A quadratic upper bound on the Chvátal rank of polytopes in the 0/1-cube

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Abstract

We show that every polytope $P \subseteq [0,1]^n$, and more generally every compact convex set, has Chvátal rank at most $12.22\,n^2 + n\log_2 n + 2n + 4$. This improves the $O(n^2 \log n)$ bound of Eisenbrand and Schulz and, together with the $\Omega(n^2)$ lower bound of Rothvoß and Sanità, shows that the maximum Chvátal rank of a polytope in $[0,1]^n$ is $\Theta(n^2)$. More precisely, if P contains an integer point, then for every $c \in \mathbb{Z}^n \setminus \{0\}$ the inequality $cx \leq \max\{cy : y \in P \cap \mathbb{Z}^n\}$ is valid for the k-th Chvátal closure of P for some $k \leq 12.22\,n^2 + 2n + 2\log_2 \|c\|_\infty + 4$. Following Eisenbrand and Schulz, we derive this inequality along a chain of coarser and coarser integer vectors, but instead of halving the vector in each step, we round τc for a scale τ chosen freely in each dyadic window $[2^{-t-1}, 2^{-t}]$. The main new ingredient is a multiscale version of Dirichlet's approximation theorem, proved by an elementary volume argument: for every $c \in \mathbb{R}^n$, the ℓ_1-distances of τc to $\mathbb{Z}^n$, minimized within each dyadic window and summed over all windows, total less than $1.222\,n^2$, independently of $\|c\|_\infty$.

Keywords: Chvátal rank, Chvátal–Gomory cuts, 0/1-polytopes, cutting planes, Diophantine approximation.
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1 Introduction

Chvátal–Gomory cuts are a basic tool of integer programming, and the Chvátal rank of a polytope measures how many rounds of them are needed to reach its integer hull. Chvátal [3] showed that the rank of every rational polytope is finite, and Schrijver [14] extended this to rational polyhedra. In general the rank can be arbitrarily large already in dimension two [15, p. 344]. For polytopes in the cube $[0,1]^n$, however, it is bounded in terms of n alone [2]. The previous best upper bound, $O(n^2 \log n)$, is due to Eisenbrand and Schulz [8], and the best lower bound, $\Omega(n^2)$, is due to Rothvoß and Sanità [12, 13]. We close this gap.

For a compact convex set $P \subseteq \mathbb{R}^n$ its *Chvátal closure* is

$$P' := \bigcap_{a \in \mathbb{Z}^n} \left\{ x \in \mathbb{R}^n : ax \leq \left\lfloor \max_{y \in P} ay \right\rfloor \right\},$$

where $ax := \sum_{\ell=1}^n a_\ell x_\ell$ denotes the standard inner product of $a, x \in \mathbb{R}^n$. We set $P' := \emptyset$ if $P = \emptyset$. Let $P_I := \operatorname{conv}(P \cap \mathbb{Z}^n)$ be the *integer hull* of P, which is a polytope since $P \cap \mathbb{Z}^n$ is finite. It is well known that $P_I \subseteq P' \subseteq P$. In particular, P' is again a compact convex set, and we can iterate:

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let $P^{(0)} := P$ and $P^{(k+1)} := (P^{(k)})'$ for every integer $k \geq 0$. We call $P^{(k)}$ the *k-th Chvátal closure* of P. The *Chvátal rank* $\mathrm{rk}(P)$ is the least k with $P^{(k)} = P_I$.

If $P_I \neq \emptyset$, then for $v \in \mathbb{Z}^n$ we write $h(v) := \max_{x \in P_I} vx$, which is an integer, and $\mathrm{depth}_P(v)$ denotes the least k such that $vx \leq h(v)$ is valid for $P^{(k)}$. Following [8], we say that v is saturated with respect to $P^{(k)}$ if $vx \leq h(v)$ is valid for $P^{(k)}$.

The Chvátal rank is usually studied for polytopes, but our main result holds more generally for compact convex sets.

Theorem 1. *Let $P \subseteq [0,1]^n$ be a compact convex set with $P_I \neq \emptyset$, and let $c \in \mathbb{Z}^n \setminus \{0\}$, $N := \|c\|_\infty$. Then*

$$\mathrm{depth}_P(c) \;\leq\; 12.22\, n^2 + 2n + 2\log_2 N + 4.$$

Moreover, for every compact convex set $P \subseteq [0,1]^n$,

$$\mathrm{rk}(P) \;\leq\; 12.22\, n^2 + n\log_2 n + 2n + 4.$$

Together with the lower bound of [13], Theorem 1 shows that the maximum Chvátal rank of a polytope in $[0,1]^n$ is $\Theta(n^2)$. We have not tried to optimize the constants, since our focus is the order of growth.

