma_r) = v_{r-1}$ and $h(a_r, \sigma_r) = v_r$ for $r = 1, \ldots, m$. For a path $p = \big(v_0, (a_1, \sigma_1), v_1, \ldots, (a_m, \sigma_m), v_m\big)$ we denote by

$$\bar{A}(p) \coloneqq \{(a_1, \sigma_1), \ldots, (a_m, \sigma_m)\} \subseteq \bar{A}$$

its set of traversals and by

$$A(p) \coloneqq \{a_1, \dots, a_m\} \subseteq A$$

the set of arcs traversed by p. Since p is simple and every arc admits at most one f-consistent orientation, the projection $\mathrm{proj} : (a, \sigma) \mapsto a$ is a bijection from $\bar{A}(p)$ onto $A(p)$.

Accordingly, an arc a crosses a cut $(S, \bar{S})$ along an f-consistent path only if $a \in \delta^{\mathrm{out}}(S)$. Finally, we define the maximal potential budget as

$$\Pi \coloneqq \max_{u \in V} \overline{\pi}_u - \min_{v \in V} \underline{\pi}_v.$$

Lemma 4.5. *Let $X, Y \subseteq V$ be disjoint, let $\mathcal{S} = (S_1, \dots, S_k)$ be a k-disjoint (X, Y)-cut sequence, and let (x, y, f, π, α) be feasible for Model (4). Let*

$$p = \big(v_0, (a_1, \sigma_1), v_1, \dots, (a_m, \sigma_m), v_m\big)$$

be an f-consistent X-to-Y path. Let

$$\tau \coloneqq \min\{r : (a_r, \sigma_r) \in \bar{\delta}^{\mathrm{out}}(S_k)\}, \qquad \rho \coloneqq \max\{r \le \tau : (a_r, \sigma_r) \in \bar{\delta}^{\mathrm{out}}(S_1)\}.$$

Define the interval-crossing subpath

$$p^{\mathrm{int}}(\mathcal{S}) \coloneqq \big(v_{\rho-1}, (a_\rho, \sigma_\rho), v_\rho, (a_{\rho+1}, \sigma_{\rho+1}), \dots, (a_\tau, \sigma_\tau), v_\tau\big).$$

Then

1. *$A(p^{\mathrm{int}}(\mathcal{S})) \subseteq A(\mathcal{S})$,*
2. *$\sum_{(a,\sigma) \in \bar{A}(p^{\mathrm{int}}(\mathcal{S}))} \eta_{(a,\sigma)}(\pi, \alpha) \le \Pi + H_{\mathcal{S}}(y)$, with $H_{\mathcal{S}}(y) = \sum_{a \in C(\mathcal{S})} \gamma_a y_a$*
3. *$A(p^{\mathrm{int}}(\mathcal{S})) \cap \delta^{\mathrm{out}}(S_i) \ne \emptyset$ for every $i = 1, \dots, k$.*

Proof. We first show that τ and ρ are well defined. Since $X \subseteq S_1 \subseteq S_k$ and $v_0 \in X$, the path starts in S_k. Since $Y \cap S_k = \emptyset$ and $v_m \in Y$, the path eventually leaves S_k. Hence τ is well defined. Moreover $v_\tau \notin S_k$ and therefore $v_\tau \notin S_1$, while $v_0 \in X \subseteq S_1$. Thus $(a_r, \sigma_r) \in \bar{\delta}^{\mathrm{out}}(S_1)$ for some $r \le \tau$, so ρ is well defined.

1.: By definition of τ we have $(a_r, \sigma_r) \notin \bar{\delta}^{\mathrm{out}}(S_k)$ for all $r < \tau$, and since $v_0 \in S_k$ it follows inductively that $v_r \in S_k$ for all $r < \tau$. Hence every traversal of $p^{\mathrm{int}}(\mathcal{S})$ has its initial node in S_k, so no arc of $A(p^{\mathrm{int}}(\mathcal{S}))$ has both endpoints in $V \setminus S_k$. Consequently $A(p^{\mathrm{int}}(\mathcal{S})) \cap A[V \setminus S_k] = \emptyset$.

By definition of ρ we have $(a_r, \sigma_r) \notin \bar{\delta}^{\mathrm{out}}(S_1)$ for all $\rho < r \le \tau$. Since $v_\rho \notin S_1$ and a path can leave S_1 only through a traversal in $\bar{\delta}^{\mathrm{out}}(S_1)$, it follows inductively that $v_r \notin S_1$ for all $\rho \le r \le \tau$. Hence no arc of $A(p^{\mathrm{int}}(\mathcal{S}))$ has both endpoints in S_1, so $A(p^{\mathrm{int}}(\mathcal{S})) \cap A[S_1] = \emptyset$. Combining both statements yields

$$A\big(p^{\mathrm{int}}(\mathcal{S})\big) \subseteq A \setminus \big(A[S_1] \cup A[V \setminus S_k]\big) = A(\mathcal{S}).$$

2.: By Definition 4.2 we have $t(a_r, \sigma_r) = v_{r-1}$ and $h(a_r, \sigma_r) = v_r$, hence

$$\sum_{(a,\sigma) \in \bar{A}(p^{\mathrm{int}}(\mathcal{S}))} \eta_{(a,\sigma)}(\pi, \alpha) = \sum_{r=\rho}^{\tau} \big(\pi_{v_{r-1}} - \pi_{v_r}\big) + \sum_{\substack{(a,\sigma) \in \bar{A}(p^{\mathrm{int}}(\mathcal{S})): \\ a \in A^{\mathrm{c}}}} \alpha_a.$$

The first sum telescopes,

$$\sum_{r=\rho}^{\tau} \big(\pi_{v_{r-1}} - \pi_{v_r}\big) = \pi_{v_{\rho-1}} - \pi_{v_\tau} \le \Pi$$

by the definition of Π and the potential bounds (4g). For the second sum, recall that the projection $\mathrm{proj} : (a, \sigma) \mapsto a$ is a bijection from $\bar{A}(p^{\mathrm{int}}(\mathcal{S}))$ onto $A(p^{\mathrm{int}}(\mathcal{S}))$. Therefore using

$A(p^{\mathrm{int}}(\mathcal{S})) \subseteq A(\mathcal{S})$ and $0 \le \alpha_a \le \gamma_a y_a$ for all $a \in A^{\mathrm{c}}$, we obtain

$$\sum_{\substack{(a,\sigma)\in \bar{A}(p^{\mathrm{int}}(\mathcal{S}))\\ a\in A^{\mathrm{c}}}} \alpha_a = \sum_{a\in A(p^{\mathrm{int}}(\mathcal{S}))\cap A^{\mathrm{c}}} \alpha_a \le \sum_{a\in C(\mathcal{S})} \gamma_a y_a = H_{\mathcal{S}}(y).$$

Combining the two estimates gives the claim.

3.: Let $i \in \{1,\dots,k\}$. Since $v_{\rho-1} \in S_1 \subseteq S_i$ and $v_\tau \notin S_k \supseteq S_i$, we have $v_\tau \notin S_i$. Hence there exists an index $r \in \{\rho,\dots,\tau\}$ with $v_{r-1} \in S_i$ and $v_r \notin S_i$, that is, $(a_r,\sigma_r) \in \bar{\delta}^{\mathrm{out}}(S_i)$. By the bijection between $\bar{\delta}^{\mathrm{out}}(S_i)$ and $\delta^{\mathrm{out}}(S_i)$, the underlying arc satisfies $a_r \in \delta^{\mathrm{out}}(S_i)$. Therefore $A(p^{\mathrm{int}}(\mathcal{S})) \cap \delta^{\mathrm{out}}(S_i) \neq \emptyset$. □

Lemma 4.5 bounds the potential consumed along a single f-consistent path crossing a given sequence of nested cuts between X and Y. By the Weymouth relation this can later be transferred into a bound on the flow that a single path can transport. To pass from a single path to a prescribed amount of flow, we use that a flow can be decomposed into flows along individual paths, and aggregate the path-wise bound over such a decomposition. The following definition makes precise which flows admit a decomposition carrying a value F from X to Y.

