Redundant objectives in multiobjective optimization

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September 22, 2026

Abstract

This work examines three different concepts of objective redundancy in multiobjective optimization. Multiobjective optimization is known to suffer from the curse of dimensionality and we aim at reducing the number of objective functions which need to be considered. In this context, a set of objectives is called redundant if the sets of weakly efficient, efficient, or properly efficient solutions remain the same after removing the selected set of objectives. We clarify the relations between these three concepts and discuss combinatorial aspects with regard to the simultaneous and successive removal of objectives. We also provide sufficient conditions for each of these concepts. Further, we briefly discuss the connections to a stronger concept of redundancy which is useful for descent-type methods for solving multiobjective optimization problems.

Keywords: multiobjective optimization, objective redundancy, redundant objectives, nonessential objectives, dimension reduction

Mathematics subject classifications (MSC 2020):
90C26: Nonconvex programming, global optimization
90C29: Multi-objective and goal programming

1 Introduction

In a multiobjective optimization problem, the objectives are typically conflicting and a feasible point that optimizes all objectives simultaneously generally does not exist. Instead, different solution concepts are used. In this paper, we consider the notions of weakly efficient, efficient, and properly efficient solutions (in the sense of Geoffrion); see the forthcoming Definition 2.1. A detailed introduction to multiobjective optimization can be found, for instance, in [13]. For an overview of this field of optimization and its relevant progress over the last decades, we refer to [14].

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Multiobjective optimization problems may involve a large number of objectives. As the number of objectives increases, determining or approximating the set of efficient solutions requires more computational effort. In [2], for instance, for combinatorial optimization problems it is empirically confirmed that the number of efficient solutions grows exponentially with the number of objectives. The main reason for this phenomenon is that if the number of objectives grows, fewer image points become comparable with respect to the componentwise order relation $\leq$. It is therefore useful to determine whether some objectives can be omitted without changing the solutions, which we refer to as objective redundancy. In this work, we examine several notions of objective redundancy, depending on whether the sets of weakly efficient, efficient, or properly efficient solutions remain unchanged when certain objectives are omitted. We investigate the relationships between these concepts of redundancy and derive conditions under which they apply.

Early work on objective redundancy focuses on linear multiobjective optimization; see, e.g., [18, 21] and [20, Chapter 9]. In particular, a sufficient condition for objective redundancy with respect to the set of efficient solutions and a computational method for identifying such redundant objectives are provided therein. Our work is based on the definitions of objective redundancy introduced in these fundamental studies. Note that in the literature the term "nonessential" is sometimes used synonymously with "redundant". Further criteria and computational methods for identifying redundant objectives in linear multiobjective optimization are proposed in [32, 39]. More recently, [27] shows that, for rank-deficient linear multiobjective problems, an equivalent problem with a reduced number of objectives can be obtained via a modified ordering cone.

On the other hand, [41] considers a nonlinear setting and studies how adding, removing, aggregating, or transforming objectives affects the set of efficient solutions. Some of these results are also discussed for the set of weakly efficient solutions. Later works study the question of objective redundancy in different directions. For example, [30] considers objective redundancy for convex nonlinear problems and in [31], the author addresses not only the preservation of efficient solutions but also that of weakly efficient and even properly efficient solutions. Closely related papers investigate general multiobjective optimization problems and how different solution sets change when an objective is added; see, e.g., [15, 29].

In addition, in [1], a different notion of objective redundancy is considered by comparing pairs of objectives only on the set of efficient solutions. It is also shown, in particular, that the concept of redundancy studied in our work does not imply the concept studied therein. In addition to a more detailed consideration of linear problems, the author also uses correlations as an indication of redundancy for nonlinear problems, following the idea of [40]. However, this does not provide an exact test. For this reason, we do not address this approach here. Further work examines the exact or approximate preservation of dominance between feasible points and sets of efficient solutions, as well as the selection, construction, or grouping of a smaller number of objectives; see, e.g., [5, 6, 7, 8, 11, 12, 25, 26, 36, 37, 42].

The existing literature therefore already provides definitions, relationships, and criteria for objective redundancy. However, much of the work on objective redundancy concerns the redundancy of a single objective function, linear problems, or approximation approaches. In contrast, the simultaneous and successive removal of several objectives, the resulting combinatorial questions, and sufficient conditions based on the relationships between the omitted and retained objectives have not yet been sufficiently studied, despite their high practical relevance. In this paper, we address these points and complement the existing results by further relationships and counterexamples.

As a related topic, a stronger concept of objective redundancy is considered in [5, 6, 7, 8]. These works focus on finding a largest possible set of redundant objectives, as well as related approximate variants. We call this stronger concept order-redundancy and distinguish a strict and a non-strict version in Definition 6.1. In Section 6, we briefly discuss these notions and relate them to the notions of objective redundancy considered in this paper.

Moreover, if a set is described by inequality constraints, redundancy of such constraints has been studied extensively; see, e.g., [4, 24, 34, 38], and also [20, Chapter 2]. In this context, constraints are said to be redundant if the set remains the same when they are removed. These works focus in particular on linear and quadratic constraints. In [33], redundant constraints are also considered specifically in linear multiobjective optimization. However, to the best of our knowledge, a relationship between redundancy of constraints and objective redundancy exists only in a very specific context; it is therefore not further discussed in this work.

The remainder of this paper is structured as follows. Basic notation and concepts of multiobjective optimization, which we will use throughout this paper, are introduced in Section 2. There, we also define the different concepts of objective redundancy. An examination of the relations between the various redundancy concepts can be found in Section 3. Section 4 discusses combinatorial and algorithmic aspects. In Section 5, we provide sufficient conditions for all concepts of objective redundancy. Related topics, including the so-called order-redundancy and the preservation of common descent directions, are briefly discussed in Section 6. Finally, Section 7 concludes the paper and provides an outlook on future research questions.

2 Notation and basic concepts

In this section, we clarify the notation that we will use throughout this paper and define the basic concepts we will rely on in the forthcoming sections.

In doing so we define $\mathbb{R}_+ := \{z \in \mathbb{R} \mid z \geq 0\}$ and $\mathbb{R}_{++} := \{z \in \mathbb{R} \mid z > 0\}$. For $\ell \in \mathbb{N}$ we set $[\ell] := \{1, \ldots, \ell\}$ and for $K \subsetneq [\ell]$ we define $[\ell]_{-K} := [\ell] \setminus K$. Furthermore, we will use for $z^1, z^2 \in \mathbb{R}^\ell$ the relations $z^1 \leq z^2 :\Leftrightarrow z^1_i \leq z^2_i$ for all $i \in [\ell]$ and $z^1 < z^2 :\Leftrightarrow z^1_i < z^2_i$ for all $i \in [\ell]$, respectively. Finally, and based on that, we define $\mathbb{R}^\ell_+ := \{z \in \mathbb{R}^\ell \mid z \geq 0\}$ and $\mathbb{R}^\ell_{++} := \{z \in \mathbb{R}^\ell \mid z > 0\}$ for $\ell \geq 1$.

