Sharp Singularity-Degree Bounds for Equality-Generated SDP–RLT Relaxations of Binary Programs

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Abstract

Singularity degree is an important measure of semidefinite programming (SDP) degeneracy, but it is generally unavailable a priori from the problem data. We augment the Shor relaxation of binary sets $\{x \in \{0,1\}^n : Ax = b\}$ with the first-level Reformulation–Linearization Technique (RLT) equations generated by the defining linear equalities. For the resulting equality-generated SDP–RLT relaxation, we determine the exact worst-case singularity degree. If $\operatorname{rank}(A) = m$ and $0 < m < n$, then the associated relaxation has singularity degree at most $\min\{m, n-m\}$, and this rank–nullity bound is attained for every possible rank in this range. Consequently, the worst-case singularity degree over this class is $\lfloor n/2 \rfloor$ for $n \geq 2$. This is strikingly smaller than the sharp general bound n for feasible SDP systems with matrix variables of order $n+1$ [13, Example 2]. Thus, for individual relaxations, rank and nullity provide an a priori bound on the otherwise inaccessible singularity degree and on the Hölder exponent in error bounds estimating distance to feasibility from constraint residuals.

Key Words: semidefinite programming, binary optimization, facial reduction, singularity degree, Shor relaxation, Reformulation–Linearization Technique

1 Introduction

The singularity degree of a semidefinite program quantifies its failure of Slater's condition: it is the minimum number of facial-reduction steps needed to reach the minimal face containing the feasible set. Although it governs the worst-case Hölder behavior of error bounds and is closely connected with the numerical sensitivity of degenerate SDPs [12,13], it is generally an a posteriori parameter. Determining it generally requires analyzing the facial-reduction certificates leading to the minimal face, and no tractable procedure is known for general spectrahedra [11]. The maximum singularity degree is the largest possible length of a facial-reduction sequence, and computing it is NP-hard for general SDPs [5]. Singularity degree is therefore used primarily to explain pathological behavior, rather than as a quantity that is readily computable from the problem data.

This makes sharp a priori bounds for important structured classes especially valuable: even when the singularity degree of a particular instance is unknown, a uniform bound for the class determines in advance its largest possible value. Through existing error-bound theory, such a bound supplies an a priori bound on the singularity-degree exponent governing the Hölder relationship

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between constraint residuals and distance to feasibility. An important structured class consists of equality-generated SDP relaxations based on the Reformulation–Linearization Technique (RLT) [9, 10]. These relaxations offer a practical modeling choice when the Shor relaxation provides insufficient bounds but imposing the full collection of McCormick inequalities is too costly. We therefore seek a sharp upper bound on the singularity degree of these relaxations.

To formalize this question, let

$$P(A, b) := \{x \in \{0, 1\}^n \mid Ax = b\}$$

be the binary set defined by the linear equalities $Ax = b$. We define its equality-generated SDP–RLT relaxation by

$$\mathcal{R}(A, b) := \left\{ Y = \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0 \;\middle|\; \operatorname{diag}(X) = x,\ Ax = b,\ AX = bx^T \right\}.$$

For a fixed number n of binary variables, we study the extremal problem

$$\max_{A,b\colon P(A,b)\neq\emptyset} \operatorname{sd}\big(\mathcal{R}(A, b)\big). \tag{1}$$

We call the value in (1) the *worst-case singularity degree* of the equality-generated SDP–RLT class in n binary variables. The maximization ranges over affine descriptions (A, b) satisfying $P(A, b) \neq \emptyset$, with each description inducing the relaxation $\mathcal{R}(A, b)$. Different descriptions of the same binary set may induce relaxations with different singularity degrees.

Our main result gives the exact answer: the value in (1) is

$$\begin{cases} 1, & n = 1, \\ \lfloor n/2 \rfloor, & n \geq 2. \end{cases}$$

The result has a sharper rank–nullity form. Let $m = \operatorname{rank}(A)$. For $0 < m < n$,

$$\operatorname{sd}\big(\mathcal{R}(A, b)\big) \leq \min\{m, n - m\},$$

and this bound is attained for every possible rank m in this range. Therefore, $\min\{m, n - m\}$ is the best possible uniform bound based only on rank and nullity. Maximizing it over m gives $\lfloor n/2 \rfloor$, which occurs when rank and nullity are as nearly balanced as possible. For $n \geq 2$, this worst-case value can be attained by a homogeneous system with $P(A, 0) = \{0\}$. Thus even a singleton binary set can yield a lifted relaxation with singularity degree $\lfloor n/2 \rfloor$.

The proof homogenizes the affine equations and uses a nullspace parametrization to obtain a reduced formulation whose matrix variable has order $r+1$, where $r := n-m$ is the nullity of A. For this formulation, we prove the stronger statement that its maximum singularity degree is at most $\min\{m, r\}$. Consequently, every facial-reduction sequence of the reduced formulation has length at most $\min\{m, r\}$, independently of the choice of exposing matrices. This stronger bound for the reduced formulation is the key mechanism behind the singularity-degree bound for the original relaxation $\mathcal{R}(A, b)$.

Relation to prior work. Facial reduction was introduced by Borwein and Wolkowicz [1,2], and Sturm introduced singularity degree and established its connection with error bounds [13]. Later developments include Pataki's treatment of facial reduction and extended duality [8], as well as the iteration bounds of Lourenço, Muramatsu, and Tsuchiya based on partial polyhedrality [7]. Facial reduction for structured conic and polynomial optimization problems is studied in [3, 12, 16]. For SDP relaxations of nonconvex sets, Tunçel relates the existence of Slater points to the dimension and affine hull of the original set [15].

Several results derive singularity-degree bounds from problem structure. For positive semidefinite matrix completion, graph structure controls the degree [14]. For SDP relaxations of binary sets defined by linear equalities, the basic Shor relaxation has singularity degree at most one [6]. By contrast, general order-$(n+1)$ SDP systems can attain singularity degree n [13, Example 2]. Our theorem gives an analogous structure-dependent bound for equality-generated SDP–RLT relaxations and places their worst-case singularity degree between these two benchmarks.

The relaxations studied here augment the Shor relaxation with the standard first-level RLT equations generated by linear equalities [9, 10]. Recent work has also used facial reduction to regularize SDP–RLT relaxations with linear equality constraints [17] and has studied $AX = 0$ facial constraints arising from equality-generated RLT and moment–SOS relaxations [4]. These works do not determine the singularity degree of the resulting formulations; the present paper establishes its sharp class-wide rank–nullity bound in the binary setting. Sturm's example, by contrast, does not arise from an SDP relaxation of a binary program and therefore does not resolve the present question. The sharpness construction developed in this paper yields a family of equality-generated first-level SDP–RLT relaxations of linearly constrained binary sets whose singularity degree grows linearly with the number of variables.

Organization. Section 2 introduces notation and the facial-reduction terminology used throughout. Section 3 defines the equality-generated SDP–RLT relaxation, derives its reduced formulation, and relates the facial-reduction sequences of the two systems. Section 4 develops the Vandermonde–Hankel construction that attains the worst-case value. Section 5 proves the sharp rank–nullity upper bound, resolves (1), and establishes attainment for every rank–nullity pair. The appendix proves a linear-span identity used to compare the original and reduced formulations.

2 SDP relaxations of binary programs and facial reduction

2.1 Notation

For a positive integer p, let $\mathbb{R}^p$ be Euclidean space, $\mathbb{S}^p$ the space of real symmetric $p \times p$ matrices, and $\mathbb{S}^p_+$ and $\mathbb{S}^p_{++}$ the positive semidefinite and positive definite cones, respectively. We use $\langle X, Y \rangle = \operatorname{tr}(XY)$ on symmetric matrix spaces. We label the coordinates of $\mathbb{R}^p$ by $1, \ldots, p$ and those of its homogenization $\mathbb{R}^{p+1}$ by $0, 1, \ldots, p$. When the ambient space is clear, e_i denotes the standard unit vector associated with coordinate i; in the homogenized space, e_0 corresponds to the homogenizing coordinate, while $e_1, \ldots, e_p$ correspond to the original coordinates. We use the symmetric basis

$$E_{ij} := \frac{1}{2}(e_i e_j^T + e_j e_i^T) \quad (i \neq j), \qquad E_{ii} := e_i e_i^T. \tag{2}$$

For a square matrix X, $\operatorname{diag}(X)$ is its diagonal vector. For a matrix or linear map T, $\operatorname{range}(T)$ and $\ker(T)$ denote its range and kernel, and T^* denotes the adjoint. For a cone K in an inner-product space, $K^* := \{z \mid \langle z, x\rangle \geq 0 \text{ for every } x \in K\}$ denotes its dual cone. For a set S in an inner-product space,

$$S^\perp := \{z \mid \langle z, s\rangle = 0 \text{ for every } s \in S\} = \operatorname{span}(S)^\perp.$$

2.2 Facial reduction and singularity degree

Let L be an affine subspace with $L \cap \mathbb{S}^p_+ \neq \emptyset$. Slater's condition is $L \cap \mathbb{S}^p_{++} \neq \emptyset$. If it fails, a theorem of the alternative yields a nonzero matrix $W \in L^\perp \cap \mathbb{S}^p_+$. Its orthogonal hyperplane exposes a proper face of the current PSD cone that still contains $L \cap \mathbb{S}^p_+$ [1,2].

