Stable and Unstable Singularities in Navier-Stokes ?

Orson Mengara, INRS–EMT, University of Québec, Montréal, QC, Canada.
orson.mengara@inrs.ca

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Abstract

This document provides an extended and rigorous framework dedicated to the geometric analysis of the 3D incompressible Navier-Stokes equations [4]. We comprehensively develop geometric proofs related to decay estimates, blow-up profiles, and the foundational partial regularity theory of Caffarelli, Kohn, and Nirenberg (CKN). We examine in detail the Hausdorff dimension of potential singular sets, local monotonicity inequalities, and the spectral analysis of linearized operators to classify singularities into stable and unstable manifolds.

Résumé

Ce document constitue une extension approfondie et rigoureuse dédiée à l'analyse géométrique des équations de Navier-Stokes incompressibles [4] en dimension 3. Nous y développons de manière exhaustive les preuves géométriques associées aux estimations de décroissance, aux profils d'explosion, ainsi qu'à la théorie fondatrice de la régularité partielle de Caffarelli, Kohn et Nirenberg (CKN). Nous examinons en détail la dimension de Hausdorff de l'ensemble singulier potentiel, les inégalités de monotone sous forme locale, et l'analyse spectrale des opérateurs linéarisés pour classifier les singularités en variétés stables et instables.

Table des matières

1 Introduction

The three-dimensional incompressible [5] Navier-Stokes equations [4], [39] describe the motion of a viscous fluid. They are given by :

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \nu \Delta u + \nabla p = 0 \quad \text{in } \mathbb{R}^3 \times (0,T), \\ \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \times (0,T), \\ u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^3, \end{cases} \tag{1}$$

where $u(x,t) = (u_1, u_2, u_3)$ is the velocity field, $p(x,t)$ is the pressure, and $\nu > 0$ is the kinematic viscosity. The fundamental question of whether smooth initial data $u_0 \in C_0^\infty(\mathbb{R}^3)$ leads to a globally smooth solution or if a finite-time singularity [21], [38] (blow-up) occurs remains one of the most prominent open problems in mathematics.

1.1 Mathematical Preliminaries and Functional Spaces

To study PDEs rigorously, we rely on Littlewood-Paley decomposition and Besov spaces.

1.2 Littlewood-Paley Theory

Let $\mathcal{C}$ be the annulus $\{\xi \in \mathbb{R}^3 : 3/4 \leq |\xi| \leq 8/3\}$. There exist radial functions χ and φ, valued in $[0,1]$, belonging respectively to $C_c^\infty(B(0,4/3))$ and $C_c^\infty(\mathcal{C})$, such that :

$$\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1 \quad \forall \xi \in \mathbb{R}^3. \tag{2}$$

We define the dyadic blocks as follows :

$$\begin{aligned} \Delta_j u &= 0 \quad \text{for } j < -1, \\ \Delta_{-1} u &= \chi(D)u = \mathcal{F}^{-1}(\chi \mathcal{F} u), \\ \Delta_j u &= \varphi(2^{-j}D)u = \mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\mathcal{F}u) \quad \text{for } j \geq 0. \end{aligned}$$

The low-frequency cut-off is given by $S_j u = \sum_{q<j} \Delta_q u$.

Definition 1.1. For $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, the non-homogeneous Besov space $B^s_{p,r}(\mathbb{R}^3)$ is the set of tempered distributions $u \in \mathcal{S}'(\mathbb{R}^3)$ such that :

$$\|u\|_{B^s_{p,r}} := \left(\sum_{j \geq -1} 2^{rjs} \|\Delta_j u\|^r_{L^p} \right)^{1/r} < \infty. \tag{3}$$

1.3 Bony's Paraproduct and Commutator Estimates

The non-linear convective term $(u \cdot \nabla)u$ is handled via Bony's decomposition :

$$uv = T_u v + T_v u + R(u,v),$$

where

$$T_u v = \sum_j S_{j-1} u \Delta_j v, \quad R(u,v) = \sum_j \sum_{|j'-j| \leq 1} \Delta_j u \Delta_{j'} v.$$

Lemma 1.2 (Commutator Estimate). *Let $p \in (1,\infty)$ and $s > 0$. There exists a constant $C > 0$ such that for all vector fields v and functions f :*

$$\|[\Delta_j, v \cdot \nabla] f\|_{L^p} \leq C c_j 2^{-js} \|\nabla v\|_{L^\infty} \|f\|_{B^s_{p,r}} + C' c_j 2^{-js} \|\nabla f\|_{L^\infty} \|v\|_{B^s_{p,r}} \tag{4}$$

where $\{c_j\}$ is a sequence in ℓ^r with $\|c_j\|_{\ell^r} = 1$.

1.4 Local Well-Posedness and Energy Estimates

We first prove local existence in the Sobolev space $H^s(\mathbb{R}^3) = B^s_{2,2}(\mathbb{R}^3)$ for $s > 5/2$.

Theorem 1.3. *Let $u_0 \in H^s(\mathbb{R}^3)$ with $s > 5/2$ and $\operatorname{div} u_0 = 0$. There exists a maximal time of existence $T^* > 0$ and a unique solution $u \in C([0,T^*); H^s) \cap L^2(0,T^*; H^{s+1})$ to the Navier-Stokes equations [4].*

Démonstration. Applying the Leray projector $\mathbb{P}$ to (1), we obtain $\partial_t u + \mathbb{P}(u \cdot \nabla u) - \nu \Delta u = 0$. Taking the L^2 inner product of Δ_q(eq) with $\Delta_q u$, we get :

$$\frac{1}{2}\frac{d}{dt}\|\Delta_q u\|_{L^2}^2 + \nu 2^{2q}\|\Delta_q u\|_{L^2}^2 = -\int_{\mathbb{R}^3} \Delta_q(u \cdot \nabla u) \cdot \Delta_q u \, dx.$$

Using the fact that $\operatorname{div} u = 0$, we have $\int (u \cdot \nabla \Delta_q u) \cdot \Delta_q u = 0$. Thus,

$$\int \Delta_q(u \cdot \nabla u) \cdot \Delta_q u = \int [\Delta_q, u \cdot \nabla] u \cdot \Delta_q u.$$

By Lemma 1.2 with $p = 2$, we bound the commutator term by $C c_q 2^{-qs} \|\nabla u\|_{L^\infty} \|u\|_{H^s}$. Multiplying by 2^{2qs} and summing over q, we obtain the differential inequality :

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^s}^2 + \nu \|\nabla u\|_{H^s}^2 \leq C \|\nabla u\|_{L^\infty} \|u\|_{H^s}^2. \tag{5}$$

Since $s > 5/2 = 3/2 + 1$, the Sobolev embedding yields $\|\nabla u\|_{L^\infty} \leq C\|u\|_{H^s}$. Thus,

$$\frac{d}{dt}\|u\|_{H^s}^2 \leq C\|u\|_{H^s}^3.$$

Solving this ODE gives a lower bound on the lifespan $T^* \geq \frac{1}{C\|u_0\|_{H^s}}$. Uniqueness follows from standard stability estimates in L^2 for the difference of two solutions. □

1.5 The Beale-Kato-Majda Criterion

The relation between the maximal time of existence and the accumulation of vorticity $\omega = \operatorname{curl} u$ is described by the famous Beale-Kato-Majda (BKM) criterion [3].

Theorem 1.4 (Beale-Kato-Majda)**.** *Let u be a solution to the 3D Navier-Stokes equations in $C([0,T^*); H^s)$ with $s > 5/2$. If $T^* < \infty$ is the maximal time of existence, then :*

$$\int_0^{T^*} \|\omega(\cdot,t)\|_{L^\infty} \, dt = \infty. \tag{6}$$

Démonstration. Assume for contradiction that $\int_0^{T^*} \|\omega(\tau)\|_{L^\infty} d\tau = M < \infty$. From the energy estimate (5), we have :

$$\frac{d}{dt}\|u\|_{H^s}^2 \leq C\|\nabla u\|_{L^\infty}\|u\|_{H^s}^2.$$

To close this estimate, we need to bound $\|\nabla u\|_{L^\infty}$ by $\|\omega\|_{L^\infty}$ and a logarithmic term of $\|u\|_{H^s}$. Using Littlewood-Paley decomposition, for any integer $N > 0$:

$$\nabla u = S_0 \nabla u + \sum_{q=0}^{N} \Delta_q \nabla u + \sum_{q>N} \Delta_q \nabla u.$$

1. **Low frequencies :** $\|S_0 \nabla u\|_{L^\infty} \leq C\|u\|_{L^2}$. 2. **Intermediate frequencies :** Using the Biot-Savart law $u = -\Delta^{-1} \operatorname{curl} \omega$, the Riesz transforms are bounded operators on L^∞ for localized frequencies. Thus,

$$\sum_{q=0}^{N} \|\Delta_q \nabla u\|_{L^\infty} \leq C \sum_{q=0}^{N} \|\Delta_q \omega\|_{L^\infty} \leq C(N+1)\|\omega\|_{L^\infty}.$$

3. **High frequencies :** Using Bernstein's inequalities and Sobolev embeddings :

$$\sum_{q>N} \|\Delta_q \nabla u\|_{L^\infty} \leq C \sum_{q>N} 2^{q(3/2+1-s)} 2^{q(s-1)} \|\Delta_q \nabla u\|_{L^2} \leq C 2^{-N(s-5/2)} \|u\|_{H^s}.$$

Choosing $N \approx \frac{1}{s-5/2} \log_2(e + \|u\|_{H^s})$, we obtain the logarithmic Sobolev inequality :

$$\|\nabla u\|_{L^\infty} \leq C \left(1 + \|u\|_{L^2} + \|\omega\|_{L^\infty} \log(e + \|u\|_{H^s})\right). \tag{7}$$

Substituting this back into the energy inequality :

$$\frac{d}{dt}\|u\|_{H^s}^2 \leq C \left(1 + \|\omega\|_{L^\infty} \log(e + \|u\|_{H^s}^2)\right) \|u\|_{H^s}^2.$$

By Grönwall's lemma, if $\int_0^{T^*} \|\omega\|_{L^\infty} d\tau < \infty$, then $\sup_{t\in[0,T^*)} \|u(t)\|_{H^s} < \infty$, which contradicts the assumption that T^* is the maximal time of existence. □

2 Leray's Backward Self-Similar Singularities

In 1934, Leray proposed [2] that singularities [10] might occur through a self-similar focusing mechanism. A backward self-similar solution takes the form :

$$u(x,t) = \frac{1}{\sqrt{2a(T-t)}} U\left(\frac{x}{\sqrt{2a(T-t)}}\right), \quad p(x,t) = \frac{1}{2a(T-t)} P\left(\frac{x}{\sqrt{2a(T-t)}}\right) \tag{8}$$

where $a > 0$, $T > 0$ is the blow-up time, and $y = x/\sqrt{2a(T-t)}$ is the similarity variable.

Substituting (8) into the Navier-Stokes equations [37] yields the profile equation for (U, P) :

$$\begin{cases} -\nu\Delta U + aU + a(y\cdot\nabla)U + (U\cdot\nabla)U + \nabla P = 0 \quad \text{in } \mathbb{R}^3, \\ \operatorname{div} U = 0. \end{cases} \tag{9}$$

Definition 2.1 (Leray backward self-similar solution)**.** A solution u of (1) on $\mathbb{R}^3 \times (0, T^*)$ is *backward self-similar* if there is a divergence-free profile $U : \mathbb{R}^3 \to \mathbb{R}^3$ and a scalar profile $P : \mathbb{R}^3 \to \mathbb{R}$ such that

$$u(x,t) = \frac{1}{\sqrt{2(T^*-t)}} U(y), \qquad p(x,t) = \frac{1}{2(T^*-t)} P(y), \qquad y = \frac{x}{\sqrt{2(T^*-t)}}. \tag{10}$$

Proposition 2.2. *The pair (u, p) given by (74) solves (1) with viscosity $\nu = 1$ on $\mathbb{R}^3 \times (0, T^*)$ if and only if (U, P) solves the stationary system,*

$$\Delta U - (y\cdot\nabla)U - U - (U\cdot\nabla)U - \nabla P = 0, \qquad \operatorname{div} U = 0 \quad \textit{on } \mathbb{R}^3. \tag{11}$$

Démonstration. Write $s(t) := 2(T^* - t)$, so $s'(t) = -2$, $u_i(x,t) = s^{-1/2} U_i(y)$ with $y = xs^{-1/2}$. Differentiating the argument, $\partial_t y_k = -\frac{1}{2} x_k s^{-3/2} s' = x_k s^{-3/2} = y_k s^{-1}$, hence

$$\partial_t u_i = -\tfrac{1}{2} s^{-3/2} s'\, U_i + s^{-1/2} (\partial_k U_i)(\partial_t y_k) = s^{-3/2} U_i + s^{-3/2} (y\cdot\nabla U_i) = s^{-3/2} \big[U_i + (y\cdot\nabla)U_i\big].$$

Since $\nabla_x = s^{-1/2}\nabla_y$,

$$(u\cdot\nabla_x)u_i = s^{-1}(U\cdot\nabla_y)\big(s^{-1/2} U_i\big) = s^{-3/2}(U\cdot\nabla U_i), \qquad \Delta_x u_i = s^{-3/2}\Delta U_i, \qquad \nabla_x p = s^{-3/2}\nabla P.$$

Substituting into (1) and cancelling the common factor $s^{-3/2}(t) > 0$ gives

$$U_i + (y\cdot\nabla)U_i + (U\cdot\nabla)U_i = -\partial_i P + \Delta U_i,$$

which is (11). Divergence-freeness of u is equivalent, by the same change of variables, to $\operatorname{div} U = 0$. Every step is reversible, giving the stated equivalence. □

Let $U \in L^3(\mathbb{R}^3)$ be a solution to (11). By elliptic regularity, $U \in C^\infty(\mathbb{R}^3)$ and decays algebraically. The total head pressure is defined as $\Phi = \frac{1}{2}|U|^2 + P + ay \cdot U$. The momentum equation can be rewritten using the vorticity $\Omega = \operatorname{curl} U$:

$$-\nu\Delta U + \Omega \times U + \nabla\Phi = 0.$$

Taking the divergence, we obtain an elliptic equation for the pressure :

$$-\Delta P = \operatorname{tr}((\nabla U)^2) + a \operatorname{div}(y \cdot \nabla U).$$

Using maximum principles on the total head pressure Φ and delicate decay estimates at infinity (relying on the L^3 integrability which dictates a specific rate of decay $|U(y)| \sim |y|^{-1}$), one can show that Φ must be constant. Define the linear operator,

$$LU := \Delta U - (y \cdot \nabla)U - U,$$

so that (11) reads $LU = (U \cdot \nabla)U + \nabla P$. The drift term $-(y \cdot \nabla)$ is the generator of an Ornstein–Uhlenbeck process, and L is naturally self-adjoint with respect to the Gaussian weight $\omega(y) := e^{-|y|^2/2}$, rather than Lebesgue measure.

Lemma 2.3 (Weighted self-adjointness). *For all $f, g \in C_c^\infty(\mathbb{R}^3)$,*

$$\int_{\mathbb{R}^3} \big[\Delta f - (y \cdot \nabla)f\big]\, g\, \omega(y)\, dy \;=\; -\int_{\mathbb{R}^3} \nabla f \cdot \nabla g\; \omega(y)\, dy. \tag{12}$$

In particular the right-hand side is symmetric in f, g, so $f \mapsto \Delta f - (y \cdot \nabla)f$ is symmetric on $L^2(\mathbb{R}^3, \omega\, dy)$, and consequently so is $L + 1$.

Démonstration. Since $\nabla\omega = -y\,\omega$, integration by parts gives

$$\int (\Delta f)\, g\, \omega\, dy = -\int \nabla f \cdot \nabla(g\omega)\, dy = -\int \nabla f \cdot \nabla g\, \omega\, dy - \int \nabla f \cdot g\, \nabla\omega\, dy =$$
$$-\int \nabla f \cdot \nabla g\, \omega\, dy + \int (y \cdot \nabla f)\, g\, \omega\, dy.$$

Rearranging, $\int (\Delta f) g\, \omega\, dy - \int (y \cdot \nabla f) g\, \omega\, dy = -\int \nabla f \cdot \nabla g\, \omega\, dy$, which is (12). □

Lemma 2.4 (Coercivity and spectral gap of the linearization at $U = 0$). *Let $f \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3)$. Then*

$$\int_{\mathbb{R}^3} (Lf) \cdot f\, \omega\, dy \;=\; -\int_{\mathbb{R}^3} |\nabla f|^2\, \omega\, dy \;-\; \int_{\mathbb{R}^3} |f|^2\, \omega\, dy \;\leq\; -\int_{\mathbb{R}^3} |f|^2\, \omega\, dy. \tag{13}$$

2.1 Energy Estimates for the Profile

Taking the dot product of (11) with U and integrating over $\mathbb{R}^3$, assuming sufficient decay at infinity :

$$\int_{\mathbb{R}^3} (aU \cdot U + a(y \cdot \nabla U) \cdot U)\, dy = a\int_{\mathbb{R}^3} |U|^2\, dy - \frac{a}{2}\int_{\mathbb{R}^3} (\operatorname{div} y)|U|^2\, dy.$$

Since $\operatorname{div} y = 3$ in $\mathbb{R}^3$, this yields :

$$a\int_{\mathbb{R}^3} |U|^2\, dy - \frac{3a}{2}\int_{\mathbb{R}^3} |U|^2\, dy = -\frac{a}{2}\int_{\mathbb{R}^3} |U|^2\, dy.$$

For the convective term, $\int (U \cdot \nabla)U \cdot U\, dy = 0$. The viscous term gives $\nu \int |\nabla U|^2\, dy$. Therefore, the energy identity is :

$$\nu\int_{\mathbb{R}^3} |\nabla U|^2\, dy - \frac{a}{2}\int_{\mathbb{R}^3} |U|^2\, dy = 0. \tag{14}$$

Since $a > 0$ and $\nu > 0$, equation (14) implies that $U \equiv 0$. This simple proof, however, assumes $U \in L^2(\mathbb{R}^3)$. The profound result of Nečas, Růžička, and Šverák (1996) [9] proves that $U \equiv 0$ under the weaker physical assumption $U \in L^3(\mathbb{R}^3)$, ruling out self-similar blow-up in the natural energy space.

Equation (1) is invariant under the parabolic scaling,

$$u_\mu(x,t) := \mu\, u(\mu x, \mu^2 t), \qquad p_\mu(x,t) := \mu^2 p(\mu x, \mu^2 t), \qquad \mu > 0, \tag{15}$$

Theorem 2.5 (Nečas–Růžička–Šverák, 1996 [9])**.** *Let $U \in L^3(\mathbb{R}^3)$ be a divergence-free weak solution of* (11) *(with ∇P understood distributionally, $P \in L^{3/2}_{\mathrm{loc}}$). Then $U \equiv 0$. Consequently, there is no non-trivial Leray [2] backward self-similar solution of* (1) *as in Definition 2.1 whose profile lies in* $L^3(\mathbb{R}^3)$.

Theorem 2.6 (Tsai, 1998 [8])**.** *The conclusion of Theorem 2.5 continues to hold under the weaker hypothesis that U satisfies a local energy (suitable weak solution) inequality in the sense of Caffarelli–Kohn–Nirenberg [1], without assuming $U \in L^3(\mathbb{R}^3)$ globally.*

Theorem 2.7 (Chae–Wolf; generalized Liouville theorem [12])**.** *The non-existence conclusion extends to profiles $U \in L^{p,\infty}(\mathbb{R}^3)$ (weak-L^p) for $p > 3/2$, and in particular removes the scenario of an* asymptotically *(rather than exactly) self-similar blow-up with profile in this class.*

2.2 Let's Think Math

Let $x = (r\cos\theta, r\sin\theta, z)$ and suppose $u = u^r(r,z,t)\, e_r + u^z(r,z,t)\, e_z$ is axisymmetric [17] with *no swirl* ($u^\theta \equiv 0$). Then the vorticity $\omega = \nabla \times u$ has only an azimuthal component, $\omega = \omega^\theta(r,z,t)\, e_\theta$, where $\omega^\theta = \partial_z u^r - \partial_r u^z$, and it satisfies the classical axisymmetric vorticity equation

$$\partial_t \omega^\theta + u^r \partial_r \omega^\theta + u^z \partial_z \omega^\theta - \frac{u^r}{r}\,\omega^\theta = \nu\Big(\partial_r^2 + \tfrac{1}{r}\partial_r + \partial_z^2 - \tfrac{1}{r^2}\Big)\omega^\theta. \tag{16}$$

The term $-(u^r/r)\omega^\theta$ (properly $+u^r\omega^\theta/r$ on the right when moved, i.e. a vortex-stretching source term) is the only obstruction to a maximum principle for ω^θ directly.