Previous work. Chvátal, Cook and Hartmann [4] gave polytopes in the cube with empty integer hull and rank at least n, which is exactly n by [2]. They also gave lower bounds, none exceeding n, for relaxations of natural combinatorial problems. Bockmayr, Eisenbrand, Hartmann and Schulz [2] proved the first polynomial upper bound, $\mathrm{rk}(P) = O(n^3 \log n)$, via $\mathrm{depth}_P(c) \leq n^2(1 + \lfloor \log_2 \|c\|_\infty \rfloor)$ for $c \in \mathbb{Z}^n \setminus \{0\}$. Eisenbrand and Schulz [8] improved both bounds to

$$\mathrm{depth}_P(c) \leq n^2 + 2n\big(1 + \lfloor \log_2 \|c\|_\infty \rfloor\big), \qquad \mathrm{rk}(P) \leq n^2\big(2 + \lfloor \log_2 n \rfloor\big),$$

and also proved $\mathrm{depth}_P(c) \leq n + \|c\|_1$ and a lower bound $(1+\varepsilon)n$ for infinitely many n. Pokutta and Stauffer [11] raised the lower bound to $(1 + 1/e - o(1))n$. Pokutta and Schulz [10] characterized the integer-empty polytopes of rank n. Rothvoß and Sanità [12, 13] constructed polytopes of rank $\Omega(n^2)$ (see also the textbook [5, Section 5.2.2]).

The $\log n$ in the upper bound comes only from the size of facet coefficients: the facet normals of P_I can require coefficients with $\Theta(n \log n)$ bits [1, 16], and [8] spends $\Theta(n)$ rounds per bit. Hence the $\log n$ cannot be removed by a better upper bound on the coefficients, and a better depth bound is needed. Cornuéjols and Lee [6], [7, Section 7] obtained $O(n^2 \log\log n)$ for polytopes that miss at most quasi-polynomially many 0/1 points, by showing that then $\log_2 \|c\|_\infty = O(n \log\log n)$ for the facet normals. The coefficients in the construction of [13] are of order $2^{\Theta(n)}$. In the concluding remarks of [13], Rothvoß and Sanità observe that beyond this range Dirichlet's theorem on simultaneous Diophantine approximation is an obstacle to extending their lower bound, and that "this insight could potentially be used to improve the upper bound on the Chvátal rank". Our proof carries out this idea, as explained in the overview below.

Overview of the proof. As in [8], we saturate c through a chain of coarser and coarser integer vectors $c = v_0, v_1, \ldots$ ending at 0, together with positive multipliers μ_t such that v_t is close in ℓ_1-norm to $\mu_t v_{t+1}$. If v_{t+1} is already saturated with respect to $P^{(k)}$, then by Lemma 2 the inequality $v_t x \leq h(v_t) + \|v_t - \mu_t v_{t+1}\|_1$ is valid for $P^{(k)}$. Each unit of the excess $\|v_t - \mu_t v_{t+1}\|_1$ costs two further applications of the closure, and the induction over the faces of the cube adds only $2n$ in total. This is Lemma 3: $\mathrm{depth}_P(c) \leq 2n + 2\sum_t E_t$, where E_t is the error of the t-th step rounded up. Lemmas 2 and 3 are proved in Section 2.

It remains to choose the chain so that its total error is small. Eisenbrand and Schulz take $v_{t+1} = \lfloor v_t/2 \rfloor$ and $\mu_t = 2$, so each of the roughly $\log_2 \|c\|_\infty$ steps can have error close to n. The errors then add up to about $n \log_2 \|c\|_\infty$, which is of order $n^2 \log n$ for the facet normals of P_I in the worst case. Instead, we take v_t to be a rounding of $\tau_t c$, where τ_t is chosen freely in the dyadic window $[2^{-t-1}, 2^{-t}]$ so that $\tau_t c$ is as close as possible to $\mathbb{Z}^n$. The main new ingredient is Theorem 5: for every $c \in \mathbb{R}^n$ and $0 < \rho \leq n$, at most $n \log_2(n/\rho)$ windows contain no τ for which the ℓ_1-distance of τc to $\mathbb{Z}^n$ is at most ρ. Consequently the smallest such distances, one for each window, sum to $O(n^2)$ independently of $\|c\|_\infty$, which is Corollary 6, and hence so does the total error along the chain. The proof of Theorem 5 is an elementary volume argument that treats all windows at once. Theorem 5 and Corollary 6 are proved in Section 3, which concerns only a vector $c \in \mathbb{R}^n$ and involves no polytope.

