SOLUTION OF BINARY-CONSTRAINED QUADRATIC-DEFINED OPTIMIZATION PROBLEMS BY A PROGRESSIVE INTEGER PROGRAMMING METHOD*

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Abstract. Extending a classic result of Giannessi and Tomasin [*Lecture Notes in Comput. Sci. 3*, Springer, 1973, pp. 437–449], this paper shows that a binary-constrained quadratic-defined optimization problem can be formulated as a binary-constrained linear program with linear complementarity constraints (Bi-LPCC). The term "quadratic-defined problems" encompasses many problems that are defined by quadratic functions in the objective, constraints, and even in a lower-level problem. The binary restriction of some variables add to the challenge in the rigorous solution of these problems. The Bi-LPCC formulation enables all these special cases to be solvable by mixed-integer linear programming, either in their full integer (denoted FMIP) formulation, or by the progressive integer programming (PIP) enhancement. The superior performance of the PIP method over the FMIP version is demonstrated by some extensive computational results on some test problems of binary-constrained quadratic programs with linear complementarity and variants of the simplex constraints. Theoretically, assuming solvability of the given problem, the solution obtained by PIP is provably to be a locally optimal solution of the problem, which is globally optimal if the full-IP version is solved to optimality.

Key words. binary-constrained nonconvex quadratic program, linear program with complementarity constraints, mixed-integer linear programming, progressive integer programming

MSC codes. 90C11, 90C20, 90C26, 90C33

1. Introduction.

A classical result of Giannessi-Tomasin (abbreviated as the GT result) [15] states that an (indefinite) quadratic program (QP) is equivalent to a linear program with linear complementarity constraints (LPCC) in the primal-dual space, provided that the QP has an optimal solution to start with. The equivalence is in terms of their globally optimal solutions. The result has been the cornerstone for solving a nonconvex QP by mixed-integer linear programming (MILP); see e.g. Gondzio and Yıldırım [16] and Hu, Mitchell, and Pang [19]. It was recognized that based on the GT result, quadratically constrained quadratic programs (QCQPs) failing the Slater constraint qualification can be formulated with the same quadratic objective but with the quadratic constraints replaced by linear complementarity constraints [2]. In turn, the global resolution to linear complementarity constraints by integer programming methods is well studied; see [20, 21, 22]. Burer and his collaborators, among others, have studied nonconvex QPs and QCQPs extensively using copositive programming method in matrix variables; see [5, 6, 7, 8, 9] for a sample of works. Growing out of this line of research, we refer to the surveys of Dür [13] and Bomze [3] for an overview. Two of its threads bear direct connections to the problems treated here. Testing copositivity is itself amenable to mixed-integer linear programming by Anstreicher [1], a methodological parallel to the approach we pursue. Closer still, Bomze and Peng [4] establish an exact completely positive reformulation of quadratic programs with complementarity constraints under conditions involving the constraints alone, whereas the route taken below is the complementary one: we

*Submitted to the editors September 13, 2026.

Funding: This work was based on research supported by the U.S. Air Force Office of Scientific Research under grant FA9550-22-1-0045. The research of Bo Peng was funded in whole or in part by the Austrian Science Fund (FWF) through the Erwin Schrödinger Fellowship [10.55776/J4969].

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retain the binary variables and exploit the disjunctive feature of complementarity.

The present work is motivated by two highlights of the literature reviewed above. One is the fundamental question of whether the classical result of GT can be extended to an (indefinite) quadratic program with (some) $\{0,1\}$-variables; that is, whether a binary-constrained QP admits an LPCC formulation with mixed integer variables. The other motivation is the recent work [28] in which a progressive integer programming (PIP) method is applied to a partial MILP formulation of the LPCC derived from a QP; the method yields considerable improvements in objective values over those obtained by publicly available implementations of well-known nonlinear programming methods that are available on the NEOS collection [11, 12, 17]; moreover, the PIP results are superior to those obtained by global solvers such as GUROBI [18] and BARON [25] in two ways: 1) for problems whose global solutions can be computed by these solvers, PIP can obtain either the matching or approximately optimal solutions in significantly less times, and 2) for problems for which these solvers fail due to excessive computational times, PIP can obtain solutions with superior quality than the solvers' best solutions in very reasonable amounts of times. We refer the reader to [14, 29] for the application of the PIP method to some Heaviside composite optimization problems with similar findings. These two motivations result in a major contribution of this work, namely, besides providing an affirmative answer to the extension of the GT result, the technique of proof opens a promising venue for solving many binary-constrained quadatic-defined nonconvex optimization problems by integer programming methods, in particular, reaping the benefits of the PIP method for obtaining superior-quality suboptimal solutions in modest times compared to applicable methods, which are presently rather limited when binary variables are present in the given problems. The term "quadratic-defined problems" encompasses many models that are defined by quadratic functions in the objective, constraints, and even in a lower-level problem.

2. The Binary-Constrained QPCC. Subsequently, we will show that many quadratic-defined problems can be unified by the following binary-constrained quadratic program with complementarity constraints (Bi-QPCC) formulation:

$$
\begin{array}{lll}
\underset{\boldsymbol{v}\in\mathbb{R}^{n+2m+\ell}}{\textbf{minimize}} & \boldsymbol{v}^\top \boldsymbol{Q}\boldsymbol{v} + \boldsymbol{q}^\top \boldsymbol{v} & \\
\textbf{subject to} & \boldsymbol{A}\boldsymbol{v} \geq e \in \mathbb{R}^k & \\
\textbf{and} & 0 \leq y \perp z \geq 0 & \text{complementarity, i.e., } y^\top z = 0; \\
\textbf{where} & \boldsymbol{Q} \triangleq \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xz} & Q_{xw} \\ Q_{yx} & Q_{yy} & Q_{yz} & Q_{yw} \\ Q_{zx} & Q_{zy} & Q_{zz} & Q_{zw} \\ Q_{wx} & Q_{wy} & Q_{wz} & Q_{ww} \end{bmatrix} & \begin{array}{l}\text{is symmetric}\\ \text{and indefinite}\end{array} \\
& \boldsymbol{A} \triangleq \begin{bmatrix} A & B & C & D \end{bmatrix} & \\
& \boldsymbol{v} \triangleq \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^{n+2m} \times \underbrace{\{0,1\}^\ell}_{\text{binary}} \quad \text{and} \quad \boldsymbol{q} \triangleq \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, &
\end{array}
\tag{2.1}
$$

and the dimensions of the submatrices in $\boldsymbol{Q}$ and $\boldsymbol{A}$ and those of the subvectors in $\boldsymbol{q}$ are determined by the corresponding subvectors in $\boldsymbol{v}$. Besides demonstrating the modeling breadth of the problem, we aim to study several major aspects of this problem:

(a) derive an equivalent formulation of the problem as a binary constrained LPCC in terms of their globally optimal solutions (Theorem 2.2);

(b) introduce, by way of the derived LPCC, a necessary condition (which albeit is not sufficient) for a "locally optimal solution" of (2.1) in terms of a fixed-point property of a mixed binary constrained linear integer program;

(c) demonstrate that the fixed-point property of (b) is necessary and sufficient for the solution on hand to be a locally optimal solution to the derived LPCC;

(d) describe the PIP method for computing a locally optimal solution to the derived LPCC obtained in (a), thus the verification of the necessary condition in (b) for (2.1) via the fixed-point property in (c) can in principle be accomplished in finite computational time; and

(e) demonstrate the effectiveness of the PIP method by several means: (i) if a globally optimal solution to (2.1) can be computed, then the PIP method can compute either a matching optimal solution or a highly accurate suboptimal solution in significantly less computational time; (ii) if a globally optimal solution to (2.1) cannot be computed, due perhaps to excessive computational time, the PIP method can yield a quantifiable suboptimal solution in acceptable computational time, and (iii), when start with a feasible solution to the derived LPCC, obtained by perhaps a nonlinear programming solver, the PIP method can improve the quality of the given solution.

Throughout, no special properties are imposed on the problem (2.1) except for its solvability, which implies in particular the consistency of the constraints; thus, the feasible set $\mathbb{V} \triangleq \{ \boldsymbol{Av} \geq e;\ 0 \leq y \perp z \geq 0;\ \text{and } w \ \text{ binary} \}$ is nonmempty. Nevertheless, to derive an equivalent binary-constrained LPCC of (2.6), which involves linearizaions of products of continuous variables with binary variables, we assume that

- **(boundedness of primal variables):** the primal triple (x, y, z) is bounded:

$$\boldsymbol{v} \in \mathbb{V} \Rightarrow \left\{ \begin{array}{ll} \underline{x}_i \leq x_i \leq \overline{x}_i & i = 1, \cdots, n \\ y_j \leq \overline{y}_j \text{ and } z_j \leq \overline{z}_j & j = 1, \cdots, m \end{array} \right\}, \tag{2.2}$$

where the upper bounds $\overline{y}$ and $\overline{z}$ are positive vectors. Under this boundedness assumption, problem (2.1) is equivalent to:

$$\left[\begin{array}{l} \underset{\boldsymbol{v} \in \mathbb{R}^{n+2m+\ell};\, s}{\textbf{minimize}} \ \boldsymbol{v}^\top \boldsymbol{Q} \boldsymbol{v} + \boldsymbol{q}^\top \boldsymbol{v} \\ \textbf{subject to } \boldsymbol{Av} \geq e \in \mathbb{R}^k \\ \qquad \left\{ \begin{array}{l} 0 \leq y_j \leq \overline{y}_j\, s_j \\ 0 \leq z_j \leq \overline{z}_j\, (1 - s_j) \end{array} \right\} j = 1, \cdots, m \\ \qquad w \in \{0,1\}^\ell \quad \text{and} \quad s \in \{0,1\}^m \end{array} \right] \text{single minimization}$$

$$\Longleftrightarrow \underset{w \in \{0,1\}^\ell;\, s \in \{0,1\}^m}{\textbf{minimize}} \left[\begin{array}{l} \underset{(x,y,z) \in \mathbb{R}^{n+2m}}{\textbf{minimize}} \ \boldsymbol{v}^\top \boldsymbol{Q} \boldsymbol{v} + \boldsymbol{q}^\top \boldsymbol{v} \\ \textbf{subject to } \ Ax + By + Cz + Dw \geq e \\ \qquad \left\{ \begin{array}{l} 0 \leq y_j \leq \overline{y}_j\, s_j \\ 0 \leq z_j \leq \overline{z}_j\, (1 - s_j) \end{array} \right\} j = 1, \cdots, m \end{array} \right] \begin{array}{l} \text{outer +} \\ \text{inner} \\ \text{minimi-} \\ \text{zation} \end{array} \tag{2.3}$$

in terms of their respective globally optimal solutions; in particular, they have the same optimum objective values. Note that the upper bound $\overline{x}_i$ and lower bound $\underline{x}_i$ of x_i do not appear explicitly in (2.3) but the upper bounds $\overline{y}_i$ and $\overline{z}_i$ do and remain so in the rest of the derivations. Thus it is important that these bounds can be derived as sharply as possible to facilitate the efficient computational solution to the problem. We have

$$\boldsymbol{v}^\top \boldsymbol{Q} \boldsymbol{v} + \boldsymbol{q}^\top \boldsymbol{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}^\top \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xz} \\ Q_{yx} & Q_{yy} & Q_{yz} \\ Q_{zx} & Q_{zy} & Q_{zz} \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} +$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}^\top \left\{ \begin{bmatrix} Q_{xw} + Q_{wx}^\top \\ Q_{yw} + Q_{wy}^\top \\ Q_{zw} + Q_{wz}^\top \end{bmatrix} w + \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\} + w^\top Q_{ww}\, w + d^\top w.$$

Hence for fixed pair (w, s), the inner minimization problem in (2.3) is a quadratic program in the triplet (x, y, z) that itself is equivalent to a linear program with linear complementarity constraints, provided that this parameterized QP has an optimal solution. Thus, throughout this paper, we make the blanket assumption that

- **(piecewise solvability)** for every pair $(w, s) \in \{0,1\}^{\ell+m}$, the quadratic program

$$\begin{array}{ll} \underset{(x,y,z)\in\mathbb{R}^{n+2m}}{\textbf{minimize}} & \boldsymbol{v}^\top \boldsymbol{Q} \boldsymbol{v} + \boldsymbol{q}^\top \boldsymbol{v} \\[1em] \textbf{subject to} & Ax + By + Cz + Dw \geq e \\[0.5em] & \left\{ \begin{array}{l} 0 \leq y_j \leq \overline{y}_j\, s_j \\ 0 \leq z_j \leq \overline{z}_j\,(1 - s_j) \end{array} \right\} \; j = 1, \cdots, m \end{array} \tag{2.4}$$

has a finite optimal solution; (or equivalently, is bounded below).

2.1. The linearization of (2.1) **via** (2.4)**.** We obtain a binary-constrained LPCC formulation of (2.4) in the following steps. **Step 1:** derive an LPCC formulation of (2.4) for each binary pair (w, s) by introducing nonnegative multipliers $\eta \in \mathbb{R}^k_+$ and $\{(\lambda_j, \mu_j)\}_{j=1}^m \subseteq \mathbb{R}^{2m}_+$ constrained by the Karush-Kuhn-Tucker (KKT) conditions; **Step 2:** bound the introduced multipliers; **Step 3:** apply McCormick bounds to (exactly) linearize the product of a continuous and a binary variable.