In the remainder of the paper we will consider multiobjective optimization problems. Therefore, in the following definition we recall three solution concepts used in this regard.

Definition 2.1. *Let $X \subseteq \mathbb{R}^n$ be a nonempty set, let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function, and consider the multiobjective optimization problem given by*

$$\min_{x \in X} f(x). \tag{MOP(f,X)}$$

We say that $\bar{x} \in X$ is

(i) a weakly efficient solution of $\mathrm{MOP}(f, X)$ *if there exists no $x \in X$ such that $f(x) < f(\bar{x})$.*

(ii) *an efficient solution of* $\mathrm{MOP}(f, X)$ *if there exists no* $x \in X$ *such that* $f(x) \leq f(\bar{x})$ *and* $f(x) \neq f(\bar{x})$*.*

(iii) *a properly efficient solution of* $\mathrm{MOP}(f, X)$ *(in the sense of Geoffrion) if* $\bar{x}$ *is an efficient solution of* $\mathrm{MOP}(f, X)$ *and there exists some real number* $M > 0$ *such that for all* $i \in [m]$ *and* $x \in X$ *with* $f_i(x) < f_i(\bar{x})$ *there exists some* $j \in [m]_{-\{i\}}$ *with* $f_j(x) > f_j(\bar{x})$ *and*

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leq M.$$

The sets containing all weakly efficient solutions, all efficient solutions, and all properly efficient solutions of $\mathrm{MOP}(f, X)$ *are denoted by* $\mathcal{E}_{\mathrm{w}}(f, X)$*,* $\mathcal{E}(f, X)$*, and* $\mathcal{E}_{\mathrm{p}}(f, X)$*, respectively.*

Note that under the assumptions of Definition 2.1 it holds

$$\mathcal{E}_{\mathrm{p}}(f, X) \subseteq \mathcal{E}(f, X) \subseteq \mathcal{E}_{\mathrm{w}}(f, X). \tag{1}$$

In this context, we would like to emphasize that in Definition 2.1 (*iii*), we explicitly assume that a properly efficient solution is indeed an efficient solution. We thus use the original definition given in [22]. However, it is known (and easy to show) that for any $\bar{x} \in X$, the existence of a real number M with the corresponding property already implies that $\bar{x} \in \mathcal{E}(f, X)$. This explicit assumption (of efficiency) can therefore be omitted. In addition, if the nonempty set $X \subseteq \mathbb{R}^n$ is convex and all objective functions f_i, $i \in [m]$, are strictly convex on X, then it holds $\mathcal{E}(f, X) = \mathcal{E}_{\mathrm{w}}(f, X)$ (see, e.g., [28, Proposition 5.13]). If $\mathrm{MOP}(f, X)$ is a linear multiobjective optimization problem, i.e., $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is given by $f(x) := Cx$ for all $x \in \mathbb{R}^n$ with $C \in \mathbb{R}^{m \times n}$ and $X \subseteq \mathbb{R}^n$ is a polyhedron, then we even have $\mathcal{E}_{\mathrm{p}}(f, X) = \mathcal{E}(f, X)$ (see, e.g., [19, Theorem 3.2]). We will apply these facts in the following, particularly in the examples.

Let now $m \geq 2$, $K \subsetneq [m]$ with $\kappa := |K| \geq 1$, and $K = \{k_1, \ldots, k_\kappa\}$ with $k_i < k_{i+1}$ for all $i \in [\kappa - 1]$. Then we define for a given function $f = (f_1, \ldots, f_m)^\top\colon \mathbb{R}^n \to \mathbb{R}^m$ the corresponding function $f_K\colon \mathbb{R}^n \to \mathbb{R}^\kappa$ by $f_K(x) := (f_{k_1}(x), \ldots, f_{k_\kappa}(x))^\top$ for all $x \in \mathbb{R}^n$. Based on this, the remaining functions f_i with $i \in [m]_{-K}$ are collected in the function $f_{-K}\colon \mathbb{R}^n \to \mathbb{R}^{m-\kappa}$, while their original order is retained here as well. For instance, let $m = 5$, $f = (f_1, f_2, f_3, f_4, f_5)^\top\colon \mathbb{R}^n \to \mathbb{R}^5$ and $K = \{1, 4\} \subsetneq [5]$. Then it holds $f_K = (f_1, f_4)^\top\colon \mathbb{R}^n \to \mathbb{R}^2$ with $f_K(x) := (f_1(x), f_4(x))^\top$ and $f_{-K} = (f_2, f_3, f_5)^\top\colon \mathbb{R}^n \to \mathbb{R}^3$ with $f_{-K}(x) := (f_2(x), f_3(x), f_5(x))^\top$ for all $x \in \mathbb{R}^n$. Using this notation, we will now formulate the concepts of redundancy that we intend to examine in this paper.

Definition 2.2. *Let* $X \subseteq \mathbb{R}^n$ *be a nonempty set, let* $f\colon \mathbb{R}^n \to \mathbb{R}^m$ *be a vector-valued function with* $m \geq 2$*, and let* $K \subsetneq [m]$ *with* $|K| \geq 1$*. Then* f_K *is called*

(i) *weakly redundant for* $\mathrm{MOP}(f, X)$ *if* $\mathcal{E}_{\mathrm{w}}(f_{-K}, X) = \mathcal{E}_{\mathrm{w}}(f, X)$*.*

(ii) *redundant for* $\mathrm{MOP}(f, X)$ *if* $\mathcal{E}(f_{-K}, X) = \mathcal{E}(f, X)$*.*

(iii) *properly redundant for* $\mathrm{MOP}(f, X)$ *if* $\mathcal{E}_{\mathrm{p}}(f_{-K}, X) = \mathcal{E}_{\mathrm{p}}(f, X)$*.*

Based on this, we call an objective function f_k *weakly redundant, redundant, and properly redundant for* $\mathrm{MOP}(f, X)$ *if the corresponding properties are fulfilled for* $K = \{k\}$*.*

We would like to point out that our terminology in Definition 2.2 is based on the fundamental work [21], and that another frequently used term for redundant functions is “nonessential functions” (see, for instance, [18]). Furthermore, these fundamental publications on this topic have in common that the basic notion of redundancy is formulated for a single objective function f_k, $k \in [m]$. To the best of our knowledge, the set notation we have chosen in Definition 2.2 was introduced in [37] using the term “redundant objective set” and later in [27] using the term “nonessential system of objectives”.

However, initial results in this context can already be found in [41]. Note that there exist simple assumptions that ensure that a single function satisfies the redundancy concepts specified in Definition 2.2. To see this, let $k \in [m]$. If f_k is constant on X, then f_k is redundant and properly redundant for $\mathrm{MOP}(f, X)$ (see the forthcoming Remark 5.12), and by definition it holds $\mathcal{E}_{\mathrm{w}}(f, X) = X$. If, on the other hand, f_i is constant on X for some $i \in [m]_{-\{k\}}$, then again by definition it holds $\mathcal{E}_{\mathrm{w}}(f_{-\{k\}}, X) = X$ and f_k is weakly redundant for $\mathrm{MOP}(f, X)$ (see the forthcoming Lemma 4.2). Thus, there are at least trivial examples for each of the redundancy concepts mentioned in Definition 2.2. More interesting examples will be provided in the next section.