In Section 4 we combine these results to prove Theorem 1. The bound on the rank follows since the facet normals of P_I have coefficients of absolute value at most $n^{n/2}$. We close this section with some known facts that we will use.

Fact A ([14, Theorem 1])**.** If P is a rational polytope, so is P'.

Fact B ([15, p. 340])**.** If F is a face of a rational polytope P, then $F' = P' \cap F$.

Fact C ([2, Lemma 3])**.** If $P \subseteq [0,1]^n$ is a rational polytope with $P_I = \emptyset$, then $P^{(n)} = \emptyset$.

Fact D ([9])**.** Let $P \subseteq [0,1]^n$ be a polytope with $P_I \neq \emptyset$. There are $a_1, \dots, a_m \in \mathbb{Z}^n \setminus \{0\}$ and $b_1, \dots, b_m \in \mathbb{Z}$ such that $P_I = \{x \in \mathbb{R}^n : a_i x \leq b_i,\ i = 1, \dots, m\}$ and $\|a_i\|_\infty \leq n^{n/2}$ for all i.

2 Chains of integer vectors

A *chain* is a sequence $v_0, v_1, \dots, v_T \in \mathbb{Z}^n$ of integer vectors together with positive multipliers $\mu_0, \dots, \mu_{T-1}$. Our first lemma shows that if w is saturated and v is close in ℓ_1-norm to a positive multiple of w, then v is almost saturated.

Lemma 2. *Let $P \subseteq [0,1]^n$ be a polytope with $P_I \neq \emptyset$ and h as above. Let $Q \subseteq [0,1]^n$ be any set, $v, w \in \mathbb{Z}^n$ and $\mu > 0$, and suppose $wx \leq h(w)$ for all $x \in Q$. Then $vx \leq h(v) + \|v - \mu w\|_1$ for all $x \in Q$.*

Proof. Put $r := v - \mu w$. For every $y \in P_I$ we have $\mu w y = vy - ry \leq h(v) - ry$. Taking the maximum over $y \in P_I$, and using $\mu > 0$, gives

$$\mu h(w) \leq h(v) + \max_{y \in P_I} -ry.$$

Hence, for every $x \in Q$,

$$vx = \mu w x + rx \leq \mu h(w) + rx \leq h(v) + \max_{y \in P_I} r(x - y) \leq h(v) + \|r\|_1,$$

where the last inequality holds because $x, y \in [0,1]^n$ implies $|x_\ell - y_\ell| \leq 1$ for all ℓ. □

Lemma 2 generalizes the bound on the integrality gap in the proof of [8, Proposition 3.2], which is the case $v \geq 0$, $w = \lfloor v/2 \rfloor$, $\mu = 2$.

The next lemma, the main result of this section, bounds the depth of an integer vector c in terms of the ℓ_1-errors $\|v_t - \mu_t v_{t+1}\|_1$ along a chain starting at c.

Lemma 3. *Let $P \subseteq [0,1]^n$ be a rational polytope with $P_I \neq \emptyset$ and h as above. Let $c \in \mathbb{Z}^n$, and let $v_0, \dots, v_T$ with multipliers $\mu_0, \dots, \mu_{T-1}$ be a chain with $v_0 = c$. Put $E_t := \lceil \|v_t - \mu_t v_{t+1}\|_1 \rceil$ for $t < T$, and $E_T := \lceil \max_{y \in P} v_T y - h(v_T) \rceil \geq 0$. Then*

$$\operatorname{depth}_P(c) \;\leq\; 2n + 2\sum_{t=0}^{T} E_t.$$

Proof. The idea is to lower the right-hand side of $v_t x \leq h(v_t) + E_t$ one unit at a time on the faces of $[0,1]^n$, for $t = T, T-1, \dots, 0$, passing from v_{t+1} to v_t by Lemma 2. Claims 1–3 give recursive bounds on the number of rounds this requires, and Claim 4 together with an induction combines them.