By the GT result applied to (2.4) for each binary pair (w, s), the original Bi-QPCC is equivalent to the mixed-integer complementarity-constrained minimization problems involving products of the $\{0,1\}$-vector w with itself and with the primal variables (x, y, z) and the products of $\{0,1\}$-vector s with the multipliers (η, λ, μ):

minimize over $(x, y, z) \in \mathbb{R}^{n+2m}$; $w \in \{0,1\}^\ell$; $s \in \{0,1\}^m$; $(\eta, \lambda, \mu) \in \mathbb{R}^{k+2m}_+$:

$$w^\top Q_{ww}\, w + d^\top w + \tfrac{1}{2} \begin{pmatrix} x \\ y \\ z \end{pmatrix}^\top \left\{ \begin{bmatrix} Q_{xw} + Q_{wx}^\top \\ Q_{yw} + Q_{wy}^\top \\ Q_{zw} + Q_{wz}^\top \end{bmatrix} w + \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\} \tag{2.5}$$

$$- \tfrac{1}{2} \left\{ \eta^\top (Dw - e) + \lambda^\top (\bar{y} \circ s) + \mu^\top (\bar{z} \circ (\mathbf{1} - s)) \right\}$$

subject to

$$\begin{array}{rcl} 0 & = & \\ 0 & \leq & \\ 0 & \leq & \end{array} 2 \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xz} \\ Q_{yx} & Q_{yy} & Q_{yz} \\ Q_{zx} & Q_{zy} & Q_{zz} \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{bmatrix} Q_{xw} + Q_{wx}^\top \\ Q_{yw} + Q_{wy}^\top \\ Q_{zw} + Q_{wz}^\top \end{bmatrix} w + \begin{pmatrix} a \\ b \\ c \end{pmatrix} -$$

$$\begin{bmatrix} A^\top \\ B^\top \\ C^\top \end{bmatrix} \eta + \begin{pmatrix} 0 \\ \lambda \\ \mu \end{pmatrix} \begin{array}{c} \\ \perp \\ \perp \end{array} \begin{pmatrix} \\ y \\ z \end{pmatrix} \begin{array}{c} \\ \geq 0 \\ \geq 0 \end{array}$$

$$0 \leq \eta \perp Ax + By + Cz + Dw - e \geq 0$$

$$0 \leq \lambda \perp \overline{y} \circ s - y \geq 0 \ \textbf{ and } \ 0 \leq \mu \perp \overline{z} \circ (\mathbf{1} - s) - z \geq 0,$$

where $\overline{y} \triangleq \{\overline{y}_j\}_{j=1}^m$ and $\overline{z} \triangleq \{\overline{z}_j\}_{j=1}^m$ are the vectors of upper bounds for y and z, respectively, and $\circ$ denotes the Hadamard product of two vectors. The following lemma linearizes the bilinear products of the primal variables.

LEMMA 2.1. The following two statements hold:

(i) A $\{0,1\}$-variable $r = r_1 r_2$ for two $\{0,1\}$-variables r_1 and r_2 if and only if the following linear conditions hold for the three $\{0,1\}$-variables r, r_1, and r_2:

$$0 \leq r \leq \min(r_1, r_2); \quad r_1 + r_2 - 1 \leq r.$$

Note that the variable r does not need to be restricted to binary; the constraints will imply this property. Moreover, if $r_1 = r_2$, then $r = r_1$.

(ii) If $\underline{a} \leq a \leq \overline{a}$ and $r \in \{0,1\}$, then the product $b = a\,r$ holds if and only if the linear conditions hold:

$$\overline{a}\,(r-1) \leq b - a \leq \underline{a}\,(r-1) \quad \text{and} \quad \underline{a}\,r \leq b \leq \overline{a}\,r.$$

Therefore, we have

$$w^\top Q_{ww}\, w = \sum_{i,j=1}^{\ell} [\,Q_{ww}\,]_{ij}\, w_i\, w_j = \sum_{i,j=1}^{\ell} [\,Q_{ww}\,]_{ij}\, w_{ij}, \text{ with } w_{ii} = w_i,$$

where each $[\,Q_{ww}\,]_{ij}\, w_{ij}$ is linear in w_{ij}, and $w_{ij} \triangleq w_i\, w_j$ satisfies:

$$0 \leq w_{ij} \leq \min(\,w_i, w_j\,) \text{ and } w_i + w_j - 1 \leq w_{ij}, \quad \forall\, i \neq j \text{ and } i,j = 1, \cdots, \ell.$$

We vectorize the submatrix Q_{ww} and the products $\{w_{ij}\}_{i,j=1}^{\ell}$ and denote the resulting vectors by $\widehat{Q}_{ww}$ and $\widehat{w}$, respectively. Similarly, for $x^\top Q_{xw}\, w$, we have

$$x^\top Q_{xw}\, w = \sum_{i=1}^{n}\sum_{j=1}^{\ell} [\,Q_{xw}\,]_{ij}\, x_i\, w_j = \sum_{i=1}^{n}\sum_{j=1}^{\ell} [\,Q_{xw}\,]_{ij}\, x_{ij},$$

where each $[\,Q_{xw}\,]_{ij}\, x_{ij}$ is linear in x_{ij}, and $x_{ij} \triangleq x_i\, w_j$ satisfies

$$\overline{x}_i(\,w_j - 1\,) \leq x_{ij} - x_i \leq \underline{x}_i\,(w_j - 1) \quad \text{and} \quad \underline{x}_i\, w_j \leq x_{ij} \leq \overline{x}_i\, w_j.$$

Hence, by vectorizing the submatrices $Q_{xw} = Q_{wx}^\top$, $Q_{yw} = Q_{wy}^\top$, and $Q_{zw} = Q_{wz}^\top$ into $\widehat{Q}_{xw}$, $\widehat{Q}_{yw}$, and $\widehat{Q}_{zw}$, respectively, and vectorizing the products $\{x_{ij}\}_{i,j=1}^{n,\ell}$ and the similarly defined $\{y_{ij}\}_{i,j=1}^{m,\ell}$ and $\{z_{ij}\}_{i,j=1}^{m,\ell}$ into $\widehat{x}$, $\widehat{y}$, and $\widehat{z}$, respectively, we have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}^\top \begin{bmatrix} Q_{xw} + Q_{wx}^\top \\ Q_{yw} + Q_{wy}^\top \\ Q_{zw} + Q_{wz}^\top \end{bmatrix} w = 2 \left(\widehat{Q}_{xw}^\top \widehat{x} + \widehat{Q}_{yw}^\top \widehat{y} + \widehat{Q}_{zw}^\top \widehat{z} \right).$$

It remains to linearize $\eta^\top Dw$, $\lambda^\top s$, and $\mu^\top(\mathbf{1}-s)$. To this end, we derive upper bounds for the multipliers η, λ, and μ satisfying the constraints in (2.5), assuming:

• **(existence of Slater points)** for every pair $p \triangleq (w,s) \in \{0,1\}^{\ell+m}$, the quadratic program (2.4) has a Slater point; i.e., there exist (x^p, y^p, z^p) feasible to (2.4) and satisfying

$$\sigma^p \triangleq Ax^p + By^p + Cz^p + Dw - e > 0$$

$$[\, y_i^p < \overline{y}_i \text{ if } s_i = 1\,] \text{ and } [\, z_i^p < \overline{z}_i \text{ if } s_i = 0\,].$$

A sufficient condition is that there exists $\widehat{x}$ such that $A\widehat{x} > 0$. Since there are only finitely many such representative Slater tuples and the strict inequality for (y^p, z^p) implies strict complementarity, i.e., $y^p + z^p > 0$, we use these tuples (x^p, y^p, z^p), one for each pair (w,s), to bound any tuple (η, λ, μ) such that $(x,y,z,w,\eta,\lambda,\mu)$ is feasible to (2.5) for some (x,y,z). We have

$$
\begin{aligned}
0 \le\;& \begin{pmatrix} x^p \\ y^p \\ z^p \end{pmatrix}^\top \left[2 \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xz} \\ Q_{yx} & Q_{yy} & Q_{yz} \\ Q_{zx} & Q_{zy} & Q_{zz} \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{bmatrix} Q_{xw} + Q_{wx}^\top \\ Q_{yw} + Q_{wy}^\top \\ Q_{zw} + Q_{wz}^\top \end{bmatrix} w + \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right] \\
&- \; [\, Ax^p + By^p + Cz^p \,]^\top \eta + \sum_{i\,:\,s_i=1} y_i^p \lambda_i + \sum_{i\,:\,s_i=0} z_i^p \mu_i \\
&- \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}^\top \left[2 \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xz} \\ Q_{yx} & Q_{yy} & Q_{yz} \\ Q_{zx} & Q_{zy} & Q_{zz} \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{bmatrix} Q_{xw} + Q_{wx}^\top \\ Q_{yw} + Q_{wy}^\top \\ Q_{zw} + Q_{wz}^\top \end{bmatrix} w + \begin{pmatrix} a \\ b \\ c \end{pmatrix} - \begin{bmatrix} A^\top \\ B^\top \\ C^\top \end{bmatrix} \eta + \begin{pmatrix} 0 \\ \lambda \\ \mu \end{pmatrix} \right]}_{=\,0,\text{ by complementarity}} \\
&\underbrace{- [\, Ax + By + Cz + Dw - e \,]^\top \eta + \sum_{i\,:\,s_i=1} (y_i - \overline{y}_i)\lambda_i + \sum_{i\,:\,s_i=0} (z_i - \overline{z}_i)\,\mu_i}_{\text{each term } =\, 0,\text{ by complementarity}} \\
=\;& \underbrace{\left[\begin{pmatrix} x^p \\ y^p \\ z^p \end{pmatrix} - \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right]^\top \left[2 \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xz} \\ Q_{yx} & Q_{yy} & Q_{yz} \\ Q_{zx} & Q_{zy} & Q_{zz} \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{bmatrix} Q_{xw} + Q_{wx}^\top \\ Q_{yw} + Q_{wy}^\top \\ Q_{zw} + Q_{wz}^\top \end{bmatrix} w + \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right]}_{\text{denoted sum}_1} \\
&- \; \eta^\top \sigma^p - \sum_{i\,:\,s_i=1} (\overline{y}_i - y_i^p)\lambda_i - \sum_{i\,:\,s_i=0} (\overline{z}_i - z_i^p)\,\mu_i.
\end{aligned}
$$

Since sum_1 is bounded, say by $\Gamma > 0$, we have, for $i = 1, \cdots, k$

$$\eta_i \le \min_{\text{all } p} \frac{\Gamma}{\sigma_i^p} \triangleq \Delta_i;$$

and for $i = 1, \cdots, m$, adopt that $\Delta_i^1 = 0$ if $s_i = 0$ and $\Delta_i^0 = 0$ if $s_i = 1$, we deduce

$$\lambda_i \le \min_{\text{all } p} \frac{\Gamma}{\overline{y}_i - y_i^p} \triangleq \Delta_i^1, \quad \text{if } s_i = 1; \qquad \mu_i \le \min_{\text{all } p} \frac{\Gamma}{\overline{z}_i - z_i^p} \triangleq \Delta_i^0, \quad \text{if } s_i = 0,$$

Hence, letting $\widehat{\eta} \triangleq \{\eta_{ij}\}_{i,j=1}^{k,\ell}$, $\widehat{\lambda} \triangleq \left\{\widehat{\lambda}_i\right\}_{i=1}^m$, and $\widehat{\mu} \triangleq \{\widehat{\mu}_i\}_{i=1}^m$, where $\eta_{ij} \triangleq \eta_i w_j$, $\widehat{\lambda}_i \triangleq \lambda_i s_i$, and $\widehat{\mu}_i \triangleq \mu_i(1-s_i)$, and vectorizing $\{D_{ij}\}_{i,j=1}^{k,l}$ into $\widehat{D}$, we have

$$0 \leq \lambda_i - \widehat{\lambda}_i \leq \Delta_i^1\,(1-s_i),\; 0 \leq \widehat{\lambda}_i \leq \Delta_i^1\,s_i; \quad 0 \leq \mu_i - \widehat{\mu}_i \leq \Delta_i^0\,s_i,\; 0 \leq \widehat{\mu}_i \leq \Delta_i^0\,(1-s_i);$$

$$\eta^\top D w = \sum_{i=1}^{k}\sum_{j=1}^{\ell} D_{ij}\eta_{ij} = \widehat{D}^\top \widehat{\eta}, \quad \text{where } 0 \leq \eta_i - \eta_{ij} \leq \Delta_i\,(1-w_j) \text{ and } 0 \leq \eta_{ij} \leq \Delta_i\,w_j.$$

Therefore the problem (2.5) becomes the following binary constrained LPCC:

$$\begin{array}{l}\textbf{minimize}\\ \text{over}\\ \text{variables}\end{array}\left\{\begin{array}{l}\underbrace{w \in \{0,1\}^\ell \text{ and } s \in \{0,1\}^m}_{\text{integer variables}};\quad \underbrace{(x,y,z) \in \mathbb{R}^{n+2m};\ (\eta,\lambda,\mu) \in \mathbb{R}_+^{k+2m}}_{\text{cont. primal vars. and multipliers, resp.}}\\ \underbrace{(\widehat{x},\widehat{y},\widehat{z},\widehat{w}) \in \mathbb{R}^{(n+2m+\ell)\ell};\ (\widehat{\eta},\widehat{\lambda},\widehat{\mu}) \in \mathbb{R}_+^{k\ell+2m}}_{\text{(continuous) product variables}}:\end{array}\right.$$

$$\text{(2.6)}\quad \textbf{objective:}\quad \underbrace{\left\{\begin{array}{l}\widehat{Q}_{ww}\,\widehat{w} + d^\top w + \widehat{Q}_{xw}^\top\widehat{x} + \widehat{Q}_{yw}^\top\widehat{y} + \widehat{Q}_{zw}^\top\widehat{z} + \frac{1}{2}\left(a^\top x + b^\top y + c^\top z\right)\\ -\frac{1}{2}\left\{\widehat{D}^\top\,\widehat{\eta} - \eta^\top e + \bar{y}^\top\widehat{\lambda} + \bar{z}^\top\widehat{\mu}\right\}\end{array}\right\}}_{\text{linear objective, denoted } \theta(\boldsymbol{\xi})}$$

subject to (mixed complementarity constraints and auxiliary linear inequalities):

$$\underbrace{\left\{\begin{array}{l}\begin{array}{c}0 \;=\\ 0 \;\leq\\ 0 \;\leq\end{array}\ 2\begin{bmatrix}Q_{xx} & Q_{xy} & Q_{xz}\\ Q_{yx} & Q_{yy} & Q_{yz}\\ Q_{zx} & Q_{zy} & Q_{zz}\end{bmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} + \begin{bmatrix}Q_{xw} + Q_{wx}^\top\\ Q_{yw} + Q_{wy}^\top\\ Q_{zw} + Q_{wz}^\top\end{bmatrix} w + \begin{pmatrix}a\\ b\\ c\end{pmatrix} -\\ \qquad\qquad \begin{bmatrix}A^\top\\ B^\top\\ C^\top\end{bmatrix}\eta + \begin{pmatrix}0\\ \lambda\\ \mu\end{pmatrix} \triangleq \begin{pmatrix}\\ u\\ v\end{pmatrix}\begin{array}{c}\\ \perp\\ \perp\end{array}\begin{pmatrix}\\ y\\ z\end{pmatrix}\begin{array}{c}\\ \geq 0\\ \geq 0\end{array}\\ 0 \leq \eta \perp \sigma^x \triangleq Ax + By + Cz + Dw - e \geq 0\\ 0 \leq \lambda \perp \sigma^y \triangleq \bar{y}\circ s - y \geq 0, \quad 0 \leq \mu \perp \sigma^z \triangleq \bar{z}\circ(\mathbf{1}-s) - z \geq 0\end{array}\right\}}_{\text{mixed complementarity conditions, obtained from the GT result}}$$

$$\underbrace{\left\{\begin{array}{l}\left\{\begin{array}{l}\text{for all } j = 1,\cdots,\ell:\\ 0 \leq w_{ij} \leq \min(w_i,w_j), \quad w_i + w_j - 1 \leq w_{ij}, \quad \forall\, i = 1,\cdots,l\\ \bar{x}_i(w_j-1) \leq x_{ij} - x_i \leq \underline{x}_i\,(w_j-1), \quad \underline{x}_i\,w_j \leq x_{ij} \leq \bar{x}_i\,w_j, \quad \forall\, i = 1,\cdots,n\\ \bar{y}_i(w_j-1) \leq y_{ij} - y_i \leq \underline{y}_i\,(w_j-1), \quad \underline{y}_i\,w_j \leq y_{ij} \leq \bar{y}_i\,w_j, \quad \forall\, i = 1,\cdots,m\\ \bar{z}_i(w_j-1) \leq z_{ij} - z_i \leq \underline{z}_i\,(w_j-1), \quad \underline{z}_i\,w_j \leq z_{ij} \leq \bar{z}_i\,w_j, \quad \forall\, i = 1,\cdots,m\\ 0 \leq \eta_i - \eta_{ij} \leq \Delta_i\,(1-w_j), \quad 0 \leq \eta_{ij} \leq \Delta_i\,w_j, \quad \forall\, i = 1,\cdots,k\end{array}\right\}\\ \left\{\begin{array}{l}\text{and for all } i = 1,\cdots,m:\\ 0 \leq \lambda_i - \widehat{\lambda}_i \leq \Delta_i^1\,(1-s_i), \quad 0 \leq \widehat{\lambda}_i \leq \Delta_i^1\,s_i,\\ 0 \leq \mu_i - \widehat{\mu}_i \leq \Delta_i^0\,s_i, \quad 0 \leq \widehat{\mu}_i \leq \Delta_i^0\,(1-s_i)\end{array}\right\}\end{array}\right.}_{\text{auxiliary constraints, due to linearization of products.}}$$

