Symmetry-dependence in Rounding of a Convex Body

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Abstract

The symmetry measure of a convex body $S \subset \mathbb{R}^n$ is given by:

$$\mathbf{sym}(S) := \max\{\alpha \geq 0 : \text{ there exists } x \in S \text{ such that } -\alpha(S - x) \subseteq S - x\},$$

where such an x is called a Minkowski center. We prove that every convex body S admits a $\sqrt{\frac{n}{\mathbf{sym}(S)}}$-rounding of S, namely, there exists an origin-centered ellipsoid E and a center c such that the following rounding holds:

$$E \subseteq S - c \subseteq \sqrt{\frac{n}{\mathbf{sym}(S)}}\, E\,.$$

This result was conjectured in 2005 by Belloni and Freund. As special cases, this recovers the known result of an n-rounding of S (since it always holds that $\mathbf{sym}(S) \geq 1/n$), and also recovers the known result of a $\sqrt{n}$-rounding when $\mathbf{sym}(S) = 1$.

In the case when S is a polytope given as the convex hull of points, the desired rounding is produced by a regularized minimum volume covering ellipsoid problem where the regularization is with respect to the Minkowski center. Similarly, when S is a polytope given as the intersection of halfspaces, such a rounding is produced by a regularized maximum volume inscribed ellipsoid problem. In both of these cases, the rounding can be computed by first solving a linear optimization problem (to compute $\mathbf{sym}(S)$ and a Minkowski center), and then solving a convex optimization problem with a logarithmic determinant objective, second-order cone constraints, and one semidefinite cone constraint.

We also show that the factor $\sqrt{\frac{n}{\mathbf{sym}(S)}}$ is nearly tight in its dependence on dimension and symmetry. When $\frac{n+1}{1+\mathbf{sym}(S)}$ is an integer, we show by explicit construction that the factor $\sqrt{\frac{n}{\mathbf{sym}(S)}}$ is tight. In the more general case, for every dimension n and every admissible symmetry value, we construct a polytope S for which every rounding is at least $\sqrt{\frac{2}{3}}\sqrt{\frac{n}{\mathbf{sym}(S)}}$.

1 Introduction

Let $S \subset \mathbb{R}^n$ be a full-dimensional convex body, namely a closed bounded convex set with nonempty interior. Let $c \in \mathbb{R}^n$ and $E \subset \mathbb{R}^n$ be an ellipsoid centered at the origin. For $\rho > 0$ we say that $c + E$ is a *ρ-rounding* of S if

$$E \subseteq S - c \subseteq \rho E\,, \tag{1}$$

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and we call ρ the *rounding factor*. A ρ-rounding of S means that $c + E$ approximates S to within a factor of ρ; under the linear map that maps E to the Euclidean unit ball, S is mapped between the Euclidean ball and its ρ-scaled copy. Roundings arise in cutting-plane methods (Grötschel et al., 1993), geometric random-walk sampling (Lovász, 1999), volume computation (Kannan et al., 1997), integer optimization (Lenstra, 1983), and the design of efficient gradient methods for linear programming (Nesterov, 2008). They are also related to ellipsoidal uncertainty models in robust optimization (Ben-Tal and Nemirovski, 1998).

Polyhedra give rise to several optimization-related roundings. If S is a bounded polyhedron described by m linear inequalities, the analytic center together with its Dikin ball yields an $(m-1)$-rounding of S (Dikin, 1967; Sonnevend, 1986). Anstreicher (1999) showed that an $O(n)$-rounding of S can be constructed using points at or near the volumetric center. Indeed, there are extended versions of these results that apply to analytic centers of any convex body via the theory of self-concordant functions (Nesterov and Nemirovskii, 1994).

For a general convex body S, the minimum-volume covering ellipsoid (MVCE, also known as the Löwner ellipsoid) and the maximum-volume inscribed ellipsoid (MVIE, also known as the John ellipsoid) provide two other classical constructions. Each yields an n-rounding for general S, and also a $\sqrt{n}$-rounding of S if S is centrally symmetric; furthermore these factors are sharp (John, 1948; Todd, 2016).

Algorithms for computing roundings via the two extremal ellipsoid problems MVCE and MVIE have also been the subject of research investigations. Khachiyan and Todd (1993) showed how to approximate the MVIE of a polytope described by linear inequalities. Khachiyan (1996) provided a $(1+\varepsilon)n$-rounding algorithm based on the dual formulation of the MVCE problem. Other computational approaches include Sun and Freund (2004); Zhang and Gao (2003). Through convex duality, the MVCE problem is also closely connected to D-optimal experimental design (Gürtuna, 2010).

The sharp factors of n- and $\sqrt{n}$-roundings for general and centrally symmetric convex bodies (respectively) raise the question of what rounding factor can be attained at an intermediate degree of symmetry? To quantify this, define the symmetry of S about $x \in S$ by

$$\mathbf{sym}(x, S) := \max\left\{\alpha \geq 0 \,|\, -\alpha(S - x) \subseteq S - x\right\} ,$$

and the symmetry of S by

$$\mathbf{sym}(S) := \max_{x \in S} \mathbf{sym}(x, S) .$$

Thus $\mathbf{sym}(x, S)$ measures how far $S - x$ can be reflected through the origin while remaining in $S - x$, and a point attaining the optimized value $\mathbf{sym}(S)$ is called a *Minkowski center*.

For any convex body S it holds that $\mathbf{sym}(S) \in [\frac{1}{n}, 1]$; the upper bound follows from the definition of $\mathbf{sym}(S)$, and the lower bound is a consequence of the existence of an n-rounding of S which implies that the center c of such a rounding must have $\mathbf{sym}(c, S) \geq 1/n$. Both limits are attained: for any centrally symmetric convex body S (such as the unit ball of a norm) we have $\mathbf{sym}(S) = 1$, and $\mathbf{sym}(S) = 1/n$ if (and only if) S is any n-dimensional simplex (Minkowski, 1897; Radon, 1916; Klee, 1953). Belloni and Freund (2008) conjectured that for any convex body S there exists a rounding with factor $\sqrt{\frac{n}{\mathbf{sym}(S)}}$:

$$E \subseteq S - c \subseteq \sqrt{\frac{n}{\mathbf{sym}(S)}}\, E . \tag{2}$$

The quantity $\sqrt{\frac{n}{\mathbf{sym}(S)}}$ interpolates between the endpoints n (for general convex bodies) and $\sqrt{n}$ (for symmetric convex bodies). Our main result is a proof of this rounding guarantee, which is formally stated below in Theorem 1.1.

The symmetry measure $\mathbf{sym}(x, S)$ occurs elsewhere in optimization. For example in conic optimization, it enters interior-point complexity bounds (Belloni and Freund, 2009) and is related to condition measures and well-posedness (Peña and Roshchina, 2020). In robust and adaptive optimization, $\mathbf{sym}(x, S)$ and $\mathbf{sym}(S)$ enter approximation guarantees (Bertsimas et al., 2011). Computing a Minkowski center has also been formulated as a robust optimization problem and has been shown computationally to improve the convergence of hit-and-run and cutting-plane methods (den Hertog et al., 2024).

Existing symmetry-dependent rounding guarantees are expressed in terms of the symmetry at a prescribed center. For an arbitrary $x \in \operatorname{int} S$, Belloni and Freund (2008) proved the existence of a $\frac{\sqrt{n}}{\mathbf{sym}(x,S)}$-rounding centered at x and obtained the sharper factor $\sqrt{\frac{n}{\mathbf{sym}(x_L,S)}}$ at the Löwner center x_L. At the John center x_J, the analogous $\sqrt{\frac{n}{\mathbf{sym}(x_J,S)}}$-rounding follows from Brandenberg and König (2015). All of these results use the local symmetry $\mathbf{sym}(x, S)$ and not the global (and optimized) symmetry $\mathbf{sym}(S)$. Indeed, the symmetry at the John center can be smaller than $\mathbf{sym}(S)$ by a factor of order n, and even a convex body with global symmetry greater than $\frac{1}{2}$ may require a rounding with factor n when using the MVIE (Brandenberg and Grundbacher, 2025, Example A.1).

Our constructive proof of the conjectured bound (2) modifies the MVCE problem for polytopes by adding a symmetry-dependent Minkowski center regularization term. In the case when S is a polytope (represented either as the convex hull of points or as the intersection of halfspaces), the desired rounding can be computed by first solving a linear optimization problem (to compute $\mathbf{sym}(S)$ and a Minkowski center), and then solving a convex optimization problem formulation of either the associated MVCE or MVIE problem depending on the polytope representation.

We also show that the bound (2) is sharp up to a universal constant for every n and every admissible symmetry value α, and is exactly sharp for an infinite class of pairs of n and α.

Figure 1 shows a 2-dimensional polytope K (in blue) along with (a) the MVCE ellipsoidal rounding, (b) the MVIE rounding, (c) our center-regularized MVCE rounding, and (d) our center-regularized MVIE rounding. Numerical details are provided in Appendix A. For this polytope, both classical volume-extremal roundings only attain the dimension-dependent factor 2, whereas both of our constructions yield rounding factors less than the bound $\sqrt{n/\,\mathbf{sym}(K)} \approx 1.6816$ and select centers visibly closer to the Minkowski center.

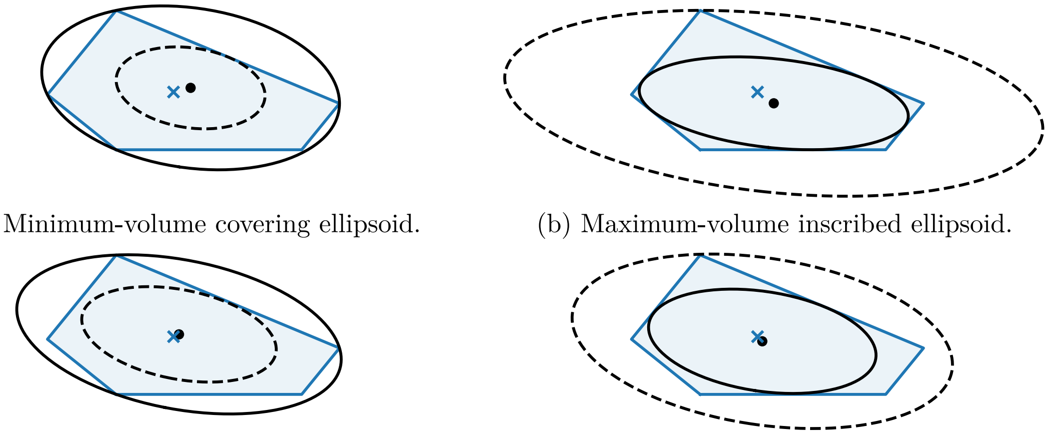
For $Q \succ 0$, define the origin-centered ellipsoid

$$E_Q := \left\{ x \in \mathbb{R}^n \,\middle|\, x^\top Q x \le 1 \right\} .$$

Our main result is the proof of the following symmetry-dependent rounding bound for every convex body.

