Attainability of properly efficient points via weighted norm scalarization

Felix Neussel* Oliver Stein#

August 6, 2026

Abstract

We show that efficient points can be obtained with the weighted norm scalarization under assumptions less restrictive than in the existing literature. For properly efficient points with a given trade-off bound, we provide an easy-to-compute lower bound on the norm parameter that allows the approximation of the desired points within a specified tolerance. We apply a technique from machine learning that allows for stable computations, even for larger values of the norm parameter, and solve some test instances of the weighted norm scalarization globally with the alpha-BB method.

Keywords: multiobjective optimization, supportedness, weighted norm scalarization, proper efficiency, trade-off bound

1 Introduction

Multiobjective optimization problems are optimization problems with multiple objective functions to be minimized over a feasible set. Unless the individual minima of each objective can be attained with a feasible point,

*Corresponding author, Institute for Operations Research (IOR), Karlsruhe Institute of Technology (KIT), Karlsruhe, Germany, felix.neussel@kit.edu, https://orcid.org/0009-0005-9579-9509

#Institute for Operations Research (IOR), Karlsruhe Institute of Technology (KIT), Karlsruhe, Germany, stein@kit.edu, https://orcid.org/0000-0001-9514-6317

such problems do not have a unique optimal value, but a set of nondominated points, which cannot be ordered objectively. The preimage of the set of nondominated points is the set of efficient points.

A scalarization of such a problem is a single-objective parametric optimization problem whose solutions yield different efficient points, depending on the parameter. Scalarizations occur naturally in a-priori methods where a decision maker combines the objectives using a personal utility function. Scalarizations also occur as subproblems in a-posteriori methods, which aim at determining the set of efficient points and leave the choice of a specific one to the decision maker, and interactive methods, in which the decision maker iteratively provides preference information during the solution process ([29]). A desirable property of a scalarization approach is that it provides a characterization of the efficient or the properly efficient points (cf. [41]) in the following sense: for every parameter choice, all minimal points of the scalarization should be (properly) efficient, and every (properly) efficient point of MOP should be attained as a minimal point of the scalarization for some parameter choice.

Unfortunately, many common scalarization techniques provide either a necessary or sufficient condition for (proper) efficiency, but not both. While the weighted sum approach does provide a characterization of the properly efficient points in the convex case, it fails to identify all such points in the general nonlinear case (cf. [7]). On the other hand, the Pascoletti-Serafini scalarization (cf. [33]) provides a necessary condition for efficiency but may yield solutions that are not efficient but only weakly efficient. A solution of the Pascoletti-Serafini problem is efficient if it is unique, but this condition is difficult to computationally verify (cf. [41]). The same holds for the ε-constraint and weighted Chebyshev scalarization (cf. [29]), which are special cases of the Pascoletti-Serafini method ([8]).

More promising results hold for scalarizations based on minimizing the distance, measured in some weighted norm, to the so-called ideal point. Two such methods are the augmented and the modified weighted Chebyshev scalarization, which are actually equivalent and characterize the set of properly efficient points ([22]). Both have a non-smooth objective function but can be reformulated as a smooth problem with the epigraph reformulation ([35]). This, however, leads to a disadvantage for a certain (but quite general) class of multiobjective problems. If the nonconvexity of the MOP stems only from the objective functions while the constraints are convex, the epigraph reformulation yields a scalar problem with nonconvex constraints. Since nonconvex constraints are more difficult to handle than convex ones, both for local and global solvers, it would be beneficial to use a scalarization that does not introduce new constraints.

A different approach is the weighted (p-)norm scalarization, which measures the distance to the ideal point in a weighted ℓ_p-norm. Like the weighted sum scalarization, it naturally has a smooth objective (given that the original objectives are smooth) and does not introduce new constraints. Gearhart ([13]) showed that it yields efficient points, and, while not every efficient point can be attained with this approach, that all efficient points can be approximated: for every efficient point and a given tolerance, there exist a finite p and some weight vector such that the image of each solution to the weighted norm problem has a distance to the image of the desired efficient point not larger than the tolerance.

White ([40]) later showed that for discrete problems, there even exists a uniform finite p such that all efficient points can be attained as solutions to the weighted norm scalarization with a certain weight. In the continuous case, Li ([24]) studied the so-called power transformation of an MOP, which raises each objective to a power of $p \geqq 1$. Applying the weighted sum to this transformed problem leads to a scalar problem equivalent to the weighted norm approach. Under the assumption that the set of nondominated points can be expressed as the graph of a twice continuously differentiable function with strictly negative first partial derivatives, [24] showed that the so-called upper image set of the transformed problem is convex for a sufficiently large finite p, hence implying that every properly nondominated point can be attained with the weighted norm scalarization.

Li's assumption is quite restrictive: even linear MOPs typically have a nonsmooth set of nondominated points. On the other hand, it is immediately clear that the assumption is not necessary, since the nondominated points of linear MOPs are, in fact, characterized by the weighted sum scalarization. Li's assumption was also used in [25] to establish a similar result for an exponential transformation. Another scalarization technique with a smooth objective that keeps the original constraints is the smooth Chebyshev scalarization from [27]. It has properties similar to the weighted norm approach, which were also shown under Li's assumption.

In our related work [31], we showed Li's result on the power transformation without his assumption. Instead we required the feasible set to be a polyhedron and the desired efficient point to fulfill some regularity conditions. In the present paper, we weaken the assumptions further and prove the result for general nonlinear smooth constraints. In contrast to our proof from [31], which used some results from single-objective Lagrangian duality, the proof in this work is self-contained.

Another open question is how to choose the value p in the weighted norm scalarization. Both [16, 26] provided a lower bound under Li's assumption. It is not suitable for practical purposes, however, since it depends on the functional representation of the set of nondominated points. In the discrete case, [40] also provided a lower bound on p, which is cumbersome to compute. For discrete problems with a specific structure, [4] provided a simple formula for a lower bound on p. In [38], this scalarization is used within a decomposition-based evolutionary algorithm, and the value p is updated adaptively with values from $[1/2, 1000]$ according to an estimated local shape of the set of nondominated points (note that we only consider $p \geqq 1$ in the present paper).

We propose a lower bound on p assuming that the decision maker is satisfied with only finding properly efficient points with fixed maximal trade-offs that they choose a-priori. We provide a bound on p that allows approximating all such points with a relative tolerance provided by the user. With the same technique, we provide a bound on the parameter used within the smooth Chebyshev scalarization. We also show the weighted norm problem can be treated in a numerically stable manner, even with large values of the power parameter. We then perform a small numerical study, demonstrating that both scalarizations can be solved globally with the αBB method.

The rest of the paper is organized as follows. We state some definitions and preliminary results in Sect. 2 before proving our main theoretical result and demonstrating the relevance of the assumptions with some examples in Sect. 3. Sect. 4 presents the parameter bounds and some computationally interesting aspects of the scalarizations. Sect. 5 summarizes the numerical results, and Sect. 6 offers concluding remarks.

2 Notation and preliminaries

Interior, boundary, and closure of a set are denoted by int, bd, and cl, respectively. In the absence of ambiguity, we write singleton sets without brackets, e.g., a set consisting of a single point $y \in \mathbb{R}^m$ is denoted by y instead of $\{y\}$. For $m \in \mathbb{N}$, we define the index set $[m] := \{1, \dots, m\}$, and for $i \in [m]$, we define $[m]_{-i} := [m] \setminus i$. The Euclidean norm is denoted by $\| \cdot \|$ and the Chebyshev norm by $\| \cdot \|_\infty$. For a point $\bar{x} \in \mathbb{R}^n$ and some $t > 0$, we define $B(\bar{x}, t) := \{x \in \mathbb{R}^n \mid \|x - \bar{x}\| < t\}$ and $B_\infty(\bar{x}, t) := \{x \in \mathbb{R}^n \mid \|x - \bar{x}\|_\infty < t\}$. For a vector $x \in \mathbb{R}^n$ and a number $a \in \mathbb{R}$, we let $x^a = (x_1^a, \dots, x_n^a)^\top$ and

$a^x = (a^{x_1}, \dots, a^{x_n})^\top$. For $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$, we denote by $\mathcal{C}(X,Y)$ the continuous and by $\mathcal{C}^k(X,Y)$ the k times continuously differentiable functions mapping from X to Y. If X and Y are clearly given from context, we may simply write $f \in \mathcal{C}^k$. For $f \in \mathcal{C}^1(X,Y)$, we denote by Df the Jacobian and by ∇f the gradient matrix, i.e., $(Df)^\top$. For $f \in \mathcal{C}^2(X,Y)$ we denote by $D^2 f$ the Hessian matrix.

2.1 Solution concepts

A general multiobjective optimization problem is of the form

$$\text{(MOP)} \qquad \min\ f(x) \quad \text{s.t.} \quad x \in X$$

with a nonempty set $X \subseteq \mathbb{R}^n$ of feasible points and a vector-valued objective function $f : X \to \mathbb{R}^m$ with $m > 1$. The image set of X under f is denoted by $Y := f(X)$. For inline references, we address an optimization problem by the tuple (f, X). If X is described via inequality constraints $g(x) \leqq 0$ and equality constraints $h(x) = 0$, we refer to an optimization problem as (f, g, h).

For the definition of multiobjective solution concepts, we use the following notation for relations between vectors (cf. [7]). The usual inequality sign $\leq$ is replaced by the sign $\leqq$ and the sign $\leq$ is redefined.

Definition 2.1. *For $y^1, y^2 \in \mathbb{R}^m$ with $m \in \mathbb{N}$ we define*

(a) $y^1 \leqq y^2 \ :\Leftrightarrow\ y^1_j \leqq y^2_j,\ j \in [m]$,

(b) $y^1 \leq y^2 \ :\Leftrightarrow\ y^1 \leqq y^2$ *and* $y^1 \neq y^2$,

(c) $y^1 < y^2 \ :\Leftrightarrow\ y^1_j < y^2_j,\ j \in [m]$.

In the case $y^1 \leq y^2$ one says that y^1 dominates y^2, and for $y^1 < y^2$ that y^1 strictly dominates y^2. The inequalities $y^1 \geqq y^2$, $y^1 \geq y^2$ and $y^1 > y^2$ are defined analogously.

Note that for scalars the inequality $y^1 \leq y^2$ is equivalent to $y^1 < y^2$, so that for scalars we shall only use the relations $y^1 \leqq y^2$ and $y^1 < y^2$. The relation $y^1 \leq y^2$ is a relevant concept only for $m \geqq 2$.

With the cones

$$\begin{aligned}
\mathbb{R}^m_{\geqq} &:= \{y \in \mathbb{R}^m \mid y \geqq 0\}, \\
\mathbb{R}^m_{\geq} &:= \{y \in \mathbb{R}^m \mid y \geq 0\} \ = \ \mathbb{R}^m_{\geqq} \setminus \{0\} \ \text{ and} \\
\mathbb{R}^m_{>} &:= \{y \in \mathbb{R}^m \mid y > 0\} \ = \ \operatorname{int} \mathbb{R}^m_{\geqq}
\end{aligned}$$

one may write a relation like $y^1 \leq y^2$ equivalently as $y^2 - y^1 \in \mathbb{R}^m_{\geq}$, etc. One may also define ordering structures on $\mathbb{R}^m$ by replacing the standard ordering cone (or Pareto cone) $\mathbb{R}^m_{\geqq}$ with other convex cones, but in the present paper we focus on the standard case of component-wise inequalities.

Definition 2.2.

(a) *For $Y \subseteq \mathbb{R}^m$ a point $\bar{y} \in Y$ is called weakly nondominated if no $y \in Y$ with $y < \bar{y}$ exists.*

(b) *For* MOP *a point $\bar{x} \in X$ is called weakly efficient if $f(\bar{x})$ is a weakly nondominated point of $f(X)$.*

(c) *We denote the sets of weakly nondominated points of Y and of weakly efficient points of* MOP *by* $\mathrm{WNd}(Y)$ *and* $\mathrm{WE}(f, X)$*, respectively.*

Definition 2.3.

(a) *For $Y \subseteq \mathbb{R}^m$ a point $\bar{y} \in Y$ is called nondominated if no $y \in Y$ with $y \leq \bar{y}$ exists.*

(b) *For* MOP *a point $\bar{x} \in X$ is called efficient if $f(\bar{x})$ is a nondominated point of $f(X)$.*

(c) *We denote the sets of nondominated points of Y and of efficient points of* MOP *by* $\mathrm{Nd}(Y)$ *and* $\mathrm{E}(f, X)$*, respectively.*

Each nondominated point of a set Y is also weakly nondominated, but not vice versa. The analogous statement is true for efficient and weakly efficient points of MOP. A third solution concept is stronger than nondominatedness and efficiency, respectively. We use the notions of proper nondominatedness and proper efficiency in the sense of Geoffrion [14], since it is tailored to the component-wise structure of the standard ordering cone $\mathbb{R}^m_{\geqq}$. Other properness concepts are due to, for example, Benson [1], Borwein [3] and Henig [19].

Definition 2.4.

(a) *For $Y \subseteq \mathbb{R}^m$ the point $\bar{y} \in Y$ is called properly nondominated, if some real number $K > 0$ exists such that for all $y \in Y$ and all $i \in [m]$ with $y_i < \bar{y}_i$ some $j \in [m]$ with $y_j > \bar{y}_j$ and*

$$\frac{\bar{y}_i - y_i}{y_j - \bar{y}_j} \leqq K$$

exists.

(b) For MOP *the point $\bar{x}$ is called properly efficient, if $f(\bar{x})$ is a properly nondominated point of $f(X)$.*

(c) We denote the sets of properly nondominated points of Y and of properly efficient points of MOP *by* $\mathrm{PNd}(Y)$ *and* $\mathrm{PE}(f, X)$*, respectively.*

For motivations and illustrations of these solution concepts, we refer to, e.g., [7, 29]. We now define properly efficient points with a fixed trade-off bound (cf. [41, 34]).

Definition 2.5. *If the trade-off bound K from Def. 2.4 is known, we say that $\bar{y}$ is K-properly nondominated and that $\bar{x}$ is K-properly efficient. For a given $K > 0$, we denote the sets of K-properly nondominated points of Y and of K-properly efficient points of* MOP *by* $\mathrm{PNd}(Y, K)$ *and* $\mathrm{PE}(f, X, K)$*, respectively.*

Throughout this work, we assume $K \geqq 1$ for the trade-off bound. This is not restrictive for our results since for $K^1 \leqq K^2$, it holds that $\mathrm{PNd}(Y, K^1) \subseteq \mathrm{PNd}(Y, K^2)$. Furthermore, we prove in App. A a conjecture from [34], stating that the set $\mathrm{PNd}(Y, K)$ contains at most a single point if $K < 1$. The following solution concepts only exist in the decision space.

Definition 2.6. *A point $\bar{x}$ is called locally properly efficient ($\bar{x} \in \mathrm{LPE}(f, X)$) if $\bar{x} \in \mathrm{PE}(f, X \cap U)$ for some neighborhood U of $\bar{x}$.*

Definition 2.7. *A point $\bar{x} \in X$ is called strictly efficient if there is no $x \in X$, $x \neq \bar{x}$, such that $f(x) \leqq f(\bar{x})$.* $\mathrm{StrE}(f, X)$ *denotes the set of strictly efficient points.*

Definition 2.8 ([21])**.** *Let $\ell \in \mathbb{N}$, $\ell \geqq 1$. A point $\bar{x} \in X$ is called strictly locally efficient of order ℓ if there exist some $a > 0$ and a neighborhood U of $\bar{x}$ such that for all $x \in X \cap U \setminus \{\bar{x}\}$, it holds that*

$$(f(x) + \mathbb{R}^m_{\geqq}) \cap B(f(\bar{x}), a\|x - \bar{x}\|^\ell) = \emptyset.$$

$\mathrm{StrLE}(f, X, \ell)$ *denotes the set of strictly locally efficient points of order ℓ.*

Finally, for a nonempty set $Y \subseteq \mathbb{R}^m$, the vectors $y^I(Y)$ and $y^A(Y)$ with extended real-valued entries

$$y^I(Y)_j \;=\; \inf_{y \in Y} y_j, \quad y^A(Y)_j \;=\; \sup_{y \in Y} y_j, \; j \in [m]$$

are called the ideal point and anti-ideal point of Y. We say that Y is bounded from below if $y^I(Y) \in \mathbb{R}^m$ and bounded from above if $y^A(Y) \in \mathbb{R}^m$. We refer to a vector $y^U(Y) \in \mathbb{R}^m$ with $y^U(Y) < y^I(Y)$ as a utopia point of Y. A utopia point is not unique. We will assume that for a given problem instance, a fixed utopia point is used.