3 Relations between redundancy concepts

With regard to the relations between the various redundancy concepts, [31, Example 4.3] shows that redundancy generally does not imply proper redundancy. Using further examples, we will now show that the three redundancy concepts defined in Definition 2.2 do not imply one another and therefore they are not directly related. In doing so we start with two linear multiobjective optimization problems that demonstrate that weak redundancy does not imply redundancy (and thus by the linearity of the problem also proper redundancy), and vice versa.

Example 3.1. *(i) Let $X = [0, 1]^2 \subseteq \mathbb{R}^2$ and the vector-valued function $f: \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $f(x) := (x_1, -x_1, x_2)^\top$ for all $x \in \mathbb{R}^2$. Then it holds for the weakly efficient solutions $\mathcal{E}_{\mathrm{w}}(f, X) = \mathcal{E}_{\mathrm{w}}(f_{-\{3\}}, X) = X$, and for the efficient solutions $\mathcal{E}(f, X) = [0, 1] \times \{0\}$ and $\mathcal{E}(f_{-\{3\}}, X) = X$. Thus, f_3 is weakly redundant, but neither redundant nor properly redundant for* $\mathrm{MOP}(f, X)$.
(ii) Let $X = [0, 1]^2$ and $f: \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $f(x) := (x_1, x_1 + x_2, x_2)^\top$ for all $x \in \mathbb{R}^2$. Then it holds $\mathcal{E}(f, X) = \mathcal{E}(f_{-\{3\}}, X) = \{(0, 0)^\top\}$, $\mathcal{E}_{\mathrm{w}}(f, X) = (\{0\} \times [0, 1]) \cup ([0, 1] \times \{0\})$, and $\mathcal{E}_{\mathrm{w}}(f_{-\{3\}}, X) = \{0\} \times [0, 1]$ (see Figure 1). Thus, f_3 is redundant and properly redundant, but not weakly redundant for $\mathrm{MOP}(f, X)$.

The following example shows that proper redundancy (even when combined with weak redundancy) generally does not imply redundancy.

Example 3.2. *Let $X = [0, 1]^2$ and $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $f(x) := (x_1^2, (x_1 - 1)^2 + x_1 x_2, x_2)^\top$ for all $x \in \mathbb{R}^2$. Then it holds*

$$\mathcal{E}_{\mathrm{w}}(f, X) = \mathcal{E}_{\mathrm{w}}(f_{-\{3\}}, X) = \mathcal{E}(f_{-\{3\}}, X) = (\{0\} \times [0, 1]) \cup ([0, 1] \times \{0\})$$

and $\mathcal{E}(f, X) = [0, 1] \times \{0\}$. Let now $\bar{x} \in \mathcal{E}_{\mathrm{p}}(f, X) \subseteq \mathcal{E}(f, X)$. Thus, $\bar{x}$ has the form $\bar{x} = (\bar{x}_1, 0)^\top$ with $\bar{x}_1 \in [0, 1]$ and, in particular, $\bar{x} \in$ $\operatorname{argmin}_{x \in X} \{f_3(x)\}$. Assume that

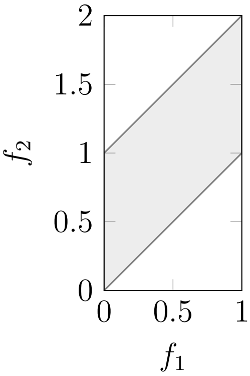


Figure 1: Image set $f_{-\{3\}}(X)$ of Example 3.1 (*ii*).

$\bar{x}_1 = 0$. Then, we obtain $f(\bar{x}) = (0,1,0)^\top$ and we define $x(\varepsilon) := (\varepsilon, 0)^\top \in X$ for all $\varepsilon \in (0,1)$ with $f(x(\varepsilon)) = (\varepsilon^2, (\varepsilon-1)^2, 0)^\top$. Hence, it holds

$$f_1(x(\varepsilon)) = \varepsilon^2 > 0 = f_1(\bar{x}) \text{ and } f_2(x(\varepsilon)) = (\varepsilon-1)^2 < 1 = f_2(\bar{x}) \tag{2}$$

for all $\varepsilon \in (0,1)$, and we obtain

$$\lim_{\varepsilon \to 0^+} \frac{f_2(\bar{x}) - f_2(x(\varepsilon))}{f_1(x(\varepsilon)) - f_1(\bar{x})} = \lim_{\varepsilon \to 0^+} \left(-1 + \frac{2}{\varepsilon}\right) = \infty. \tag{3}$$

Thus, $(0,0)^\top \notin \mathcal{E}_{\mathrm{p}}(f,X)$. In a similar way, it can be shown that $(1,0)^\top \notin \mathcal{E}_{\mathrm{p}}(f,X)$. Hence, we may assume that $\bar{x}_1 \in (0,1)$ with $f(\bar{x}) = (\bar{x}_1^2, (\bar{x}_1-1)^2, 0)^\top$. Then, we obtain for all $x \in X$ with $f_1(x) = x_1^2 < \bar{x}_1^2 = f_1(\bar{x})$ that $x_1 - \bar{x}_1 < 0$, $x_1 + \bar{x}_1 < 2$, $f_2(x) > f_2(\bar{x})$, and

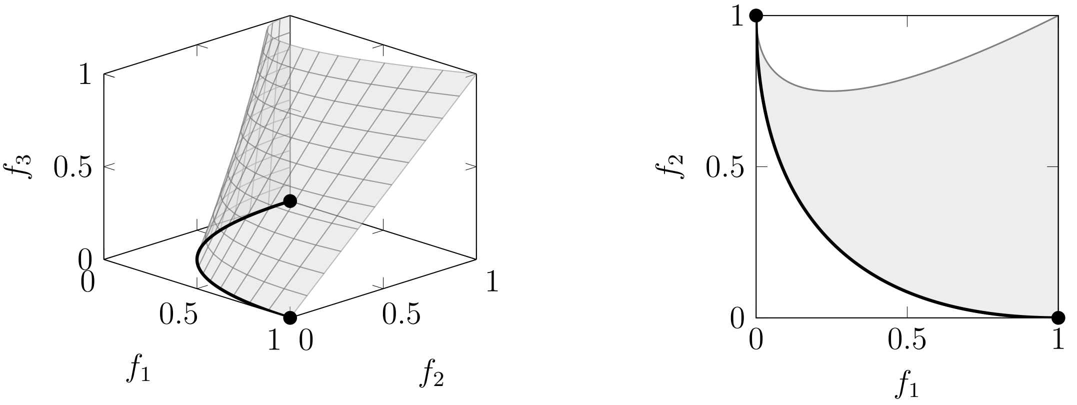
$$\frac{f_1(\bar{x}) - f_1(x)}{f_2(x) - f_2(\bar{x})} \leq \frac{\bar{x}_1}{1-\bar{x}_1}.$$