Theorem 1.1. *Let $S \subset \mathbb{R}^n$ be a convex body. Then there exists a rounding of S with factor $\sqrt{\frac{n}{\mathbf{sym}(S)}}$, namely there exist $c \in \mathbb{R}^n$ and a matrix $Q \succ 0$ such that*

$$E_Q \subseteq S - c \subseteq \sqrt{\frac{n}{\mathbf{sym}(S)}}\, E_Q \,. \tag{3}$$



(a) Minimum-volume covering ellipsoid.

(b) Maximum-volume inscribed ellipsoid.

(c) Our center-regularized minimum-volume covering ellipsoid (Lemma 2.1).

(d) Our center-regularized maximum-volume inscribed ellipsoid (Lemma 4.1).

Figure 1: Four ellipsoidal roundings of the polytope K. The blue cross denotes the Minkowski center of K. The black dots denote ellipsoid centers.

In Section 2 we treat the case when S is a polytope. We introduce a center-regularized version of the MVCE problem, we derive an equivalent convex formulation and its dual, and we prove that the resulting ellipsoid yields the desired rounding stated in Theorem 1.1. In Section 3 we prove the general case of Theorem 1.1 from the polyhedral case by approximating S by a sequence of polytopes and taking limits. In Section 4 we further address the case when S is a polytope but is described as the intersection of finitely many halfspaces. In this representation, we present a center-regularized version of the MVIE problem and use this problem to (re-)prove the rounding guarantee of Theorem 1.1. In Section 5 we show that the rounding factor in (2) is nearly tight in its dependence on dimension and symmetry. When $\frac{n+1}{1+\mathbf{sym}(S)}$ is an integer, we show by explicit construction that the factor $\sqrt{\frac{n}{\mathbf{sym}(S)}}$ is tight. In the more general case, for every $n \geq 1$ and admissible $\mathbf{sym}(S)$, we construct a polytope for which every rounding factor is at least $\sqrt{\frac{2n}{3\,\mathbf{sym}(S)}}$.

Concurrent work of Brandenberg et al. (2026) in the context of Banach–Mazur distance establishes a related nonconcentric ellipsoidal containment guarantee. Stating their result in our notation, they prove that for every convex body S there exists an origin-centered ellipsoid E and possibly distinct centers c, d such that $c + E \subseteq S \subseteq d + \sqrt{\frac{n}{\mathbf{sym}(S)}}E$. In their construction, the outer ellipsoid is the MVCE of S, and the inner ellipsoid is a translated and scaled copy of the outer ellipsoid. They also present an example showing that the translation cannot in general be removed while retaining the MVCE as the outer ellipsoid. Nevertheless, their result implies a $2\sqrt{\frac{n}{\mathbf{sym}(S)}}$-rounding in the sense of (1).

Notation. For $d \geq 1$, let $\mathbb{R}^d_+$ denote the nonnegative orthant, let e denote the vector of all ones with dimension determined by context, and let I_d denote the $d \times d$ identity matrix. The symbols $\langle \cdot, \cdot \rangle$ and $\|\cdot\|_2$ denote the Euclidean inner product and norm, respectively. For $w \in \mathbb{R}^d$, $\mathrm{Diag}(w)$ is the diagonal matrix whose diagonal is w. The relations $Q \succeq 0$ and $Q \succ 0$ mean that Q is positive semidefinite and positive definite, respectively. For $Q \succ 0$, $Q^{\frac{1}{2}}$ denotes its unique positive definite square root. For $X \succ 0$, we use the notation $X^{-2} := (X^2)^{-1}$.

For a set $\Omega \subseteq \mathbb{R}^n$, a vector $c \in \mathbb{R}^n$, and a scalar t, write $\operatorname{int}\Omega$ for the interior of Ω, $\Omega - c := \{x - c : x \in \Omega\}$, and $t\Omega := \{tx : x \in \Omega\}$. The operator **conv** denotes the convex hull, and **sym** denotes the measure of symmetry defined above. We write B_2^n for the Euclidean unit ball and vol_n for n-dimensional volume. A *convex body* is a compact convex subset of $\mathbb{R}^n$ with nonempty interior.

2 Center-regularized MVCE for a polytope

In this section we prove Theorem 1.1 in the case when S is a polytope. We construct a center-regularized version of the MVCE optimization problem, and we show that a solution to this optimization problem achieves the desired $\sqrt{\frac{n}{\mathbf{sym}(S)}}$-rounding.

As every point in a polytope can be expressed as a weighting of finitely many points, we consider the following *W-representation* of S:

$$S = \mathbf{conv}\{a_1, \ldots, a_m\}\,, \tag{4}$$

where $a_1, \ldots, a_m$ are the listed points. Since S is assumed to be a convex body, we further assume without loss of generality that the origin is in the interior of S (which can be accomplished by choosing any interior point of S and translating to the origin). While the representation in (4) is commonly referred to as a V-representation, we prefer to call it a W-representation, with W standing for "weighting of points" since we do not need to assume that each of the a_i is a vertex.

First of all, we recall the ordinary MVCE problem. For an ellipsoid with center c and shape matrix Q, denoted by $c + E_Q$, its volume is $\mathrm{vol}_n(c + E_Q) = \mathrm{vol}_n(B_2^n)(\det Q)^{-\frac{1}{2}}$ (Grötschel et al., 1993). Because an ellipsoid is convex, it contains S if and only if it contains $a_1, \ldots, a_m$. Thus the MVCE solves the optimization problem

$$\begin{array}{lll} \underset{Q,c}{\text{minimize}} & -\log\det Q & \\ \text{subject to} & (a_i - c)^\top Q(a_i - c) \le 1\,, & i = 1, \ldots, m\,, \\ & Q \succ 0, \qquad c \in \mathbb{R}^n\,. & \end{array} \tag{5}$$

This formulation is standard (Todd, 2016).

To motivate our center-regularized version of the MVCE problem, suppose for the sake of argument that a point x^h with a high symmetry value $\mathbf{sym}(x^h, S)$ is known (e.g., x^h could be a Minkowski center of S), and that S has been translated by x^h so that $\mathbf{sym}(0, S)$ is high. Then the origin lies "deep" inside S. This suggests regularizing the ellipsoid center of the MVCE toward the origin while allowing the ellipsoid center and its shape to adjust jointly. For a regularization parameter $\lambda \ge 0$, we therefore consider the following center-regularized MVCE problem:

$$\begin{array}{lll} \underset{Q,c}{\text{minimize}} & -\log\det Q & \\ \text{subject to} & (a_i - c)^\top Q(a_i - c) + \lambda c^\top Q c \le 1\,, & i = 1, \ldots, m\,, \\ & Q \succ 0, \qquad c \in \mathbb{R}^n\,. & \end{array} \tag{P_λ}$$

Compared with (5), the additional term in each constraint discourages the center c from moving away from the origin. For every feasible (Q, c) it still holds that $S - c \subseteq E_Q$, and the objective function still minimizes the volume of E_Q. At $\lambda = 0$ (P_λ) specializes to (5), while at $\lambda = +\infty$ (P_λ) is equivalent to fixing $c = 0$.

It will be useful to set the regularization parameter λ in accordance with the symmetry of the origin:

$$\alpha := \mathbf{sym}(0, S)\,, \qquad \lambda_\alpha := \frac{1+\alpha}{1-\alpha}\,, \tag{7}$$

where in the case when $\alpha = 1$ we define $\lambda_1 := +\infty$. Geometrically, a larger value of α means that the origin is more centrally located in S. We therefore treat higher symmetry α as stronger geometric information that the origin should be a good ellipsoidal rounding center, and so penalize deviations from the origin more heavily. When S is centrally symmetric about the origin, $\lambda_\alpha = +\infty$, and the ellipsoid center c is fixed at the origin in (P_λ). We now present the following rounding results regarding optimal solutions of (P_λ), the proof of which will be the subject of most of this section.

Lemma 2.1. *Let S be a polytope defined as in (4) and suppose that $0 \in \operatorname{int} S$, and let α and λ_α be defined as in (7). The center-regularized problem (P_λ) with $\lambda = \lambda_\alpha$ attains its optimal value. Every optimal solution $(Q^\star, c^\star)$ satisfies*

$$\sqrt{\frac{\alpha}{n}}\, E_{Q^\star} \subseteq S - c^\star \subseteq E_{Q^\star}\,.$$

Moreover, the center $c^\star$ satisfies $c^\star \in \frac{1-\alpha}{2} S$. Furthermore, when $\alpha < 1$, the ellipsoid of the outer containment can be tightened to $\sqrt{1 - \lambda_\alpha (c^\star)^\top Q^\star c^\star}\, E_{Q^\star}$.

Consider the case when S has been translated so that the origin is a Minkowski center (so that $\alpha = \mathbf{sym}(S)$). Then applying Lemma 2.1 yields a $\sqrt{\frac{n}{\mathbf{sym}(S)}}$-rounding of S and thus proves Theorem 1.1 when S is a polytope. Lemma 2.1 is more general than Theorem 1.1 because it allows the origin to be *any* interior point of S, not necessarily a Minkowski center. Indeed, if $\mathbf{sym}(0, S) > \frac{1}{n}$, the center-regularized MVCE problem yields a $\sqrt{\frac{n}{\mathbf{sym}(0,S)}}$-rounding of S whose rounding guarantee is strictly better than the general n-rounding guarantee of the MVCE.

Note that the optimized center $c^\star$ of the center-regularized MVCE might not be a Minkowski center, nor the origin, nor the center of the ordinary MVCE. Belloni and Freund (2008) show that the MVCE center x_L yields a $\sqrt{\frac{n}{\mathbf{sym}(x_L,S)}}$-rounding, whereas Lemma 2.1 gives a $\sqrt{\frac{n}{\mathbf{sym}(0,S)}}$-rounding of S centered at $c^\star$ where notably, $c^\star$ need not equal 0. Intuitively, the greater the symmetry of the origin is, the closer $c^\star$ will be to the origin, which is embodied in the property that $c^\star \in \frac{1-\alpha}{2} S$.