Definition 4.6 (Dominated X-to-Y flow)**.** Let $X, Y \subseteq V$ be disjoint, let $f \in \mathbb{R}^A$ and let $F > 0$. We say that f *dominates an X-to-Y flow of value* F if there exist a finite set P of f-consistent X-to-Y paths and values $\lambda_p \ge 0$, $p \in P$, such that

$$\sum_{p\in P} \lambda_p = F \qquad \text{and} \qquad \sum_{\substack{p\in P\\ a\in A(p)}} \lambda_p \le |f_a| \qquad \forall\, a \in A.$$

For an arc $a \in A$ with $f_a \neq 0$ let $\sigma(a)$ denote the unique orientation in $O(a)$ with $\sigma(a) f_a > 0$ and write $\eta_a(\pi,\alpha) := \eta_{(a,\sigma(a))}(\pi,\alpha)$. For $f_a = 0$ we set $\eta_a(\pi,\alpha) := 0$.

In the remainder of this section we derive two bounds based on the term

$$\sum_{i=1}^{k} \sum_{a\in\delta^{\mathrm{out}}(S_i)} \beta_a \big(\hat{f}_a^i\big)^3. \tag{6}$$

We first show in Lemma 4.7 and Lemma 4.8 that this term (6) can be bounded from above by the available potential budget, and then derive in Lemma 4.9 a lower bound for each of its summands in terms of the installed conductance. Summing the latter over $i = 1,\dots,k$ places the friction term between the two bounds,

$$F^3 \sum_{i=1}^{k} \frac{1}{\Phi_i(z)^2} \le \sum_{i=1}^{k} \sum_{a\in\delta^{\mathrm{out}}(S_i)} \beta_a \big(\hat{f}_a^i\big)^3 \le F B_{\mathcal{S}}(y).$$

Dividing by $F > 0$ yields the nonlinear condition of Proposition 4.10.

Lemma 4.7. *Let $\mathcal{S} = (S_1,\dots,S_k)$ be a k-disjoint (X,Y)-cut sequence and set $B_{\mathcal{S}}(y) := \Pi + H_{\mathcal{S}}(y)$. Let (x,y,f,π,α) be feasible for Model (4) and suppose that f dominates an X-to-Y flow of value $F > 0$. Then there exist values*

$$\hat{f}_a^i \ge 0, \qquad a \in \delta^{\mathrm{out}}(S_i),\ i = 1,\dots,k,$$

such that

$$\sum_{a\in\delta^{\mathrm{out}}(S_i)} \hat{f}_a^i = F, \qquad \hat{f}_a^i \le |f_a| \quad \forall\, a \in \delta^{\mathrm{out}}(S_i), \qquad i = 1,\dots,k,$$

and

$$\sum_{i=1}^{k} \sum_{a\in\delta^{\mathrm{out}}(S_i)} \hat f_a^i \eta_a(\pi,\alpha) \le F B_{\mathcal{S}}(y).$$

Proof. Let P and $(\lambda_p)_{p\in P}$ be as in Definition 4.6. For every $p \in P$ choose for every $i \in \{1,\dots,k\}$ an arc

$$e_i(p) \in A\big(p^{\mathrm{int}}(\mathcal{S})\big) \cap \delta^{\mathrm{out}}(S_i).$$

These are pairwise distinct, because the cutsets $\delta^{\mathrm{out}}(S_1),\dots,\delta^{\mathrm{out}}(S_k)$ are pairwise disjoint. Every arc of $a \in A(p^{\mathrm{int}}(\mathcal{S}))$ carries flow along p, hence $z_a = 1$. By Observation 4.3 we therefore have $\eta_a(\pi,\alpha) = \beta_a f_a^2 \ge 0$ for all $a \in A(p^{\mathrm{int}}(\mathcal{S}))$. Using this nonnegativity together with Part 2 of Lemma 4.5 and the distinctness of the selected arcs,

$$\sum_{i=1}^{k} \eta_{e_i(p)}(\pi,\alpha) \le \sum_{a\in A(p^{\mathrm{int}}(\mathcal{S}))} \eta_a(\pi,\alpha) \le B_{\mathcal{S}}(y). \tag{7}$$

Now define, for $i = 1,\dots,k$ and $a \in \delta^{\mathrm{out}}(S_i)$,

$$\hat f_a^i := \sum_{p\in P: e_i(p)=a} \lambda_p \ge 0.$$

Then

$$\sum_{a\in\delta^{\mathrm{out}}(S_i)} \hat f_a^i = \sum_{p\in P} \lambda_p = F, \qquad i = 1,\dots,k,$$

and, since $e_i(p) = a$ implies $a \in A(p)$,

$$\hat f_a^i = \sum_{p\in P: e_i(p)=a} \lambda_p \le \sum_{\substack{p\in P\\ a\in A(p)}} \lambda_p \le |f_a|.$$

Finally, multiplying (7) by $\lambda_p \ge 0$ and summing over $p \in P$ gives

$$\sum_{p\in P} \lambda_p \sum_{i=1}^{k} \eta_{e_i(p)}(\pi,\alpha) \le \sum_{p\in P} \lambda_p B_{\mathcal{S}}(y) = F B_{\mathcal{S}}(y).$$

Reordering yields

$$\sum_{p\in P} \lambda_p \sum_{i=1}^{k} \eta_{e_i(p)}(\pi,\alpha) = \sum_{i=1}^{k} \sum_{a\in\delta^{\mathrm{out}}(S_i)} \hat f_a^i \eta_a(\pi,\alpha).$$

□

Lemma 4.8. *Let $\mathcal{S} = (S_1,\dots,S_k)$ be a k-disjoint (X,Y)-cut sequence and let (x,y,f,π,α) be feasible for Model (4) such that f dominates an X-to-Y flow of value $F > 0$. Let $\hat f$ be the crossing flow of Lemma 4.7. Then*

$$\sum_{i=1}^{k} \sum_{a\in\delta^{\mathrm{out}}(S_i)} \beta_a \big(\hat f_a^i\big)^3 \le F B_{\mathcal{S}}(y).$$

Proof. Fix $i \in \{1,\dots,k\}$ and $a \in \delta^{\mathrm{out}}(S_i)$. If $\hat f_a^i = 0$, then $\hat f_a^i \eta_a(\pi,\alpha) = 0 = \beta_a(\hat f_a^i)^3$. If $\hat f_a^i > 0$, then $0 < \hat f_a^i \le |f_a|$ by Lemma 4.7, so $f_a \ne 0$ and the arc is built by (4e) and (4f). Observation 4.3 applied to the orientation $\sigma(a)$ gives $\eta_a(\pi,\alpha) = \beta_a f_a^2$, hence

$$\hat f_a^i \eta_a(\pi,\alpha) = \beta_a \hat f_a^i |f_a|^2 \ge \beta_a \big(\hat f_a^i\big)^3,$$

where the inequality uses $\hat f^i_a \le |f_a|$ and $\beta_a > 0$. In both cases $\hat f^i_a \eta_a(\pi,\alpha) \ge \beta_a(\hat f^i_a)^3$. Summing over all $a \in \delta^{\mathrm{out}}(S_i)$ and $i = 1,\dots,k$ and combining with Lemma 4.7 yields the claim. □

Lemma 4.9. *Let S be a node set and let $\hat f_a \ge 0$, $a \in \delta^{\mathrm{out}}(S)$, satisfy $\sum_{a\in\delta^{\mathrm{out}}(S)} \hat f_a = F > 0$ and $\hat f_a = 0$ whenever $z_a = 0$. Then $\Phi(S,z) > 0$ and*

$$\sum_{a\in\delta^{\mathrm{out}}(S)} \beta_a \hat f_a^3 \ge \frac{F^3}{\Phi(S,z)^2}.$$