By Fact A, every $P^{(k)}$ is a rational polytope. A *d-face* of $[0,1]^n$ is a face of dimension d. For integers $0 \leq t \leq T$, $0 \leq d \leq n$ and $0 \leq j \leq E_t$ let $S(t,d,j)$ be the statement *"$v_t x \leq h(v_t) + E_t - j$ holds on $P^{(k)} \cap G$ for every d-face G of $[0,1]^n$"*, and let $f(t,d,j) \in \{0,1,2,\dots\} \cup \{\infty\}$ be the least k for which it holds. Since $P^{(k+1)} \subseteq P^{(k)}$, it then holds for all larger k.

Claim 1. *$f(t,0,j) = 0$.*

Proof of claim. A 0-face of $[0,1]^n$ is a set $\{x\}$ with $x \in \{0,1\}^n$; if $x \in P$, then $x \in P_I$, so $v_t x \leq h(v_t) \leq h(v_t) + E_t - j$. ◇

Claim 2. *$f(t,d,0) \leq f(t+1,d,E_{t+1})$ for $t < T$, and $f(T,d,0) = 0$.*

Proof of claim. Let $t < T$ and assume that $k := f(t+1,d,E_{t+1})$ is finite. Let G be a d-face of $[0,1]^n$. By $S(t+1,d,E_{t+1})$, $v_{t+1}x \leq h(v_{t+1})$ for all $x \in Q := P^{(k)} \cap G$, so Lemma 2 with $v = v_t$, $w = v_{t+1}$, $\mu = \mu_t$ gives $v_t x \leq h(v_t) + \|v_t - \mu_t v_{t+1}\|_1 \leq h(v_t) + E_t$ on Q, which is $S(t,d,0)$. Moreover $f(T,d,0) = 0$, since for every $x \in P$ we have $v_T x \leq \max_{y \in P} v_T y \leq h(v_T) + E_T$ by definition of E_T. ◇

Claim 3. *For $d \geq 1$ and $0 \leq j < E_t$: $f(t,d,j+1) \leq \max\{f(t,d,j), f(t,d-1,j+1)\} + 2$.*

Proof of claim. Assume that $k := \max\{f(t,d,j), f(t,d-1,j+1)\}$ is finite. Let G be a d-face of $[0,1]^n$, and let $\gamma := h(v_t) + E_t - j \in \mathbb{Z}$. By $S(t,d,j)$, since $k \geq f(t,d,j)$, we have $v_t x \leq \gamma$ on $P^{(k)} \cap G$. Let $F := P^{(k)} \cap G \cap \{x \in \mathbb{R}^n : v_t x = \gamma\}$. If $F \neq \emptyset$, it is a face of $P^{(k)} \cap G$, and so of $P^{(k)}$.

We have $F \subseteq \operatorname{relint} G$. Indeed, as $d \geq 1$, the relative boundary of G is the union of its facets, which are $(d-1)$-faces of $[0,1]^n$. For each such facet G', $v_t x \leq \gamma - 1$ holds on $P^{(k)} \cap G'$ by $S(t,d-1,j+1)$, since $k \geq f(t,d-1,j+1)$. As $v_t x = \gamma$ on F, no point of F lies on the relative boundary of G.

We show that $P^{(k+1)} \cap F = \emptyset$. This is clear if $F = \emptyset$. Otherwise F is a face of $P^{(k)}$, so Fact B gives $P^{(k+1)} \cap F = F'$, and it suffices to show $F' = \emptyset$. Pick a coordinate x_ℓ that is not fixed on G, which exists since $d \geq 1$. As F is a nonempty rational polytope contained in $\operatorname{relint} G$, we have $0 < \min_{x \in F} x_\ell \leq \max_{x \in F} x_\ell < 1$, so $x_\ell \leq 0$ and $x_\ell \geq 1$ are valid for F', and $F' = \emptyset$.

Thus $v_t x < \gamma$ on $P^{(k+1)} \cap G$. If $P^{(k+1)} \cap G = \emptyset$, then $P^{(k+2)} \cap G = \emptyset$ as well. Otherwise the maximum of $v_t x$ over the polytope $P^{(k+1)} \cap G$ is attained, hence smaller than γ, and since $P^{(k+1)} \cap G$ is a face of $P^{(k+1)}$, Fact B and the integrality of γ give $v_t x \leq \gamma - 1$ on $(P^{(k+1)} \cap G)' = P^{(k+2)} \cap G$. In both cases $v_t x \leq \gamma - 1$ on $P^{(k+2)} \cap G$, and since $\gamma - 1 = h(v_t) + E_t - (j+1)$, this is $S(t,d,j+1)$ with $k+2$ in place of k. ◇

To combine Claims 1–3, we list all pairs (t,j) with $0 \le t \le T$ and $0 \le j \le E_t$ in the order $(T,0),(T,1),\dots,(T,E_T),(T-1,0),\dots,(0,E_0)$, and number them $i=1,\dots,M$, where $\Sigma := \sum_{t=0}^{T} E_t$ and $M := \Sigma + T + 1$. Let $g(i,d) := f(t,d,j)$ for the i-th pair (t,j), let $g(0,d) := 0$, and let $\sigma(i)$ be the number of pairs with $j \ge 1$ among the first i.