The center-regularized MVCE is a generalization of the MVCE. If 0 is a Minkowski center of S and λ is chosen as in (7), then the center-regularized MVCE recovers the MVCE at the lower and upper extrema of $\mathbf{sym}(S)$. When $\mathbf{sym}(S) = 1$, then S is centrally symmetric about the origin (which is the Minkowski center). When $\mathbf{sym}(S) = \frac{1}{n}$, S must be a simplex, whose Minkowski center (the origin) is also the center of the MVCE. In both cases, the center of the MVCE must be the origin. Thus, the optimal solution $(Q^\star, c^\star)$ of (5) is still feasible for (P_λ) because $c^\star = 0$. This means $(Q^\star, c^\star)$ is also an optimal solution of (P_λ) for both extrema of $\mathbf{sym}(S)$. Furthermore, since every feasible solution of (P_λ) is also feasible for (5), the optimal solution of (P_λ) must be $(Q^\star, c^\star)$ as well for both extrema of $\mathbf{sym}(S)$. Additionally, Lemma 2.1 also recovers the classical MVCE rounding results for both general polytopes as well as centrally symmetric polytopes.

The center-regularized problem (P_λ) can be transformed into a convex problem by a change of variables. For $0 \le \lambda < +\infty$, make the change of variables

$$M := Q^{\frac{1}{2}}\,, \quad \text{and} \quad z := Mc\,, \tag{8}$$

which is the standard convexifying transformation used for the MVCE as in Sun and Freund (2004). Under this change of variables, problem (P_λ) is equivalent to the convex problem

$$
\begin{aligned}
\underset{M,z}{\text{minimize}} \quad & -2\log\det M \\
\text{subject to} \quad & \|Ma_i - z\|_2^2 + \lambda\|z\|_2^2 \le 1\,, \qquad i = 1,\dots,m\,, \\
& M \succ 0\,, \qquad z \in \mathbb{R}^n\,.
\end{aligned}
\tag{$\widetilde{\mathrm{P}}_\lambda$}
$$

From any solution (M, z) of ($\widetilde{\mathrm{P}}_\lambda$), the original variables are recovered using $Q = M^2$ and $c = M^{-1}z$. At the endpoint $\lambda = +\infty$, one sets $z = 0$ in ($\widetilde{\mathrm{P}}_\lambda$), leaving M as the only free variable.

We prove Lemma 2.1 in the rest of this section. For notational convenience, collect the listed points in the column-stacked matrix

$$
A := [a_1, \dots, a_m] \in \mathbb{R}^{n\times m}\,.
$$

2.1 Lagrange dual and optimality conditions of ($\widetilde{\mathrm{P}}_\lambda$)

Let $0 \le \lambda < +\infty$, and let $e \in \mathbb{R}^m$ denote the vector of all ones. Assigning a multiplier $u_i \ge 0$ to the ith constraint and using standard Lagrangian duality yields the following dual problem of ($\widetilde{\mathrm{P}}_\lambda$):

$$
\begin{aligned}
\underset{u\in\mathbb{R}^m_+}{\text{maximize}} \quad & n + \log\det\left(A\,\mathrm{Diag}(u)A^\top - \frac{(Au)(Au)^\top}{(1+\lambda)e^\top u}\right) - e^\top u \\
\text{subject to} \quad & e^\top u > 0\,, \\
& A\,\mathrm{Diag}(u)A^\top - \frac{(Au)(Au)^\top}{(1+\lambda)e^\top u} \succ 0\,,
\end{aligned}
\tag{D_λ}
$$

which is a minor extension of the MVCE dual formulations in Sun and Freund (2004); Gürtuna (2010). The primal problem ($\widetilde{\mathrm{P}}_\lambda$) attains its optimal value because its sublevel sets are compact. In addition, the Slater condition holds since sufficiently small positive multiples of $(I, 0)$ are strictly feasible for ($\widetilde{\mathrm{P}}_\lambda$), whereby strong duality and dual attainment both hold, and the Karush–Kuhn–Tucker optimality conditions are both necessary and sufficient.

Accordingly, (M, z) and the dual multiplier u are optimal for ($\widetilde{\mathrm{P}}_\lambda$) if and only if the following are satisfied:

$$
\text{Primal feasibility:} \quad M \succ 0\,, \quad z \in \mathbb{R}^n\,, \quad \|Ma_i - z\|_2^2 + \lambda\|z\|_2^2 \le 1 \quad \text{for } i = 1,\dots,m\,, \tag{11}
$$

$$
\text{Dual feasibility:} \quad u \in \mathbb{R}^m_+\,, \quad e^\top u > 0\,, \quad A\,\mathrm{Diag}(u)A^\top - \frac{(Au)(Au)^\top}{(1+\lambda)e^\top u} \succ 0\,, \tag{12}
$$

$$
\text{Stationarity:} \quad z = \frac{MAu}{(1+\lambda)e^\top u}\,, \qquad M^2 = \left(A\,\mathrm{Diag}(u)A^\top - \frac{(Au)(Au)^\top}{(1+\lambda)e^\top u}\right)^{-1}\,, \tag{13}
$$

$$
\text{Complementary slackness:} \quad u_i\left(\|Ma_i - z\|_2^2 + \lambda\|z\|_2^2 - 1\right) = 0 \quad \text{for } i = 1,\dots,m\,. \tag{14}
$$

At $\lambda = +\infty$, the same conditions apply after omitting z and all terms containing z, and replacing the matrix $A\,\mathrm{Diag}(u)A^\top - \frac{(Au)(Au)^\top}{(1+\lambda)e^\top u}$ in (12) and (13) by $A\,\mathrm{Diag}(u)A^\top$.

Similar to the MVCE problem, an immediate consequence of the optimality conditions is that $\sum_i u_i^\star$ is not free. Let $(M^\star, z^\star, u^\star)$ be an optimal primal-dual triple. Summing the complementary-slackness equations (14) and then using stationarity (13) yields

$$0 = \sum_{i=1}^{m} u_i^\star(\|M^\star a_i - z^\star\|_2^2 + \lambda\|z^\star\|_2^2 - 1) = \operatorname{tr}\left(M^\star \left[A\operatorname{Diag}(u^\star)A^\top - \frac{(Au^\star)(Au^\star)^\top}{(1+\lambda)e^\top u^\star}\right] M^\star\right) - e^\top u^\star$$

$$= \operatorname{tr}\left(\left[A\operatorname{Diag}(u^\star)A^\top - \frac{(Au^\star)(Au^\star)^\top}{(1+\lambda)e^\top u^\star}\right] (M^\star)^2\right) - e^\top u^\star = n - e^\top u^\star\,. \tag{15}$$

Hence any optimal $u^\star$ must satisfy $e^\top u^\star = n$. The same calculation proves this identity at $\lambda = +\infty$. An analogous identity holds for the MVCE problem (Sun and Freund, 2004).

Thus every dual optimizer lies on the slice $e^\top u = n$. For u on this slice, write

$$\tilde{u} := \frac{u}{n} \qquad \text{and} \qquad \widetilde{U} := \operatorname{Diag}(\tilde{u})\,,$$

and then $e^\top \tilde{u} = 1$. Since every dual optimizer must satisfy $e^\top\tilde{u} = 1$, restricting the dual in this way leaves its optima unchanged and so we consider the "reduced dual" problem:

$$\begin{aligned} \underset{\tilde{u}\in\mathbb{R}^m_+}{\text{maximize}} \quad & n\log n + \log\det\left(A\widetilde{U}A^\top - \frac{1}{1+\lambda}(A\tilde{u})(A\tilde{u})^\top\right) \\ \text{subject to} \quad & e^\top\tilde{u} = 1\,, \quad A\widetilde{U}A^\top - \frac{1}{1+\lambda}(A\tilde{u})(A\tilde{u})^\top \succ 0\,. \end{aligned} \tag{$\mathrm{D}^{\mathrm{red}}_\lambda$}$$

We use the convention $\frac{1}{1+\infty} := 0$ to define $(\mathrm{D}^{\mathrm{red}}_\infty)$ for consistency. Let us set the notation:

$$Q_{\tilde{u},\lambda} := A\widetilde{U}A^\top - \frac{1}{1+\lambda}(A\tilde{u})(A\tilde{u})^\top$$

for $0 \le \lambda \le +\infty$. Then a pair (M, z) and a vector $\tilde{u}$ are optimal for $(\widetilde{\mathrm{P}}_\lambda)$ and $(\mathrm{D}^{\mathrm{red}}_\lambda)$ if and only if (M, z) and $u = n\tilde{u}$ satisfy the optimality conditions (11)–(14). In particular, the stationarity relations (13) become

$$z = \frac{1}{1+\lambda}MA\tilde{u}\,, \qquad M^2 = \left(nQ_{\tilde{u},\lambda}\right)^{-1}.$$

Using $Q = M^2$ and $c = M^{-1}z$, these relations become

$$c = \frac{1}{1+\lambda}A\tilde{u}\,, \qquad Q^{-1} = nQ_{\tilde{u},\lambda}\,. \tag{17}$$

At $\lambda = +\infty$, the same statement holds after omitting z, c, and all terms where they appear.

Remark 2.2 (Connection with D-optimal design)**.** *The duality between MVCE problems and D-optimal experimental design is classical (Titterington, 1975; Gürtuna, 2010). For the origin-centered MVCE problem, the dual is the standard D-optimal-design problem on the points $a_1, \ldots, a_m$, with objective $\det(A\widetilde{U}A^\top)$. For the free-center MVCE problem, after the affine lifting $\widehat{a}_i := (a_i^\top, 1)^\top$, the dual problem is also a standard D-optimal-design problem on the lifted points $\widehat{a}_1, \ldots, \widehat{a}_m$.*

More generally, let $\widehat{A}$ denote the column-stacked matrix of $\widehat{a}_1, \ldots, \widehat{a}_m$. Then, for finite λ, it holds that $\det\left(\widehat{A}\widetilde{U}\widehat{A}^\top + \lambda e_{n+1}e_{n+1}^\top\right) = (1+\lambda)\det Q_{\tilde{u},\lambda}$ (where e_{n+1} is the unit vector in $\mathbb{R}^{n+1}$ with 1 in its last component). Thus $(D^{\mathrm{red}}_\lambda)$ can be viewed as a modified D-optimal-design problem with an additional fixed information term in the last lifted coordinate. In this way $(D^{\mathrm{red}}_\lambda)$ interpolates *between the D-optimal design on the lifted points at $\lambda = 0$ and the D-optimal design on the original points as $\lambda \to +\infty$.*

The reduced dual (D^{red}_λ) also has nice properties for algorithms. For every finite λ, (D^{red}_λ) falls within the framework of log-homogeneous-barrier problems of Dvurechensky et al. (2023); Zhao and Freund (2023); Zhao (2026), so the Frank–Wolfe method applies directly. The earlier Frank–Wolfe method of Ahipaşaoğlu et al. (2008) also applies to the D-optimal-design problems at the two extrema of λ.