Proof. Since $F > 0$ there is an arc $a \in \delta^{\mathrm{out}}(S)$ with $\hat f_a > 0$, hence $z_a = 1$ and $\Phi(S,z) > 0$. As z is binary,

$$\hat f_a = \Big(\frac{z_a}{\sqrt{\beta_a}}\Big)^{2/3}\big(\beta_a \hat f_a^3\big)^{1/3} \qquad \text{for all } a \in \delta^{\mathrm{out}}(S),$$

because the right-hand side equals $z_a^{2/3}\hat f_a$, which is $\hat f_a$ if $z_a = 1$ and $0 = \hat f_a$ if $z_a = 0$. Applying Hölder's inequality [10] with the conjugate exponents $3/2$ and 3 gives

$$F = \sum_{a\in\delta^{\mathrm{out}}(S)} \hat f_a \le \Big(\sum_{a\in\delta^{\mathrm{out}}(S)} \frac{z_a}{\sqrt{\beta_a}}\Big)^{2/3}\Big(\sum_{a\in\delta^{\mathrm{out}}(S)} \beta_a \hat f_a^3\Big)^{1/3} = \Phi(S,z)^{2/3}\Big(\sum_{a\in\delta^{\mathrm{out}}(S)} \beta_a \hat f_a^3\Big)^{1/3}.$$

Cubing and dividing by $\Phi(S,z)^2 > 0$ yields the claim. □

Proposition 4.10. *Let $\mathcal{S} = (S_1,\dots,S_k)$ be a k-disjoint (X,Y)-cut sequence and let (x,y,f,π,α) be feasible for model (4) such that f dominates an X-to-Y flow of value $F > 0$. Then $\Phi_i(z) > 0$ for $i = 1,\dots,k$ and*

$$F^2 \sum_{i=1}^{k} \frac{1}{\Phi_i(z)^2} \le \Pi + H_{\mathcal{S}}(y).$$

Proof. Let $\hat f$ be as in Lemma 4.8. Fix $i \in \{1,\dots,k\}$. By (4e) and (4f) we have $f_a = 0$ whenever $z_a = 0$, and $\hat f^i_a \le |f_a|$, so $\hat f^i_a = 0$ whenever $z_a = 0$. Together with $\sum_{a\in\delta^{\mathrm{out}}(S_i)} \hat f^i_a = F > 0$, Lemma 4.9 applies and gives $\Phi_i(z) > 0$ and

$$\sum_{a\in\delta^{\mathrm{out}}(S_i)} \beta_a\big(\hat f^i_a\big)^3 \ge \frac{F^3}{\Phi_i(z)^2}.$$

Summing over $i = 1,\dots,k$ and combining with Lemma 4.8 yields

$$F^3 \sum_{i=1}^{k} \frac{1}{\Phi_i(z)^2} \le \sum_{i=1}^{k} \sum_{a\in\delta^{\mathrm{out}}(S_i)} \beta_a\big(\hat f^i_a\big)^3 \le F\big(\Pi + H_{\mathcal{S}}(y)\big),$$

and division by $F > 0$ gives the claim. □

The condition of Proposition 4.10 is valid but nonlinear in the design variables on both sides. We treat the two sources of nonlinearity in the next section.

4.2 Linearization of the condition

Throughout this subsection the terminal sets $X, Y \subseteq V$ and the required flow value $F = F_{X,Y} > 0$ are fixed, and we suppress the dependence on X and Y in the notation. It is convenient to rewrite Proposition 4.10 in terms of the *effective conductance* of the cut sequence $\mathcal{S}$,

$$U_{\mathcal{S}}(z) := \Big(\sum_{i=1}^{k} \frac{1}{\Phi_i(z)^2}\Big)^{-1/2},$$

in which form the condition reads

$$U_{\mathcal{S}}(z) \geq \frac{F}{\sqrt{\Pi + H_{\mathcal{S}}(y)}}. \tag{8}$$

The effective conductance is itself nonlinear in the cutsets conductances. The following elementary estimate bounds it from above by their sum.

Lemma 4.11. *Let $\Phi_1, \ldots, \Phi_k > 0$. Then*

$$\Big(\sum_{i=1}^{k} \frac{1}{\Phi_i^2}\Big)^{-1/2} \leq \frac{1}{k\sqrt{k}} \sum_{i=1}^{k} \Phi_i.$$

Proof. By the Cauchy–Schwarz inequality, applied to $\sqrt{\Phi_i}$ and $1/\sqrt{\Phi_i}$,

$$k^2 = \Big(\sum_{i=1}^{k} \sqrt{\Phi_i}\frac{1}{\sqrt{\Phi_i}}\Big)^2 \leq \Big(\sum_{i=1}^{k} \Phi_i\Big)\Big(\sum_{i=1}^{k} \frac{1}{\Phi_i}\Big),$$

and, applied to 1 and $1/\Phi_i$,

$$\Big(\sum_{i=1}^{k} \frac{1}{\Phi_i}\Big)^2 \leq k \sum_{i=1}^{k} \frac{1}{\Phi_i^2}.$$

Combining both estimates yields

$$k^3 \leq \Big(\sum_{i=1}^{k} \Phi_i\Big)^2 \Big(\sum_{i=1}^{k} \frac{1}{\Phi_i^2}\Big),$$

and taking square roots and rearranging gives the claim. □

Since $U_{\mathcal{S}}(z)$ is bounded above by the linear expression of Lemma 4.11, condition (8) implies

$$\sum_{i=1}^{k} \Phi_i(z) \geq R_k\big(H_{\mathcal{S}}(y)\big), \qquad \text{where} \qquad R_k(H) := \frac{k\sqrt{k}F}{\sqrt{\Pi + H}} \quad (H \geq 0), \tag{9}$$

We abbreviate $R_k := R_k(0) = k\sqrt{k}F/\sqrt{\Pi}$. Next, we will focus on the right side of (9) to obtain a total linear inequality.

A compressor increases the available potential budget and thereby reduces the conductance required to transport the flow across the cut sequence. We express this reduction as an additive credit per compressor arc. For $H \geq 0$ let

$$\Delta_k(H) := R_k(0) - R_k(H) = k\sqrt{k}F\Big(\frac{1}{\sqrt{\Pi}} - \frac{1}{\sqrt{\Pi + H}}\Big)$$

denote the reduction in required total conductance caused by a total boost H, and define for an individual compressor arc $a \in A^{\mathrm{c}}$

$$\varepsilon_{k,a} := \Delta_k(\gamma_a).$$

Thus $\varepsilon_{k,a}$ measures how much less total cutsets conductance is required when compressor a supplies its full potential boost γ_a. The following lemma shows that summing these individual credits yields a valid bound on the reduction caused by several compressors simultaneously.

Lemma 4.12. *The function Δ_k is nonnegative, increasing and concave on $\mathbb{R}_+$ with $\Delta_k(0) = 0$, and hence subadditive, that is,*

$$\Delta_k(H_1 + H_2) \leq \Delta_k(H_1) + \Delta_k(H_2) \qquad \textit{for all } H_1, H_2 \geq 0.$$

Consequently, for every subset $A' \subseteq A^{c}$,

$$\Delta_k\Big(\sum_{a\in A'} \gamma_a\Big) \le \sum_{a\in A'} \Delta_k(\gamma_a) = \sum_{a\in A'} \varepsilon_{k,a}.$$

Proof. Write $K := k\sqrt{k}F > 0$, so that $\Delta_k(H) = K\big(\Pi^{-1/2} - (\Pi+H)^{-1/2}\big)$ and $\Delta_k(0) = 0$. Then

$$\Delta_k'(H) = \tfrac{K}{2}(\Pi+H)^{-3/2} > 0, \qquad \Delta_k''(H) = -\tfrac{3K}{4}(\Pi+H)^{-5/2} < 0,$$

so Δ_k is increasing and concave on $\mathbb{R}_+$, and nonnegative because $\Delta_k(0) = 0$. A concave function on $\mathbb{R}_+$ with $\Delta_k(0) \ge 0$ is subadditive. The second claim follows by repeated application of subadditivity. □

The linear lower bound on the required total conductance and the additive compressor credits together turn the nonlinear condition of Proposition 4.10 into a linear inequality in the design variables. It is valid for every feasible solution of model (4) that transports the required flow from X to Y, and is therefore a valid inequality for the model.