Claim 4. *Let $1 \le i \le M$ and let (t,j) be the i-th pair. Then $g(i,0) = 0$. If $j = 0$, then $g(i,d) \le g(i-1,d)$ for $0 \le d \le n$. If $j \ge 1$, then $g(i,d) \le \max\{g(i-1,d), g(i,d-1)\} + 2$ for $1 \le d \le n$.*

Proof of claim. The equality $g(i,0) = f(t,0,j) = 0$ is Claim 1. Let $j = 0$. If $i = 1$, then $(t,j) = (T,0)$ and $g(1,d) = f(T,d,0) = 0$ by Claim 2. If $i \ge 2$, then $t < T$ and the $(i-1)$-st pair is $(t+1, E_{t+1})$, so Claim 2 gives $g(i,d) = f(t,d,0) \le f(t+1,d,E_{t+1}) = g(i-1,d)$. If $j \ge 1$, then $(t,j-1)$ is the $(i-1)$-st pair, and Claim 3 with $j-1$ in place of j gives the inequality. ⋄

Induction on $i+d$ gives $g(i,d) \le 2(\sigma(i)+d)$ for $0 \le i \le M$ and $0 \le d \le n$. This holds if $i = 0$, since $g(0,d) = 0$ and $\sigma(0) = 0$, and if $d = 0$, since $g(i,0) = 0$ by Claim 4. For $i,d \ge 1$ it follows from Claim 4, since $\sigma(i) = \sigma(i-1)$ if the i-th pair has $j = 0$, and $\sigma(i) = \sigma(i-1)+1$ otherwise.

The last pair is $(0,E_0)$ and $\sigma(M) = \Sigma$, so $f(0,n,E_0) = g(M,n) \le 2(\Sigma+n)$. The only n-face of $[0,1]^n$ is $[0,1]^n \supseteq P^{(k)}$, so $S(0,n,E_0)$ says that $cx \le h(c)$ on $P^{(k)}$. Hence $\mathrm{depth}_P(c) \le 2(\Sigma+n)$. □

Corollary 4. *Let $P \subseteq [0,1]^n$ be a rational polytope with $P_I \neq \emptyset$. Then $\mathrm{depth}_P(c) \le 2n(\lfloor \log_2 \|c\|_\infty \rfloor + 2)$ for every $c \in \mathbb{Z}^n \setminus \{0\}$.*

Proof. Let v_{t+1} be the vector obtained from $v_t/2$ by rounding each coordinate toward 0, and let $\mu_t := 2$. Then $v_t - 2v_{t+1} \in \{-1,0,1\}^n$, so $E_t \le n$, and $v_T = 0$ for $T = \lfloor \log_2 \|c\|_\infty \rfloor + 1$, so $E_T = 0$. Apply Lemma 3. □

Lemma 3 refines the argument in the proof of [8, Proposition 3.2], which, for $c \ge 0$, uses the chain of vectors $\lfloor c/2^t \rfloor$, $t = 0,1,\dots$, with all multipliers equal to 2. There are two differences. First, Lemma 3 applies to an arbitrary chain, and its bound depends on the actual errors E_t, while the argument in [8] uses, at every step, the worst-case bound n on the integrality gap that holds for the halving chain. This flexibility is what we exploit in Section 4. Second, in [8] the bound on $v_t x$ over $P^{(k)}$ starts to be lowered only after $v_t x \le h(v_t)$ holds on $P^{(k)} \cap G$ for every facet G of $[0,1]^n$. This ensures the hypothesis of [8, Lemma 3.1], and it is established by induction on the dimension. In Claim 3, instead, to lower the bound on a face G of $[0,1]^n$ by one it suffices that the bound on the facets of G has already been lowered by one. Thus the bounds on faces of all dimensions decrease simultaneously, and the induction over the dimension contributes only the additive term $2n$ in Lemma 3, instead of n^2 in [8, Proposition 3.2].