Every face of $\mathbb{S}^p_+$ can be written as

$$F = \{Y \succeq 0 \mid \operatorname{range}(Y) \subseteq \mathcal{V}\} = \{VRV^T \mid R \in \mathbb{S}^r_+\},$$

where the columns of $V \in \mathbb{R}^{p\times r}$ span $\mathcal{V}$. Let F be a current face containing $L \cap \mathbb{S}^p_+$. A matrix $W \in L^\perp \cap (F^* \setminus F^\perp)$ is an exposing matrix for a facial-reduction step on F. The condition $W \in L^\perp$ ensures that the new face still contains the feasible set. Moreover, $W \in F^*$ gives $V^TWV \succeq 0$, while $W \notin F^\perp$ makes this matrix nonzero. Hence $F \cap W^\perp$ is a proper face of F. A *facial-reduction sequence of length* d is constructed as follows. Set $F_0 := \mathbb{S}^p_+$. For $j = 1, \ldots, d$, the jth facial-reduction step selects an exposing matrix

$$W_j \in L^\perp \cap (F^*_{j-1} \setminus F^\perp_{j-1})$$

and set $F_j := F_{j-1} \cap W_j^\perp$. The terminal condition is $F_d = F_{\min}$, where $F_{\min}$ is the minimal face containing the feasible set. Consequently,

$$\mathbb{S}^p_+ = F_0 \supsetneq F_1 \supsetneq \cdots \supsetneq F_d = F_{\min}.$$

Thus every step properly reduces the current face. A *partial facial-reduction sequence* satisfies the same step conditions but is not required to satisfy the terminal condition $F_d = F_{\min}$.

More generally, if F is a face of $\mathbb{S}^p_+$ and $L \cap F \neq \emptyset$, a facial-reduction sequence for $L \cap F$ is defined by taking $F_0 := F$ and applying the same step condition. Its terminal face is the minimal face of F containing $L \cap F$.

The minimum possible length d of a facial-reduction sequence is the singularity degree, denoted by $\operatorname{sd}(L \cap \mathbb{S}^p_+)$. The maximum possible length of a facial-reduction sequence is the *maximum singularity degree*, denoted by $\operatorname{MSD}(L \cap \mathbb{S}^p_+)$ [5, Section 2.1]. Therefore,

$$\operatorname{sd}(L \cap \mathbb{S}^p_+) \leq \operatorname{MSD}(L \cap \mathbb{S}^p_+). \tag{3}$$

3 Equality-generated SDP–RLT relaxations and facial-reduction tools

3.1 Definition and omitted redundant equations

Let $A \in \mathbb{R}^{q\times n}$ and let $b \in \mathbb{R}^q$. Recall that the associated linearly constrained binary set is

$$P := P(A, b) = \{x \in \{0,1\}^n \mid Ax = b\}. \tag{4}$$

Its equality-generated SDP–RLT relaxation is

$$\mathcal{R}(A,b) := \left\{ Y = \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathbb{S}^{n+1}_+ \;\middle|\; \operatorname{diag}(X) = x, \quad \begin{pmatrix} -b & A \end{pmatrix} Y = 0 \right\}. \tag{5}$$

We index the rows and columns of Y by $0, 1, \ldots, n$, where index 0 corresponds to the constant monomial. Thus, $Y_{00} = 1$ is the normalization equation, and $\operatorname{diag}(X) = x$ is equivalently the family of binary arrow equations $Y_{ii} = Y_{0i}$, $i = 1, \ldots, n$. The matrix equation in (5) is equivalent to $Ax = b$ and $AX = bx^T$. These equations have the standard degree-one RLT interpretation. Indeed, for every row a_i^T of A and every $j = 1, \ldots, n$, multiplying $a_i^T x = b_i$ by x_j and linearizing gives

$$(AX)_{ij} = b_i x_j.$$

Throughout the remainder of the paper, we assume that $P \neq \emptyset$. Every $x \in P$ yields the feasible rank-one matrix $(1, x^T)^T (1, x^T) \in \mathcal{R}(A,b)$, so $\mathcal{R}(A,b) \neq \emptyset$. The same assumption supplies the binary feasible point used in the reduction to homogeneous equations in the next subsection.

The following operations leave both the SDP relaxation and its singularity and maximum singularity degrees unchanged because they do not change the span of the homogeneous constraint matrices:

- including the RLT equations generated using $1 - x_j$;
- including the linearized products between pairs of rows of $Ax = b$;
- removing redundant rows of $Ax = b$.

We therefore retain none of these redundant equations and assume that $A \in \mathbb{R}^{m \times n}$ has full row rank, where $m = \operatorname{rank}(A)$. By contrast, the McCormick inequalities in the full first-level SDP–RLT relaxation generally strengthen the relaxation and, when represented using nonnegative slacks, produce a conic formulation over the product of the positive semidefinite cone and a nonnegative orthant. Their singularity degree requires a separate analysis.

3.2 Reduction of affine equations to homogeneous form

We next apply the standard *switching operation*, also called binary-variable complementation, to move a binary feasible point to the origin without leaving the binary cube; see, e.g., [18]. Fix $\overline{x} \in P$. We switch precisely the variables for which $\overline{x}_i = 1$: if $\overline{x}_i = 0$, set $w_i = x_i$, whereas if $\overline{x}_i = 1$, set $w_i = 1 - x_i$. Equivalently, set

$$D := \operatorname{Diag}(\mathbf{1} - 2\overline{x}), \qquad w = D(x - \overline{x}), \qquad x = \overline{x} + Dw.$$

Thus $x \in \{0,1\}^n$ if and only if $w \in \{0,1\}^n$, and $\overline{x}$ is mapped to $w = 0$. Define

$$\widehat{A} := AD, \qquad \widehat{P} := \{ w \in \{0,1\}^n \mid \widehat{A} w = 0 \}.$$

Since $A\overline{x} = b$,

$$Ax = b \quad \Longleftrightarrow \quad \widehat{A} w = 0.$$

Hence the change of variables maps P bijectively onto $\widehat{P}$, whose equality-generated SDP–RLT relaxation is $\mathcal{R}(\widehat{A}, 0)$. The next lemma shows that this change of variables extends to an automorphism of the PSD cone that maps the affine subspace defined by the equality constraints of $\mathcal{R}(\widehat{A}, 0)$ onto that defined by the equality constraints of $\mathcal{R}(A,b)$. Consequently, it preserves facial-reduction sequences and singularity degree.

Lemma 3.1. *With the notation above,*

$$\mathrm{sd}(\mathcal{R}(A, b)) = \mathrm{sd}(\mathcal{R}(\widehat{A}, 0)).$$

Proof. Set

$$Q := \begin{pmatrix} 1 & 0 \\ \overline{x} & D \end{pmatrix}, \qquad \Psi(\widehat{Y}) := Q\widehat{Y}Q^T.$$

The diagonal entries of D belong to $\{-1, 1\}$, so Q is invertible. Hence Ψ is an invertible linear map.

Let $\widehat{L}$ and L denote the affine subspaces defined by the equality constraints of $\mathcal{R}(\widehat{A}, 0)$ and $\mathcal{R}(A, b)$, respectively. We show that $\Psi(\widehat{L}) = L$. For $Y := \Psi(\widehat{Y})$ and every $H \in \mathbb{S}^{n+1}$,

$$\langle H, Y\rangle = \langle Q^T HQ, \widehat{Y}\rangle. \tag{6}$$

Thus the normalization and arrow equations can be compared by transforming their coefficient matrices by $H \mapsto Q^T HQ$. Because $\overline{x}_i \in \{0, 1\}$ and $D_{ii} = 1 - 2\overline{x}_i$, direct calculation gives

$$Q^T E_{00} Q = E_{00}, \qquad Q^T (E_{ii} - E_{0i})Q = E_{ii} - E_{0i} \quad (i = 1, \ldots, n).$$

Thus the normalization equation and the binary arrow equations, including their right-hand sides, correspond exactly under Ψ.

Let $B := [-b \;\; A]$ and $\widehat{B} := [0 \;\; \widehat{A}]$. Then

$$BQ = \begin{pmatrix} -b + A\overline{x} & AD \end{pmatrix} = \widehat{B}.$$

Consequently, $BY = BQ\widehat{Y}Q^T = \widehat{B}\widehat{Y}Q^T$. Since Q^T is invertible, $BY = 0$ if and only if $\widehat{B}\widehat{Y} = 0$. Together with the identities for the normalization and arrow equations, this proves $\Psi(\widehat{L}) = L$.

Since Ψ maps the PSD cone onto itself and $\Psi(\widehat{L}) = L$, the standard invariance of facial reduction under cone automorphisms gives a length-preserving bijection between the facial-reduction sequences of the two formulations. They therefore have equal singularity degree. $\square$

The proof also shows that $\mathcal{R}(A, b) = \Psi(\mathcal{R}(\widehat{A}, 0))$. Hence every affine system with a binary feasible point can be reduced to a homogeneous system by complementing binary variables. We therefore work below with $b = 0$ and reuse A for the resulting homogeneous constraint matrix.