2.2 Outer and inner ellipsoids from primal and dual feasibility

In this subsection we present outer and inner ellipsoidal containments obtained from (P_λ) and (D_λ).

Lemma 2.3. *Let $0 \le \lambda < +\infty$. Every feasible (Q, c) of (P_λ) satisfies*

$$S - c \subseteq \sqrt{1 - \lambda c^\top Q c}\, E_Q \subseteq E_Q\,.$$

In the case of $\lambda = +\infty$, the center c is fixed at zero and $S \subseteq E_Q$.

Proof. For finite λ, primal feasibility yields $(a_i - c)^\top Q(a_i - c) \le 1 - \lambda c^\top Q c$ for every i. The right-hand side is nonnegative because the left-hand side is nonnegative, and it is at most 1 because $\lambda c^\top Q c \ge 0$. Hence the middle ellipsoid contains every $a_i - c$ and therefore their convex hull which is $S - c$. Also the middle ellipsoid is itself contained in E_Q. For $\lambda = +\infty$, the center is fixed at zero and the constraints directly yield $a_i \in E_Q$ for every i, hence $S \subseteq E_Q$. $\square$

As in (7) let us assign $\alpha := \mathbf{sym}(0, S)$ and $\lambda := \lambda_\alpha$, from which it follows that

$$\frac{1}{1 + \lambda_\alpha} = \frac{1 - \alpha}{2}\,. \tag{18}$$

Lemma 2.4. *Let $\tilde u$ be feasible for ($\mathrm{D}^{\text{red}}_{\lambda_\alpha}$), and assign $c := \frac{1-\alpha}{2} A\tilde u$. Then $Q_{\tilde u, \lambda_\alpha} \succ 0$. Write $\widehat{Q} := Q_{\tilde u, \lambda_\alpha}$. We have*

$$\sqrt{\alpha}\, E_{\widehat{Q}^{-1}} = \left\{x \in \mathbb{R}^n :\ x^\top \widehat{Q}^{-1} x \le \alpha\right\} \subseteq S - c\,. \tag{19}$$

Moreover, the center c satisfies $c \in \frac{1-\alpha}{2} S$.

Proof. Since $\tilde u \ge 0$ and $e^\top \tilde u = 1$, we have $A\tilde u \in S$. Consequently, $c = \frac{1-\alpha}{2} A\tilde u \in \frac{1-\alpha}{2} S$.

Feasibility for ($\mathrm{D}^{\text{red}}_{\lambda_\alpha}$) implies $\widehat{Q} \succ 0$. By the definition of $\widehat{Q}$ and (18), we also have $\widehat{Q} = A\widetilde{U}A^\top - \frac{1-\alpha}{2}(A\tilde u)(A\tilde u)^\top$. To prove (19), it follows from a separating hyperplane argument that it is sufficient to show that for every $y \in \mathbb{R}^n$ the following inequality holds:

$$\max_{x \in \sqrt{\alpha}\, E_{\widehat{Q}^{-1}}} y^\top x \le \max_{x \in S - c} y^\top x\,. \tag{20}$$

The case $y = 0$ is immediate. For $y \ne 0$, the quantity $\max_i y^\top a_i$ is positive because $0 \in \operatorname{int} S$. Both sides of (20) are positively homogeneous in y, so, after replacing y by $\frac{y}{\max_i y^\top a_i}$, we may assume without loss of generality that $\max_i y^\top a_i = 1$.

For every x in the ellipsoid on the left-hand side in (20), the Cauchy–Schwarz inequality yields

$$y^\top x = (\widehat{Q}^{\frac{1}{2}} y)^\top (\widehat{Q}^{-\frac{1}{2}} x) \leq \sqrt{y^\top \widehat{Q} y}\sqrt{x^\top \widehat{Q}^{-1} x} \leq \sqrt{\alpha y^\top \widehat{Q} y}\,,$$

with equality attained above for $x = \frac{\sqrt{\alpha}\,\widehat{Q}y}{\sqrt{y^\top \widehat{Q} y}}$. Hence the left-hand side of (20) is

$$\max_{x \in \sqrt{\alpha}\, E_{\widehat{Q}^{-1}}} y^\top x = \sqrt{\alpha y^\top \widehat{Q} y}\,. \tag{21}$$

For the right-hand side of (20),

$$\max_{x \in S - c} y^\top x = \max_{s \in S} y^\top (s - c) = \max_i y^\top a_i - y^\top c = 1 - \frac{1-\alpha}{2} y^\top A\tilde{u}\,, \tag{22}$$

where the last equality uses $\max_i y^\top a_i = 1$ and $c = \frac{1-\alpha}{2} A\tilde{u}$.

For a given index i it holds that $y^\top a_i \leq 1$. On the other hand, since $-\alpha S \subseteq S$, one also has $-\alpha a_i \in S$, so $-\alpha y^\top a_i \leq \max_j y^\top a_j = 1$ and hence $y^\top a_i \geq -\frac{1}{\alpha}$. Write $t_i := y^\top a_i$. Then

$$\alpha t_i^2 + (1-\alpha) t_i \leq 1\,, \tag{23}$$

because $1 - (\alpha t_i^2 + (1-\alpha)t_i) = (1 - t_i)(1 + \alpha t_i) \geq 0$ for $-\frac{1}{\alpha} \leq t_i \leq 1$. Multiplying (23) by $\tilde{u}_i$ and summing over $i = 1, \ldots, m$ yields

$$\alpha \sum_{i=1}^m \tilde{u}_i (y^\top a_i)^2 + (1-\alpha) \sum_{i=1}^m \tilde{u}_i y^\top a_i \leq \sum_{i=1}^m \tilde{u}_i = 1\,,$$

where the last equality uses $e^\top \tilde{u} = 1$. In matrix notation, using $A\tilde{u} = \sum_i \tilde{u}_i a_i$ and $A\widetilde{U}A^\top = \sum_i \tilde{u}_i a_i a_i^\top$, this is

$$\alpha y^\top A\widetilde{U}A^\top y + (1-\alpha) y^\top A\tilde{u} \leq 1\,. \tag{24}$$

Therefore

$$\begin{aligned}
\alpha y^\top \widehat{Q} y &= \alpha y^\top A\widetilde{U}A^\top y - \frac{\alpha(1-\alpha)}{2}(y^\top A\tilde{u})^2 \leq 1 - (1-\alpha) y^\top A\tilde{u} - \frac{\alpha(1-\alpha)}{2}(y^\top A\tilde{u})^2 \\
&= \left(1 - \frac{1-\alpha}{2} y^\top A\tilde{u}\right)^2 - \frac{1-\alpha^2}{4}(y^\top A\tilde{u})^2 \leq \left(1 - \frac{1-\alpha}{2} y^\top A\tilde{u}\right)^2 .
\end{aligned} \tag{25}$$

Here the first equality expands $\widehat{Q} = A\widetilde{U}A^\top - \frac{1-\alpha}{2}(A\tilde{u})(A\tilde{u})^\top$, the first inequality is (24), and the last inequality uses $0 < \alpha \leq 1$. It follows from (21) that the left-hand side of (25) is the square of the left-hand side of (20). And it follows from (22) that the right-hand side of (25) is the square of the right-hand side of (20). The first maximum in (20) is nonnegative; also $A\tilde{u} \in S$ and the normalization $\max_i y^\top a_i = 1$ imply that $1 - \frac{1-\alpha}{2} y^\top A\tilde{u} \geq \frac{1+\alpha}{2} > 0$ and so the second maximum in (20) is nonnegative. Therefore taking square roots in (25) yields (20), and hence (19). □

2.3 Proof of Lemma 2.1

Proof of Lemma 2.1. Let $(Q^\star, c^\star)$ be an optimal solution of $(\mathrm{P}_{\lambda_\alpha})$. By the change of variables (8) it follows that $M := (Q^\star)^{\frac{1}{2}}$ and $z := Mc^\star$ are optimal solutions of $(\widetilde{\mathrm{P}}_\lambda)$ whereby there exists u for which the triplet (M, z, u) satisfies the optimality conditions (11), (12), (13), and (14), and in fact as shown earlier it holds that $e^\top u = n$. Therefore $\tilde{u} := \frac{u}{n}$ is feasible for the reduced dual problem ($\mathrm{D}^{\mathrm{red}}_{\lambda_\alpha}$), and also it follows from (13), (17), and the notation $\widehat{Q} := Q_{\tilde{u},\lambda_\alpha}$ from Lemma 2.4 that

$$c^\star = \frac{1}{1+\lambda_\alpha} A\tilde{u}\,, \qquad (Q^\star)^{-1} = M^{-2} = nQ_{\tilde{u},\lambda_\alpha} = n\widehat{Q}\,. \tag{26}$$

When $\alpha = 1$, (26) remains valid: indeed, $\lambda_\alpha = +\infty$ fixes $c^\star = 0$, and the endpoint stationarity condition yields $(Q^\star)^{-1} = nQ_{\tilde{u},\infty} = n\widehat{Q}$.

It therefore follows from Lemma 2.4 that $\sqrt{\alpha}E_{\widehat{Q}^{-1}} \subset S - c^\star$. Now note from (26) that $\widehat{Q}^{-1} = nQ^\star$ whereby $\sqrt{\alpha}E_{\widehat{Q}^{-1}} = \sqrt{\frac{\alpha}{n}}E_{Q^\star}$ and therefore

$$\sqrt{\frac{\alpha}{n}}E_{Q^\star} \subset S - c^\star\,,$$

which is the inscribed inclusion in Lemma 2.1.

The outer containment statements of Lemma 2.1 follow by applying Lemma 2.3 to $(Q^\star, c^\star)$. It also follows from (18) and (26) that $c^\star \in \frac{1-\alpha}{2}S$. $\square$

3 Proof of Theorem 1.1

Lemma 2.1 provides the guaranteed $\sqrt{\frac{n}{\mathbf{sym}(S)}}$-rounding of Theorem 1.1 in the case when S is a polytope. When S is a general convex body we prove Theorem 1.1 by approximating S by a sequence of polytopes and taking limits.