Proposition 4.13. *Let X, $Y \subseteq V$ be disjoint and let $\mathcal{S} = (S_1, \dots, S_k)$ be a k-disjoint (X,Y)-cut sequence. Then every feasible solution (x, y, f, π, α) of model (4) where f dominates an X-to-Y flow of value $F > 0$ satisfies*

$$\sum_{i=1}^{k} \sum_{a\in\delta^{\mathrm{out}}(S_i)} \frac{z_a}{\sqrt{\beta_a}} + \sum_{a\in C(\mathcal{S})} \varepsilon_{k,a} y_a \ge R_k. \tag{10}$$

Proof. Since y is binary, (9) and Lemma 4.12 yields

$$\sum_{i=1}^{k} \Phi_i(z) + \sum_{a\in C(\mathcal{S})} \varepsilon_{k,a} y_a \ge R_k.$$

Substituting $\sum_i \Phi_i(z) = \sum_i \sum_{a\in\delta^{\mathrm{out}}(S_i)} z_a/\sqrt{\beta_a}$ gives the claim. □

Remark 4.14. If no compressor arc of the interval region is built, then $H_{\mathcal{S}}(y) = 0$ and $\delta(v) = \delta^{out}(v)$ for all $v \in V$, and therefore (10) reduces to

$$\sum_{i=1}^{k} \sum_{a\in\delta(S_i)} \frac{x_a}{\sqrt{\beta_a}} \ge \frac{k\sqrt{k}F}{\sqrt{\Pi}}.$$

This is the analogue of the potential-based flow cut in Börner et al. [3] for the purely passive case $A = A^{\mathrm{p}}$, for degree $r = 2$ and signed flows.

5 Separation

Let $(\bar{x}, \bar{y})$ be the design projection of a solution of the relaxation of model (4), and set $\bar{z}_a := \bar{x}_a$ for $a \in A^{\mathrm{p}}$ and $\bar{z}_a := \bar{y}_a$ for $a \in A^{\mathrm{c}}$. Throughout this section the terminal sets $X, Y \subseteq V$ and the length k are fixed, together with the required flow value $F = F_{X,Y} > 0$. Evaluating the compressor-aware potential cut (10) at $(\bar{x}, \bar{y})$ for a disjoint (X,Y)-cut sequence $\mathcal{S} = (S_1, \dots, S_k)$ gives the left-hand side value

$$\mathrm{LHS}_{\mathcal{S}}(\bar{x}, \bar{y}) := \sum_{i=1}^{k} \sum_{a\in\delta^{\mathrm{out}}(S_i)} \frac{\bar{z}_a}{\sqrt{\beta_a}} + \sum_{a\in C(\mathcal{S})} \varepsilon_{k,a} \bar{y}_a.$$

An inequality (10) induced by a sequence of length k is violated if and only if

$$\mathrm{OPT}_k^{X,Y}(\bar{x},\bar{y}) := \min_k\big\{\mathrm{LHS}_{\mathcal{S}}(\bar{x},\bar{y}) : S \text{ a disjoint } (X,Y)\text{-cut sequence of length } k\big\} < R_k. \quad (11)$$

For brevity we write $w_a := \bar{z}_a/\sqrt{\beta_a}$ for an arc $a \in A$, and $\theta_a := \varepsilon_{k,a}\bar{y}_a$ for a compressor arc $a \in A^{\mathrm{c}}$; both are nonnegative.

We show that the separation problem (11) for fixed sets X and Y can be solved by a single minimum s-t cut computation in an auxiliary graph. The reduction proceeds in two steps. First we replace the sequence S by a single level function assigning the number of cuts that separate it from X to every node. Additionally we transfer the conditions on the cut sequence to equivalent conditions on the level function. In a second step, we encode this function by binary threshold variables, which turns the resulting minimization into a minimum s-t cut problem.

5.1 A level-function reformulation

Definition 5.1 (Level function)**.** A *k-level function* is a map $\ell : V \to \{0,\dots,k\}$, written $\ell_i := \ell(i)$. Its *sublevel sets* are

$$S_r(\ell) := \{i \in V : \ell_i < r\}, \qquad r = 1,\dots,k.$$

We call ℓ *(X,Y)-admissible* if

$$\ell_i = 0 \;\forall\, i \in X, \qquad \ell_i = k \;\forall\, i \in Y,$$

and *disjoint* if

$$\ell_{h(a,\sigma)} \le \ell_{t(a,\sigma)} + 1 \qquad \forall\,(a,\sigma) \in \bar{A}. \quad (12)$$

The sublevel sets are nested by construction, $S_1(\ell) \subseteq \cdots \subseteq S_k(\ell)$, and admissibility states precisely that each of them induces an (X,Y)-cut: $\ell_i = 0$ for $i \in X$ gives $X \subseteq S_r(\ell)$, and $\ell_i = k$ for $i \in Y$ gives $Y \cap S_r(\ell) = \emptyset$ for every r. Condition (12) is justified by the following lemma.

Lemma 5.2. *Let ℓ be a k-level function.*

1. *For every traversal $(a,\sigma) \in \bar{A}$ and every $r \in \{1,\dots,k\}$,*

$$(a,\sigma) \in \bar{\delta}^{\mathrm{out}}\big(S_r(\ell)\big) \iff \ell_{t(a,\sigma)} < r \le \ell_{h(a,\sigma)},$$

   and hence

$$\big|\{r : (a,\sigma) \in \bar{\delta}^{\mathrm{out}}(S_r(\ell))\}\big| = \max\big\{0,\; \ell_{h(a,\sigma)} - \ell_{t(a,\sigma)}\big\}.$$

2. *For every arc $a = (i,j) \in A$,*

$$\sum_{r=1}^{k} \mathbb{1}\big\{a \in \delta^{\mathrm{out}}(S_r(\ell))\big\} = \big|\{r : a \in \delta^{\mathrm{out}}(S_r(\ell))\}\big| = \begin{cases} |\ell_j - \ell_i|, & a \in A^{\mathrm{p}},\\ \max\{0,\; \ell_j - \ell_i\}, & a \in A^{\mathrm{c}}. \end{cases}$$

3. *The outflow cutsets $\delta^{\mathrm{out}}(S_1(\ell)),\dots,\delta^{\mathrm{out}}(S_k(\ell))$ are pairwise disjoint if and only if ℓ is disjoint in the sense of (12).*

Proof. Let $S_r := S_r(\ell)$, so that $v \in S_r \iff \ell_v < r$.

1.: By definition, $(a,\sigma) \in \bar{\delta}^{\mathrm{out}}(S_r)$ if and only if $t(a,\sigma) \in S_r$ and $h(a,\sigma) \notin S_r$, that is, $\ell_{t(a,\sigma)} < r$ and $\ell_{h(a,\sigma)} \ge r$. The number of integers $r \in \{1,\dots,k\}$ with $u < r \le v$ equals $\max\{0, v-u\}$ for $u, v \in \{0,\dots,k\}$, which gives the stated count.