3.3 Reduction to the facially reduced formulation

This subsection shows that one facial-reduction step for $\mathcal{R}(A, 0)$ exposes the face forced by $Ax = 0$ and $AX = 0$. Parametrizing this face produces the smaller system $\mathcal{T}(C)$, whose constraints are precisely the remaining normalization and arrow equations. We first establish the exact relation between the two formulations and then compare their singularity degrees. This explains why the subsequent analysis may focus on $\mathcal{T}(C)$.

Assume in this subsection that $m \geq 1$, so $A \neq 0$. If $m = 0$, there are no linear equations and $\mathcal{R}(0, 0)$ is strictly feasible, so its singularity degree is zero. Indeed, take $x = \frac{1}{2}\mathbf{1}$ and $X = \frac{1}{4}\mathbf{1}\mathbf{1}^T + \frac{1}{4}I$. Then $\mathrm{diag}(X) = x$, and the Schur complement is $X - xx^T = \frac{1}{4}I \succ 0$.

Let $A \in \mathbb{R}^{m\times n}$ have full row rank and set

$$r := n - m.$$

Let $C \in \mathbb{R}^{n\times r}$ have full column rank with

$$\ker A = \operatorname{range}(C).$$

Thus $m = \operatorname{rank} A$ is the number of independent equations in $Ax = 0$, $r = \dim\ker A$ is the reduced dimension, and $n = m + r$. We abbreviate

$$\mathcal{R}(A) := \mathcal{R}(A, 0) = \left\{ \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathbb{S}^{n+1}_+ \;\middle|\; \operatorname{diag}(X) = x, \quad Ax = 0, \quad AX = 0 \right\}. \tag{7}$$

We introduce the following notation for an arbitrary matrix H. For any integer $q \geq 0$ and any $h \in \mathbb{R}^q$, define

$$\Phi(h) := \begin{pmatrix} 0 & -\frac{1}{2}h^T \\ -\frac{1}{2}h & hh^T \end{pmatrix}$$

and, for any matrix $H \in \mathbb{R}^{n\times q}$ with rows h_i^T, set

$$\mathcal{T}(H) := \left\{ Z \in \mathbb{S}^{q+1}_+ \;\middle|\; Z_{00} = 1, \quad \langle \Phi(h_i), Z\rangle = 0, \quad i = 1, \dots, n \right\}. \tag{8}$$

Writing the rows of C as c_i^T, the choice $H = C$ defines the candidate reduced system $\mathcal{T}(C)$. To identify it with the formulation obtained after the facial-reduction step constructed below, set

$$V := \begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix}, \qquad \widetilde{A} := \begin{pmatrix} 0 & A \end{pmatrix}, \qquad W := \widetilde{A}^T\widetilde{A} = \begin{pmatrix} 0 & 0 \\ 0 & A^TA \end{pmatrix}. \tag{9}$$

Lemma 3.2. *The matrix W is an exposing matrix for a first facial-reduction step of $\mathcal{R}(A)$. The face exposed by this step is*

$$F_A := \mathbb{S}^{n+1}_+ \cap W^\perp = \{VZV^T : Z \in \mathbb{S}^{r+1}_+\}.$$

Under the parametrization $Z \mapsto VZV^T$, restricting the defining SDP system to F_A yields $\mathcal{T}(C)$. In particular,

$$\mathcal{R}(A) = \{VZV^T : Z \in \mathcal{T}(C)\}. \tag{10}$$

Proof. The equations $Ax = 0$ and $AX = 0$ are precisely

$$Y\widetilde{A}^T = 0.$$

The linear span of their constraint matrices contains $W = \widetilde{A}^T\widetilde{A}$. Since $W \succeq 0$ and $W \neq 0$, it is an exposing matrix for a first facial-reduction step of $\mathcal{R}(A)$.

It remains to identify the exposed face. Since $\ker A = \operatorname{range}(C)$,

$$\ker(\widetilde{A}) = \operatorname{range}(V) = \ker(W).$$

The characterization of faces of $\mathbb{S}^{n+1}_+$ therefore gives

$$\mathbb{S}^{n+1}_+ \cap W^\perp = \{VZV^T : Z \in \mathbb{S}^{r+1}_+\} = F_A.$$

Hence every $Y \in \mathcal{R}(A)$ can be written as $Y = VZV^T$ for some $Z \succeq 0$.

The preceding argument proves the assertions about W and F_A. It remains to identify the restricted SDP system as $\mathcal{T}(C)$. Since $\widetilde{A}V = 0$, every constraint matrix H associated with $Ax = 0$ and $AX = 0$ satisfies $V^T H V = 0$. It therefore suffices to compare the normalization and arrow equations under this parametrization. Using the symmetric basis from (2), a direct calculation gives

$$V^T(E_{ii} - E_{0i})V = \Phi(c_i).$$

Hence, for $Y = VZV^T$,

$$Y_{ii} - Y_{0i} = \langle E_{ii} - E_{0i}, Y\rangle = \langle \Phi(c_i), Z\rangle.$$

Likewise, the coefficient matrix $E_{00} \in \mathbb{S}^{n+1}$ of $Y_{00} = 1$ maps to $E_{00} \in \mathbb{S}^{r+1}$, the coefficient matrix of $Z_{00} = 1$. Thus the restricted defining equations are exactly those of $\mathcal{T}(C)$, which in particular proves (10) and completes the proof. □

We next record the general mechanism that allows two facial-reduction steps to be combined in the present setting.

Lemma 3.3. *Let $L \subseteq \mathbb{S}^p$ be an affine subspace such that $L \cap \mathbb{S}^p_+$ is nonempty. Suppose that $W_1 \in L^\perp \cap \mathbb{S}^p_+$ exposes a proper face $F := \mathbb{S}^p_+ \cap W_1^\perp$ containing $L \cap \mathbb{S}^p_+$, and assume that*

$$F^\perp \subseteq L^\perp. \tag{11}$$

Then, for every $W_2 \in L^\perp \cap (F^* \setminus F^\perp)$, there exists $\overline{W} \in L^\perp \cap \mathbb{S}^p_+$ such that

$$\mathbb{S}^p_+ \cap \overline{W}^\perp = F \cap W_2^\perp.$$

Proof. For every face F of the positive semidefinite cone, the standard identity $F^* = \mathbb{S}^p_+ + F^\perp$ holds. Hence we may write $W_2 = \widetilde{W}_2 + N$, where $\widetilde{W}_2 \succeq 0$ and $N \in F^\perp$. By (11), $N \in L^\perp$. Since also $W_2 \in L^\perp$, it follows that $\widetilde{W}_2 \in L^\perp$. Set $\overline{W} := W_1 + \widetilde{W}_2$, which belongs to $L^\perp \cap \mathbb{S}^p_+$. Since both summands are positive semidefinite, and since $N \in F^\perp$ implies that W_2 and $\widetilde{W}_2$ agree on F, we obtain

$$\mathbb{S}^p_+ \cap \overline{W}^\perp = F \cap \widetilde{W}_2^\perp = F \cap W_2^\perp.$$

This proves the result. □

The condition (11) is equivalent to $L \subseteq \operatorname{span}(F)$. Thus the entire affine subspace, rather than only its positive semidefinite part, already lies in the linear span of the exposed face. In the application below, the equations $Ax = 0$ and $AX = 0$ enforce this property for F_A. Consequently, the first facial-reduction step replaces the ambient cone by a face whose linear span is already imposed by the affine equations. This structural property is what allows the first step to be combined with a subsequent facial-reduction step.

The next proposition compares the lengths of facial-reduction sequences for $\mathcal{R}(A)$ and its reduced formulation $\mathcal{T}(C)$. We state the stronger bound on maximum singularity degree because the later upper-bound argument in Section 5 must control arbitrary facial-reduction sequences, not only shortest ones.

Proposition 1. *Assume $r \geq 1$. Then*

$$\mathrm{MSD}(\mathcal{T}(C)) \leq r \qquad \textit{and} \qquad \mathrm{sd}(\mathcal{R}(A)) = \max\{1, \mathrm{sd}(\mathcal{T}(C))\} \leq r.$$

Proof. Since $E_{00} \in \mathcal{T}(C)$, the standard matrix-order bound for a nonzero feasible system over $\mathbb{S}^{r+1}_+$ gives $\mathrm{MSD}(\mathcal{T}(C)) \le r$.

We next prove the formula for $\mathrm{sd}(\mathcal{R}(A))$. Let L be the affine subspace defined by the constraints in (7). By Lemma 3.2, the matrix W in (9) belongs to $L^\perp \cap \mathbb{S}^{n+1}_+$ and exposes F_A. Let L_A denote the linear span of the constraint matrices for $Ax = 0$ and $AX = 0$. A direct linear-algebra calculation gives

$$L_A = \{N \in \mathbb{S}^{n+1} : V^T N V = 0\} = F_A^\perp; \tag{12}$$

see Appendix A. Every coefficient matrix of $Ax = 0$ and $AX = 0$ is orthogonal to every $Y \in L$, because these constraints have zero right-hand sides. Hence $L_A \subseteq L^\perp$, and $F_A^\perp \subseteq L^\perp$, which is precisely the assumption in Lemma 3.3.

