Fixed charges of arbitrary sign: what survives and what fails

Duy T. Nguyen*
Independent Researcher, Ho Chi Minh City, Vietnam

Abstract

For integer activities, conditioning on the support makes a fixed-charge objective affine. If every support-conditioned cell is integral, an optimal solution is a vertex of its cell for arbitrary fixed charges and marginal rates. A totally unimodular constraint matrix with integral data guarantees this condition. This surviving property is weaker than the classical conclusions. A negative fixed charge rewards opening an activity at its minimum positive level, which the vertex property does not constrain. We give unique-optimum instances that violate the fractional-period property in capacitated lot sizing and the positive-support forest properties in fixed-charge transportation and flow. A four-period lot-sizing instance further has unrestricted optimum -8, while every schedule in the subplan state space used by a classical $O(T^4)$ recursion costs at least -5. Finally, the one-dimensional charge-if-positive function is piecewise concave in Zangwill's sense exactly for a non-negative fixed charge. Thus integrality preserves a weak vertex statement across the sign change, while the stronger structures and an algorithmic state space built on them can fail.

Keywords: fixed-charge problems, negative prices, total unimodularity, lot sizing, integral polyhedra

1. Introduction

What remains of classical fixed-charge structure when a charge becomes a payment? The question is too broad without a precise cost model. We study the discrete charge-if-positive function

$$\varphi_j(0) = 0, \qquad \varphi_j(x) = s_j + c_j x \quad (x > 0), \tag{1}$$

where s_j is a fixed charge and c_j is a marginal rate. Both can inherit the sign of one price: $s_j = p_j\sigma_j$ and $c_j = p_j\gamma_j$, with $\sigma_j, \gamma_j > 0$. Wholesale electricity provides the motivating case. Its price can fall below zero, and these events grow with renewable capacity [15]. The fixed charge is then received rather than paid.

The sign of s_j decides the curvature. For $x > 0$ and $0 < \lambda < 1$,

$$\varphi_j(\lambda x) - \lambda\varphi_j(x) = s_j(1-\lambda), \tag{2}$$

so φ_j is concave on $[0, u_j]$ exactly when $s_j \geq 0$. When $s_j < 0$ it lies strictly below its own chord, which makes it *convex*; what (1) loses at the origin is not convexity but *lower semicontinuity*, since $\liminf_{x\downarrow 0} \varphi_j(x) = s_j < 0 = \varphi_j(0)$.

*Corresponding author. ORCID: 0009-0004-0732-2093
Email address: duynguyen.das@gmail.com (Duy T. Nguyen)

This sign change removes two hypotheses used by the structure theory: concavity and lower semicontinuity. Florian and Klein [3] assume concavity; Love [11] and Swoveland [17] assume piecewise concavity. Koca, Yaman and Aktürk [9] assume piecewise concavity together with lower semicontinuity. Akbalik, Penz and Rapine [2] invoke Love's property explicitly under concave costs. The same model has a polyhedral literature [1], which does not use the structural property at all; and the concave case with arbitrary capacities is NP-hard [4], so the polynomial results are all for restricted capacities. Ou [13] imposes no sign restriction in his model statement, but the structural property he uses is attributed to Swoveland and is proved there for piecewise concave costs, so the hypothesis is inherited by the proof.

Two qualifications prevent an overclaim. First, [9] already give a repair: their §3.4 models minimum order quantities by setting $p_t(x) = \infty$ on $(0, L) \cup (C, \infty)$, and solves that case in $O(n^6)$. With $L = 1$ the effective domain becomes $\{0\} \cup [1, C]$, which restores lower semicontinuity and leaves a piecewise concave cost whose breakpoints $\{0, 1, C\}$ are time-invariant and fixed in number. Their polynomial algorithm requires precisely this condition. Negative charges are therefore not outside their framework; they are outside the framework as it is usually instantiated. The polynomial solvability of the $L = 1$ case in [9] is not at issue here. What fails, and what Sections 4–5 are about, is the transfer of the classical structural properties, and of the algorithms built on them, to charges of either sign. Second, [11] is closer than it first appears. His model carries production bounds $x_i^L \le x_i \le x_i^U$, and his conclusion excepts production equal to "zero or its upper or lower bound". Setting $x_i^L = 1$ gives the exception set $\{0, 1, x_i^U\}$, exactly the conclusion of Corollary 4 below. What the corollary adds is that this holds *without* concavity. Love assumes his cost concave on $[0, \infty)$, which (1) is not when $s_j < 0$. The classical property in the usual $x_i^L = 0$ setting is a different statement, and Section 4 shows it fails.