2. Since the projection proj:$(a,\sigma) \mapsto a$ is a bijection from $\bar{\delta}^{\mathrm{out}}(S_r)$ onto $\delta^{\mathrm{out}}(S_r)$, the arc a lies in $\delta^{\mathrm{out}}(S_r)$ if and only if $(a,\sigma) \in \bar{\delta}^{\mathrm{out}}(S_r)$ for some $\sigma \in O(a)$, whereby σ is unique. Summing the count of (1) over $\sigma \in O(a)$ therefore gives the total number of sublevel sets whose outflow cutset contains a. For $a \in A^{\mathrm{c}}$ we have $O(a) = \{+1\}$ and the sum consists of the single term

$\max\{0, \ell_j - \ell_i\}$. For $a \in A^{\mathrm{p}}$ we have $O(a) = \{+1, -1\}$ and the two terms are $\max\{0, \ell_j - \ell_i\}$ and $\max\{0, \ell_i - \ell_j\}$, of which at most one is nonzero; their sum is $|\ell_j - \ell_i|$.

3. Since the projection proj:$(a, \sigma) \mapsto a$ is a bijection the outflow cutsets $\delta^{\mathrm{out}}(S_r)$ are pairwise disjoint if no traversal belongs to more than one of the sets $\bar{\delta}^{\mathrm{out}}(S_r)$. By 1. this holds if and only if $\max\{0, \ell_{h(a,\sigma)} - \ell_{t(a,\sigma)}\} \le 1$ for every $(a, \sigma) \in \bar{A}$, which is (12). □

It remains to express the compressor region $C(\mathcal{S})$ of the induced cut sequence in terms of the level function. Since $S_1(\ell) = \{i \in V : \ell_i = 0\}$ and $V \setminus S_k(\ell) = \{i \in V : \ell_i = k\}$, and since the arc set $A[U]$ induced by a node set depends on both endpoints of an arc, we obtain for every $a = (i, j) \in A$

$$a \in A\big(\mathcal{S}(\ell)\big) \iff (\ell_i, \ell_j) \ne (0, 0) \quad\text{and}\quad (\ell_i, \ell_j) \ne (k, k), \tag{13}$$

where $\mathcal{S}(\ell) \coloneqq (S_1(\ell), \ldots, S_k(\ell))$. In particular, a compressor arc belongs to $C(\mathcal{S}(\ell)) = A^{\mathrm{c}} \cap A(\mathcal{S}(\ell))$ if and only if its endpoints are neither both at level 0 nor both at level k. Note that (13) involves no orientation, since $A[\cdot]$ is symmetric in the two endpoints.

For a k-level function ℓ define

$$\begin{aligned}\Lambda_k(\ell; \bar{x}, \bar{y}) \coloneqq \sum_{a=(i,j)\in A^{\mathrm{p}}} w_a |\ell_j - \ell_i| + \sum_{a=(i,j)\in A^{\mathrm{c}}} w_a \max\{0, \ell_j - \ell_i\} \\ + \sum_{a=(i,j)\in A^{\mathrm{c}}} \theta_a \mathbb{1}\big\{(\ell_i, \ell_j) \ne (0,0), (\ell_i, \ell_j) \ne (k, k)\big\}.\end{aligned} \tag{14}$$

Proposition 5.3. *The map $\ell \mapsto \mathcal{S}(\ell)$ is a bijection between the disjoint, (X, Y)-admissible k-level functions and the disjoint (X, Y)-cut sequences of length k, and*

$$\mathrm{LHS}_{\mathcal{S}(\ell)}(\bar{x}, \bar{y}) = \Lambda_k(\ell; \bar{x}, \bar{y})$$

for every such ℓ. Consequently

$$\mathrm{OPT}_k^{X,Y}(\bar{x}, \bar{y}) = \min\big\{\Lambda_k(\ell; \bar{x}, \bar{y}) : \ell \text{ a disjoint, } (X, Y)\text{-admissible } k\text{-level function}\big\}.$$

Proof. The map is well defined: Let ℓ be disjoint and (X, Y)-admissible. The sublevel sets are nested, each is an (X, Y)-cut by admissibility, and their outflow cutsets are pairwise disjoint by Lemma 5.2(3). Hence $\mathcal{S}(\ell)$ is a disjoint (X, Y)-cut sequence of length k.

The map is bijective: Let $\mathcal{S} = (S_1, \ldots, S_k)$ be a disjoint (X, Y)-cut sequence and set $\ell_i \coloneqq k - |\{r : i \in S_r\}|$. Since the sets are nested, $i \notin S_r$ is equivalent to $i \notin S_t\ \forall t \le r$. Hence

$$\ell_i = k - |\{r : i \in S_r\}| \ge k - (k - r) = r$$

and therefore both $i \notin S_r \Leftrightarrow l_i \ge r$ and $i \in S_r \Leftrightarrow l_i < r$ holds. Then by definition follows immediately $S_r(\ell) = S_r$. Admissibility of ℓ follows from $X \subseteq S_r$ and $Y \cap S_r = \emptyset$ for all r, and disjointness of ℓ by Lemma 5.2(3). The level function ℓ is uniquely determined by $\mathcal{S}$, since ℓ_i is determined by the number of sets containing i, and $\ell \mapsto \mathcal{S}(\ell)$ and $\mathcal{S} \mapsto \ell$ are mutually inverse.

The objectives are equivalent: Let ℓ be disjoint and (X, Y)-admissible and write $\mathcal{S} \coloneqq \mathcal{S}(\ell)$. Exchanging the order of summation together with Lemma 5.2 yields

$$\begin{aligned}\sum_{i=1}^{k} \sum_{a\in\delta^{\mathrm{out}}(S_i)} w_a &= \sum_{a\in A} w_a \sum_{r=1}^{k} \mathbb{1}\big\{a \in \delta^{\mathrm{out}}(S_r(\ell))\big\} \\ &= \sum_{a=(i,j)\in A^{\mathrm{p}}} w_a |\ell_j - \ell_i| + \sum_{a=(i,j)\in A^{\mathrm{c}}} w_a \max\{0, \ell_j - \ell_i\},\end{aligned}$$

which corresponds to the first part of $\mathrm{LHS}_{\mathcal{S}(\ell)}(\bar{x}, \bar{y})$ or the first two terms of $\Lambda_k(\ell)$, respectively. Overall then together with (13), the two expressions coincide. □

5.2 Minimum-cut separation

By Proposition 5.3, separating the compressor-aware potential cuts of length k amounts to minimizing the function Λ_k over all disjoint, (X,Y)-admissible k-level functions. In this form the problem is an energy-minimization problem over integer labels $\ell\colon V \to \{0,\dots,k\}$ with unary and pairwise terms. Problems of this type are exactly solvable by a single minimum s-t cut in a layered graph with one copy of each node per level, provided that every pairwise term is a convex function of the level difference [1, 13]. In the remainder of this section we carry out this construction for the function Λ_k of Proposition 5.3.

Definition 5.4 (Threshold representation)**.** Let ℓ be a k-level function. For $i \in V$ and $h = 1,\dots,k$ set

$$r_{i,h} \coloneqq \begin{cases} 1, & \ell_i < h, \\ 0, & \ell_i \geq h, \end{cases} \qquad \text{equivalently} \qquad r_{i,h} = 1 \iff i \in S_h(\ell).$$

For every node i the vector $(r_{i,1},\dots,r_{i,k})$ is nondecreasing, that is,

$$r_{i,h} \leq r_{i,h+1}, \qquad h = 1,\dots,k-1, \tag{15}$$

and conversely every binary vector satisfying (15) arises from the unique level

$$\ell_i = k - \sum_{h=1}^{k} r_{i,h}. \tag{16}$$

Thresholds and levels are therefore equivalent encodings. The conditions of Definition 5.1 translate as follows.