Let $d := \mathrm{sd}(\mathcal{T}(C))$. By Lemma 3.2, after the initial step exposing F_A, the resulting formulation is $\mathcal{T}(C)$. Hence a shortest facial-reduction sequence for $\mathcal{T}(C)$ supplies d further steps. If $d = 0$, then F_A is the minimal face containing $\mathcal{R}(A)$, and the step exposed by W gives $\mathrm{sd}(\mathcal{R}(A)) = 1$. If $d \ge 1$, Lemma 3.3 combines the exposure by W with the first of these d additional steps. Applying the remaining $d - 1$ steps gives

$$\mathrm{sd}(\mathcal{R}(A)) \le d.$$

Conversely, intersect the faces in a shortest facial-reduction sequence for $\mathcal{R}(A)$ with F_A. After repeated faces are discarded, the remaining faces form a facial-reduction sequence for the formulation on F_A, of length at most $\mathrm{sd}(\mathcal{R}(A))$. Under $Z \mapsto VZV^T$, this becomes a facial-reduction sequence for $\mathcal{T}(C)$. Therefore,

$$d \le \mathrm{sd}(\mathcal{R}(A)).$$

Combining the two cases yields

$$\mathrm{sd}(\mathcal{R}(A)) = \max\{1, \mathrm{sd}(\mathcal{T}(C))\}.$$

Finally, (3) gives $\mathrm{sd}(\mathcal{T}(C)) \le \mathrm{MSD}(\mathcal{T}(C)) \le r$. Since $r \ge 1$, the displayed maximum is at most r. $\square$

3.4 Facial reduction under restriction to a face

The preceding proof used the fact that, after each face in a facial-reduction sequence is intersected with a fixed face, every strict inclusion that remains is a valid facial-reduction step for the restricted problem. We now record the general form of this restriction principle for later use. Here the face need not contain every feasible matrix, and the restriction is simply an intersection in the original matrix space.

Lemma 3.4 (Restriction of a facial-reduction sequence)**.** *Let $L \subseteq \mathbb{S}^p$ be affine, let F be a face of $\mathbb{S}^p_+$ such that $L \cap F \ne \emptyset$, and let $F_0, \dots, F_d$ be a facial-reduction sequence for $L \cap \mathbb{S}^p_+$. For $j = 1, \dots, d$, let W_j be the exposing matrix used in the step from F_{j-1} to F_j. For $j = 0, \dots, d$, set*

$$\overline{F}_j := F_j \cap F.$$

Then each $\overline{F}_j$ is a face of F containing $L \cap F$, and

$$\overline{F}_j = \overline{F}_{j-1} \cap W_j^\perp \qquad (j = 1, \dots, d).$$

After repeated faces are omitted from $\overline{F}_0, \dots, \overline{F}_d$, the remaining faces and corresponding matrices W_j form a partial facial-reduction sequence for $L \cap F$.

Proof. Since F_j and F are faces of $\mathbb{S}^p_+$, their intersection $\overline{F}_j$ is a face of F containing $L \cap F$. Moreover,

$$\overline{F}_j = (F_{j-1} \cap W_j^\perp) \cap F = \overline{F}_{j-1} \cap W_j^\perp.$$

For each retained strict inclusion, the original sequence gives $W_j \in L^\perp \cap F_{j-1}^*$. Since $\overline{F}_{j-1} \subseteq F_{j-1}$, we have $W_j \in \overline{F}_{j-1}^*$, while strictness gives $W_j \notin \overline{F}_{j-1}^\perp$. Hence $W_j \in L^\perp \cap (\overline{F}_{j-1}^* \setminus \overline{F}_{j-1}^\perp)$. Thus W_j exposes $\overline{F}_j$ from $\overline{F}_{j-1}$. Removing the indices j for which $\overline{F}_j = \overline{F}_{j-1}$, together with the corresponding matrices W_j, yields the asserted partial facial-reduction sequence. □

The restriction principle also gives the following bound on maximum singularity degree.

Lemma 3.5 (Maximum singularity degree under restriction)**.** *Let $L \subseteq \mathbb{S}^p$ be affine. Let $\mathcal{G} \subseteq \mathbb{R}^p$ be a subspace of dimension q, and let $F_\mathcal{G}$ be the face of $\mathbb{S}^p_+$ associated with $\mathcal{G}$. Assume that $L \cap F_\mathcal{G} \neq \emptyset$, choose $U \in \mathbb{R}^{p\times q}$ with orthonormal columns spanning $\mathcal{G}$, and define*

$$L_\mathcal{G} := \{Z \in \mathbb{S}^q : UZU^T \in L\}.$$

Then

$$\mathrm{MSD}(L \cap \mathbb{S}^p_+) \leq \dim(\mathcal{G}^\perp) + \mathrm{MSD}(L_\mathcal{G} \cap \mathbb{S}^q_+).$$

Proof. Consider an arbitrary facial-reduction sequence of length d for $L \cap \mathbb{S}^p_+$, with associated faces $F_0, \ldots, F_d$. Let $\mathbb{R}^p = \mathcal{V}_0 \supsetneq \cdots \supsetneq \mathcal{V}_d$ be their associated subspaces, and set $\overline{F}_j := F_j \cap F_\mathcal{G}$. By Lemma 3.4, the strict inclusions among the faces $\overline{F}_j$ form a partial facial-reduction sequence for $L \cap F_\mathcal{G}$. Under the face parametrization $Z \mapsto UZU^T$, this is a partial facial-reduction sequence for $L_\mathcal{G} \cap \mathbb{S}^q_+$. This partial sequence can be extended to a facial-reduction sequence. Hence at most $\mathrm{MSD}(L_\mathcal{G} \cap \mathbb{S}^q_+)$ indices satisfy $\overline{F}_j \subsetneq \overline{F}_{j-1}$.

It remains to count the indices for which $\overline{F}_j = \overline{F}_{j-1}$. Set $\mathcal{G}_j := \mathcal{V}_j \cap \mathcal{G}$, let $P_{\mathcal{G}^\perp}$ denote the orthogonal projector onto $\mathcal{G}^\perp$, and define

$$\delta_j := \dim\big(P_{\mathcal{G}^\perp}(\mathcal{V}_j)\big).$$

The face $\overline{F}_j$ is associated with $\mathcal{G}_j$. Moreover, the kernel of $P_{\mathcal{G}^\perp}$ restricted to $\mathcal{V}_j$ is $\mathcal{G}_j$, so rank–nullity gives

$$\dim \mathcal{V}_j = \dim \mathcal{G}_j + \delta_j.$$

If $\overline{F}_j = \overline{F}_{j-1}$, then $\mathcal{G}_j = \mathcal{G}_{j-1}$, whereas $\mathcal{V}_j \subsetneq \mathcal{V}_{j-1}$. Therefore, $\delta_j \leq \delta_{j-1} - 1$. Since $\delta_0 = \dim(\mathcal{G}^\perp)$, at most $\dim(\mathcal{G}^\perp)$ indices yield repeated faces. Combining the two counts proves the result. □

4 A high-singularity-degree equality-generated SDP–RLT relaxation

Semidefinite programs can be highly ill-conditioned when Slater's condition fails, and high singularity degree is a structural source of this difficulty [12, 13]. Although general SDP systems with high singularity degree are known, those examples do not establish whether the same behavior can occur in SDP relaxations of linearly constrained binary programs. Such relaxations contain highly structured equations tied to individual variables, such as the arrow constraints $Y_{ii} = Y_{0i}$ in (5), which might appear to preclude long facial-reduction sequences. In particular, Sturm's classical example [13, Example 2] does not directly answer this question, because it does not include the binary arrow constraints and its data matrices arise from linearizing the quadratic equations $x_i^2 = x_{i-1}$, rather than from linear constraints on binary variables.

Here we answer the question positively. For every $n \geq 2$, we construct an SDP relaxation of a linearly constrained binary set in n variables whose singularity degree is exactly $\lfloor n/2 \rfloor$. Thus, there exist such relaxations whose singularity degree grows linearly with the number of binary variables.

The proof proceeds in three stages. We first construct a linearly constrained binary singleton whose nullspace has a Vandermonde basis. We then characterize the positive semidefinite matrices in the linear span of its reduced arrow-constraint matrices. Finally, we combine these ingredients with the reduction in Section 3 to compute the exact singularity degree.

Throughout this section, fix $n \geq 2$ and set

$$m := \left\lceil \frac{n}{2} \right\rceil, \qquad r := n - m = \left\lfloor \frac{n}{2} \right\rfloor.$$

For $s = 1, \ldots, r$, define the column vector $c_s(t) \in \mathbb{R}^s$ by

$$c_s(t) := \left(t, t^2, \ldots, t^s\right)^T.$$

4.1 A Vandermonde construction of a binary singleton

Define $C \in \mathbb{R}^{n \times r}$ by

$$C_{kj} := k^j, \qquad k = 1, \ldots, n, \quad j = 1, \ldots, r. \tag{13}$$

Thus, the k-th row of C is $c_r(k)^T$. The matrix C has full column rank. To give an explicit linear description of its range, partition it as

$$C = \begin{pmatrix} C_{\text{top}} \\ C_{\text{bot}} \end{pmatrix}, \qquad C_{\text{top}} \in \mathbb{R}^{r \times r},$$

where C_{top} consists of the first r rows. The matrix C_{top} is a row-scaled ordinary Vandermonde matrix and is nonsingular. Set

$$A := \begin{pmatrix} -C_{\text{bot}} C_{\text{top}}^{-1} & I_m \end{pmatrix} \in \mathbb{R}^{m \times n}. \tag{14}$$

Then A has full row rank and

$$\ker A = \operatorname{range}(C). \tag{15}$$

Consider the linearly constrained binary set

$$P_n := \{x \in \{0,1\}^n \mid Ax = 0\}. \tag{16}$$

Thus P_n is defined using only linear equalities and binarity. Since the equations are homogeneous, $0 \in P_n$, so P_n is nonempty. We prove the stronger fact that P_n is a singleton. Thus the high singularity degree established below occurs even though the underlying binary feasible set is trivial.