The surviving statement has a short proof. Conditioning on the support leaves an *affine* function on a bounded integral polyhedron. We claim no depth for this step. The point is that the classical route is closed, while this weaker route remains open. Proposition 5 makes the boundary exact. Zangwill [19] is the canonical reference here, and it is how [11] reaches the lot-sizing result. His theory is not confined to continuous costs: his Theorem 6 minimises any *piecewise concave* function over its dominant set, and such functions may jump. Proposition 5 shows that (1) lies in that class precisely when $s_j \ge 0$, so the sign of the charge is the exact boundary of the theory rather than an obstruction we happen to meet.

Contribution.. Four claims answer the opening question. First, Theorem 2 and Corollary 4 identify the weak vertex statement that survives arbitrary signs. Second, Section 4 gives three unique-optimum witnesses that violate the stronger classical statements. Third, Section 5 shows that the $O(T^4)$ state space of [2], correct under its assumptions, excludes the unique optimum of a four-period signed-charge instance. Finally, Proposition 6 shows why support-cell integrality is needed for our proof, and Remark 7 records the associated modelling trap.

2. The problem class

Let A be a real matrix, let b be a real right-hand side, let $u \in \mathbb{Z}^n_{>0}$, and consider

$$\text{(FC)} \qquad \min\ \Phi(x) = \sum_{j=1}^{n} \varphi_j(x_j) \quad \text{over} \quad x \in \mathbb{Z}^n, \quad Ax \lesseqgtr b, \quad 0 \le x \le u, \tag{3}$$

with φ_j as in (1) and $s_j, c_j \in \mathbb{R}$ *of arbitrary sign.* Assume throughout that (3) is feasible. Write $\operatorname{supp}(x) = \{j : x_j > 0\}$, and for $S \subseteq \{1, \dots, n\}$ define the *cell*

$$P(S) = \bigl\{x \in \mathbb{R}^n : \; Ax \lesseqqgtr b, \quad x_j = 0 \; (j \notin S), \quad 1 \le x_j \le u_j \; (j \in S)\bigr\}. \tag{4}$$

The constant 1 in (4) is where integrality enters, and it is the source of everything in Section 4: for continuous variables the set $\{x_j > 0, \; j \in S\}$ is not closed and carries no vertex guarantee.

3. The vertex theorem

Lemma 1. *Let x^* be feasible for* (3) *and $S = \operatorname{supp}(x^*)$. Then $x^* \in P(S)$; every integral point of $P(S)$ is feasible for* (3) *with support exactly S; Φ agrees on $P(S)$ with the affine function $\sum_{j\in S} s_j + \sum_{j\in S} c_j x_j$; and $P(S)$ is a non-empty bounded polyhedron.*

Proof. x^* is integral with $x^*_j \ge 1$ on S, so $x^* \in P(S)$. Conversely any $x \in P(S)$, integral or not, has $x_j \ge 1 > 0$ exactly on S, hence $\Phi(x) = \sum_{j\in S}(s_j + c_j x_j)$; the first sum is constant on $P(S)$, so this is affine there. Boundedness is immediate from $0 \le x \le u$. □

The switch at the origin prevents Φ from being convex or concave as a whole. Conditioning on the support freezes this switch: the fixed charges become a constant, and nothing non-linear remains. No curvature is needed, so no sign restriction is needed either.

Theorem 2. *Suppose every non-empty support cell $P(S)$ is integral. Problem* (3) *has an optimal solution $\hat{x}$ that is a vertex of $P(\operatorname{supp}(\hat{x}))$, for fixed charges and marginal rates of arbitrary sign.*

Proof. Let x^* be optimal and $S = \operatorname{supp}(x^*)$. By Lemma 1, $P(S)$ is a non-empty bounded polyhedron on which Φ is affine, so Φ attains its minimum over $P(S)$ at a vertex $\hat{x}$, and $\Phi(\hat{x}) \le \Phi(x^*)$. Cell integrality makes $\hat{x}$ integral, and Lemma 1 makes it feasible with support S, so $\Phi(\hat{x}) \ge \mathrm{OPT} = \Phi(x^*)$. Hence $\hat{x}$ is optimal and is a vertex of $P(S) = P(\operatorname{supp}(\hat{x}))$. □