Corollary 4, which uses essentially the halving chain, slightly sharpens the bound $n^2 + 2n(1 + \lfloor \log_2 \|c\|_\infty \rfloor)$ of [8] for $n \ge 3$. Combined with Facts C and D and the reduction at the start of Section 4, it gives $\mathrm{rk}(P) \le n^2 \log_2 n + 4n$ for every compact convex set $P \subseteq [0,1]^n$, slightly better than the bound $n^2(2 + \lfloor \log_2 n \rfloor)$ of [8] for $n \ge 3$.

3 A multiscale Dirichlet theorem

For integers $j \ge 0$ let $I_j := [2^{-j-1}, 2^{-j}]$ be the *dyadic windows*. In Section 4, the vectors of our chain are roundings of $\tau_t c$ with $\tau_t \in I_t$. In this section we show that the scales τ_t can be chosen so that the ℓ_1-distances of the vectors $\tau_t c$ to $\mathbb{Z}^n$ sum to $O(n^2)$, independently of $\|c\|_\infty$.

For $c \in \mathbb{R}^n$ and $\tau \in \mathbb{R}$ let $D_c(\tau) := \min_{z\in\mathbb{Z}^n} \|\tau c - z\|_1$ be the ℓ_1-distance of τc to $\mathbb{Z}^n$. Since every real number is within $\frac{1}{2}$ of an integer, $D_c(\tau) \in [0, n/2]$. The function D_c is continuous and even, i.e., $D_c(-\tau) = D_c(\tau)$.

The following theorem shows that, for every c, only few dyadic windows I_j contain no τ for which τc is close to $\mathbb{Z}^n$ in ℓ_1-norm.

Theorem 5. *Let $c \in \mathbb{R}^n$ and $0 < \rho \leq n$, and let $B_\rho := \{j \geq 0 : D_c(\tau) > \rho$ for all $\tau \in I_j\}$. Then*

$$|B_\rho| \;\leq\; n \log_2(n/\rho).$$

Proof. We write $\mathrm{vol}(Y)$ for the Lebesgue measure of a measurable set $Y \subseteq \mathbb{R}$ or $Y \subseteq \mathbb{R}^n$, and we let $\sigma := \rho/n \in (0, 1]$.

We first construct a set $X \subseteq [0, 1)$ with $\mathrm{vol}(X) \geq \sigma^n$, and then show that it satisfies

$$|\tau - \tau'| \notin \textstyle\bigcup_{j\in B_\rho} I_j \qquad \text{for all } \tau, \tau' \in X. \tag{1}$$

For $z \in \mathbb{R}^n$ let $X_z := \{\tau \in [0, 1) : \tau c - z \in [0, \sigma)^n + \mathbb{Z}^n\}$. The set of pairs $(\tau, z) \in [0, 1) \times [0, 1)^n$ with $\tau c - z \in [0, \sigma)^n + \mathbb{Z}^n$ is the union over $k \in \mathbb{Z}^n$ of the convex sets of pairs with $\tau c - z - k \in [0, \sigma)^n$. Only finitely many of these are nonempty, since $\tau c - z$ ranges over a bounded set. Hence this set is measurable, and exchanging the order of integration (Fubini's theorem) gives $\int_{[0,1)^n} \mathrm{vol}(X_z)\, dz = \int_0^1 \mathrm{vol}\{z \in [0, 1)^n : \tau c - z \in [0, \sigma)^n + \mathbb{Z}^n\}\, d\tau$. For fixed τ, the condition on z splits into the conditions $z_\ell \in (\tau c_\ell - \sigma, \tau c_\ell] + \mathbb{Z}$, $\ell = 1, \ldots, n$. As $\sigma \leq 1$, each of them is satisfied by a subset of $[0, 1)$ of length σ. So the integrand on the right-hand side equals σ^n for every τ, and $\int_{[0,1)^n} \mathrm{vol}(X_z)\, dz = \sigma^n$. Since $[0, 1)^n$ has volume 1, there is $z \in [0, 1)^n$ such that $X := X_z$ satisfies $\mathrm{vol}(X) \geq \sigma^n$.

We now show that X satisfies (1). Let $\tau, \tau' \in X$. Then $\tau c - z = q + k$ and $\tau' c - z = q' + k'$ with $q, q' \in [0, \sigma)^n$ and $k, k' \in \mathbb{Z}^n$. Subtracting, $(\tau - \tau')c - (k - k') = q - q'$, so $D_c(\tau - \tau') \leq \|q - q'\|_1 < n\sigma = \rho$. Since D_c is even, this gives $D_c(|\tau - \tau'|) < \rho$. So, by the definition of B_ρ, no window I_j with $j \in B_\rho$ contains $|\tau - \tau'|$, which is (1).