Lemma 5.5. *Let ℓ be a k-level function with the associated threshold representation r.*

1. *ℓ is (X,Y)-admissible if and only if*

$$r_{i,h} = 1 \;\forall\, i \in X, h = 1,\dots,k, \qquad r_{i,h} = 0 \;\forall\, i \in Y, h = 1,\dots,k.$$

2. *ℓ is disjoint if and only if*

$$r_{t(a,\sigma),h} \leq r_{h(a,\sigma),h+1}, \qquad h = 1,\dots,k-1,\ (a,\sigma) \in \bar{A}. \tag{17}$$

Proof. 1.: This follows straight from (16).

2.: Fix $(a,\sigma) \in \bar{A}$ and abbreviate $u \coloneqq t(a,\sigma)$, $v \coloneqq h(a,\sigma)$. Assume $\ell_v \leq \ell_u + 1$ and let $h \in \{1,\dots,k-1\}$ with $r_{u,h} = 1$, i.e. $\ell_u < h$. Since the levels are integral, $\ell_u + 1 \leq h$, hence $\ell_v \leq h < h+1$ and $r_{v,h+1} = 1$. Conversely, assume (17) and suppose $\ell_v > \ell_u + 1$. Choosing $h \coloneqq \ell_u + 1 \leq k-1$ gives $\ell_u < h$, hence $r_{u,h} = 1$, while $\ell_v \geq h+1$ gives $r_{v,h+1} = 0$, contradicting (17). Thus $\ell_v \leq \ell_u + 1$. □

We now construct the auxiliary graph. Its nodes are the pairs (i,h), one for each node $i \in V$ and threshold $h \in \{1,\dots,k\}$; a minimum s-t cut will place (i,h) on the source side exactly when $r_{i,h} = 1$. Conditions (15), Lemma 5.5(1) and Lemma 5.5(2) are enforced by arcs of prohibitively large capacity M, while the terms of the objective Λ_k are represented by arcs of finite capacity.

Definition 5.6 (Layered auxiliary graph)**.** Fix $k \in \mathbb{N}$, disjoint sets $X, Y \subseteq V$, weights $w_a \geq 0$ for $a \in A$, credits $\theta_a \geq 0$ for $a \in A^{\mathrm{c}}$, and a constant $M > 0$. The layered auxiliary graph $G_k^{\mathrm{aux}} = (V_k^{\mathrm{aux}}, A_k^{\mathrm{aux}}, u_k^{\mathrm{aux}})$ contains

- the node set

$$V_k^{\mathrm{aux}} \coloneqq \{s,t\} \cup \{(i,h) : i \in V,\ h = 1,\dots,k\},$$

- the arc set $A_k^{\text{aux}} \coloneqq A_k^{\text{mon}} \cup A_k^{\text{term}} \cup A_k^{\text{jump}} \cup A_k^{\text{cross}} \cup A_k^{\text{comp}}$, where

$$\begin{aligned}
A_k^{\text{mon}} &\coloneqq \big\{\big((i,h),(i,h+1)\big) : i \in V,\ h = 1,\dots,k-1\big\},\\
A_k^{\text{term}} &\coloneqq \big\{\big(s,(i,h)\big) : i \in X,\ h = 1,\dots,k\big\} \cup \big\{\big((i,h),t\big) : i \in Y,\ h = 1,\dots,k\big\},\\
A_k^{\text{jump}} &\coloneqq \big\{\big((t(a,\sigma),h),(h(a,\sigma),h+1)\big) : (a,\sigma) \in \bar{A},\ h = 1,\dots,k-1\big\},\\
A_k^{\text{cross}} &\coloneqq \big\{\big((t(a,\sigma),h),(h(a,\sigma),h)\big) : (a,\sigma) \in \bar{A},\ h = 1,\dots,k\big\},
\end{aligned}$$

and, for every compressor arc $a = (i,j) \in A^{\text{c}}$,

$$\begin{aligned}
A_k^{\text{comp},1}(a) &\coloneqq \big\{\big(s,(i,1)\big),\ \big(s,(j,1)\big),\ \big((i,1),(j,1)\big),\ \big((j,1),(i,1)\big)\big\},\\
A_k^{\text{comp},k}(a) &\coloneqq \big\{\big((i,k),t\big),\ \big((j,k),t\big),\ \big((i,k),(j,k)\big),\ \big((j,k),(i,k)\big)\big\},
\end{aligned}$$

with $A_k^{\text{comp}} \coloneqq \bigcup_{a\in A^{\text{c}}} \big(A_k^{\text{comp},1}(a) \cup A_k^{\text{comp},k}(a)\big)$,

- the capacity function $u_k^{\text{aux}} : A_k^{\text{aux}} \to \mathbb{R}_{\geq 0}$ is defined additively. For $e \in A_k^{\text{aux}}$,

$$\begin{aligned}
u_k^{\text{aux}}(e) \coloneqq{}& M\mathbb{1}\{e \in A_k^{\text{mon}}\} + M\mathbb{1}\{e \in A_k^{\text{term}}\} + M\mathbb{1}\{e \in A_k^{\text{jump}}\}\\
&+ \sum_{\substack{(a,\sigma)\in\bar{A}\\ h=1,\dots,k}} w_a \mathbb{1}\Big\{e = \big((t(a,\sigma),h),(h(a,\sigma),h)\big)\Big\}\\
&+ \sum_{a\in A^{\text{c}}} \frac{\theta_a}{2}\mathbb{1}\Big\{e \in A_k^{\text{comp},1}(a) \cup A_k^{\text{comp},k}(a)\Big\}.
\end{aligned}$$

If the same auxiliary arc is generated by several arc classes or by several traversals, then u_k^{aux} adds all corresponding capacity contributions.

Throughout, an s-t cut in G_k^{aux} is identified with the set $W \subseteq V_k^{\text{aux}}$ of nodes on the source side, so $s \in W$ and $t \notin W$, and its value is $u_k^{\text{aux}}(\delta^+(W)) = \sum_{e\in\delta^+(W)} u_k^{\text{aux}}(e)$. We call an arc of G_k^{aux} *prohibitive* if its capacity contains a contribution M, and we let C^{np} denote the total non-prohibitive capacity in G_k^{aux}, that is,

$$C^{\text{np}} \coloneqq k\sum_{a\in A}|O(a)|w_a + 4\sum_{a\in A^{\text{c}}}\theta_a.$$

From now on we fix $M > C^{\text{np}}$. Then an s-t cut has value less than M if and only if it cuts no prohibitive arc.

Proposition 5.7. *Let $M > C^{\text{np}}$.*

1. *Let W be an s-t cut in G_k^{aux} with $u_k^{\text{aux}}(\delta^+(W)) < M$. Set $r_{i,h} \coloneqq \mathbb{1}\{(i,h) \in W\}$ and $\ell_i \coloneqq k - \sum_{h=1}^{k} r_{i,h}$. Then ℓ is a disjoint, (X,Y)-admissible k-level function with threshold representation r.*

2. *Conversely, let ℓ be a disjoint, (X,Y)-admissible k-level function and set $W(\ell) \coloneqq \{s\} \cup \{(i,h) \in V_k^{\text{aux}} : \ell_i < h\}$. Then $W(\ell)$ is an s-t cut with $u_k^{\text{aux}}(\delta^+(W(\ell))) < M$.*

Moreover, the two maps are mutually inverse on the sets described.

Proof. 1. Since the cut value is smaller than M, no prohibitive arc is cut. Let $i \in V$ and $h \in \{1,\dots,k-1\}$. Since no arc $((i,h),(i,h+1)) \in A_k^{\text{mon}}$ can be part of the s-t cut, $r_{i,h} \leq r_{i,h+1}$ holds. Hence $(r_{i,1},\dots,r_{i,k})$ is non decreasing and $\ell_i \in \{0,\dots,k\}$ is well defined with $r_{i,h} = 1 \iff \ell_i < h$. Therefore r is the threshold representation of ℓ.