Proof of Theorem 1.1. For any $\varepsilon > 0$ there exists a polytope P that approximates S in the sense that $S \subset P \subset S + \varepsilon B_2^n$. Let $\{P_k\}$ be a sequence of polytopes converging to S such that $S \subseteq P_k \subseteq S + \varepsilon_k B_2^n$ for some sequence $\varepsilon_k \downarrow 0$. After translating each P_k so that a Minkowski center lies at the origin and applying Lemma 2.1 directly to P_k, it follows for each k that there exist $Q_k \succ 0$ and $c_k \in P_k$ such that

$$\sqrt{\frac{\mathbf{sym}(P_k)}{n}}\, E_{Q_k} \subseteq P_k - c_k \subseteq E_{Q_k}\,. \tag{27}$$

The centers c_k are uniformly bounded, and the ellipsoids E_{Q_k} are uniformly bounded and nondegenerate. Therefore, there exists a subsequence, indexed by k_j, such that $c_{k_j} \to c$ and $Q_{k_j} \to Q$ for some c and Q. Furthermore, since $P_{k_j} \to S$, it holds that $\mathbf{sym}(P_{k_j}) \to \mathbf{sym}(S)$. Taking limits in (27) along this subsequence yields $\sqrt{\frac{\mathbf{sym}(S)}{n}}\, E_Q \subseteq S - c \subseteq E_Q$. $\square$

4 Center-regularized MVIE for a polytope in H-representation

In Section 2, we constructed the rounding ellipsoid (the center-regularized MVCE) for a polytope in W-representation. This construction involves first translating the polytope to the origin by a Minkowski center, and then solving (P_λ) to obtain the ellipsoid. A Minkowski center and the value $\mathbf{sym}(S)$ can be computed by linear programming when S is given as the convex hull (see Belloni and Freund, 2008). After translating such a center to the origin, the change of variables turns (P_λ) into ($\widetilde{\mathrm{P}}_\lambda$), a convex problem with a logarithmic determinant objective, m second-order cone constraints, and one positive semidefinite cone constraint. This problem ($\widetilde{\mathrm{P}}_\lambda$) can be addressed by an interior point method (Vandenberghe et al., 1998).

However, in many cases in real-world applications, a polytope is given in H-representation, i.e., as the intersection of finitely many halfspaces. Transferring a polytope from an H-representation to a W-representation is generally a hard task and impractical. In this section, we construct a center-regularized version of the MVIE optimization problem for polytopes in H-representation. The construction is analogous to the center-regularized MVCE problem, but is naturally better suited to polytopes in H-representation.

Suppose that S contains the origin as an interior point and is described by the intersection of ℓ halfspaces, i.e., it is given in H-representation as

$$S = \left\{ x \in \mathbb{R}^n \,\middle|\, h_i^\top x \le 1, \quad i = 1, \dots, \ell \right\}, \tag{28}$$

where $h_1, \dots, h_\ell$ are the listed nonzero halfspace normals. Apparently, the origin is an interior point.

We first recall the ordinary MVIE optimization problem. For $Q \succ 0$, because $\max_{x \in E_Q} h_i^\top x = \sqrt{h_i^\top Q^{-1} h_i}$, we have:

$$\begin{aligned} c + E_Q \subseteq S \quad &\iff \quad h_i^\top c + \sqrt{h_i^\top Q^{-1} h_i} \le 1 && \text{for } i = 1, \dots, \ell, \\ &\iff \quad h_i^\top Q^{-1} h_i \le (1 - h_i^\top c)^2 \quad \text{and} \quad h_i^\top c \le 1 && \text{for } i = 1, \dots, \ell. \end{aligned} \tag{29}$$

Therefore, the ordinary MVIE is obtained from

$$\begin{aligned} \underset{X, c}{\text{minimize}} \quad & -2 \log \det X \\ \text{subject to} \quad & \|X h_i\|_2^2 \le (1 - h_i^\top c)^2, \qquad h_i^\top c \le 1, \quad i = 1, \dots, \ell, \\ & X \succ 0, \qquad c \in \mathbb{R}^n. \end{aligned} \tag{30}$$

The corresponding ellipsoid is $c + E_{X^{-2}}$. This convex formulation and a polynomial-time algorithm appear in Khachiyan and Todd (1993). Related algorithms for solving it include Anstreicher (2002); Zhang and Gao (2003).

To regularize the center, we rewrite the first constraint in (30) as $\|X h_i\|_2^2 - (h_i^\top c)^2 + 2 h_i^\top c \le 1$ and add the center penalty $\lambda (h_i^\top c)^2$, which yields the following problem.

$$\begin{aligned} \underset{X, c}{\text{minimize}} \quad & -2 \log \det X \\ \text{subject to} \quad & \|X h_i\|_2^2 + (\lambda - 1)(h_i^\top c)^2 + 2 h_i^\top c \le 1, \quad i = 1, \dots, \ell, \\ & X \succ 0, \qquad c \in \mathbb{R}^n. \end{aligned} \tag{J$_\lambda$}$$

The objective maximizes the volume of $E_{X^{-2}}$. We choose $\lambda \ge 1$, so feasibility implies $h_i^\top c < 1$ for every i, so $c \in \operatorname{int} S$. The objective and all constraints in (J$_\lambda$) are convex. At $\lambda = +\infty$, we fix $c = 0$ and impose $\|X h_i\|_2^2 \le 1$ for every i.

Lemma 4.1. *Let S be the polytope defined as in (28), and let α and λ_α be defined as in (7). The center-regularized problem (J$_\lambda$) with $\lambda = \lambda_\alpha$ attains its optimal value. Every optimal solution $(X^\star, c^\star)$, with $Q^\star := (X^\star)^{-2}$, satisfies*

$$E_{Q^\star} \subseteq S - c^\star \subseteq \sqrt{\frac{n}{\alpha}}\, E_{Q^\star}.$$

Moreover, the ellipsoid in the inner containment can be enlarged to $\left(\min_i \frac{1 - h_i^\top c^\star}{\|X^\star h_i\|_2}\right) E_{Q^\star}$.

For any polytope in H-representation with nonempty interior, translating a Minkowski center to the origin and then applying Lemma 4.1 yields a $\sqrt{\frac{n}{\mathbf{sym}(S)}}$-rounding and proves Theorem 1.1 for polytopes.

As with the center-regularized MVCE, if 0 is a Minkowski center of S and λ is chosen as in (7), then the center-regularized MVIE problem (J_λ) recovers the MVIE at both endpoints of $\mathbf{sym}(S)$. Lemma 4.1 likewise recovers the classical MVIE rounding guarantees at both endpoints of $\mathbf{sym}(S)$.

Computing the center-regularized MVIE involves first computing the Minkowski center and then solving (J_λ). A Minkowski center and the value $\mathbf{sym}(S)$ can be computed by linear programming (see Belloni and Freund, 2008). After translating such a center to the origin, (J_λ) is a convex problem with a logarithmic determinant objective, ℓ second-order cone constraints, and one positive semidefinite cone constraint. It can be addressed by an interior point method.

We prove Lemma 4.1 in the rest of this section. For notational convenience, collect the halfspace normals in the column-stacked matrix

$$H := [h_1, \ldots, h_\ell] \in \mathbb{R}^{n\times\ell}\,.$$

4.1 Lagrange dual and optimality conditions of (J_λ)

In this subsection, we derive the Lagrange dual of (J_λ) and state its optimality conditions. For $1 < \lambda < +\infty$, assign a multiplier $v_i \geq 0$ to the ith constraint of (J_λ). The Lagrangian is

$$L(X, c, v) = -2\log\det X + \mathrm{tr}\Big(XH\,\mathrm{Diag}(v)H^\top X\Big) + (\lambda-1)c^\top H\,\mathrm{Diag}(v)H^\top c + 2c^\top Hv - e^\top v\,.$$

The primal problem can be written as $\min_{X\succ 0,\, c\in\mathbb{R}^n}\max_{v\geq 0} L(X,c,v)$. The Lagrange dual interchanges these two optimizations and maximizes the dual function

$$g(v) := \min_{X\succ 0,\, c\in\mathbb{R}^n} L(X,c,v)$$

over $v \geq 0$. We therefore compute this minimum by minimizing first over X and then over c. If $H\,\mathrm{Diag}(v)H^\top$ is singular, the objective is unbounded below along a null direction. Thus only positive definite matrices can occur in the domain of g. When $H\,\mathrm{Diag}(v)H^\top \succ 0$, the matrix part is

$$\min_{X\succ 0}\Big\{-2\log\det X + \mathrm{tr}\Big(XH\,\mathrm{Diag}(v)H^\top X\Big)\Big\} = n + \log\det\Big(H\,\mathrm{Diag}(v)H^\top\Big)\,,$$

and its unique minimizer is $X = (H\,\mathrm{Diag}(v)H^\top)^{-\frac{1}{2}}$. Next, minimization over c gives $c = -(\lambda-1)^{-1}(H\,\mathrm{Diag}(v)H^\top)^{-1}Hv$. Substitution gives the Lagrange dual

$$\begin{aligned} \underset{v\in\mathbb{R}^\ell_+}{\text{maximize}}\quad & n + \log\det\Big(H\,\mathrm{Diag}(v)H^\top\Big) - e^\top v - \frac{1}{\lambda-1}(Hv)^\top\Big(H\,\mathrm{Diag}(v)H^\top\Big)^{-1}Hv \\ \text{subject to}\quad & H\,\mathrm{Diag}(v)H^\top \succ 0\,. \end{aligned} \tag{$\mathrm{D}^{\mathrm{J}}_\lambda$}$$

For the rest of the proof, we write

$$G_v := H\,\mathrm{Diag}(v)H^\top\,. \tag{33}$$

The objective function in ($\mathrm{D}^{\mathrm{J}}_\lambda$) is concave in v on the domain $G_v \succ 0$.