For $\lambda > 0$ let $\Phi(\lambda) := \sup_{\xi\in\mathbb{R}} \mathrm{vol}(X \cap [\xi, \xi + \lambda])$. Since $X \subseteq [0, 1)$, we have $\mathrm{vol}(X) \leq \Phi(1)$. Clearly $\Phi(\lambda) \leq \lambda$, and $\Phi(2\lambda) \leq 2\Phi(\lambda)$ since $[\xi, \xi+2\lambda] = [\xi, \xi+\lambda] \cup [\xi+\lambda, \xi+2\lambda]$. We now show that $\Phi(2^{-j}) \leq \Phi(2^{-j-1})$ for every $j \in B_\rho$. Let $j \in B_\rho$, $\xi \in \mathbb{R}$, and $Y := X \cap [\xi, \xi + 2^{-j}]$. Any two points of Y are at distance at most 2^{-j}. Since $j \in B_\rho$, by (1) this distance does not lie in $I_j = [2^{-j-1}, 2^{-j}]$, so it is less than 2^{-j-1}. Hence Y lies in an interval of length 2^{-j-1}, and so $\mathrm{vol}(Y) \leq \Phi(2^{-j-1})$. Taking the supremum over ξ gives $\Phi(2^{-j}) \leq \Phi(2^{-j-1})$.

Finally, we bound $|B_\rho|$. Let $L \geq 0$ be an integer and $\beta_L := |\{0, \ldots, L\} \cap B_\rho|$. Going from $\Phi(1)$ to $\Phi(2^{-L-1})$ in $L+1$ halvings, we use $\Phi(2\lambda) \leq 2\Phi(\lambda)$ for the $L + 1 - \beta_L$ indices $j \in \{0, \ldots, L\} \setminus B_\rho$ and $\Phi(2^{-j}) \leq \Phi(2^{-j-1})$ for the β_L indices $j \in \{0, \ldots, L\} \cap B_\rho$. Together with $\Phi(2^{-L-1}) \leq 2^{-L-1}$ this gives

$$\sigma^n \leq \mathrm{vol}(X) \leq \Phi(1) \leq 2^{L+1-\beta_L}\, \Phi(2^{-L-1}) \leq 2^{-\beta_L}.$$

Therefore $\beta_L \leq n \log_2(1/\sigma) = n \log_2(n/\rho)$ for all L. Since $|B_\rho| = \sup_L \beta_L$, this proves the theorem. □

For $j \geq 0$ let $m_j := \min_{\tau\in I_j} D_c(\tau)$, which exists since D_c is continuous and I_j is compact. The next corollary shows that the sum of these minima over all windows is $O(n^2)$, independently of $\|c\|_\infty$.

Corollary 6. *For every $c \in \mathbb{R}^n$:* $\displaystyle\sum_{j\geq 0} m_j \;\leq\; \Big(\frac{1}{2} + \frac{1}{2\ln 2}\Big) n^2 \;<\; 1.222\, n^2.$

Proof. We have

$$\sum_{j\geq 0} m_j = \sum_j \int_0^\infty \mathbf{1}[m_j > \rho]\,d\rho = \int_0^{n/2} |B_\rho|\,d\rho$$
$$\leq \int_0^{n/2} n\log_2 \frac{n}{\rho}\,d\rho = n^2\int_0^{1/2} \log_2\frac{1}{u}\,du = n^2\Big(\frac{1}{2}+\frac{1}{2\ln 2}\Big).$$

The first equality holds since $a = \int_0^\infty \mathbf{1}[a > \rho]\,d\rho$ for every $a \geq 0$. The second follows by exchanging sum and integral, which is allowed since all terms are nonnegative (Tonelli's theorem), since $m_j > \rho$ if and only if $j \in B_\rho$ because the minimum is attained, and since $m_j \leq n/2$ because D_c takes values in $[0, n/2]$. The inequality is Theorem 5, applied for $0 < \rho \leq n/2$. The next equality follows from the substitution $\rho = nu$, and the last one from $\int_0^{1/2} \log_2 \frac{1}{u}\,du = \frac{1}{\ln 2}\big[u - u\ln u\big]_0^{1/2}$. □