Note that the complementarity $y_i\, z_i = 0$ must hold for any feasible tuple to (2.6). Moreover, u and v are derived variables that are complementary to y and z respectively; σ^x, σ^y, and σ^z are slack variables of the constraints in (2.4). Lastly, the upper bound $\overline{x}$ and lower bound $\underline{x}$ of x appear in the auxiliary constraints.

2.2. A touch of optimality theory. In this subsection, we present three basic results for the two problems: (2.1) and (2.6). The first establish the equivalence of their globally optimal solutions.

THEOREM 2.2. Suppose that problem (2.1) is feasible and the following three conditions hold: boundedness of primal variables, piecewise solvability, and existence of Slater points. Then, a tuple $(\bar{x}, \bar{y}, \bar{z}, \bar{w})$ is a globally optimal solution to (2.1) if and only if there exists a tuple (s, η, λ, μ) such that the tuple:

$$\left(\bar{x}, \bar{y}, \bar{z}, \bar{w}, \widehat{x}, \widehat{y}, \widehat{z}, \widehat{w}, s, \eta, \lambda, \mu, \widehat{\eta}, \widehat{\lambda}, \widehat{\mu}\right) \tag{2.7}$$

is a globally optimal solution to (2.6). Consequently, the two problems attain identical optimal objective values. □

Analyzing locally optimal solution is more nuanced. Because both problems (2.1) and (2.6) contain binary variables and since the values 0 and 1 are isolated, we say that the tuple $\bar{\boldsymbol{v}} \triangleq (\bar{x}, \bar{y}, \bar{z}, \bar{w})$ is a locally optimal solution to (2.1) if $(\bar{x}, \bar{y}, \bar{z})$ is a locally optimal solution to the problem w fixed at $\bar{w}$. Similarly, tuple (2.7) is a locally optimal solution of (2.6) if it is a locally optimal solution when the binary pair (w, s) is fixed at $(\bar{w}, \bar{s})$.

THEOREM 2.3. Suppose that problem (2.1) is feasible and the following three conditions hold: boundedness of primal variables, piecewise solvability, and existence of Slater points. If a tuple $(\bar{x}, \bar{y}, \bar{z}, \bar{w})$ is a locally optimal solution to (2.1), then there exists a tuple $(\bar{s}, \bar{\eta}, \bar{\lambda}, \bar{\mu})$ such that the tuple:

$$\bar{\boldsymbol{\xi}} \triangleq \left(\bar{x}, \bar{y}, \bar{z}, \bar{w}, \widehat{x}, \widehat{y}, \widehat{z}, \widehat{w}, \bar{s}, \bar{\eta}, \bar{\lambda}, \bar{\mu}, \widehat{\eta}, \widehat{\lambda}, \widehat{\mu}\right) \tag{2.8}$$

is a locally optimal solution to (2.6). In general, the converse does not hold.

Proof. Define $\bar{s} \triangleq (\bar{s}_i)_{i=1}^m$ componentwise by $\bar{s}_i \left\{\begin{array}{ll} = 1 & \text{if } \bar{y}_i > 0 \\ = 0 & \text{if } \bar{z}_i > 0 \\ \in \{0,1\} & \text{if } \bar{y}_i = \bar{z}_i = 0 \end{array}\right\}$. Following the derivation of the problem (2.6), we define the multipliers $(\bar{\eta}, \bar{\lambda}, \bar{\mu})$ and the respective producuts $(\widehat{x}, \widehat{y}, \widehat{z}, \widehat{w})$ and $(\widehat{\eta}, \widehat{\lambda}, \widehat{\mu})$ associated with the pair $(\bar{w}, \bar{s})$. The tuple (2.8) is easily seen to be feasible for (2.6). To show that it is a locally optimal solution to this LPCC, let

$$\boldsymbol{\xi}' \triangleq \left(x', y', z', w', \widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{w}, s', \eta', \lambda', \mu', \widetilde{\eta}, \widetilde{\lambda}, \widetilde{\mu}\right) \tag{2.9}$$

be an arbitrary feasible tuple for (2.6) that is sufficiently close to the tuple (2.8). Since w and s are binary variables, it follows that $w' = \bar{w}$ and $s' = \bar{s}$. It then follows that

$$\begin{aligned} &\widetilde{x} \triangleq x' \otimes \bar{w}, \quad \widetilde{y} \triangleq y' \otimes \bar{w}, \quad \widetilde{z} \triangleq z' \otimes \bar{w}, \quad \widetilde{y} \triangleq w' \otimes \bar{w}, \quad \widetilde{\eta} \triangleq \eta' \otimes \bar{w} \\ &\widetilde{\lambda} \triangleq \lambda' \circ \bar{s}, \quad \text{and } \widetilde{\mu} \triangleq \mu' \circ (\mathbf{1} - \bar{s}), \end{aligned}$$

where $\otimes$ denotes the notation for the Kronecker product of two vectors, and $\circ$ denotes the Hadamard product. Hence, the objective value of (2.6) at the tuple (2.9) is equal to that of (2.1) at $(x', y', z', \bar{w})$, which is also feasible to (2.1). This establish that (2.8) is a locally optimal solution to (2.6). The failure of the converse in general stems from known properties of indefinite QPs, where a local optimum of an LPCC is not necessarily a local optimum of the corresponding QP. □

Our next result provides a necessary and sufficient condition for the tuple (2.8) to be a locally optimal solution to the binary-constrained LPCC (2.6) in terms of a fixed-point property. In light of Theorem 2.3, this condition serves as a necessary (i.e., stationary) condition for a tuple $(\bar{x}, \bar{y}, \bar{z}, \bar{w})$ to be a locally optimal solution to the binary constrained QPCC (2.1). To define the fixed-point property, we let $\mathcal{F}$ denote the feasible region of (2.6) and introduce some index sets associated with the given tuple (2.8). These index sets are used to fix a subset of the binary components of $\bar{w}$ and $\bar{s}$ to either 0 or 1, as well as to fix the complementary conditions for a subset of the pairs $(\bar{y}_i, \bar{u}_i)$, $(\bar{z}_i, \bar{v}_i)$, $(\bar{\eta}_i, \bar{\sigma}^x_i)$, $(\bar{\lambda}_i, \bar{\sigma}^y_i)$, and $(\bar{\mu}_i, \bar{\sigma}^z_i)$ in (2.6), where

$$\left.\begin{cases} \bar{u} \triangleq 2\left[Q_{yx}\bar{x} + Q_{yy}\bar{y} + Q_{yz}\bar{z}\right] + (Q_{yw} + Q_{wy}^\top)\bar{w} + b - B^\top\bar{\eta} + \bar{\lambda} \\ \bar{v} \triangleq 2\left[Q_{zx}\bar{x} + Q_{zy}\bar{y} + Q_{zz}\bar{z}\right] + (Q_{zw} + Q_{wz}^\top)\bar{w} + c - C^\top\bar{\eta} + \bar{\mu} \end{cases}\right\} \text{ dual slacks at } \bar{\boldsymbol{\xi}} \text{ corresponding to } \bar{y} \text{ and } \bar{z}, \text{ respectively;}$$

$$\left.\begin{cases} \bar{\sigma}^x \triangleq A\bar{x} + B\bar{y} + C\bar{z} + D\bar{w} - e \\ \bar{\sigma}^y \triangleq \bar{y} \circ \bar{s} - \bar{y}, \quad \text{and} \quad \bar{\sigma}^z \triangleq \bar{z} \circ (1 - \bar{s}) - \bar{z} \end{cases}\right\} \text{ primal slacks at } \bar{\boldsymbol{\xi}} \text{ with dual variables } \bar{\eta}, \bar{\lambda}, \text{ and } \bar{\mu}, \text{ respectively.}$$

Specifically, let

- $(\bar{\mathcal{I}}^w_0, \bar{\mathcal{I}}^w_1, \bar{\mathcal{I}}^w_c)$ be a partition of $\{1, \cdots, \ell\}$ such that

$$\bar{\mathcal{I}}^w_0 \subseteq \{j \mid \bar{w}_j = 0\}, \ \bar{\mathcal{I}}^w_1 \subseteq \{j \mid \bar{w}_j = 1\}, \ \bar{\mathcal{I}}^w_c \triangleq \{1, \cdots, \ell\} \setminus \left(\bar{\mathcal{I}}^w_0 \cup \bar{\mathcal{I}}^w_1\right); \tag{2.10}$$

- $(\bar{\mathcal{I}}^s_0, \bar{\mathcal{I}}^s_1, \bar{\mathcal{I}}^s_c)$ be a partition of $\{1, \cdots, m\}$ such that

$$\bar{\mathcal{I}}^s_0 \subseteq \{i \mid \bar{s}_i = 0\}, \ \bar{\mathcal{I}}^s_1 \subseteq \{i \mid \bar{s}_i = 1\}, \ \bar{\mathcal{I}}^s_c \triangleq \{1, \cdots, m\} \setminus \left(\bar{\mathcal{I}}^s_0 \cup \bar{\mathcal{I}}^s_1\right); \tag{2.11}$$

- $\bar{\mathcal{I}}^y_+$, $\bar{\mathcal{I}}^u_+$, and $\bar{\mathcal{I}}^{y\,\cup\,u}_c$ be three index sets partitioning $\{1, \cdots, m\}$ such that

$$\bar{\mathcal{I}}^y_+ \subseteq \{i \mid \bar{y}_i > 0\}, \quad \bar{\mathcal{I}}^u_+ \subseteq \{i \mid \bar{u}_i > 0\}, \tag{2.12}$$

  and $\bar{\mathcal{I}}^{y\,\cup\,u}_c \triangleq \{1, \cdots, m\} \setminus \left(\bar{\mathcal{I}}^y_+ \cup \bar{\mathcal{I}}^u_+\right)$; note that $\bar{\mathcal{I}}^{u\,\cup\,y}_c \supseteq \{i \mid \bar{u}_i = \bar{y}_i = 0\}$;
- $\bar{\mathcal{I}}^z_+$, $\bar{\mathcal{I}}^v_+$, and $\bar{\mathcal{I}}^{z\,\cup\,v}_c$ be three index sets partitioning $\{1, \cdots, m\}$ such that

$$\bar{\mathcal{I}}^z_+ \subseteq \{i \mid \bar{z}_i > 0\}, \quad \bar{\mathcal{I}}^v_+ \subseteq \{i \mid \bar{v}_i > 0\} \tag{2.13}$$

  and $\bar{\mathcal{I}}^{v\,\cup\,z}_c \triangleq \{1, \cdots, m\} \setminus \left(\bar{\mathcal{I}}^z_+ \cup \bar{\mathcal{I}}^v_+\right)$; note that $\bar{\mathcal{I}}^{v\,\cup\,z}_c \supseteq \{i \mid \bar{v}_i = \bar{z}_i = 0\}$;
- $\bar{\mathcal{I}}^\eta_+$, $\bar{\mathcal{I}}^{\sigma^x}_+$, and and $\bar{\mathcal{I}}^{\eta\,\cup\,\sigma^x}_c$ be three index sets partitioning $\{1, \cdots, k\}$ such that

$$\bar{\mathcal{I}}^\eta_+ \subseteq \{i \mid \bar{\eta}_i > 0\}, \quad \bar{\mathcal{I}}^{\sigma^x}_+ \subseteq \{i \mid \bar{\sigma}^x_i > 0\}, \tag{2.14}$$

  and $\bar{\mathcal{I}}^{\eta\,\cup\,\sigma^x}_c \triangleq \{1, \cdots, k\} \setminus \left(\bar{\mathcal{I}}^\eta_+ \cup \bar{\mathcal{I}}^{\sigma^x}_+\right)$; note that $\bar{\mathcal{I}}^{\eta\,\cup\,\sigma^x}_c \supseteq \{i \mid \bar{\eta}_i = \bar{\sigma}^x_i = 0\}$
- $\bar{\mathcal{I}}^\lambda_+$, $\bar{\mathcal{I}}^{\sigma^y}_+$, and $\bar{\mathcal{I}}^{\lambda\,\cup\,\sigma^y}_c$ be three index sets partitioning $\{1, \cdots, m\}$ such that

$$\bar{\mathcal{I}}^\lambda_+ \subseteq \{i \mid \bar{\lambda}_i > 0\}, \quad \bar{\mathcal{I}}^{\sigma^y}_+ \subseteq \{i \mid \bar{\sigma}^y_i > 0\}, \tag{2.15}$$

  and $\bar{\mathcal{I}}^{\lambda\,\cup\,\sigma_y}_c \triangleq \{1, \cdots, m\} \setminus \left(\bar{\mathcal{I}}^\lambda_+ \cup \bar{\mathcal{I}}^{\sigma^y}_+\right)$; note that $\bar{\mathcal{I}}^{\lambda\,\cup\,\sigma^y}_c \supseteq \{i \mid \bar{\lambda}_i = \bar{\sigma}^y_i = 0\}$
- $\bar{\mathcal{I}}^\mu_+$, $\bar{\mathcal{I}}^{\sigma^z}_+$, and $\bar{\mathcal{I}}^{\mu\,\cup\,\sigma^z}_c$ be three index sets partitioning $\{1, \cdots, m\}$ such that

$$\bar{\mathcal{I}}^\mu_+ \subseteq \{i \mid \bar{\mu}_i > 0\}, \quad \bar{\mathcal{I}}^{\sigma^z}_+ \subseteq \{i \mid \bar{\sigma}^z_i > 0\} \tag{2.16}$$

  and $\bar{\mathcal{I}}^{\mu\,\cup\,\sigma^z}_c \triangleq \{1, \cdots, m\} \setminus \left(\bar{\mathcal{I}}^\mu_+ \cup \bar{\mathcal{I}}^{\sigma^z}_+\right)$; note that $\bar{\mathcal{I}}^{\mu\,\cup\,\sigma^z}_c \supseteq \{i \mid \bar{\mu}_i = \bar{\sigma}^z_i = 0\}$.