2.2 Optimality conditions

We recall a first-order necessary condition for proper efficiency and a characterization of strict local efficiency of order 1. Let the feasible set of MOP be given by

$$X := \{x \in \mathbb{R}^n \mid g_j(x) \leqq 0, j \in [q], h_k(x) = 0, k \in [r]\},$$

with $g : \mathbb{R}^n \to \mathbb{R}^q$ and $h : \mathbb{R}^n \to \mathbb{R}^r$. We say $(f, g, h) \in \mathcal{C}^k$ if $g \in \mathcal{C}^k(\mathbb{R}^n, \mathbb{R}^q)$, $h \in \mathcal{C}^k(\mathbb{R}^n, \mathbb{R}^r)$, and $f \in \mathcal{C}^k(X, \mathbb{R}^m)$.

Definition 2.9. *Let $(f, g, h) \in \mathcal{C}^1$. A point $\bar{x} \in X$ is called a proper Kuhn-Tucker point ($\bar{x} \in \mathrm{KT}_>(f, g, h)$) if there exist $\kappa \in \mathbb{R}^m_>$, $\lambda \in \mathbb{R}^q_{\geqq}$, $\mu \in \mathbb{R}^r$ such that*

$$\sum_{i=1}^{m} \kappa_i \nabla f_i(\bar{x}) + \sum_{j=1}^{q} \lambda_j \nabla g_j(\bar{x}) + \sum_{k=1}^{r} \mu_k \nabla h_k(\bar{x}) = 0, \tag{2.1}$$

$$\lambda_j g_j(\bar{x}) = 0, \; j \in [q]. \tag{2.2}$$

We define the tangent cone

$$T(\bar{x}, X) := \{d \in \mathbb{R}^n \mid \exists (d^k, t^k)_{k \in \mathbb{N}}, d^k \to d, t^k \searrow 0 \text{ with } \bar{x} + t^k d^k \in X \; \forall\, k \in \mathbb{N}\}$$

and, with the set of indices of active inequality constraints,

$$I_0(\bar{x}) = \{i \in [q] \mid g_i(\bar{x}) = 0\},$$

the linearization cone

$$L_{\leqq}(\bar{x}, X) = \{d \in \mathbb{R}^n \mid \nabla g_i(\bar{x})^\top d \leqq 0, \; i \in I_0(\bar{x}), \; \nabla h_j(\bar{x})^\top d = 0, \; j \in [r]\},$$

as well as

$$L_{\leqq}(\bar{x}, f) := \{d \in \mathbb{R}^n \mid \nabla f_i(\bar{x})^\top d \leqq 0, \; i \in [m]\}.$$

If $T(\bar{x}, X) = L_{\leqq}(\bar{x}, X)$ is fulfilled, the Abadie constraint qualification (ACQ) is said to hold at $\bar{x}$. We denote the set of points $\bar{x} \in X$ at which the ACQ is fulfilled by $\mathrm{ACQ}(X)$.

Proposition 2.10.

(a) $\mathrm{ACQ}(X) \cap \mathrm{LPE}(f, X) \subseteq \mathrm{KT}_>(f, g, h)$ *if* $(f, g, h) \in \mathcal{C}^1$

(b) For $f \in \mathcal{C}^1$ and $\bar{x} \in X$, $\bar{x} \in \mathrm{StrLE}(f, X, 1)$ if and only if

$$T(\bar{x}, X) \cap L_{\leqq}(\bar{x}, f) = \{0\} \tag{2.3}$$

(c) For $(f, g, h) \in \mathcal{C}^1$ and $\bar{x} \in \mathrm{ACQ}(X)$, $\bar{x} \in \mathrm{StrLE}(f, X, 1)$ if and only if

$$L_{\leqq}(\bar{x}, X) \cap L_{\leqq}(\bar{x}, f) = \{0\} \tag{2.4}$$

(d) $\mathrm{StrLE}(f, X, 1) \subseteq \mathrm{LPE}(f, X)$ *if* $(f, g, h) \in \mathcal{C}^1$

(e) $\mathrm{StrE}(f, X) \cap \mathrm{LPE}(f, X) \subseteq \mathrm{PE}(f, X)$ *if* X *is compact and* $f \in \mathcal{C}(X, \mathbb{R}^m)$

(f) $\mathrm{StrE}(f, X) \cap \mathrm{StrLE}(f, X, 1) \subseteq \mathrm{PE}(f, X)$ *for compact* X *and* $(f, g, h) \in \mathcal{C}^1$

Parts (a) and (d) were shown in [36], part (b) was shown in [21], and part (c) directly follows from (b). We prove part (e) in App. B, and (f) directly follows from (d) and (e).

A different type of optimality condition is supportedness, which is related to the weighted sum scalarization

$$(\mathrm{WS}(w)) \qquad \min\ \psi_w^{\mathrm{WS}}(y) := w^\top y \quad \text{s.t.} \quad y \in Y,$$

with some $w \in \mathbb{R}^m_{\geq}$. We point out that $(\mathrm{WS}(w))$ is formulated in the image space. For practical purposes, one can instead consider the decision space formulation $(\psi_w^{\mathrm{WS}} \circ f, X)$. The set of supported points $\mathrm{Supp}(Y)$ consists of those $y \in Y$ that are a minimal point of $(\mathrm{WS}(w))$ for some $w \in \mathbb{R}^m_{\geq}$. A preimage point $x \in X$ is called supported ($x \in \mathrm{Supp}(f, X)$) if $f(x) \in \mathrm{Supp}(f(X))$. It is well-known that all supported points are weakly nondominated. If f and X are convex, the weakly nondominated points are even characterized by the supported points, while in the general nonlinear setting, unsupported weakly nondominated points may exist. The motivation for the paper present was to transform a general nonlinear MOP in such a way that more of the (weakly) nondominated points become supported. We thus first introduce so-called image space transformations, after which we discuss the concept of supportedness in greater detail.

2.3 Image space transformations

In our related work [30], we study how MOP can be equivalently reformulated by a coordinate transformation of the image space. We call a set $\mathcal{Y}$ a box if it can be written as $\mathcal{Y} = \mathcal{Y}_1 \times \ldots \times \mathcal{Y}_m$ with not necessarily closed or bounded intervals $\mathcal{Y}_j \subseteq \mathbb{R}^1$, $j \in [m]$.

Definition 2.11. *For boxes $\mathcal{Y}, \mathcal{Z} \subseteq \mathbb{R}^m$ we call a bijective mapping $\Phi : \mathcal{Y} \to \mathcal{Z}$ a component-wise monotone transformation (CMT) if it is of the form*

$$\Phi(y) = P\left(\varphi_1(y_1), \cdots, \varphi_m(y_m)\right)^\top$$

with a permutation matrix P and strictly increasing functions $\varphi_j : \mathcal{Y}_j \to \mathcal{Z}_j$, $j \in [m]$. If additionally $\mathcal{Y}$ and $\mathcal{Z}$ are open and Φ is a $\mathcal{C}^1$-diffeomorphism, Φ is called a component-wise monotone $\mathcal{C}^1$-transformation, or $\mathcal{C}^1$-CMT for short.

Given a CMT $\Phi : \mathcal{Y} \to \mathcal{Z}$ and f, X with $f(X) \subseteq \mathcal{Y}$, one can write down a transformed problem

$$(\mathrm{MOP}_\Phi) \qquad \min\,(\Phi \circ f)(x) \quad \text{s.t.} \quad x \in X.$$

It is well-known (e.g., [39, Prop. 10.2]) that $\mathrm{E}(f, X) = \mathrm{E}(\Phi \circ f, X)$ holds if $P = I$. We include the permutation matrix P in [30] to establish that under certain conditions, CMTs are in fact the only image space transformations that ensure $\mathrm{E}(f, X) = \mathrm{E}(\Phi \circ f, X)$ to hold. In the present paper, we simply assume $P = I$.

Proposition 2.12. *Let $\Phi : \mathcal{Y} \to \mathcal{Z}$ be a CMT, $X \subseteq \mathbb{R}^m$, and $f : X \to \mathbb{R}^m$ such that $f(X) \subseteq \mathcal{Y}$. It then holds that*

(a) $\mathrm{E}(f, X) = \mathrm{E}(\Phi \circ f, X)$

(b) $\mathrm{WE}(f, X) = \mathrm{WE}(\Phi \circ f, X)$

(c) $\mathrm{StrE}(f, X) = \mathrm{StrE}(\Phi \circ f, X)$

(d) $\mathrm{StrLE}(f, X, 1) = \mathrm{StrLE}(\Phi \circ f, X, 1)$ *if $\Phi, f \in \mathcal{C}^1$ and $\varphi_i' > 0$ on $\mathcal{Y}_i$, $i \in [m]$*

(e) $\mathrm{PE}(f, X) = \mathrm{PE}(\Phi \circ f, X)$ *if $\Phi \in \mathcal{C}^1$ and X is compact*

(f) $\mathrm{PE}(f, X) = \mathrm{PE}(\Phi \circ f, X)$ *if $\mathcal{Y} = \mathbb{R}^m$, $\Phi \in \mathcal{C}^1$, Y is bounded from below, and $\varphi_i' > 0$ and increasing on $(y_i^I(Y), y_i^A(Y))$*

(g) $\mathrm{Supp}(f, X) \subseteq \mathrm{Supp}(\Phi \circ f, X)$ *if $\varphi_i, i \in [m]$ are convex*

Proof. Parts (a), (b) are easy to prove, e.g. with the arguments from [30, Cor. 3.1, Cor. 3.2]. Part (c) holds since Φ is bijective. Part (d) holds since

$x \in \mathrm{StrLE}(f, X, 1)$ if and only if (2.3) is fulfilled and $L_{\leqq}(\bar{x}, f) = L_{\leqq}(\bar{x}, \Phi \circ f)$ due to

$$\forall i \in [m], x \in X : \nabla(\varphi_i \circ f_i)(x)^\top d = \varphi_i'(x) \nabla f_i(x)^\top d \leqq 0 \Leftrightarrow \nabla f_i(x)^\top d \leqq 0.$$

Part (e) was shown in [30, Th. 4.1], and part (f) follows from [43, Th. 2]. Part (g) follows from our analysis of so-called supportedness respecting transformation in [31, Def. 3.1, Th. 3.4]. □

Examples of CMTs include the shifting transformation $\Phi(y) := y - y^U$, which ensures $\Phi(Y) \subseteq \mathbb{R}^m_>$ and the normalizing transformation with components $\varphi_i(y) := (y_i - y_i^U)/(y_i^A - y_i^U)$, which ensures $\Phi(Y) \subseteq (0, 1]^m$.

2.4 Power transformation and weighted norm

For a given $p \geqq 1$, the function $\varphi_p : \mathbb{R}_> \to \mathbb{R}_>, \varphi_p(y) := y^p$ is strictly increasing, smooth, and convex. Thus, the power transformation

$$\Phi_p : \mathbb{R}^m_> \to \mathbb{R}^m_>, \Phi_p(y) = (\varphi_p(y_1), \ldots, \varphi_p(y_m))^\top =: y^p$$

is a CMT that fulfills every assumption in Prop. 2.12. When $Y = f(X) \subseteq \mathbb{R}^m_>$, the problem (f^p, X) is called the p-th power transformation of (f, X). If $Y \subseteq \mathbb{R}^m_>$ does not hold but Y is bounded from below, we can first apply the shifting transformation from the previous section.

The power transformation was introduced by Li in [24]. He assumed that $\mathrm{Nd}(Y)$ is the graph of some $\mathcal{C}^2$-function from $\mathbb{R}^{m-1}$ to $\mathbb{R}$ with strictly negative first partial derivatives. Under this assumption, he showed that for the transformed image set $Y^p := \{y^p \mid y \in Y\}$ for some sufficiently large p, it holds that $\mathrm{Nd}(Y^p) = \mathrm{Supp}(Y^p)$. Specifically, for each $\bar{y} \in \mathrm{Nd}(Y)$, there exists some $w \in \mathbb{R}^m_\geq$ such that $\bar{y}$ is a minimal point of

$$(\mathrm{WN}_p(w)) \qquad \min\ \psi^{\mathrm{WN}}_{p,w}(y) := w^\top y^p \quad \text{s.t.} \quad y \in Y.$$

WN is an abbreviation of weighted norm. The name is motivated by the fact that $\mathrm{WN}_p(w)$ can also be derived by minimizing the distance of image points to a utopia point in a weighted ℓ_p-norm

$$\|y\|_{w,p} := \left(\sum_{i=1}^m |w_i y_i^p| \right)^{\frac{1}{p}}.$$

Under our assumption $Y \subseteq \mathbb{R}^m_>$, the origin can be used as a utopia point. The absolute value in the definition of $\|y\|_{w,p}$ can be left out since the argument is positive, and one may replace $\|y\|_{w,p}$ with $\|y\|^p_{w,p}$. The resulting scalarization is precisely $\mathrm{WN}_p(w)$ and was originally introduced under the name compromise programming in [42, 44]. Miettinen ([29]) used the term "weighted metric," while other authors (e.g., [18]) refer to it as "weighted norm," as we do in the present paper.

With the weighted Chebyshev norm,

$$\|y\|_{w,\infty} := \max_{i \in [m]} |w_i y_i|,$$

one can also define $\mathrm{WN}_p(w)$ for $p = \infty$. This approach is known as the weighted Chebyshev scalarization (cf. [29]). We point out that some authors (e.g., [13]) use transformed weights w_i^p in $\|y\|_{w,p}$ for $p \in [1, \infty)$, which simplifies the interpretation of the weights since, according to [13],

$$\forall y \in \mathbb{R}^m : \left(\sum_{i=1}^m |w_i^p y_i^p| \right)^{\frac{1}{p}} \xrightarrow{p \to \infty} \|y\|_{w,\infty}.$$

For theoretical results on the existence of certain weight vectors, the distinction between w and w^p is irrelevant since $(\mathbb{R}^m_\geq)^p = \mathbb{R}^m_\geq$ and $(\mathbb{R}^m_>)^p = \mathbb{R}^m_>$.

To summarize, there is no difference between the supported points of the power transformation and minimal points to the weighted norm scalarization. The following concepts exist independently of the two interpretations.

Definition 2.13. *We say that $\bar{y} \in Y$ is*

(a) a p-supported point of Y ($\bar{y} \in \mathrm{Supp}(Y, p)$) if there exists some $w \in \mathbb{R}^m_\geq$ such that $\bar{y}$ is a minimal point of $\mathrm{WN}_p(w)$.

(b) a positively p-supported point of Y ($\bar{y} \in \mathrm{PSupp}(Y, p)$) if there exists some $w \in \mathbb{R}^m_>$ such that $\bar{y}$ is a minimal point of $\mathrm{WN}_p(w)$.

(c) a uniquely p-supported point of Y ($\bar{y} \in \mathrm{USupp}(Y, p)$) if there exists some $w \in \mathbb{R}^m_>$ such that $\bar{y}$ is the unique minimal point of $\mathrm{WN}_p(w)$.

A point $\bar{x} \in X$ is called a p-supported point of (f, X) ($\bar{x} \in \mathrm{Supp}(f, X, p)$) if $f(\bar{x}) \in \mathrm{Supp}(f(X), p)$. The sets $\mathrm{PSupp}(f, X, p)$ and $\mathrm{USupp}(f, X, p)$ of positively and uniquely supported points $\bar{x} \in X$ are defined accordingly. If $p = 1$, we speak of supported points instead of 1-supported points and omit the argument for p, e.g., $\mathrm{Supp}(Y)$ instead of $\mathrm{Supp}(Y, 1)$.

Without explicit mention, we will characterize $\mathrm{Supp}(Y, p)$ by

$$\bar{y} \in \mathrm{Supp}(Y, p) \Leftrightarrow \bar{y} \in Y \text{ and } \exists w \in \mathbb{R}^m_{\geq} \, \forall y \in Y : w^\top (y^p - \bar{y}^p) \geqq 0.$$

$\mathrm{PSupp}(Y, p)$ and $\mathrm{USupp}(Y, p)$ are characterized similarly with $w \in \mathbb{R}^m_{>}$. In the case of $\mathrm{USupp}(Y, p)$, the inequality is strict for all $y \in Y \setminus \bar{y}$.

It is known (e.g., [13, 30]) that for all closed $Y \subseteq \mathbb{R}^m_{>}$ and $p \geqq 1$, it holds that $\mathrm{Supp}(Y, p) \subseteq \mathrm{WNd}(Y)$ and $\mathrm{PSupp}(Y, p) \subseteq \mathrm{PNd}(Y)$. The inclusions become equalities if the upper image $Y + \mathbb{R}^m_{\geqq}$ is convex. Furthermore, if $1 \leqq p_1 < p_2$, then $\mathrm{PSupp}(Y, p_1) \subseteq \mathrm{USupp}(Y, p_2)$ (cf. [18, 30]).