The primal problem has compact sublevel sets and therefore attains its optimal value. The point $(X, c) = (\varepsilon I, 0)$ is strictly feasible for every sufficiently small $\varepsilon > 0$, so Slater's condition gives strong duality and dual attainment. The Karush–Kuhn–Tucker optimality conditions are therefore necessary and sufficient (see also Gürtuna, 2010, for the dual problem of the ordinary MVIE). Thus (X, c) and v are optimal for (J_λ) and ($\mathrm{D}^{\mathrm{J}}_\lambda$) if and only if

$$\begin{aligned}
&\text{Primal feasibility:}\quad X \succ 0, \quad c \in \mathbb{R}^n, \quad \|Xh_i\|_2^2 + (\lambda - 1)(h_i^\top c)^2 + 2h_i^\top c \le 1 \quad (i = 1, \ldots, \ell)\,.\\
&\text{Dual feasibility:}\quad v \in \mathbb{R}^\ell_+, \quad G_v \succ 0\,.\\
&\text{Stationarity:}\quad X = G_v^{-1/2}, \qquad Hv = -(\lambda - 1)G_v c\,.\\
&\text{Complementary slackness:}\quad v_i(\|Xh_i\|_2^2 + (\lambda - 1)(h_i^\top c)^2 + 2h_i^\top c - 1) = 0 \quad (i = 1, \ldots, \ell)\,.
\end{aligned} \tag{34}$$

At $\lambda = +\infty$, one replaces the primal and complementary slackness expressions by their fixed center forms $\|Xh_i\|_2^2 \le 1$, fixes $c = 0$, and deletes the second stationarity equation.

From the optimality conditions, we also obtain the following equality for every optimal solution $v^\star$ of ($\mathrm{D}^{\mathrm{J}}_\lambda$).

Lemma 4.2. *For $1 < \lambda < +\infty$, every optimizer $v^\star$ of ($\mathrm{D}^{\mathrm{J}}_\lambda$) satisfies*

$$e^\top v^\star + \frac{1}{\lambda - 1}(Hv^\star)^\top G_{v^\star}^{-1} Hv^\star = n\,. \tag{35}$$

At $\lambda = +\infty$, every dual optimizer satisfies $e^\top v^\star = n$.

Proof. If $v^\star$ is dual optimal, then $tv^\star$ is dual feasible for every $t > 0$. Along this ray, the dual objective is

$$n + \log\det G_{v^\star} + n\log t - t\left(e^\top v^\star + \frac{1}{\lambda - 1}(Hv^\star)^\top G_{v^\star}^{-1} Hv^\star\right). \tag{36}$$

The derivative of (36) at the maximizer $t = 1$ is zero, which gives (35). At $\lambda = +\infty$, the final term in parentheses is absent and the same argument gives $e^\top v^\star = n$. □

4.2 Inner and outer ellipsoids from primal and dual feasibility

The inner containment uses only primal feasibility.

Lemma 4.3. *Let $1 < \lambda \le +\infty$. Every feasible (X, c) of (J_λ) satisfies*

$$E_{X^{-2}} \subseteq \left(\min_i \frac{1 - h_i^\top c}{\|Xh_i\|_2}\right) E_{X^{-2}} \subseteq S - c\,.$$

Proof. For finite λ, rearranging primal feasibility yields $\|Xh_i\|_2^2 + \lambda(h_i^\top c)^2 \le (1 - h_i^\top c)^2$, and hence $\|Xh_i\|_2^2 \le (1 - h_i^\top c)^2$. Feasibility and $\lambda > 1$ also imply $1 - h_i^\top c > 0$. Thus every nonzero h_i satisfies $\frac{1 - h_i^\top c}{\|Xh_i\|_2} \ge 1$. At $\lambda = +\infty$, the center is zero and the constraints $\|Xh_i\|_2^2 \le 1$ yield the same inequality. The factor $\min_i \frac{1 - h_i^\top c}{\|Xh_i\|_2}$ is therefore at least 1, which proves the first containment. Moreover, for every nonzero h_i,

$$h_i^\top c + \left(\min_j \frac{1 - h_j^\top c}{\|Xh_j\|_2}\right) \|Xh_i\|_2 \le 1\,,$$

so (29), applied with $Q = \left(\min_j \frac{1 - h_j^\top c}{\|Xh_j\|_2}\right)^{-2} X^{-2}$, proves the second containment. □

Lemma 4.4. *Let $0 < \alpha < 1$ and $\lambda = \lambda_\alpha$. Suppose that v is feasible for ($\mathrm{D}^{\mathrm{J}}_{\lambda}$) and also satisfies*

$$e^\top v + \frac{1}{\lambda - 1}(Hv)^\top G_v^{-1} Hv \le n\,. \tag{37}$$

Set $c_v := -(\lambda-1)^{-1} G_v^{-1} Hv$. Then

$$S - c_v \subseteq \sqrt{\frac{n}{\alpha}}\, E_{G_v}\,. \tag{38}$$

At $\alpha = 1$, every $v \in \mathbb{R}^\ell_+$ satisfying $G_v \succ 0$ and $e^\top v \le n$ satisfies $S \subseteq \sqrt{n}\, E_{G_v}$.

Proof. Suppose first that $0 < \alpha < 1$. By the definition of E_{G_v}, (38) is equivalent to proving that every $x \in S$ satisfies

$$(x - c_v)^\top G_v (x - c_v) \le \frac{n}{\alpha}\,. \tag{39}$$

The symmetry assumption first gives a scalar inequality for every halfspace normal. If $x \in S$, then $h_i^\top x \le 1$. Since $-\alpha x \in S$, we also have $h_i^\top x \ge -\frac{1}{\alpha}$. Consequently, $(1 - h_i^\top x)(1 + \alpha h_i^\top x) \ge 0$, and expanding it gives

$$\alpha (h_i^\top x)^2 + (1-\alpha) h_i^\top x \le 1\,. \tag{40}$$

Multiplying these inequalities by the nonnegative weights v_i and summing yields

$$\sum_{i=1}^{\ell} v_i \big(\alpha (h_i^\top x)^2 + (1-\alpha) h_i^\top x\big) = \alpha x^\top G_v x + (1-\alpha)(Hv)^\top x \le e^\top v\,. \tag{41}$$

Here the equality follows from (33), and the inequality follows from (40) and $v \ge 0$.

To recover the quadratic form in (39), use the definition of c_v and $(1-\alpha)(\lambda_\alpha - 1) = 2\alpha$ (by the definition of λ_α) to obtain: $(1-\alpha)Hv = -2\alpha G_v c_v$. Substituting this identity into (41) and adding $\alpha c_v^\top G_v c_v$ to both sides yields

$$\alpha (x - c_v)^\top G_v (x - c_v) \le e^\top v + \alpha c_v^\top G_v c_v\,. \tag{42}$$

Since $\lambda = \lambda_\alpha$, we have $\lambda_\alpha - 1 = \frac{2\alpha}{1-\alpha} \ge \alpha$. Therefore

$$e^\top v + \alpha c_v^\top G_v c_v \le e^\top v + (\lambda_\alpha - 1) c_v^\top G_v c_v = e^\top v + \frac{1}{\lambda - 1}(Hv)^\top G_v^{-1} Hv \le n\,.$$

Here the equality uses $Hv = -(\lambda - 1) G_v c_v$, which follows from the definition of c_v, and the final inequality is (37). Combining this bound with (42) and dividing by α proves (39). As $x \in S$ was arbitrary, this is the desired containment.

At $\alpha = 1$, the assumed bound is $e^\top v \le n$, and the relation $-S \subseteq S$ implies $|h_i^\top x| \le 1$ for every $x \in S$ and every i. Then $x^\top G_v x = \sum_{i=1}^{\ell} v_i (h_i^\top x)^2 \le e^\top v \le n$. This is exactly $S \subseteq \sqrt{n}\, E_{G_v}$. □

4.3 Proof of Lemma 4.1

Proof of Lemma 4.1. Let $(X^\star, c^\star)$ be an arbitrary optimal solution of ($\mathrm{J}_{\lambda_\alpha}$), and set $Q^\star := (X^\star)^{-2}$. Strong duality and dual attainment yield a dual optimizer $v^\star$ that satisfies the optimality conditions with $(X^\star, c^\star)$. By Lemma 4.2, $v^\star$ also satisfies the additional condition in Lemma 4.4.

Suppose first that $0 < \alpha < 1$. The stationarity conditions in (34) yield

$$c^\star = -\frac{1}{\lambda_\alpha - 1} G_{v^\star}^{-1} H v^\star = c_{v^\star}\,, \qquad X^\star = G_{v^\star}^{-\frac{1}{2}}\,.$$

In particular, $G_{v^\star} = (X^\star)^{-2} = Q^\star$. The two inner containments follow from Lemma 4.3, the outer containment follows from Lemma 4.4, and the two stationarity identities show that the inner and outer ellipsoids have the same center and shape.

At $\alpha = 1$, the center is fixed at zero. The stationarity condition gives $G_{v^\star} = (X^\star)^{-2} = Q^\star$, and Lemmas 4.3 and 4.4 give the same sandwich with factor $\sqrt{n}$. □

5 Sharpness of the symmetry-dependence in rounding

In this section, we show that the factor $\sqrt{\frac{n}{\mathbf{sym}(S)}}$ is nearly tight in its dependence on dimension and symmetry. When $\frac{n+1}{1+\mathbf{sym}(S)}$ is an integer, we show by explicit construction that the factor $\sqrt{\frac{n}{\mathbf{sym}(S)}}$ is tight. More generally, for every dimension n and every admissible symmetry value, the factor is sharp up to a universal constant.

Fix an integer $n \geq 1$, and let $\Delta_n := \mathbf{conv}\{a_1, \ldots, a_{n+1}\}$ where $a_1, \ldots, a_{n+1} \in \mathbb{R}^n$ satisfy

$$\|a_i\|_2 = 1 \quad \text{for all } i, \qquad \langle a_i, a_j\rangle = -\frac{1}{n} \quad \text{for all } i \neq j\,. \tag{43}$$

It is easy to show that with (43), $\sum_{i=1}^{n+1} a_i = 0$ and all edges have the same length (Lemma 5.2). We therefore call Δ_n the *normalized regular simplex*. For $n = 1$, $a_1, \ldots, a_{n+1}$ could be the two points $a_1 = 1$ and $a_2 = -1$. For $n = 2$, they could be the vertices $(1, 0)$ and $(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$ of an equilateral triangle on the unit circle.