Corollary 3. *If A is totally unimodular, b is integral, and u is integral, the hypothesis of Theorem 2 holds.*

Proof. Orient greater-than rows by multiplying them by -1. Adjoining the signed identity rows that define $x_j = 0$ and $1 \le x_j \le u_j$ preserves total unimodularity. Every right-hand side is integral, so Hoffman–Kruskal makes each non-empty $P(S)$ integral. □

Corollary 4. *Let $\hat{x}$ be as in Theorem 2, let A_{t} be the rows of A tight at $\hat{x}$, and let $Q = \{j : 1 < \hat{x}_j < u_j\}$. Then the columns of A_{t} indexed by Q are linearly independent; in particular $|Q| \le \operatorname{rank} A_{\mathrm{t}}$.*

Proof. Suppose $d \ne 0$ is supported on Q with $A_{\mathrm{t}} d = 0$. Strict interiority and the positive slack of the finitely many inactive rows give $\varepsilon > 0$ for which $\hat{x} \pm \varepsilon d$ satisfy every row and bound; coordinates in Q are strictly interior; and coordinates outside S are untouched, since $Q \subseteq S$. So $\hat{x}$ is the midpoint of a segment in $P(S)$, contradicting that it is a vertex. □

Proposition 5. *Let $X = [0, u_j]$ and let φ_j be as in* (1)*. Then φ_j is piecewise concave in the sense of [19]* if and only if $s_j \ge 0$. *His Theorem 6 therefore applies to a fixed charge that is paid and to no fixed charge that is received.*

Proof. Zangwill calls F *continuous piecewise concave* (CPC) on X when $F = \max_i g_i$ for finitely many continuous concave g_i. Its *basic sets* are $B_i = \{z \in X : F(z) = g_i(z)\}$, and its *terminal set* is $T = \bigcup_{a \neq b}(B_a \cap B_b)$. The function F is *piecewise concave* (PC) when it is the pointwise limit of CPC functions $\{F_k\}$ that share one terminal set T and satisfy $F_k = F_1$ on $E[X] \cup T$.

Suppose $\{F_k\}$ is such a sequence for φ_j. Each F_k is a maximum of finitely many continuous functions, hence continuous; and $F_k = F_1$ on $E[X] \cup T$ together with $F_k \to \varphi_j$ pointwise gives $F_1 = \varphi_j$ there.

Since $0 \in E[X]$ we get $F_1(0) = \varphi_j(0) = 0$. As F_1 is continuous and $s_j + c_j x \to s_j \neq 0$ as $x \downarrow 0$, choose $\delta \in (0, u_j)$ with $|F_1| < |s_j|/2$ and $|s_j + c_j x| > |s_j|/2$ on $[0, \delta]$. Any $x \in T \cap (0, \delta)$ would satisfy $F_1(x) = \varphi_j(x) = s_j + c_j x$, violating both bounds; hence $T \cap (0, \delta) = \emptyset$.

Fix k. Restricted to $(0, \delta)$, the basic sets form a finite, disjoint, relatively closed cover. Connectedness therefore leaves exactly one non-empty member, so F_k agrees there with a single continuous concave g^k. By continuity, $g^k(0) = F_k(0) = F_1(0) = 0$. Concavity with $g^k(0) = 0$ gives, for $x \in (0, \delta)$ and $\lambda \in (0, 1)$,

$$g^k(\lambda x) \;\geq\; \lambda g^k(x) + (1 - \lambda) g^k(0) \;=\; \lambda g^k(x).$$

Letting $k \to \infty$ and using $g^k \to \varphi_j$ pointwise on $(0, \delta)$,

$$s_j + c_j \lambda x \;\geq\; \lambda(s_j + c_j x), \qquad \text{that is} \qquad s_j(1 - \lambda) \;\geq\; 0,$$

so $s_j \geq 0$, contradicting $s_j < 0$.