Let $i \in X$. Since $s \in W$ and no terminal arc $(s,(i,h))$ is cut, we have $(i,h) \in W$ and hence $r_{i,h} = 1$ for all h, so $\ell_i = 0$. Let $i \in Y$. Since $t \notin W$ and no terminal arc $((i,h),t)$ is cut, we have $(i,h) \notin W$ and hence $r_{i,h} = 0$ for all h, so $\ell_i = k$. By Lemma 5.5(1), ℓ is (X,Y)-admissible.

Finally, let $(a,\sigma) \in \bar{A}$ and $h \in \{1,\dots,k-1\}$. The arc $((t(a,\sigma),h),(h(a,\sigma),h+1)) \in A_k^{\mathrm{jump}}$ is not cut, so $r_{t(a,\sigma),h} \le r_{h(a,\sigma),h+1}$. By Lemma 5.5(2), ℓ is disjoint.

2. Since $t \notin W(\ell)$ and $s \in W(\ell)$, the set $W(\ell)$ is an s-t cut, and $(i,h) \in W(\ell) \iff \ell_i < h \iff r_{i,h} = 1$ for the threshold representation r of ℓ.

No monotonicity arc is cut: if $(i,h) \in W(\ell)$ then $\ell_i < h < h+1$, so $(i,h+1) \in W(\ell)$. No terminal arc is cut: for $i \in X$ admissibility gives $\ell_i = 0 < h$ for every h, so $(i,h) \in W(\ell)$ and the arcs $(s,(i,h))$ have both endpoints in $W(\ell)$; for $i \in Y$ it gives $\ell_i = k \ge h$, so $(i,h) \notin W(\ell)$ and the arcs $((i,h),t)$ have both endpoints outside $W(\ell)$. No no-forward-jump arc is cut: let $(a,\sigma) \in \bar{A}$ and $h \le k-1$ with $(t(a,\sigma),h) \in W(\ell)$, i.e. $\ell_{t(a,\sigma)} < h$. Since the levels are integral, $\ell_{t(a,\sigma)}+1 \le h$, and disjointness gives $\ell_{h(a,\sigma)} \le \ell_{t(a,\sigma)}+1 \le h < h+1$, so $(h(a,\sigma),h+1) \in W(\ell)$.

Hence no prohibitive arc is cut and the cut value is at most $C^{\mathrm{np}} < M$. That the two maps are mutually inverse follows from $(i,h) \in W \iff r_{i,h} = 1 \iff \ell_i < h$. □

Lemma 5.8. *Let $M > C^{\mathrm{np}}$, let W be an s-t cut in G_k^{aux} with $u_k^{\mathrm{aux}}(\delta^+(W)) < M$, and let ℓ be the level function induced by W according to Proposition 5.7(1). Then*

$$u_k^{\mathrm{aux}}\big(\delta^+(W)\big) = \Lambda_k(\ell) + \Theta_k, \qquad \textit{where} \quad \Theta_k := \sum_{a \in A^{\mathrm{c}}} \theta_a.$$

Proof. No prohibitive arc is cut, so the capacity contributions of A_k^{mon}, A_k^{term} and A_k^{jump} do not enter the cut value. Since capacities are additive, it suffices to evaluate the crossing-cost and the compressor-gadget contributions separately.

Crossing costs. Fix $(a,\sigma) \in \bar{A}$ and $h \in \{1,\dots,k\}$, and abbreviate $u := t(a,\sigma)$, $v := h(a,\sigma)$. The contribution w_a attached to $((u,h),(v,h))$ is cut if and only if $r_{u,h} = 1$ and $r_{v,h} = 0$, that is, $\ell_u < h \le \ell_v$. By Lemma 5.2(1), the number of such h equals $\max\{0, \ell_v - \ell_u\}$, so the traversal (a,σ) contributes $w_a \max\{0,\ell_v - \ell_u\}$ in total. Summing over $\sigma \in O(a)$ and using Lemma 5.2(2), the arc $a = (i,j)$ contributes $w_a|\ell_j - \ell_i|$ if $a \in A^{\mathrm{p}}$ and $w_a \max\{0, \ell_j - \ell_i\}$ if $a \in A^{\mathrm{c}}$. This is the sum of the first two terms of $\Lambda_k(\ell)$.

Compressor gadgets. Fix $a = (i,j) \in A^{\mathrm{c}}$ and write $\theta := \theta_a$. Consider first $A_k^{\mathrm{comp},1}(a)$, whose four arcs each carry a contribution $\theta/2$. Since $s \in W$, the arcs $(s,(i,1))$ and $(s,(j,1))$ are cut exactly when $r_{i,1} = 0$ resp. $r_{j,1} = 0$, and the arcs $((i,1),(j,1))$, $((j,1),(i,1))$ are cut exactly when $r_{i,1} \ne r_{j,1}$. Hence the contribution of $A_k^{\mathrm{comp},1}(a)$ equals 0 if $r_{i,1} = r_{j,1} = 1$, and θ in the three remaining cases; that is, it equals $\theta(1 - r_{i,1}r_{j,1})$. Since $r_{v,1} = 1 \iff \ell_v = 0$, this vanishes exactly when $(\ell_i,\ell_j) = (0,0)$.

Symmetrically, for $A_k^{\mathrm{comp},k}(a)$ we have $t \notin W$, so $((i,k),t)$ and $((j,k),t)$ are cut exactly when $r_{i,k} = 1$ resp. $r_{j,k} = 1$, and the two arcs between (i,k) and (j,k) are cut exactly when $r_{i,k} \ne r_{j,k}$. The contribution equals $\theta\big(1 - (1-r_{i,k})(1-r_{j,k})\big)$, and since $r_{v,k} = 0 \iff \ell_v = k$, it vanishes exactly when $(\ell_i,\ell_j) = (k,k)$.

As $\ell_i = \ell_j = 0$ and $\ell_i = \ell_j = k$ cannot both hold for $k \ge 1$, at most one of the two contributions vanishes, so their sum equals

$$\theta_a + \theta_a \mathbb{1}\big\{(\ell_i,\ell_j) \ne (0,0),\ (\ell_i,\ell_j) \ne (k,k)\big\}.$$

Summing over $a \in A^{\mathrm{c}}$ gives Θ_k plus the third term of $\Lambda_k(\ell)$, which proves the claim. □

Theorem 5.9 (Minimum-cut separation)**.** *Let $(\bar{x},\bar{y})$ be given, let $X, Y \subseteq V$ be disjoint, let $k \in \mathbb{N}$, and let $M > C^{\mathrm{np}}$. Denote by $\mathrm{MC}_k^{X,Y}(\bar{x},\bar{y})$ the value of a minimum s-t cut in G_k^{aux}.*

1. *If $\mathrm{MC}_k^{X,Y}(\bar{x},\bar{y}) < M$, then*

$$\mathrm{OPT}_k^{X,Y}(\bar{x},\bar{y}) = \mathrm{MC}_k^{X,Y}(\bar{x},\bar{y}) - \Theta_k,$$

and a minimum s-t cut induces, via Proposition 5.7(1) and Proposition 5.3, a cut sequence attaining the minimum in (11). In particular, a violated compressor-aware potential cut of *length k for the terminal sets X, Y exists if and only if*

$$\mathrm{MC}_k^{X,Y}(\bar{x}, \bar{y}) - \Theta_k < R_k.$$

2. *If* $\mathrm{MC}_k^{X,Y}(\bar{x}, \bar{y}) \geq M$*, then no disjoint* (X, Y)*-cut sequence of length* k *exists.*

Proof. *1.:* Let W^* be a minimum s-t cut, of value less than M, and let ℓ^* be the induced level function, which is disjoint and (X, Y)-admissible by Proposition 5.7(1). By Lemma 5.8 and Proposition 5.3,

$$\mathrm{MC}_k^{X,Y}(\bar{x}, \bar{y}) = \Lambda_k(\ell^*) + \Theta_k \geq \mathrm{OPT}_k^{X,Y}(\bar{x}, \bar{y}) + \Theta_k.$$

Conversely, let $\hat{\ell}$ attain the minimum of Λ_k over the disjoint, (X, Y)-admissible level functions. By Proposition 5.7(2) the cut $W(\hat{\ell})$ has value less than M, so Lemma 5.8 applies and gives

$$\mathrm{MC}_k^{X,Y}(\bar{x}, \bar{y}) \leq u_k^{\mathrm{aux}}\big(\delta^+(W(\hat{\ell}))\big) = \Lambda_k(\hat{\ell}) + \Theta_k = \mathrm{OPT}_k^{X,Y}(\bar{x}, \bar{y}) + \Theta_k.$$

Both inequalities together give the stated equality, and show that ℓ^* attains the minimum. The violation criterion is (11).