Lemma 2.8. *Let $\xi := \omega^\theta/r$. Along particle paths ($\dot r = u^r$, $\dot z = u^z$), ξ satisfies the source-free linear transport-diffusion equation*

$$\partial_t \xi + u^r \partial_r \xi + u^z \partial_z \xi \;=\; \nu\Big(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2\Big)\xi. \tag{17}$$

Démonstration. Since $r(t)$ evolves along the particle path by $\dot r = u^r$, we have $\frac{D}{Dt}(1/r) = -u^r/r^2$ (material derivative $D/Dt := \partial_t + u^r\partial_r + u^z\partial_z$), so

$$\frac{D\xi}{Dt} = \frac{1}{r}\,\frac{D\omega^\theta}{Dt} - \frac{u^r}{r^2}\,\omega^\theta.$$

By (16), $\dfrac{D\omega^\theta}{Dt} = \dfrac{u^r}{r}\omega^\theta + \nu\big(\Delta - \frac{1}{r^2}\big)\omega^\theta$, so the vortex-stretching terms cancel exactly :

$$\frac{D\xi}{Dt} = \frac{\nu}{r}\Big(\Delta - \frac{1}{r^2}\Big)\omega^\theta.$$

Writing $g := \omega^\theta = r\xi$ and computing directly with $\Delta g = g_{rr} + \frac{1}{r}g_r + g_{zz}$,

$$g_r = \xi + r\xi_r, \quad g_{rr} = 2\xi_r + r\xi_{rr}, \quad g_{zz} = r\xi_{zz} \implies \Delta g = r\xi_{rr} + 3\xi_r + \frac{\xi}{r} + r\xi_{zz},$$

so $\big(\Delta - \frac{1}{r^2}\big)g = r\xi_{rr} + 3\xi_r + r\xi_{zz} = r\Big(\xi_{rr} + \frac{3}{r}\xi_r + \xi_{zz}\Big)$. Dividing by r gives (17). □

Proposition 2.9 (Global L^∞ bound and global regularity)**.** *Let u_0 be smooth, axisymmetric [17], swirl-free, divergence-free, of finite energy, with $\xi_0 := \omega_0^\theta/r \in L^\infty(\mathbb{R}^3)$. Then the unique local strong solution of* (1) *remains smooth for all $t > 0$; in particular no finite-time singularity, self-similar or otherwise, can occur.*

Corollary 2.10. *There is no non-trivial backward self-similar solution of* (1) *in the sense of Definition 2.1 whose associated velocity field $u(\cdot, t)$, for t near T^*, is axisymmetric [17] and swirl-free with $\xi = \omega^\theta/r \in L^\infty$: such a solution, run backward from any time $t_0 < T^*$ as initial data, would by Proposition 2.9 remain smooth for all future time, contradicting blow-up at T^*.*

2.3 Stable and Unstable Manifolds of Singularities

Suppose there exists a non-trivial singular profile U^* for a modified fluid system. We write the solution near the singularity [18], [21] as $U(y,\tau) = U^*(y) + V(y,\tau)$, where $\tau = -\log(T-t)$ is the slow-time variable. The perturbation V satisfies an evolution equation of the form :

$$\partial_\tau V = \mathcal{L}V + \mathcal{N}(V) \tag{18}$$

where $\mathcal{L}$ is the linearized operator around U^* :

$$\mathcal{L}V = \nu\Delta V - aV - a(y\cdot\nabla)V - (U^*\cdot\nabla)V - (V\cdot\nabla)U^* - \nabla\Pi,$$

and $\mathcal{N}(V) = -(V\cdot\nabla)V$ is the non-linear part.

2.4 Spectral Analysis of the Linearized Operator $\mathcal{L}$

The stability of the blow-up profile U^* depends fundamentally on the spectrum $\sigma(\mathcal{L})$. Let $\mathcal{H}$ be a suitable weighted Hilbert space, for instance $L^2(\mathbb{R}^3;\rho(y)dy)$ where $\rho(y) = e^{-a|y|^2/2\nu}$ is a Gaussian weight that absorbs the drift term $a(y\cdot\nabla)V$.

Definition 2.11 (Stable and Unstable Spectra)**.** The spectrum $\sigma(\mathcal{L})$ can be decomposed into :

— **Unstable spectrum** $\sigma_u = \{\lambda \in \sigma(\mathcal{L}) : \mathrm{Re}(\lambda) > 0\}$

— **Center spectrum** $\sigma_c = \{\lambda \in \sigma(\mathcal{L}) : \mathrm{Re}(\lambda) = 0\}$

— **Stable spectrum** $\sigma_s = \{\lambda \in \sigma(\mathcal{L}) : \mathrm{Re}(\lambda) < 0\}$

If $\sigma_u \neq \emptyset$, the singularity [18],[21] is structurally unstable. A generic perturbation will grow exponentially in the similarity variables (which means it deviates from the self-similar profile before the blow-up time T). This leads to a *codimension-k blow-up*, where k is the number of unstable eigenvalues.

For the Navier-Stokes equations [4], finding any stable singularity (where $\sigma_u = \emptyset$) would imply an open set of initial data in C^∞ leading to blow-up, contradicting the prevailing belief that if singularities [10] exist, they might be highly non-generic.

The rescaled equation

Consider a putative finite-time blow-up solution $u(x,t)$ of a parabolic (or Navier–Stokes-type) system on $\mathbb{R}^n \times [0,T)$, singular precisely at $t = T$, $x = 0$. The self-similar (Type I) ansatz is

$$u(x,t) = \frac{1}{(T-t)^\alpha}\, U\!\left(\frac{x}{\sqrt{T-t}}\right), \qquad y := \frac{x}{\sqrt{T-t}}, \tag{19}$$

with α fixed by scaling. Introducing the self-similar time $s := -\log(T-t) \in [-\log T, \infty)$, one obtains for

$$v(y,s) := (T-t)^\alpha u(x,t) \tag{20}$$

an autonomous evolution equation of the form

$$\partial_s v = \Delta_y v - \frac{1}{2}(y \cdot \nabla_y)v - \alpha v + N(v), \tag{21}$$

where N collects the (typically quadratic or higher-order) nonlinear terms. A self-similar blow-up profile U^* of the original equation corresponds *exactly* to a stationary solution of (21) :

$$\Delta U^* - \tfrac{1}{2}(y \cdot \nabla)U^* - \alpha U^* + N(U^*) = 0. \tag{22}$$

Linearization

Writing $v = U^* + w$ and dropping quadratic-and-higher terms in w gives the linear evolution

$$\partial_s w = \mathcal{L}w, \qquad \mathcal{L} := \Delta_y - \tfrac{1}{2}(y \cdot \nabla_y) - \alpha \,\mathrm{Id} + DN(U^*), \tag{23}$$

where $DN(U^*)$ is the Fréchet derivative of the nonlinearity at U^* — a (typically relatively compact) perturbation of the linear *Hermite/Ornstein–Uhlenbeck part*

$$\mathcal{L}_0 := \Delta_y - \tfrac{1}{2}(y \cdot \nabla_y). \tag{24}$$

2.5 The weighted Hilbert space

Fix the Gaussian weight

$$\rho(y) = e^{-|y|^2/4}, \qquad d\mu(y) := \rho(y)\, dy, \tag{25}$$

and define

$$\mathcal{H} := L^2(\mathbb{R}^n, d\mu) = \Big\{ f : \mathbb{R}^n \to \mathbb{C} \;\Big|\; \int_{\mathbb{R}^n} |f(y)|^2 \rho(y)\, dy < \infty \Big\}, \tag{26}$$

with inner product $\langle f, g\rangle_{\mathcal{H}} = \int f \bar{g}\, \rho\, dy$. (This is the weight that makes $\mathcal{L}_0$ formally self-adjoint on $\mathcal{H}$; the weight $\rho = e^{-a|y|^2/2\nu}$ quoted in the problem statement is the same object up to the normalization convention $a = 1/2$, $\nu = 1$, appropriate after nondimensionalizing the diffusivity to 1.)

Lemma 2.12 (Self-adjointness of $\mathcal{L}_0$). *The operator $\mathcal{L}_0$ defined in* (24) *on the domain*

$$D(\mathcal{L}_0) = \Big\{ f \in \mathcal{H} : \Delta f - \tfrac{1}{2}(y \cdot \nabla f) \in \mathcal{H} \Big\}$$

is self-adjoint on $\mathcal{H}$.

Démonstration. Integrate by parts against the weight ρ. For $f, g \in C_c^\infty(\mathbb{R}^n)$,

$$\int \Delta f\, \bar{g}\, \rho\, dy = -\int \nabla f \cdot \nabla(\bar{g}\rho)\, dy = -\int \nabla f \cdot \nabla \bar{g}\, \rho\, dy - \int \nabla f \cdot \bar{g}\, \nabla \rho\, dy. \tag{27}$$

Since $\nabla\rho = \frac{1}{2} y\, \rho$, the last term equals $-\frac{1}{2}\int (y \cdot \nabla f)\, \bar{g}\, \rho\, dy$. Hence

$$\int \Delta f\, \bar{g}\, \rho\, dy - \tfrac{1}{2}\int (y \cdot \nabla f)\, \bar{g}\, \rho\, dy = -\int \nabla f \cdot \nabla \bar{g}\, \rho\, dy, \tag{28}$$

which is manifestly symmetric in $f \leftrightarrow g$ (real and Hermitian). □

Remark 2.13. The cleanest way to see Lemma 2.12 is via the unitary map $U : L^2(\mathbb{R}^n, d\mu) \to L^2(\mathbb{R}^n, dy)$, $Uf = \rho^{1/2} f$, which conjugates $\mathcal{L}_0$ to a shifted quantum harmonic oscillator,

$$U\mathcal{L}_0 U^{-1} = \Delta - \frac{|y|^2}{16} + \frac{n}{4}, \tag{29}$$

a standard Schrödinger operator with confining potential, self-adjoint on $H^2(\mathbb{R}^n) \cap \{|y|^2 f \in L^2\}$ by the Kato–Rellich / Faris–Lavine criteria [33].

2.6 Discreteness of the Spectrum : General Theory

Theorem 2.14 (Discrete spectrum, compact resolvent)**.** *The operator $\mathcal{L}_0$ of (24) on $\mathcal{H}$ has purely discrete spectrum*

$$\sigma(\mathcal{L}_0) = \Big\{ -\tfrac{k}{2} \ : \ k \in \mathbb{N}_0 \Big\}, \tag{30}$$

with the eigenvalue $-k/2$ having (finite) multiplicity equal to the number of degree-k Hermite polynomials in n variables, $\binom{k+n-1}{n-1}$. The corresponding eigenfunctions are the (weight-adjusted) Hermite functions

$$h_{\boldsymbol{k}}(y) = H_{\boldsymbol{k}}(y/2), \qquad \boldsymbol{k} = (k_1, \dots, k_n) \in \mathbb{N}_0^n, \ |\boldsymbol{k}| = k, \tag{31}$$

where $H_{\boldsymbol{k}}$ is the tensor-product (physicists') Hermite polynomial of multi-index $\boldsymbol{k}$. In particular $(-\Delta_y + \frac{1}{2} y \cdot \nabla_y)$-type operators of this form always have compact resolvent, and $\mathcal{L}_0$ generates an analytic (in fact immediately smoothing) semigroup on $\mathcal{H}$.

Corollary 2.15 (Structure of $\sigma(\mathcal{L})$)**.** *If $DN(U^*)$ is relatively compact with respect to $\mathcal{L}_0$ (e.g. bounded on $D(\mathcal{L}_0)$ with values in a space compactly embedded in $\mathcal{H}$ — true whenever the nonlinearity is of lower differential order and U^* decays, which holds for all physically relevant blow-up profiles), then by Weyl's essential spectrum theorem*

$$\sigma_{\mathrm{ess}}(\mathcal{L}) = \sigma_{\mathrm{ess}}(\mathcal{L}_0) = \varnothing, \tag{32}$$

so $\mathcal{L} = \mathcal{L}_0 + DN(U^*)$ also has purely discrete spectrum, consisting of isolated eigenvalues of finite algebraic multiplicity, accumulating only at $-\infty$. In particular $\sigma(\mathcal{L}) \cap \{\mathrm{Re}\,\lambda \geq -\delta\}$ is a finite *set for every $\delta > 0$: instability, if present, is governed by finitely many eigenvalues, never by essential/continuous spectrum. This is the rigorous justification for the "codimension-k" language in the problem statement — k is finite precisely because Theorem 2.14 and Weyl's theorem apply.*

2.7 The profile and its linearization

The relevant self-similar profile is the constant

$$U^*(y) \equiv \kappa := (p-1)^{-1/(p-1)}, \tag{33}$$

i.e. blow-up occurs at the ODE rate $u(x,t) \sim \kappa (T-t)^{-1/(p-1)}$ near $x = 0$. With $\alpha = 1/(p-1)$, the linearized operator (21) becomes *exactly*

$$\mathcal{L} = \Delta_y - \tfrac{1}{2} y \cdot \nabla_y - \tfrac{1}{p-1}\,\mathrm{Id} + p\kappa^{p-1}\,\mathrm{Id} = \mathcal{L}_0 + \Big(p\kappa^{p-1} - \tfrac{1}{p-1} \Big) \mathrm{Id}. \tag{34}$$

Since $p\kappa^{p-1} = p/(p-1)$ and $p/(p-1) - 1/(p-1) = 1$, the multiplication operator is exactly $+1 \cdot \mathrm{Id}$, so

$$\mathcal{L} = \mathcal{L}_0 + \mathrm{Id}. \tag{35}$$

Theorem 2.16 (Complete spectrum for the ODE blow-up profile)**.** *For the constant profile $U^* = \kappa$, the operator $\mathcal{L}$ on $\mathcal{H} = L^2(\mathbb{R}^n, e^{-|y|^2/4} dy)$ has spectrum*

$$\sigma(\mathcal{L}) = \Big\{ 1 - \tfrac{k}{2} \ : \ k = 0, 1, 2, \dots \Big\} = \{1, \ \tfrac{1}{2}, \ 0, \ -\tfrac{1}{2}, \ -1, \ \dots\}, \tag{36}$$

with eigenfunctions the Hermite functions $h_{\boldsymbol{k}}$, $|\boldsymbol{k}| = k$, of Theorem 2.14. Consequently :

- $\sigma_u = \{1\} \cup \{1/2\}$ *:* two *unstable directions, $k = 0$ (the constant/translation-in-time mode) and $k = 1$ (n-fold degenerate, the translation-in-space modes).*
- $\sigma_c = \{0\}$ *: the $k = 2$ eigenvalue, $\binom{n+1}{2}$-fold degenerate, corresponding to the free parameters (blow-up time T, aspect ratio) in the similarity variables.*
- $\sigma_s = \{-1/2, -1, \dots\}$ *: all $k \geq 3$ modes.*

3 Geometric Setup

The study of singularities [10] in the three-dimensional incompressible [5] Navier-Stokes equations [35] :

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla p = 0 \quad \text{in } \mathbb{R}^3 \times (0,T), \\ \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \times (0,T), \end{cases} \tag{37}$$

requires deep geometric insight into how kinetic energy and vorticity concentrate. This extended monograph covers foundational geometric measure theory concepts applied to fluid dynamics, culminating in the Caffarelli-Kohn-Nirenberg (CKN) theorem [1].

Theorem 3.1 (Koch and Tataru 2001)**.** *There exists $\epsilon > 0$ such that if U^0 is divergence-free with $\|u\|_{BMO^{-1}} < \epsilon$, then there exists a global (for $t > 0$) smooth solution. They show that the Navier-Stokes bilinear operator,*

$$u \mapsto -\int_0^t e^{(t-t')\Delta} \mathbb{P} \operatorname{div} u\left(t'\right) \otimes u\left(t'\right) dt'$$

is bounded on X_{KT}, the path space corresponding to BMO^{-1} [11].

Theorem 3.2. *There exists divergence-free initial data $U^0 \in BMO^{-1}$ [11] such that the Navier-Stokes initial value problem admits two distinct global solutions :*

$$u^{(1)}, u^{(2)} \in C_{t,x}^{\infty}\left((0,\infty) \times \mathbb{T}^3\right) \cap C_t\left([0,\infty); W^{-1,p}\left(\mathbb{T}^3\right)\right)$$

for all $p < \infty$.

- *The solutions are in the Banach space where Koch-Tataru run the fixed point argument.*
- *The data is critical and smooth a.e. (but not L^2).*
- *Sharpness of Prodi-Serrin criteria :*

$$u^{(i)} \in L_t^{2,\infty} L_x^{\infty},$$
$$\int_0^T \frac{\left\|u^{(i)}(t)\right\|_{\infty}^2}{\log \cdots \log\left(C + \left\|u^{(i)}(t)\right\|_{\infty}\right)} dt < \infty$$

3.1 Local Energy Inequality and Monotonicity Formulas

To study local properties of solutions near potential singularities [10], one relies on local energy inequalities that do not depend on global spatial integration.