Proposition 2. *The binary set P_n defined in* (16) *is the singleton* $\{0\}$.

Proof. Let $x \in P_n$. By (15), there is a unique $y \in \mathbb{R}^r$ such that $x = Cy$. Define the polynomial

$$p(t) := c_r(t)^T y.$$

It has degree at most r and satisfies $p(0) = 0$. Since $p(k) = x_k \in \{0, 1\}$,

$$p(k)\big(p(k) - 1\big) = 0, \qquad k = 1, \ldots, n.$$

Thus the polynomial $p(t)(p(t) - 1)$, whose degree is at most $2r$, vanishes at the n distinct positive evaluation points and also at $t = 0$. Since $n + 1 > 2r$, it must vanish identically. Therefore p is identically 0 or 1. The equality $p(0) = 0$ rules out the latter, so $p \equiv 0$. Hence $y = 0$ and $x = 0$. $\square$

For the matrix A in (14), consider the equality-generated SDP–RLT relaxation $\mathcal{R}(A)$ in (7). The RLT strengthening $AX = 0$ is essential for the high singularity degree established below. Indeed, omitting $AX = 0$ leaves the basic Shor relaxation associated with $Ax = 0$, whose singularity degree is at most one [6]. Here $r \geq 1$, A has full row rank, C has full column rank, and $\ker A = \operatorname{range}(C)$ by (15). Thus $\mathcal{R}(A)$ and $\mathcal{T}(C)$ are precisely the pair of systems covered by Proposition 1, which gives

$$\operatorname{sd}(\mathcal{R}(A)) = \max\{1, \operatorname{sd}(\mathcal{T}(C))\}.$$

Consequently, determining the singularity degree of $\mathcal{R}(A)$ reduces to analyzing the smaller system $\mathcal{T}(C)$.

4.2 Positive semidefinite matrices in the linear span of the reduced arrow-constraint matrices

We first characterize the positive semidefinite matrices in the linear span of the arrow-constraint matrices $\Phi(c_r(k))$, $k = 1, \ldots, n$, of $\mathcal{T}(C)$. We state it for every $s = 1, \ldots, r$, because the subsequent facial-reduction argument successively lowers s. The specialization of Φ to $c_s(t)$ is the symmetric matrix polynomial

$$\Phi_s(t) := \Phi(c_s(t)) = \begin{pmatrix} 0 & -\frac{1}{2}c_s(t)^T \\ -\frac{1}{2}c_s(t) & c_s(t)c_s(t)^T \end{pmatrix} \in \mathbb{S}^{s+1}. \tag{17}$$

The entries of $\Phi_s(t)$ involve exactly the consecutive powers $t, t^2, \ldots, t^{2s}$.

Lemma 4.1. *For every $s = 1, \ldots, r$,*

$$\operatorname{span}\{\Phi_s(k) : k = 1, \ldots, n\} \cap \mathbb{S}^{s+1}_+ = \{\lambda E_{ss} : \lambda \geq 0\}.$$

Here $E_{ss} = e_s e_s^T$ is the diagonal matrix unit in $\mathbb{S}^{s+1}$, as defined in (2)*.*

Proof. Let

$$W := \sum_{k=1}^{n} \mu_k \Phi_s(k), \qquad a_\ell := \sum_{k=1}^{n} \mu_k k^\ell, \quad \ell = 1, \ldots, 2s,$$

and suppose that $W \succeq 0$. Since $W_{00} = 0$, positive semidefiniteness forces row and column 0 of W to vanish. The entries of that row are

$$W_{0j} = -\frac{1}{2}a_j, \qquad j = 1, \ldots, s,$$

so

$$a_1 = \cdots = a_s = 0. \tag{18}$$

The lower-right block of W is Hankel, with

$$W_{ij} = a_{i+j}, \qquad i, j = 1, \dots, s. \tag{19}$$

We use the standard zero-propagation argument for positive semidefinite Hankel matrices; see also [16, Lemma 4.6]. Suppose for some $\ell \in \{1, \dots, s-1\}$ that $a_1 = \cdots = a_{s+\ell-1} = 0$. Then

$$W_{\ell\ell} = a_{2\ell} = 0,$$

because $2\ell \leq s + \ell - 1$. A zero diagonal entry of a positive semidefinite matrix forces the corresponding row and column to vanish. In particular, $W_{\ell s} = a_{s+\ell} = 0$. If $s = 1$, (18) already gives the desired conclusion. If $s \geq 2$, starting from (18) and applying this argument for $\ell = 1, \dots, s-1$ gives the same conclusion. Thus, in either case,

$$a_1 = \cdots = a_{2s-1} = 0.$$

Consequently, $W = a_{2s} E_{ss}$, and $a_{2s} \geq 0$ because $W \succeq 0$.

Conversely, the integers $k = 1, \dots, n$ are distinct and nonzero. Since $n \geq 2r \geq 2s$, the matrix $(k^\ell)_{k=1,\dots,n,\ \ell=1,\dots,2s}$ has full column rank. Hence there exist $\nu_1, \dots, \nu_n$ such that

$$\sum_{k=1}^{n} \nu_k k^\ell = \begin{cases} 0, & \ell = 1, \dots, 2s-1, \\ 1, & \ell = 2s. \end{cases}$$

It follows from (17) that

$$\sum_{k=1}^{n} \nu_k \Phi_s(k) = E_{ss}.$$

Thus every λE_{ss} with $\lambda \geq 0$ belongs to the intersection, which proves the stated equality. □

Example 4.1. *Let $n = 4$ and $s = r = 2$. Then*

$$c_2(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}, \qquad \Phi_2(t) = \begin{pmatrix} 0 & -\frac{1}{2}t & -\frac{1}{2}t^2 \\ -\frac{1}{2}t & t^2 & t^3 \\ -\frac{1}{2}t^2 & t^3 & t^4 \end{pmatrix}.$$

For

$$W := \sum_{k=1}^{4} \mu_k \Phi_2(k), \qquad a_\ell := \sum_{k=1}^{4} \mu_k k^\ell, \quad \ell = 1, \dots, 4,$$

we have

$$W = \begin{pmatrix} 0 & -\frac{1}{2}a_1 & -\frac{1}{2}a_2 \\ -\frac{1}{2}a_1 & a_2 & a_3 \\ -\frac{1}{2}a_2 & a_3 & a_4 \end{pmatrix}.$$

If $W \succeq 0$, then $W_{00} = 0$ forces $a_1 = a_2 = 0$. Since $W_{11} = a_2 = 0$, row and column 1 must also vanish, giving $a_3 = 0$. Therefore, $W = a_4 E_{22}$ with $a_4 \geq 0$. Conversely, direct calculation gives

$$E_{22} = -\frac{1}{6}\Phi_2(1) + \frac{1}{4}\Phi_2(2) - \frac{1}{6}\Phi_2(3) + \frac{1}{24}\Phi_2(4).$$

Thus, this example realizes both inclusions in Lemma 4.1.

4.3 Exact singularity degree of the construction

For $s = 1, \ldots, r$, define

$$\mathcal{T}_s := \left\{ Z \in \mathbb{S}^{s+1}_+ \;\middle|\; Z_{00} = 1, \quad \langle \Phi_s(k), Z \rangle = 0,\ k = 1, \ldots, n \right\}. \tag{20}$$

By (13), the rows of C are $c_r(k)^T$. Comparing (20) with (8) therefore gives

$$\mathcal{T}(C) = \mathcal{T}_r.$$

We next compute its singularity degree by analyzing the family $\{\mathcal{T}_s\}_{s=1}^r$.

Lemma 4.2. *For every $s = 1, \ldots, r$,*

$$\mathrm{sd}(\mathcal{T}_s) = \mathrm{MSD}(\mathcal{T}_s) = s. \tag{21}$$

Proof. Set $\mathcal{T}_0 := \{1\}$. At the first step of any facial-reduction sequence for $\mathcal{T}_s$, the exposing matrix has the form

$$W = \sum_{k=1}^{n} \mu_k \Phi_s(k) \succeq 0,$$

because the multiplier of the normalization must be zero. By Lemma 4.1, every such nonzero matrix is a positive multiple of E_{ss}, and E_{ss} itself belongs to this linear span. This step therefore restricts the problem to the face $\mathbb{S}^{s+1}_+ \cap E_{ss}^\perp$, on which row and column s vanish. Removing this last row and column replaces $c_s(t)$ by $c_{s-1}(t)$ and reduces $\mathcal{T}_s$ to $\mathcal{T}_{s-1}$. Thus every facial-reduction sequence makes the same reduction. Since $\mathcal{T}_0$ is strictly feasible in $\mathbb{S}^1_+$, induction proves both equalities in (21). □

We now transfer this calculation back to the original SDP–RLT relaxation.