By analogy, we obtain for all $x \in X$ with $f_2(x) = (x_1-1)^2 + x_1x_2 < (\bar{x}_1-1)^2 = f_2(\bar{x})$ that $x_1 - \bar{x}_1 > 0$, $f_1(x) = x_1^2 > \bar{x}_1^2 = f_1(\bar{x})$, and

$$\frac{f_2(\bar{x}) - f_2(x)}{f_1(x) - f_1(\bar{x})} \leq \frac{1-\bar{x}_1}{\bar{x}_1}.$$

Thus, by choosing $M := \max\left\{\frac{\bar{x}_1}{1-\bar{x}_1}, \frac{1-\bar{x}_1}{\bar{x}_1}\right\}$, it follows $\bar{x} = (\bar{x}_1, 0)^\top \in \mathcal{E}_{\mathrm{p}}(f,X)$ for $\bar{x}_1 \in (0,1)$, and we obtain $\mathcal{E}_{\mathrm{p}}(f,X) = (0,1) \times \{0\}$.
Furthermore, based on the preceding considerations it is easy to see that $(0,1) \times \{0\} \subseteq \mathcal{E}_{\mathrm{p}}(f_{-\{3\}},X)$, $(0,0)^\top \notin \mathcal{E}_{\mathrm{p}}(f_{-\{3\}},X)$, and $(1,0)^\top \notin \mathcal{E}_{\mathrm{p}}(f_{-\{3\}},X)$. Finally, assume now that there exists $\bar{x} = (0,\bar{x}_2)^\top \in \mathcal{E}_{\mathrm{p}}(f_{-\{3\}},X) \subseteq \mathcal{E}(f_{-\{3\}},X)$ with $\bar{x}_2 \in (0,1]$. Then, we obtain $f_{-\{3\}}(\bar{x}) = (0,1)^\top$ and we define again $x(\varepsilon) := (\varepsilon,0)^\top \in X$ for all $\varepsilon \in (0,1)$ with $f_{-\{3\}}(x(\varepsilon)) = (\varepsilon^2, (\varepsilon-1)^2)^\top$. Therefore, as at the beginning of the example, (2) holds for all $\varepsilon \in (0,1)$, and we also obtain (3). Thus, it holds $\mathcal{E}_{\mathrm{p}}(f,X) = \mathcal{E}_{\mathrm{p}}(f_{-\{3\}},X) = (0,1) \times \{0\}$. In summary, this means that f_3 is properly redundant and weakly redundant but not redundant for $\mathrm{MOP}(f,X)$ *(see also Figure 2 for an illustration).*

This example is of particular interest, as it shows that a single function, here f_3, can be weakly and properly redundant without being redundant. This shows that one



(a) (b)

Figure 2: Image sets of Example 3.2: (a) $f(X)$ and (b) $f_{-\{3\}}(X)$. In both illustrations, the black curve represents the image of the efficient solutions, and the two marked endpoints are image points of efficient solutions that are not properly efficient.

needs to clearly define which optimality notion is of interest before examining functions on redundancy.

We would like to note here that in [31, Proposition 4.4], it is shown that, under additional compactness and convexity assumptions on X and f, a properly redundant function f_k for $\mathrm{MOP}(f, X)$ is also redundant. The following lemma generalizes this result to a set K of objective functions.

Lemma 3.3. *Let $X \subseteq \mathbb{R}^n$ be a nonempty, compact and convex set, let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function with $m \geq 2$, let $K \subsetneq [m]$ with $|K| \geq 1$, and let f_i be continuous and strictly convex on X for all $i \in [m]$. If f_K is properly redundant for* $\mathrm{MOP}(f, X)$*, then f_K is both redundant and weakly redundant for* $\mathrm{MOP}(f, X)$.

Proof. By [13, (3.78)], it holds $\mathcal{E}_{\mathrm{p}}(f, X) \subseteq \mathcal{E}(f, X) \subseteq \mathrm{cl}(\mathcal{E}_{\mathrm{p}}(f, X))$. In addition, we obtain by [3, Theorem 3.4] that $\mathcal{E}(f, X)$ is closed and thus $\mathcal{E}_{\mathrm{p}}(f, X) \subseteq \mathcal{E}(f, X) = \mathrm{cl}(\mathcal{E}_{\mathrm{p}}(f, X))$. Applying the same arguments to f_{-K} in the case $|K| \leq m-2$ yields

$$\mathcal{E}_{\mathrm{p}}(f_{-K}, X) \subseteq \mathcal{E}(f_{-K}, X) = \mathrm{cl}(\mathcal{E}_{\mathrm{p}}(f_{-K}, X)). \tag{4}$$

Note that for $|K| = m-1$ we have $f_{-K}\colon \mathbb{R}^n \to \mathbb{R}$ and thus $\mathcal{E}_{\mathrm{w}}(f_{-K}, X) = \mathcal{E}(f_{-K}, X) = \mathcal{E}_{\mathrm{p}}(f_{-K}, X) = \mathrm{argmin}_{x \in X} f_{-K}(x)$. Moreover, by the convexity of X and since f_{-K} is strictly convex on X we have that $\mathrm{argmin}_{x \in X} f_{-K}(x)$ is a singleton, and (4) is also fulfilled in this case. By the proper redundancy of f_K for $\mathrm{MOP}(f, X)$ it follows now that

$$\mathcal{E}(f_{-K}, X) = \mathrm{cl}(\mathcal{E}_{\mathrm{p}}(f_{-K}, X)) = \mathrm{cl}(\mathcal{E}_{\mathrm{p}}(f, X)) = \mathcal{E}(f, X).$$

Thus, f_K is redundant for $\mathrm{MOP}(f, X)$, and, since $\mathcal{E}(f, X) = \mathcal{E}_{\mathrm{w}}(f, X)$ and $\mathcal{E}(f_{-K}, X) = \mathcal{E}_{\mathrm{w}}(f_{-K}, X)$ hold for instance by [28, Proposition 5.13], we are done. $\square$

It should be noted that for all redundancy concepts, the existence of a weakly efficient solution, an efficient solution, or a properly efficient solution of $\mathrm{MOP}(f, X)$ is not required.

Example 3.4. *Let* $X = \mathbb{R}^2$ *and* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *be defined by* $f(x) := (x_1^2, 2x_1x_2)^\top$ *for all* $x \in \mathbb{R}^2$. *Then it holds*

$$f(\mathbb{R}^2) = \{0\} \cup \left\{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 > 0\right\},$$

$\mathcal{E}(f, X) = \left\{(0, x_2)^\top \mid x_2 \in \mathbb{R}\right\}$, $\mathcal{E}_\mathrm{p}(f, X) = \emptyset$, $\mathcal{E}(f_{-\{1\}}, X) = \emptyset$, *and therefore also* $\mathcal{E}_\mathrm{p}(f_{-\{1\}}, X) = \emptyset$. *Hence,* f_1 *is by definition properly redundant but not redundant, even though there are no properly efficient solutions for neither* $\mathrm{MOP}(f, X)$ *nor* $\mathrm{MOP}(f_{-\{1\}}, X)$.