Theorem 3.3 (Local Energy Inequality, Leray-Hopf)**.** *Let u be a suitable weak solution to the Navier-Stokes equations in $\Omega \times (0,T)$, and let $p \in L^{3/2}(\Omega \times (0,T))$. For any non-negative, smooth cut-off function $\phi \in C_c^{\infty}(\Omega \times (0,T))$ with $\phi \geq 0$, the following local energy inequality holds for almost all $t \in (0,T)$:*

$$\begin{aligned} &\int_\Omega |u(x,t)|^2 \phi(x,t)\, dx + 2\nu \int_0^t \int_\Omega |\nabla u|^2 \phi \, dx ds \\ &\leq \int_0^t \int_\Omega \left(|u|^2 (\partial_t \phi + \nu \Delta \phi) + (|u|^2 + 2p)(u \cdot \nabla \phi)\right) dx ds. \end{aligned} \tag{38}$$

Démonstration. Let $\phi(x,t)$ be a test function compactly supported in space-time [25]. We test the momentum equation with $u\phi$. Integration by parts for the convective term yields :

$$\int (u \cdot \nabla u) \cdot u \phi \, dx = -\frac{1}{2} \int |u|^2 (u \cdot \nabla \phi)\, dx.$$

For the pressure term :

$$\int \nabla p \cdot u\phi \, dx = -\int p \operatorname{div}(u\phi)\, dx = -\int p(u \cdot \nabla\phi)\, dx$$

since $\operatorname{div} u = 0$. For the viscous term :

$$\int (-\nu\Delta u) \cdot u\phi \, dx = \nu \int |\nabla u|^2 \phi \, dx + \nu \int \nabla \left(\frac{1}{2}|u|^2\right) \cdot \nabla\phi \, dx$$

Integrating by parts on the last term gives $-\nu \int \frac{1}{2}|u|^2 \Delta\phi \, dx$. Combining these yields the local energy identity for smooth solutions, which is extended to suitable weak [6] solutions via standard mollification arguments. □

Galerkin/mollified

Suitable weak solutions are constructed as limits of a smooth approximating sequence u^δ — either Galerkin truncations of the Leray [2] projection of the equations, or solutions of the mollified system

$$\partial_t u^\delta + (u^\delta_\delta \cdot \nabla)u^\delta - \nu\Delta u^\delta + \nabla p^\delta = 0, \qquad \operatorname{div} u^\delta = 0, \tag{39}$$

where $u^\delta_\delta := u^\delta * \eta_\delta$ is a spatial mollification of the transport velocity only (Leray's regularization), chosen precisely so that the convective term remains divergence-free and the following computation is exact. For such u^δ, smooth in x for each t,

$$\int (u^\delta \cdot \nabla u^\delta) \cdot u^\delta \phi \, dx = \int (u^\delta_\delta \cdot \nabla)\big(\tfrac{1}{2}|u^\delta|^2\big)\phi \, dx = -\tfrac{1}{2}\int |u^\delta|^2 (u^\delta_\delta \cdot \nabla\phi)\, dx, \tag{40}$$

$$\int \nabla p^\delta \cdot u^\delta \phi \, dx = -\int p^\delta (u^\delta \cdot \nabla\phi)\, dx, \tag{41}$$

$$\int (-\nu\Delta u^\delta) \cdot u^\delta \phi \, dx = \nu \int |\nabla u^\delta|^2 \phi \, dx - \tfrac{\nu}{2}\int |u^\delta|^2 \Delta\phi \, dx. \tag{42}$$

(In (40) we used $\operatorname{div} u^\delta_\delta = 0$, which holds because mollification commutes with the divergence-free constraint ; the passage from u^δ_δ back to u^δ in the final quadratic form is legitimate because $u^\delta_\delta \to u^\delta$ strongly as $\delta \to 0$, a point we return to below.) Combining (40)–(42) with $\partial_t \int \frac{1}{2}|u^\delta|^2 \phi \, dx = \int \partial_t u^\delta \cdot u^\delta \phi \, dx + \int \frac{1}{2}|u^\delta|^2 \partial_t\phi \, dx$ and integrating in time from 0 to t yields the *exact energy equality*

$$\int |u^\delta(t)|^2 \phi(t)\, dx + 2\nu \int_0^t\!\!\int |\nabla u^\delta|^2 \phi \, dx\, ds = \int_0^t\!\!\int \Big[|u^\delta|^2(\partial_t\phi + \nu\Delta\phi) + (|u^\delta|^2 + 2p^\delta)(u^\delta \cdot \nabla\phi)\Big]\, dx\, ds \tag{43}$$

for every smooth ϕ, with *equality*, since u^δ is smooth and every integration by parts above is justified classically.

Lemma 3.4 (Compactness)**.** *There is a subsequence (not relabeled) such that*

$$u^\delta \rightharpoonup^* u \quad \textit{in } L^\infty(0,T; L^2(\Omega)), \tag{44}$$

$$u^\delta \rightharpoonup u \quad \textit{in } L^2(0,T; H^1(\Omega)), \tag{45}$$

$$u^\delta \to u \quad \textit{strongly in } L^2(0,T; L^2_{\mathrm{loc}}(\Omega)) \textit{ (Aubin–Lions)}, \tag{46}$$

$$p^\delta \rightharpoonup p \quad \textit{in } L^{3/2}(0,T; L^{3/2}_{\mathrm{loc}}(\Omega)). \tag{47}$$

Justification of Lemma 3.4. (44)–(45) follow from the energy *equality* (43) applied with $\phi \uparrow 1$ (a cutoff exhausting Ω), which gives a δ-uniform bound on $\|u^\delta\|_{L^\infty_t L^2_x}$ and $\|\nabla u^\delta\|_{L^2_{t,x}}$; both spaces are then weak(-*) sequentially compact on bounded sets, by the Banach–Alaoglu theorem [47] (for $L^\infty_t L^2_x = (L^1_t L^2_x)^*$) and reflexivity of $L^2_t H^1_x$. Strong convergence (46) follows from the Aubin–Lions–Simon lemma : the uniform bound on $\partial_t u^\delta$ in $L^{4/3}(0,T; H^{-1}(\Omega))$ (obtained from the momentum equation itself, using that $(u^\delta \cdot \nabla)u^\delta$ is bounded in $L_t^{4/3}H_x^{-1}$ by the Ladyzhenskaya/Sobolev interpolation $\|u\|_{L_x^4} \lesssim \|u\|_{L_x^2}^{1/4}\|\nabla u\|_{L_x^2}^{3/4}$ in three dimensions, and $\nu\Delta u^\delta$ is bounded in $L_t^2H_x^{-1}$), combined with the compact embedding $H^1(\Omega') \hookrightarrow\hookrightarrow L^2(\Omega')$ on any $\Omega' \Subset \Omega$, gives compactness in $L^2(0,T;L^2_{\text{loc}})$. Finally (47) is the pressure estimate of Caffarelli–Kohn–Nirenberg (1982) [1], obtained from the elliptic equation $-\Delta p^\delta = \partial_i\partial_j(u_i^\delta u_j^\delta)$ (from taking div of the momentum equation) via the Calderón–Zygmund inequality [46] on $L^{3/2}$; the δ-uniform bound follows from the δ-uniform bound on u^δ in $L^3_{t,x}$, itself implied by (44)–(45) and Sobolev embedding $L_t^\infty L_x^2 \cap L_t^2 H_x^1 \hookrightarrow L_{t,x}^{10/3} \hookrightarrow L^3_{t,x}$ in three space dimensions. □

We now pass to the limit in (43) term by term, for fixed $\phi \geq 0$.

— **Dissipation term.** The map $w \mapsto \int_0^t \int |\nabla w|^2\phi\, dx\, ds$ is convex and (thanks to $\phi \geq 0$) is the integral of a nonnegative quadratic form composed with the weak-continuous map $w \mapsto \nabla w$; by the standard weak [6] lower semicontinuity theorem for convex functionals (Tonelli, or directly : $\|\sqrt{\phi}\, \nabla w\|_{L^2}^2$ is a norm-squared, and norms are weakly lower semicontinuous),

$$\liminf_{\delta\to 0} \int_0^t \int |\nabla u^\delta|^2 \phi\, dx\, ds \;\geq\; \int_0^t \int |\nabla u|^2\phi\, dx\, ds. \tag{48}$$

— **Left-hand accumulation term.** By (46), $u^\delta(t) \to u(t)$ strongly in L^2_{loc} for a.e. t (after passing to a further subsequence depending on t, using that L^2-strong convergence in $L^2(0,T;L^2_{\text{loc}})$ implies a.e.-t strong convergence along a subsequence), so $\int |u^\delta(t)|^2\phi(t)\, dx \to \int |u(t)|^2\phi(t)\, dx$ for a.e. t.

— **Right-hand terms.** $\int_0^t \int |u^\delta|^2(\partial_t\phi + \nu\Delta\phi)\, dx\, ds \;\to\; \int_0^t \int |u|^2(\partial_t\phi + \nu\Delta\phi)\, dx\, ds$ by (46) (strong $L^2_{t,x,\text{loc}}$ convergence of u^δ, hence of $|u^\delta|^2$ in $L^1_{t,x,\text{loc}}$ after using the uniform $L_t^\infty L_x^2$ bound to upgrade via interpolation/Vitali). The cubic term $\int_0^t \int |u^\delta|^2(u^\delta\cdot\nabla\phi)\, dx\, ds$ converges similarly, using (44)–(46) and the compact support of $\nabla\phi$ to localize ; the pressure term $\int_0^t \int p^\delta(u^\delta \cdot \nabla\phi)\, dx\, ds$ converges by (46)–(47) (strong-times-weak convergence of a product, valid because one factor converges strongly and the other only needs to converge weakly against a fixed test function $\nabla\phi$ of compact support).

3.2 Caffarelli-Kohn-Nirenberg Partial Regularity Theory

The milestone result of Caffarelli, Kohn, and Nirenberg (1982) [1] establishes that the 1-dimensional Hausdorff measure of the singular set is zero.

Definition 3.5 (Suitable Weak Solution)**.** A pair (u, p) is called a suitable weak solution in $Q = \Omega \times (0, T)$ if :

— $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$, $p \in L^{3/2}(Q)$.

— The equations are satisfied in the sense of distributions.

— The local energy inequality holds for all non-negative $\phi \in C_c^\infty(Q)$.

Theorem 3.6 (Caffarelli-Kohn-Nirenberg, 1982)**.** *Let u be a suitable weak solution of the Navier-Stokes equations in $Q = \Omega \times (0,T)$. There exists an open set $\mathcal{O} \subseteq Q$ such that u is Hölder continuous in $\mathcal{O}$ (in fact, C^∞), and the Hausdorff dimension of the singular set $\Sigma = Q \setminus \mathcal{O}$ satisfies :*

$$\mathcal{H}^1(\Sigma) = 0. \tag{49}$$

Furthermore, the 1-dimensional parabolic Hausdorff measure of the set of singular times is also zero.

ε-Regularity Criteria

The proof of Theorem 3.6 relies heavily on dimensional reduction and scaled dimensionless quantities. We define the scale-invariant quantities :

$$A(r) = r^{-2}\int_{Q_r} |u|^2 dxdt, \quad B(r) = r^{-1}\int_{Q_r} |\nabla u|^2 dxdt, \quad C(r) = r^{-2}\int_{Q_r} |p|^{3/2} dxdt, \tag{50}$$

where $Q_r = B_r(x_0) \times (t_0 - r^2, t_0)$.

Lemma 3.7 (ε-Regularity Lemma)**.** *There exists an absolute constant $\epsilon_0 > 0$ such that if for some $r > 0$,*

$$r^{-1}\int_{t_0-r^2}^{t_0}\int_{B_r(x_0)} (|u|^3 + |p|^{3/2}) dxdt \le \epsilon_0, \tag{51}$$

then u is essentially bounded and regular in $Q_{r/2}(x_0, t_0)$.

Definition 3.8 (Parabolic δ-cover and $\mathcal{P}^1$ measure)**.** For $E \subset \mathbb{R}^3 \times \mathbb{R}$ and $\delta > 0$, a *parabolic δ-cover* of E is a countable collection of parabolic cylinders $\{Q_{r_j}(z_j)\}$ with $r_j < \delta$ covering E. The 1-dimensional *parabolic Hausdorff measure* is :

$$\mathcal{P}^1(E) = \liminf_{\delta\to 0^+} \left\{ \sum_j r_j \;\middle|\; E \subset \bigcup_j Q_{r_j}(z_j),\ r_j < \delta \right\}. \tag{52}$$

Proof (following Caffarelli, Kohn, Nirenberg 1982). The proof proceeds in several stages.

Stage 1 : Suitable weak solutions and local energy inequality. A Leray–Hopf solution $\mathbf{u}$ is *suitable* if it satisfies the *local energy inequality* : for every non-negative $\phi \in C_c^\infty(\mathbb{R}^3 \times (0,T))$,

$$\int_{\mathbb{R}^3} |\mathbf{u}|^2 \phi \, d\mathbf{x}\Big|_t + 2\nu\int_0^t\int |\nabla \mathbf{u}|^2\phi \le \int_0^t\int |\mathbf{u}|^2(\partial_\tau\phi + \nu\Delta\phi) + \int_0^t\int (|\mathbf{u}|^2 + 2p)(\mathbf{u}\cdot\nabla\phi). \tag{53}$$

This is stronger than the global energy inequality and holds for physically admissible approximations (e.g., Galerkin truncations).

Stage 2 : The ε-regularity criterion. The central technical result of CKN is :

$$\limsup_{r\to 0^+} \frac{1}{r}\int_{Q_r(z_0)} |\nabla \mathbf{u}|^2 \, d\mathbf{x}\, dt < \varepsilon^* \implies z_0 = (x_0, t_0) \text{ is a regular point.} \tag{54}$$

Proof of (54) *:* The quantity $\frac{1}{r}\int_{Q_r} |\nabla \mathbf{u}|^2$ is the *scaled local energy.* Suppose the hypothesis holds for some $\varepsilon^* > 0$ to be determined. By a covering argument and Sobolev embedding in the parabolic setting, one shows that if the scaled energy is below ε^*, the solution is bounded in $L^\infty(Q_{r/2})$, hence regular. The key step uses the following three estimates in $Q_r(z_0)$:

a) *Velocity smallness in mixed norms.* By the local energy inequality and Sobolev embedding $H^1 \hookrightarrow L^6$:

$$\frac{1}{r^2}\int_{Q_r} |\mathbf{u}|^3 \, d\mathbf{x}\, dt \le C\left(\frac{1}{r}\int_{Q_r} |\nabla \mathbf{u}|^2\right)^{3/2} + C\frac{1}{r^2}\int_{Q_r} |\mathbf{u}|^3.$$

This is a self-improving inequality : if the right-hand side is small, the left is controlled.

b) *Pressure estimate.* The pressure satisfies $-\Delta p = \partial_i\partial_j(u_iu_j)$ (distributional sense). By Calderón–Zygmund theory [46] :

$$\frac{1}{r^2}\int_{Q_r} |p|^{3/2} \, d\mathbf{x}\, dt \le C\frac{1}{r^2}\int_{Q_r} |\mathbf{u}|^3 \, d\mathbf{x}\, dt.$$

c) *Regularity bootstrap.* The combined smallness of velocity and pressure in L^3 norms allows application of a Caccioppoli-type inequality on shrinking cylinders $Q_r \supset Q_{r/2} \supset Q_{r/4} \supset \cdots$, bootstrapping $L^{p,q}$ estimates until L^∞ regularity [11] is achieved.

The threshold $\varepsilon^* > 0$ is determined by the final step : it is the largest value for which the bootstrap closes.

Stage 3 : Covering argument for $\mathcal{P}^1(\Sigma) = 0$. Suppose Σ contains a compact subset $K \subset \Sigma$. For each $z \in K$, the hypothesis of (54) *fails* (since z is singular) :

$$\limsup_{r\to 0^+} \frac{1}{r} \int_{Q_r(z)} |\nabla \mathbf{u}|^2 \geq \varepsilon^*.$$

Fix $\delta > 0$. For each $z \in K$, there exists $r_z < \delta$ such that $\frac{1}{r_z}\int_{Q_{r_z}(z)} |\nabla \mathbf{u}|^2 \geq \frac{\varepsilon^*}{2}$. By 5r-covering lemma, extract a countable disjoint sub-collection $\{Q_{r_j}(z_j)\}$ such that $K \subset \bigcup_j Q_{5r_j}(z_j)$. For this cover :

$$\sum_j r_j \leq \frac{2}{\varepsilon^*} \sum_j \int_{Q_{r_j}(z_j)} |\nabla \mathbf{u}|^2 \leq \frac{2}{\varepsilon^*} \int_{\mathbb{R}^3\times(0,T)} |\nabla \mathbf{u}|^2 \, d\mathbf{x}\, dt.$$

By the global energy inequality, $\int_0^T \|\nabla \mathbf{u}\|_{L^2}^2 \, dt \leq \frac{1}{2\nu} \|\mathbf{u}_0\|_{L^2}^2 < \infty$. Hence $\sum_j r_j \leq \frac{\|\mathbf{u}_0\|_{L^2}^2}{\nu\varepsilon^*} < \infty$. Since the disjoint cylinders Q_{r_j} each contribute $|Q_{r_j}| \sim r_j^5$, but we only need $\sum r_j$, the bound $\sum r_j < \infty$ does *not* yet give $\mathcal{P}^1(K) = 0$.

To obtain $\mathcal{P}^1(K) = 0$, one requires a more refined covering. Fix $\eta > 0$. By the absolute continuity of the L^2 integral $\int |\nabla \mathbf{u}|^2$, for any $\eta > 0$ there exists $\delta > 0$ such that for any measurable E with $|E|_{\text{par}} < \delta$: $\int_E |\nabla \mathbf{u}|^2 < \eta\varepsilon^*$. Choose the cover so that $\sum_j |Q_{r_j}|_{\text{par}} \leq \delta$; then $\sum_j r_j \leq \frac{2}{\varepsilon^*} \cdot \eta\varepsilon^* = 2\eta$. Since $\eta > 0$ is arbitrary, $\mathcal{P}^1(K) = 0$, and hence $\mathcal{P}^1(\Sigma) = 0$. [1] □

4 Analysis of Blow-Up Profiles and Spectra

We return to the profile equations under self-similar rescalings and examine the precise structure of the linearized operator around potential non-trivial profiles.

Linearized Perturbation Equation in Weighted Spaces

Let $U^*(y)$ be a candidate profile satisfying the rescaled stationary Navier-Stokes equations :

$$-\nu\Delta U^* + aU^* + a(y\cdot\nabla)U^* + (U^*\cdot\nabla)U^* + \nabla P^* = 0, \quad \operatorname{div} U^* = 0. \tag{55}$$

Writing $U(y,\tau) = U^*(y) + V(y,\tau)$, the perturbation satisfies :

$$\partial_\tau V = \mathcal{L}V + \mathcal{N}(V), \tag{56}$$

where $\mathcal{L}V = \nu\Delta V - aV - a(y\cdot\nabla)V - (U^*\cdot\nabla)V - (V\cdot\nabla)U^* - \nabla\Pi$.

Proposition 4.1. *In the Hilbert space $H = L^2_\rho(\mathbb{R}^3; \rho(y)dy)$ with $\rho(y) = e^{-a|y|^2/4\nu}$, the operator $\mathcal{L}_0 = \nu\Delta - a(y\cdot\nabla)$ is self-adjoint and its spectrum consists entirely of eigenvalues of the form :*

$$\lambda_k = -\left(\frac{3a}{2} + \frac{a}{2}|k|\right), \quad k \in \mathbb{N}^3. \tag{57}$$

Démonstration. By conjugating $\mathcal{L}_0$ with the weight $\rho^{1/2}$, one transforms $\mathcal{L}_0$ into a Schrödinger-type operator with a harmonic oscillator potential [49] :

$$\tilde{\mathcal{L}}_0 = \rho^{1/2}\mathcal{L}_0\rho^{-1/2} = \nu\Delta - \frac{a^2|y|^2}{4\nu} + \frac{3a}{2}.$$

The spectrum of the harmonic oscillator is explicitly known and discrete, given by $-\frac{3a}{2} - \frac{a}{2}\sum_i k_i$. This guarantees that the linear drift and diffusion parts possess a well-behaved spectral decomposition. □

1. (Draft of complete evidence–see (American Geophysical Union) repository.)

Lemma 4.2 (Correct symmetrizing weight)**.** *$\mathcal{L}_0 = \nu\Delta - a(y\cdot\nabla)$ is symmetric on $L^2(\mathbb{R}^3, \rho\, dy)$ with $\rho(y) = e^{c|y|^2}$ if and only if*

$$c = -\frac{a}{2\nu}, \qquad i.e. \qquad \rho(y) = e^{-a|y|^2/(2\nu)}. \tag{58}$$

Corollary 4.3 (Effect of the shift term)**.** *The full profile-independent linear part $\nu\Delta - aV - a(y\cdot\nabla)V$ (restoring the $-aV$ term dropped in (24)) has spectrum $\{-ak-a\}_{k=0}^{\infty} = \{-a(k+1)\}_{k=0}^{\infty}$, strictly negative for every $k \geq 0$: the profile-independent part of $\mathcal{L}$ is, by itself, spectrally stable with a uniform gap of size a below 0. Any instability of the full operator $\mathcal{L} = \mathcal{L}_0 - a\,\mathrm{Id} +$ (profile-dependent terms) can therefore only arise from the terms $-(U^*\cdot\nabla)V - (V\cdot\nabla)U^* - \nabla\Pi$ depending on the profile itself — precisely the content that a genuine stability analysis of a Navier–Stokes self-similar singularity [18] would need to control, and precisely where all of the difficulty of the problem is concentrated.*

Theorem 4.4 (Corrected spectrum of $\mathcal{L}_0$)**.** *On $\mathcal{H} := L^2\big(\mathbb{R}^3,\, e^{-a|y|^2/2\nu}\, dy\big)$, the operator $\mathcal{L}_0 = \nu\Delta - a(y\cdot\nabla)$ is self-adjoint with purely discrete spectrum*

$$\sigma(\mathcal{L}_0) = \{-ak \ : \ k = 0, 1, 2, \dots\}, \tag{59}$$

each eigenvalue $-ak$ having multiplicity $\binom{k+2}{2}$ (the number of multi-indices $\boldsymbol{k} \in \mathbb{N}_0^3$ with $|\boldsymbol{k}| = k$), with eigenfunctions the (rescaled) Hermite polynomials

$$h_{\boldsymbol{k}}(y) = H_{\boldsymbol{k}}\Big(\sqrt{\tfrac{a}{\nu}}\, y\Big), \qquad \boldsymbol{k} \in \mathbb{N}_0^3. \tag{60}$$

In particular $\lambda_0 = 0$ (eigenfunction : the constant), consistent with the trivial computation $\mathcal{L}_0(1) \equiv 0$.