The family of convex bodies we study is the convex hull of Δ_n and its scaled reflection:

$$S_{n,\alpha} := \mathbf{conv}(\Delta_n \cup (-\alpha\Delta_n)) = \mathbf{conv}\{a_i, -\alpha a_i : i = 1, \ldots, n+1\} \subset \mathbb{R}^n\,.$$

Theorem 5.1. *For every $n \geq 1$ and every $\alpha \in [\frac{1}{n}, 1]$, the body $S_{n,\alpha}$ satisfies $\mathbf{sym}(0, S_{n,\alpha}) = \mathbf{sym}(S_{n,\alpha}) = \alpha$. Moreover, every $c \in \mathbb{R}^n$, origin-centered ellipsoid E, and $\rho \geq 1$ satisfying $E \subseteq S_{n,\alpha} - c \subseteq \rho E$ obey*

$$\rho \geq \sqrt{\frac{2n}{3\alpha}}\,. \tag{44}$$

If, in addition, $\frac{n+1}{1+\alpha}$ is an integer, then

$$\rho \geq \sqrt{\frac{n}{\alpha}}\,. \tag{45}$$

Brandenberg et al. (2026) also independently construct equality cases for the related Banach–Mazur distance bound when $\frac{1}{\alpha}$ is an integer divisor of n. Since the Banach–Mazur distance to the Euclidean ball is a lower bound on every rounding factor, their examples also yield the lower bound $\sqrt{n/\alpha}$ for those parameter pairs. The family $S_{n,\alpha}$ defined above for every admissible pair (n, α) yields the uniform lower bound $\sqrt{2n/(3\alpha)}$ throughout the full parameter range, and establishes exact sharpness whenever $\frac{n+1}{1+\alpha}$ is an integer.

The proof uses the symmetry of $S_{n,\alpha}$ and the radii of its extremal covering and inscribed ellipsoids. We begin with two facts about the normalized regular simplex.

Lemma 5.2. *For the normalized regular simplex $\Delta_n = \mathbf{conv}\{a_1, \dots, a_{n+1}\} \subset \mathbb{R}^n$ defined above, one has $\sum_{i=1}^{n+1} a_i = 0$, all edges of Δ_n have the same length $\sqrt{2(n+1)/n}$, and $\sum_{i=1}^{n+1} a_i a_i^\top = \frac{n+1}{n} I_n$.*

Proof. Let $s := \sum_i a_i$. For every j, $\langle s, a_j \rangle = 1 + \sum_{i \neq j} \langle a_i, a_j \rangle = 1 - n(1/n) = 0$. Since the vertices span $\mathbb{R}^n$, $s = 0$. Moreover, for $i \neq j$, $\|a_i - a_j\|_2^2 = 2 - 2\langle a_i, a_j \rangle = 2(n+1)/n$, so all edges have the same length.

Let $H := \sum_i a_i a_i^\top$. Fix j. In Ha_j, the term with $i = j$ equals a_j, whereas each term with $i \neq j$ equals $-a_i/n$. Since $\sum_i a_i = 0$, one has $\sum_{i \neq j} a_i = -a_j$. Therefore $Ha_j = a_j - \frac{1}{n} \sum_{i \neq j} a_i = \frac{n+1}{n} a_j$. The vertices span $\mathbb{R}^n$, so $H = \frac{n+1}{n} I_n$. □

Let Π denote the group of all permutations of $\{1, \dots, n+1\}$.

Lemma 5.3. *For every permutation $\pi \in \Pi$, there exists a unique linear map $T_\pi : \mathbb{R}^n \to \mathbb{R}^n$ such that $T_\pi a_i = a_{\pi(i)}$ for every i. Moreover, T_π is orthogonal, $T_\pi S_{n,\alpha} = S_{n,\alpha}$, and $\sum_{\pi \in \Pi} T_\pi = 0$.*

Proof. First, $a_1, \dots, a_n$ form a basis of $\mathbb{R}^n$. Fix $\pi \in \Pi$. There is therefore a unique linear map T_π satisfying $T_\pi a_i = a_{\pi(i)}$ for $i = 1, \dots, n$. Since $\sum_{i=1}^{n+1} a_i = 0$, linearity gives $T_\pi a_{n+1} = -\sum_{i=1}^n T_\pi a_i = -\sum_{i=1}^n a_{\pi(i)} = a_{\pi(n+1)}$, so $T_\pi a_i = a_{\pi(i)}$ for every i. Since T_π permutes both sets of vertices $\{a_i\}$ and $\{-\alpha a_i\}$, taking convex hulls gives $T_\pi S_{n,\alpha} = S_{n,\alpha}$.

The permutation preserves every inner product between vertices. More explicitly, if $x = \sum_{i=1}^n c_i a_i$ and $y = \sum_{j=1}^n d_j a_j$, then

$$\langle T_\pi x, T_\pi y \rangle = \sum_{i,j=1}^n c_i d_j \langle a_{\pi(i)}, a_{\pi(j)} \rangle = \sum_{i,j=1}^n c_i d_j \langle a_i, a_j \rangle = \langle x, y \rangle .$$

This proves that T_π is orthogonal.

Finally, for each i, every vertex occurs exactly $n!$ times among the vectors $a_{\pi(i)}$. Hence $(\sum_{\pi \in \Pi} T_\pi) a_i = n! \sum_{j=1}^{n+1} a_j = 0$. The vectors $a_1, \dots, a_{n+1}$ span $\mathbb{R}^n$, and hence the linear map $\sum_{\pi \in \Pi} T_\pi$ is zero. □

5.1 Symmetry of $S_{n,\alpha}$

In this subsection, we show the symmetry of $S_{n,\alpha}$.

Lemma 5.4. *The origin is a Minkowski center of $S_{n,\alpha}$, and*

$$\mathbf{sym}(0, S_{n,\alpha}) = \mathbf{sym}(S_{n,\alpha}) = \alpha .$$

A short consequence of the general asymmetry comparison in Brandenberg and König (2015, Theorem 6.1) is the following: if K is Minkowski centered at the origin and $\tau \in [\mathbf{sym}(K), 1]$, then the origin is a Minkowski center of $\mathbf{conv}(K \cup (-\tau K))$ and $\mathbf{sym}(\mathbf{conv}(K \cup (-\tau K))) = \tau$. Taking $K = \Delta_n$ and $\tau = \alpha$ gives Lemma 5.4. For completeness, we include the following short self-contained proof.

Proof of Lemma 5.4. Since $0 \in \Delta_n$ and $0 < \alpha \leq 1$, one has $\alpha^2 \Delta_n \subseteq \Delta_n$. Scaling the defining convex hull by $-\alpha$ gives the convex hull of $-\alpha \Delta_n$ and $\alpha^2 \Delta_n$. Therefore $-\alpha S_{n,\alpha} \subseteq S_{n,\alpha}$. Thus $\mathbf{sym}(0, S_{n,\alpha}) \geq \alpha$.

For the reverse inequality, fix any vertex a_i. By (43),

$$\min_{x\in S_{n,\alpha}} \langle a_i, x\rangle = \min\left\{\min_j\langle a_i, a_j\rangle, \min_j\langle a_i, -\alpha a_j\rangle\right\} = \min\left\{-\frac{1}{n}, -\alpha\right\} = -\alpha\,.$$

Here the last equality uses $\alpha \geq \frac{1}{n}$. Suppose that $c \in \mathbb{R}^n$ and $\beta > 0$ satisfy

$$(1+\beta)c - \beta S_{n,\alpha} \subseteq S_{n,\alpha}\,.$$

Since $a_i \in S_{n,\alpha}$, the point $(1+\beta)c - \beta a_i$ belongs to $S_{n,\alpha}$. The preceding minimum therefore gives $(1+\beta)\langle a_i, c\rangle - \beta \geq -\alpha$ for each $i = 1,\ldots,n+1$. Summing these inequalities and using $\sum_i a_i = 0$ gives $-(n+1)\beta \geq -(n+1)\alpha$, and hence $\beta \leq \alpha$. Thus no center has symmetry greater than α. Together with $\mathbf{sym}(0, S_{n,\alpha}) \geq \alpha$, this finishes the proof. $\square$

5.2 MVCE and MVIE of $S_{n,\alpha}$

Lemma 5.5. *For every $n \geq 1$ and $\alpha \in [\frac{1}{n}, 1]$, let E_L denote the MVCE of $S_{n,\alpha}$. Then $E_L = B_2^n$. Let E_J denote the MVIE of $S_{n,\alpha}$ (also called the John ellipsoid), and write $E_J = r_J B_2^n$. Its radius satisfies $\sqrt{\frac{\alpha}{n}} \leq r_J \leq \sqrt{\frac{3\alpha}{2n}}$. If, in addition, $\frac{n+1}{1+\alpha}$ is an integer, then $r_J = \sqrt{\frac{\alpha}{n}}$.*

Before proving this lemma, we first show that both extremal ellipsoids are spherical.

Lemma 5.6. *For every $n \geq 1$ and $\alpha \in [\frac{1}{n}, 1]$, let E_J and E_L denote the MVIE and MVCE of $S_{n,\alpha}$, respectively. There exist radii $r_J, r_L > 0$ such that $E_J = r_J B_2^n$ and $E_L = r_L B_2^n$.*

Proof. The MVIE and MVCE are unique (Todd, 2016). Let E denote either one. By Lemma 5.3, $T_\pi E$ is an ellipsoid of the same volume satisfying the same inner or outer containment as E. Uniqueness therefore gives

$$T_\pi E = E \qquad \text{for all } \pi \in \Pi\,. \tag{46}$$

Let c be the center of E. The center of $T_\pi E$ is $T_\pi c$, so (46) implies $T_\pi c = c$ for every π. Averaging these identities and using Lemma 5.3 gives $c = \frac{1}{(n+1)!}\sum_{\pi\in\Pi} T_\pi c = 0$. Thus both ellipsoids are centered at the origin.

It remains to determine their shape. Write the origin-centered ellipsoid as $E = \{x \in \mathbb{R}^n : x^\top Q x \leq 1\}$ with $Q \succ 0$. Because T_π is orthogonal, (46) is equivalent to $T_\pi^\top Q T_\pi = Q$, or $QT_\pi = T_\pi Q$, for all $\pi \in \Pi$. Fix i. If $\pi(i) = i$, then $T_\pi(Qa_i) = QT_\pi a_i = Qa_i$. Thus Qa_i is fixed by every T_π for which $\pi(i) = i$. This invariance forces Qa_i to be a multiple of a_i. The n vectors $\{a_j : j \neq i\}$ form a basis of $\mathbb{R}^n$, so write $Qa_i = \sum_{j\neq i}\gamma_j a_j$. If $\gamma_j \neq \gamma_k$ for some $j, k \neq i$, let π be the transposition that interchanges j and k and fixes all other indices. Then T_π changes this basis representation, contradicting $T_\pi(Qa_i) = Qa_i$. Thus all γ_j are equal. Since $\sum_{j\neq i} a_j = -a_i$, there is a scalar d_i such that $Qa_i = d_i a_i$. If $\pi(i) = j$, then $Qa_j = QT_\pi a_i = T_\pi Q a_i = d_i a_j$. Comparing this identity with $Qa_j = d_j a_j$ gives $d_i = d_j$. Consequently, all d_i are equal to a common scalar d. Since $a_1, \ldots, a_n$ form a basis of $\mathbb{R}^n$, it follows that $Q = dI_n$. Positive definiteness of Q gives $d > 0$, so E is a Euclidean ball centered at the origin. $\square$

Proof of Lemma 5.5. We first compute the radius of the MVCE. By Lemma 5.6, this ellipsoid is $r_L B_2^n$. Every a_i has unit norm and lies on the boundary of B_2^n, whereas every $-\alpha a_i$ has norm α and belongs to B_2^n. Hence B_2^n is the smallest origin-centered ball containing $S_{n,\alpha}$, and therefore $r_L = 1$.