4 Proof of Theorem 1

Proof. We first reduce to the case that P is a rational polytope. If $P = \emptyset$, then $\mathrm{rk}(P) = 0$, so let $P \neq \emptyset$. Let $y \in \{0,1\}^n \setminus P$. Since P is compact and convex, there is $a \in \mathbb{R}^n$ with $ay > \max_{x\in P} ax$. Both sides depend continuously on a, so we may take $a \in \mathbb{Q}^n$. Choosing a rational β with $\max_{x\in P} ax < \beta < ay$, the rational half-space $\{x \in \mathbb{R}^n : ax \leq \beta\}$ contains P but not y. Let Q be the intersection of $[0,1]^n$ with one such half-space for each $y \in \{0,1\}^n \setminus P$ (see also [8, Section 2]). Then Q is a rational polytope with $P \subseteq Q \subseteq [0,1]^n$. Since $[0,1]^n \cap \mathbb{Z}^n = \{0,1\}^n$, we have $Q \cap \mathbb{Z}^n = P \cap \mathbb{Z}^n$, and hence $Q_I = P_I$. Since $P \subseteq Q$ implies $P' \subseteq Q'$, we have $P^{(k)} \subseteq Q^{(k)}$ for all k. Hence $\mathrm{depth}_P(c) \leq \mathrm{depth}_Q(c)$ if $P_I \neq \emptyset$, and $\mathrm{rk}(P) \leq \mathrm{rk}(Q)$ because $P_I \subseteq P^{(k)} \subseteq Q^{(k)}$ for all k. So we may assume that P is a rational polytope.

Next we construct the chain to which we apply Lemma 3. Let $\tau_0 := 1$ and, for each integer $t \geq 1$, let $\tau_t \in I_t$ be such that $D_c(\tau_t) = m_t$. For $t \geq 0$ let $v_t \in \mathbb{Z}^n$ be such that $\|v_t - \tau_t c\|_1 = D_c(\tau_t)$, for instance a coordinatewise nearest integer vector to $\tau_t c$. Since $1 \in I_0$ and $c \in \mathbb{Z}^n$, we have $D_c(\tau_0) = 0 = m_0$. Hence $v_0 = c$ and $\|v_t - \tau_t c\|_1 = m_t$ for all $t \geq 0$. Let T be the least $t \geq 1$ with $2^{-t}N < \frac{1}{2}$, i.e. $T = \lfloor \log_2 N\rfloor + 2$. Then, since $\tau_T \leq 2^{-T}$ and $|c_\ell| \leq N$, we have $|\tau_T c_\ell| \leq 2^{-T}N < \frac{1}{2}$ for all ℓ, so $v_T = 0$. Our chain consists of $v_0, \ldots, v_T$ and the multipliers $\mu_t := \tau_t/\tau_{t+1}$ for $0 \leq t < T$, as in Lemma 3. Note that $0 < \mu_t \leq 4$ for $0 \leq t < T$, since $\tau_t \leq 2^{-t}$ and $\tau_{t+1} \geq 2^{-t-2}$.

Then we bound the errors E_t. Since $v_T = 0$, we have $E_T = 0$. For $t < T$, since $\mu_t\tau_{t+1}c = \tau_t c$ and $\mu_t \leq 4$,

$$\|v_t - \mu_t v_{t+1}\|_1 = \|(v_t - \tau_t c) - \mu_t(v_{t+1} - \tau_{t+1}c)\|_1 \leq m_t + 4m_{t+1}.$$

Hence $E_t \leq 1 + m_t + 4m_{t+1}$ for $t < T$, and by Lemma 3 and Corollary 6,

$$\mathrm{depth}_P(c) \leq 2n + 2\sum_{t=0}^{T-1}\big(1 + m_t + 4m_{t+1}\big) \leq 2n + 2T + 10\sum_{j\geq 1} m_j \leq 12.22\,n^2 + 2n + 2\log_2 N + 4.$$

It remains to bound the rank. If $P_I = \emptyset$, then $\mathrm{rk}(P) \leq n$ by Fact C. Otherwise take the system $a_i x \leq b_i$ of Fact D. Since $P_I \subseteq P^{(k)}$ for all k, and $a_i x \leq h(a_i) \leq b_i$ is valid for $P^{(k)}$ whenever $k \geq \mathrm{depth}_P(a_i)$, we get $P^{(k)} = P_I$ for $k = \max_i \mathrm{depth}_P(a_i)$. Using $\log_2\|a_i\|_\infty \leq \frac{n}{2}\log_2 n$ gives $\mathrm{rk}(P) \leq 12.22\,n^2 + n\log_2 n + 2n + 4$. □

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