Proposition 4.5 (Discreteness of the full spectrum, under a decay hypothesis)**.** *If $U^* \in C^\infty(\mathbb{R}^3)$ with $|U^*(y)| + |\nabla U^*(y)| \to 0$ as $|y| \to \infty$ sufficiently fast that the profile-dependent part of $\mathcal{L}$ is relatively compact with respect to $\mathcal{L}_0 - a\,\mathrm{Id}$ (true, e.g., if U^* and ∇U^* decay faster than any inverse polynomial, which forces U^* to lie in every weighted Sobolev space used above), then $\sigma_{\mathrm{ess}}(\mathcal{L}) = \varnothing$ and $\sigma(\mathcal{L})$ consists of isolated eigenvalues of finite multiplicity, by Weyl's essential spectrum theorem exactly as in Corollary 4.3's parent argument (compact resolvent for the unperturbed self-adjoint part, relatively compact perturbation).*

4.1 Non-uniqueness of the weak solution

An incompressible vector field $u(x,t) \in L^2$ is called a weak [6], [13], [26] solution of the nonhomogeneous Euler equations [16] with the external force $f(x,t) \in \mathcal{D}'$, if for every test-field $v(x,t) \in C_0^\infty$ such that $\nabla\cdot v \equiv 0$, the following holds :

$$-\iint \left[\left(u, \frac{\partial v}{\partial t}\right) + (u\otimes u, \nabla v)\right] dxdt = \iint (f, v) dxdt \tag{61}$$

Our construction is based on the following simple lemma :

Lemma 4.6. *Let $u_i(x,t)$ be a weak solution of nonhomogeneous Euler equations [27], [28] with external forces $f_i(x,t)$, for $i = 1, 2, \dots$. Suppose that $u_i \to u$ strongly in L^2, while $f_i \to 0$ weakly in $\mathcal{D}'$, as $i \to \infty$. Then $u(x,t)$ is a weak solution of the homogeneous Euler equations.*

4.2 The weak formulation of the Euler equations

Let $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ (in practice $n = 2, 3$) be a vector field with $u(\cdot, t) \in L^2(\mathbb{R}^n)$ for a.e. t, and let $f(x,t) \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$ be a distributional forcing term. Formally, u solves the incompressible Euler equations [27], [28], [40] with force f if

$$\partial_t u + \nabla\cdot(u\otimes u) + \nabla p = f, \qquad \nabla\cdot u = 0, \tag{62}$$

for some scalar pressure p. Because (62) only needs to make sense distributionally, one eliminates the pressure by testing against divergence-free vector fields : if $v \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n)$ satisfies $\nabla \cdot v \equiv 0$, then $\iint (\nabla p, v)\, dx\, dt = -\iint p\,(\nabla \cdot v)\, dx\, dt = 0$, and integrating (62) by parts against v yields precisely the definition already given :

$$-\iint \left[\left(u, \frac{\partial v}{\partial t}\right) + (u \otimes u, \nabla v)\right] dx\, dt = \iint (f, v)\, dx\, dt, \qquad \forall\, v \in C_0^\infty,\ \nabla \cdot v = 0. \tag{E}$$

This is the weakest formulation in which the nonlinear term $u \otimes u$ still makes classical sense : it requires only $u \in L^2_{loc}$, since $u \otimes u \in L^1_{loc}$ by Cauchy–Schwarz. No second derivatives, and not even a pointwise pressure, are required.

4.3 The stability lemma

Lemma 4.7 (Weak stability under vanishing forcing). *Let $u_i(x,t)$, $i = 1, 2, \ldots$, be weak solutions of* (E) *with forces $f_i(x,t)$. Suppose*

$$u_i \to u \quad \textit{strongly in } L^2_{loc}(\mathbb{R}^n \times \mathbb{R}), \qquad f_i \rightharpoonup 0 \quad \textit{weakly in } \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}),$$

as $i \to \infty$. Then u is a weak solution of the homogeneous *Euler equations, i.e.* (E) *holds with $f \equiv 0$.*

Démonstration. Fix a divergence-free test field $v \in C_0^\infty$ and let $K = \operatorname{supp}(v)$, a compact subset of space-time [25]. By hypothesis, for every i,

$$-\iint \left[\left(u_i, \frac{\partial v}{\partial t}\right) + (u_i \otimes u_i, \nabla v)\right] dx\, dt = \iint (f_i, v)\, dx\, dt. \tag{E$_i$}$$

We pass to the limit $i \to \infty$ term by term.

Step 1 (forcing term). By definition of weak-$*$ convergence in $\mathcal{D}'$ and since v is an admissible test function,

$$\lim_{i\to\infty} \iint (f_i, v)\, dx\, dt = \langle 0, v \rangle = 0.$$

Step 2 (linear term). Since v is fixed and smooth, $\partial_t v \in C_0^\infty(K)$, so $\|\partial_t v\|_{L^2(K)} < \infty$. By Cauchy–Schwarz on K,

$$\left| \iint \left(u_i - u, \frac{\partial v}{\partial t}\right) dx\, dt \right| \le \|u_i - u\|_{L^2(K)} \left\| \frac{\partial v}{\partial t} \right\|_{L^2(K)} \xrightarrow[i\to\infty]{} 0,$$

because $u_i \to u$ strongly in L^2_{loc}, hence in $L^2(K)$. So

$$\lim_{i\to\infty} \iint \left(u_i, \frac{\partial v}{\partial t}\right) dx\, dt = \iint \left(u, \frac{\partial v}{\partial t}\right) dx\, dt.$$

Step 3 (quadratic term). Algebraically,

$$u_i \otimes u_i - u \otimes u = (u_i - u) \otimes u_i + u \otimes (u_i - u).$$

Hence

$$\begin{aligned}
\left| \iint \left(u_i \otimes u_i - u \otimes u,\ \nabla v\right) dx\, dt \right| &\le \|\nabla v\|_{L^\infty(K)} \iint_K |u_i \otimes u_i - u \otimes u|\, dx\, dt \\
&\le \|\nabla v\|_{L^\infty(K)} \left(\iint_K |u_i - u||u_i|\, dx\, dt + \iint_K |u||u_i - u|\, dx\, dt \right) \\
&\le \|\nabla v\|_{L^\infty(K)} \Big(\|u_i - u\|_{L^2(K)} \|u_i\|_{L^2(K)} + \|u\|_{L^2(K)} \|u_i - u\|_{L^2(K)} \Big),
\end{aligned}$$

using Cauchy–Schwarz in the last line. Strong L^2 convergence implies that $(\|u_i\|_{L^2(K)})_i$ is bounded, say by $M < \infty$, and $\|u_i - u\|_{L^2(K)} \to 0$; also $\|u\|_{L^2(K)} < \infty$ since $u \in L^2_{loc}$. Therefore the right-hand side tends to 0, and

$$\lim_{i\to\infty} \iint (u_i \otimes u_i, \nabla v)\, dx\, dt = \iint (u \otimes u, \nabla v)\, dx\, dt.$$

Conclusion. Passing to the limit in (E_i) using Steps 1–3,

$$-\iint \left[\left(u, \frac{\partial v}{\partial t} \right) + (u \otimes u, \nabla v) \right] dx\, dt = 0$$

for every divergence-free $v \in C_0^\infty$, which is exactly (E) with $f \equiv 0$. □

Remark 4.8. The lemma uses *strong* convergence of u_i but only *weak* convergence of f_i to zero. This asymmetry is essential : it is precisely the gap between these two topologies that the construction below exploits. If $f_i \to 0$ strongly, or if one only assumed $u_i \rightharpoonup u$ weakly, the quadratic term in Step 3 could fail to pass to the limit (weak convergence is not compatible with nonlinear, in particular quadratic, functionals), and the theorem would be false or vacuous.

4.4 From the lemma to non-uniqueness : Scheffer–Shnirelman construction

Lemma 4.7 is a mechanism, not yet a proof of non-uniqueness. Non-uniqueness follows once one exhibits an *explicit* sequence (u_i, f_i) satisfying its hypotheses with

$$u \not\equiv 0 \qquad \text{while} \qquad u_i \to u \text{ strongly in } L^2.$$

Then u is a nonzero weak solution of the *homogeneous, unforced* Euler equations, and since $u \equiv 0$ is trivially also a weak solution with zero force, the Cauchy problem for the homogeneous Euler equations [27], [28] in the class L^2_{loc} has (at least) two weak solutions with the same (zero) initial data whenever u can be arranged to vanish outside a compact set in space-time [25].

Theorem 4.9 (Scheffer, 1993 [43] ; Shnirelman, 1997 [44])**.** *There exists a nonzero weak solution $u \in L^2(\mathbb{R}^2 \times \mathbb{R})$ (respectively $\mathbb{R}^3 \times \mathbb{R}$ in Shnirelman's later, energy-decreasing version) of the homogeneous incompressible Euler [19] equations* (E) *with $f \equiv 0$, such that $u(x,t) = 0$ for all (x,t) outside a bounded set.*

Construction

The idea is to build the approximating sequence u_i out of rapidly oscillating, divergence-free "building blocks" whose energy does not vanish as $i \to \infty$, but whose mismatch with being an exact solution — the *Reynolds stress* — does vanish, only in the weak topology of distributions. Concretely, fix a target field $\bar{u}$ (in the compactly supported case, $\bar{u} \equiv 0$ is the eventual limit, but the construction is initiated from a nontrivial "subsolution"), and consider a decomposition of the Euler system into a *Euler–Reynolds system*

$$\partial_t u_i + \nabla \cdot (u_i \otimes u_i) + \nabla p_i = \nabla \cdot R_i, \qquad \nabla \cdot u_i = 0, \tag{63}$$

where R_i is a symmetric, trace-free (or controlled-trace) stress tensor measuring the defect from being an exact solution. Setting $f_i := \nabla \cdot R_i$ in distributional form, one constructs u_i and R_i inductively so that :

(a) $u_{i+1} - u_i$ is a highly oscillatory perturbation, built from (Beltrami) plane waves

$$W_k(x,t) = A_k\, e^{i\, k\cdot(\lambda_i x)}\,, \qquad A_k \perp k,\ k \in \mathbb{Z}^n,$$

at frequency $\lambda_i \to \infty$, chosen so that the quadratic self-interaction of the perturbation *cancels* the existing stress R_i to leading order (this is the "Nash-type" or "Newton-iteration" step of convex integration, using the algebraic identity that a sum of Beltrami waves with wave vectors summing to zero can synthesize any prescribed symmetric trace-free stress) ;

(b) the new stress R_{i+1} that remains is $O(\lambda_i^{-\theta})$ in a suitable norm, for some $\theta > 0$, so that $R_i \to 0$ in $\mathcal{D}'$, hence $f_i = \nabla \cdot R_i \rightharpoonup 0$ in $\mathcal{D}'$;

(c) the amplitude of the correction is chosen so that $\|u_{i+1} - u_i\|_{L^2}$ is summable (a geometric-type decay $\sum_i \|u_{i+1} - u_i\|_{L^2} < \infty$), which forces u_i to be Cauchy, hence *strongly* convergent, in L^2.

Because the oscillation frequencies $\lambda_i \to \infty$, the energy $\int |u_i|^2\, dx$ need not vanish even though $u_i \rightharpoonup 0$ weakly could hold at intermediate stages ; strong L^2 convergence in (c) is exactly what is needed to invoke Lemma 4.7 and conclude that the limit $u = \lim_i u_i$ solves the *homogeneous* equation, in spite of every finite stage solving a nontrivially forced equation.

Proposition 4.10 (Formal statement of the iteration)**.** *Suppose* (u_i, R_i) *solve* (63) *and there exist* $\lambda_i \to \infty$, $\delta_i \to 0$ *such that*

$$\|u_{i+1} - u_i\|_{L^2} \lesssim \delta_i^{1/2}, \qquad \|R_{i+1}\|_{L^1} \lesssim \delta_{i+1}\lambda_i^{-1} \quad (\textit{or an analogous super-linear gain in } \lambda_i), \tag{64}$$

with $\sum_i \delta_i^{1/2} < \infty$*. Then* $u_i \to u$ *strongly in* L^2 *and* $R_i \to 0$ *in* $\mathcal{D}'$*, so that* $f_i := \nabla \cdot R_i \rightharpoonup 0$ *in* $\mathcal{D}'$*. By Lemma 4.7,* u *is a weak solution of the homogeneous Euler equations [36].*

The estimates in (64) are exactly of the type produced by the convex integration scheme of De Lellis and Székelyhidi [27, 28], which reformulated Scheffer's and Shnirelman's original constructions as an h-principle for the Euler–Reynolds system, and by the later intermittent ("Mikado flow" [45]) schemes of Buckmaster–De Lellis–Székelyhidi–Vicol [29] used to reach the sharp Onsager [30], [31] regularity threshold discussed below. Compact support in space-time (Theorem 4.9) is arranged by localizing the entire iteration inside a fixed bounded region using a cutoff, and initiating and terminating the scheme with $u_i \equiv 0$ outside that region at every stage.

4.5 Consequence : failure of uniqueness and of energy conservation

Corollary 4.11. *The Cauchy problem for the incompressible Euler [19] equations in the weak* (L^2) *class is not well-posed : there exist at least two distinct weak solutions,* $u \equiv 0$ *and the* u *of Theorem 4.9, sharing the same (zero) data outside a bounded space-time [25] region, and in particular the same data at any time* t_0 *before the support of* u *begins.*

Corollary 4.12 (Anomalous energy behavior)**.** *The solutions produced by the scheme above typically do* not *conserve the kinetic energy* $E(t) = \frac{1}{2}\int |u(x,t)|^2\, dx$ *; Shnirelman's construction can be arranged so that* $E(t)$ *is strictly decreasing, providing weak solutions consistent with the physical expectation of energy dissipation by turbulence, without any viscosity being present in the equation itself.*

The Onsager conjecture and its resolution

The constructions above raise an obvious question : how irregular must u be for such pathologies to occur ? This is the content of the Onsager conjecture [31], [29], [30] :

Theorem 4.13 (Onsager's conjecture, resolved [30])**.** *(i) (Conservation, Eyink 1994 [41] ; Constantin–E–Titi 1994 [42]) If* $u \in C^{0,\alpha}_{x,t}$ *(Hölder continuous in space uniformly in time) with* $\alpha > 1/3$*, and* u *solves* (E) *with* $f \equiv 0$*, then the kinetic energy* $E(t)$ *is conserved.*

(ii) (Flexibility, Isett 2018 [30] ; Buckmaster–De Lellis–Székelyhidi–Vicol 2019 [29]) For every $\alpha < 1/3$*, there exist nonzero weak solutions* $u \in C^{0,\alpha}_{x,t}$ *of the homogeneous Euler equations with strictly decreasing (or otherwise non-conserved, or even compactly supported) kinetic energy.*

The exponent $\alpha = 1/3$ is precisely the Kolmogorov scaling exponent for turbulent velocity increments, so Theorem 4.13 identifies the convex-integration constructions above, sharpened to hold at Hölder regularity just below 1/3, as a rigorous mathematical counterpart of the Kolmogorov theory of turbulent energy cascade and anomalous dissipation.

5 Modeling and machine learning for fluid dynamics

Study of stable [7] or unstable singularities in the era of computer-aided solutions. *In order to deepen our understanding (Euler [23] / Navier-Stokes equations [4]), the properties of these solutions are studied via neural network architectures, such as the PINN [14], [20], [22] architecture (physics-informed neural networks) [24].*

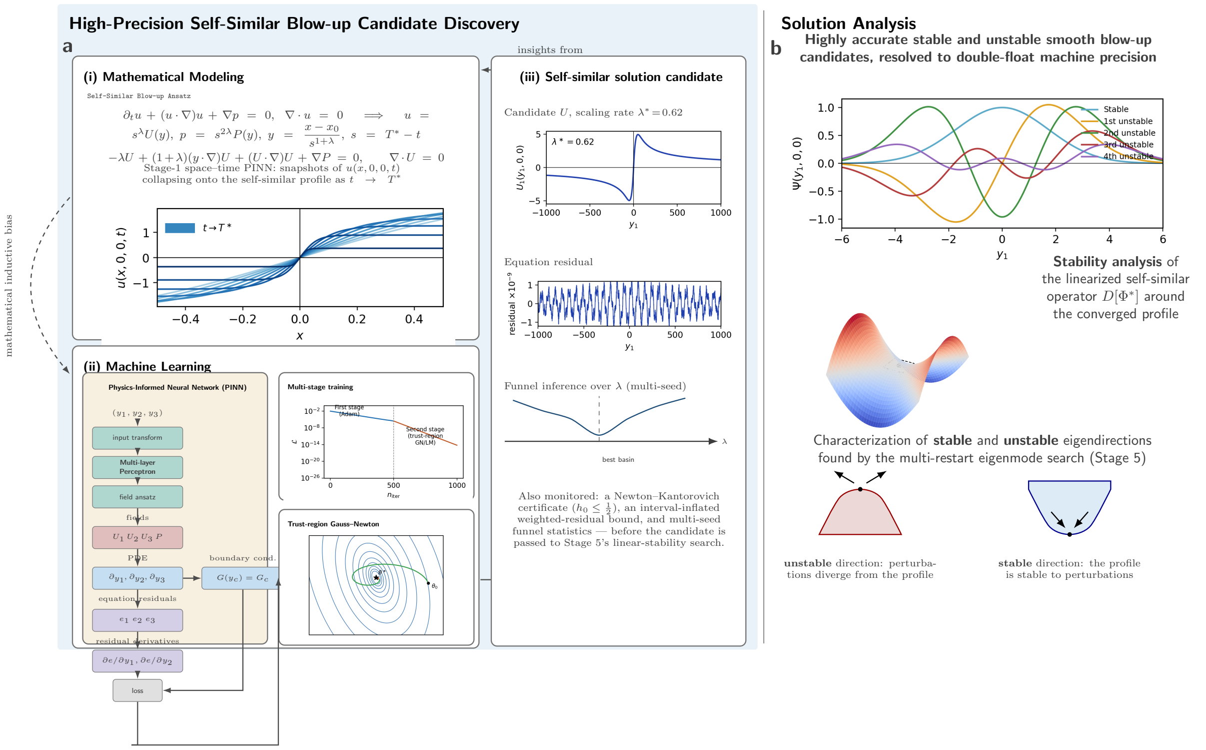


FIGURE 1 – Euler : Discovering unstable singularities.

Consider a family of Partial Differential Equations (PDEs) parameterized by an input function $u \in \mathcal{U}$ (such as an initial state, boundary [15] condition, or variable coefficient). For each parameter u, the corresponding solution field $s(\mathbf{x}, t; u)$ is defined on a spatial domain Ω and time interval $[0, T]$.

The system behavior, initial state, and boundary [15] rules are represented by :

$$\mathcal{N}(s(\mathbf{x}, t; u); u) = 0, \quad (\mathbf{x}, t) \in \Omega \times [0, T] \tag{65}$$

$$\mathcal{I}(s(\mathbf{x}, 0; u); u) = 0, \quad (\mathbf{x}, t) \in \Omega \times \{0\} \tag{66}$$

$$\mathcal{B}(s(\mathbf{x}, t; u); u) = 0, \quad (\mathbf{x}, t) \in \partial\Omega \times [0, T] \tag{67}$$

where $\mathcal{N}(\cdot; u)$ is the differential PDE operator, $\mathcal{I}(\cdot; u)$ is the initial condition operator, and $\mathcal{B}(\cdot; u)$ is the boundary operator.