To close this section, we finally give an overview of the non-implications of the three types of objective redundancy, and where the corresponding counterexamples can be found, in Figure 3.

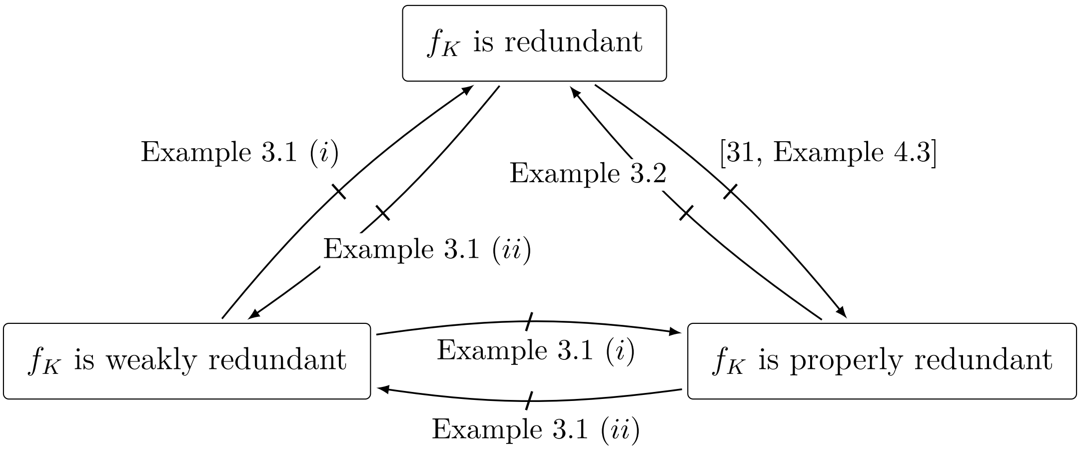


Figure 3: Pairwise non-implications between the three notions of objective redundancy. Example 3.2 additionally shows that weak and proper redundancy together also do not imply redundancy.

4 Combinatorial and algorithmic aspects

In this section, we will discuss algorithmic aspects of determining a maximal set of (weakly/properly) redundant objective functions for a given multiobjective optimization problem $\mathrm{MOP}(f, X)$. Here, we define an index set $\bar{K} \subsetneq [m]$ of (weakly/properly) redundant objective functions $f_{\bar{K}}$ as a *maximal index set* if there exists no other index set $K \subsetneq [m]$ of (weakly/properly) redundant objective functions f_K with $\bar{K} \subsetneq K$. In contrast, such a maximal index set $\bar{K}$ is called a *maximum index set* if there exists no other maximal index set $K \subsetneq [m]$ such that $|\bar{K}| < |K|$. Thus, a maximum index set is a maximal index set with the highest possible cardinality. The starting point for our examinations is the following lemma, which follows directly from the corresponding definitions.

Lemma 4.1. *Let* $X \subseteq \mathbb{R}^n$ *be a nonempty set, let* $f \colon \mathbb{R}^n \to \mathbb{R}^m$ *be a vector-valued function with* $m \geq 3$, *and let* K *and* $\tilde{K}$ *be two nonempty subsets of* $[m]$ *with* $\tilde{K} \subseteq [m]_{-K}$ *and* $K \cup \tilde{K} \subsetneq [m]$. *Then it holds:*

(i) If f_K *is weakly redundant for* $\mathrm{MOP}(f, X)$ *and* $f_{\tilde{K}}$ *is weakly redundant for* $\mathrm{MOP}(f_{-K}, X)$, *then* $f_{K \cup \tilde{K}}$ *is weakly redundant for* $\mathrm{MOP}(f, X)$.

(ii) *If* f_K *is redundant for* $\mathrm{MOP}(f, X)$ *and* $f_{\tilde{K}}$ *is redundant for* $\mathrm{MOP}(f_{-K}, X)$*, then* $f_{K\cup\tilde{K}}$ *is redundant for* $\mathrm{MOP}(f, X)$*.*

(iii) *If* f_K *is properly redundant for* $\mathrm{MOP}(f, X)$ *and* $f_{\tilde{K}}$ *is properly redundant for* $\mathrm{MOP}(f_{-K}, X)$*, then* $f_{K\cup\tilde{K}}$ *is properly redundant for* $\mathrm{MOP}(f, X)$*.*

Recall that for notational convenience, when considering f_{-K}, we continue to label its components by their original indices $[m]_{-K}$. Thus for $\tilde{K} \subsetneq [m]_{-K}$ the notations $(f_{-K})_{-\tilde{K}}$ and $(f_{-\tilde{K}})_{-K}$ are understood as $f_{-(K\cup\tilde{K})}$.

In the following we will show that without additional assumptions the converse implications in Lemma 4.1 hold only for the case of weak redundancy (see the forthcoming Theorem 4.3 and Example 4.5). In doing so, we start by stating the following result, which in turn follows directly from the definition of weakly efficient solutions.

Lemma 4.2. *Let* $X \subseteq \mathbb{R}^n$ *be a nonempty set, let* $f\colon \mathbb{R}^n \to \mathbb{R}^m$ *be a vector-valued function with* $m \geq 2$*, and let* $K \subsetneq [m]$ *with* $|K| \geq 1$*. Then it holds*

$$\mathcal{E}_{\mathrm{w}}(f_{-K}, X) \subseteq \mathcal{E}_{\mathrm{w}}(f, X).$$

Based on that, we can now prove the equivalence result for weak redundancy mentioned above.

Theorem 4.3. *Let* $X \subseteq \mathbb{R}^n$ *be a nonempty set, let* $f\colon \mathbb{R}^n \to \mathbb{R}^m$ *be a vector-valued function with* $m \geq 3$*, and let* K *and* $\tilde{K}$ *be two nonempty subsets of* $[m]$ *with* $\tilde{K} \subseteq [m]_{-K}$ *and* $K \cup \tilde{K} \subsetneq [m]$*. Then* $f_{K\cup\tilde{K}}$ *is weakly redundant for* $\mathrm{MOP}(f, X)$ *if and only if* f_K *is weakly redundant for* $\mathrm{MOP}(f, X)$ *and* $f_{\tilde{K}}$ *is weakly redundant for* $\mathrm{MOP}(f_{-K}, X)$*.*

Proof. Let $f_{K\cup\tilde{K}}$ be weakly redundant for $\mathrm{MOP}(f, X)$, i.e.,

$$\mathcal{E}_{\mathrm{w}}(f, X) = \mathcal{E}_{\mathrm{w}}(f_{-(K\cup\tilde{K})}, X). \tag{5}$$