THEOREM 2.4. Let (2.7) be an arbitrary feasible tuple to the binary-constrained LPCC (2.6). The following two statements hold.

(a) For an arbitrary tuple of index sets

$$\left\{ \mathcal{I}_0^w, \mathcal{I}_1^w, \mathcal{I}_0^s, \mathcal{I}_1^s, \mathcal{I}_+^y, \mathcal{I}_+^u, \mathcal{I}_+^z, \mathcal{I}_+^v, \mathcal{I}_+^\eta, \mathcal{I}_+^{\sigma^x}, \mathcal{I}_+^\lambda, \mathcal{I}_+^{\sigma^y}, \mathcal{I}_+^\mu, \mathcal{I}_+^{\sigma^z} \right\} \tag{2.17}$$

satisfying the conditions (2.10)–(2.16), the feasible set of (2.6) appended by the following variable fixings

$$\begin{aligned}
&w_j = 0,\ j \in \bar{\mathcal{I}}_0^w;\ w_j = 1,\ j \in \bar{\mathcal{I}}_1^w;\ s_i = 0,\ \ i \in \bar{\mathcal{I}}_0^s;\ \ s_i = 1,\ i \in \bar{\mathcal{I}}_1^s \\
&u_i = 0,\ i \in \bar{\mathcal{I}}_+^y;\ \ y_i = 0,\ i \in \bar{\mathcal{I}}_+^u;\ \ v_i = 0,\ \ i \in \bar{\mathcal{I}}_+^z;\ z_i = 0,\ i \in \bar{\mathcal{I}}_+^v \\
&\sigma_i^x = 0,\ i \in \bar{\mathcal{I}}_+^\eta;\ \eta_i = 0,\ i \in \bar{\mathcal{I}}_+^{\sigma^x}; \\
&\sigma_i^y = 0,\ \ i \in \bar{\mathcal{I}}_+^\lambda;\ \lambda_i = 0,\ i \in \bar{\mathcal{I}}_+^{\sigma^y};\ \sigma_i^z = 0,\ \ i \in \bar{\mathcal{I}}_+^\mu;\ \mu_i = 0,\ i \in \bar{\mathcal{I}}_+^{\sigma^z},
\end{aligned} \tag{2.18}$$

which we denote by $\mathcal{F}(\bar{\boldsymbol{\xi}})$, must contain $\bar{\boldsymbol{\xi}}$ and satisfies the following three properties:

(i) there exists a neighborhood $\mathcal{N}$ of $\bar{\boldsymbol{\xi}}$ such that $\mathcal{N} \cap \mathcal{F} \subseteq \mathcal{F}(\bar{\boldsymbol{\xi}})$,

(ii) the neighborhood $\mathcal{N}$ in (i) can be chosen so that for any tuple $\boldsymbol{\xi}' \in \mathcal{N} \cap \mathcal{F}$, the tuple $\bar{\boldsymbol{\xi}} + \tau(\boldsymbol{\xi}' - \bar{\boldsymbol{\xi}})$ remains in the set $\mathcal{F}(\bar{\boldsymbol{\xi}})$ for all $\tau > 0$ sufficiently small;

(iii) if the index tuple (2.17) is such that the inclusions in (2.10)–(2.16) all hold as equalities, then for any tuple $\boldsymbol{\xi}' \in \mathcal{F}$, the tuple $\bar{\boldsymbol{\xi}} + \tau(\boldsymbol{\xi}' - \bar{\boldsymbol{\xi}})$ remains in the set $\mathcal{F}(\bar{\boldsymbol{\xi}})$ for all $\tau > 0$ sufficiently small.

(b) The tuple (2.7) is a locally optimal solution to the binary-constrained LPCC (2.6) if and only if there exists a tuple of index sets (2.17) satisfying the conditions (2.10)–(2.16) such that the tuple (2.7) is a globally optimal solution to the binary-constrained LPCC (2.6) with the variable fixings (2.18), which we term the *restricted binary-constrained LPCC* associated with the tuple (2.7).

Proof. The adjective "restricted" binary-constrained LPCC is justified because $\mathcal{F}(\bar{\boldsymbol{\xi}})$ is clearly a subset of $\mathcal{F}$. The fact that $\mathcal{F}(\bar{\boldsymbol{\xi}})$ contains $\bar{\boldsymbol{\xi}}$ is also clear. The first two properties (i) and (ii) of $\mathcal{F}(\bar{\boldsymbol{\xi}})$ can be proved based on the observations that for a tuple $\boldsymbol{\xi}'$ sufficiently close to $\bar{\boldsymbol{\xi}}$, we must have $w' = \bar{w}$, $s' = \bar{s}$, and for any pair of complementary variables, a positive member of the pair in $\bar{\boldsymbol{\xi}}$ must remain positive in $\boldsymbol{\xi}'$. The third property (iii) holds because $\bar{\boldsymbol{\xi}}+\tau(\boldsymbol{\xi}'-\bar{\boldsymbol{\xi}})$ remains in the set $\mathcal{F}(\bar{\boldsymbol{\xi}})$ must be close to $\bar{\boldsymbol{\xi}}$ and belong to $\mathcal{F}$ for all $\tau > 0$ sufficiently small since all inclusions in (2.10)–(2.16) hold as equalities; therefore property (ii) can be applied to complete the proof of (iii).

To prove the "only if" assertion in statement (b), it suffices to choose the index sets (2.17) such that the inclusions in (2.10)–(2.16) all hold as equalities. To show that $\bar{\boldsymbol{\xi}}$ has the desired global optimality, let $\boldsymbol{\xi}' \in \mathcal{F}(\bar{\boldsymbol{\xi}})$ be arbitrary. By property (iii), the tuple $\bar{\boldsymbol{\xi}}+\tau(\boldsymbol{\xi}'-\bar{\boldsymbol{\xi}})$ remains in $\mathcal{F}(\bar{\boldsymbol{\xi}})$ for all $\tau > 0$ sufficiently small. Therefore, by the local optimality of $\bar{\boldsymbol{\xi}}$, we have

$$\theta(\bar{\boldsymbol{\xi}} + \tau(\boldsymbol{\xi}' - \bar{\boldsymbol{\xi}})) \geq \theta(\bar{\boldsymbol{\xi}})$$

for all $\tau > 0$ sufficiently small. By the linearity of θ in its argument, we deduce $\theta(\boldsymbol{\xi}') \geq \theta(\bar{\boldsymbol{\xi}})$ as desired. Conversely, suppose that there is an index tuple (2.17) satisfying the conditions (2.10)–(2.16) for which $\bar{\boldsymbol{\xi}}$ is a globally optimal solution to the (restricted) binary constrained LPCC (2.6) subject to the additional variable fixings (2.18). Property (i) then implies $\bar{\boldsymbol{\xi}}$ is an optimal solution to (2.6) within the neighborhood $\mathcal{N}$. $\square$

Before discussing the role of Theorem 2.4 in the progressive algorithm to be presented later for solving the binary-constrained QPCC (2.1), we show in the next section the breadth of this problem as a unified formulation for many binary-constrained quadratic-defined optimization problems.

3. Unification of Problem Classes. The formulation (2.1) includes many special cases; several of these are fairly straightforward to demonstrate while others are not. It should be pointed out that even for the straightforward classes, their reformulation as an equivalent binary-constrained LPCC is not known in the literature when there are binary variables present in the given problem. In principle, the resulting binary-constrained LPCC overcomes a major difficulty in the design of rigorous algorithms for solving the original binary-constrained nonconvex quadratic-defined problem, to a certain degree. Indeed, the derived LPCC is computationally effective for problems of small-to-medium sizes, credit to the state-of-the-art IP solvers (see numerical results to be reported later); nevertheless, for problems of larger sizes, a straightforward solution to the LPCC formulation by state-of-the-art implementation of IP methods, as by the acclaimed GUROBI solver, is handicapped by the solver's ineffectiveness in handling the large number of integer variables and auxiliary constraints. This is where the PIP approach (to be presented in the next section) is promising as a practical way to remedy the lack of efficiency of the IP solver(s) for solving such problems.

For all the problem classes discussed in this section, the objective function is nonconvex. We start with two well-known problems followed by some less known ones whose formulation as a binary-constrained QPCC requires some manipulations.

- Binary-constrained quadratic programs (i.,e., $m = 0$).

- QP with complementarity constraints without binary variables (i.e., $\ell = 0$).

- Cardinality constrained QP (CCQP): for simplicity, we drop the binary variable w in presenting this class of problems,

$$\begin{aligned}
&\underset{(x,y)\in\mathbb{R}^{n+m}}{\textbf{minimize}} \; \begin{pmatrix} x \\ y \end{pmatrix}^{\top} \underbrace{\begin{bmatrix} Q_{xx} & Q_{xy} \\ Q_{yx} & Q_{yy} \end{bmatrix}}_{\text{symmetric}} \begin{pmatrix} x \\ y \end{pmatrix} + a^{\top}x + b^{\top}y \\
&\textbf{subject to } Ax + By \geq e \;\; \textbf{and} \;\; \sum_{j=1}^{m} |\, y_j \,|_0 \leq K \; \left(\begin{array}{l} \text{integer constraint} \\ \text{with bounded } y \end{array} \right),
\end{aligned}$$

where $|t|_0 \triangleq \begin{cases} 1 \text{ if } t \neq 0 \\ 0 \text{ if } t = 0 \end{cases}$. With $M > 0$ being an upper bound for $|y_j|$, the CCQP is equivalent to the following (mixed) binary-constrained QP

$$\begin{array}{lll}
\underset{(x,y)\,\in\,\mathbb{R}^{n+m};\, t\,\in\,\{\,0,1\,\}^m}{\textbf{minimize}} & \multicolumn{2}{l}{\begin{pmatrix} x \\ y \end{pmatrix}^{\top} \begin{bmatrix} Q_{xx} & Q_{xy} \\ Q_{yx} & Q_{yy} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + a^{\top}x + b^{\top}y} \\
\textbf{subject to} & Ax + By \geq e, & \displaystyle\sum_{j=1}^{m} t_j \leq K \\
\textbf{and} & |\, y_j \,| \leq M\, t_j, & j = 1, \cdots, m.
\end{array}$$

- Quadratic-constrained quadratic program (QCQP) failing the Slater constraint qualification:

$$\left[\begin{array}{lll}
\underset{x\in\mathbb{R}^n}{\textbf{minimize}} & x^{\top}Q^{\,0}x + x^{\top}q^{\,0} & \\
\textbf{subject to} & Ax \geq e & \\
\textbf{and} & x^{\top}Q^{\,i}x + x^{\top}q^{\,i} \leq b_i, & i = 1, \cdots, m \\
\text{where} & \text{each } Q^{\,i} \text{ is symmetric} &
\end{array}\right] \quad \text{assumed feasible}$$

$$\Longleftrightarrow \underbrace{\left[\begin{array}{l} \underset{x \in \mathbb{R}^n}{\textbf{minimize}} \; x^\top Q^0 x + x^\top q^0 \\ \textbf{subject to} \, x \in \underset{x' \,:\, Ax' \geq e}{\textbf{argmin}} \; \displaystyle\sum_{i=1}^m \left[(x')^\top Q^i x' + (x')^\top q^i - b_i \right] \end{array}\right]}_{\substack{\text{a bilevel QP for which the optimal obj. value} \\ \text{of the lower-level problem is equal to zero}}} \quad \begin{array}{l} \text{assuming that} \\ Ax \geq e \text{ implies} \\ x^\top Q^i x + x^\top q^i \geq b_i \\ \forall\, i = 1, \cdots, m \end{array}$$

$$\Longleftrightarrow \left[\begin{array}{ll} \underset{x \in \mathbb{R}^n;\, \lambda}{\textbf{minimize}} \; x^\top Q^0 x + x^\top q^0 & \\ \textbf{subject to} \, 0 = \displaystyle\sum_{i=1}^m \left[2\, Q^i x + q^i \right] - A^\top \lambda & \\ \qquad\qquad 0 \leq \lambda \perp Ax - e \geq 0 & \\ \textbf{and} \qquad \tfrac{1}{2} \left[\displaystyle\sum_{i=1}^m (q^i)^\top x + \lambda^\top e \right] = \displaystyle\sum_{i=1}^m b_i & \end{array}\right] \quad \begin{array}{l} \text{optimum objective value of} \\ \text{lower-level quadratic program} \\ \text{is equal to zero.} \end{array}$$

An open problem: It is not known at this time whether a QCQP whose constraints satisfy the Slater constraint qualification can be formulated as an equivalent LPCC.