Def. 2.13 is inspired by [18], but some differences should be pointed out. Part (a) was not defined in [18]. Instead, they call the points from part (b) p-supported. The points from (c) are called extremely p-supported in [18]. Originally, extremely supported points were defined as those supported points that are also extreme points (i.e., they cannot be written as a convex combination of other image points). In the discrete setting from [18], the notions of extreme and unique supportedness coincide. In the general case, however, this does not hold anymore.

Example 2.14. *Let $Y \subseteq \mathbb{R}^2$ be the union of two closed balls with radius 1 and centers $(0,1)^\top$ and $(1,0)^\top$, respectively. Then $y^1 = (-\sqrt{2}/2, 1 - \sqrt{2}/2)^\top$ and $y^2 = (1 - \sqrt{2}/2, -\sqrt{2}/2)^\top$ are extreme points of Y and $y^1, y^2 \in \mathrm{PSupp}(Y)$, but $y^1, y^2 \notin \mathrm{USupp}(Y)$.*

We further remark that the definition of supportedness ($p = 1$) from Def. 2.13 is not the only one in the literature. In fact, there exist many different definitions of supportedness, either via scalarization (similarly to Def. 2.13), or via geometry or convex combinations of image points. [5] organizes these definitions in different layers such that definitions in the same layer are equivalent, and the layers are connected via a chain of implications. In this classification, supported points belong to the most general and positively supported points to the most restrictive layer.

3 Sufficient conditions for p-supportedness

The following results will be required to prove Lem. 3.5, which establishes that under certain regularity conditions, a point $\bar{x} \in X$ is locally uniquely p_0-supported for a sufficiently large but fixed $p_0 \geqq 1$, i.e., $\bar{x} \in \mathrm{USupp}(X \cap U, f, p_0)$ for some neighborhood U of $\bar{x}$. Under the additional assumptions that X is compact and $\bar{x}$ strictly efficient, Lem. 3.6 shows that $f(\bar{x}) \in$

USupp$(Y \cap V, p_0)$ for a neighborhood V of $f(\bar{x})$ as well. In Lem. 3.7, we show that $f(\bar{x}) \in \mathrm{USupp}(Y \cap V, p)$ holds for all $p \geqq p_0$. Finally, the main result Th. 3.9 establishes $\bar{x} \in \mathrm{USupp}(f, X, p)$ for sufficiently large p.

3.1 Local p-supportedness

Definition 3.1. *Let $r_p, s : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. We say that $r_p \in \mathcal{O}_p(s)$ for $x \to a$ if for all $p \geqq 1$, there exist some $C_p > 0$ and $\varepsilon_p > 0$ such that*

$$\forall x \in (a - \varepsilon_p, a + \varepsilon_p) : |r_p(x)| \leqq C_p |s(x)|.$$

The proofs of the technical preliminary results Lem. 3.2-3.4 are stated in App. B. From now on, we abbreviate the boundary of the unit sphere by $S^{n-1} := \mathrm{bd}\, B(0,1)$.

Lemma 3.2. *Let $p \geqq 1$ and $f \in \mathcal{C}^2(X, \mathbb{R})$. Then for all $d \in S^{n-1}$ and $t \searrow 0$, it holds that*

$$\left(\frac{f(\bar{x} + td)}{f(\bar{x})}\right)^p = 1 + p\frac{\nabla f(\bar{x})^\top d}{f(\bar{x})}t + \frac{p(p-1)}{2}\left(\frac{\nabla f(\bar{x})^\top d}{f(\bar{x})}t\right)^2 + pr(t) + \bar{r}_p(t), \tag{3.1}$$

where $r \in \mathcal{O}(t^2)$ and $\bar{r}_p \in \mathcal{O}_p(t^3)$ for $t \searrow 0$.

We now define

$$D(\bar{t}, \bar{x}, X) = \{d \in S^{n-1} \mid \exists t \in (0, \bar{t}] : \bar{x} + td \in X\}.$$

Lemma 3.3. *Let $\bar{x} \in \mathrm{ACQ}(X) \cap \mathrm{StrLE}(f, X, 1)$. Then, there exists some $\bar{t} > 0$ such that for all $d \in D(\bar{t}, \bar{x}, X)$, there exists some $i \in [m]$ such that $\nabla f_i(\bar{x})^\top d > 0$.*

Lemma 3.4. *Let $\bar{x} \in \mathrm{ACQ}(X)$ and $\bar{t} > 0$. Then $D(\bar{t}, \bar{x}, X) \cup L_{\leqq}(\bar{x}, X)$ is closed.*

For the point $\bar{x}$ in the next lemma, $\bar{x} \in \mathrm{KT}_{>}(f, g, h)$ holds in view of Prop. 2.10(a),(d). With the multiplier $\kappa > 0$, $p \geqq 1$, and $\bar{y} := f(\bar{x})$, we define $w(p) \in \mathbb{R}^m_{>}$ by

$$w_i(p) = \frac{\kappa_i}{p\bar{y}_i^{p-1}}, \quad i \in [m]. \tag{3.2}$$

Lemma 3.5. *Let $(f, g, h) \in \mathcal{C}^2$ with $f(X) \subseteq \mathbb{R}^m_{>}$, where X is the feasible set described by g and h. Let $\bar{x} \in \mathrm{ACQ}(X) \cap \mathrm{StrLE}(f, X, 1)$. Then there exist some $p_0 \geqq 1$ and a neighborhood U of $\bar{x}$ such that*

$$\forall x \in X \cap U \setminus \bar{x} : w(p_0)^\top (f(x)^{p_0} - f(\bar{x})^{p_0}) > 0, \tag{3.3}$$

where $w(p_0)$ is defined as in (3.2).

Proof. We will show that (3.3) holds with $U := B(\bar{x}, \bar{t})$ for some $\bar{t} > 0$. Note that we can write $X \cap U \setminus \bar{x} = \{\bar{x} + td \mid t \in (0, \bar{t}), d \in S^{n-1}, \bar{x} + td \in X\}$, and that

$$X \cap U \setminus \bar{x} \subseteq \{\bar{x} + td \mid t \in (0, \bar{t}), d \in D(\bar{t}, \bar{x}, X)\}. \tag{3.4}$$

By Prop. 2.10(d) we have $\bar{x} \in \mathrm{StrLE}(f, X, 1) \subseteq \mathrm{LPE}(f, X)$ and together with $\bar{x} \in \mathrm{ACQ}(X)$, Prop. 2.10(a) implies $\bar{x} \in \mathrm{KT}_{>}(f, g, h)$. With the corresponding multiplier $\kappa \in \mathbb{R}^m_{>}$, we can thus define $w(p) \in \mathbb{R}^m_{>}$ as in (3.2) for $p \geqq 1$. Let $t > 0$ and $d \in S^{n-1}$. We have

$$\begin{aligned} w(p)^\top (f(\bar{x} + td)^p - f(\bar{x})^p) =& \sum_{i=1}^{m} \frac{\kappa_i}{p f_i(\bar{x})^{p-1}} (f_i(\bar{x} + td)^p - f_i(\bar{x})^p) \\ =& \frac{1}{p} \sum_{i=1}^{m} \kappa_i \left(\left(\frac{f_i(\bar{x} + td)}{f_i(\bar{x})} \right)^p - 1 \right) f_i(\bar{x}). \end{aligned} \tag{3.5}$$

We abbreviate $\alpha_i := (\nabla f_i(\bar{x})^\top d)/f_i(\bar{x})$. For $i \in [m]$, we know from Lem. 3.2 that

$$\left(\frac{f_i(\bar{x} + td)}{f_i(\bar{x})} \right)^p = 1 + p\alpha_i t + \frac{p(p-1)}{2} (\alpha_i t)^2 + p r_i(t) + \bar{r}_{p,i}(t), \tag{3.6}$$

with $r_i \in \mathcal{O}(t^2)$ and $\bar{r}_{p,i} \in \mathcal{O}_p(t^2)$ for all $i \in [m]$ and $t \searrow 0$. We substitute the formulae for $(f_i(\bar{x} + td)/f_i(\bar{x}))^p$ from (3.6) in (3.5), resulting in

$$\begin{aligned} w(p)^\top (f(\bar{x} + td)^p - f(\bar{x})^p) =& \left(\sum_{i=1}^{m} \kappa_i \nabla f_i(\bar{x})^\top d \right) t \\ &+ \frac{p-1}{2} \left(\sum_{i=1}^{m} \kappa_i \frac{(\nabla f_i(\bar{x})^\top d)^2}{f_i(\bar{x})} \right) t^2 + r(t) + \bar{r}_p(t), \end{aligned} \tag{3.7}$$

with $r(t) := \sum_{i=1}^m \kappa_i f_i(\bar{x}) r_i(t) \in \mathcal{O}(t^2)$ and $\bar{r}_p(t) := (1/p) \sum_{i=1}^m \kappa_i f_i(\bar{x}) \bar{r}_{p,i}(t) \in \mathcal{O}_p(t^3)$. Using (2.1), we rewrite the first term as

$$(\sum_{i=1}^{m} \kappa_i \nabla f_i(\bar{x}))^\top dt = - \left(\sum_{j=1}^{q} \lambda_j \nabla g_j(\bar{x})^\top dt + \sum_{k=1}^{r} \mu_k \nabla h_k(\bar{x})^\top dt \right). \tag{3.8}$$

For each $j \in [q]$ with $\lambda_j > 0$, we know that $g_j(\bar{x}) = 0$ and thus, for all sufficiently small $t > 0$ and $d \in S^{n-1}$ with $\bar{x} + td \in X$, we have that

$$0 \geqq g_j(\bar{x} + td) = \nabla g_j(\bar{x})^\top dt + r_j^g(t) \Leftrightarrow -\nabla g_j(\bar{x})^\top dt \geqq r_j^g(t),$$

where $r_j^g(t) \in \mathcal{O}(t^2)$, $j \in [q]$. Similarly, we can show that for all $k \in [r]$, it holds that

$$-\nabla h_k(\bar{x})^\top dt = r_k^h(t),$$

with $r_k^h(t) \in \mathcal{O}(t^2)$, $k \in [r]$. We have thus found a lower bound on the right-hand side of (3.8), resulting in

$$(\sum_{i=1}^{m} \kappa_i \nabla f_i(\bar{x}))^\top dt \geqq \sum_{j\in[m]:\lambda_j>0} \lambda_j r_j^g(t) + \sum_{k=1}^{r} \mu_k r_k^h(t) =: \widetilde{r}(t) \in \mathcal{O}(t^2). \quad (3.9)$$

We substitute $\widetilde{r}(t)$ in (3.7) and define $\widehat{r}(t) := r(t) + \widetilde{r}(t) \in \mathcal{O}(t^2)$, which gives us

$$\begin{aligned} &w(p)^\top (f(\bar{x}+td)^p - f(\bar{x})^p) \\ &\qquad \geqq \frac{p-1}{2} \left(\sum_{i=1}^{m} \kappa_i \frac{(\nabla f_i(\bar{x})^\top d)^2}{f_i(\bar{x})} \right) t^2 + \widehat{r}(t) + \bar{r}_p(t). \end{aligned} \quad (3.10)$$

We define $\psi(d) := \max_{i\in[m]} \nabla f_i(\bar{x})^\top d$ for $d \in \mathbb{R}^n$. It then holds that

$$\sum_{i=1}^{m} (\nabla f_i(\bar{x})^\top d)^2 \geqq \psi(d)^2. \quad (3.11)$$

We now show that there is some $\bar{t} > 0$ such that there exists a positive lower bound on $\psi(d)$ for all $t \in (0, \bar{t})$ and $d \in S^{n-1}$ with $\bar{x} + td \in X \cap B(\bar{x}, \bar{t}) \setminus \bar{x}$. Since (2.4) holds, for all $d \in L_{\leqq}(\bar{x}, X) \setminus 0$, there exists some $i \in [m]$ with $\nabla f_i(\bar{x})^\top d > 0$. Furthermore, we know from Lem. 3.3 that there exists some $\bar{t} > 0$ such that for all $d \in D(\bar{t}, \bar{x}, X)$, there exists an $i \in [m]$ with $\nabla f_i(\bar{x})^\top d > 0$. Thus, for all $d \in (D(\bar{t}, \bar{x}, X) \cup L_{\leqq}(\bar{x}, X)) \cap S^{n-1}$, we know that $\psi(d) > 0$. The function ψ is continuous as a maximum of continuous functions. Note that the set $(D(\bar{t}, \bar{x}, X) \cup L_{\leqq}(\bar{x}, X)) \cap S^{n-1}$ is nonempty, bounded, and closed according to Lem. 3.4 and thus, from the Weierstrass Theorem, we know that

$$b := \min\{\psi(d) \mid d \in (D(\bar{t}, \bar{x}, X) \cup L_{\leqq}(\bar{x}, X)) \cap S^{n-1}\} > 0.$$

Thus by (3.4), we know in particular that for all $t \in (0, \bar{t})$ and $d \in S^{n-1}$ with $\bar{x} + td \in B(\bar{x}, \bar{t}) \cap X \setminus \bar{x}$,

$$\sum_{i=1}^{m} (\nabla f_i(\bar{x})^\top d)^2 \geqq \psi(d)^2 \geqq b^2 > 0 \quad (3.12)$$

is fulfilled, and in consequence, we have

$$\frac{1}{2} \left(\sum_{i=1}^{m} \kappa_i \frac{(\nabla f_i(\bar{x})^\top d)^2}{f_i(\bar{x})} \right) \geqq \underbrace{\frac{\min_{i\in[m]} \kappa_i b^2}{2 \max_{i\in[m]} f_i(\bar{x})}}_{=:c} > 0,$$

which provides the lower bound

$$w(p)^\top (f(\bar{x}+td)^p - f(\bar{x})^p) \geqq c(p-1)t^2 + \widehat{r}(t) + \bar{r}_p(t) \tag{3.13}$$

on (3.10). Since $\widehat{r} \in \mathcal{O}(t^2)$, there exists some $C > 0$ such that after possibly shrinking $\bar{t}$, we have $\widehat{r}(t) \geqq -Ct^2$ for $t \in (0, \bar{t})$. We choose some $p_0 \geqq 1$ such $c(p_0 - 1) > C$. Since $\bar{r}_{p_0} \in \mathcal{O}_{p_0}(t^3)$, there is some $C_{p_0} > 0$ such that after possible shrinking $\bar{t}$, we have $\bar{r}_{p_0}(t) \geqq -C_{p_0} t^3$ for $t \in (0, \bar{t})$. For $p = p_0$ and $t \in (0, \bar{t})$, we have found a lower bound on the right-hand side of (3.13), resulting in

$$w(p_0)^\top (f(\bar{x}+td)^{p_0} - f(\bar{x})^{p_0}) \geqq c(p_0-1)t^2 - Ct^2 - C_{p_0} t^3. \tag{3.14}$$

We now choose some $\widetilde{t} \in (0, \bar{t}]$ such that $(c(p_0-1) - C)t^2 > C_{p_0} t^3$ for all $t \in (0, \widetilde{t})$. If we define $U := B(\bar{x}, \widetilde{t})$, then (3.3) is fulfilled. □

Lemma 3.6. *Let X be compact and $\bar{x} \in \mathrm{StrE}(f, X)$. Assume that there exist some open neighborhood U of $\bar{x}$ and a $w \in \mathbb{R}^m_>$ such that*

$$\forall x \in X \cap U \setminus \bar{x} : w^\top (f(x) - f(\bar{x})) > 0 \tag{3.15}$$

holds. Then, there exists some open neighborhood V of $\bar{y} := f(\bar{x})$ such that

$$\forall y \in f(X) \cap V \setminus \bar{y} : w^\top (y - \bar{y}) > 0. \tag{3.16}$$

Proof. If $X \subseteq U$, the statement obviously holds. Otherwise, assume the statement were false. Then there exists some sequence $(y^k) \subset f(X) \setminus \bar{y}$ with $y^k \to \bar{y}$ and

$$\forall k \in \mathbb{N} : w^\top (y^k - \bar{y}) \leqq 0. \tag{3.17}$$

There exists some sequence $(x^k) \subset X \setminus \bar{x}$ with $y^k = f(x^k)$, $k \in \mathbb{N}$. In particular, we have

$$\forall k \in \mathbb{N} : w^\top (f(x^k) - f(\bar{x})) \leqq 0. \tag{3.18}$$

Because of (3.15), we know that $x^k \in X \setminus U$ for all $k \in \mathbb{N}$. Since $X \setminus U$ is nonempty and compact, we can redefine (x^k) as some suitable subsequence such that $x^k \to \widetilde{x} \in X \setminus U$. By the continuity of f, we know that $f(\widetilde{x}) = f(\bar{x})$, which contradicts $\bar{x} \in \mathrm{StrE}(f, X)$. □

Lemma 3.7. *Let $f(X) \subseteq \mathbb{R}^m_>$, $\bar{y} \in f(X)$ and assume that there exist some neighborhood V of $\bar{y}$ and a $p_0 \geqq 1$ such that*

$$\forall y \in f(X) \cap V \setminus \bar{y} : w(p_0)^\top (y^{p_0} - \bar{y}^{p_0}) \geqq 0. \tag{3.19}$$

Then for all $p > p_0$, we have

$$\forall y \in f(X) \cap V \setminus \bar{y} : w(p)^\top (y^p - \bar{y}^p) > 0, \tag{3.20}$$

where $w(p)$ is defined as in (3.2) with some arbitrary $\kappa \in \mathbb{R}^m_>$.