Since $K \subsetneq [m]$ and $\tilde{K} \subsetneq [m]_{-K}$, it holds by Lemma 4.2 that

$$\mathcal{E}_{\mathrm{w}}(f_{-K}, X) \subseteq \mathcal{E}_{\mathrm{w}}(f, X) \tag{6}$$

and

$$\mathcal{E}_{\mathrm{w}}(f_{-(K\cup\tilde{K})}, X) \subseteq \mathcal{E}_{\mathrm{w}}(f_{-K}, X), \tag{7}$$

respectively. Now, we obtain by (5) and (7) that

$$\mathcal{E}_{\mathrm{w}}(f, X) = \mathcal{E}_{\mathrm{w}}(f_{-(K\cup\tilde{K})}, X) \subseteq \mathcal{E}_{\mathrm{w}}(f_{-K}, X),$$

and the weak redundancy of f_K for $\mathrm{MOP}(f, X)$ follows with (6). Furthermore, it follows by (6) and (5) that

$$\mathcal{E}_{\mathrm{w}}(f_{-K}, X) \subseteq \mathcal{E}_{\mathrm{w}}(f, X) = \mathcal{E}_{\mathrm{w}}(f_{-(K\cup\tilde{K})}, X),$$

and the weak redundancy of $f_{\tilde{K}}$ for $\mathrm{MOP}(f_{-K}, X)$ follows with (7). Together with Lemma 4.1 this proves the result. □

In other words, Theorem 4.3 shows that in the case of weakly redundant objective functions the determination of a maximal index set can be done iteratively, removing objective functions one by one. If an index set $K \subsetneq [m]$ of weakly redundant objective functions f_K has already been determined, then it suffices in the next step to determine (if possible) a weakly redundant objective function f_k for the reduced problem $\mathrm{MOP}(f_{-K}, X)$. An index set $\bar{K}$ of weakly redundant objectives is a maximal index set for $\mathrm{MOP}(f, X)$ if and only if no weakly redundant objective function exists for the corresponding reduced problem.

Note that Theorem 4.3 also shows that if $\bar{K}$ is an index set of weakly redundant objective functions with $|\bar{K}| \geq 2$, then it can be obtained by the prescribed iterative approach considering $|\bar{K}|-1$ corresponding reduced problems. In this context the order of the functions f_k, $k \in \bar{K}$, that are selected in each iteration step for the construction of the set $\bar{K}$, can be chosen arbitrarily. The reason for this is that the subsets K and $\tilde{K}$ in Theorem 4.3 can be interchanged. Of course, this iterative approach does not guarantee that a maximum index set is found. The determination of a maximum index set constitutes a more complicated combinatorial problem.

Example 4.4. *Let $X = [0,1]^2$ and $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $f(x) := (x_1, x_1 + x_2, x_1 - x_2)^\top$ for all $x \in \mathbb{R}^2$. Then it holds $\mathcal{E}_{\mathrm{w}}(f, X) = \{0\} \times [0,1]$ and $\mathcal{E}_{\mathrm{w}}(f, X) = \mathcal{E}_{\mathrm{w}}(f_{-\{k\}}, X)$ is fulfilled for $k \in \{1,2,3\}$. Hence, f_1, f_2 and f_3 are weakly redundant for $\mathrm{MOP}(f, X)$. Moreover, with $\operatorname{argmin}_{x \in X} f_{-\{1,2\}}(x) = \{(0,1)^\top\}$ and $\operatorname{argmin}_{x \in X} f_{-\{1,3\}}(x) = \{(0,0)^\top\}$ we obtain that $\bar{K}_1 := \{1\}$ is a maximal index set of weakly redundant objective functions for $\mathrm{MOP}(f, X)$. Finally, $\operatorname{argmin}_{x \in X} f_{-\{2,3\}}(x) = \{0\} \times [0,1]$ shows that f_3 is weakly redundant for $\mathrm{MOP}(f_{-\{2\}}, X)$ and f_2 is weakly redundant for $\mathrm{MOP}(f_{-\{3\}}, X)$. Thus, $\bar{K}_2 := \{2,3\}$ is a maximum index set of weakly redundant objective functions for $\mathrm{MOP}(f, X)$.*

This example shows that a collection of individually weakly redundant objective functions need not be jointly weakly redundant. Moreover, it shows that the iterative approach may terminate at a maximal index set which is not necessarily a maximum index set (see $\bar{K}_1 := \{1\}$).

The following example shows that individual (proper) redundancy of objective functions generally does not imply their joint (proper) redundancy. More importantly, a corresponding result to Theorem 4.3 does not hold in this case. This means that, in general, not every maximal index set of (properly) redundant objective functions can be determined by successive single-objective removals, as it is possible for weak redundancy. In particular, the iterative approach may terminate at an index set that is not even maximal. All of this is illustrated with the next example.

Example 4.5. *We combine the two linear problems from Example 3.1 (i) and Example 3.1 (ii). The feasible set remains $X = [0,1]^2$, and we consider $f : \mathbb{R}^2 \to \mathbb{R}^4$ given by $f(x) := (x_1, -x_1, x_2, x_1 + x_2)^\top$ for all $x \in \mathbb{R}^2$. Thus, this is again a linear problem, which means that properly efficient and efficient solutions coincide. For this reason, we will limit ourselves to the efficient solutions in the following.*

It holds $\mathcal{E}(f, X) = [0,1] \times \{0\}$ and $\mathcal{E}(f, X) = \mathcal{E}(f_{-\{k\}}, X)$ is fulfilled only for $k \in \{1,3,4\}$. Furthermore, $\mathcal{E}(f, X) = \mathcal{E}(f_{-K}, X)$ and $|K| = 2$ is only satisfied in the case $K = \{1,3\}$. Finally, the uniquely determined maximum index set of redundant objective functions is given by $\bar{K} := \{1,2,4\}$, even though neither $f_{\{1,2\}}$, $f_{\{1,4\}}$, nor $f_{\{2,4\}}$ are redundant.

5 Sufficient conditions for redundancy

If f_K is weakly redundant for MOP(f, X), then it holds $\mathcal{E}_{\mathrm{w}}(f_{-K}, X) = \mathcal{E}_{\mathrm{w}}(f, X)$ by definition, and this is equivalent to the fact that both inclusions $\mathcal{E}_{\mathrm{w}}(f_{-K}, X) \subseteq \mathcal{E}_{\mathrm{w}}(f, X)$ and $\mathcal{E}_{\mathrm{w}}(f, X) \subseteq \mathcal{E}_{\mathrm{w}}(f_{-K}, X)$ are fulfilled. The first subset relation is always satisfied according to Lemma 4.2, i.e., without any additional assumptions. A corresponding result does not apply to (properly) efficient solutions, as the following example shows.

Example 5.1. *We consider again the linear multiobjective optimization problem presented in Example 3.1 (i). Here, as mentioned above, we have* $\mathcal{E}(f, X) = [0, 1] \times \{0\}$ *and* $\mathcal{E}(f_{-\{3\}}, X) = X$*, and we observe* $\mathcal{E}(f_{-\{3\}}, X) \not\subseteq \mathcal{E}(f, X)$.

In order to prove a sufficient condition for the corresponding subset relation in the case of efficient solutions, we will use the following auxiliary result, which is a generalization of the first statement in [15, Theorem 2].