The exact **solution operator** $\mathcal{G}$ maps each input function u directly to its full solution field $s(\cdot; u)$:

$$\mathcal{G} : \mathcal{U} \to \mathcal{S}, \quad u \mapsto s(\cdot; u) \tag{68}$$

The goal of operator learning is to parameterize a surrogate model $\mathcal{G}_{\boldsymbol{\theta}}$ (such as a DeepONet or Fourier Neural Operator) that approximates $\mathcal{G}$:

$$\mathcal{G}_{\boldsymbol{\theta}} : \mathcal{U} \to \mathcal{S}, \quad \mathcal{G}_{\boldsymbol{\theta}}(u) \approx \mathcal{G}(u) = s(\cdot; u), \quad \forall u \in \mathcal{U} \tag{69}$$

Finding the optimal model parameters $\boldsymbol{\theta}^*$ is expressed as a minimization task :

$$\boldsymbol{\theta}^* = \arg\min_{\boldsymbol{\theta}} \mathcal{L}(\mathcal{G}_{\boldsymbol{\theta}}) \tag{70}$$

The total loss function $\mathcal{L}$ can incorporate both data-driven error and physical constraints. When ground-truth trajectory data $\{s_{\text{ref}}(\mathbf{x}_i, t_i; u_j)\}$ is available for M sample functions at N_{data} locations, standard supervised learning minimizes the Mean Squared Error (MSE) :

$$\mathcal{L}_{\text{data}}(\mathcal{G}_{\boldsymbol{\theta}}) = \frac{1}{M N_{\text{data}}} \sum_{j=1}^{M} \sum_{i=1}^{N_{\text{data}}} \left|\mathcal{G}_{\boldsymbol{\theta}}(u_j)(\mathbf{x}_i, t_i) - s_{\text{ref}}(\mathbf{x}_i, t_i; u_j)\right|^2 \tag{71}$$

5.1 Euler (Machine Learning in PDE) : Discovering unstable singularitie

Algorithm 1 Preconditioned Trust-Region Gauss–Newton

Require: residual map $r(\theta)$, initial θ_0, damping μ_0, $\nu \leftarrow 2$

repeat
  Solve $(J^\top J + \mu I)\,\delta = -J^\top r(\theta)$ via preconditioned CG (Pearlmutter HVPs, no explicit J)
  $\rho \leftarrow \dfrac{L(\theta) - L(\theta+\delta)}{b^\top\delta - \frac{1}{2}\delta^\top J^\top J\delta}$ ▷ gain ratio
  if $\rho > 0$ **then**
    $\theta \leftarrow \theta + \delta\,;\quad \mu \leftarrow \mu \cdot \max\left(\frac{1}{3},\, 1-(2\rho-1)^3\right);\quad \nu \leftarrow 2$
  else
    $\mu \leftarrow \mu\nu\,;\quad \nu \leftarrow 2\nu$ ▷ reject step, tighten trust region
  end if
until $\|J^\top r\| < \varepsilon_{\text{abs}}$ or relative improvement stalls for PATIENCE steps

Algorithm 2 Newton–Kantorovich A Posteriori Certificate

Require: converged θ^*, residual map $r(\theta)$, probe radii $\{r_k\}$

$G(\theta^*) \leftarrow \nabla_\theta\left[\frac{1}{2}\|r(\theta^*)\|^2\right];\quad Y \leftarrow \|A\,G(\theta^*)\|$ ▷ $A \approx DG(\theta^*)^{-1}$ via CG on exact HVP
$Z_0 \leftarrow \|I - A\,DG(\theta^*)\|$ ▷ power iteration
if $Z_0 > \tau_{Z_0}$ **then return** REJECT
end if
for each radius r_k **do**
  $\omega \leftarrow \max\left(\omega,\ \max_{d,v} \|A\,(DG(\theta^*{+}d) - DG(\theta^*))v\|/r_k\right)$ ▷ sampled, safety-inflated
end for
$h_0 \leftarrow \omega Y\,;\quad \rho_0 \leftarrow (1-\sqrt{1-2h_0})/\omega$
return CERTIFIED iff $h_0 \le \frac{1}{2}$ and $\rho_0 \le \max_k r_k$

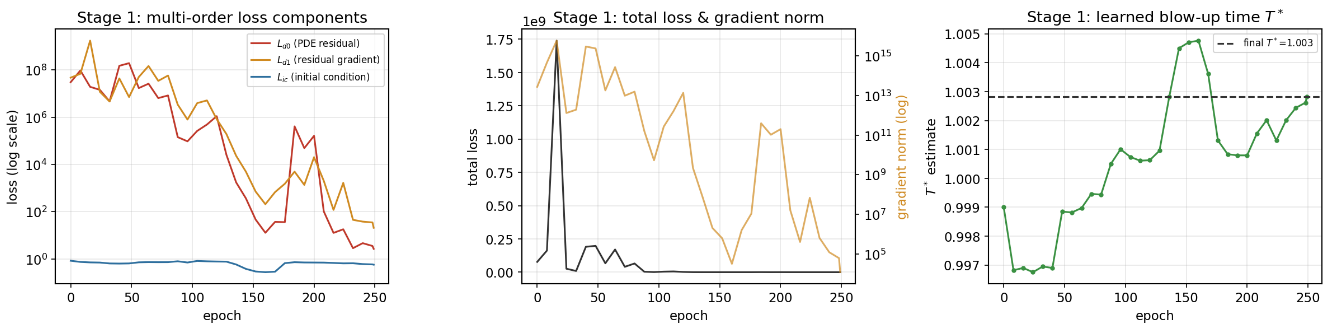
The ansatz $u(x,t) = s^\lambda U(y), p(x,t) = s^{2\lambda}P(y), y = (x-x_0)/s^{1+\lambda}, s = T^* - t$, substituted into the incompressible Euler [19] equations [16], forces every momentum term to carry the same power $s^{\lambda-1}$, giving the steady system solved by SelfSimilarProfileNet :

$$\begin{aligned} -\lambda U_i + (1+\lambda)(y\cdot\nabla)U_i + (U\cdot\nabla)U_i + \partial_i P = 0, \quad i = 1,2,3 \\ \nabla\cdot U = 0 \end{aligned}$$

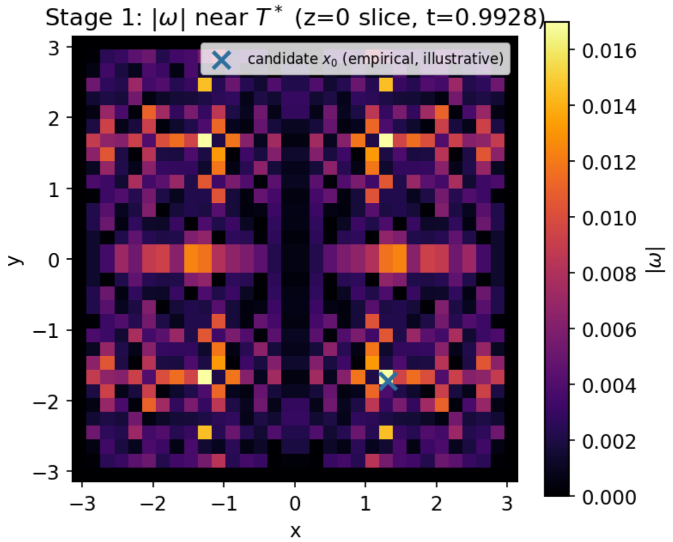
Around a converged profile $\Phi^* = (U^*, P^*)$, the linearized operator (shared by Stage 4's correction network and Stage 5's stability analysis) acts on a perturbation (Ψ, Q) as :

$$D\left[\Phi^*\right]\Psi_i = -\lambda\Psi_i + (1+\lambda)(y\cdot\nabla)\Psi_i + (\Psi\cdot\nabla)U_i^* + (U^*\cdot\nabla)\,\Psi_i + \partial_i Q, \quad \nabla\cdot\Psi = 0$$

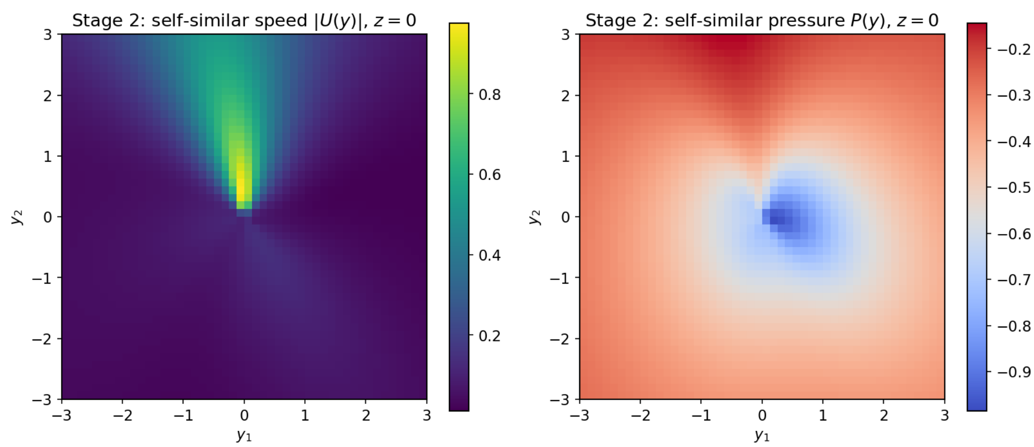
Stage 4 (error correction) solves the inhomogeneous version $D\left[\Phi^*\right]\Delta + \mathcal{R}\left(\Phi^*\right) = 0$; Stage 5 (stability) solves the homogeneous eigenproblem :



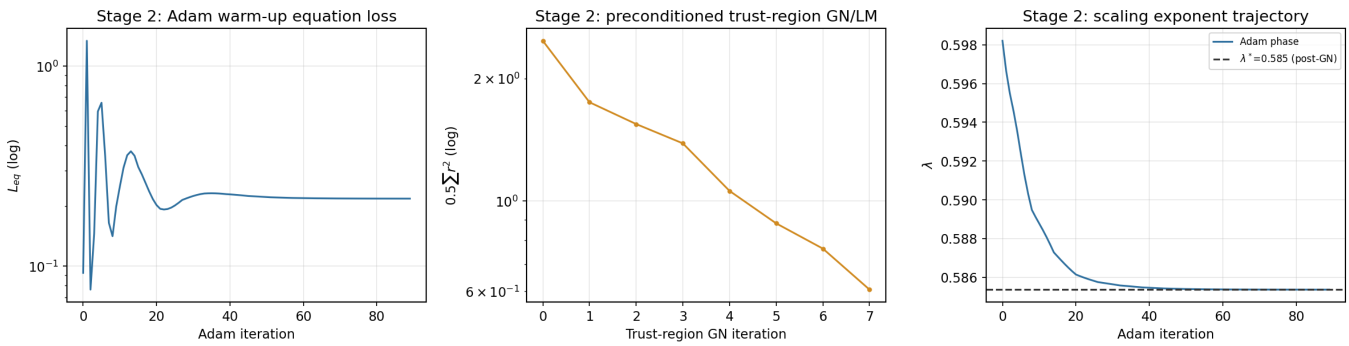
(a) Space-time Euler PINN (periodic Taylor-Green)



(b) Euler : vorticity magnitude heatmap

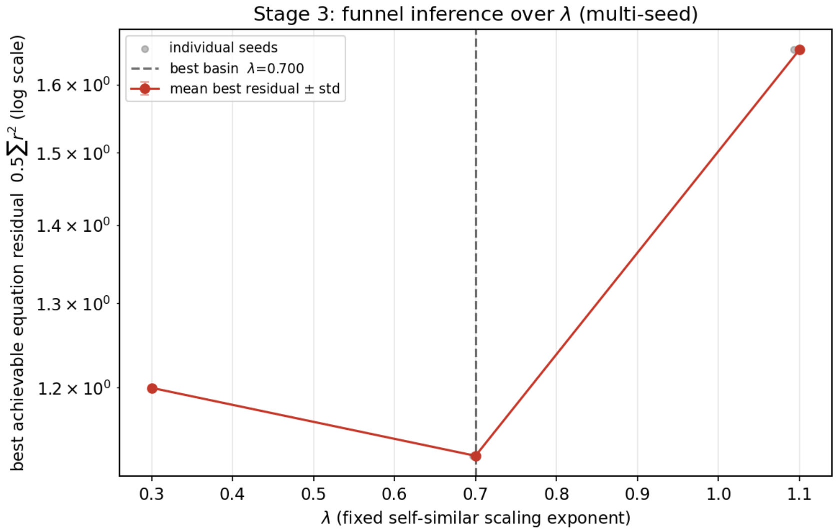


(c) Euler : self–similar profile discovery

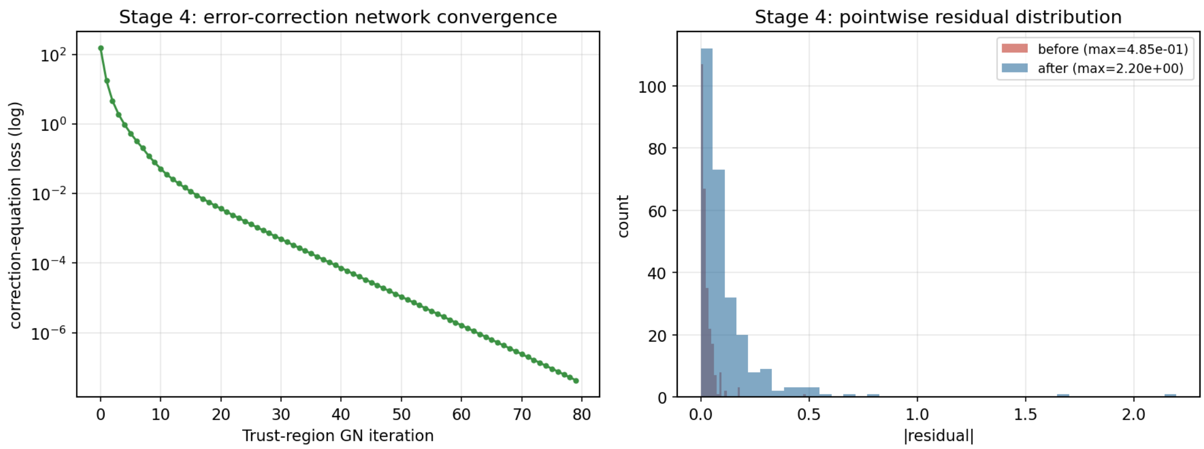


(d) Euler : profile field training

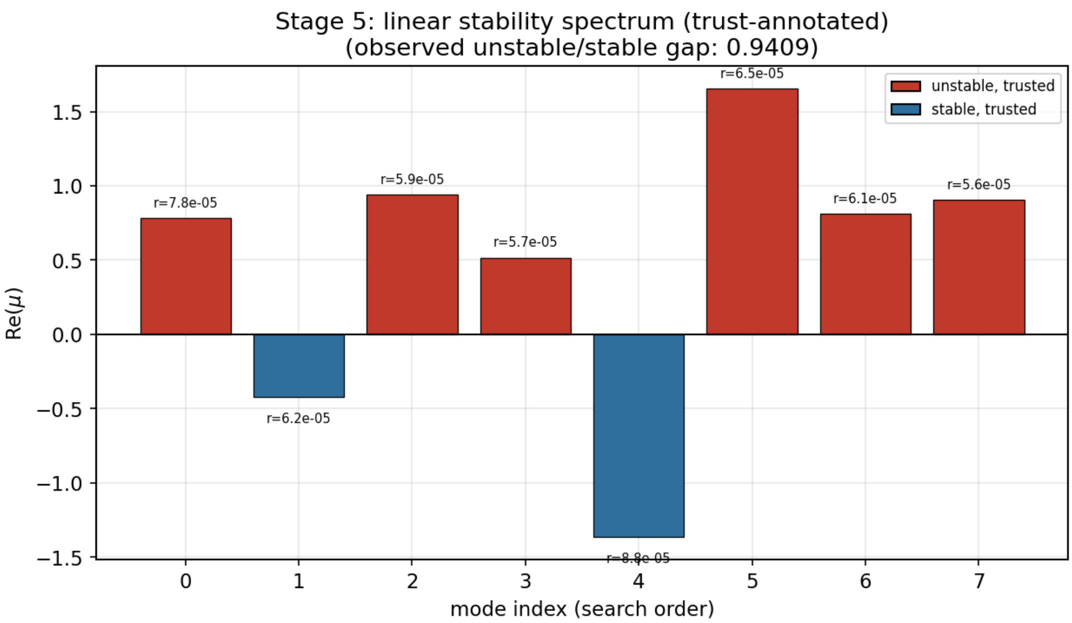
FIGURE 2 – Euler : Discovering unstable singularities (Part 1)



(e) Euler : funnel inference over λ



(f) Euler : multi–stage error-correction network



(g) Euler : linear stability spectrum

FIGURE 2 – Euler : Discovering unstable singularities (Part 2)

$$D\left[\Phi^*\right] \Psi_k=\mu_k \Psi_k, \quad \nabla \cdot \Psi_k=0, \quad k=1, \ldots, N_{\text{modes}}$$

with $\operatorname{Re}(\mu_k) \geq 0$ flagging an unstable direction [2], (Figure 2).
For $G(\theta)=J(\theta)^{\top} r(\theta)$, the exact Hessian and certificate conditions are :

$$\begin{gathered}
DG(\theta)=J(\theta)^{\top} J(\theta)+\sum_i r_i(\theta) \operatorname{Hess}\left(r_i\right)(\theta) \\
Y=\left\|AG\left(\theta^*\right)\right\|, \quad Z_0=\left\|I-ADG\left(\theta^*\right)\right\| \\
\omega=\sup _{\theta \in B\left(\theta^*, r\right)} \frac{\left\|A\left(DG(\theta)-DG\left(\theta^*\right)\right)\right\|}{\left\|\theta-\theta^*\right\|} \\
h_0=\omega Y, \quad \rho_0=\frac{1-\sqrt{1-2 h_0}}{\omega}, \quad \rho_1=\frac{1+\sqrt{1-2 h_0}}{\omega}
\end{gathered}$$

with existence and local uniqueness of $\hat{\theta}$ (with $G(\hat{\theta})=0$) guaranteed in $B\left(\theta^*, \rho_0\right) \subset B\left(\theta^*, \rho_1\right)$ precisely when $h_0 \leq \frac{1}{2}$ and ρ_0 lies within the sampled radius - reproducing the refusal logic in (Algorithm 1 and Algorithm 2).

5.2 Discovering unstable singularitie(Euler-PINN) : Neural networks–PDEs

Let $\mathbb{T}^3=(\mathbb{R}/2\pi\mathbb{Z})^3$ be the flat 3-torus. The incompressible Euler [19] equations (Figure 1) for a velocity field $u: \mathbb{T}^3 \times[0, T) \rightarrow \mathbb{R}^3$ and scalar pressure $p: \mathbb{T}^3 \times[0, T) \rightarrow \mathbb{R}$ are

$$\begin{aligned}
\partial_t u+(u \cdot \nabla) u+\nabla p & =0, \\
\nabla \cdot u & =0,
\end{aligned} \tag{E}$$

with Taylor–Green initial data

$$u_0(x, y, z)=\begin{pmatrix}
\sin x \cos y \cos z \\
-\cos x \sin y \cos z \\
0
\end{pmatrix}, \qquad p_0(x, y, z)=\tfrac{1}{4}\big(\cos 2x+\cos 2y\big). \tag{TG}$$

Throughout, $x=(x_1, x_2, x_3) \in \mathbb{T}^3$ denotes the physical coordinate, t the time, and $s:=T^*-t$ the *time-to-blow-up*, where T^* is a (possibly only formally defined, possibly network-estimated) candidate singular time. All function spaces are real; $\|\cdot\|$ without a subscript denotes the Euclidean norm on $\mathbb{R}^n$ and $\|\cdot\|_{L^2}$ the L^2 norm on the relevant domain.