It remains to compute the radius of the MVIE. By Lemma 5.6, this ellipsoid is $r_J B_2^n$, and hence $r_J = \max\{r \geq 0 : rB_2^n \subseteq S_{n,\alpha}\}$. We use the separating hyperplane theorem to characterize this inclusion. The inclusion $rB_2^n \subseteq S_{n,\alpha}$ holds if and only if

$$r = \max_{x \in rB_2^n} \langle u, x \rangle \leq \max_{x \in S_{n,\alpha}} \langle u, x \rangle \,,$$

for every unit vector u. Let $g(u) := \max_{x \in S_{n,\alpha}} \langle u, x \rangle$. Since g is continuous on the compact unit sphere, taking the largest r gives

$$r_J = \max\{r \geq 0 : rB_2^n \subseteq S_{n,\alpha}\} = \min_{\|u\|_2 = 1} g(u). \tag{47}$$

To obtain an exact formula for r_J, fix a unit vector u and set $s_i := \langle u, a_i \rangle$. By $\sum_i a_i = 0$ and Lemma 5.2,

$$\sum_{i=1}^{n+1} s_i = 0 \,, \qquad \sum_{i=1}^{n+1} s_i^2 = \frac{n+1}{n} \,. \tag{48}$$

Since $0 \in \operatorname{int} S_{n,\alpha}$, one has $g(u) > 0$. By the definition of g, the numbers $y_i := \frac{s_i}{g(u)}$ satisfy $\sum_i y_i = 0$ and $-\frac{1}{\alpha} \leq y_i \leq 1$. The second identity in (48) then gives $g(u)^2 = \frac{n+1}{n \sum_i y_i^2}$. Conversely, for any numbers $y_1, \ldots, y_{n+1}$, not all zero, satisfying these same constraints, let $w := \sum_i y_i a_i$. The simplex identities give

$$\langle w, a_i \rangle = \frac{n+1}{n} y_i \quad (i = 1, \ldots, n+1), \qquad \|w\|_2^2 = \frac{n+1}{n} \sum_{i=1}^{n+1} y_i^2 \,.$$

Thus, for $u := \frac{w}{\|w\|_2}$, $g(u) \leq \sqrt{\frac{n+1}{n}} \left(\sum_i y_i^2\right)^{-\frac{1}{2}}$, because $\max_i y_i \leq 1$ and $-\alpha \min_i y_i \leq 1$. The first construction proves one direction of the following identity, while choosing a maximizer y in the converse construction proves the reverse direction:

$$r_J^2 = \frac{n+1}{n} \left(\max \left\{ \sum_{i=1}^{n+1} y_i^2 : \sum_{i=1}^{n+1} y_i = 0, \quad -\frac{1}{\alpha} \leq y_i \leq 1 \text{ (for all } i = 1, \ldots, n+1) \right\} \right)^{-1} . \tag{49}$$

It remains to evaluate this maximum. The feasible set in (49) is a compact polytope, and the objective is convex, so it has a maximizer at an extreme point. At such a point, at most one coordinate lies strictly between $-\frac{1}{\alpha}$ and 1. If an extreme point has ℓ coordinates equal to 1 and $n - \ell$ equal to $-\frac{1}{\alpha}$, its remaining coordinate is fixed by the zero-sum constraint. Up to a permutation, a maximizer therefore has $\lfloor \frac{n+1}{1+\alpha} \rfloor$ coordinates equal to 1, $n - \lfloor \frac{n+1}{1+\alpha} \rfloor$ coordinates equal to $-\frac{1}{\alpha}$, and one remaining coordinate fixed by the zero-sum constraint. Let $\{x\} := x - \lfloor x \rfloor$ denote the fractional part of x. Direct substitution into (49) gives

$$r_J^2 = \frac{\alpha}{n} \left(1 - \frac{(1+\alpha)^2}{\alpha(n+1)} \left\{ \frac{n+1}{1+\alpha} \right\} \left(1 - \left\{ \frac{n+1}{1+\alpha} \right\} \right) \right)^{-1} . \tag{50}$$

The claimed bounds follow from this formula. The term subtracted from 1 in (50) is nonnegative, so $r_J^2 \geq \frac{\alpha}{n}$. For the reverse bound, it suffices to show that this term is at most $\frac{1}{3}$. First suppose that $n \geq 3$. Since $\alpha \in [\frac{1}{n}, 1]$ and the function $t \mapsto \frac{t}{(1+t)^2}$ is increasing on $[0, 1]$, we have

$\frac{(1+\alpha)^2}{\alpha(n+1)} \le \frac{(1+\frac{1}{n})^2}{(\frac{1}{n})(n+1)} = \frac{n+1}{n}$. Moreover, the product $\left\{\frac{n+1}{1+\alpha}\right\}\left(1-\left\{\frac{n+1}{1+\alpha}\right\}\right)$ is at most $\frac{1}{4}$. Hence the term subtracted from 1 is at most $\frac{n+1}{4n} \le \frac{1}{3}$.

When $n = 2$, a direct calculation gives $\frac{(1+\alpha)^2}{3\alpha}\left\{\frac{3}{1+\alpha}\right\}\left(1-\left\{\frac{3}{1+\alpha}\right\}\right) = \frac{1}{3} - \frac{2(1-\alpha)^2}{3\alpha} \le \frac{1}{3}$. Thus, for every $n \ge 2$, the parenthesized factor in (50) is at least $\frac{2}{3}$, and therefore $r_J^2 \le \frac{3\alpha}{2n}$. When $n = 1$, necessarily $\alpha = 1$ and $r_J = 1$.

Finally, if $\frac{n+1}{1+\alpha}$ is an integer, then $\left\{\frac{n+1}{1+\alpha}\right\} = 0$ and $r_J^2 = \frac{\alpha}{n}$. This completes the proof. □

Proof of Theorem 5.1. The symmetry identity is Lemma 5.4. Let $E_J = r_J B_2^n$ and $E_L = B_2^n$ denote the MVIE and MVCE of $S_{n,\alpha}$, respectively. For any rounding $E \subseteq S_{n,\alpha} - c \subseteq \rho E$, the extremal-volume properties of E_J and E_L give

$$\rho^n = \frac{\mathrm{vol}_n(\rho E)}{\mathrm{vol}_n(E)} \ge \frac{\mathrm{vol}_n(E_L)}{\mathrm{vol}_n(E_J)} = \frac{\mathrm{vol}_n(B_2^n)}{\mathrm{vol}_n(r_J B_2^n)} = \left(\frac{1}{r_J}\right)^n .$$

The uniform upper bound $r_J \le \sqrt{\frac{3\alpha}{2n}}$ in Lemma 5.5 therefore proves (44). If $\frac{n+1}{1+\alpha}$ is an integer, the same lemma gives $r_J = \sqrt{\frac{\alpha}{n}}$, which proves (45). □

A Numerical details for Figure 1

The polytope in Figure 1 is

$$K := \mathbf{conv}\{(0,0), (2,0), (2.407407, 0.498538), (0, \tfrac{3}{2}), (-0.740741, 0.593567)\}. \tag{51}$$

Numerically, $\mathbf{sym}(K) \approx 0.7072$, and hence $\sqrt{\frac{2}{\mathbf{sym}(K)}} \approx 1.6816$, strictly smaller than the factor 2 guaranteed by the MVCE and MVIE.

The center and shape in Figure 1(a) are obtained from the MVCE problem (5), and those in Figure 1(b) are obtained from the MVIE problem (30). In Figure 1(a), the solid ellipsoid is the MVCE and the dashed ellipsoid is its $\frac{1}{2}$-scaled copy. In Figure 1(b), the solid ellipsoid is the MVIE and the dashed ellipsoid is its 2-scaled copy. In both figures the two displayed ellipsoids are tangent to K.

For Figure 1(c), we translate a Minkowski center to the origin and solve (P_λ) with $\lambda = \lambda_\alpha$. The dashed inner ellipsoid is $c^\star + \sqrt{\frac{\alpha}{n}}\, E_{Q^\star}$, and the solid outer ellipsoid is $c^\star + \sqrt{1 - \lambda_\alpha (c^\star)^\top Q^\star c^\star}\, E_{Q^\star}$. These are the inner ellipsoid and the refined outer ellipsoid in Lemma 2.1. The rounding factor is $\sqrt{\frac{n}{\alpha}(1 - \lambda_\alpha (c^\star)^\top Q^\star c^\star)} \approx 1.6679$. The outer ellipsoid is tangent to K, whereas the inner ellipsoid has positive slack despite appearing tangent.

For Figure 1(d), we normalize the translated halfspace description as in (28) and solve (J_λ) with $\lambda = \lambda_\alpha$. The solid inner ellipsoid and the dashed outer ellipsoid are, respectively, $c^\star + \left(\min_i \frac{1-h_i^\top c^\star}{\|X^\star h_i\|_2}\right) E_{Q^\star}$ and $c^\star + \sqrt{\frac{n}{\alpha}}\, E_{Q^\star}$, where $Q^\star = (X^\star)^{-2}$. These are the refined inner ellipsoid and the outer ellipsoid in Lemma 4.1. The rounding factor is $\sqrt{\frac{n}{\alpha}}\left(\min_i \frac{1-h_i^\top c^\star}{\|X^\star h_i\|_2}\right)^{-1} \approx$ 1.6720. The inner ellipsoid is tangent to K, whereas the outer ellipsoid has positive slack despite appearing tangent.

The optimization models are implemented in Julia/JuMP using the solver MOSEK.

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