Proof. Inserting definitions in (3.19) gives us

$$\forall y \in f(X) \cap V \setminus \bar{y} : \sum_{i=1}^{m} \frac{\kappa_i}{p_0} \left(\left(\frac{y_i}{\bar{y}_i} \right)^{p_0} - 1 \right) \bar{y}_i \geqq 0. \tag{3.21}$$

We now consider the function $\eta_c : [1, \infty) \to \mathbb{R}, \eta_c(p) := (c^p - 1)/p$ for some $c > 0$ and show that η_c is monotonically increasing. The derivative of η_c is given by $\eta_c'(p) = p^{-2}(c^p(\log(c)p - 1) + 1)$. The function $p \mapsto c^{-p}$ is convex and thus bounded from below by its linearization at $p = 0$, i.e., it holds that

$$\forall p \in \mathbb{R} : c^{-p} \geqq 1 - \log(c)p. \tag{3.22}$$

Rearranging (3.22) gives us

$$c^p(\log(c)p - 1) + 1 \geqq 0. \tag{3.23}$$

For $p \geqq 1$, we can multiply (3.23) by $p^{-2} > 0$, which results in $\eta_c'(p) \geqq 0$. If $c \neq 1$, the function $p \mapsto c^{-p}$ is even strictly convex and (3.22) holds strictly for $p \neq 0$ (see [20, Th. 4.1.1]). This results in $\eta_c'(p) > 0$, i.e., η_c is strictly increasing. Now, let $y \in f(X) \cap V \setminus \bar{y}$ and define $c_i := y_i/\bar{y}_i$, $i \in [m]$. Then, (3.21) reads

$$\sum_{i=1}^{m} \kappa_i \bar{y}_i \eta_{c_i}(p_0) \geqq 0.$$

All η_{c_i} are monotonically increasing, and since $y \neq \bar{y}$, there exists some $j \in [m]$ with $c_j \neq 0$, and thus, η_{c_j} is strictly increasing. It follows that

$$\sum_{i=1}^{m} \kappa_i \bar{y}_i \eta_{c_i}(p) > 0$$

for all $p > p_0$. □

3.2 Global p-supportedness

Lemma 3.8. *Let $K \geqq 1$, $\bar{y} \in \mathrm{PNd}(Y, K)$, and $\varepsilon > 0$. Then there exists some $k \in [m]$ such that for all $y \in Y \setminus B_\infty(\bar{y}, \varepsilon)$, we have*

$$\frac{y_k}{\bar{y}_k} \geqq 1 + \frac{\varepsilon}{K \bar{y}_k}. \tag{3.24}$$

Proof. For all $y \in Y \setminus B_\infty(\bar{y}, \varepsilon)$ there is some $i \in [m]$ with $|y_i - \bar{y}_i| \geqq \varepsilon$. If $y_i \geqq \bar{y}_i + \varepsilon$, we know that

$$\frac{y_i}{\bar{y}_i} \geqq 1 + \frac{\varepsilon}{\bar{y}_i}.$$

If $y_i \leqq \bar{y}_i - \varepsilon$, since $\bar{y}_i$ is K-properly nondominated, there exists some $j \in [m]$ with $y_j > \bar{y}_j$ and
$$\frac{\varepsilon}{y_j - \bar{y}_j} \leqq \frac{\bar{y}_i - y_i}{y_j - \bar{y}_j} \leqq K.$$
Some rearranging yields
$$\frac{y_j}{\bar{y}_j} \geqq 1 + \frac{\varepsilon}{K\bar{y}_j}.$$
Since $K \geqq 1$, (3.24) is fulfilled for some $k \in \{i, j\}$. □

Theorem 3.9. *Let $(f, g, h) \in \mathcal{C}^2$ and assume that the set X described by g and h is compact and $f(X) \subseteq \mathbb{R}^m_>$. Let $\bar{x} \in \mathrm{ACQ}(X) \cap \mathrm{StrLE}(f, X, 1) \cap \mathrm{StrE}(f, X)$. Then there exists some $\bar{p} \geqq 1$ such that for all $p \geqq \bar{p}$, $\bar{x} \in \mathrm{USupp}(f, X, p)$.*

Proof. Let $w(p) \in \mathbb{R}^m_>$ be defined as in (3.2). We know from Lem. 3.5 that there exist some $p_0 \geqq 1$ and an open neighborhood U of $\bar{x}$ such that (3.3) holds. It follows from Lem. 3.6 that there exists some open neighborhood V of $\bar{y} := f(\bar{x})$ such that (3.19) holds. According to Lem. 3.7, (3.20) is fulfilled for all $p \geqq p_0$. We choose some $\varepsilon > 0$ such that $B_\infty(f(\bar{x}), \varepsilon) \subseteq V$. We know from Prop. 2.10(f) that $\bar{x} \in \mathrm{PE}(f, X)$. Therefore, there exists some $K \geqq 1$ such that $f(\bar{x}) \in \mathrm{PNd}(f(X), K)$. It follows from Lem. 3.8 that there exists some $k \in [m]$ such that
$$\frac{f_k(x)}{f_k(\bar{x})} \geqq 1 + \frac{\varepsilon}{K f_k(\bar{x})} \geqq 1 + \frac{\varepsilon}{K \max_{j\in[m]} f_j(\bar{x})} =: \bar{c} \tag{3.25}$$
holds for all $f(x) \in Y \cap V^c$. We have already seen in (3.5) that
$$w(p)^\top (f(x)^p - f(\bar{x})^p) = \frac{1}{p} \sum_{i=1}^m \kappa_i \left(\left(\frac{f_i(x)}{f_i(\bar{x})} \right)^p - 1 \right) f_i(\bar{x}).$$
For all $f(x) \in Y \cap V^c$, we thus have
$$w(p)^\top (f(x)^p - f(\bar{x})^p) \geqq \frac{1}{p} \left(\min_{i\in[m]} \kappa_i (\bar{c}^p - 1) f_i(\bar{x}) - (m-1) \max_{i\in[m]} \kappa_i f_i(\bar{x}) \right),$$
with $\bar{c} > 1$. Thus, $\bar{c}^p \xrightarrow{p\to\infty} \infty$ and there exist some $p_1 \geqq 1$ such that for all $p \geqq p_1$, it holds that
$$\forall f(x) \in f(X) \cap V^c : w(p)^\top (f(x)^p - f(\bar{x})^p) > 0.$$
If we now choose $\bar{p} := \max\{p_0, p_1\}$, then for all $p \geqq \bar{p}$, we have that
$$\forall x \in X \setminus \bar{x} : w(p)^\top (f(x)^p - f(\bar{x})^p) > 0,$$
i.e. $\bar{x} \in \mathrm{USupp}(f, X, p)$. □

Example 3.10. *Let $X = [-1, 0.5^{1/3}]$, $f_1(x) = x^3 + 4/3$, $f_2(x) = -0.5x^3 + 4/3$ for $x \leqq 0$, and $f_2(x) = -2x^3 + 4/3$ for $x > 0$. Since f_1 is strictly increasing and f_2 strictly decreasing on X, we have $\mathrm{StrE}(f, X) = X$. Furthermore, $f_1'(x) = 3x^2$ and $f_2'(x) = -1.5x^2$ for $x < 0$, $f_2'(0) = 0$, and $f_2'(x) = -6x^2$ for $x > 0$. In particular, $f_1'(x) > 0$ and $f_2'(x) < 0$ for $x \neq 0$ and thus, $L_{\leqq}(x, f) = 0$ for $x \neq 0$, which by Prop. 2.10(b) means that $X \setminus 0 \subseteq \mathrm{StrLE}(f, X, 1)$. On the other hand, $L_{\leqq}(0, f) = T(0, X) = \mathbb{R}$, i.e., $0 \notin \mathrm{StrLE}(f, X, 1)$. It is easily verified that all other assumptions of Th. 3.9 hold everywhere in X. We let $Y_1 := f([-1, 0]) = \{(u, 2 - u/2) \mid u \in [1/3, 4/3]\}$ and $Y_2 := f([0, 0.5^{1/3}]) = \{(2 - u/2, u) \mid u \in [1/3, 4/3]\}$. The image set $Y := f(X) = Y_1 \cup Y_2$ is depicted in Fig. 1. One sees that $\mathrm{Supp}(Y) = \{f(-1), f(0.5^{1/3})\}$. For $\bar{x} \in X \setminus 0$, we will derive a bound $\bar{p}$ such that $\bar{x} \in \mathrm{USupp}(f, X, p)$ for all $p \geqq \bar{p}$. This bound is specific to the present example. We do not propose a general lower bound for the value $\bar{p}$ from Th. 3.9. Note that Y is symmetric in the sense that $Y = \{(y_2, y_1) \mid y \in Y\}$. Thus, if for some $\bar{y} \in Y$, it holds that $\bar{y} \in \mathrm{USupp}(Y, p)$, then $(\bar{y}_2, \bar{y}_1) \in \mathrm{USupp}(Y, p)$. Hence, and since $-1 \in \mathrm{Supp}(f, X)$, it is sufficient to consider $\bar{x} \in (-1, 0)$. For all such $\bar{x}$ no constraint is active and* (2.1) *becomes*

$$3\kappa_1 x^2 - 1.5\kappa_2 x^2 = 0. \tag{3.26}$$

Thus, $\bar{x} \in \mathrm{KT}_{>}(f, X)$ with multipliers $\kappa_1 = \kappa_2/2$. We choose $\kappa_1 = f_1(\bar{x})^{-1}$ and $\kappa_2 = 2\kappa_1$. For $p \geqq 1$, we define $w(p)$ as in (3.2) *and obtain*

$$pw(p)^\top (f(x)^p - f(\bar{x})^p) = \left(\frac{f_1(x)}{f_1(\bar{x})}\right)^p - 1 + 2\frac{f_2(\bar{x})}{f_1(\bar{x})}\left(\left(\frac{f_2(x)}{f_2(\bar{x})}\right)^p - 1\right). \tag{3.27}$$

For all $x \in [-1, 0]$, one computes $w(1)^\top (f(x) - f(\bar{x})) = 0$. This is also justified geometrically since Y_1 is a line segment and thus, $\mathrm{Supp}(Y_1) = Y_1$. According to Lem. 3.7 with $p_0 = 1$ and V such that $f(X) \cap V = Y_1$, we have $pw(p)^\top (f(x)^p - f(\bar{x})^p) > 0$ for $p > 1$ and $x \in [-1, 0] \setminus \bar{x}$. For $x \in (0, 0.5^{1/3}]$, we have

$$\begin{aligned} &pw(p)^\top (f(x)^p - f(\bar{x})^p) \\ &\quad = \left(\frac{x^3 + 4/3}{\bar{x}^3 + 4/3}\right)^p - 1 + \underbrace{\frac{-\bar{x}^3 + 8/3}{\bar{x}^3 + 4/3}}_{\leqq 11}\left(\underbrace{\left(\frac{-2x^3 + 4/3}{-0.5\bar{x}^3 + 4/3}\right)^p - 1}_{>-1}\right) \\ &\quad > \left(\frac{4/3}{\bar{x}^3 + 4/3}\right)^p - 12. \end{aligned}$$

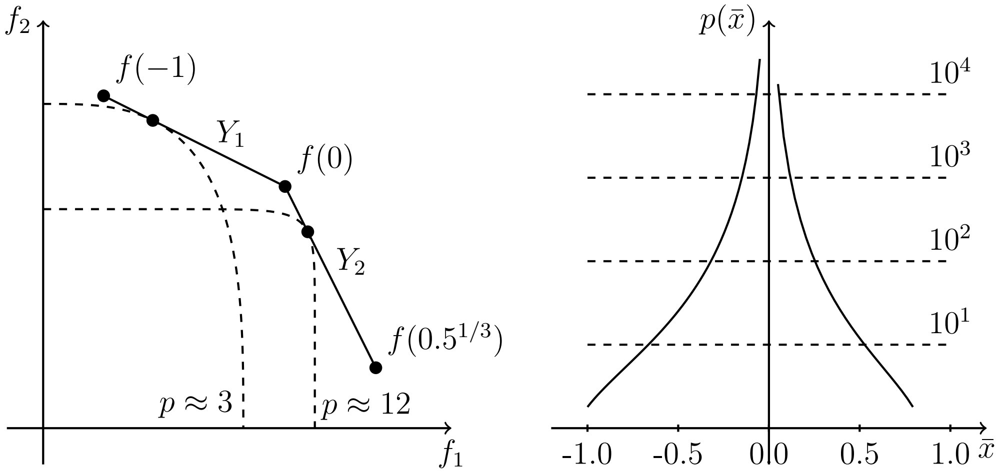

Figure 1: Illustration of Ex. 3.10. Left: Image set Y and images of two p-supported points. Right: Lower bound $p(\bar{x})$ on p for $\bar{x} \in X \setminus 0$.

We impose that the last line be positive and isolate p, resulting in

$$p > \frac{\log(12)}{\log(4/3) - \log(\bar{x}^3 + 4/3)} =: \bar{p}(\bar{x}). \tag{3.28}$$

Thus, for $\bar{x} \in [-1, 0)$, if $p \geqq \bar{p}(\bar{x})$, then $\bar{x} \in \mathrm{USupp}(f, X, p)$. By the aforementioned symmetry, for $\bar{x} \in (0, 0.5^{1/3}]$, we have $\bar{x} \in \mathrm{USupp}(f, X, p)$ if $(f_2(\bar{x}), f_1(\bar{x})) \in \mathrm{USupp}(Y, p)$. Since $(f_2(\bar{x}), f_1(\bar{x})) = f(-2^{1/3}\bar{x})$, it is sufficient to put $p \geqq \bar{p}(-2^{1/3}\bar{x})$ in order to ensure that $\bar{x} \in \mathrm{USupp}(f, X, p)$. Fig. 1 shows two points $\bar{x}$ and the level lines $w(p)^\top y^p = w(p)^\top f(\bar{x})^p$ with $p = p(\bar{x})$. Note that the bound is not tight in the sense that $\bar{x} \in \mathrm{USupp}(f, X, p)$ if p is slightly smaller than $\bar{p}(\bar{x})$. This is due to the coarse estimates made during the derivation. Also depicted in Fig. 1 is $p(\bar{x})$ for $\bar{x} \in X \setminus 0$, with logarithmic y-axis. One sees that $\bar{p}(\bar{x}) \xrightarrow{\bar{x} \to 0} \infty$. In fact, there is no $p \geqq 1$ such that $0 \in \mathrm{Supp}(f, X, p)$, see also Ex. 3.12.