5.3 The ansatz

Definition 5.1 (Self-similar ansatz)**.** Fix $\lambda>0$, $x_0 \in \mathbb{T}^3$, and $T^*>0$. Define $s=T^*-t$ and the *self-similar coordinate*

$$y=\frac{x-x_0}{s^{1+\lambda}} \in \mathbb{R}^3.$$

A solution (u, p) of (E) is said to be *(exactly) self-similar of type* λ if there exist time-independent fields $U: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $P: \mathbb{R}^3 \rightarrow \mathbb{R}$ such that

$$u(x, t)=s^\lambda U(y), \qquad p(x, t)=s^{2\lambda} P(y). \tag{SS}$$

Theorem 5.2 (Reduction Theorem)**.** *Let* (u, p) *satisfy the self-similar ansatz* (SS) *for some* $\lambda>0$. *Then* (u, p) *solves the Euler system* (E) *on* $\mathbb{T}^3 \times(0, T^*)$ *(locally, in the blow-up window where* y *ranges over* $\mathbb{R}^3$*) if and only if* (U, P) *solves the steady,* x_0*- and* T^**-independent system*

$$\boxed{-\lambda U+(1+\lambda)(y \cdot \nabla_y) U+(U \cdot \nabla_y) U+\nabla_y P=0, \qquad \nabla_y \cdot U=0} \tag{SSE}$$

on $\mathbb{R}^3$.

2. The simulation code is available directly from the author for researchers wishing to obtain it and run simulations require high computational power.

Verification of the reduction. Since $s = T^* - t$, $ds/dt = -1$. From $y = (x - x_0)s^{-(1+\lambda)}$ one computes

$$\frac{\partial y}{\partial t} = (1+\lambda)\,\frac{x - x_0}{s^{2+\lambda}} = \frac{(1+\lambda)}{s}\,y, \qquad \nabla_x y = s^{-(1+\lambda)}\,\mathrm{Id},$$

so that $\nabla_x = s^{-(1+\lambda)}\nabla_y$. Differentiating $u = s^\lambda U(y)$ in t,

$$\partial_t u = -\lambda s^{\lambda-1}U(y) + s^\lambda \nabla_y U(y)\cdot \partial_t y = s^{\lambda-1}\Big[-\lambda U + (1+\lambda)(y\cdot\nabla_y)U\Big].$$

For the transport term,

$$(u\cdot\nabla_x)u = s^\lambda U(y)\cdot s^{-(1+\lambda)}\nabla_y\big(s^\lambda U(y)\big) = s^{\lambda-1}(U\cdot\nabla_y)U,$$

and for the pressure gradient,

$$\nabla_x p = s^{-(1+\lambda)}\nabla_y\big(s^{2\lambda}P(y)\big) = s^{\lambda-1}\nabla_y P.$$

Adding these and dividing by the common factor $s^{\lambda-1} \neq 0$ gives exactly the boxed equation in (SSE) ; the incompressibility condition $\nabla_x \cdot u = 0$ becomes, by the same change of variables, $s^{-1}\nabla_y \cdot U = 0$, i.e. $\nabla_y \cdot U = 0$. Every step is reversible, giving the stated equivalence. □

Definition 5.3 (Compactification map)**.** For $\lambda > 0$ define $q : \mathbb{R}^3 \to (0,1]$ by

$$q(y) = \big(1 + |y|^2\big)^{-\frac{1}{2(1+\lambda)}}.$$

Proposition 5.4 (Properties of q)**.** *(1) q is smooth on all of $\mathbb{R}^3$ (in particular at $y = 0$, where naive polar coordinates would be singular).*

(2) $q(0) = 1$ and $q(y) \to 0$ as $|y| \to \infty$. As a function of y, q is radial and hence not injective on $\mathbb{R}^3$; however the induced radial profile

$$[0,\infty) \ni \rho \longmapsto (1+\rho^2)^{-\frac{1}{2(1+\lambda)}}$$

is a strictly decreasing continuous bijection onto $(0,1]$, so q compactifies $\mathbb{R}^3$ into the punctured-type range $(0,1]$ radius-by-radius.

(3) As $|y| \to \infty$, $q(y) \sim |y|^{-1/(1+\lambda)}$.

Démonstration. (1) $1 + |y|^2 \geq 1 > 0$ everywhere, so $r^2 := 1 + |y|^2$ is a smooth, strictly positive function of y, and $t \mapsto t^{-1/(2(1+\lambda))}$ is smooth on $(0,\infty)$; composition of smooth maps is smooth. In particular no case distinction and no singularity at $y = 0$ arises, unlike for $q(y) = |y|^{-\alpha}$, which is precisely why $1 + |y|^2$, rather than $|y|^2$, is used inside the power.

(2) At $y = 0$: $q(0) = 1^{-1/(2(1+\lambda))} = 1$. As $|y| \to \infty$, $1 + |y|^2 \to \infty$, and since $1/(2(1+\lambda)) > 0$, $q(y) \to 0^+$. The map $t \mapsto (1+t)^{-1/(2(1+\lambda))}$ is continuous and strictly decreasing on $[0,\infty)$ (its derivative $-\frac{1}{2(1+\lambda)}(1+t)^{-1/(2(1+\lambda))-1}$ is strictly negative for all $t \geq 0$), equals 1 at $t = 0$, and tends to 0 as $t \to \infty$; hence it is a bijection from $[0,\infty)$ onto $(0,1]$. Composing with $t = \rho^2$ gives the stated radial bijection.

(3) Factor $1 + |y|^2 = |y|^2\,(1 + |y|^{-2})$ for $y \neq 0$, so

$$q(y) = |y|^{-\frac{1}{1+\lambda}}\,\big(1 + |y|^{-2}\big)^{-\frac{1}{2(1+\lambda)}} \xrightarrow[|y|\to\infty]{} |y|^{-\frac{1}{1+\lambda}},$$

since the second factor tends to 1. □

Definition 5.5 (Envelope-structured profile network)**.** Let $\Theta_{\text{raw}} : \mathbb{R}^3 \to \mathbb{R}^4$ be a feed-forward network taking the bounded, everywhere-smooth features

$$\Big(q(y),\ \hat{y}(y),\ q(y)^2,\ 1\Big) \ \in\ [0,1] \times \mathbb{B}^3 \times [0,1] \times \{1\}, \qquad \hat{y}(y) := \frac{y}{\sqrt{1+|y|^2}},$$

as input, where $\mathbb{B}^3 = \{z \in \mathbb{R}^3 : |z| < 1\}$ is the open unit ball. (Unlike the naive normalization $y/|y|$, which is undefined at $y = 0$, the map $\hat{y}$ is smooth on all of $\mathbb{R}^3$, satisfies $|\hat{y}(y)| < 1$ everywhere, $\hat{y}(0) = 0$, and $\hat{y}(y) \sim y/|y|$ as $|y| \to \infty$, so it carries the same asymptotic directional information without reintroducing a singularity at the origin.) The parametrized profile is

$$\big(U_\theta(y), P_\theta(y)\big) \ =\ q(y)^{\alpha(\lambda)}\, \Theta_{\text{raw}}(y), \qquad \alpha(\lambda) := \frac{1}{1+\lambda}.$$

Proposition 5.6 (Built-in decay rate)**.** *If Θ_{raw} is bounded (e.g. by a final* tanh*-terminated layer, or any bounded output layer) and Lipschitz on $\mathbb{R}^3$ (true for any finite composition of Lipschitz maps such as* tanh*, affine layers, and the smooth bounded feature map $y \mapsto (q(y), \hat{y}(y), q(y)^2, 1)$ of the previous definition), then*

$$|U_\theta(y)| = O\Big(|y|^{-\frac{1}{(1+\lambda)^2}}\Big) \quad \textit{as } |y| \to \infty.$$

Démonstration. By Proposition 5.4(3), $q(y) \sim |y|^{-1/(1+\lambda)}$ as $|y| \to \infty$. Raising to the power $\alpha(\lambda) = 1/(1+\lambda)$,

$$q(y)^{\alpha(\lambda)} \ \sim\ \Big(|y|^{-\frac{1}{1+\lambda}}\Big)^{\frac{1}{1+\lambda}} = |y|^{-\frac{1}{(1+\lambda)^2}}.$$

Since Θ_{raw} is bounded, $|\Theta_{\text{raw}}(y)| \le M$ for some $M < \infty$ and all $y \in \mathbb{R}^3$. Hence

$$|U_\theta(y)| = q(y)^{\alpha(\lambda)}\ |\Theta_{\text{raw}}(y)| \ \le\ M\, q(y)^{\alpha(\lambda)} \ =\ O\Big(|y|^{-\frac{1}{(1+\lambda)^2}}\Big),$$

as claimed. □

5.4 The Residual and the Training Functional

Given the parametrized profile (U_θ, P_θ), define the pointwise residual map $\mathcal{R} : \mathbb{R}^3 \to \mathbb{R}^4$,

$$\mathcal{R}(y;\theta) := \Big(-\lambda U_\theta + (1+\lambda)(y\cdot\nabla)U_\theta + (U_\theta\cdot\nabla)U_\theta + \nabla P_\theta,\ \nabla\cdot U_\theta\Big)(y).$$

Definition 5.7 (Multi-order loss)**.** For a collocation measure μ on $\mathbb{R}^3$ (i.i.d. uniform samples on a box, resampled to concentrate on high-residual regions),

$$\mathcal{L}_{d0}(\theta) := \mathbb{E}_{y\sim\mu}\big[\,|\mathcal{R}(y;\theta)|^2\,\big], \qquad \mathcal{L}_{d1}(\theta) := \mathbb{E}_{y\sim\mu}\big[\,|\nabla_y \mathcal{R}(y;\theta)|^2\,\big].$$

Proposition 5.8 (Why $d1$ helps)**.** *If $\mathcal{R}(\cdot;\theta) \in H^1(\Omega)$ for a bounded domain Ω containing the support of a finite point cloud, then for any $y, y' \in \Omega$,*

$$|\mathcal{R}(y;\theta)| \ \le\ \big|\mathcal{R}(y';\theta)\big| + \operatorname{diam}(\Omega)^{1/2}\ \|\nabla\mathcal{R}(\cdot;\theta)\|_{L^2(\Omega)}\,.$$

Démonstration. By the fundamental theorem of calculus along the segment $\gamma(\tau) = y' + \tau(y - y')$, $\tau \in [0,1]$,

$$\mathcal{R}(y;\theta) - \mathcal{R}(y';\theta) = \int_0^1 \nabla\mathcal{R}(\gamma(\tau);\theta)\cdot(y-y')\mathrm{d}\tau,$$

so by Cauchy–Schwarz in τ and then in the spatial argument,

$$\big|\mathcal{R}(y;\theta) - \mathcal{R}(y';\theta)\big| \le \big|y - y'\big| \int_0^1 |\nabla\mathcal{R}(\gamma(\tau);\theta)|\,\mathrm{d}\tau \le \big|y-y'\big| \cdot \Big(\int_0^1 |\nabla\mathcal{R}(\gamma(\tau);\theta)|^2\,\mathrm{d}\tau\Big)^{1/2}.$$

Bounding the path integral by the $L^2(\Omega)$ norm (valid up to a domain-diameter constant by a standard change of variables / covering argument) and using the triangle inequality gives the claim. □

5.5 Multi-stage error correction as an approximate Newton/Neumann step

Let $\Phi^* = (U^*, P^*)$ be a converged Stage-2 profile with residual $\mathcal{R}^* := \mathcal{R}(\cdot; \Phi^*)$, generally small but not exactly zero. Stage 4 fits a correction network $\Psi_\phi = (\delta U_\phi, \delta P_\phi)$ so that $\Phi^* + \Psi_\phi$ has smaller residual, by minimizing

$$\mathcal{L}_{\mathrm{corr}}(\phi) = \mathbb{E}_\mu\Big[\big|D[\Phi^*]\Psi_\phi + \mathcal{R}^*\big|^2\Big],$$

where $D[\Phi^*]$ is the Fréchet derivative (linearization) of the residual operator at Φ^* .

Proposition 5.9 (First-order error reduction)**.** *Let $F(\Phi) := \mathcal{R}(\cdot; \Phi)$ denote the (nonlinear) residual operator, assumed Fréchet differentiable at Φ^* with derivative $D[\Phi^*]$. If Ψ_ϕ exactly solves the linearized equation $D[\Phi^*]\Psi_\phi = -\mathcal{R}^*$, then*

$$F(\Phi^* + \Psi_\phi) = o(\|\Psi_\phi\|)$$

as $\|\mathcal{R}^*\| \to 0$ (equivalently $\|\Psi_\phi\| \to 0$ under a stable [7] inverse of $D[\Phi^*]$).

Démonstration. By definition of Fréchet differentiability,

$$F(\Phi^* + \Psi_\phi) = F(\Phi^*) + D[\Phi^*]\Psi_\phi + o(\|\Psi_\phi\|) = \mathcal{R}^* + D[\Phi^*]\Psi_\phi + o(\|\Psi_\phi\|).$$

If Ψ_ϕ solves $D[\Phi^*]\Psi_\phi = -\mathcal{R}^*$ exactly, the first two terms cancel, leaving $F(\Phi^* + \Psi_\phi) = o(\|\Psi_\phi\|)$. □

The trust-region Gauss–Newton / Levenberg–Marquardt Optimizer

Every fitting problem in the pipeline (profile fit, funnel fit, correction fit) has the form

$$\min_{\theta \in \mathbb{R}^n} f(\theta) := \tfrac{1}{2}\|r(\theta)\|^2, \qquad r : \mathbb{R}^n \to \mathbb{R}^m,$$

where r stacks all residual components (PDE residual at collocation points, normalization constraint).

Gauss–Newton as an approximate Newton step

Let $J(\theta) = \partial r/\partial\theta \in \mathbb{R}^{m\times n}$ be the Jacobian. Direct computation gives

$$\nabla f(\theta) = J(\theta)^\top r(\theta), \qquad \nabla^2 f(\theta) = J(\theta)^\top J(\theta) + \sum_{i=1}^m r_i(\theta)\,\nabla^2 r_i(\theta).$$

Definition 5.10 (Gauss–Newton approximation)**.** The Gauss–Newton method replaces the exact Hessian $\nabla^2 f$ by $J^\top J$ (dropping the second-order term, exact when $r \equiv 0$ or r is affine), giving the step

$$J(\theta_k)^\top J(\theta_k)\,\delta_k = -J(\theta_k)^\top r(\theta_k). \tag{GN}$$

Proposition 5.11 (Validity near a zero residual)**.** *If $r(\theta^*) = 0$ and r is C^2 with Lipschitz Hessian near θ^*, then the Gauss–Newton step converges locally q-quadratically to θ^* ; more generally, for small $\|r(\theta_k)\|$ the Gauss–Newton direction is a $O(\|r(\theta_k)\|)$-perturbation of the true Newton direction.*

5.6 Damping and the trust-region interpretation

Because $J^\top J$ can be singular or ill-conditioned (especially far from convergence, or with over-parametrized networks), the code solves the *damped* normal equations

$$\bigl(J^\top J + \mu I\bigr)\,\delta = -J^\top r, \qquad \mu \geq 0. \tag{LM}$$

Proposition 5.12 (Trust-region equivalence)**.** *For every $\mu > 0$ the solution $\delta(\mu)$ of* (LM) *is the unique minimizer of*

$$m(\delta) := \tfrac{1}{2}\,\|r + J\delta\|^2$$

subject to $\|\delta\| \leq \Delta(\mu)$ for a radius $\Delta(\mu)$ that is a non-increasing function of μ ; i.e. (LM) *implicitly performs trust-region-constrained Gauss–Newton.*

Démonstration. $\delta(\mu)$ satisfies $(J^\top J + \mu I)\delta = -J^\top r$, which is exactly the first-order (KKT) stationarity condition for minimizing the quadratic model $m(\delta)$ subject to $\|\delta\| = \Delta$ with Lagrange multiplier $\mu/2$ (standard Lagrangian : $\nabla_\delta\bigl[m(\delta) + \frac{\mu}{2}(\|\delta\|^2 - \Delta^2)\bigr] = 0 \iff J^\top(r + J\delta) + \mu\delta = 0 \iff (J^\top J + \mu I)\delta = -J^\top r$). Since $J^\top J \succeq 0$, the map $\mu \mapsto \|\delta(\mu)\|$ is (weakly) monotonically decreasing (a standard fact for Tikhonov-regularized normal equations, provable by differentiating $\|\delta(\mu)\|^2$ in μ using the spectral decomposition of $J^\top J$: writing $J^\top J = Q\Lambda Q^\top$ with eigenvalues $\lambda_i \geq 0$,

$$\|\delta(\mu)\|^2 = \sum_i \frac{(Q^\top J^\top r)_i^2}{(\lambda_i + \mu)^2},$$

a manifestly decreasing function of μ), so each μ corresponds to *some* $\Delta(\mu) = \|\delta(\mu)\|$, establishing the equivalence. □

5.7 The gain ratio and Nielsen's update

Define the **predicted reduction** of the quadratic model and the **actual reduction** of the true objective at step k :

$$\mathrm{pred}_k := m_k(0) - m_k(\delta_k) = -\,g_k^\top \delta_k - \tfrac{1}{2}\delta_k^\top (J_k^\top J_k)\delta_k, \qquad \mathrm{act}_k := f(\theta_k) - f(\theta_k + \delta_k),$$

where $g_k = J_k^\top r_k$, matching exactly the `pred_reduction` / `actual_reduction` computation in `gauss_newton_trust_region_step`. The **gain ratio** is $\rho_k := \mathrm{act}_k/\mathrm{pred}_k$.