Proposition 3. *For every $n \geq 2$, let A be the matrix defined in* (14)*. Then*

$$\mathrm{sd}(\mathcal{R}(A)) = r = \left\lfloor \frac{n}{2} \right\rfloor.$$

Proof. The reduced system satisfies $\mathcal{T}(C) = \mathcal{T}_r$. Therefore, Proposition 1 and Lemma 4.2 give

$$\mathrm{sd}(\mathcal{R}(A)) = \max\{1, \mathrm{sd}(\mathcal{T}_r)\} = \max\{1, r\} = r.$$

□

For every $n \geq 2$, the construction uses n binary variables and an $(n+1) \times (n+1)$ matrix, while its singularity degree is

$$r = \left\lfloor \frac{n}{2} \right\rfloor.$$

The point is nontrivial at the formulation level: although the binary set is defined solely by linear equalities, the standard equality-generated SDP–RLT strengthening produces a facial-reduction sequence whose length grows linearly with the problem dimension. Thus high singularity degree, a structural source of weak error bounds and numerical sensitivity, can arise in SDP relaxations of linearly constrained binary sets.

The construction above supplies the lower bound $\lfloor n/2 \rfloor$. The next section proves that this value is globally optimal within the equality-generated SDP–RLT class.

5 The sharp upper bound for the equality-generated SDP–RLT class

This section proves that, for every full-column-rank $C \in \mathbb{R}^{n\times r}$, the maximum singularity degree of $\mathcal{T}(C)$ is at most $\min\{n-r, r\}$, and hence so is its singularity degree. Corollary 5.1 then resolves the extremal problem stated in (1).

For a full-column-rank matrix, we call the number of rows minus the number of columns its *row redundancy.* It is the number of rows beyond the minimum required for full column rank. In particular, the row redundancy of C is $m := n - r$.

The case $r = 0$ is immediate because $\mathcal{T}(C) = \{1\}$. For $r \geq 1$, Proposition 1 gives the upper bound $\mathrm{MSD}(\mathcal{T}(C)) \leq r$, so the main task is to prove the complementary bound by the row redundancy m. The proof has two ingredients. Subsection 5.1 analyzes the first facial-reduction step and derives the row-redundancy bound needed for the induction. Subsection 5.2 introduces an auxiliary restriction and bounds the number of steps that disappear under this restriction. Subsection 5.3 combines these ingredients by induction on row redundancy.

5.1 The first facial-reduction step and its row-redundancy bound

Let $C \in \mathbb{R}^{n\times r}$ have full column rank, and set $m := n - r$. Since $E_{00} \in \mathcal{T}(C)$, orthogonality to the affine constraint set forces the multiplier of the normalization equation $Z_{00} = 1$ in any exposing matrix to be zero. Consider the exposing matrix at the first step of an arbitrary facial-reduction sequence,

$$W := \sum_{i=1}^{n} \mu_i \Phi(c_i) = \begin{pmatrix} 0 & -\frac{1}{2}\mu^T C \\ -\frac{1}{2}C^T\mu & C^T\mathrm{Diag}(\mu)C \end{pmatrix} \succeq 0,$$

where $W \neq 0$. Since $W_{00} = 0$, positive semidefiniteness forces the zeroth row and column of W to vanish. Hence

$$C^T\mu = 0,$$

and therefore

$$W = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}, \qquad B := C^T\mathrm{Diag}(\mu)C \succeq 0.$$

Thus the special structure of $\Phi(c_i)$ makes the first exposing matrix simultaneously produce a linear dependence among the rows of C and a positive semidefinite matrix B determining the exposed face. Set

$$\rho := \mathrm{rank}(W) = \mathrm{rank}(B), \qquad k := r - \rho.$$

Let the columns of $\Gamma \in \mathbb{R}^{r\times k}$ span $\ker B$. Set

$$D := C\Gamma = \begin{pmatrix} d_1^T \\ \vdots \\ d_n^T \end{pmatrix}, \qquad V := \begin{pmatrix} 1 & 0 \\ 0 & \Gamma \end{pmatrix}.$$

By the characterization of faces of the positive semidefinite cone in Section 2.2, W exposes the face associated with $\mathrm{span}\{e_0\} \oplus \ker B$. Moreover,

$$V^T\Phi(c_i)V = \Phi(d_i), \qquad i = 1, \ldots, n.$$

Consequently,

$$\mathcal{T}(C) = \left\{ VZV^T \;\middle|\; Z \in \mathcal{T}(D) \right\}. \tag{22}$$

Thus $\mathcal{T}(D)$ is precisely the formulation obtained by parametrizing the face exposed at this first facial-reduction step.

To state the row-redundancy bound, let $S := \{i \in \{1, \dots, n\} : \mu_i \neq 0\}$ be the index set of the nonzero entries of the multiplier vector μ. Set $S^c := \{1, \dots, n\} \setminus S$. Let D_S denote the submatrix of D formed by the rows indexed by S. Define $p := |S|$, let $\mathbf{1} \in \mathbb{R}^p$ denote the all-ones vector, and let $\mu_S \in \mathbb{R}^p$ denote the subvector of μ indexed by S. Set

$$s := \operatorname{rank}(D_S), \quad h := \operatorname{rank}\begin{pmatrix} \mathbf{1} & D_S \end{pmatrix}.$$

Since D_S has rank s, exactly $k-s$ additional independent rows are needed to span the k-dimensional row space of D. Among the $|S^c|$ available rows, this leaves

$$\eta := |S^c| - (k - s)$$

remaining rows after a rank-extending subset has been selected. We use this notation in the lemma below and in the proof of Theorem 5.1.

Lemma 5.1. *Under the first-step notation above,*

$$p = |S| \geq h + s + \rho. \tag{23}$$

Equivalently,

$$0 \leq \eta \leq m - h. \tag{24}$$

In particular, $h + \eta \leq m$.

Proof. Choose $R \in \mathbb{R}^{r \times \rho}$ whose columns span range(B). Since B is symmetric, range$(B) = (\ker B)^\perp$, so $Q = (\Gamma\ R)$ is invertible. Set $Z := CR$, and let Z_S denote the submatrix of Z consisting of the rows indexed by S. Then

$$CQ = (D\ Z).$$

Since $B \succeq 0$ and range$(R) =$ range(B), $P := R^T B R \in \mathbb{S}^\rho$ is positive definite. Moreover, $B\Gamma = 0$, so the change of basis $Q = (\Gamma\ R)$ gives

$$Q^T B Q = \begin{pmatrix} \Gamma^T B \Gamma & \Gamma^T B R \\ R^T B \Gamma & R^T B R \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & P \end{pmatrix}.$$

Let

$$\widetilde{D}_S := \begin{pmatrix} \mathbf{1} & D_S \end{pmatrix}.$$

Put $J = \operatorname{Diag}(\mu_S)$, which is nonsingular, and define

$$\mathcal{E} := \operatorname{range}(\widetilde{D}_S), \qquad \mathcal{G} := \operatorname{range}\begin{pmatrix} D_S & Z_S \end{pmatrix}.$$

By definition, $\dim \mathcal{E} = h$.

Because μ is supported on S and $J\mathbf{1} = \mu_S$, the equation $C^T \mu = 0$ gives

$$\mathbf{1}^T J D_S = 0, \qquad \mathbf{1}^T J Z_S = 0. \tag{25}$$

Moreover,

$$\begin{pmatrix} D_S^T J D_S & D_S^T J Z_S \\ Z_S^T J D_S & Z_S^T J Z_S \end{pmatrix} = (CQ)^T \mathrm{Diag}(\mu)(CQ) = Q^T B Q = \begin{pmatrix} 0 & 0 \\ 0 & P \end{pmatrix}. \tag{26}$$

The $(2,2)$ block of (26) gives $Z_S^T J Z_S = P \succ 0$, so Z_S has full column rank ρ. Moreover, suppose $w = D_S a = Z_S b$. The $(1,1)$ and $(2,2)$ blocks of (26) give, respectively,

$$w^T J w = a^T D_S^T J D_S a = 0,$$

and

$$w^T J w = b^T Z_S^T J Z_S b = b^T P b.$$

Since $P \succ 0$, we have $b = 0$, and hence $w = 0$. Consequently,

$$\mathrm{range}(D_S) \cap \mathrm{range}(Z_S) = \{0\}.$$

Since $\mathrm{rank}(D_S) = s$, it follows that

$$\dim \mathcal{G} = s + \rho.$$

The identities in (25), together with the $(1,1)$ and $(1,2)$ blocks of (26), combine into

$$\widetilde{D}_S^T J \begin{pmatrix} D_S & Z_S \end{pmatrix} = 0.$$

Thus, with respect to the ordinary Euclidean inner product on $\mathbb{R}^p$,

$$\mathcal{E} \perp J\mathcal{G}.$$

Since J is nonsingular, $\dim(J\mathcal{G}) = s + \rho$. The two orthogonal subspaces must fit in $\mathbb{R}^p$, and hence

$$h + s + \rho = \dim \mathcal{E} + \dim(J\mathcal{G}) \le p.$$

This proves (23). Equivalently, $s + \rho \le p - h$.