Conversely let $s_j \geq 0$, put $K_0 = s_j/u_j + |c_j| + 1$ and

$$F_k(x) \;=\; \min\bigl\{(k + K_0)x,\; s_j + c_j x\bigr\}, \qquad k \geq 1.$$

Each F_k is a minimum of two affine functions, hence continuous and concave, hence CPC as a family with the single member F_k itself. That family has no second member, so no two basic sets meet and the terminal set is $\emptyset$ for every k; condition (2) then only asks for agreement on $E[X] = \{0, u_j\}$. There $F_k(0) = \min\{0, s_j\} = 0 = \varphi_j(0)$, using $s_j \geq 0$, and $(k + K_0)u_j > s_j + c_j u_j$ for every $k \geq 1$ by the choice of K_0, so $F_k(u_j) = s_j + c_j u_j$ independently of k. For fixed $x \in (0, u_j]$, the inequality $(k + K_0)x > s_j + c_j x$ holds once k is large. Hence $F_k(x) \to s_j + c_j x$, while $F_k(0) = 0$. Thus $F_k \to \varphi_j$ pointwise, and $\{F_k\}$ is a defining sequence. □

The sign of the fixed charge is the boundary of the class covered by Zangwill's theory. The classical route applies to a paid charge and closes when the charge is received.

4. What does not survive

Losing a hypothesis is not losing a conclusion. A received charge is not piecewise concave, so [3, 11] and the recursions built on them no longer *apply*; that alone would leave the classical properties unproven rather than false. What follows is stronger: in each setting the conclusion itself is violated.

Corollary 4 constrains only the coordinates with $1 < \hat{x}_j < u_j$. The classical structural properties constrain those with $0 < \hat{x}_j < u_j$. Under negative charges this difference is decisive: opening is rewarded and positive volumes can be driven to their minimum level. Each witness below has a *unique* optimum, so no choice of optimal solution repairs the classical statement, and each respects the coupling $s_j = p_j \sigma_j$, $c_j = p_j \gamma_j$ with $\sigma_j, \gamma_j > 0$ of Section 1. Thus the failures do not arise from letting the fixed charge and the marginal rate carry opposite signs.

Lot sizing.. Let x_t be production in period t, $u_t = C$, and let the rows of A be the prefix sums bounded below by cumulative demand and above by cumulative demand plus a storage bound; A is an interval matrix, hence totally unimodular. Take $T = 3$, $C = 3$, zero demand, storage bound 2, and prices $p = (-1, -1, 1)$ with $\sigma_t = \gamma_t = 1$, so $s_t = c_t = p_t$. The unique optimum is $x = (1, 1, 0)$, with prefix sums $(1, 2, 2)$ and value -4. Rows 2 and 3 are tight and row 1 is not, so no inventory point separates periods 1 and 2, both of which produce strictly between 0 and C: the fractional-period property of [3, 11] fails on the unique optimum. Here $Q = \emptyset$, so Corollary 4 holds vacuously.

Fixed-charge transportation.. Let A be the 2×2 bipartite incidence matrix with supplies and demands $(2, 2)$, $u_{ij} = 2$ and $s_{ij} = c_{ij} = -1$. The unique optimum, of value -8, sends one unit on each of the four arcs. No arc is strictly interior, so again Corollary 4 is vacuous, while the four positive arcs exceed the classical bound $m + n - 1 = 3$ and form a cycle.

Fixed-charge min-cost flow.. Read the same instance as a four-node flow problem. Its four positive arcs exceed the spanning-tree bound $n - 1 = 3$.

So the sign hypothesis cannot be removed from the classical statements; it can only be removed from the vertex statement, which is weaker. Section 5 shows the difference is not academic.

5. A classical state space that does not transfer

Corollary 4 might suggest that an algorithm based only on a vertex argument transfers to arbitrary signs. The recursion of [2] gives a concrete test: it uses the stronger classical property rather than the surviving vertex property.

The recursion of [2] is $O(T^4)$ and is correct under the concave costs it assumes. Its state space does not, however, transfer unchanged to arbitrary signs. It is built directly on Love's property, which it states, after [11], in terms of periods that are *fractional* in the classical sense $0 < x_t < P$: within a subplan, every period produces either zero or P, apart from at most one fractional period. Take $T = 4$, $C = 5$, storage bound 4, demand $(0, 0, 0, 4)$, zero initial and terminal inventory, and $s_t = c_t = -1$. Conservation fixes total production at four, so every feasible plan costs $-4 - |\operatorname{supp}(x)|$. The unique optimum is $(1, 1, 1, 1)$, of value -8. Before final demand, inventory cannot return to zero after positive production; reaching the upper bound four also consumes all production. A plan decomposable into the subplans of [2] therefore has at most one positive period, necessarily of size four, and costs -5. Thus the state space cannot express the optimum outside its concavity assumptions; the absolute gap is three cost units.