- Affine variational inequality (AVI) constrained QP:

$$\left[\begin{array}{ll} \underset{(x,y) \in \mathbb{R}^{n+m}}{\textbf{minimize}} & \begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{bmatrix} Q_{xx} & Q_{xy} \\ Q_{yx} & Q_{yy} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + a^\top x + b^\top y \\ \textbf{subject to} & Ax + By \geq e; \quad A'x + B'y \geq e' \\ \textbf{and} & [\, c + Ex + Q'y \,]^\top (y' - y) \geq 0, \quad \text{for all } y' \text{ satisfying } A'x + B'y' \geq e' \\ \text{where} & Q' \text{ is not necessarily symmetric} \end{array}\right]$$

$$\Longleftrightarrow \left[\begin{array}{ll} \underset{(x,y) \in \mathbb{R}^{n+m}}{\textbf{minimize}} & \begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{bmatrix} Q_{xx} & Q_{xy} \\ Q_{yx} & Q_{yy} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + a^\top x + b^\top y \\ \textbf{subject to} & Ax + By \geq e \\ \textbf{and} & \left\{ \begin{array}{l} 0 = c + Ex + Q'y - (B')^\top \lambda \\ 0 \leq \lambda \perp A'x + B'y - e' \geq 0 \end{array} \right. \end{array}\right] \quad \begin{array}{l} \text{equivalent linear} \\ \text{complementarity} \\ \text{formulation of AVI} \end{array}$$

$$\Longleftrightarrow \underbrace{\left[\begin{array}{ll} \underset{(x,y) \in \mathbb{R}^{n+m}}{\textbf{minimize}} & \begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{bmatrix} Q_{xx} & Q_{xy} \\ Q_{yx} & Q_{yy} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + a^\top x + b^\top y \\ \textbf{subject to} & Ax + By \geq e \\ \textbf{and} & \left\{ \begin{array}{l} y \in \underset{y' \in \mathbb{R}^m}{\textbf{argmin}} \quad (c + Ex)^\top y' + \tfrac{1}{2} (y')^\top Q' y' \\ \qquad \textbf{subject to} \; A'x + B'y' \geq e' \end{array} \right. \end{array}\right]}_{\text{assuming that } Q' \text{ is symmetric positive semidefinite}}.$$

The AVI provides a necessary and sufficient formulation for a non-cooperative game with a finite number of individual decision-making players each solving a convex quadratic program;

thus the AVI-constrained QP models a leader-follower game of the Stackelberg kind with a quadratic upper objective and lower-level players solving convex QPs.

- Quadratic value-function optimization:

$$\left[\begin{array}{ll} \underset{x\in P}{\textbf{minimize}} & x^\top Qx + q^\top x + v(x), \qquad \text{where } P \text{ is a polyhedron} \\ \text{and } v(x) & = \underset{y}{\textbf{minimum}}\ (c+Ex)^\top y' + y^\top Q'y \\ & \quad\ \textbf{subject to}\ \ A'x + B'y \geq e' \end{array}\right]$$

$$\Longleftrightarrow \left[\begin{array}{ll} \underset{x\in P;\, y}{\textbf{minimize}} & x^\top Qx + q^\top x + \tfrac{1}{2} \underbrace{\left[(c+Ex)^\top y + \lambda^\top (A'x - e')\right]}_{\text{a quadratic function}} \\ \textbf{subject to}\ 0 & = c + Ex + 2Q'y - (B')^\top \lambda \\ \textbf{and} \qquad\ \ 0 & \leq \lambda \perp A'x + B'y - e' \geq 0 \end{array}\right].$$

Note that in the last two problems, both lower-level QPs are linearly bi-parameterized by the upper-level variable x; i.e., both the linear term in the objective and that in the constraint "right-hand" side are linearly parameterized by x. Note further that in the last problem, the second-level variable y enters into the first-level objective function only through the value-function $v(x)$, a noteworthy remark about the resulting QPCC is that it is derived from a lower-level minimization problem that is not required to be convex.

- Sum of products of PA functions:

The value-function $v(x)$ is piecewise linear-quadratic (PLQ) [10]. A large class of PLQ functions is given by the sum of finitely many products of piecewise affine (PA) functions:

$$r(x) \triangleq \sum_{i=1}^{I} p_i(x)\, q_i(x),$$

where all the functions $p_i(x)$ and $q_i(x)$ are piecewise functions of the variable $x \in \mathbb{R}^n$. By a known result [26], every PA function can be written as the difference of two convex piecewise functions:

$$p_i(x) = \underbrace{\max_{1\leq j\leq J_i^p} \left[(a^{ij})^\top x + \alpha_{ij} \right]}_{\text{denoted } t_i^p} - \underbrace{\max_{1\leq \ell\leq L_i^p} \left[(b^{i\ell})^\top x + \beta_{i\ell} \right]}_{\text{denoted } s_i^p}$$

$$q_i(x) = \underbrace{\max_{1\leq j\leq J_i^q} \left[(c^{ij})^\top x + \gamma_{ij} \right]}_{\text{denoted } t_i^q} - \underbrace{\max_{1\leq \ell\leq L_i^q} \left[(d^{i\ell})^\top x + \delta_{i\ell} \right]}_{\text{denoted } s_i^q},$$

for some vectors a^{ij}, $b^{i\ell}$, c^{ij}, and $d^{i\ell}$ and scalars α_{ij}, $\beta_{i\ell}$, γ_{ij}, and $\delta_{i\ell}$ and positive integers J_i^p, L_i^p, J_i^q, and L_i^q. Hence,

$$r(x) \triangleq \sum_{i=1}^{I} \underbrace{[\, t_i^p - s_i^p \,]\, [\, t_i^q - s_i^q \,]}_{\text{a quadratic function}}$$

subject to (assuming that x-variable is bounded)

$$\begin{array}{ll}
0 \leq t_i^p - \left[\, (a^{ij})^\top x + \alpha_{ij} \, \right] \leq M \, u_{ij}^p, & \forall \, j = 1, \cdots, J_i^p \\
0 \leq s_i^p - \left[\, (b^{i\ell})^\top x + \beta_{i\ell} \, \right] \leq M \, v_{i\ell}^p, & \forall \, \ell = 1, \cdots, L_i^p \\
0 \leq t_i^q - \left[\, (c^{ij})^\top x + \gamma_{ij} \, \right] \leq M \, u_{ij}^q, & \forall \, j = 1, \cdots, J_i^q \\
0 \leq s_i^q - \left[\, (d^{i\ell})^\top x + \delta_{i\ell} \, \right] \leq M \, v_{i\ell}^q, & \forall \, \ell = 1, \cdots, L_i^q
\end{array}$$

$$\sum_{j=1}^{J_i^p} u_{ij}^p \leq J_i^p - 1; \quad \sum_{\ell=1}^{L_i^p} v_{i\ell}^p \leq L_i^p - 1; \quad \sum_{j=1}^{J_i^q} u_{ij}^q \leq J_i^q - 1; \quad \sum_{\ell=1}^{L_i^q} v_{i\ell}^q \leq L_i^q - 1;$$

and u_{ij}^p, $v_{i\ell}^p$, u_{ij}^q, and $v_{i\ell}^q$ are all $\{0,1\}$-variables. Hence a minimization problem with $r(x)$ as the objective can be formulated as a binary-constrained QP and is thus amenable to the treatment by the LPCC reformulation.

3.1. A counterexample. The successful derivation of the Bi-LPCC (2.6) as an equivalent formulation of the given Bi-QPCC (2.1) can be attributed to the "pulling-out" of the binary variables w. In what follows, we provide a counterexample to illustrate that such a pull-out is not valid if the binary variables occur in a lower-level problem. This example therefore raises the question of whether a (general) bilevel QP admits an equivalent binary-constrained LPCC formulation; at this time, this is possible only for three cases: (i) the lower-level problem is convex, (ii) the lower-level variable appears in the upper-level only through the (lower-level) value function; or (iii) for a nonconvex lower-level problem, one restricts the associated lower-level constraints to be the stationarity conditions of the problem.

Consider a trivial bilevel quadratic program with the lower-level being the minimization of a concave quadratic function:

$$\underset{x \in \mathbb{R}}{\textbf{minimize}} \ \frac{1}{2} x^2 \quad \textbf{subject to} \quad x \in \underset{0 \leq y \leq 1}{\textbf{argmin}} \ -\frac{1}{2} y^2.$$

There is a unique feasible solution: $x^* = 1$, which is therefore the unique optimal solution to the problem. Writing out the LPCC formulation of the lower-level QP and introducing binary variables to express the complementary slackness conditions, we obtain the reformulation of the bilevel program where the lower-level problem is now a mixed-integer linear program:

$$\begin{array}{ll}
\underset{x \in \mathbb{R}}{\textbf{minimize}} & \frac{1}{2} x^2 \\[1em]
\textbf{subject to} & \left\{ \begin{array}{lll}
(x, \lambda) \in & \underset{y, \lambda', z, z'}{\textbf{argmin}} & -\lambda' \\
& \textbf{subject to} & 0 \leq y \leq z \\
& & 0 \leq -y + \lambda' \leq 1 - z \\
& & 0 \leq \lambda' \leq z' \\
& & 0 \leq 1 - y \leq 1 - z' \\
& \textbf{and} & z, z' \in \{0, 1\}.
\end{array} \right\}.
\end{array}$$

Pulling out the two binary variables z and z' from the lower-level problem yields a modified lower-level problem which is a linear program in (y, λ') parameterized the pair (z, z'):

$$\operatorname*{\mathbf{minimize}}_{z,\, z' \in \{0,1\}} \left[\begin{array}{ll} \operatorname*{\mathbf{minimize}}_{x \in \mathbb{R}} & \frac{1}{2} x^2 \\ \mathbf{subject\ to} & \left\{ \begin{array}{lll} (x, \lambda) \in & \operatorname*{\mathbf{argmin}}_{y,\, \lambda'} & -\lambda' \\ & \mathbf{subject\ to} & 0 \leq y \leq z \\ & & 0 \leq -y + \lambda' \leq 1 - z \\ & & 0 \leq \lambda' \leq z' \\ & & 0 \leq 1 - y \leq 1 - z'. \end{array} \right\} \end{array} \right].$$

One can check that with $(z, z') = (1, 0)$, the pair $(y, \lambda') = (0, 0)$ is the only feasible, thus optimal, solution to the inner minimization problem, and therefore $x = 0$ is optimal to the outer minimization problem since its objective is nonnegative.

4. The Progressive Integer Programming Method. We employ the PIP method of [28] to (2.6), which is the full complementarity-constrained, mixed-integer program (FMIP) formulation of the Bi-QPCC (2.1). One feature that distinguishes (2.6) from the (much simpler) QP-derived LPCC treated in the reference is that (2.6) carries two kinds of disjunction. One kind is the integrality of the binaries $w \in \{0, 1\}^\ell$ and $s \in \{0, 1\}^m$, and the other kind is the five families of complementary pairs:

$$\mathcal{P} \triangleq \left\{ \underbrace{(\eta, \sigma^x)}_{2k \text{ variables}}, \underbrace{(\lambda, \sigma^y),\ (\mu, \sigma^z),\ (u, y),\ (v, z)}_{\text{each pair has } 2m \text{ variables}} \right\}, \tag{4.1}$$

Together, there are $\ell + k + 5m$ disjunctions; solved inclusively, they can present great difficulty for a straightforward application of state-of-the-art IP methodology (this will be confirmed subsequently in the computational results). Instead, PIP replaces the FMIP by a sequence of restricted problems, in each of which a controlled number of the disjunctions is fixed at an incumbent, so that every subproblem carries only a fraction of the whole set of disjunctions. We describe below the restricted Bi-LPCC being solved at each iteration, given a feasible point and fixing plans for the family of variables in $\mathcal{P}$.

4.1. The restricted binary-constrained LPCC. Let $\bar{\boldsymbol{\xi}}$ be a feasible tuple of (2.6), composed of the primal variables $(\bar{x}, \bar{y}, \bar{z}, \bar{w})$, the complementarity encoder $\bar{s}$, multipliers $(\bar{\eta}, \bar{\lambda}, \bar{\mu}, \bar{u}, \bar{v})$, and the primal slacks $(\bar{\sigma}^x, \bar{\sigma}^y, \bar{\sigma}^z)$. Feasibility of $\bar{\boldsymbol{\xi}}$ resolves every disjunction, meaning that each $\bar{w}_j$ and each $\bar{s}_i$ equals 0 or 1, and at least one member vanishes in each complementary pair in the family (4.1).

A fixing plan $\bar{\boldsymbol{\mathcal{I}}}$ at $\bar{\boldsymbol{\xi}}$ records, for each disjunction, whether to commit the incumbent's resolution. It is the tuple of index sets (2.17), each one being a subset of the coordinates at which $\bar{\boldsymbol{\xi}}$ supplies that resolution (conditions (2.10)–(2.16)):

$$\begin{aligned} &\bar{\mathcal{I}}_0^w \subseteq \{ j \mid \bar{w}_j = 0 \}, \quad \bar{\mathcal{I}}_1^w \subseteq \{ j \mid \bar{w}_j = 1 \}; \\ &\bar{\mathcal{I}}_0^s \subseteq \{ i \mid \bar{s}_i = 0 \}, \quad \bar{\mathcal{I}}_1^s \subseteq \{ i \mid \bar{s}_i = 1 \}; \\ &\bar{\mathcal{I}}_+^a \subseteq \{ i \mid \bar{a}_i > 0 \}, \quad \bar{\mathcal{I}}_+^b \subseteq \left\{ i \mid \bar{b}_i > 0 \right\} \quad \text{for each pair } (a, b) \in \mathcal{P}. \end{aligned} \tag{4.2}$$

Committing the multiplier side $\bar{\mathcal{I}}_+^a$ of a pair (a, b) drives its partner b to zero, and committing the partner side $\bar{\mathcal{I}}_+^b$ drives a to zero. The uncommitted coordinates form the central sets: $\bar{\mathcal{I}}_c^w$ and $\bar{\mathcal{I}}_c^s$ for the two binary families, and $\bar{\mathcal{I}}_c^{(a,b)}$ for the pair (a, b). The latter necessarily contains every doubly degenerate index, one for which $\bar{a}_i = \bar{b}_i = 0$ that is not resolved by the incumbent.

The restricted binary-constrained LPCC $R(\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\mathcal{I}}})$ is (2.6) augmented by the variable fixings (2.18) that the plan $\bar{\boldsymbol{\mathcal{I}}}$ prescribes. The restricted problem depends on both the incumbent and the plan, and distinct plans at the same $\bar{\boldsymbol{\xi}}$ yield distinct restricted problems:

$$
\begin{array}{rll}
R(\bar{\boldsymbol{\xi}}, \bar{\mathcal{I}}): & \underset{\boldsymbol{\xi} \in \mathcal{F}_{(2.6)}}{\textbf{minimize}} & \theta(\boldsymbol{\xi}) \\
& \textbf{subject to} & w_j = 0,\ j \in \bar{\mathcal{I}}_0^w; \quad w_j = 1,\ j \in \bar{\mathcal{I}}_1^w \\
& & s_i = 0,\ i \in \bar{\mathcal{I}}_0^s; \quad s_i = 1,\ i \in \bar{\mathcal{I}}_1^s \\
& & \sigma_i^x = 0,\ i \in \bar{\mathcal{I}}_+^\eta; \quad \eta_i = 0,\ i \in \bar{\mathcal{I}}_+^{\sigma^x} \\
& & \sigma_i^y = 0,\ i \in \bar{\mathcal{I}}_+^\lambda; \quad \lambda_i = 0,\ i \in \bar{\mathcal{I}}_+^{\sigma^y} \\
& & \sigma_i^z = 0,\ i \in \bar{\mathcal{I}}_+^\mu; \quad \mu_i = 0,\ i \in \bar{\mathcal{I}}_+^{\sigma^z} \\
& & y_i = 0,\ i \in \bar{\mathcal{I}}_+^u; \quad u_i = 0,\ i \in \bar{\mathcal{I}}_+^y \\
& & z_i = 0,\ i \in \bar{\mathcal{I}}_+^v; \quad v_i = 0,\ i \in \bar{\mathcal{I}}_+^z
\end{array}
\tag{4.3}
$$

where $\mathcal{F}_{(2.6)}$ is the feasible set of (2.6); write $\mathcal{F}(\bar{\boldsymbol{\xi}}, \bar{\mathcal{I}}) \subseteq \mathcal{F}_{(2.6)}$ for the feasible set of $R(\bar{\boldsymbol{\xi}}, \bar{\mathcal{I}})$, consisting of the tuples of $\mathcal{F}_{(2.6)}$ that meet the fixings above. Three facts make (4.3) the desirable workhorse.