3.3 Role of the assumptions for p-supportedness

This section establishes that the seemingly technical assumptions of Th. 3.9 are not superfluous. For each assumption, we provide an example where it is violated individually and some efficient point that is not p-supported for any $p \geqq 1$ exists. All examples are biobjective with a set of nondominated points that is the graph of a function $\psi : \mathbb{R}_> \to \mathbb{R}_>$ that is one-sided directionally

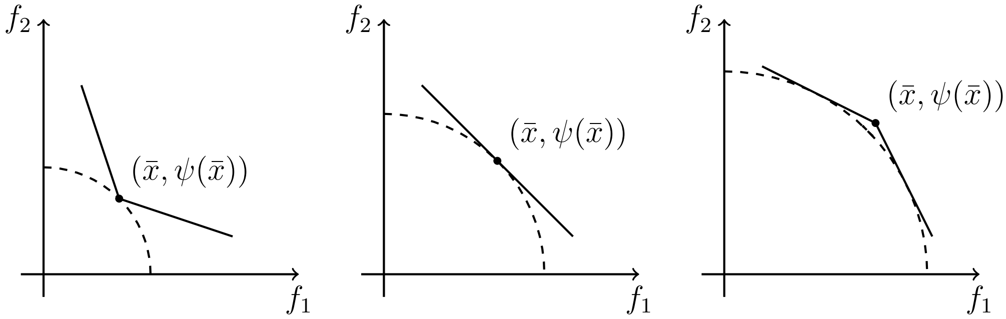

Figure 2: Nondominated sets as graphs of different functions ψ (solid), level lines of $w^\top f^p$ at optimal points of $\mathrm{WN}_p(w)$ with $w = (1,1)^\top$ and $p = 2$ (dashed), and points $\bar{y} := (\bar{x}, \psi(\bar{x}))$. Left: $\psi'(\bar{x}^-) < \psi'(\bar{x}^+)$ and $\bar{y} \in \mathrm{Supp}(Y,p)$. Middle: $\psi'(\bar{x}^-) = \psi'(\bar{x}^+)$ and $\bar{y} \in \mathrm{Supp}(Y,p)$. Right: $\psi'(\bar{x}^-) > \psi'(\bar{x}^+)$ and $\bar{y} \notin \mathrm{Supp}(Y,p)$.

differentiable at a point $\bar{x}$. Fig. 2 depicts three different problems where the left derivative $\psi'(\bar{x}^-)$ of ψ at $\bar{x}$ is either larger, equal to, or smaller than the right derivative $\psi'(\bar{x}^+)$. Since the level lines of the scalar objective function $\psi^{\mathrm{WN}}_{p,w}$ are smooth for all $p \geqq 1$ and $w \in \mathbb{R}^m_>$, it is geometrically clear that $(\bar{x}, \psi(\bar{x}))$ cannot be p supported for any $p \geqq 1$ if $\psi'(\bar{x}^-) > \psi'(\bar{x}^+)$. This is shown formally in the next result.

Proposition 3.11. *For $U \subseteq \mathbb{R}_>$, let $\psi := U \to \mathbb{R}_>$ be strictly decreasing and one-sided directionally differentiable at $\bar{x} \in U$. Let $Y \subseteq \mathbb{R}^m$ with $\mathrm{Nd}(Y) = \{(x, \psi(x)) \mid x \in U\}$. If there exists a $p \geqq 1$ such that $(\bar{x}, \psi(\bar{x})) \in \mathrm{Supp}(Y,p)$, then $\psi'(\bar{x}^-) \leqq \psi'(\bar{x}^+)$ must hold.*

Proof. Choose some $\varepsilon > 0$ such that $(\bar{x} - \varepsilon, \bar{x} + \varepsilon) \subseteq U$. Under the given assumptions, there exist some $w \in \mathbb{R}^2_\geq$ and a $p \geqq 1$ such that

$$\forall h \in (-\varepsilon, \varepsilon) : w^\top \left(\begin{pmatrix} \bar{x} + h \\ \psi(\bar{x} + h) \end{pmatrix}^p - \begin{pmatrix} \bar{x} \\ \psi(\bar{x}) \end{pmatrix}^p \right) \geqq 0. \tag{3.29}$$

If $w_2 = 0$, we have $w_1 > 0$, and (3.29) yields for all $h \in (-\varepsilon, 0)$ the contradiction $h \geqq 0$. We thus have $w_2 > 0$. Dividing both sides of (3.29) by $h > 0$, letting $h \searrow 0$, and using the chain rule yields

$$\begin{aligned} \lim_{h \searrow 0} w_1 \frac{(\bar{x}+h)^p - \bar{x}^p}{h} &+ w_2 \frac{\psi(\bar{x}+h)^p - \psi(\bar{x})^p}{h} \\ &= w_1 p \bar{x}^{p-1} + w_2 p \psi(\bar{x})^{p-1} \psi'(\bar{x}^+) \geqq 0. \end{aligned} \tag{3.30}$$

Similarly, if we divide both sides by $h < 0$ and let $h \nearrow 0$, we obtain

$$-pw_1\bar{x}^{p-1} - w_2 p\psi(\bar{x})^{p-1}\psi'(\bar{x}^-) \geqq 0. \tag{3.31}$$

If we add (3.30) and (3.31) and divide the result by $w_2 p\psi(\bar{x})^{p-1} > 0$, we obtain $\psi'(\bar{x}^+) - \psi'(\bar{x}^-) \geqq 0$. $\square$

Example 3.12 ($\bar{x} \notin \mathrm{StrLE}(f, X, 1)$)**.** *Let X and f be defined as in Ex. 3.10. We have already shown that $0 \notin \mathrm{StrLE}(f, X, 1)$ while all other assumptions of Th. 3.9 are fulfilled. By substituting $u := f_1(x)$, one sees that $f(X) = \{(u, \psi(u)) \mid u \in [1/3, 11/6]\}$, where $\psi(u) = 2 - 0.5u$ for $u \leqq 4/3$ and $\psi(u) = 4 - 2u$ for $u > 4/3$. It holds that $f(0) = (4/3, 4/3) = (4/3, \psi(4/3))$, and $\psi'((4/3)^-) = -0.5 > -2 = \psi'((4/3)^+)$. Thus, by Prop. 3.11, $0 \notin \mathrm{Supp}(f, X, p)$ for any $p \geqq 1$.*

Example 3.13 (X not compact)**.** *We only provide an example with an unbounded but closed set X. A similar example with an open but bounded X can be easily constructed. Let $g(x) = -(x - 1/3)(x - 4/3)(x - 2)$, then $X = \{x \in \mathbb{R} \mid g(x) \leqq 0\} = [1/3, 4/3] \cup [2, \infty)$ is not bounded. We define f such that $f_1(x) = x$ and $f_2(x) = 2 - 0.5x$ for $x \leqq 4/3$, $f_1(x) = 4/3 + 1/x$ and $f_2(x) = 4/3 - 2/x$ for $x \geqq 2$, and f_1, f_2 are extended to $(4/3, 2)$ such that $f \in \mathcal{C}^2(X, \mathbb{R}^2)$. Again, by substituting $u := f_1(x)$, one sees that $f(X) = \{(u, \psi(u)) \mid u \in [1/3, 11/6]\}$, with ψ from Ex. 3.12. Since (with f from the current example) $f(4/3) = (4/3, \psi(4/3))$, we know that $4/3 \notin \mathrm{Supp}(f, X, p)$ for any $p \geqq 1$. At the same time, $(f, g) \in \mathcal{C}^2$ by construction and $4/3 \in \mathrm{ACQ}(X)$ is implied by the stronger linear independence constraint qualification since $g'(4/3) \neq 0$. Furthermore, $4/3 \in \mathrm{StrE}(f, X)$ since $f_1(x) < 4/3$ for all $x < 4/3$ and $f_1(x) > 4/3$ for all $x \geqq 2$. Lastly, $f_1'(4/3) > 0$ while $f_2'(4/3) < 0$ and thus, $L_{\leqq}(4/3, f) = 0$, which by Prop. 2.10(c) means that $4/3 \in \mathrm{StrLE}(f, X, 1)$.*

Example 3.14 ($\bar{x} \notin \mathrm{ACQ}(X)$)**.** *Let $h(x) := x_1^2 + x_2^3 - 2x_2^2$ and $X = \{x \in \mathbb{R}^2 \mid h(x) = 0, x_2 \geqq -1\}$. Then $\nabla h(0) = 0$ and thus, $L_{\leqq}(0, X) = \mathbb{R}^2$, while $T(0, X) = \{d \in \mathbb{R}^2 \mid d_2 = \pm(\sqrt{2}/2)d_1\}$ (see Fig. 3 left), so $0 \notin \mathrm{ACQ}(X)$. Let $f_1(x) = (x_1 + x_2)/2 + 2$ and $f_2(x) = (x_2 - x_1)/2 + 2$. The set $f(X)$ is depicted in Fig. 3 (right). It is evident that $f(0) \notin \mathrm{Supp}(f, X, p)$ for any $p \geqq 1$. At the same time, we have $0 \in \mathrm{StrE}(f, X)$ since f is bijective. We furthermore have $0 \in \mathrm{StrLE}(f, X, 1)$ by Prop. 2.10(c), since $L_{\leqq}(0, f) = \{d \in \mathbb{R}^2 \mid d_1 + d_2 \leqq 0, -d_1 + d_2 \leqq 0\}$, and thus, $T(0, X) \cap L_{\leqq}(0, f) = 0$ (see Fig. 3 left). The other assumptions of Th. 3.9 are easily verified.*

Example 3.15 ($\bar{x} \notin \mathrm{StrE}(f, X)$)**.** *The set X from Ex. 3.14 is a curve that can also be parametrized as $\{x(t) \mid t \in \widetilde{X}\}$ with $x_1(t) = 2t - t^3$, $x_2(t) = 2 - t^2$,*

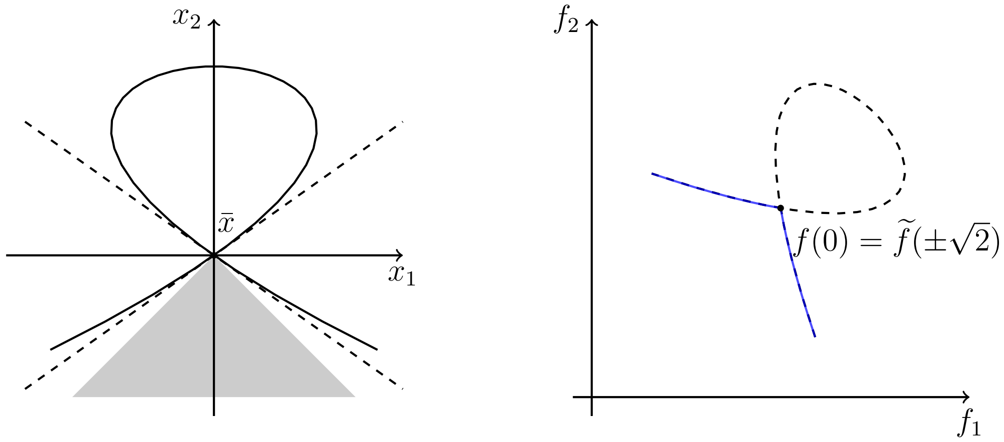

Figure 3: Left: Set X from Ex. 3.14 (solid curve), tangent cone $T(0, X)$ (dashed lines), and (closed) cone $L_{\leqq}(0, f)$ (gray area). Right: Sets $f(X) = \widetilde{f}(\widetilde{X})$ from Examples 3.14 and 3.15 (dashed curve) and $\mathrm{Nd}(f(X)) = \mathrm{Nd}(\widetilde{f}(\widetilde{X}))$ (blue, solid curve).

and $\widetilde{X} := [-\sqrt{3}, \sqrt{3}]$. *With* f *from Ex. 3.14, we define* $\widetilde{f}(t) := f(x(t))$ *and obtain* $\widetilde{f}(\widetilde{X}) = f(X)$. *For* $t = \pm\sqrt{2}$, *we have* $\widetilde{f}(t) = f(0)$, *so* $t \notin \mathrm{StrE}(\widetilde{f}, \widetilde{X})$ *and* $t \notin \mathrm{Supp}(\widetilde{f}, \widetilde{X}, p)$ *for any* $p \geqq 1$. *On the other hand,* $t \in \mathrm{ACQ}(\widetilde{X})$ *since* $t \notin \mathrm{bd}\, \widetilde{X}$ *and* $t \in \mathrm{StrLE}(\widetilde{f}, \widetilde{X}, 1)$ *since* $\widetilde{f}_1'(t) < 0$, $\widetilde{f}_2'(t) > 0$, $t = \pm\sqrt{2}$, *and thus,* $L_{\leqq}(t, \widetilde{f}) = 0$. *The other assumptions of Th. 3.9 are easily verified.*

4 Practical aspects

Ex. 3.10 illustrated that no uniform bound on $\bar{p}$ may exist if the assumptions of Th. 3.9 are violated at a single point. In this section, we thus expand on Gearhart's approximation idea ([13]). The decision maker has to specify a trade-off bound K and a relative approximation tolerance $\varepsilon > 0$. We derive a lower bound on p such that any $\bar{y} \in \mathrm{PNd}(Y, K)$ can be approximated by a point $y \in \mathrm{PSupp}(Y, p)$ in the sense that $\|\bar{y} - y\|_\infty < \varepsilon$. In order to avoid overflows due to large values of p in the exponent, we derive similar results for two related scalarizations.

The following results only require Y to be compact. In order to be able to interpret ε as a relative approximation tolerance, we assume $Y \subseteq (0, 1]^m$. If this is not already the case, it can easily be enforced by a scaling transformation. The only reasonable choices for the tolerance are hence $\varepsilon \in (0, 1]$.

4.1 A lower bound on p

Theorem 4.1. *Let $K \geqq 1$, $\emptyset \neq Y \subseteq (0,1]^m$ be closed, $\varepsilon \in (0,1)$, and*

$$p > \frac{\log(m)}{\log(1+\varepsilon/K)}. \tag{4.1}$$

Then for all $\bar{y} \in \mathrm{PNd}(Y, K)$ there exists some $w \in \mathbb{R}^m_>$ such that $\mathrm{WN}_p(w)$ is solvable and $\|\bar{y} - y\|_\infty < \varepsilon$ holds for any minimal point y of $\mathrm{WN}_p(w)$.

Proof. $\mathrm{WN}_p(w)$ fulfills the assumptions of the Weierstrass Theorem for all $w \in \mathbb{R}^m_>, p \geqq 1$ and is thus solvable. Let p fulfill (4.1) and put $w(p) = (1/\bar{y}_i^p)_{i\in[m]}$. It follows from (4.1) that

$$(1 + \frac{\varepsilon}{K})^p - m > 0.$$

Furthermore, for all $y \in Y$ with $\|\bar{y} - y\|_\infty \geqq \varepsilon$, it follows from Lem. 3.8 and $\bar{y} \in (0,1]^m$ that there exists some $k \in [m]$ with $y_k/\bar{y}_k \geqq 1 + \varepsilon/K$, and hence,

$$w(p)^\top (y^p - \bar{y}^p) = \left(\sum_{i=1}^m \left(\frac{y_i}{\bar{y}_i} \right)^p \right) - m \geqq (1 + \frac{\varepsilon}{K})^p - m > 0.$$

We thus have $w(p)^\top y^p > w(p)^\top \bar{y}^p$ for all $y \in Y$ with $\|\bar{y} - y\|_\infty \geqq \varepsilon$. □

We point out that for the feasible parameters $m \geqq 2$, $\varepsilon \in (0,1]$, and $K \geqq 1$, the lower bound on p in (4.1) is not smaller than 1. A bound with a similar structure is presented in [4] for discrete image sets $Y \subseteq \mathbb{Z}^m$. Furthermore, in a different notion of approximation, which was popularized by [32], a point $y \in Y$ α-approximates $\bar{y} \in Y$ with some $\alpha \geqq 1$ if $y \leqq \alpha\bar{y}$. Similar to the above result, [15, 17] provide for a given p the approximation tolerance α such that all $\bar{y} \in Y$ are α-approximated by some $y \in \mathrm{PSupp}(Y, p)$.

Example 4.2. *Let f and X be defined as in Ex. 3.10. We scale f such that $Y = f(X) \subseteq (0,1]^2$. Since $Y = \mathrm{PNd}(Y, 2)$ holds, we can put $K = 2$. We choose $\varepsilon = 0.1$ and obtain the bound $p = \log(2)/\log(1 + \varepsilon/K) \approx 14$ from Th. 4.1. With $w = (1,1)$, $\mathrm{WN}_p(w)$ has two solutions y^1, y^2, and we have $\|f(0) - y^i\|_\infty < \varepsilon$ for $i = 1, 2$ (see Fig. 4).*

Even for conservative values of m, K, and ε, the lower bound on p becomes quite large. For example, with $m = 2$, $K = 10$, and $\varepsilon = 0.1$, we have to put $p \approx 70$. In order to avoid overflows due to large values of p while solving $\mathrm{WN}_p(w)$, the next section shows how first applying an exponential transformation to MOP allows performing numerically stable computations with large powers.

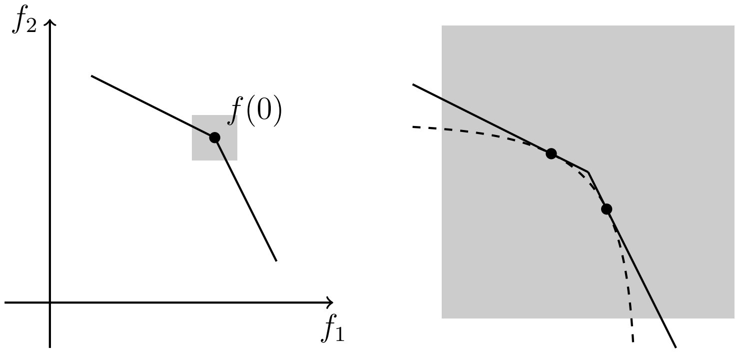

Figure 4: Illustration of Ex. 4.2. Left: Set Y, point $f(0)$ and $B_\infty(f(0), \varepsilon)$. Right: $B_\infty(f(0), \varepsilon)$ magnified, images of two solutions of $\mathrm{WN}_p(w)$, and level line of $\psi^{\mathrm{WN}}_{p,w}$ at optimal value level.