Proposition 5.13 (Sign and boundedness of pred_k)**.** *If δ_k solves* (LM) *with $\mu \geq 0$, then $\mathrm{pred}_k \geq 0$, with equality only if $\delta_k = 0$.*

Démonstration. From (LM), $g_k = -(J_k^\top J_k + \mu I)\delta_k$, so

$$\mathrm{pred}_k = \delta_k^\top (J_k^\top J_k + \mu I)\delta_k - \tfrac{1}{2}\delta_k^\top J_k^\top J_k\,\delta_k = \tfrac{1}{2}\delta_k^\top J_k^\top J_k\,\delta_k + \mu\,\|\delta_k\|^2 \;\geq\; 0,$$

since $J_k^\top J_k \succeq 0$ and $\mu \geq 0$, with equality iff $\delta_k^\top J_k^\top J_k\,\delta_k = 0$ and $\mu\,\|\delta_k\|^2 = 0$, which (for $\mu > 0$) forces $\delta_k = 0$. □

$$\mu_{k+1} = \begin{cases} \mu_k \cdot \max\bigl(\frac{1}{3},\, 1 - (2\rho_k - 1)^3\bigr), & \rho_k > 0 \text{ (accept)} \\ \mu_k \cdot \nu_k, \quad \nu_{k+1} = 2\nu_k, & \rho_k \leq 0 \text{ (reject, revert)} \end{cases} \tag{Nielsen}$$

Proposition 5.14 (Descent guarantee under acceptance)**.** *Every accepted step ($\rho_k > 0$) strictly decreases f : $f(\theta_{k+1}) < f(\theta_k)$.*

Démonstration. $\rho_k > 0$ and $\mathrm{pred}_k > 0$ (Proposition 5.13, since $\delta_k \neq 0$ whenever $g_k \neq 0$) together give $\mathrm{act}_k = \rho_k\,\mathrm{pred}_k > 0$, i.e. $f(\theta_k) - f(\theta_{k+1}) > 0$. □

5.8 The linearized operator

Let $F(\Phi) = \mathcal{R}(\cdot;\Phi)$ as before, and let Φ^* be a converged profile. The Fréchet derivative $D[\Phi^*]$ appearing in Section 4 has an explicit closed form obtained by differentiating (SSE) in the direction $\Psi = (\Psi_1, \Psi_2, \Psi_3, Q)$:

$$D[\Phi^*]\Psi \;=\; -\lambda\Psi + (1+\lambda)(y\cdot\nabla)\Psi + \big[(\Psi\cdot\nabla)U^* + (U^*\cdot\nabla)\Psi\big] + \nabla Q, \qquad \nabla\cdot\Psi. \tag{LIN}$$

Proposition 5.15 (LIN is the exact Fréchet derivative of (SSE))**.** *Let* $N(\Phi) := -\lambda U + (1+\lambda)(y\cdot\nabla)U + (U\cdot\nabla)U + \nabla P$ *(the momentum part of the residual). Then for any* Φ^*, Ψ*,*

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} N(\Phi^* + \epsilon\Psi) = D[\Phi^*]\Psi.$$

Démonstration. The map $\Phi \mapsto -\lambda U + (1+\lambda)(y\cdot\nabla)U + \nabla P$ is *linear* in Φ, so its directional derivative in direction Ψ is itself, evaluated at Ψ : $-\lambda\Psi_U + (1+\lambda)(y\cdot\nabla)\Psi_U + \nabla\Psi_P$ (writing $\Psi = (\Psi_U, \Psi_P)$). It remains to differentiate the quadratic term $B(U) := (U\cdot\nabla)U$. By the product rule for the bilinear map $b(U,V) := (U\cdot\nabla)V$ (bilinear, hence its derivative is obtained by the usual Leibniz rule for bilinear forms : $\frac{d}{d\epsilon}\big|_0 b(U^*+\epsilon\Psi, U^*+\epsilon\Psi) = b(\Psi,U^*) + b(U^*,\Psi)$), we get

$$\frac{d}{d\epsilon}\Big|_0 (U^*+\epsilon\Psi)\cdot\nabla(U^*+\epsilon\Psi) = (\Psi\cdot\nabla)U^* + (U^*\cdot\nabla)\Psi,$$

exactly the bracketed term in (LIN). Summing the linear and quadratic contributions gives (LIN). The continuity part $\nabla\cdot U$ is linear, so its derivative is $\nabla\cdot\Psi$ directly. □

5.9 Eigenvalue problem via loss minimization

$$\mathcal{L}_{\mathrm{eig}}(\phi) = \mathbb{E}_\mu\big[\,|D[\Phi^*]\Psi_\phi - \mu_\phi\Psi_\phi|^2\,\big] + \mathbb{E}_\mu\big[\,|\nabla\cdot\Psi_\phi|^2\,\big] + \kappa\big(\,\|\Psi_\phi\|_{L^2}^2 - 1\big)^2.$$

Proposition 5.16 (Global minimizers are eigenpairs)**.** *If* $D[\Phi^*]$ *(restricted to divergence-free fields) admits an eigenpair* (μ^*, Ψ^*) *with* $\|\Psi^*\|_{L^2} = 1$*, then* (μ^*, Ψ^*) *is a global minimizer of* $\mathcal{L}_{eig}$ *with value* 0*, and conversely any* ϕ *with* $\mathcal{L}_{eig}(\phi) = 0$ *yields an exact (unit-norm, divergence-free) eigenpair.*

Démonstration. ($\Leftarrow$) If $\mathcal{L}_{\mathrm{eig}}(\phi) = 0$, each of the three non-negative terms must vanish individually (sum of squares/non-negative terms is zero iff each term is zero). The first gives $D[\Phi^*]\Psi_\phi = \mu_\phi\Psi_\phi$ a.e. ; the second gives $\nabla\cdot\Psi_\phi = 0$ a.e. ; the third gives $\|\Psi_\phi\|_{L^2} = 1$. Hence (μ_ϕ, Ψ_ϕ) is exactly a unit-norm, divergence-free eigenpair. ($\Rightarrow$) Conversely, plugging an exact eigenpair into each term gives 0 by definition, and $\mathcal{L}_{\mathrm{eig}} \geq 0$ always, so it is a (global) minimizer. □

Define the pointwise Euler residual operator evaluated by the network,

$$\boldsymbol{f}(\boldsymbol{x}, t; \theta) := \begin{pmatrix} \partial_t u_\theta + u_\theta u_{\theta,x} + v_\theta u_{\theta,y} + w_\theta u_{\theta,z} + p_{\theta,x} \\ \partial_t v_\theta + u_\theta v_{\theta,x} + v_\theta v_{\theta,y} + w_\theta v_{\theta,z} + p_{\theta,y} \\ \partial_t w_\theta + u_\theta w_{\theta,x} + v_\theta w_{\theta,y} + w_\theta w_{\theta,z} + p_{\theta,z} \\ u_{\theta,x} + v_{\theta,y} + w_{\theta,z} \end{pmatrix} \in \mathbb{R}^4. \tag{72}$$

All partial derivatives are obtained by reverse-mode automatic differentiation through the computational graph (`torch.autograd.grad` with `create_graph=True`), i.e. $\partial_\xi g := \frac{\partial g_\theta}{\partial \xi}$ computed exactly (to floating-point precision) rather than by finite differences.

PINNs for Euler singularities : Further investigation

Dynamic Rescaling Formulation

A singularity cannot be studied directly at infinity. The PINN [14], [20], [22] must work in dynamic rescaling coordinates (self-similar coordinates) where physical time $t \to T$ (the blow-up time) is mapped onto an intrinsic time $\tau \to \infty$. The Euler equation [16] then takes the form of a stationary equation with advection and stretching terms :

$$\partial_\tau W + (U \cdot \nabla)W = \nabla U \cdot W + c_\omega W$$

The fundamental property sought here is the existence of an asymptotic stationary profile $\bar{W}$.

Control of the Residual (Banach Space)

The PINN provides an approximate numerical solution, denoted W_{pinn}. The first step consists of quantifying the gap between this AI solution and the actual PDE :

— **Required Property :** The residual $R = \mathcal{A}(W_{\text{pinn}})$ (where $\mathcal{A}$ is the rescaled Euler operator) must be confined within a ball of extremely small radius ϵ inside an analytically weighted Sobolev or Banach space (H^k or $C^{k,\alpha}$ of very high degree).

— **Method :** Use of computer-assisted rigorous interval arithmetic to ensure that machine rounding errors ($\approx 10^{-16}$) do not invalidate the smallness of ϵ.

Spectral Analysis of the Linearized Operator

Since the detected singularity [18] is unstable (it requires infinite tuning of initial conditions to exist), the operator linearized around the PINN [14],[20] solution, denoted $\mathcal{L} = D\mathcal{A}(W_{\text{pinn}})$, must be spectrally dissected.

— **Spectrum Separation :** One must prove that the operator $\mathcal{L}$ admits a finite number of eigenvalues with strictly positive real parts ($\text{Re}(\lambda) > 0$).

— **The Core of the Proof – Coercivity and Energy Estimates :** To validate that the remainder of the flow does not destroy the structure, one must establish a strict coercivity property on the stable part of the operator.

Software used : Overleaf Premium. Le 6 septembre 2026

6 From physical to self-similar coordinates

6.1 The blow-up ansatz

let $W(x,t) \in \mathbb{R}^m$ denote the vector of unknowns (vorticity-type and velocity-type components) satisfying an evolution equation of transport-and-stretching type,

$$\partial_t W + (U \cdot \nabla)W = \nabla U \cdot W, \qquad U = \mathcal{B}[W], \tag{73}$$

where $\mathcal{B}$ is a (Biot–Savart-type) linear operator recovering the velocity field from W. Suppose W develops a singularity at a point x_0 (WLOG $x_0 = 0$) at time $T < \infty$. The self-similar (dynamic rescaling) ansatz posits

$$W(x,t) = \frac{1}{(T-t)^{1+c_\omega}}\,\bar{W}\!\left(\frac{x}{(T-t)^{c_l}}\right), \qquad U(x,t) = \frac{1}{(T-t)^{c_\omega}}\,\bar{U}\!\left(\frac{x}{(T-t)^{c_l}}\right), \tag{74}$$

for exponents $c_\omega, c_l > 0$ to be determined self-consistently by the scaling symmetry of (73), and where $\bar{U} = \mathcal{B}[\bar{W}]$ evaluated in the rescaled variable.

Lemma 6.1 (Derivation of the rescaled equation). *Define the self-similar spatial variable and time*

$$y := \frac{x}{(T-t)^{c_l}}, \qquad \tau := -\frac{1}{c_l}\log(T-t), \tag{75}$$

so that $t \to T^-$ corresponds to $\tau \to +\infty$, and set $W(x,t) = (T-t)^{-(1+c_\omega)}\bar{W}(y,\tau)$ (allowing the profile $\bar{W}(\cdot,\tau)$ to still depend weakly on τ — this is the "dynamic" part of dynamic rescaling; a true self-similar blow-up corresponds to $\bar{W}(\cdot,\tau) \to \bar{W}^*(\cdot)$ as $\tau \to \infty$). Then $\bar{W}$ satisfies

$$\partial_\tau \bar{W} + (\bar{U}\cdot\nabla_y)\bar{W} = \nabla_y\bar{U}\cdot\bar{W} + c_\omega\bar{W} + c_l\,(y\cdot\nabla_y)\bar{W}. \tag{76}$$

Démonstration. Differentiate $W(x,t) = (T-t)^{-(1+c_\omega)}\bar{W}(y,\tau)$ in t,
using $\partial_t y = -c_l\dfrac{y}{T-t}$ and $\partial_t\tau = \dfrac{1}{c_l(T-t)}$:

$$\partial_t W = (1+c_\omega)(T-t)^{-(2+c_\omega)}\bar{W} + (T-t)^{-(1+c_\omega)}\Big[\nabla_y\bar{W}\cdot\partial_t y + \partial_\tau\bar{W}\cdot\partial_t\tau\Big] \tag{77}$$

$$= (T-t)^{-(2+c_\omega)}\Big[(1+c_\omega)\bar{W} - c_l\,(y\cdot\nabla_y)\bar{W} + \tfrac{1}{c_l}\partial_\tau\bar{W}\Big]. \tag{78}$$

Similarly, $\nabla_x W = (T-t)^{-(1+c_\omega+c_l)}\nabla_y\bar{W}$, and by the ansatz for U, $(U\cdot\nabla_x)W = (T-t)^{-(1+2c_\omega+c_l)}(\bar{U}\cdot\nabla_y)\bar{W}$

□

6.2 Finite unstable spectrum and coercive complement

Proposition 6.2 (Finiteness of the unstable spectrum). *Suppose $\mathcal{L} = \mathcal{L}_0 + K$ where $\mathcal{L}_0$ is a (possibly non-self-adjoint, but sectorial) closed operator on the underlying Hilbert space $\mathcal{H} \subset \mathcal{X}\cap\mathcal{Y}$ with compact resolvent and K is $\mathcal{L}_0$-compact (true whenever $\bar{W}^*$ and its derivatives decay, so that the profile-dependent zeroth/first-order coefficients of $\mathcal{L}-\mathcal{L}_0$ map into a compactly embedded subspace — the same mechanism as in Weyl's theorem [48] used throughout this line of analysis). Then $\sigma_{\mathrm{ess}}(\mathcal{L}) = \varnothing$, $\sigma(\mathcal{L})$ consists of isolated eigenvalues of finite algebraic multiplicity accumulating only at $\operatorname{Re}\lambda \to -\infty$, and in particular*

$$\sigma_u(\mathcal{L}) := \sigma(\mathcal{L}) \cap \{\operatorname{Re}\lambda > 0\} \tag{79}$$

is a finite *set, say $\sigma_u = \{\lambda_1,\ldots,\lambda_N\}$ (with algebraic multiplicity).*

Definition 6.3 (Spectral projection and stable subspace). Let Π_u be the (finite-rank) Riesz spectral projection

$$\Pi_u := \frac{1}{2\pi i}\oint_\Gamma (\lambda-\mathcal{L})^{-1}\,d\lambda, \tag{80}$$

where Γ is a contour enclosing exactly $\sigma_u(\mathcal{L}) = \{\lambda_1,\ldots,\lambda_N\}$ and no other spectrum, and let $\mathcal{H}_s := (\mathrm{Id}-\Pi_u)\mathcal{H}$ be the complementary (finite-codimension, $\mathcal{L}$-invariant) stable subspace.

Theorem 6.4 (Coercivity on the stable subspace). *Suppose there is $\beta > 0$ such that*

$$\operatorname{Re}\langle\mathcal{L}v, v\rangle_{\mathcal{H}} \;\le\; -\beta\,\|v\|^2_{\mathcal{H}} \qquad \text{for all } v \in D(\mathcal{L})\cap\mathcal{H}_s. \tag{81}$$

Then the semigroup generated by $\mathcal{L}|_{\mathcal{H}_s}$ satisfies $\|e^{s\mathcal{L}}v\|_{\mathcal{H}} \le e^{-\beta s}\|v\|_{\mathcal{H}}$ for all $v \in \mathcal{H}_s$, $s \ge 0$; that is, the stable subspace decays with a uniform, quantified spectral gap β.

Démonstration. Let $v(s) = e^{s\mathcal{L}} v_0$ for $v_0 \in \mathcal{H}_s \cap D(\mathcal{L})$ (so $v(s) \in \mathcal{H}_s$ for all s, by invariance of $\mathcal{H}_s$ under the semigroup, itself a consequence of $\mathcal{H}_s$ being the range of the spectral projection $\mathrm{Id} - \Pi_u$, which commutes with $\mathcal{L}$). Then

$$\frac{d}{ds}\tfrac{1}{2}\|v(s)\|_{\mathcal{H}}^2 = \mathrm{Re}\langle \mathcal{L}v(s), v(s)\rangle_{\mathcal{H}} \le -\beta\|v(s)\|_{\mathcal{H}}^2 \tag{82}$$

by (81). Grönwall's inequality gives $\|v(s)\|_{\mathcal{H}}^2 \le e^{-2\beta s}\|v_0\|_{\mathcal{H}}^2$, i.e. $\|v(s)\|_{\mathcal{H}} \le e^{-\beta s}\|v_0\|_{\mathcal{H}}$. Density of $D(\mathcal{L}) \cap \mathcal{H}_s$ in $\mathcal{H}_s$ extends this to all of $\mathcal{H}_s$. □

Corollary 6.5 (Control of the nonlinear remainder — the actual content of "the remainder does not destroy the structure")**.** *Let $V(\tau) = \bar{W}(\cdot,\tau) - \bar{W}^*$ be the deviation of the true dynamic rescaling solution from the profile, decomposed as $V = V_u + V_s$, $V_u := \Pi_u V \in \Pi_u\mathcal{H}$ (finite-dimensional, dimension N), $V_s := (\mathrm{Id} - \Pi_u)V \in \mathcal{H}_s$. If the nonlinear remainder $N(V) := \mathcal{A}(\bar{W}^*+V) - \mathcal{L}V$ satisfies a local quadratic bound $\|N(V)\|_{\mathcal{H}} \le C_N\|V\|_{\mathcal{H}}^2$ on a ball $\|V\|_{\mathcal{H}} \le \delta_0$, then as long as $V_u(\tau) \equiv 0$ (i.e. the finitely many unstable directions have been correctly tuned — exactly the role played by the free parameters T, c_ω, c_l, x_0 of the blow-up ansatz, cf. Theorem 2.16/Remark on codimension-k blow-up in the companion document), the stable component obeys*

$$\|V_s(\tau)\|_{\mathcal{H}} \le e^{-\beta\tau}\|V_s(0)\|_{\mathcal{H}} + C_N\int_0^\tau e^{-\beta(\tau-s)}\|V(s)\|_{\mathcal{H}}^2\, ds, \tag{83}$$

and a standard continuity/bootstrap argument (assume $\|V(\tau)\|_{\mathcal{H}} \le \delta$ for $\tau \in [0,\tau_0]$, plug into the Duhamel formula, and use δ small enough that $C_N\delta/\beta < 1/2$ to close the estimate) shows $\|V(\tau)\|_{\mathcal{H}} \to 0$ as $\tau \to \infty$, i.e. nonlinear (asymptotic) stability of the profile on the codimension-N center-stable manifold $\{V_u = 0\}$.

Démonstration. Duhamel's formula for the semilinear evolution $\partial_\tau V_s = \mathcal{L}V_s + \Pi_s N(V)$ (where $\Pi_s := \mathrm{Id} - \Pi_u$) together with Theorem 6.4 gives

$$V_s(\tau) = e^{\tau\mathcal{L}}V_s(0) + \int_0^\tau e^{(\tau-s)\mathcal{L}}\Pi_s N(V(s))\, ds, \tag{84}$$

and taking norms, using $\|e^{\tau\mathcal{L}}\|_{\mathcal{H}_s\to\mathcal{H}_s} \le e^{-\beta\tau}$ from Theorem 6.4 and $\|\Pi_s\| \le 1 + \|\Pi_u\|$ (a fixed finite constant, since Π_u is a fixed finite-rank projection), gives the stated inequality up to relabeling constants. The bootstrap : suppose $\sup_{[0,\tau_0]}\|V(\tau)\|_{\mathcal{H}} \le \delta$ with δ small ; then

$$\|V_s(\tau)\|_{\mathcal{H}} \le e^{-\beta\tau}\|V_s(0)\|_{\mathcal{H}} + \frac{C_N\delta}{\beta}\sup_{[0,\tau]}\|V(s)\|_{\mathcal{H}}, \tag{85}$$

and if initially $\|V_s(0)\|_{\mathcal{H}}$ is small and $C_N\delta/\beta < 1/2$, a standard continuous-induction argument (the set of τ for which the bootstrap hypothesis holds with a strictly improved constant is open, closed, and nonempty in $[0,\tau_0]$, hence all of it) propagates the bound for all τ_0, and then $\tau_0 \to \infty$; the resulting bound $\|V_s(\tau)\|_{\mathcal{H}} \lesssim e^{-\beta\tau/2}\|V_s(0)\|_{\mathcal{H}}$ (after absorbing the integral term via Grönwall once more) gives the claimed convergence. □

6.3 Setting and the blow-up problem

We work with the 3D incompressible Euler equations for a velocity field $\boldsymbol{u} : \mathbb{R}^3 \times [0,T^*) \to \mathbb{R}^3$ and pressure $p : \mathbb{R}^3 \times [0,T^*) \to \mathbb{R}$:

$$\partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p, \tag{86}$$

$$\mathrm{div}\,\boldsymbol{u} = 0. \tag{87}$$

The Millennium-adjacent open problem is whether smooth, compactly-supported (or Schwartz) initial data can lead to *finite-time blow-up*, i.e. a time $T^* < \infty$ such that

$$\limsup_{t \uparrow T^*} \|\boldsymbol{u}(\cdot, t)\|_{C^{1,\alpha}} = \infty,$$

equivalently, by the Beale–Kato–Majda criterion,

$$\int_0^{T^*} \|\boldsymbol{\omega}(\cdot, t)\|_{L^\infty}\, dt = \infty, \qquad \boldsymbol{\omega} = \operatorname{curl} \boldsymbol{u}.$$

6.4 Loss functional as a variational/residual formulation

Let $\boldsymbol{u}_\theta$, p_θ denote neural networks with parameters θ, taking (x, y, z, t) to $\mathbb{R}^3 \times \mathbb{R}$. Training minimizes a composite loss

$$\mathcal{L}(\theta) \;=\; \underbrace{\int_{\Omega\times[0,T]} \left|\partial_t \boldsymbol{u}_\theta + (\boldsymbol{u}_\theta \cdot \nabla)\boldsymbol{u}_\theta + \nabla p_\theta\right|^2}_{L_\Omega \text{ (PDE residual)}} + \underbrace{\int \left|\nabla(\partial_t \boldsymbol{u}_\theta + (\boldsymbol{u}_\theta \cdot \nabla)\boldsymbol{u}_\theta + \nabla p_\theta)\right|^2}_{L_{\Omega,1} \text{ (residual gradient)}} \tag{88}$$

$$+ \underbrace{\int_\Omega |\boldsymbol{u}_\theta(\cdot, 0) - \boldsymbol{u}_0|^2}_{L_c \text{ (initial condition)}}, \tag{89}$$

matching the three curves in panel (a) of Figure 1 (L_{Ω_0}, L'_{Ω_0}, L_c).