Lemma 5.2. *Let* $X \subseteq \mathbb{R}^n$ *be a nonempty set, let* $f\colon \mathbb{R}^n \to \mathbb{R}^m$ *be a vector-valued function with* $m \geq 2$*, and let* $K \subsetneq [m]$ *with* $|K| \geq 1$*. Then it holds*

$$\mathcal{E}(f_{-K}, X) \setminus \mathcal{E}(f, X) = \left\{ \bar{x} \in \mathcal{E}(f_{-K}, X) \;\middle|\; \begin{array}{l} \exists \tilde{x} \in X\colon \\ f_{-K}(\tilde{x}) = f_{-K}(\bar{x}) \text{ and} \\ f_K(\tilde{x}) \leq f_K(\bar{x}) \text{ and} \\ f_K(\tilde{x}) \neq f_K(\bar{x}) \end{array} \right\}.$$

Proof. We follow the proof of the corresponding statement in [15, Theorem 2] for $|K| = 1$. Let $\bar{x} \in \mathcal{E}(f_{-K}, X) \setminus \mathcal{E}(f, X)$. Then, by $\bar{x} \in \mathcal{E}(f_{-K}, X)$ there exists no $x \in X$ such that $f_{-K}(x) \leq f_{-K}(\bar{x})$ and $f_j(x) < f_j(\bar{x})$ for at least one $j \in [m]_{-K}$. Hence, it follows for all $x \in X$ that $f_j(x) > f_j(\bar{x})$ for at least one $j \in [m]_{-K}$ or $f_{-K}(x) \geq f_{-K}(\bar{x})$. This, in turn, leads to

$$f_j(x) > f_j(\bar{x}) \text{ for at least one } j \in [m]_{-K} \text{ or } f_{-K}(x) = f_{-K}(\bar{x}) \tag{8}$$

for all $x \in X$. Furthermore, by using $\bar{x} \notin \mathcal{E}(f, X)$, there must exist $\tilde{x} \in X$ such that

$$f(\tilde{x}) \leq f(\bar{x}) \text{ and } f(\tilde{x}) \neq f(\bar{x}), \tag{9}$$

which implies $f_K(\tilde{x}) \leq f_K(\bar{x})$. Assume now that $f_{-K}(\tilde{x}) \neq f_{-K}(\bar{x})$. Then it follows by (8) that $f_j(\tilde{x}) > f_j(\bar{x})$ for at least one $j \in [m]_{-K}$ – contradicting the first statement in (9). Hence, we obtain $f_{-K}(\tilde{x}) = f_{-K}(\bar{x})$, and $f_K(\tilde{x}) \neq f_K(\bar{x})$ follows together with the second statement in (9). Since the other inclusion is trivially fulfilled, we are done. □

The following lemma formulates a sufficient condition for the desired subset relation for efficient solutions.

Lemma 5.3. *Let* $X \subseteq \mathbb{R}^n$ *be a nonempty set, let* $f\colon \mathbb{R}^n \to \mathbb{R}^m$ *be a vector-valued function with* $m \geq 2$*, and let* $K \subsetneq [m]$ *with* $|K| \geq 1$*. If for all* $k \in K$ *there exists a scalar-valued function* $\varphi_k\colon \mathbb{R}^{m-|K|} \to \mathbb{R}$ *with* $f_k(x) = \varphi_k(f_{-K}(x))$ *for all* $x \in X$*, then it holds*

$$\mathcal{E}(f_{-K}, X) \subseteq \mathcal{E}(f, X). \tag{10}$$

Proof. Let $\bar{x} \in \mathcal{E}(f_{-K}, X)$ and assume that $\bar{x} \notin \mathcal{E}(f, X)$. Then by Lemma 5.2 there exist $\tilde{x} \in X$ and $k \in K$ such that $f_{-K}(\tilde{x}) = f_{-K}(\bar{x})$ and $f_k(\tilde{x}) < f_k(\bar{x})$. Here, the first equation and the fact that $f_k(x) = \varphi_k(f_{-K}(x))$ holds for all $x \in X$ lead to $f_k(\tilde{x}) = \varphi_k(f_{-K}(\tilde{x})) = \varphi_k(f_{-K}(\bar{x})) = f_k(\bar{x})$, which contradicts the second strict inequality. □

At this point, we would like to note that, for $k \in K$, the scalar-valued functions φ_k introduced in Lemma 5.3 need not to be uniquely determined, as the following example illustrates.

Example 5.4. *Let $X = [0,1]$ and $f = (f_1, f_2, f_3)^\top : \mathbb{R} \to \mathbb{R}^3$ with $f_1(x) := x$, $f_2(x) := 1-x$, and $f_3(x) := \ln(1+\exp(x)+\exp(1-x))$ for all $x \in \mathbb{R}$. Moreover, let the scalar-valued functions $\varphi, \hat{\varphi} \colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $\varphi(y) := \ln(1+\exp(y_1)+\exp(1-y_1))$ for all $y \in \mathbb{R}^2$ and by $\hat{\varphi}(y) := \ln(1+\exp(y_1)+\exp(y_2))$ for all $y \in \mathbb{R}^2$, respectively. Then, it holds $f_3(x) = \varphi(y) = \hat{\varphi}(y)$ where $y = (f_1(x), f_2(x))^\top$.*

However, to prove a similar result for properly efficient solutions, we have to make additional assumptions regarding the monotonicity and continuity of the scalar-valued functions φ_k, $k \in K$. In this context, we will use the following concepts of monotonicity, see for instance [23, Definition 5.1].

Definition 5.5. *Let $Y \subseteq \mathbb{R}^\ell$ be a nonempty set. A scalar-valued function $\varphi \colon \mathbb{R}^\ell \to \mathbb{R}$ is called*

(i) monotonically increasing on Y if the implication

$$y^1 \leq y^2 \ \Rightarrow\ \varphi(y^1) \leq \varphi(y^2)$$

holds for all $y^1, y^2 \in Y$.

(ii) strictly monotonically increasing on Y if the implication

$$y^1 < y^2 \ \Rightarrow\ \varphi(y^1) < \varphi(y^2)$$

holds for all $y^1, y^2 \in Y$.

Using these concepts of monotonicity, we can now formulate a sufficient condition for the desired subset relation w.r.t. properly efficient solutions. For this, we need the following product set

$$Y_{-K} := \prod_{\ell \in [m]_{-K}} f_\ell(X) \subseteq \mathbb{R}^{m-|K|}, \tag{11}$$

which is the Cartesian product of the image sets of the objectives f_ℓ for $\ell \in [m]_{-K}$.