4.2 Exponential weighted norm scalarization

We apply an exponential transformation $\exp(y) := (\exp(y_1), \ldots, \exp(y_m))^\top$ to (f, X), resulting in the transformed problem $(\exp(f), X)$. It is easily verified that (f, X) and $(\exp(f), X)$ are equivalent in the sense of Prop. 2.12. Applying the scalar problem $\mathrm{WN}_p(w)$ with some $p \geqq 1$ to the transformed image set $\exp(Y)$ results in

$$\min w^\top \exp(py) \quad \text{s.t.} \quad y \in Y,$$

whose objective can be expressed more neatly if we parametrize the set $\mathbb{R}^m_>$ of weights by

$$v \mapsto w(v) := \exp(pv), v \in \mathbb{R}^m. \tag{4.2}$$

This enables us to write down the problem in the form

$$\min \sum_{i=1}^{m} \exp(p(v_i + y_i)) \quad \text{s.t.} \quad y \in Y.$$

We may scale the objective by $\log(\cdot)/p$, resulting in

$$\min \frac{1}{p} \log \left(\sum_{i=1}^{m} \exp(p(v_i + y_i)) \right) \quad \text{s.t.} \quad y \in Y.$$

With the log-sum-exp function $\mathrm{LSE}(z) = \log\left(\sum_{i=1}^m e^{z_i}\right)$, one can express the objective more compactly and obtains

$$(\mathrm{EWN}_p(v)) \qquad \min \psi^{\mathrm{EWN}}_{p,v}(y) := \frac{1}{p} \mathrm{LSE}\left(p\left(v + y\right)\right) \quad \text{s.t.} \quad y \in Y,$$

which we refer to as exponential weighted norm scalarization. The advantage of this formulation is that we can use the log-sum-exp trick to evaluate LSE at a point $z \in \mathbb{R}^m$: We can choose a number M and multiply the term $\sum_{i=1}^m e^{z_i}$ by $e^M e^{-M}$. Rearranging results in $\mathrm{LSE}(z) = M + \log\left(\sum_{i=1}^m e^{z_i - M}\right)$. By choosing $M := M(z) := \max_i z_i$ we can ensure that exp is evaluated at no number larger than 0, thus preventing overflows. This technique is often used in machine learning. A numerical analysis of the log-sum-exp trick is provided in [2].

The next result provides a bound on the parameter p in $\mathrm{EWN}_p(v)$. It is based on the trade-off and approximation tolerance in the original problem (f, X), which is easier to interpret for a decision maker.

Proposition 4.3. *Let $K \geqq 1$, $\emptyset \neq Y \subseteq (0,1]^m$ be closed, $\varepsilon \in (0,1)$, and*

$$p > \frac{\log(m)K}{\varepsilon}. \tag{4.3}$$

Then for all $\bar{y} \in \mathrm{PNd}(Y, K)$, there exists some $v \in \mathbb{R}^m$ such that $\mathrm{EWN}_p(v)$ is solvable and $\|\bar{y} - y\|_\infty < \varepsilon$ holds for any minimal point y of $\mathrm{EWN}_p(v)$.

Proof. $\mathrm{EWN}_p(v)$ fulfills the assumptions of the Weierstrass Theorem for all $v \in \mathbb{R}^m, p \geqq 1$ and is thus solvable. Let p fulfill (4.3) and put $v = -\bar{y}$. It follows from (4.3) that

$$\exp\left(\frac{p\varepsilon}{K}\right) - m > 0.$$

Furthermore, for all $y \in Y \setminus B_\infty(\bar{y}, \varepsilon)$, it follows from Lem. 3.8 and a simple rearrangement of (3.24) that there exists some $k \in [m]$ with $y_k - \bar{y}_k \geqq \varepsilon/K$, and thus,

$$\sum_{i=1}^m \left(e^{p(v_i + y_i)} - e^{p(v_i + \bar{y}_i)}\right) = \left(\sum_{i=1}^m e^{p(y_i - \bar{y}_i)}\right) - m \geqq \exp\left(\frac{p\varepsilon}{K}\right) - m > 0.$$

Rearranging and applying $\log(\cdot)/p$ yields $\psi_{p,v}^{\mathrm{EWN}}(y) > \psi_{p,v}^{\mathrm{EWN}}(\bar{y})$ for all $y \in Y$ with $\|\bar{y} - y\|_\infty \geqq \varepsilon$. $\square$

For computational aspects, we investigate the gradient and the Hessian matrix of $\Psi_{p,v}^{\mathrm{EWN}}(x) := \psi_{p,v}^{\mathrm{EWN}}(f(x))$. The calculations are carried out in detail in App. C. Let

$$\sigma : \mathbb{R}^n \to [0,1]^m, \sigma_i(x) = \frac{\exp(p(v_i + f_i(x)))}{\sum_{j=1}^m \exp(p(v_j + f_j(x)))}.$$

Note that for a given $x \in \mathbb{R}^n$, $\sigma(x)$ is the softmax function of the vector $p(v + f(x))$. In particular, it holds that $\sigma_i(x) \in [0, 1]$ and $\sum_{i=1}^m \sigma_i(x) = 1$. The gradient

$$\nabla \Psi_{p,v}^{\mathrm{EWN}}(x) = \sum_{i=1}^m \sigma_i(x) \nabla f_i(x)$$

is thus a certain weighted sum of the gradients of $f_1, \ldots, f_m$. For the Hessian matrix, we omit the argument x in all terms for the sake of brevity. The Hessian matrix is given by

$$D^2 \Psi_{p,v}^{\mathrm{EWN}} := pV + \sum_{i=1}^m \sigma_i D^2 f_i, \tag{4.4}$$

where

$$V = \sum_{i=1}^m \sigma_i \nabla f_i \nabla f_i^\top - \left(\sum_{i=1}^m \sigma_i \nabla f_i \right) \left(\sum_{i=1}^m \sigma_i \nabla f_i \right)^\top .$$

We can show that V is always positive semidefinite. Let Z be a discrete random vector with possible outcomes $\{\nabla f_i \mid i \in [m]\}$ and probabilities $P(z = \nabla f_i) = \sigma_i$, $i \in [m]$. The covariance matrix of Z is given by $E(ZZ^\top) - E(Z)E(Z)^\top = V$. It is well-known that covariance matrices are positive semidefinite.

This special structure of $D^2 \Psi_{p,v}^{\mathrm{EWN}}(x)$ is particularly helpful for the αBB method for deterministic global optimization. We only discuss the relevant aspects of the αBB method and refer to [11] for details. The αBB method, applied to $\mathrm{EWN}_p(v)$ with a box-constrained feasible set $X = \{x \in \mathbb{R}^n \mid x^L \leqq x \leqq x^U\}$, determines lower bounds on the objective using convex underestimating functions

$$\Psi_{p,v}^{\mathrm{EWN}}(x) + \frac{\alpha}{2}(x^L - x)^\top (x^U - x)$$

with $\alpha \geqq 0$ such that its Hessian

$$D^2 \Psi_{p,v}^{\mathrm{EWN}}(x) + \alpha I = pV(x) + \sum_{i=1}^m \sigma_i(x) D^2 f_i(x) + \alpha I$$

is positive semidefinite. Since $\forall x \in X : pV(x) \succeq 0$, and since the sum of positive semidefinite matrices is positive semidefinite, it suffices to choose α such that $\forall x \in X : \sum_{i=1}^m \sigma_i(x) D^2 f_i(x) + \alpha I \succeq 0$. Such an α is obtained by using interval arithmetic in order to compute a matrix whose entries are intervals containing the respective entries of $\sum_{i=1}^m \sigma_i(x) D^2 f_i(x)$ for all $x \in X$.

One can then apply Gerschgorin's Theorem to this interval valued matrix, which yields

$$\alpha \geqq \max\left\{0, -\min_{x\in X} \lambda_{\min}(x)\right\},$$

where $\lambda_{\min}(x)$ denotes the minimal eigenvalue of $\sum_{i=1}^m \sigma_i(x)D^2 f_i(x)$. Since $\forall x \in X \forall i \in [m] : \sigma_i(x) \in [0,1]$, we can replace $\sigma_i(x)$ by $[0,1]$ when computing the interval extension matrix of $\sum_{i=1}^m \sigma_i(x)D^2 f_i(x)$. In particular, the thus obtained α does not depend on the power parameter p and does not require evaluations of V or σ. A further advantage is that the dependency effect, which causes the bounds obtained by interval arithmetic to be coarser when the same variable appears more often in an expression, to be less severe. Preliminary tests performed before our experiments in Sect. 5 confirmed that the thus obtained α is smaller than the α computed with the standard method by multiple orders of magnitude.

4.3 Smooth Chebyshev scalarization

We discuss the smooth Chebyshev scalarization from [27], since its objective is similar to $\psi_{p,v}^{\mathrm{EWN}}(y)$. It is a smoothed version of the weighted Chebyshev scalarization with the objective

$$\widetilde{\psi}_{\mu,w}^{\mathrm{SCH}}(y) := \mu \log\left(\sum_{i=1}^m \exp\left(\frac{w_i(y_i - y^U(Y))}{\mu}\right)\right),$$

with a parameter $\mu > 0$ and a fixed utopia point $y^U(Y)$. Under our assumption $Y \subseteq \mathbb{R}^m_>$, we can simply choose $y^U(Y) = 0$. Furthermore, the main results from [27] hold for $\mu \to 0$, and all tests were performed with $\mu \leqq 1$. For the sake of consistent notation, we thus let $\mu = 1/p$ with $p \geqq 1$ and use the formulation

$$(\mathrm{SCH}_p(w)) \qquad \min \psi_{p,w}^{\mathrm{SCH}}(y) := \frac{1}{p}\log\left(\sum_{i=1}^m \exp(pw_i y_i)\right) \quad \text{s.t.} \quad y \in Y.$$

It was shown in [27] that minimal points of $\mathrm{SCH}_p(w)$ are weakly nondominated and nondominated if the minimal point is unique or all $w_i > 0$. The next result shows that the latter assumption even implies proper efficiency.

Proposition 4.4. *Let $Y \subseteq \mathbb{R}^m_>$, $p \geqq 1$, and $w \in \mathbb{R}^m_>$. Then,*

$$\operatorname{argmin}_{y\in Y}\left(\psi_{p,w}^{SCH}(y)\right) \subseteq \mathrm{PNd}(Y). \tag{4.5}$$

Proof. Let Φ be the component-wise monotone transformation with components $\varphi_i : \mathbb{R} \to \mathbb{R}, \varphi_i(y_i) = \exp(pw_i y_i)$. Let $Z = \Phi(Y)$. All φ_i and Y fulfill the assumptions of Prop. 2.12(f) and thus, $\mathrm{PNd}(Z) = \Phi(\mathrm{PNd}(Y))$ holds. With $\bar{y} \in \operatorname{argmin}_{y\in Y} \left(\psi^{\mathrm{SCH}}_{p,w}(y)\right)$ and $\bar{z} = \Phi(\bar{y})$ we obtain

$$\begin{aligned}\psi^{\mathrm{SCH}}_{p,w}(\bar{y}) &= \min_{y\in Y} \psi^{\mathrm{SCH}}_{p,w}(y)\\ \Leftrightarrow \sum_{i=1}^{m} \exp(pw_i\bar{y}_i) &= \min_{y\in Y} \sum_{i=1}^{m} \exp(pw_i y_i)\\ \Leftrightarrow \sum_{i=1}^{m} \bar{z}_i &= \min_{z\in Z} \sum_{i=1}^{m} z_i.\end{aligned}$$

Thus, $\bar{z}$ minimizes a weighted sum with positive weights over Z and by [7], we have $\bar{z} \in \mathrm{PNd}(Z) = \Phi(\mathrm{PNd}(Y))$, and thus $\bar{y} \in \mathrm{PNd}(Y)$. $\square$

Under Li's assumption that $\mathrm{Nd}(Y)$ is the graph of a $\mathcal{C}^2$-function ([24]), it was also shown in [27] that there exists some $\bar{p} \geqq 1$ such that for all $p \geqq \bar{p}$ and $\bar{y} \in \mathrm{Nd}(Y)$, there exists some $w \in \mathbb{R}^m_{\geq}$ such that $\bar{y} \in \operatorname{argmin}_{y\in Y} \psi^{\mathrm{SCH}}_{p,w}(y)$. We show an approximation result similar to Th. 4.1 and Prop. 4.3.

Proposition 4.5. *Let $K \geqq 1$, $\emptyset \neq Y \subseteq (0,1]^m$ be closed, $\varepsilon \in (0,1)$, and*

$$p > \frac{\log(m)K}{\varepsilon}. \tag{4.6}$$

Then for all $\bar{y} \in \mathrm{PNd}(Y,K)$, there exists some $w \in \mathbb{R}^m_{>}$ such that $\mathrm{SCH}_p(w)$ is solvable and $\|\bar{y} - y\|_\infty < \varepsilon$ holds for any minimal point y of $\mathrm{SCH}_p(w)$.

Proof. $\mathrm{SCH}_p(w)$ fulfills the assumptions of the Weierstrass Theorem for all $w \in \mathbb{R}^m_{>}, p \geqq 1$ and is thus solvable. Let p fulfill (4.6), put $w_i = 1/\bar{y}_i$ for $i \in [m]$, and let $y \in Y \setminus B_\infty(\bar{y}, \varepsilon)$. From (4.6) we obtain

$$\exp\left(p\left(1 + \frac{\varepsilon}{K}\right)\right) - m\exp(p) > 0.$$

Using the estimate from Lem. 3.8 as in the proof of Th. 4.1 then yields

$$\begin{aligned}\sum_{i=1}^{m} \left(\exp(pw_i y_i) - \exp(pw_i\bar{y}_i)\right) &= \left(\sum_{i=1}^{m} \exp\left(p\frac{y_i}{\bar{y}_i}\right)\right) - m\exp(p)\\ &\geqq \exp\left(p\left(1 + \frac{\varepsilon}{K}\right)\right) - m\exp(p) > 0.\end{aligned}$$

Rearranging and applying $\log(\cdot)/p$ yields $\psi^{\mathrm{SCH}}_{p,w}(y) > \psi^{\mathrm{SCH}}_{p,w}(\bar{y})$ for all $y \in Y$ with $\|\bar{y} - y\|_\infty \geqq \varepsilon$. $\square$

The gradient and Hessian of $\Psi^{\mathrm{SCH}}_{p,w} := \psi^{\mathrm{SCH}}_{p,w} \circ f$ can be computed similarly to App. C. Their structure is similar to that of the gradient and Hessian of $\Psi^{\mathrm{EWN}}_{p,v}$. In particular, the parameter α in the αBB method can be estimated independently of p, which we use for our numerical experiments in Sect. 5.

5 Numerical Experiments

We tested the exponential weighted norm (EWN) and the smooth Chebyshev scalarization (SCH) on a set of nine multiobjective test instances. The test problems are stated in Tab. 3 in App. D. The instances NEST2 and NEST3 are the problems from Examples 3.10 and 3.15, and the instance NEST is also proposed in this work. The other six problems are known from literature.

The tests were implemented in Python. We used the package sympy ([28]) to symbolically compute the required gradients and Hessian matrices and for parsing the symbolic expressions to Python functions that can be evaluated efficiently. We implemented the αBB method in which we compute the value α independently of p as described in Sect. 4.2. We also implemented a small module to perform the required interval arithmetic operations. The package scipy ([37]) was used to perform the local optimizations required in each iteration of αBB. The code is available at `https://github.com/felixneussel/p_power_mop`.

Both EWN and SCH are able to approximate any K-nondominated point of an instance with m objectives with a relative tolerance of ε, if for the parameter p we have $p > \log(m)K/\varepsilon$. If we fix $K = 20$, we get $p > 60.3$ for $m = 2$ and $\varepsilon = 0.1$ and $p > 955$ for $m = 3$ and $\varepsilon = 0.01$. We thus test EWN and SCH with values of $p \in \{50, 100, \dots, 950, 1000\}$.

For each value of p, we solved each instance with different weight vectors. The weight vectors for SCH were computed as a uniform lattice on

$$W := \{w \in \mathbb{R}^m_> \mid e^\top w = 1\}$$

with 20 different weight vectors for biobjective problems and 210 for problems with three objectives. Motivated by the proof of Prop. 4.3, where we chose $v = -\bar{y}$, the parameters v for EWN were computed as a uniform lattice on $-W$.