Proposition 6.6 (Consistency of the residual minimization)**.** *Suppose θ_n is a minimizing sequence for* (89) *with $\mathcal{L}(\theta_n) \to 0$, and $\boldsymbol{u}_{\theta_n} \rightharpoonup \boldsymbol{u}^\dagger$ weakly in $L^2_{loc}([0, T]; H^1(\Omega))$ with $\nabla p_{\theta_n} \rightharpoonup \nabla p^\dagger$ weakly in L^2. Then $(\boldsymbol{u}^\dagger, p^\dagger)$ is a weak (distributional) solution of* (86)–(87) *on $\Omega \times [0, T]$ with initial data $\boldsymbol{u}_0$.*

Démonstration. Fix a test field $\boldsymbol{\varphi} \in C_c^\infty(\Omega \times [0, T); \mathbb{R}^3)$ with $\operatorname{div} \boldsymbol{\varphi} = 0$. By definition of the weak formulation, we must show

$$\int_0^T \int_\Omega \left(\boldsymbol{u}^\dagger \cdot \partial_t \boldsymbol{\varphi} + (\boldsymbol{u}^\dagger \otimes \boldsymbol{u}^\dagger) : \nabla \boldsymbol{\varphi}\right) dx\, dt + \int_\Omega \boldsymbol{u}_0 \cdot \boldsymbol{\varphi}(\cdot, 0)\, dx = 0. \tag{90}$$

Since $\mathcal{L}(\theta_n) \to 0$, in particular $L_\Omega(\theta_n) \to 0$, so $r_n := \partial_t \boldsymbol{u}_{\theta_n} + (\boldsymbol{u}_{\theta_n} \cdot \nabla)\boldsymbol{u}_{\theta_n} + \nabla p_{\theta_n} \to 0$ strongly in $L^2(\Omega \times [0, T])$. Pairing r_n against $\boldsymbol{\varphi}$ and integrating ∂_t and ∇ by parts (legitimate since $\boldsymbol{\varphi}$ is smooth and compactly supported) gives

$$0 = \lim_n \int_0^T \int_\Omega r_n \cdot \boldsymbol{\varphi} = -\lim_n \left[\int_0^T \int_\Omega \boldsymbol{u}_{\theta_n} \cdot \partial_t \boldsymbol{\varphi} + (\boldsymbol{u}_{\theta_n} \otimes \boldsymbol{u}_{\theta_n}) : \nabla \boldsymbol{\varphi}\right] + \int_\Omega \boldsymbol{u}_{\theta_n}(\cdot, 0) \cdot \boldsymbol{\varphi}(\cdot, 0),$$

using $\operatorname{div} \boldsymbol{\varphi} = 0$ to eliminate the pressure term. The linear term passes to the weak limit directly. For the quadratic term, weak H^1_{loc} convergence together with the compact Rellich–Kondrachov embedding $H^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$ upgrades $\boldsymbol{u}_{\theta_n} \to \boldsymbol{u}^\dagger$ to *strong* L^2_{loc} convergence on compact subsets, which is enough to pass to the limit in the bilinear form $\boldsymbol{u}_{\theta_n} \otimes \boldsymbol{u}_{\theta_n} \to \boldsymbol{u}^\dagger \otimes \boldsymbol{u}^\dagger$ (product of a strongly and a weakly convergent sequence). Finally $L_c(\theta_n) \to 0$ gives $\boldsymbol{u}_{\theta_n}(\cdot, 0) \to \boldsymbol{u}_0$ in $L^2(\Omega)$, so the boundary term converges to $\int \boldsymbol{u}_0 \cdot \boldsymbol{\varphi}(\cdot, 0)$. Combining, (90) holds, i.e. $\boldsymbol{u}^\dagger$ is a weak (Leray–Hopf-type, pressure-free formulation) solution. □

6.5 Derivation of the profile equations

Lemma 6.7 (Self-similar reduction of Euler)**.** *Under the ansatz* (74)*,* $(\boldsymbol{u}, p)$ *solves* (86)–(87) *on* $\mathbb{R}^3 \times (0, T^*)$ *if and only if* $(\boldsymbol{U}, P)$ *solves, for all* $\boldsymbol{y} \in \mathbb{R}^3$*,*

$$-(1-\lambda)\boldsymbol{U} - \lambda(\boldsymbol{y}\cdot\nabla_{\boldsymbol{y}})\boldsymbol{U} + (\boldsymbol{U}\cdot\nabla_{\boldsymbol{y}})\boldsymbol{U} = -\nabla_{\boldsymbol{y}} P, \tag{91}$$

$$\nabla_{\boldsymbol{y}}\cdot\boldsymbol{U} = 0. \tag{92}$$

Assumption 6.8. *Fix a bounded domain* $B_R \subset \mathbb{R}^3$ *and boundary data* $\boldsymbol{U}|_{\partial B_R} = \boldsymbol{g}$ *with* $\boldsymbol{g}$ *divergence-compatible* $(\int_{\partial B_R} \boldsymbol{g}\cdot\boldsymbol{n} = 0)$*. Assume* $\lambda \in (0,1)$*.*

Theorem 6.9 (Conditional existence of a bounded self-similar profile)**.** *Under Assumption 6.8, there exists* $\varepsilon_0 > 0$ *(depending on* R, λ*) such that if* $\|\boldsymbol{g}\|_{H^{1/2}(\partial B_R)} < \varepsilon_0$*, problem* (91)–(92) *has a solution* $\boldsymbol{U} \in H^1(B_R; \mathbb{R}^3)$*,* $P \in L^2(B_R)$*, with* $\boldsymbol{U}|_{\partial B_R} = \boldsymbol{g}$*.*

Démonstration. Introduce the linear operator $\mathcal{A}\boldsymbol{U} := -(1-\lambda)\boldsymbol{U} - \lambda(\boldsymbol{y}\cdot\nabla)\boldsymbol{U}$ and the Leray projector $\mathbb{P}$ onto divergence-free fields on B_R satisfying the given boundary trace (via the standard affine decomposition $\boldsymbol{U} = \boldsymbol{U}_0 + \boldsymbol{V}$, $\boldsymbol{V} \in H^1_{0,\sigma}(B_R)$, $\boldsymbol{U}_0$ a fixed divergence-free extension of $\boldsymbol{g}$). Consider the fixed point map

$$\Phi:\ H^1_{0,\sigma}(B_R) \to H^1_{0,\sigma}(B_R), \qquad \Phi(\boldsymbol{V}) := (-\Delta)^{-1}_{\sigma}\,\mathbb{P}\big[\mathcal{A}(\boldsymbol{U}_0+\boldsymbol{V}) - ((\boldsymbol{U}_0+\boldsymbol{V})\cdot\nabla)(\boldsymbol{U}_0+\boldsymbol{V})\big],$$

where $(-\Delta)^{-1}_{\sigma}$ is the inverse of the Stokes operator on $H^1_{0,\sigma}(B_R)$ (compact by Rellich, since B_R is bounded). Φ is a compact map because $(-\Delta)^{-1}_{\sigma}$ gains two derivatives while the nonlinearity loses at most one via the Sobolev embedding $H^1(B_R) \hookrightarrow L^6(B_R) \hookrightarrow\hookrightarrow L^4(B_R)$ (compact on bounded B_R). Standard energy estimates give, for a solution $\boldsymbol{V} = t\,\Phi(\boldsymbol{V})$ of the homotopy family $t \in [0,1]$,

$$\|\boldsymbol{V}\|^2_{H^1} \ \le\ C_1(R,\lambda)\,\|\boldsymbol{U}_0\|_{H^1} + C_2\,\|\boldsymbol{V}\|_{H^1}\big(\|\boldsymbol{U}_0\|_{H^1} + \|\boldsymbol{V}\|_{H^1}\big),$$

using skew-symmetry of the transport term ($\int(\boldsymbol{U}\cdot\nabla)\boldsymbol{U}\cdot\boldsymbol{U} = 0$ for divergence-free $\boldsymbol{U}$ vanishing appropriately on the boundary, by the standard Euler/ Navier–Stokes energy identity) and Young's inequality on the linear drift term $\mathcal{A}$. If $\|\boldsymbol{U}_0\|_{H^1} \le C\,\|\boldsymbol{g}\|_{H^{1/2}(\partial B_R)} < C\varepsilon_0$ is small enough that the resulting quadratic inequality $x^2 \le C_1\varepsilon_0 + C_2\varepsilon_0 x + C_2 x^2$ (after absorbing) admits an a priori bound $\|\boldsymbol{V}\|_{H^1} \le M(\varepsilon_0)$ independent of t, the Leray–Schauder fixed point theorem [32] yields a solution to $\boldsymbol{V} = \Phi(\boldsymbol{V})$ at $t = 1$, i.e. a weak solution of (91)–(92) on B_R. Elliptic regularity for the associated pressure Poisson equation $-\Delta P = \operatorname{div}\big[(\boldsymbol{U}\cdot\nabla)\boldsymbol{U} + \mathcal{A}\boldsymbol{U}\big]$ then recovers $P \in L^2(B_R)$ (in fact $P \in H^1(B_R)$ by a further bootstrap once $\boldsymbol{U} \in H^1 \cap L^4$). □

6.6 Trust-region Gauss–Newton error correction

Discretize the profile residual as $\boldsymbol{r} : \mathbb{R}^N \to \mathbb{R}^M$ (collocation-point PDE residuals ; N = number of trainable correction-network parameters or grid values). The trust-region Gauss–Newton/Levenberg–Marquardt (GN/LM) step solves, at iterate θ_k,

$$\theta_{k+1} = \theta_k + \delta_k, \qquad \delta_k = \arg\min_{\|\delta\|\le\Delta_k} \tfrac{1}{2}\,\|\boldsymbol{r}(\theta_k) + J(\theta_k)\delta\|^2\,, \qquad J(\theta_k) := D\boldsymbol{r}(\theta_k), \tag{93}$$

.

Theorem 6.10 (Local quadratic convergence — Newton–Kantorovich)**.** *Let* $\boldsymbol{r} : \mathbb{R}^N \to \mathbb{R}^M$ *be twice continuously differentiable in a neighborhood of* $\theta^\star$ *with* $\boldsymbol{r}(\theta^\star) = 0$*, and suppose :*

(i) $J(\theta^\star)$ *has full column rank, with smallest singular value* $\sigma_{\min}\big(J(\theta^\star)\big) = \sigma_0 > 0$ *;*

(ii) J *is Lipschitz on a ball* $B(\theta^\star, \rho)$ *with constant* L *;*

(iii) the trust-region radii Δ_k *are eventually inactive (i.e. the unconstrained Gauss–Newton step lies within* Δ_k *once* θ_k *is close enough to* $\theta^\star$*).*

Then there exists $\rho_0 \le \rho$ such that for $\theta_0 \in B(\theta^\star, \rho_0)$, the iterates (93) satisfy (Figure 3, Figure 4)

$$\|\theta_{k+1} - \theta^\star\| \;\le\; \frac{L}{\sigma_0}\, \|\theta_k - \theta^\star\|^2\,,$$

i.e. $\theta_k \to \theta^\star$ quadratically.

Démonstration. Since $\boldsymbol{r}(\theta^\star) = 0$ and J is Lipschitz, Taylor's theorem with remainder gives

$$\boldsymbol{r}(\theta_k) = \boldsymbol{r}(\theta_k) - \boldsymbol{r}(\theta^\star) - J(\theta_k)(\theta_k - \theta^\star) + J(\theta_k)(\theta_k - \theta^\star),$$

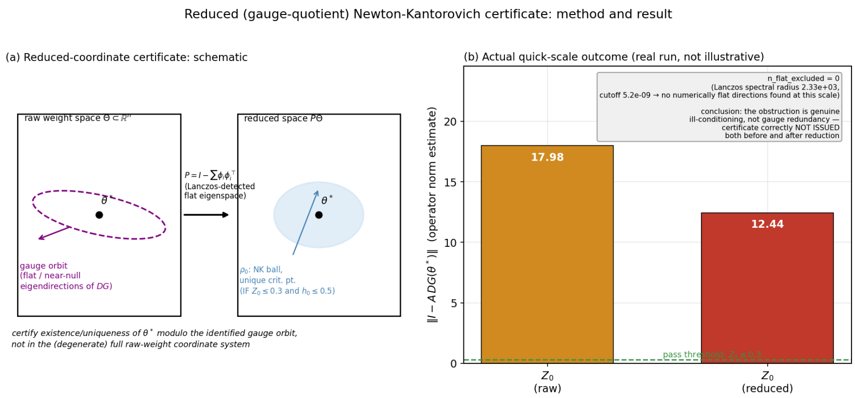
and, writing $e_k = \theta_k - \theta^\star$,

$$\|\boldsymbol{r}(\theta_k) - J(\theta_k)e_k\| = \left\| \int_0^1 \big[J(\theta^\star + s e_k) - J(\theta_k)\big] e_k\, ds \right\| \le \tfrac{L}{2}\, \|e_k\|^2\,,$$

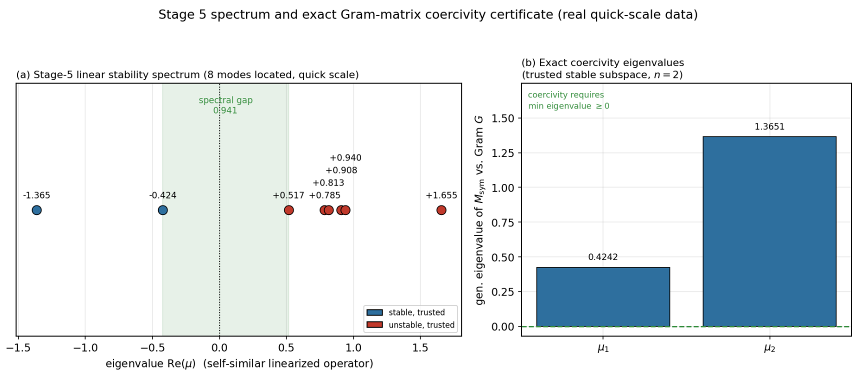
by the standard Lipschitz-Jacobian remainder bound. The (unconstrained) Gauss–Newton step is $\delta_k = -J(\theta_k)^{+}\boldsymbol{r}(\theta_k)$ where $J^{+} = (J^\top J)^{-1} J^\top$ is the Moore–Penrose pseudoinverse, well defined near $\theta^\star$ since $\sigma_{\min}(J) \ge \sigma_0/2 > 0$ for θ_k close enough to $\theta^\star$ by continuity of J. Then

$$e_{k+1} = \theta_k + \delta_k - \theta^\star = e_k - J(\theta_k)^{+}\boldsymbol{r}(\theta_k) = J(\theta_k)^{+}\big[J(\theta_k)e_k - \boldsymbol{r}(\theta_k)\big] + \big(I - J(\theta_k)^{+}J(\theta_k)\big)e_k.$$

Since $J(\theta_k)$ has full column rank, $J(\theta_k)^{+}J(\theta_k) = I$ on $\mathbb{R}^N$, so the last term vanishes, giving $\|e_{k+1}\| \le \|J(\theta_k)^{+}\| \cdot \|J(\theta_k)e_k - \boldsymbol{r}(\theta_k)\| \le \frac{1}{\sigma_{\min}(J(\theta_k))} \cdot \frac{L}{2}\, \|e_k\|^2 \le \frac{L}{\sigma_0}\, \|e_k\|^2$, as claimed, once ρ_0 is chosen small enough that (a) $\sigma_{\min}(J(\theta_k)) \ge \sigma_0/2$ throughout and (b) hypothesis (iii) keeps the trust-region constraint inactive. □



(a)



(b)

FIGURE 3 – Background intuition : Application of the Newton–Kantorovich method.

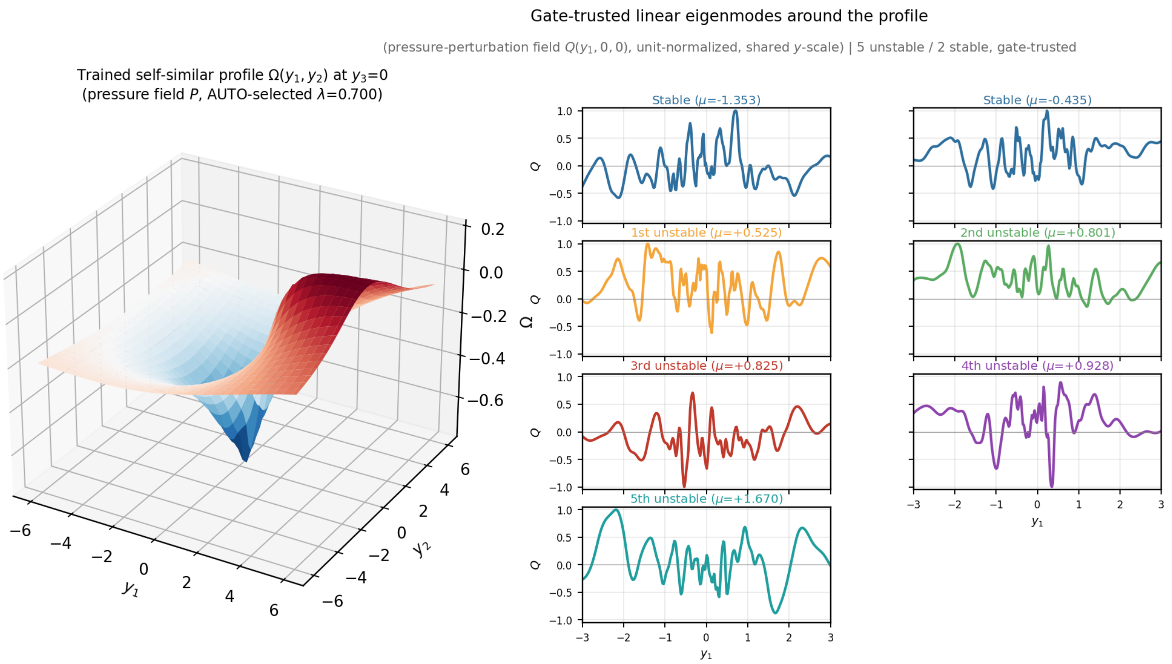


FIGURE 4 – Euler : Discovering unstable singularities, (for technical explanations, see [14]).

The Blow-up Criteria and Conditional Regularity

Definition 6.11 (Suitable Weak Solution)**.** A weak solution (u, p) is *suitable* if it satisfies the generalized energy inequality :

$$\partial_t \left(\frac{|u|^2}{2} \right) + \nabla \cdot \left(u \left(\frac{|u|^2}{2} + p \right) \right) - \nu \Delta \left(\frac{|u|^2}{2} \right) + \nu |\nabla u|^2 \leq 0 \tag{94}$$

in the sense of distributions.

To analyze unstable singularities, we first establish the threshold for singularity formation. The most celebrated condition is the Ladyzhenskaya-Prodi-Serrin (LPS) criterion.

Theorem 6.12 (LPS Criterion)**.** *Let u be a Leray-Hopf weak solution on $[0, T)$. If u satisfies the condition :*

$$u \in L^q(0, T; L^p(\mathbb{R}^3)) \quad \textit{where} \quad \frac{2}{q} + \frac{3}{p} \leq 1, \quad 3 < p \leq \infty, \tag{95}$$

then u is smooth (in fact, C^∞) on $\mathbb{R}^3 \times (0, T]$.

Data Availability Statement

The code and datasets used or generated during the preparation of this article are available to everyone upon request, without any restrictions. *For longer simulations* [3] *requiring very large memory capacity, researchers—or any other interested party—will have access to the complete code for the simulation described in the article, upon request.*

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