(R1) Simplification/Size reduction. Fixing w_j removes it from the integer set and naturally simplifies every product carrying it: the lifted variables w_{ij}, x_{ij}, y_{ij}, z_{ij}, and η_{ij} collapse to 0 (if $\bar{w}_j = 0$) or to the plain variable (if $\bar{w}_j = 1$), thus removing the corresponding McCormick inequalities. Fixing s_i likewise simplifies $\widehat{\lambda}_i$ and $\widehat{\mu}_i$ and the corresponding McCormick inequalities. Thus (4.3) carries $|\bar{\mathcal{I}}_c^w| + |\bar{\mathcal{I}}_c^s|$ binary variables and $\sum_{(a,b)\in\mathcal{P}} |\bar{\mathcal{I}}_c^{(a,b)}|$ complementarity conditions, against $\ell + m$ and $k + 4m$ respectively in the FMIP.

(R2) Incumbent feasibility, hence nonincreasing objective. For every plan $\bar{\boldsymbol{\mathcal{I}}}$ the fixings are derived from the current incumbent $\bar{\boldsymbol{\xi}}$, so that $\bar{\boldsymbol{\xi}} \in \mathcal{F}(\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\mathcal{I}}})$; hence the optimal value of $R(\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\mathcal{I}}})$ never exceeds $\theta(\bar{\boldsymbol{\xi}})$, and every iterate is feasible to (2.6) since $\mathcal{F}(\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\mathcal{I}}}) \subseteq \mathcal{F}_{(2.6)}$. The objective is therefore nonincreasing along the PIP iterates.

(R3) A local-optimality certificate. By Theorem 2.4(a)(i), any plan $\bar{\boldsymbol{\mathcal{I}}}$ (4.2) admits a neighborhood $\mathcal{N}$ of $\bar{\boldsymbol{\xi}}$ with $\mathcal{N} \cap \mathcal{F} \subseteq \mathcal{F}(\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\mathcal{I}}})$: locally the committed binary variables cannot change and the strictly positive member of each committed pair stays positive (forcing its partner to 0 in $\mathcal{F}_{(2.6)}$), so no feasible direction of (2.6) is lost. Hence if $\bar{\boldsymbol{\xi}}$ is a global optimum of $R(\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\mathcal{I}}})$, it is a local optimum of (2.6) (Theorem 2.4(b)); and, by Theorem 2.3, its projection $(\bar{x}, \bar{y}, \bar{z}, \bar{w})$ satisfies the necessary condition for local optimality of (2.1). The plan needs not commit every resolved disjunction for this, which is exactly why a cheap, heavily-fixed subproblem can still certify local optimality.

4.2. Progressive control of the committed fraction. The PIP method trades the cost of $R(\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\mathcal{I}}})$ against its effectiveness by a single parameter, the committed fraction $\rho \in (0, 1]$. Given $\bar{\boldsymbol{\xi}}$ and ρ, a fixing rule builds the plan $\bar{\boldsymbol{\mathcal{I}}}(\bar{\boldsymbol{\xi}}, \rho)$, committing in each family the ρ-fraction that is resolved with the widest margin; this is the analogue of [28, Eq. (19)]. For the five pairs, rank by the magnitude of the positive member: with $\mathcal{J}^y \triangleq \{i : \bar{y}_i > 0\}$,

$$
\bar{\mathcal{I}}_+^y = \left\{ \text{indices of the } \lfloor \rho\, |\mathcal{J}^y| \rfloor \text{ largest } \bar{y}_i,\ i \in \mathcal{J}^y \right\},
\tag{4.4}
$$

where $\lfloor \bullet \rfloor$ is the floor of a scalar; and similarly for $\bar{\mathcal{I}}_+^z$, $\bar{\mathcal{I}}_+^{\sigma^x}$, $\bar{\mathcal{I}}_+^{\sigma^y}$, $\bar{\mathcal{I}}_+^{\sigma^z}$, $\bar{\mathcal{I}}_+^u$, $\bar{\mathcal{I}}_+^v$, $\bar{\mathcal{I}}_+^\eta$, $\bar{\mathcal{I}}_+^\lambda$, $\bar{\mathcal{I}}_+^\mu$ off the respective variables $\bar{z}$, $\bar{\sigma}^x$, $\bar{\sigma}^y$, $\bar{\sigma}^z$, $\bar{u}$, $\bar{v}$, $\bar{\eta}$, $\bar{\lambda}$, and $\bar{\mu}$. The binaries w are never committed (see discussion in the paragraph before the next section): every w_j remains a variable of each subproblem. The binaries s follow the committed rules: the index i joins $\bar{\mathcal{I}}_1^s$ when $i \in \bar{\mathcal{I}}_+^y$, joins $\bar{\mathcal{I}}_0^s$ when $i \in \bar{\mathcal{I}}_+^z$, and stays uncommitted otherwise; at most one of the first two can occur, since $\bar{y}_i$ and $\bar{z}_i$ are complementary. Committed indices are frozen as in (4.3); everything else, and every degenerate index, is left uncommitted.

The overall progressive control mirrors the strategy in [28]. Fix a scalar $\alpha \in (0,1)$, a floor $\rho_{\min} \in (0,1)$, an initial $\rho_0 \in (\rho_{\min}, 1]$ (e.g. $\rho_0 = 0.8$), and a cap $r_{\max}$ (e.g. 3) on how long ρ may stay put before being updated. After each solve of (4.3): if there is improvement in objective, keep ρ unchanged, but lower it only after $r_{\max}$ consecutive improvements; if there is no improvement, reduce ρ by α at once, freeing more disjunctions so that the next subproblem can reach farther; once ρ has been decreased to the floor, it is held there, and the algorithm stops at the first solve that then fails to improve. Because every solve leaves the incumbent feasible and no worse (by (R2)), and because each subproblem is a genuine integer/complementarity program in only its central variables, PIP bypasses the FMIP entirely while retaining, at termination, the certificate of (R3): the loop exits only through a solve that fails to improve, and at such a solve $\bar{\boldsymbol{\xi}}$ is feasible for $R(\bar{\boldsymbol{\xi}}, \bar{\mathcal{I}})$ by (R2), so the optimal value is at most $\theta(\bar{\boldsymbol{\xi}})$, while the absence of improvement makes it at least $\theta(\bar{\boldsymbol{\xi}})$; thus the two values agree, so $\bar{\boldsymbol{\xi}}$ is itself a global optimum of its own restricted problem and (R3) certifies it. We summarize this termination in Proposition 4.1.

Algorithm 4.1 PIP($\rho_{\min}$) for the binary-constrained LPCC (2.6)

Input: feasible $\bar{\boldsymbol{\xi}}$; parameters $\rho_0 \in (\rho_{\min}, 1]$, $\alpha \in (0,1)$, and $r_{\max} \in \mathbb{N}_{++}$.
$\rho \leftarrow \rho_0$ and $r \leftarrow 0$.
repeat
  build the plan $\bar{\boldsymbol{\mathcal{I}}} = \bar{\boldsymbol{\mathcal{I}}}(\bar{\boldsymbol{\xi}}, \rho)$ satisfying (4.2) at fraction ρ via (4.4) and the binary rule
  $\boldsymbol{\xi}^\star \leftarrow$ solve $R(\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\mathcal{I}}})$ in (4.3) ▷ warm-started at $\bar{\boldsymbol{\xi}}$
  if $\theta(\boldsymbol{\xi}^\star) < \theta(\bar{\boldsymbol{\xi}})$ **then** ▷ improvement: keep ρ unchanged, up to $r_{\max}$ times
    $\bar{\boldsymbol{\xi}} \leftarrow \boldsymbol{\xi}^\star, \;\; r \leftarrow r+1$
    if $r = r_{\max}$ **then** $\rho \leftarrow \max(\rho - \alpha, \rho_{\min}), \;\; r \leftarrow 0$
    end if
  else if $\rho \leq \rho_{\min}$ **then** ▷ no improvement at the floor: certified
    return $\bar{\boldsymbol{\xi}}$
  else ▷ no improvement: enrich the next subproblem
    $\rho \leftarrow \max(\rho - \alpha, \rho_{\min}), \;\; r \leftarrow 0$
  end if
until terminated

4.3. Finite termination and local optimality. The theoretical support of the above algorithm is summarized in the result below.

PROPOSITION 4.1. Assume the hypotheses of Theorem 2.4 and that every subproblem (4.3) is solved to global optimality. Then Algorithm 4.1 terminates finitely with a non-increased incumbent objective, and the returned $\bar{\boldsymbol{\xi}}$ at termination is a locally optimal solution to (2.6); consequently $(\bar{x}, \bar{y}, \bar{z}, \bar{w})$ satisfies the necessary condition for the local optimality of (2.1).

Proof. Monotonicity is by (R2). Above the floor the cap forces progress: ρ is lowered by α at least once every $r_{\max}$ solves, so it reaches $\rho_{\min}$ within $\lceil (\rho_0 - \rho_{\min})/\alpha \rceil \, r_{\max}$ solves. At the floor $\max(\rho - \alpha, \rho_{\min})$, ρ is left unchanged, and termination rests on the objective instead: each solve there either fails to improve, in which case the algorithm stops, or lowers $\theta(\bar{\boldsymbol{\xi}})$ strictly. The latter can occur only finitely often, since the fixings in (4.3) depend on $\bar{\boldsymbol{\xi}}$ only through $\bar{\boldsymbol{\mathcal{I}}}$ and the admissible plans are finite in number, so the attainable objective values form a finite set. The loop therefore halts, and only through the no-improvement branch at $\rho = \rho_{\min}$. Writing $\bar{\boldsymbol{\mathcal{I}}}$ for the plan of that final solve, $\bar{\boldsymbol{\xi}}$ is feasible for $R(\bar{\boldsymbol{\xi}}, \bar{\mathcal{I}})$ by (R2) and the solve found nothing better, so it attains the optimal value there. By Theorem 2.4(a)(i),(b) it is then a local optimum of (2.6), and the last claim is Theorem 2.3. ∎

An open research problem: The two binary vectors w and s are of different kinds; the former is from the original Bi-QPCC (2.1) whereas the latter is induced by the complementarity conditions: $0 \le y \perp z \ge 0$ where y and z are continuous variable. As such, the binary variables in s can be committed (or uncommitted) based on the rule (4.4) for the y and z variables. At this time, we do not have a good rule to commit the original binary variables in w. More research is needed in this direction to further enhance the PIP method for such binary variables.

5. Numerical Experiments. This section evaluates the PIP method of Section 4 on a concrete member of the class (2.1). We first introduce the working example and place it in Bi-LPCC form, then derive in closed form the multiplier bounds that its McCormick linearization requires. The experimental design and the results follow.

5.1. The working example: the Bi-StQPCC. The standard quadratic program (StQP), an indefinite quadratic minimized over the unit simplex, is a hard instance and the principal testbed of [28]. Augmenting it with the two structures that (2.1) admits, a complementarity pair $y \perp z$ and a binary vector w under a cardinality budget, yields the member of that class on which we test PIP. With $\boldsymbol{v} = (x, y, z, w)$, $x \in \mathbb{R}^n$ with $n \ge 1$, $y, z \in \mathbb{R}^m_+$, and $w \in \{0,1\}^\ell$, this *binary-constrained standard QP with complementarity constraints* (Bi-StQPCC) is

$$
\begin{array}{lll}
\underset{(x,y,z,w)\in\mathbb{R}^{n+2m+\ell}}{\textbf{minimize}} & \boldsymbol{v}^\top \boldsymbol{Q} \boldsymbol{v} + \boldsymbol{q}^\top \boldsymbol{v} & \\
\textbf{subject to} & \mathbf{1}_n^\top x + \mathbf{1}_m^\top y + \mathbf{1}_m^\top z = \rho, \quad \rho \triangleq \dfrac{n+2m}{2}, & \\
& \mathbf{1}_\ell^\top w = N, \quad N \triangleq \lfloor \ell/2 \rfloor, \quad x \ge 0, & \\
& 0 \le y \perp z \ge 0, \quad w \in \{0,1\}^\ell, &
\end{array} \tag{5.1}
$$

with $\boldsymbol{Q}$ being symmetric and indefinite, partitioned as in Section 2 into the blocks $Q_{\xi\xi}, Q_{\xi w}$, and Q_{ww} of $\xi \triangleq (x, y, z)$ against itself and against w, and $\boldsymbol{q}$ into q_ξ and q_w.

We now adapt the reformulation of Section 2 to place (5.1) in the form (2.6). We observe that of its constraints only the simplex equality and $x \ge 0$ involve ξ; the budget row $\mathbf{1}_\ell^\top w = N$ involves w alone and stays in the outer minimization, undualized. Ordering the equality first,

$$
[\,A \;\; B \;\; C \;\; D \;\; e\,] = \begin{bmatrix} \mathbf{1}_n^\top & \mathbf{1}_m^\top & \mathbf{1}_m^\top & 0 & \rho \\ I_n & 0 & 0 & 0 & 0_n \end{bmatrix}, \quad k = n+1, \quad \overline{y} = \overline{z} = \rho\,\mathbf{1}_m, \tag{5.2}
$$

so that $\eta \in \mathbb{R} \times \mathbb{R}^n_+$: the component η_1 carries the equality and is sign free, while η_{1+i} carries $x_i \ge 0$. Writing

$$
g \triangleq 2\,Q_{\xi\xi}\,\xi + q_\xi + 2\,Q_{\xi w}\,w = (g_x, g_y, g_z) \tag{5.3}
$$

for the objective gradient in ξ, the conditions of (2.6) read, at a pair (w, s),

$$
\begin{array}{lll}
(g_x)_i = \eta_1 + \eta_{1+i}, & 0 \le \eta_{1+i} \perp x_i \ge 0, & \forall\, i = 1, \ldots, n \\
u = g_y - \eta_1 \mathbf{1}_m + \lambda, & 0 \le \lambda \perp \sigma^y \ge 0, & 0 \le u \perp y \ge 0 \\
v = g_z - \eta_1 \mathbf{1}_m + \mu, & 0 \le \mu \perp \sigma^z \ge 0, & 0 \le v \perp z \ge 0,
\end{array} \tag{5.4}
$$

where $\sigma^y = \rho\, s - y$ and $\sigma^z = \rho\,(\mathbf{1}_m - s) - z$. A tuple $(\xi, \eta, \lambda, \mu, u, v)$ satisfying (5.4) together with $\mathbf{1}^\top \xi = \rho$ is a KKT tuple, and every feasible tuple of (2.6) for (5.1) determines one.