By its definition, $\eta \ge 0$. Since $n = m + r = k + \rho + m$,

$$\eta = (n - p) - (k - s) = s + \rho + m - p \le m - h.$$

□

5.2 The auxiliary restricted system

The proof of the upper bound below uses two successive operations, which have different roles. After this first facial-reduction step for $\mathcal{T}(C)$, parametrizing the exposed face with $V = \mathrm{Diag}(1, \Gamma)$ gives

$$C \xrightarrow{\Gamma} D = C\Gamma;$$

this parametrizes the face reached by the facial-reduction step, and the remaining steps form a facial-reduction sequence for $\mathcal{T}(D)$. To bound its length, we restrict $\mathcal{T}(D)$ to the face associated with $\mathrm{span}\{e_0\} \oplus G$, for a chosen subspace $G \subseteq \mathbb{R}^k$. If the columns of U form an orthonormal basis of G, then, in this basis, the restricted system is $\mathcal{T}(E)$, where

$$D \xrightarrow{U} E = DU.$$

This is the restriction construction described in Subsection 3.4, specialized to the structured system $\mathcal{T}(D)$; it is not another facial-reduction step. This restriction makes specified rows of D identical. The lemma below bounds the length of a facial-reduction sequence for $\mathcal{T}(D)$ in terms of that of the restricted system $\mathcal{T}(E)$.

Lemma 5.2 (Auxiliary restriction lemma). *Let $D \in \mathbb{R}^{n\times k}$ have full column rank. Let $G \subseteq \mathbb{R}^k$ be a subspace, set $q := \dim G$, and choose $U \in \mathbb{R}^{k\times q}$ with orthonormal columns spanning G. Define*

$$E := DU \in \mathbb{R}^{n\times q}.$$

Then E has full column rank, and

$$\mathrm{MSD}(\mathcal{T}(D)) \le \dim(G^\perp) + \mathrm{MSD}(\mathcal{T}(E)).$$

Proof. Since D has full column rank and U has full column rank, $E = DU$ has full column rank. Set $\widehat{U} := \mathrm{Diag}(1, U)$, which has orthonormal columns, and let F_G be the face associated with $\mathrm{span}\{e_0\} \oplus G = \mathrm{range}(\widehat{U})$. Since $E_{00} \in \mathcal{T}(D) \cap F_G$, this intersection is nonempty. Applying the same block-diagonal congruence calculation used in deriving (22), with U in place of Γ, gives

$$\mathcal{T}(D) \cap F_G = \{\widehat{U} Z \widehat{U}^T : Z \in \mathcal{T}(E)\}.$$

Since $(\mathrm{span}\{e_0\} \oplus G)^\perp = \{0\} \oplus G^\perp$, the subspace associated with F_G has orthogonal complement of dimension $\dim(G^\perp)$. Applying Lemma 3.5 to the affine system defining $\mathcal{T}(D)$ now gives the claimed inequality. □

5.3 The sharp upper bound

Lemmas 5.1 and 5.2 provide the two ingredients for the induction. The former gives the row-redundancy bound for the first facial-reduction step, while the latter bounds the number of steps that disappear under the auxiliary restriction. We now combine them to prove the sharp upper bound.

The use of maximum singularity degree in the theorem below deserves a brief explanation. Although our final target is $\mathrm{sd}(\mathcal{T}(C))$, the preceding restriction argument must control arbitrary facial-reduction sequences, not only a shortest one. By Lemma 3.4, restricting the feasible set to a face of the positive semidefinite cone may turn some steps into repetitions, but the remaining strict steps form a partial facial-reduction sequence for the restricted problem. This partial sequence can be extended to a facial-reduction sequence, so its length is bounded by maximum singularity degree, whereas it need not be bounded by singularity degree. This is why the theorem establishes the stronger bound on maximum singularity degree.

Theorem 5.1. *For every full-column-rank $C \in \mathbb{R}^{n\times r}$,*

$$\mathrm{MSD}(\mathcal{T}(C)) \le \min\{n - r, r\}.$$

Proof. If $r = 0$, then C has no columns and $\mathcal{T}(C) = \{1\}$, so its maximum singularity degree is zero. Assume henceforth that $r \ge 1$.

The row redundancy of C is $m := n - r$. Proposition 1 already gives $\mathrm{MSD}(\mathcal{T}(C)) \le r$, so it remains to prove

$$\mathrm{MSD}(\mathcal{T}(C)) \le m \tag{27}$$

by strong induction on the row redundancy m.

If $m = 0$, then C is square and nonsingular. A first exposing multiplier must satisfy $C^T\mu = 0$, because the exposing matrix has zero zeroth row. Thus $\mu = 0$, and there is no facial-reduction step.

Let $m \geq 1$, and assume that the maximum singularity degree is at most the row redundancy for every full-column-rank matrix whose row redundancy is smaller than m. Consider an arbitrary facial-reduction sequence of length d for $\mathcal{T}(C)$. If $d = 0$, there is nothing to prove. Assume $d \geq 1$, and apply the first-step notation introduced before Lemma 5.1 to the first step of this sequence. By (22), the remaining $d - 1$ steps form a facial-reduction sequence for $\mathcal{T}(D)$.

Recall that $D \in \mathbb{R}^{n \times k}$ and

$$s = \operatorname{rank}(D_S), \qquad h = \operatorname{rank}\begin{pmatrix} \mathbf{1} & D_S \end{pmatrix}.$$

Thus s is the dimension of the linear span of $\{d_i : i \in S\}$, whereas $h - 1$ is the dimension of their affine hull. Let G be the orthogonal complement of the direction space of the affine hull of $\{d_i : i \in S\}$; that is,

$$G := \big(\operatorname{span}\{d_i - d_j : i, j \in S\}\big)^{\perp}.$$

Consequently,

$$\dim(G^{\perp}) = h - 1, \qquad \dim G = k - h + 1.$$

Apply the auxiliary restriction of Subsection 5.2 to the present D and G. Set $q := \dim G = k - h + 1$, choose $U \in \mathbb{R}^{k \times q}$ with orthonormal columns spanning G, and define

$$E := DU \in \mathbb{R}^{n \times q}, \qquad \bar{d}_i := U^T d_i, \quad i = 1, \dots, n,$$

so the ith row of E is $\bar{d}_i^T$. Since every $d_i - d_j$, $i, j \in S$, belongs to $G^{\perp}$,

$$\bar{d}_i - \bar{d}_j = U^T(d_i - d_j) = 0.$$

Thus the rows of E indexed by S have a common value a^T. Geometrically, the projection onto G either sends all points d_i, $i \in S$, to zero, when their affine hull contains the origin, or sends them all to the same nonzero point, when their affine hull does not contain the origin. This is precisely the dichotomy

$$s = h - 1 \quad \text{and} \quad a = 0, \qquad \text{or} \qquad s = h \quad \text{and} \quad a \neq 0.$$

If $s = h - 1$, then $a = 0$, so the rows of E indexed by S are zero, and we delete them. If $s = h$, then $a \neq 0$, so these rows are identical and nonzero, and we retain one copy. Denote the resulting matrix by $\widehat{E}$. Because $\Phi(0) = 0$ and repeated rows give identical constraint matrices, deleting these equations does not change the affine constraint system. Hence $\mathcal{T}(E)$ and $\mathcal{T}(\widehat{E})$ have the same exposing matrices and facial-reduction sequences; in particular,

$$\operatorname{MSD}(\mathcal{T}(E)) = \operatorname{MSD}(\mathcal{T}(\widehat{E})).$$

The matrix $E = DU$ has full column rank, and deleting zero rows and repeated copies does not change its row span. Hence $\widehat{E}$ has full column rank $q = \dim G$. Recall from the definition of η that $|S^c| = k - s + \eta$. Let n_E denote the number of rows of $\widehat{E}$. If $s = h - 1$,

$$n_E = |S^c| = k - s + \eta = \dim G + \eta.$$

If $s = h$,

$$n_E = 1 + |S^c| = 1 + k - s + \eta = \dim G + \eta.$$

Thus, in either case, the row redundancy of $\widehat{E}$ is η. Since $h \geq 1$, Lemma 5.1 gives

$$0 \leq \eta \leq m - h < m.$$

By Lemma 5.2 and the induction hypothesis applied to $\widehat{E}$, which has row redundancy $\eta < m$, we obtain

$$\mathrm{MSD}(\mathcal{T}(D)) \leq \dim(G^{\perp}) + \mathrm{MSD}(\mathcal{T}(E)) = (h-1) + \mathrm{MSD}(\mathcal{T}(\widehat{E})) \leq h - 1 + \eta.$$