6. Support-cell integrality cannot be discarded

Proposition 6. *There is an instance of* (3) *with a non-integral support cell whose unique optimum is not a vertex of that cell, so the conclusion of Theorem 2 fails.*

Proof. Take the odd-cycle system $x_1 + x_2 \leq 6$, $x_2 + x_3 \leq 7$, $x_1 + x_3 \leq 6$, whose constraint matrix has determinant 2, with $u = (5, 5, 5)$, $s = (1.32,\ 2.51,\ 0.55)$ and $c = (-2.02,\ -2.06,\ -1.19)$. Exhaustive enumeration over the 113 feasible integer points gives the unique optimum $\hat{x} = (2, 4, 3)$ of value -11.47, the runner-up being $(3, 3, 3)$ at -11.43. Its support is $\{1, 2, 3\}$, so its cell is $\{x :\ Ax \leq b,\ 1 \leq x \leq 5\}$. The direction $d = (1, -1, 1)$ preserves the first two rows, and $\hat{x} \pm d/2$ satisfy the third row and every cell bound. These two distinct cell points have midpoint $\hat{x}$, so $\hat{x}$ is not a vertex. □

This shows that support-cell integrality cannot be removed without replacement; it does not make total unimodularity necessary.

Remark 7 (A modelling consequence). The natural mixed-integer formulation of (1) writes $x_j \leq u_j y_j$ with y_j binary. For $s_j \geq 0$ that suffices, because setting $y_j = 1$ with $x_j = 0$ only adds cost. For $s_j < 0$ it does not: the solver sets $y_j = 1$ and $x_j = 0$ and banks the charge. Over integer x the constraint $y_j \leq x_j$ is required. Consider four periods with prices $(1, -5, -5, -5)$, two units due by the horizon, $u = 2$, and unit coefficients. The guarded formulation returns -20, whereas the unguarded one returns -25. The latter creates an artificial credit of 5 by "using" idle activities to collect the charge. The object it optimises is the lower semicontinuous hull of φ_j, which is the function [9] assume and (1) is not.

The unguarded pattern is the standard one. [2] write $0 \leq x_t \leq P y_t$ with no lower link, and [10] write $x_t \leq M y_t$ having just *defined* "$y_t = 1$ if $x_t > 0$, and 0 otherwise", a two-sided definition implemented by a one-sided constraint. Both are correct in their own settings, where the setup cost is non-negative and $y_t = 1$ with $x_t = 0$ is never optimal; we report no error in either.

Two qualifications keep this in proportion. First, energy models already use the repair. The minimum-load constraint $p_t \geq P^{\min} z_t^{\mathrm{on}} + P^{\mathrm{sb}} z_t^{\mathrm{sb}}$ in [14] forbids an activated unit from consuming nothing. What is missing from the lot-sizing formulations above is a constraint the energy literature writes as a matter of course. Second, the general shape is widespread and not confined to (1). The same paper [14], whose numerical study is explicitly about nonpositive prices, defines its start-up binary by $z_t^{\mathrm{su}} \geq z_{t-1}^{\mathrm{off}} + z_t^{\mathrm{on}} + z_t^{\mathrm{sb}} - 1$ with no upper pin, which is correct only because the start-up cost is a positive constant rather than a price. Our point is about (1); the pattern is broader.

7. Computational verification

The supplement reproduces every finite witness by exhaustive enumeration. This includes the instance in Section 5. It also checks the vertex theorem on 1,085 independently generated matrices verified to be totally unimodular. A negative control allows supports to vary and breaks the affine identity on 217 of 220 instances. These computations test the implementations and transcriptions; the general claims follow from the proofs above.

The supplement also contains a Lean 4 formalization of the support-conditioning identity, the TU support-cell integrality argument, the vertex theorem, and the active-column corollary. The TU proof selects a nonsingular active minor and derives integrality from its unit determinant. The formal theorem takes an integer TU constraint matrix, integral right-hand sides and upper bounds, and feasibility as inputs. It produces an optimal integer solution that is a vertex of its support cell. The general theorem chain contains no native evaluation or project-added axioms.

8. Concluding remarks

The opening question has a narrow answer. A vertex optimum survives arbitrary signs after conditioning on an integral support cell. The stronger classical structures do not. Zangwill's class ends at the same sign change.