We define the iteration number required to solve a given instance with one of the scalarization EWN or SCH with a certain value of p as the sum of iteration numbers required for each choice of the parameter w or v, respectively. Run times are defined analogously. Thus, with nine instances, two scalarizations,

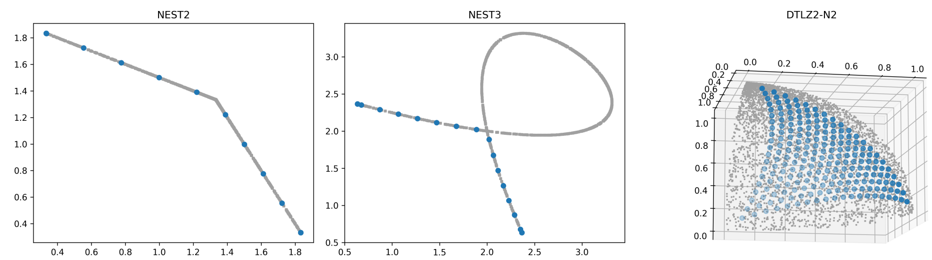

Figure 5: Approximation of $\mathrm{Nd}(Y)$ for NEST2, NEST3, and DTLZ2-N2.

and 20 different choices of p, we obtain a dataset with 360 rows, where each row contains iterations and run time required to solve one instance with a given scalarization and value of p. For illustrative purposes, Fig. 5 depicts the thus generated approximation of the set of nondominated points with EWN and $p = 1000$ for three of the test instances.

The test results are summarized in Tab. 1. For each instance (inst.) and scalarization (scal.), we report the maximum and minimum run time as well as the maximum and minimum iteration number ($t^+, t^-, \mathrm{it}^+, \mathrm{it}^-$) across all $p \in \{50, 100, \ldots, 1000\}$. We also state for which value of p they were attained ($p_t^+, p_t^-, p_i^+, p_i^-$).

All instances could be solved with both scalarizations and all values of p. We now investigate the effect of p on run times and iteration numbers separately for both scalarizations. For each instance and each value of p, we compute the ratio between the run time at the given p and the average run time across all p. We then take the average of these ratios across all instances. The iteration numbers are treated in the same way. These average ratios are depicted in Fig. 6. Surprisingly, the iteration numbers decrease with p for both scalarizations. On the other hand, the run times increase with p. While we have no explanation for the decreasing iteration numbers, the increasing run times indicate that the local minimization that is performed in each iteration of the αBB method becomes more difficult for larger p. The overall effect is not dramatic however, with the average ratios ranging between 0.94 and 1.08.

We now compare the performances of EWN and SCH. For each given instance and p, we compare iteration numbers required with EWN and SCH as follows. We compute the ratio between the iterations required with EWN and SCH. The mean of these ratios is 1.169, i.e. EWN required more iterations than

inst.	scal.	t^+	p_t^+	t^-	p_t^-	it$^+$	p_i^+	it$^-$	p_i^-
DEB2DK	EWN	6.09	1000	4.19	50	2516	650	2348	50
	SCH	2.65	350	2.20	300	1270	50	1192	1000
DTLZ2-N2	EWN	22.50	1000	19.50	200	9348	50	7514	1000
	SCH	24.72	400	21.77	50	9942	50	8234	1000
DTLZ2-N3	EWN	227.16	1000	199.77	100	63928	50	54440	1000
	SCH	269.34	1000	254.75	450	88648	50	73428	1000
FF2D	EWN	7.36	900	6.60	550	4026	50	3898	750
	SCH	8.41	450	5.64	150	3606	50	3488	950
NEST	EWN	0.70	1000	0.50	100	318	500	307	100
	SCH	0.61	700	0.39	50	252	800	245	200
NEST2	EWN	0.24	300	0.16	100	154	50	139	600
	SCH	0.35	750	0.24	400	136	50	125	550
NEST3	EWN	0.33	950	0.26	50	222	50	214	200
	SCH	0.81	850	0.29	1000	226	50	218	100
SHEKEL	EWN	10.21	850	8.38	550	5030	50	4902	1000
	SCH	8.27	900	8.00	50	4738	50	4618	700
VFM	EWN	178.36	1000	168.66	100	91331	750	89206	50
	SCH	96.68	50	88.34	700	53918	50	48635	1000

Table 1: Test results

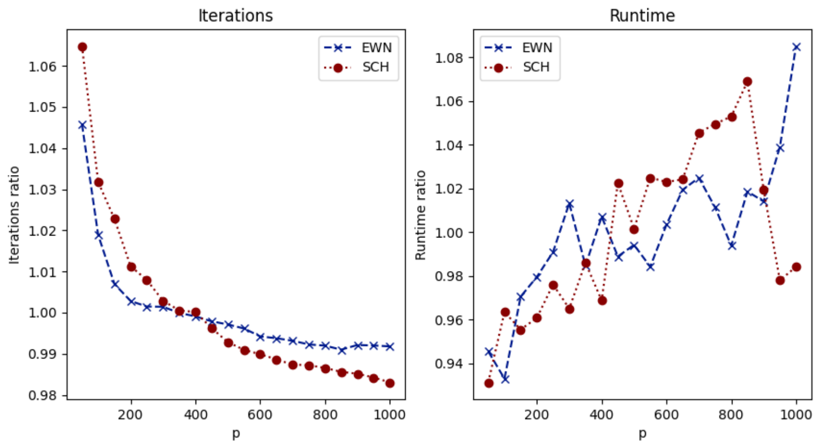

Figure 6: Average iteration and run time ratios.

SCH on average. We also perform one-sided t-test on the logarithm of the ratios with the alternative hypothesis that the mean log ratio is greater than zero. The results are stated in Tab. 2. With a p-value smaller than 10^{-9}, the observed difference is significant.

The run times are compared in the same manner. The results are depicted in Tab. 2 as well. The mean ratio also indicates that EWN required longer run times, but with a p-value of 0.173, the difference is not significant.

	mean log ratio	mean ratio	test statistic	p-val
Iterations	0.156	1.169	6.633	$< 10^{-9}$
Run time	0.031	1.031	0.945	0.173

Table 2: Results of the t-test.

6 Conclusion

In this work, we have gained new insights on p-supported points. We have shown that under standard assumptions, any feasible point that is both strictly efficient and locally strictly efficient of order 1 becomes uniquely p-supported when p is sufficiently large. Our examples indicate that these assumptions are not merely technical, but prohibit a certain geometry in the set of nondominated points that prevents efficient points from ever becoming p-supported. We have further provided an easy-to-compute lower bound on the parameter p that ensures that any properly efficient point with a desired trade-off bound can be approximated with a specified relative approximation tolerance.

In order to compute p-supported points, one may use the weighted norm scalarization. Using an exponential transformation, we have provided a numerically stable way to perform computations with said scalarization, even with very large values of the power parameter p. The thus tractable scalarization is an attractive alternative to solve nonconvex MOPs. Compared to other scalarizations suitable for solving such problems, it has a smooth objective and moves none of the original objectives to the constraints, which is desirable if the objective functions are nonconvex, while the constraints are convex.

Another scalarization with this desirable property is the smooth Chebyshev scalarization from [27], for which we have also derived a similar lower bound on the parameter p (or an upper bound on the parameter $\mu = 1/p$ in the notation of [27]). For both scalarizations, we have found a special structure of the Hessian matrix, which allows computing lower bounds in the αBB method independently of p.

Numerical experiments have confirmed that both scalarizations can be efficiently solved to global optimality, even with large values of p. While run times do increase with p, the effect is not dramatic and likely caused by the local minimization performed to obtain lower bounds. It is therefore an interesting question whether the special structure of the Hessians of the two scalarizations can also be exploited by local solvers that rely on second order information, e.g. Newton's method or trust region methods. Furthermore, it should be investigated if smaller bounds on p can be found for problems with special structure.

A possible application of our results are algorithms that compute a discrete representation of the nondominated set by using scalarizations and updating the parameters adaptively, see e.g. [8, 23]. In such a method that uses the weighted norm scalarization, our lower bound on p could possibly indicate if certain regions in the image set do not contain any properly nondominated points with the desired maximal trade-off.

Acknowledgments

This work was supported by Deutsche Forschungsgemeinschaft (German Research Foundation) under grant 528525668. The authors express their gratitude to Gabriele Eichfelder and Christian Kugelmann for fruitful discussions and helpful suggestions during the development of this work.

Declarations

Competing interests The authors have no relevant financial or non-financial interests to disclose.

Data availability The datasets generated during the numerical experiments are available at `https://github.com/felixneussel/p_power_mop`.

Code availability The source code of all numerical experiments is available at `https://github.com/felixneussel/p_power_mop`.

A Result on properly efficient points

We prove the following result, which was stated as a conjecture in [34].

Proposition A.1. *Let $Y \subseteq \mathbb{R}^m$ and $0 < K < 1$. Then* $|\operatorname{PNd}(Y, K)| \leqq 1$.

Proof. Assume on the contrary that there existed $y, z \in \operatorname{PNd}(Y, K)$ with $y \neq z$. Since $z \nleqq y$, there exists some $i^1 \in [m]$ with $y_{i^1} < z_{i^1}$. Since $z \in \operatorname{PNd}(Y, K)$, there exist some $j^1 \in [m]$ with $z_{j^1} < y_{j^1}$ such that

$$\frac{z_{i^1} - y_{i^1}}{y_{j^1} - z_{j^1}} \leqq K < 1.$$

But since $z_{j^1} < y_{j^1}$, and $y \in \operatorname{PNd}(Y, K)$, there exists some $i^2 \in [m]$ with $y_{i^2} < z_{i^2}$ such that

$$\frac{y_{j^1} - z_{j^1}}{z_{i^2} - y_{i^2}} < 1.$$

Iterating over these arguments yields infinite sequences $(i^k), (j^k) \subset [m]$ with

$$\forall k \in \mathbb{N} : y_{i^k} < z_{i^k},\, z_{j^k} < y_{j^k},\, \frac{z_{i^k} - y_{i^k}}{y_{j^k} - z_{j^k}} < 1,\, \frac{y_{j^k} - z_{j^k}}{z_{i^{k+1}} - y_{i^{k+1}}} < 1.$$

Multiplying the last two inequalities by the denominators of the fractions on the left-hand sides, respectively, yields

$$z_{i^{k+1}} - y_{i^{k+1}} > y_{j^k} - z_{j^k} > z_{i^k} - y_{i^k},$$

proving that $i^k \neq i^{k+1}$ and by induction that no $i \in [m]$ appears more than once in (i^k), which leads to a contradiction since m is finite. $\square$

B Proofs of auxiliary results

Proof of Prop. 2.10(e) First note that proper nondominance is easily characterized as follows. It holds that $\bar{y} \in \operatorname{PNd}(Y)$ if and only if $\bar{y} \in Nd(Y)$ and if there exists some $K > 0$ such that for all $y \in Y$ and $i \in [m]$ with $y_i < \bar{y}_i$, we have that

$$\frac{\bar{y}_i - y_i}{\max_{j \in [m]}(y_j - \bar{y}_j)} \leqq K. \tag{B.1}$$

Now let U be an open neighborhood of $\bar{x}$ such that $\bar{x} \in \operatorname{StrE}(f, X) \cap \operatorname{PE}(f, X \cap U)$. In the case $X \subseteq U$ the assertion clearly holds. Otherwise, $X \setminus U$ is nonempty and compact. Assume that $\bar{x} \notin \operatorname{PE}(f, X)$. Since $\operatorname{StrE}(f, X) \subseteq \operatorname{E}(f, x)$, there exists for all $k \in \mathbb{N}$ some $x^k \in X$ and $i^k \in [m]$ with $f_{i^k}(x^k) < f_{i^k}(\bar{x})$ and

$$\frac{f_{i^k}(\bar{x}) - f_{i^k}(x^k)}{\max_{j\in[m]}(f_j(x^k) - f_j(\bar{x}))} > k. \tag{B.2}$$

Since $\bar{x} \in \mathrm{PE}(f, X\cap U)$, we know that $x^k \in X\setminus U$ for k sufficiently large. Since $f(X)$ is compact, we can choose some $C > 0$ such that $f(X) \subseteq B_\infty(0, C)$. It then follows from (B.2) that $\max_{j\in[m]}(f_j(x^k) - f_j(\bar{x})) < 2C/k$. Thus, (x^k) has a cluster point $\widetilde{x} \in X \setminus U$ with $\max_{j\in[m]}(f_j(\widetilde{x}) - f_j(\bar{x})) \leqq 0$, due to the continuity of f. Thus, $f(\widetilde{x}) \leqq f(\bar{x})$, which contradicts $\bar{x} \in \mathrm{StrE}(f, X)$. $\square$

Proof of Lem. 3.2. By a Taylor expansion of $u \mapsto (1+u)^p$ around 0, we have that

$$(1+u)^p = 1 + pu + \frac{1}{2}p(p-1)u^2 + r_p(u), \tag{B.3}$$

where $r_p \in \mathcal{O}_p(|u|^3)$ for $u \to 0$. By a Taylor expansion of $f(\bar{x}+td)/f(\bar{x})$ for $t \geqq 0$ with $d \in S^{n-1}$ around $t = 0$, letting $\alpha := (\nabla f(\bar{x})^\top d)/f(\bar{x})$, we get

$$\frac{f(\bar{x}+td)}{f(\bar{x})} = 1 + \alpha t + r(t)$$

with $r \in \mathcal{O}(t^2)$. We substitute $\alpha t + r(t)$ for u in (B.3), resulting in

$$\left(\frac{f(\bar{x}+td)}{f(\bar{x})}\right)^p = 1+p(\alpha t+r(t))+\frac{p(p-1)}{2}(\alpha t+r(t))^2+r_p(\alpha t+r(t)). \tag{B.4}$$

By defining

$$\bar{r}_p(t) := p(p-1)(\alpha t r(t) + \frac{1}{2}r(t)^2) + r_p(\alpha t + r(t)) \in \mathcal{O}_p(t^3), \tag{B.5}$$

we can rearrange (B.4) to (3.1). $\square$

Proof of Lem. 3.3 Assume the statement were false. Then there exist sequences $t^k \searrow 0$ and $d^k \subseteq S^{n-1}$ with $\bar{x} + t^k d^k \in X$ for all $k \in \mathbb{N}$, but $\nabla f_i(\bar{x})^\top d^k \leqq 0$ for all $i \in [m]$ and $k \in \mathbb{N}$. Since the set S^{n-1} is compact, the sequence d^k has a convergent subsequence, according to the Bolzano-Weierstrass-Theorem. We redefine d^k as this subsequence and t^k accordingly. Thus, d^k has a limit point $\bar{d} \in S^{n-1}$, and $t^k \searrow 0$. It follows from the definition that $\bar{d} \in T(\bar{x}, X)$ and since $\bar{x} \in \mathrm{ACQ}(X)$ that $\bar{d} \in L_{\leqq}(\bar{x}, X)$. Furthermore we have $\bar{d} \in L_{\leqq}(\bar{x}, f)$. Since $\bar{x} \in \mathrm{StrLE}(f, X, 1)$, (2.4) holds, which yields the contradiction $\bar{d} = 0$. $\square$

Proof of Lem. 3.4 For a less involved notation, we abbreviate $D := D(\bar{t}, \bar{x}, X)$, $L := L_{\leqq}(\bar{x}, X)$ and $T := T(\bar{x}, X)$. By the ACQ, we know that $T = L$ holds and in particular that T is closed. Let $(d^k) \subset (D \cup T)$ with $d^k \to \bar{d} \in \mathbb{R}^n$. For all $k \in \mathbb{N}$, we have $d^k \in D$ or $d^k \in T$. Thus, d^k has a subsequence in T or D.

Case 1: d^k has a subsequence in T. We redefine d^k to itself be this subsequence. Since T is closed, it then holds that $\bar{d} \in T$.

Case 2: d^k has a subsequence in D. We redefine d^k to itself be this subsequence. Then for all $k \in \mathbb{N}$, there exists some $t^k \in (0, \bar{t}]$ such that $\bar{x} + t^k d^k \in X$. The sequence t^k has a convergent subsequence with limit $\widetilde{t} \in [0, \bar{t}]$. We redefine t^k to itself be this subsequence and d^k accordingly. If $\widetilde{t} = 0$, then $\bar{d} \in T$. If $\widetilde{t} \in (0, \bar{t}]$ then $\bar{d} \in D$ (since $\bar{x} + \widetilde{t}\bar{d} \in X$ by the closedness of X).