Including the first step and using (24) gives

$$\mathrm{MSD}(\mathcal{T}(C)) \leq 1 + (h-1) + \eta = h + \eta \leq m.$$

This proves (27) and hence the theorem. □

Since the singularity degree is no larger than the maximum singularity degree, Theorem 5.1 also gives

$$\mathrm{sd}(\mathcal{T}(C)) \leq \min\{n-r, r\}.$$

Corollary 5.1. *The worst-case singularity degree over all equality-generated SDP–RLT relaxations $\mathcal{R}(A, b)$ in n binary variables for which $P \neq \emptyset$ is*

$$\max_{A,b\colon P\neq\emptyset} \mathrm{sd}(\mathcal{R}(A,b)) = \begin{cases} 1, & n = 1, \\ \lfloor n/2 \rfloor, & n \geq 2. \end{cases}$$

Proof. By Lemma 3.1, it suffices to consider $b = 0$. After removing redundant rows as in Subsection 3.1, assume that A has full row rank, and put $m := \mathrm{rank}(A)$ and $r := n - m$. If $m = 0$, the basic arrow system is strictly feasible and has singularity degree zero. If $r = 0$, the linear equations restrict the feasible matrices to the ray generated by E_{00}, which is exposed in one step. For $n = 1$, these are the only two cases, and $A = [1]$, $b = 0$ gives a relaxation with singularity degree one. Assume henceforth that $n \geq 2$, $m > 0$, and $r > 0$. Then Proposition 1, Theorem 5.1, and (3) give

$$\mathrm{sd}(\mathcal{R}(A)) = \max\{1, \mathrm{sd}(\mathcal{T}(C))\} \leq \min\{n-r, r\}.$$

The two boundary cases have singularity degree at most one, while

$$\max_{1\leq r\leq n-1} \min\{n-r, r\} = \left\lfloor \frac{n}{2} \right\rfloor.$$

The construction in Section 4 attains this value. □

5.4 Attainment for every parameter pair

The preceding corollary maximizes over r. The upper bound in Theorem 5.1 is, in fact, attained for each fixed pair (n, r).

Corollary 5.2. *For every integer $n \geq 1$ and every r with $0 \leq r \leq n$, there is a full-column-rank matrix $C \in \mathbb{R}^{n\times r}$ such that*

$$\mathrm{sd}(\mathcal{T}(C)) = \mathrm{MSD}(\mathcal{T}(C)) = \min\{n-r, r\}.$$

Proof. If $r = 0$, then $\mathcal{T}(C) = \{1\}$. If $r = n$, take $C = I_n$; the resulting basic arrow system is strictly feasible. Thus both singularity degrees are zero in either boundary case. Suppose that $0 < r < n$, and set $m := n - r$. If $n \geq 2r$, the consecutive-power construction in Section 4 has singularity degree r. More explicitly, $n \geq 2r$ implies $r \leq \lfloor n/2 \rfloor$, and taking $s = r$ in Lemma 4.2 gives the matrix $C_{ij} = i^j$, $i = 1, \ldots, n$, $j = 1, \ldots, r$, with singularity degree r. Since $n - r \geq r$, this attains the bound $\min\{n - r, r\}$.

It remains to consider $m < r$. For these fixed m and r, let $C_{\text{pow}} \in \mathbb{R}^{2m \times m}$ be the consecutive-power matrix defined in (13); explicitly, $(C_{\text{pow}})_{ij} = i^j$ for $i = 1, \ldots, 2m$ and $j = 1, \ldots, m$. Define

$$C := \begin{pmatrix} C_{\text{pow}} & 0 \\ 0 & I_{r-m} \end{pmatrix} \in \mathbb{R}^{n \times r}.$$

Indeed, C has

$$2m + (r - m) = m + r = n$$

rows and $m + (r - m) = r$ columns, and it has full column rank.

The last $r - m$ rows are the standard unit vectors $e_{m+1}^T, \ldots, e_r^T$ in $\mathbb{R}^r$. If ν_j is the multiplier associated with the constraint matrix $\Phi(e_j)$, then the $(0, j)$ entry of the exposing matrix is $-\nu_j/2$, because no other row of C has a nonzero jth component. At the first step, the exposing matrix is positive semidefinite, and its zero $(0, 0)$ entry forces its zeroth row to vanish; hence $\nu_j = 0$. The same argument applies at each subsequent step: the subspace associated with the current face still contains $\operatorname{span}\{e_0, e_{m+1}, \ldots, e_r\}$. Therefore, the principal submatrix of the next exposing matrix indexed by $0, m + 1, \ldots, r$ is positive semidefinite. Its $(0, 0)$ entry is zero, so its zeroth row vanishes and $\nu_j = 0$ for every $j = m+1, \ldots, r$. Thus the appended rows do not participate in facial reduction, and the first m coordinates reproduce the forced m-step sequence of the consecutive-power construction. After these steps, the basic arrow system in the remaining $r - m$ coordinates is strictly feasible. Thus the resulting system has singularity degree $m = n - r$. Since $m \leq r$, this again attains the bound $\min\{n - r, r\}$. In both cases, Theorem 5.1 and the fact that singularity degree is no larger than maximum singularity degree force maximum singularity degree to attain the same value. $\square$

For $0 < r < n$, choose a full-row-rank matrix A satisfying $\ker(A) = \operatorname{range}(C)$. Then $P(A, 0) \neq \emptyset$, and Proposition 1 gives $\operatorname{sd}(\mathcal{R}(A, 0)) = \min\{n - r, r\}$. Hence the rank–nullity bound is attained for every pair with $0 < r < n$. At the boundary, $r = n$ gives a strictly feasible basic arrow system with singularity degree zero, whereas $r = 0$ gives the ray generated by E_{00}, whose singularity degree is one.

The construction for $m < r$ is used only to establish sharpness within the class $\mathcal{T}(C)$; unlike the construction in Section 4, it need not define a singleton binary set.

6 Conclusion

This paper determines the exact worst-case singularity degree of the equality-generated SDP–RLT relaxations of nonempty binary sets defined by $Ax = b$. If $\operatorname{rank}(A) = m$ and $0 < m < n$, then the associated relaxation has singularity degree at most $\min\{m, n - m\}$, and this bound is attained for every such rank–nullity pair. Consequently, the worst-case singularity degree over all these relaxations is 1 for $n = 1$ and $\lfloor n/2 \rfloor$ for $n \geq 2$. For $n \geq 2$, the latter value is attained even when the binary feasible set is a singleton.

Unlike singularity degree itself, the rank and nullity of the equality system are available directly from the formulation. The bound therefore certifies in advance that systems with either few independent equalities or small nullity have small singularity degree; the largest worst-case degree occurs only when rank and nullity are nearly balanced. Through general SDP error-bound theory, the rank–nullity bound yields a more favorable Hölder exponent in estimates of the distance to feasibility from constraint residuals.

A Linear span of the lifted equality-constraint matrices

This appendix proves the identity for the linear span of the lifted equality-constraint matrices used in Proposition 1. We first state the underlying linear-algebra identity in dimensions matching the application and then apply it to the lifted equations $Ax = 0$ and $AX = 0$.

Proposition 4. *Let $V \in \mathbb{R}^{(n+1)\times(r+1)}$ have full column rank, and let $\widetilde{A} \in \mathbb{R}^{m\times(n+1)}$ have full row rank with $\ker(\widetilde{A}) = \operatorname{range}(V)$. Define*

$$\mathcal{B} : \mathbb{S}^{n+1} \to \mathbb{R}^{(n+1)\times m}, \qquad \mathcal{B}(Y) := Y\widetilde{A}^T.$$

Then

$$\operatorname{range}(\mathcal{B}^*) = \{N \in \mathbb{S}^{n+1} \mid V^T N V = 0\}.$$

Proof. Since $\operatorname{range}(\widetilde{A}^T) = \ker(\widetilde{A})^\perp = \operatorname{range}(V)^\perp$ and Y is symmetric,

$$\ker(\mathcal{B}) = \{V Z V^T \mid Z \in \mathbb{S}^{r+1}\}.$$

Indeed, $Y\widetilde{A}^T = 0$ is equivalent to $\operatorname{range}(V)^\perp \subseteq \ker(Y)$, and hence to $\operatorname{range}(Y) \subseteq \operatorname{range}(V)$. Therefore

$$\operatorname{range}(\mathcal{B}^*) = \ker(\mathcal{B})^\perp = \{N \in \mathbb{S}^{n+1} \mid V^T N V = 0\},$$

as claimed. □

Corollary A.1. *Let $A \in \mathbb{R}^{m\times n}$ have full row rank, let $C \in \mathbb{R}^{n\times r}$ have full column rank with $\ker(A) = \operatorname{range}(C)$, and set*

$$V := \begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix}.$$

If L_A is the linear span of the constraint matrices for the lifted equations $Ax = 0$ and $AX = 0$, then

$$L_A = \{N \in \mathbb{S}^{n+1} \mid V^T N V = 0\}.$$

Proof. Set $\widetilde{A} := \begin{pmatrix} 0 & A \end{pmatrix}$. The lifted equations are precisely $Y\widetilde{A}^T = 0$, so their coefficient matrices span the range of the adjoint of $Y \mapsto Y\widetilde{A}^T$. Moreover,

$$\ker(\widetilde{A}) = \mathbb{R} \oplus \ker(A) = \mathbb{R} \oplus \operatorname{range}(C) = \operatorname{range}(V).$$

The result now follows from Proposition 4. □

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