The vertex result follows from known integrality. It continues the vertex viewpoint of Hirsch and Hoffman [6], Hirsch and Dantzig [5], and Murty [12]. It also relates to results for nonconvex production costs [8, 16] and implied integrality [7]. We claim no new source of integrality.

Signed setup coefficients also appear in Van Hoesel and Wagelmans: [18] allow a negative setup cost and note it is then profitable to set up even with no production, absorbing such terms as constants. Their formulation lets a setup be chosen while $x = 0$, which is a different object from the charge-if-positive function studied here.

The stronger classical statements fail in all three settings. A published polynomial recursion, correct under its own hypotheses, loses optimality when its state space is transferred unchanged to a four-period signed-charge instance. The practical conclusion is direct. A negative fixed charge changes which plans are optimal: few-and-large can become many-and-minimal.

The intrinsic condition in Theorem 2 is integrality of the support cells; characterising broader matrix classes that guarantee it is left open. Of the formulations we checked, the two lot-sizing ones [2, 10] omit the guard of Remark 7. This omission is harmless under their assumptions but unsafe under signed charges. The energy model [14] carries its continuous analogue as a minimum-load constraint. The exposure we describe is therefore a property of how these models would be combined, rather than one we exhibit in print.

References

[1] A. Atamtürk, S. Küçükyavuz, Lot sizing with inventory bounds and fixed costs: polyhedral study and computation, *Operations Research* 53 (4) (2005) 711–730.

[2] A. Akbalik, B. Penz, C. Rapine, Capacitated lot sizing problems with inventory bounds, *Annals of Operations Research* 229 (1) (2015) 1–18.

[3] M. Florian, M. Klein, Deterministic production planning with concave costs and capacity constraints, *Management Science* 18 (1) (1971) 12–20.

[4] M. Florian, J. K. Lenstra, A. H. G. Rinnooy Kan, Deterministic production planning: algorithms and complexity, *Management Science* 26 (7) (1980) 669–679.

[5] W. M. Hirsch, G. B. Dantzig, The fixed charge problem, *Naval Research Logistics Quarterly* 15 (3) (1968) 413–424.

[6] W. M. Hirsch, A. J. Hoffman, Extreme varieties, concave functions, and the fixed charge problem, *Communications on Pure and Applied Mathematics* 14 (3) (1961) 355–369.

[7] R. van der Hulst, M. Walter, Implied integrality in mixed-integer optimization, *Mathematical Programming* (2026), in press.

[8] R. Jagannathan, M. R. Rao, A class of deterministic production planning problems, *Management Science* 19 (11) (1973) 1295–1300.

[9] E. Koca, H. Yaman, M. S. Aktürk, Lot sizing with piecewise concave production costs, *INFORMS Journal on Computing* 26 (4) (2014) 767–779.

[10] C.-Y. Lee, S. Çetinkaya, A. P. M. Wagelmans, A dynamic lot-sizing model with demand time windows, *Management Science* 47 (10) (2001) 1384–1395.

[11] S. F. Love, Bounded production and inventory models with piecewise concave costs, *Management Science* 20 (3) (1973) 313–318.

[12] K. G. Murty, Solving the fixed charge problem by ranking the extreme points, *Operations Research* 16 (2) (1968) 268–279.

[13] J. Ou, Improved exact algorithms to economic lot-sizing with piecewise linear production costs, *European Journal of Operational Research* 256 (3) (2017) 777–784.

[14] E. Raheli, Y. Werner, J. Kazempour, A conic model for electrolyzer scheduling, *Computers & Chemical Engineering* 179 (2023) 108450.

[15] J. Seel, D. Millstein, A. Mills, M. Bolinger, R. Wiser, Plentiful electricity turns wholesale prices negative, *Advances in Applied Energy* 4 (2021) 100073.

[16] S. Sadagopan, A. Ravindran, A vertex ranking algorithm for the fixed-charge transportation problem, *Journal of Optimization Theory and Applications* 37 (2) (1982) 221–230.

[17] C. Swoveland, A deterministic multi-period production planning model with piecewise concave production and holding-backorder costs, *Management Science* 21 (9) (1975) 1007–1013.

[18] C. P. M. Van Hoesel, A. P. M. Wagelmans, Sensitivity analysis of the economic lot-sizing problem, *Discrete Applied Mathematics* 45 (3) (1993) 291–312.

[19] W. I. Zangwill, The piecewise concave function, *Management Science* 13 (11) (1967) 900–912.