Since $e = (\rho, 0_n)$, $D = 0$ and $\overline{y} = \overline{z} = \rho \mathbf{1}_m$, the dual part of the objective collapses to

$$
-\frac{1}{2}\Big\{ \widehat{D}^\top \widehat{\eta} - \eta^\top e + \overline{y}^\top \widehat{\lambda} + \overline{z}^\top \widehat{\mu} \Big\} = \frac{\rho}{2}\Big(\eta_1 - \mathbf{1}_m^\top \widehat{\lambda} - \mathbf{1}_m^\top \widehat{\mu} \Big), \tag{5.5}
$$

so that η_1 and the lifted $\widehat{\lambda}$ and $\widehat{\mu}$ are the only variables it contains. These variables are relatively few. That of η against w vanishes with D, and the diagonal s-coupling leaves λ and μ only the $2m$ entries $\widehat{\lambda} = \lambda \circ s$ and $\widehat{\mu} = \mu \circ (\mathbf{1}_m - s)$, so the McCormick machinery reduces to the structural products $\xi_i w_j$ and $w_j w_{j'}$.

5.2. Closed-form dual bounds. The McCormick blocks of (2.6) require a uniform upper bound Δ on the multipliers (η, λ, μ). For the Bi-StQPCC one is available in closed form, computable from the problem data alone without solving a linear program. The derivation runs in three steps: an envelope for the objective gradient g over the feasible set; a two-sided bound on the single free multiplier η_1; and bounds for every remaining multiplier in terms of that interval, which together give Δ.

For $N \le \ell$ let τ_N^+ and τ_N^- denote the sum of the N largest and of the N smallest components of a vector in $\mathbb{R}^\ell$; let $(Q_{\xi\xi})_{j\bullet}$ and $(Q_{\xi w})_{j\bullet}$ denote row j.

LEMMA 5.1. For every feasible point of (5.1) and every $j \in \{1, \dots, n{+}2m\}$, $\underline{g}_j \le g_j \le \overline{g}_j$, where

$$\begin{aligned} \underline{g}_j &\triangleq 2\rho \min_t (Q_{\xi\xi})_{jt} + 2\,\tau_N^-\big((Q_{\xi w})_{j\cdot}\big) + (q_\xi)_j, \\ \overline{g}_j &\triangleq 2\rho \max_t (Q_{\xi\xi})_{jt} + 2\,\tau_N^+\big((Q_{\xi w})_{j\cdot}\big) + (q_\xi)_j. \end{aligned}$$

Proof. $\xi = \rho\,\upsilon$ with υ in the unit simplex, so $\sum_t (Q_{\xi\xi})_{jt}\xi_t = \rho \sum_t (Q_{\xi\xi})_{jt}\upsilon_t$ is linear on a simplex and lies in $[\rho \min_t (Q_{\xi\xi})_{jt},\ \rho \max_t (Q_{\xi\xi})_{jt}]$; and $\sum_p (Q_{\xi w})_{jp} w_p$ is a sum of exactly N entries of $(Q_{\xi w})_{j\cdot}$. Add the two brackets to (5.3). ∎

It is worth noticing that Lemma 5.1 bounds each g_j on both sides, confining it to the interval $[\underline{g}_j, \overline{g}_j]$, whose endpoints may be of either sign, rather than merely bounding the magnitude $|g_j|$. We next move on to derive the bounds on η_1.

PROPOSITION 5.2. For every KKT tuple,

$$\underline{\eta}_1 \triangleq \min_{1 \le j \le n+2m} \underline{g}_j \ \le\ \eta_1 \ \le\ \min_{1 \le i \le n} \overline{g}_i \triangleq \bar{\eta}_1.$$

Proof. We first show the upper bound. The x-block of (5.4) reads $\eta_{1+i} = (g_x)_i - \eta_1$, and $\eta_{1+i} \ge 0$; hence $\eta_1 \le (g_x)_i \le \overline{g}_i$ for every $i = 1, \dots, n$, and minimizing over i gives $\eta_1 \le \bar{\eta}_1$.

We turn to the lower bound. Since $\mathbf{1}^\top \xi = \rho > 0$ and $\xi \ge 0$, some component ξ_{j^*} is strictly positive. Complementarity then forces the multiplier of the sign constraint on that component to vanish, and stationarity (5.4) turns this into a lower bound on η_1:

$$\begin{aligned} &\xi_{j^*} = x_i: && \eta_{1+i} = 0, && \eta_1 = (g_x)_i; \\ &\xi_{j^*} = y_\kappa: && u_\kappa = 0, && \eta_1 = (g_y)_\kappa + \lambda_\kappa \ \ge\ (g_y)_\kappa; \\ &\xi_{j^*} = z_\kappa: && v_\kappa = 0, && \eta_1 = (g_z)_\kappa + \mu_\kappa \ \ge\ (g_z)_\kappa, \end{aligned}$$

the last two inequalities by $\lambda \ge 0$ and $\mu \ge 0$. In every case $\eta_1 \ge g_{j^*} \ge \underline{g}_{j^*} \ge \underline{\eta}_1$, the last step because $\underline{\eta}_1$ is the least of the $\underline{g}_j$. ∎

PROPOSITION 5.3. Let $(\bullet)^+ \triangleq \max(\bullet, 0)$. For every KKT tuple, every $i = 1, \dots, n$ and every $\kappa = 1, \dots, m$, the following three statements hold:

(i) $\eta_{1+i} = (g_x)_i - \eta_1$ is uniquely determined, and $0 \le \eta_{1+i} \le \bar{\eta}_{1+i}$, where $\bar{\eta}_{1+i} \triangleq \overline{g}_i - \underline{\eta}_1 \ge 0$.

(ii) The pair $(\lambda_\kappa, u_\kappa)$ is uniquely determined if and only if $s_\kappa = 1$, and (μ_κ, v_κ) if and only if $s_\kappa = 0$: each is indeterminate exactly where its own entry $\widehat{\lambda}_\kappa$, resp. $\widehat{\mu}_\kappa$, of (5.5) vanishes. Where determined, the pair takes the canonical values

$$\lambda_\kappa^{\mathrm{c}} = \big(\eta_1 - (g_y)_\kappa\big)^+,\ u_\kappa^{\mathrm{c}} = \big((g_y)_\kappa - \eta_1\big)^+,\ \mu_\kappa^{\mathrm{c}} = \big(\eta_1 - (g_z)_\kappa\big)^+,\ v_\kappa^{\mathrm{c}} = \big((g_z)_\kappa - \eta_1\big)^+; \tag{5.6}$$

where indeterminate, it is free up to adding $t\,(1,1)$ with $t \ge 0$, which leaves that entry at zero. Substituting (5.6) into any KKT tuple therefore returns a KKT tuple of the same objective value in (2.6), and one obeying

$$\lambda_\kappa \le \bar{\lambda}_\kappa \triangleq \big(\bar{\eta}_1 - \underline{g}_{y,\kappa}\big)^+,\ \mu_\kappa \le \bar{\mu}_\kappa \triangleq \big(\bar{\eta}_1 - \underline{g}_{z,\kappa}\big)^+,\ u_\kappa \le \big(\overline{g}_{y,\kappa} - \underline{\eta}_1\big)^+,\ v_\kappa \le \big(\overline{g}_{z,\kappa} - \underline{\eta}_1\big)^+.$$

(iii) The McCormick constant of (2.6) may be taken as

$$\Delta \triangleq \max\{\, |\underline{\eta}_1|,\ |\bar{\eta}_1|,\ \max_i \bar{\eta}_{1+i},\ \max_\kappa \bar{\lambda}_\kappa,\ \max_\kappa \bar{\mu}_\kappa \,\}.$$

The componentwise bounds above may replace it row by row.

Proof. (i) The x-block of (5.4) determines η_{1+i}, and $\eta_{1+i} \geq 0$ there; the upper bound follows from $(g_x)_i \leq \bar{g}_i$ and $\eta_1 \geq \underline{\eta}_1$, and $\bar{\eta}_{1+i} \geq 0$ from $\bar{g}_i \geq \underline{g}_i \geq \underline{\eta}_1$.

(ii) It suffices to treat the y-block, the z-block being its image under $s \mapsto \mathbf{1}_m - s$, which exchanges $(\lambda, u, \widehat{\lambda})$ with $(\mu, v, \widehat{\mu})$. Let $s_\kappa = 1$, so that $\sigma^y_\kappa = \rho - y_\kappa$; complementarity then leaves three cases,

$$\begin{array}{lll} y_\kappa = \rho : & u_\kappa = 0, & \lambda_\kappa = \eta_1 - (g_y)_\kappa \geq 0; \\ 0 < y_\kappa < \rho : & \lambda_\kappa = u_\kappa = 0, & (g_y)_\kappa = \eta_1; \\ y_\kappa = 0 : & \lambda_\kappa = 0, & u_\kappa = (g_y)_\kappa - \eta_1 \geq 0, \end{array}$$

each pinning both members, at values agreeing with (5.6). Let $s_\kappa = 0$ instead. Then $\sigma^y_\kappa = -y_\kappa$ with both sides nonnegative, so $y_\kappa = \sigma^y_\kappa = 0$; neither complementarity binds, and (5.4) pins only the difference $\lambda_\kappa - u_\kappa = \eta_1 - (g_y)_\kappa$, leaving the pair free along $(1, 1)$. This proves the equivalence, and since $\widehat{\lambda}_\kappa = \lambda_\kappa s_\kappa$ vanishes throughout the second case, the free shift never reaches the objective.

At each κ, therefore, exactly one of $\widehat{\lambda}_\kappa$, $\widehat{\mu}_\kappa$ can be nonzero, and it belongs to the pair that (5.4) determines; the canonical tuple agrees with the given one there, so the objective is unchanged. The stated bounds follow by monotonicity of $(\bullet)^+$ in (5.6), using $\eta_1 \leq \bar{\eta}_1$ and $g \geq \underline{g}$ for λ, μ, and $\eta_1 \geq \underline{\eta}_1$ and $g \leq \bar{g}$ for u, v.

(iii) is immediate from (i), (ii) and Proposition 5.2. ∎

5.3. Experimental design. Instances of the form (5.1) at a given size are determined by the Hessian $\boldsymbol{Q} \in \mathbb{R}^{n_{\text{tot}} \times n_{\text{tot}}}$ and the linear term $\boldsymbol{q} \in \mathbb{R}^{n_{\text{tot}}}$, where $n_{\text{tot}} \triangleq n + 2m + \ell$ is the total dimension. We adopt the deterministic pseudorandom generator used for StQP instances in [28], which in turn follows the indefinite-quadratic-over-a-simplex test problems of [24]. The Hessian is $\boldsymbol{Q} = F_l - F_c$, where F_l is given by

$$(F_l)_{ij} = \begin{cases} d_i, & i = j, \\ \frac{1}{2}\,(d_i + d_j), & i \neq j, \end{cases} \tag{5.7}$$

with $d_1, \ldots, d_{n_{\text{tot}}}$ independent and uniform on $[0, 2]$, and F_c is symmetric with zero diagonal and with the entries $(F_c)_{ij}$, $i < j$, drawn independently from the mixture

$$(F_c)_{ij} \ \sim\ p\,\mathcal{U}[0, 10] \,+\, (1 - p)\,\mathcal{U}[-10, 0],$$

where $\mathcal{U}$ denotes the uniform distribution and p is the density parameter. The linear term $\boldsymbol{q}$ is drawn uniformly from $[-1, 1]^{n_{\text{tot}}}$. Consequently

$$\boldsymbol{Q}_{ii} = d_i, \qquad \boldsymbol{Q}_{ij} = \frac{1}{2}\,(d_i + d_j) \,-\, (F_c)_{ij}, \qquad i \neq j.$$

We use three sizes and two values of p, five seeds each (Table 1), for thirty instances in all.

Two directions establish the indefiniteness of $\boldsymbol{Q}$ without recourse to its spectrum. First, $e_i^\top \boldsymbol{Q} e_i = d_i$, which is positive almost surely. Second, the averaging rule in (5.7) makes the terms in d cancel along an edge of the simplex,

$$(e_i - e_j)^\top \boldsymbol{Q}\,(e_i - e_j) \;=\; \boldsymbol{Q}_{ii} + \boldsymbol{Q}_{jj} - 2\,\boldsymbol{Q}_{ij} \;=\; 2\,(F_c)_{ij}, \tag{5.8}$$

so that any pair with $(F_c)_{ij} < 0$ supplies a direction of negative curvature. The latter is absent only when all $\binom{n_{\text{tot}}}{2}$ off-diagonal draws fall in the branch $[0, 10]$, an event of probability

$p^{\binom{n_{\text{tot}}}{2}} \le 10^{-2486}$ at the sizes used; in fact each of the thirty instances contains thousands of entries from the negative branch, and roughly half of the eigenvalues of each $\boldsymbol{Q}$ are negative.

Identity (5.8) also fixes the role of p: the objective restricted to that edge is convex precisely when $(F_c)_{ij} \ge 0$, so p is the probability that an edge is convex. Raising p does not drive $\boldsymbol{Q}$ toward convexity, since the directions $e_i - e_j$ span the tangent space of the simplex but a quadratic form positive on a spanning set need not be positive semidefinite on its span; what it does is move the local minimizers off the vertices onto faces of higher dimension, the minimum along a concave edge being attained at an endpoint. These are the harder instances reported in [24, 28].

size	n_{tot}	m	ℓ	n	$N = \lfloor \ell/2 \rfloor$	# disjunctions
S	200	20	10	150	5	260
M	500	50	20	380	10	650
L	1000	100	50	750	25	1300

TABLE 1
The three instance sizes, each generated at $p \in \{0.50, 0.75\}$ *over seeds* 1–5.