2.: Suppose a disjoint (X, Y)-cut sequence of length k exists. By Proposition 5.3 it corresponds to a disjoint, (X, Y)-admissible level function ℓ, and by Proposition 5.7(2) the cut $W(\ell)$ has value less than M. Hence $\mathrm{MC}_k^{X,Y}(\bar{x}, \bar{y}) < M$, contradicting the assumption. □

Corollary 5.10. *For fixed disjoint sets $X, Y \subseteq V$, the separation problem for compressor-aware potential cuts can be solved in polynomial time by one minimum s-t cut computation in G_k^{aux}.*

Proof. By Theorem 5.9 the separation problem reduces to one minimum s-t cut computation in G_k^{aux}, so it suffices to bound the size of this graph. It has $k|V| + 2$ nodes and at most

$$(k-1)|V| + k\big(|X| + |Y|\big) + (2k-1)|\bar{A}| + 8|A^{\mathrm{c}}|$$

arcs, where $|\bar{A}| = 2|A^{\mathrm{p}}| + |A^{\mathrm{c}}|$. Therefore this can be solved in polynomial time in the size of G for all k. □

5.3 A no-cut certificate

During branch-and-cut the separation routine is called at a large number of LP solutions whose design variables differ only marginally. Solving the minimum-cut problem of Theorem 5.9 at every such point is wasteful whenever the preceding call already established that all inequalities of the family were satisfied. We therefore use a certificate-based filter that can prove, by a cheap closed-form bound, that no violated compressor-aware potential cut of a fixed sequence length k can exist at the new point. If this proof succeeds, the corresponding minimum-cut separation is skipped.

Within this subsection we make the dependence on the terminal sets explicit and write $R_{X,Y,k}$ and $\varepsilon_{X,Y,k,a}$ for the right-hand side and the conductance-equivalent credits of Section 4.2.

Proposition 5.11 (No-cut certificate for a fixed triple)**.** *Let $X, Y \subseteq V$ be disjoint, let $k \in \mathbb{N}$, and let $(\bar{x}^{\mathrm{old}}, \bar{y}^{\mathrm{old}})$ and $(\bar{x}^{\mathrm{new}}, \bar{y}^{\mathrm{new}})$ be design projections of two solutions of the relaxation of model (4). Suppose that no compressor-aware potential cut for the triple (X, Y, k) is violated at $(\bar{x}^{\mathrm{old}}, \bar{y}^{\mathrm{old}})$, and let*

$$s_{X,Y,k}^{\mathrm{old}} := \mathrm{OPT}_k^{X,Y}\big(\bar{x}^{\mathrm{old}}, \bar{y}^{\mathrm{old}}\big) - R_{X,Y,k} \geq 0$$

denote the corresponding margin. Define

$$B_{X,Y,k} := \sum_{a \in A} \frac{1}{\sqrt{\beta_a}} \max\big\{0,\ \bar{z}_a^{\mathrm{old}} - \bar{z}_a^{\mathrm{new}}\big\} + \sum_{a \in A^{\mathrm{c}}} \varepsilon_{X,Y,k,a} \max\big\{0,\ \bar{y}_a^{\mathrm{old}} - \bar{y}_a^{\mathrm{new}}\big\}.$$

If $B_{X,Y,k} \le s^{\mathrm{old}}_{X,Y,k}$, then no compressor-aware potential cut for the triple (X,Y,k) is violated at $(\bar{x}^{\mathrm{new}}, \bar{y}^{\mathrm{new}})$.

Proof. Let $\mathcal{S} = (S_1, \ldots, S_k)$ be an arbitrary disjoint (X,Y)-cut sequence of length k. Its left-hand side value at a design projection $(\bar{x}, \bar{y})$ is

$$\mathrm{LHS}_{\mathcal{S}}(\bar{x}, \bar{y}) = \sum_{i=1}^{k} \sum_{a \in \delta^{\mathrm{out}}(S_i)} \frac{\bar{z}_a}{\sqrt{\beta_a}} + \sum_{a \in C(\mathcal{S})} \varepsilon_{X,Y,k,a} \bar{y}_a.$$

Since the cutsets $\delta^{\mathrm{out}}(S_1), \ldots, \delta^{\mathrm{out}}(S_k)$ are pairwise disjoint, every arc occurs in at most one of the inner sums, and every compressor arc occurs at most once in the second sum. All coefficients $1/\sqrt{\beta_a}$ and $\varepsilon_{X,Y,k,a}$ are nonnegative, so

$$\mathrm{LHS}_{\mathcal{S}}\big(\bar{x}^{\mathrm{old}}, \bar{y}^{\mathrm{old}}\big) - \mathrm{LHS}_{\mathcal{S}}\big(\bar{x}^{\mathrm{new}}, \bar{y}^{\mathrm{new}}\big) \le B_{X,Y,k},$$

because the left-hand side can decrease from the old to the new point only through decreases of the design variables, each of which is accounted for in $B_{X,Y,k}$ with its coefficient, and $B_{X,Y,k}$ sums over all arcs rather than only those occurring in $\mathcal{S}$.

By the definition of $s^{\mathrm{old}}_{X,Y,k}$ as the minimum margin over all admissible sequences,

$$\mathrm{LHS}_{\mathcal{S}}\big(\bar{x}^{\mathrm{old}}, \bar{y}^{\mathrm{old}}\big) - R_{X,Y,k} \ge s^{\mathrm{old}}_{X,Y,k}.$$

Combining the two estimates yields

$$\mathrm{LHS}_{\mathcal{S}}\big(\bar{x}^{\mathrm{new}}, \bar{y}^{\mathrm{new}}\big) - R_{X,Y,k} \ge s^{\mathrm{old}}_{X,Y,k} - B_{X,Y,k} \ge 0,$$

so the inequality induced by $\mathcal{S}$ is satisfied at the new point. □

6 Outlook

This work extends the potential-based valid inequalities presented in [3] from purely passive networks to networks that also contain directed compressor arcs. The resulting compressor-aware inequalities combine the conductance installed across nested cuts with the additional potential provided by compressors. For fixed terminal sets the separation problem can be solved in polynomial time using minimum *s*-*t* cuts in an auxiliary graph.

Several directions remain open. First, the practical value of the inequalities has yet to be assessed: The next step will be to discover whether integrating this separation procedure into a spatial branching framework reduces the overall solving procedure and can outperform existing approaches. Furthermore, how to choose the sequence length k and the terminal sets in practice to obtain the best results is yet to be determined.

Additionally, compressors are commonly modeled with three operating modes: *active*, in which the potential is increased along the prescribed direction; *bypass*, in which flow passes without a potential increase; and *closed*, in which no flow is permitted. These modes need to be included in model (4), and it requires careful consideration how the separation procedure would change, or whether it carries over at all. If the inequalities can be shown to handle these operating modes, an evident next step is to include valves as a further active component in the model.

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