In conclusion, it holds in all cases that $\bar{d} \in D \cup T = D \cup L$ and thus, $D \cup L$ is closed. $\square$

C Hessian of exponential weighted norm

We define $\Psi(x) := \psi_{p,v}^{\mathrm{EWN}}(f(x))$. With the abbreviations

$$E_i(x) := \exp(p(v_i + f_i(x))), \quad S(x) := \sum_{i=1}^{m} E_i(x), \quad \sigma_i(x) := \frac{E_i(x)}{S(x)}$$

we can write $\Psi(x) = \frac{1}{p} \log(S(x))$. We compute

$$\begin{aligned} \nabla E_i(x) &= p E_i(x) \nabla f_i(x), \\ \nabla \sigma_i(x) &= \frac{S(x)\nabla E_i(x) - E_i(x)\nabla S(x)}{S^2(x)} \\ &= p\,\frac{S(x)E_i(x)\nabla f_i(x) - E_i(x)\sum_{j=1}^{m} E_j(x)\nabla f_j(x)}{S^2(x)} \\ &= p\left(\sigma_i(x)\nabla f_i(x) - \sigma_i(x)\sum_{j=1}^{m}\sigma_j(x)\nabla f_j(x)\right), \end{aligned}$$

and therefore

$$\nabla \Psi(x) = \frac{1}{p}\frac{1}{S(x)}\sum_{i=1}^{m} p E_i(x)\nabla f_i(x) = \sum_{i=1}^{m}\sigma_i(x)\nabla f_i(x)$$

as well as

$$\begin{aligned} D^2\Psi(x) &= \sum_{i=1}^{m} \nabla f_i(x) \nabla \sigma_i(x)^\top + \sum_{i=1}^{m} \sigma_i(x) D^2 f_i(x) \\ &= pV(x) + \sum_{i=1}^{m} \sigma_i(x) D^2 f_i(x) \end{aligned}$$

with

$$V(x) := \sum_{i=1}^{m} \sigma_i(x) \nabla f_i(x) \nabla f_i(x)^\top - \left(\sum_{i=1}^{m} \sigma_i(x) \nabla f_i(x) \right) \left(\sum_{j=1}^{m} \sigma_j(x) \nabla f_j(x) \right)^\top .$$

D Test Problems

Name	Objectives	Bounds	Reference
DEB2DK	$r(x) = \left(5 + 10(x_1 - 0.5)^2 + \cos(4\pi x_1)\right)(1 + 9x_2)$, $f_1(x) = r(x)\sin\left(\frac{\pi x_1}{2}\right)$, $f_2(x) = r(x)\cos\left(\frac{\pi x_1}{2}\right)$	$x_i \in [0, 1],\ i = 1, 2$	[9]
DTLZ2-N2	$f_1(x) = \cos\left(\frac{\pi x_1}{2}\right)\cos\left(\frac{\pi x_2}{2}\right)$, $f_2(x) = \cos\left(\frac{\pi x_1}{2}\right)\sin\left(\frac{\pi x_2}{2}\right)$, $f_3(x) = \sin\left(\frac{\pi x_1}{2}\right)$	$x_i \in [0, 1],\ i = 1, 2$	[6]
DTLZ2-N3	$g(x) = (x_3 - 0.5)^2$, $f_1(x) = (1 + g(x))\cos\left(\frac{\pi x_1}{2}\right)\cos\left(\frac{\pi x_2}{2}\right)$, $f_2(x) = (1 + g(x))\cos\left(\frac{\pi x_1}{2}\right)\sin\left(\frac{\pi x_2}{2}\right)$, $f_3(x) = (1 + g(x))\sin\left(\frac{\pi x_1}{2}\right)$	$x_i \in [0, 1],\ i = 1, 2, 3$	[6]
FF2D	$f_1(x) = 1 - \exp\left(-\sum_{i=1}^{2}\left(x_i - \frac{1}{\sqrt{2}}\right)^2\right)$, $f_2(x) = 1 - \exp\left(-\sum_{i=1}^{2}\left(x_i + \frac{1}{\sqrt{2}}\right)^2\right)$	$x_i \in [-4, 4],\ i = 1, 2$	[12]
NEST	$f_1(x) = x_1 + \frac{4}{3}$, $f_2(x) = \frac{7}{3} - \frac{2}{3}x_1^3 x_2 - \frac{4}{3}x_2$	$x_1 \in [-1, 1]$, $x_2 \in [0, 1]$	This work
NEST2	$f_1(x) = x^3 + \frac{4}{3}$, $f_2(x) = \begin{cases} -\frac{x^3}{2} + \frac{4}{3}, & x \le 0, \\ -2x^3 + \frac{4}{3}, & x > 0 \end{cases}$	$x \in \left[-1, \sqrt[3]{0.5}\right]$	This work
NEST3	$x_1(t) = 2t - t^3, x_2(t) = 2 - t^2$, $f_1(t) = \frac{x_1(t) + x_2(t)}{2} + 2$, $f_2(t) = \frac{x_2(t) - x_1(t)}{2} + 2$	$t \in \left[-\sqrt{3}, \sqrt{3}\right]$	This work
SHEKEL	$f_1(x) = -\frac{0.1}{0.1 + (x_1 - 0.1)^2 + 2(x_2 - 0.1)^2} - \frac{0.1}{0.14 + 20(x_1 - 0.45)^2 + 20(x_2 - 0.55)^2}$, $f_2(x) = -\frac{0.1}{0.15 + 40(x_1 - 0.55)^2 + 40(x_2 - 0.45)^2} - \frac{0.1}{0.1 + (x_1 - 0.3)^2 + (x_2 - 0.95)^2}$	$x_i \in [0, 1],\ i = 1, 2$	[9]
VFM	$f_1(x) = \frac{1}{2}(x_1^2 + x_2^2)^2 + \sin(x_1^2 + x_2^2)$, $f_2(x) = \frac{(3x_1 - 2x_2 + 4)^2}{8} + \frac{(x_1 - x_2 + 1)^2}{27} + 15$, $f_3(x) = \frac{1}{x_1^2 + x_2^2 + 1} - 1.1\exp(-x_1^2 - x_2^2)$	$x_i \in [-3, 3],\ i = 1, 2$	[10]

Table 3: Multiobjective optimization test problems.

References

[1] H. P. Benson, "An Improved Definition of Proper Efficiency for Vector Maximization with Respect to Cones," *Journal of Mathematical Anal-*

ysis and Applications 71 (1979), pp. 232–241, DOI: 10.1016/0022-247X(79)90226-9.

[2] P. Blanchard, D. J. Higham, and N. J. Higham, “Accurately Computing the Log-Sum-Exp and Softmax Functions,” *IMA Journal of Numerical Analysis* 41 (2021), pp. 2311–2330, DOI: 10.1093/imanum/draa038.

[3] J. Borwein, “Proper Efficient Points for Maximizations with Respect to Cones,” *SIAM Journal on Control and Optimization* 15 (1977), pp. 57–63, DOI: 10.1137/0315004.

[4] C. Büsing, K.-S. Goetzmann, J. Matuschke, and S. Stiller, “Reference Points and Approximation Algorithms in Multicriteria Discrete Optimization,” *European Journal of Operational Research* 260 (2017), pp. 829–840, DOI: 10.1016/j.ejor.2016.05.027.

[5] F. Chlumsky-Harttmann, “Robust Multi-Objective Optimization: Analysis and Algorithmic Approaches,” PhD thesis, Rheinland-Pfälzische Technische Universität Kaiserslautern-Landau, 2024, DOI: https://doi.org/10.26204/KLUEDO/8859.

[6] K. Deb, L. Thiele, M. Laumanns, and E. Zitzler, “Scalable Multi-Objective Optimization Test Problems,” *Proceedings of the 2002 Congress on Evolutionary Computation.* Vol. 1, 2002, pp. 825–830, DOI: 10.1109/CEC.2002.1007032.

[7] M. Ehrgott, *Multicriteria Optimization*, 2nd ed, Berlin ; New York: Springer, 2005.

[8] G. Eichfelder, *Adaptive Scalarization Methods in Multiobjective Optimization*, ed. by J. Jahn, Vector Optimization, Berlin, Heidelberg: Springer, 2008, DOI: 10.1007/978-3-540-79159-1.

[9] G. Eichfelder, P. Kirst, L. Meng, and O. Stein, “A General Branch-and-Bound Framework for Continuous Global Multiobjective Optimization,” *Journal of Global Optimization* 80 (2021), pp. 195–227, DOI: 10.1007/s10898-020-00984-y.

[10] G. Eichfelder and O. Stein, “Limit Sets in Global Multiobjective Optimization,” *Optimization* 73 (2024), pp. 1–27, DOI: 10.1080/02331934.2022.2092479.

[11] C. A. Floudas, *Deterministic Global Optimization: Theory, Methods and Applications*, Nonconvex Optimization and Its Applications, Dordrecht Boston: Kluwer Academic Publishers, 2000.

[12] C. M. Fonseca and P. J. Fleming, "An Overview of Evolutionary Algorithms in Multiobjective Optimization," *Evolutionary Computation* 3 (1995), pp. 1–16, DOI: 10.1162/evco.1995.3.1.1.

[13] W. B. Gearhart, "Compromise Solutions and Estimation of the Noninferior Set," *Journal of Optimization Theory and Applications* 28 (1979), pp. 29–47, DOI: 10.1007/BF00933599.

[14] A. M. Geoffrion, "Proper Efficiency and the Theory of Vector Maximization," *Journal of Mathematical Analysis and Applications* 22 (1968), pp. 618–630, DOI: 10.1016/0022-247X(68)90201-1.

[15] C. Glaßer, C. Reitwießner, H. Schmitz, and M. Witek, *Hardness and Approximability in Multi-Objective Optimization*, Preprint, 2010, Electronic Colloquium on Computational Complexity: TR10-031, https://eccc.weizmann.ac.il/report/2010/031/.

[16] C. J. Goh and X. Q. Yang, "Convexification of a Noninferior Frontier," *Journal of Optimization Theory and Applications* 97 (1998), pp. 759–768, DOI: 10.1023/A:1022654528902.

[17] S. Helfrich, A. Herzel, S. Ruzika, and C. Thielen, "Using Scalarizations for the Approximation of Multiobjective Optimization Problems: Towards a General Theory," *Mathematical Methods of Operations Research* 100 (2024), pp. 27–63, DOI: 10.1007/s00186-023-00823-2.

[18] S. Helfrich, K. Prinz, and S. Ruzika, "The Weighted P-Norm Weight Set Decomposition for Multiobjective Discrete Optimization Problems," *Journal of Optimization Theory and Applications* 202 (2024), pp. 1187–1216, DOI: 10.1007/s10957-024-02481-8.

[19] M. I. Henig, "Proper Efficiency with Respect to Cones," *Journal of Optimization Theory and Applications* 36 (1982), pp. 387–407, DOI: 10.1007/BF00934353.

[20] J.-B. Hiriart-Urruty and C. Lemaréchal, *Convex Analysis and Minimization Algorithms*, Grundlehren Der Mathematischen Wissenschaften, Berlin New York Paris [etc.]: Springer, 1993.

[21] B. Jiménez and V. Novo, "First and Second Order Sufficient Conditions for Strict Minimality in Nonsmooth Vector Optimization," *Journal of Mathematical Analysis and Applications* 284 (2003), pp. 496–510, DOI: 10.1016/s0022-247x(03)00337-8.

[22] I. Kaliszewski, *Quantitative Pareto Analysis by Cone Separation Technique*, Boston, MA: Springer US, 1994, DOI: 10.1007/978-1-4615-2772-5.

[23] I. Kim and O. de Weck, "Adaptive Weighted-Sum Method for Bi-Objective Optimization: Pareto Front Generation," *Structural and Multidisciplinary Optimization* 29 (2005), pp. 149–158, DOI: `10.1007/s00158-004-0465-1`.

[24] D. Li, "Convexification of a Noninferior Frontier," *Journal of Optimization Theory and Applications* 88 (1996), pp. 177–196, DOI: `10.1007/BF02192028`.

[25] D. Li and M. P. Biswal, "Exponential Transformation in Convexifying a Noninferior Frontier and Exponential Generating Method," *Journal of Optimization Theory and Applications* 99 (1998), pp. 183–199, DOI: `10.1023/A:1021708412776`.

[26] D. Li, J.-B. Yang, and M. Biswal, "Quantitative Parametric Connections between Methods for Generating Noninferior Solutions in Multiobjective Optimization," *European Journal of Operational Research* 117 (1999), pp. 84–99, DOI: `10.1016/S0377-2217(98)00018-6`.

[27] X. Lin et al., *Smooth Tchebycheff Scalarization for Multi-Objective Optimization*, Preprint, 2024, DOI: `10.48550/arXiv.2402.19078`, arXiv: `2402.19078 [cs]`.

[28] A. Meurer et al., "SymPy: Symbolic Computing in Python," *PeerJ Computer Science* 3 (2017), e103, DOI: `10.7717/peerj-cs.103`.

[29] K. Miettinen, *Nonlinear Multiobjective Optimization*, ed. by F. S. Hillier, vol. 12, International Series in Operations Research & Management Science, Boston, MA: Springer US, 1998, DOI: `10.1007/978-1-4615-5563-6`.

[30] F. Neussel and O. Stein, *On Image Space Transformations in Multiobjective Optimization*, Preprint, 2025, `https://optimization-online.org/?p=30125`.

[31] F. Neussel and O. Stein, "On Supportedness-Promoting Image Space Transformations in Multiobjective Optimization," *Optimization* (2026), DOI: `10.1080/02331934.2026.2680422`.

[32] C. Papadimitriou and M. Yannakakis, "On the Approximability of Trade-Offs and Optimal Access of Web Sources," *Proceedings 41st Annual Symposium on Foundations of Computer Science*, Redondo Beach, CA, USA: IEEE Comput. Soc, 2000, pp. 86–92, DOI: `10.1109/SFCS.2000.892068`.

[33] A. Pascoletti and P. Serafini, "Scalarizing Vector Optimization Problems," *Journal of Optimization Theory and Applications* 42 (1984), pp. 499–524, DOI: `10.1007/BF00934564`.

[34] P. K. Shukla, J. Dutta, K. Deb, and P. Kesarwani, "On a Practical Notion of Geoffrion Proper Optimality in Multicriteria Optimization," *Optimization* 69 (2020), pp. 1513–1539, DOI: 10.1080/02331934.2019.1613403.

[35] O. Stein, "A Tutorial on Properties of the Epigraph Reformulation," *EURO Journal on Computational Optimization* 13 (2025), p. 100109, DOI: 10.1016/j.ejco.2025.100109.

[36] O. Stein and M. Volk, "Generalized Polarity and Weakest Constraint Qualifications in Multiobjective Optimization," *Journal of Optimization Theory and Applications* 198 (2023), pp. 1156–1190, DOI: 10.1007/s10957-023-02256-7.

[37] P. Virtanen et al., "SciPy 1.0: Fundamental Algorithms for Scientific Computing in Python," *Nature Methods* 17 (2020), pp. 261–272, DOI: 10.1038/s41592-019-0686-2.

[38] R. Wang, Q. Zhang, and T. Zhang, "Pareto Adaptive Scalarising Functions for Decomposition Based Algorithms," *Evolutionary Multi-Criterion Optimization*, ed. by A. Gaspar-Cunha, C. Henggeler Antunes, and C. C. Coello, vol. 9018, Cham: Springer International Publishing, 2015, pp. 248–262, DOI: 10.1007/978-3-319-15934-8_17.

[39] P. Weidner, "Complete Efficiency and Interdependencies between Objective Functions in Vector Optimization," *ZOR Zeitschrift für Operations Research Methods and Models of Operations Research* 34 (1990), pp. 91–115, DOI: 10.1007/BF01415973.

[40] D. White, "Weighting Factor Extensions for Finite Multiple Objective Vector Minimization Problems," *European Journal of Operational Research* 36 (1988), pp. 256–265, DOI: 10.1016/0377-2217(88)90433-X.

[41] A. P. Wierzbicki, "On the Completeness and Constructiveness of Parametric Characterizations to Vector Optimization Problems," *Operations-Research-Spektrum* 8 (1986), pp. 73–87, DOI: 10.1007/BF01719738.

[42] P. L. Yu, "A Class of Solutions for Group Decision Problems," *Management Science* 19 (1973), pp. 936–946, DOI: 10.1287/mnsc.19.8.936.

[43] M. Zarepisheh and P. M. Pardalos, "An Equivalent Transformation of Multi-Objective Optimization Problems," *Annals of Operations Research* 249 (2017), pp. 5–15, DOI: 10.1007/s10479-014-1782-4.

[44] M. Zeleny, "Compromise Programming," *Multiple Criteria Decision Making*, ed. by M. Zeleny and J. Cochrane, Columbia, South Carolina: University of South Carolina Press, 1973, pp. 262–301.