Methods. Seven methods are run on each instance. Two solve it from scratch: the original Bi-QPCC, namely (5.1), and the FMIP, namely (2.6) handed to Gurobi's spatial branch-and-bound. The third is the NLP warm start, which proceeds in two steps. The first relaxes $\boldsymbol{w} \in \{0,1\}^{\ell}$ and $\boldsymbol{s} \in \{0,1\}^{m}$ of (2.3) to their box relaxations and solves the resulting continuous QP with Ipopt under a CPU-time budget, a penalty $\rho_{\text{pen}} \sum_i y_i z_i$ ($\rho_{\text{pen}} = 100$) added to the objective to press each (y_i, z_i) pair toward complementarity; the fractional solution is then rounded to a candidate $(\bar{w}, \bar{s})$ — top-N selection on w, with the $N = \lfloor \ell/2 \rfloor$ largest entries set to one, and the sign of $y_i - z_i$ on s. The second step fixes $(w, s) = (\bar{w}, \bar{s})$ in (2.6), which collapses every bilinear McCormick term to a linear one, and solves the resulting LPCC with Gurobi under its own time limit, with complementarity pairs enforced through SOS1 constraints. The result is an (5.1)-feasible incumbent $\bar{\xi} = (\bar{x}, \bar{y}, \bar{z})$ together with $(\bar{w}, \bar{s})$.

The remaining four are seeded at that incumbent: the warm Bi-QPCC, the warm FMIP under the full and under a short time budget, and PIP. Seeding all warm methods identically separates the effect of the progressive scheme from that of the starting point; the reported times of the warm methods exclude the shared warm start, whose cost is reported as a method of its own. All objectives are reported as the primal value $\boldsymbol{v}^{\top}\boldsymbol{Q}\boldsymbol{v} + \boldsymbol{q}^{\top}\boldsymbol{v}$, comparable across methods by Theorem 2.2.

Implementation. All experiments run on a workstation with an Intel Xeon Platinum 8270 CPU and 64 GB of RAM. Models are built in Julia with JuMP 1.30.1 [23]; the discrete models (namely the Bi-QPCCs, the FMIPs, the warm start's second step, and the PIP subproblems) are solved by Gurobi 13.0.2 [18] with relative gap tolerance 10^{-6} and 10 threads, while the penalized relaxation of the warm start's first step is solved by Ipopt 3.14.19 [27].

PIP follows Algorithm 4.1 with $\rho_0 = 0.8$, $\rho_{\min} = 0.4$, $\alpha = 0.2$ and $r_{\max} = 3$. Each subproblem runs under a time limit of its own and is terminated early if its incumbent has not improved for a prescribed length of time; both limits are listed in Table 2, where all budgets are collected.

size	Bi-QPCC / FMIP	NLP st. 1	NLP st. 2	FMIP (short)	PIP sub.	PIP no-impr.
S	1800	25	50	300	60	40
M	3600	50	150	600	150	100
L	7200	100	300	1200	300	200

TABLE 2
Time budgets in seconds. The small, medium, and large settings are shared by both densities.

5.4. Computational results. Tables 3–8 report the six tests, one table each. A row gives, for one method, its incumbent objective on each of the five seeds, its mean wall-clock time in seconds, its mean optimality gap in percent, and

$$\bar{\delta} \triangleq \frac{1}{5}\sum_{\sigma=1}^{5} \frac{\theta_\sigma - \theta_\sigma^*}{|\theta_\sigma^*|} \times 100\%,$$

the mean relative gap to the best incumbent of the instance, where θ_σ is the method's objective on seed σ and θ_σ^* is the smallest objective returned by any tested method on that seed. Objectives are divided by 10^3 and rounded to one decimal, the rounding absorbing differences the comparison should ignore; the bold entries are those attaining the best rounded value of their column. The time, gap and $\bar{\delta}$ columns are computed from the unrounded solver output. The last two methods are those run under a short budget, and are separated by a rule from the four run under the full budget of Table 2; in the $\bar{\delta}$ column the best entry of each of the two blocks is set in bold. A superscript $^{\text{w}}$ marks a method started from the warm start of Section 5.3, whose own cost is the row NLP-WS, and a star marks a certificate of optimality. Times and gaps are rounded to integers. The gap column carries – for NLP-WS and PIP$^{\text{w}}$: both are primal heuristics, returning a feasible point and no lower bound, so no optimality gap is defined for them; – in an objective column marks a seed on which no incumbent was returned (see Table 8).

	objective by seed							
method	1	2	3	4	5	time	gap	$\bar{\delta}$
Bi-QPCC	−56.6	−49.9	−51.2	−57.6	−56.2	1803	3345	11.43
FMIP	**−58.9**	**−62.8**	−61.0	**−61.2***	**−62.7***	1608	10	0.13
NLP-WS	−57.7	−61.3	−58.8	−58.6	−59.8	4	–	3.50
Bi-QPCC$^{\text{w}}$	**−58.9**	**−62.8**	−59.6	−59.2	−60.9	1803	3006	1.82
FMIP$^{\text{w}}$	**−58.9***	**−62.8**	**−61.4**	**−61.2***	**−62.7***	1194	4	**0.00**
FMIP$^{\text{w}}_{\text{short}}$	**−58.9**	**−62.8**	−59.7	−60.8	**−62.7**	301	48	**0.70**
PIP$^{\text{w}}$	**−58.9**	**−62.8**	−59.7	−59.3	−61.0	56	–	1.73

TABLE 3
Size S, $p = 0.50$.

	objective by seed							
method	1	2	3	4	5	time	gap	$\bar{\delta}$
Bi-QPCC	−364.1	−388.8	−361.8	−364.6	−373.1	3601	10365	9.70
FMIP	−373.1	−366.8	−392.4	−380.4	−393.4	3607	212	7.03
NLP-WS	−397.3	−406.9	−398.4	−389.0	−403.6	87	–	2.72
Bi-QPCC$^{\text{w}}$	−397.3	−406.9	−398.4	−389.0	−403.6	3601	9753	2.72
FMIP$^{\text{w}}$	**−410.8**	**−419.7**	**−404.0**	**−398.3**	−418.2	3608	175	**0.02**
FMIP$^{\text{w}}_{\text{short}}$	**−410.8**	−416.4	**−404.0**	−397.1	−418.2	605	332	0.23
PIP$^{\text{w}}$	**−410.8**	**−419.7**	**−404.0**	**−398.3**	**−418.5**	271	–	**0.00**

TABLE 4
Size M, $p = 0.50$.

Four patterns stand out.

• **The warm-started Bi-LPCC methods dominate the Bi-QPCC.** On the four medium and large tests FMIP$^{\text{w}}$ holds $\bar{\delta}$ at 0.02–0.07% and PIP$^{\text{w}}$ at 0.00–0.15%, against 1.16–2.72% for Bi-QPCC$^{\text{w}}$ and 4.66–9.70% for the Bi-QPCC solved from scratch: an improvement of at least a factor of eleven over the warm Bi-QPCC and of at least thirty over the cold one. On the two small tests, which a direct solve can still reach, the margin narrows to $\bar{\delta}$

	objective by seed							
method	1	2	3	4	5	time	gap	$\bar{\delta}$
Bi-QPCC	−1464.4	−1495.8	−1522.2	−1553.7	−1542.0	7195	28508	6.91
FMIP	−1391.4	−1428.2	−1456.2	−1436.9	−1509.8	7213	387	11.27
NLP-WS	−1595.0	−1607.1	−1573.4	−1596.4	−1597.8	373	–	2.11
Bi-QPCC$^{\mathrm{w}}$	−1595.0	−1607.1	−1573.4	−1596.4	−1597.8	7200	27091	2.11
FMIP$^{\mathrm{w}}$	−1626.0	−1658.6	−1612.2	−1622.7	**−1617.9**	7198	332	**0.05**
FMIP$^{\mathrm{w}}_{\mathrm{short}}$	−1622.9	−1649.1	−1612.2	−1622.7	−1611.1	1203	336	0.29
PIP$^{\mathrm{w}}$	**−1626.3**	**−1658.7**	**−1615.0**	**−1623.8**	−1613.0	1029	–	**0.06**

TABLE 5
Size L, $p = 0.50$.

	objective by seed							
method	1	2	3	4	5	time	gap	$\bar{\delta}$
Bi-QPCC	−61.3	−62.8	−65.9	−61.8	−64.5	1815	3574	3.29
FMIP	**−63.3**	**−66.0**	−66.0	−63.5	**−66.0**	1802	21	0.73
NLP-WS	**−63.3**	−65.4	−64.7	**−65.4**	−64.2	6	–	1.24
Bi-QPCC$^{\mathrm{w}}$	**−63.3**	−65.4	−65.8	**−65.4**	−64.5	1928	3656	0.82
FMIP$^{\mathrm{w}}$	**−63.3**	−65.9	**−66.5**	**−65.4***	−65.5	1724	16	**0.17**
FMIP$^{\mathrm{w}}_{\mathrm{short}}$	**−63.3**	−65.4	−65.9	**−65.4**	−64.6	302	84	0.78
PIP$^{\mathrm{w}}$	**−63.3**	−65.4	**−66.5**	**−65.4**	−64.6	76	–	**0.59**

TABLE 6
Size S, $p = 0.75$.

of 0.00% and 0.17% for FMIP$^{\mathrm{w}}$ and 1.73% and 0.59% for PIP$^{\mathrm{w}}$, against 1.82% and 0.82% for Bi-QPCC$^{\mathrm{w}}$. The gain is not the reformulation alone: solved from scratch the FMIP is the weakest method on the large tests ($\bar{\delta}$ of 11.27% and 11.12%, with two seeds returning no incumbent), so it is the combination with the warm start, and with the progressive scheme, that carries the advantage.

• **Only the Bi-LPCC methods exploit the warm start.** Measured as the improvement of the mean objective over the tuple all four warm methods are seeded at, Bi-QPCC$^{\mathrm{w}}$ gains 0.00, 0.00, 0.00 and 0.08 percent on the four medium and large tests: it returns the seed unchanged on every one of the five seeds of three tests and on four of the five of the fourth, after an hour or two of spatial branch-and-bound. On the same tests FMIP$^{\mathrm{w}}$ gains 1.14–2.80% and PIP$^{\mathrm{w}}$ gains 1.17–2.81%. On the two small tests the Bi-QPCC does move, by 1.75% and 0.43%, but the Bi-LPCC methods still gain more, FMIP$^{\mathrm{w}}$ by 3.64% and 1.10%.

• **PIP outperforms the short-budget FMIP at comparable cost.** Within the short-budget block, PIP$^{\mathrm{w}}$ has the smaller $\bar{\delta}$ on five of the six tests, at 0.00, 0.06, 0.59, 0.00 and 0.15 percent against 0.23, 0.29, 0.78, 0.09 and 0.50, and it reaches these values in strictly less time; the exception is $(\mathrm{S}, 0.50)$, where FMIP$^{\mathrm{w}}_{\mathrm{short}}$ leads by one percent. Over the thirty instances PIP$^{\mathrm{w}}$ attains the best rounded objective 21 times against 9. Against the full-budget FMIP$^{\mathrm{w}}$ it gives up at most 0.15% of $\bar{\delta}$ on the four medium and large tests while running 6–13 times faster.

• **The Bi-LPCC formulation also yields far stronger bounds.** In the tables, the gap is the relative MIP optimality gap reported by Gurobi, expressed as a percentage. Across the six tables, the ratio of the Bi-QPCC gap to the FMIP gap ranges from approximately 49 to 331 for the cold runs and from approximately 56 to 752 for the warm runs. Moreover, every certificate of optimality obtained in these experiments comes from the FMIP: two of

method	objective by seed 1	2	3	4	5	time	gap	$\bar{\delta}$
Bi-QPCC	−406.5	−408.0	−399.0	−400.7	−411.3	3601	17791	6.10
FMIP	−409.5	−413.4	−400.0	−384.1	−388.0	3602	114	7.53
NLP-WS	−429.5	−432.3	−424.5	−418.0	−428.0	157	–	1.16
Bi-QPCC$^{\mathrm{w}}$	−429.5	−432.3	−424.5	−418.0	−428.0	3601	16533	1.16
FMIP$^{\mathrm{w}}$	**−436.0**	**−437.4**	−425.7	−425.2	**−432.3**	3602	94	**0.03**
FMIP$^{\mathrm{w}}_{\mathrm{short}}$	**−436.0**	−436.6	−425.7	−424.7	**−432.3**	601	302	0.09
PIP$^{\mathrm{w}}$	**−436.0**	**−437.4**	**−426.1**	**−425.4**	**−432.3**	450	–	**0.00**

TABLE 7
Size M, $p = 0.75$.

method	objective by seed 1	2	3	4	5	time	gap	$\bar{\delta}$
Bi-QPCC	−1592.4	−1627.2	−1712.9	−1705.5	−1643.4	7204	39245	4.66
FMIP	–	−1493.1	–	−1563.6	−1574.7	7214	355	11.12[a]
NLP-WS	−1699.4	−1679.2	−1706.1	−1729.3	−1722.5	368	–	1.71
Bi-QPCC$^{\mathrm{w}}$	−1699.4	−1679.2	−1712.9	−1729.3	−1722.5	7200	37988	1.64
FMIP$^{\mathrm{w}}$	**−1731.4**	−1718.0	**−1743.6**	−1741.2	**−1744.5**	7200	304	**0.07**
FMIP$^{\mathrm{w}}_{\mathrm{short}}$	−1722.1	−1711.3	−1736.5	−1734.0	−1737.5	1212	308	0.50
PIP$^{\mathrm{w}}$	−1723.3	**−1718.9**	−1742.6	**−1746.8**	−1741.0	1177	–	**0.15**

[a] For FMIP, $\bar{\delta}$ is averaged over the three successful seeds.

TABLE 8
Size L, $p = 0.75$.

five seeds cold and three of five warm at $(\mathrm{S}, 0.50)$, one of five warm at $(\mathrm{S}, 0.75)$, and none elsewhere.

6. Conclusion. We have extended the Giannessi-Tomasin reformulation to binary-constrained quadratic-defined optimization problems and developed a PIP algorithm for solving the resulting Bi-LPCC. Under the stated assumptions, PIP terminates finitely at a local optimal solution of the Bi-LPCC whose projection onto the space of the primary variables satisfies a necessary condition for local optimality of the original problem. The computational results indicate that the advantage of PIP becomes more pronounced as the problem dimension increases: within practical time limits, it tends to produce better feasible solutions than solving the full mixed-integer formulation directly, while requiring substantially less computational time. Future work includes developing effective commitment rules for the original binary variables and sharper multiplier bounds.

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