Lemma 5.6. *Let $X \subseteq \mathbb{R}^n$ be a nonempty set, $f \colon \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function with $m \geq 2$, $K \subsetneq [m]$ with $|K| \geq 1$, and Y_{-K} be defined as in (11). If for all $k \in K$ there exists a scalar-valued function $\varphi_k \colon \mathbb{R}^{m-|K|} \to \mathbb{R}$ with $f_k(x) = \varphi_k(f_{-K}(x))$ for all $x \in X$, which is monotonically increasing on Y_{-K} and Lipschitz continuous on an open neighbourhood of Y_{-K}, then it holds*

$$\mathcal{E}_{\mathrm{p}}(f_{-K}, X) \subseteq \mathcal{E}_{\mathrm{p}}(f, X). \tag{12}$$

Proof. Let $\bar{x} \in \mathcal{E}_{\mathrm{p}}(f_{-K}, X)$. Then, it holds $\bar{x} \in \mathcal{E}(f_{-K}, X)$ by definition and thus $\bar{x} \in \mathcal{E}(f, X)$ follows by Lemma 5.3. Furthermore, by definition, there exists a real number $\bar{M} > 0$ such that for all $i \in [m]_{-K}$ and $x \in X$ with $f_i(x) < f_i(\bar{x})$ there exists some $j \in [m]_{-(K \cup \{i\})}$ with

$$f_j(x) > f_j(\bar{x}) \text{ and } \frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leq \bar{M}. \tag{13}$$

If now

$$\bar{K} := \{i \in K \mid \; \exists z \in X\colon\, f_i(z) < f_i(\bar{x})\} = \emptyset,$$

then we are already done. Hence, we assume in the following $\bar{K} \neq \emptyset$. Let $i \in K$ and $x \in X$ with $f_i(x) = \varphi_i\left(f_{-K}(x)\right) < \varphi_i\left(f_{-K}(\bar{x})\right) = f_i(\bar{x})$. Then, since φ_i is monotonically increasing on Y_{-K}, there exists $\ell \in [m]_{-K}$ such that $f_\ell(x) < f_\ell(\bar{x})$. We define

$$\varrho := \max_{\ell \in [m]_{-K}} \{f_\ell(\bar{x}) - f_\ell(x)\} > 0.$$

Based on this, let $\tilde{\ell} \in [m]_{-K}$ with $f_{\tilde{\ell}}(\bar{x}) - f_{\tilde{\ell}}(x) = \varrho$. Furthermore, we define the function $w\colon X \to \mathbb{R}^m$ by

$$w_t(x) := \min\{f_t(x), f_t(\bar{x})\},\; t \in [m] \text{ for all } x \in X.$$

Then, it holds $w(x) \leq f(x)$ and, since φ_i is monotonically increasing on Y_{-K}, we obtain

$$\varphi_i\left(w_{-K}(x)\right) \leq \varphi_i\left(f_{-K}(x)\right).$$

Since φ_i is Lipschitz continuous on an open neighbourhood of Y_{-K}, there exists $L_i > 0$ such that for all u and v in this neighbourhood it holds $|\varphi_i(u) - \varphi_i(v)| \leq L_i\|u - v\|_\infty$. This leads to

$$\begin{array}{rcl}
0 & < & f_i(\bar{x}) - f_i(x) \\
& = & \varphi_i\left(f_{-K}(\bar{x})\right) - \varphi_i\left(f_{-K}(x)\right) \\
& \leq & \varphi_i\left(f_{-K}(\bar{x})\right) - \varphi_i\left(w_{-K}(x)\right) \\
& \leq & \left|\varphi_i\left(f_{-K}(\bar{x})\right) - \varphi_i\left(w_{-K}(x)\right)\right| \\
& \leq & L_i \cdot \|f_{-K}(\bar{x}) - w_{-K}(x)\|_\infty \\
& = & L_i \cdot \max_{\ell \in [m]_{-K}} \{f_\ell(\bar{x}) - w_\ell(x)\} \\
& = & L_i \cdot \max_{\ell \in [m]_{-K}} \{f_\ell(\bar{x}) - \min\{f_\ell(x), f_\ell(\bar{x})\}\} \\
& = & L_i \cdot \max_{\ell \in [m]_{-K}} \{\max\{0, f_\ell(\bar{x}) - f_\ell(x)\}\} \\
& = & L_i \cdot \max_{\ell \in [m]_{-K}} \{f_\ell(\bar{x}) - f_\ell(x)\} \\
& = & L_i \cdot \varrho \\
& = & L_i \cdot (f_{\tilde{\ell}}(\bar{x}) - f_{\tilde{\ell}}(x))
\end{array}$$

and thus $f_{\tilde{\ell}}(x) < f_{\tilde{\ell}}(\bar{x})$. Hence, by (13), there exists some $j \in [m]_{-(K \cup \{\tilde{\ell}\})}$ with

$$f_j(x) > f_j(\bar{x}) \text{ and } \frac{f_{\tilde{\ell}}(\bar{x}) - f_{\tilde{\ell}}(x)}{f_j(x) - f_j(\bar{x})} \leq \bar{M},$$

and we obtain

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leq L_i \cdot \frac{f_{\tilde{\ell}}(\bar{x}) - f_{\tilde{\ell}}(x)}{f_j(x) - f_j(\bar{x})} \leq L_i \cdot \bar{M}.$$

By choosing

$$M := \max\left\{\bar{M}, \max_{i \in \bar{K}}\{L_i \cdot \bar{M}\}\right\},$$

it follows $\bar{x} \in \mathcal{E}_{\mathrm{p}}(f, X)$, and thus (12). □

Based on the results obtained so far, sufficient conditions for redundancy will now be formulated. The following theorem presents a sufficient condition for weak redundancy and generalizes a known result given in [31, Theorem 3.5] presented there for $|K| = 1$.

Theorem 5.7. *Let $X \subseteq \mathbb{R}^n$ be a nonempty set, let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function with $m \geq 2$, and let $K \subsetneq [m]$ with $|K| \geq 1$. If, for all $k \in K$ there exists a scalar-valued function $\varphi_k\colon \mathbb{R}^{m-|K|} \to \mathbb{R}$ with $f_k(x) = \varphi_k(f_{-K}(x))$ for all $x \in X$, which is strictly monotonically increasing on $f_{-K}(X)$, then f_K is weakly redundant for* $\mathrm{MOP}(f, X)$.

Proof. Since $\mathcal{E}_{\mathrm{w}}(f_{-K}, X) \subseteq \mathcal{E}_{\mathrm{w}}(f, X)$ holds by Lemma 4.2, it remains to show that $\mathcal{E}_{\mathrm{w}}(f, X) \subseteq \mathcal{E}_{\mathrm{w}}(f_{-K}, X)$. In doing so, we assume that there exists $\bar{x} \in \mathcal{E}_{\mathrm{w}}(f, X) \setminus \mathcal{E}_{\mathrm{w}}(f_{-K}, X)$. Then, by definition, there exist $\tilde{x} \in X$ and $k \in K$ such that $f_{-K}(\tilde{x}) < f_{-K}(\bar{x})$ and $f_k(\tilde{x}) \geq f_k(\bar{x})$. Here, since $f_k(x) = \varphi_k(f_{-K}(x))$ holds for all $x \in X$ and φ_k is strictly monotonically increasing on $f_{-K}(X)$, the first strict inequality leads to $f_k(\tilde{x}) = \varphi_k(f_{-K}(\tilde{x})) < \varphi_k(f_{-K}(\bar{x})) = f_k(\bar{x})$, which contradicts the second inequality. □

The following theorem formulates a corresponding sufficient condition for redundancy. Note that in contrast to the weak redundancy in the previous case, monotonicity is assumed here instead of strict monotonicity for the scalar-valued functions φ_k, $k \in K$.

Theorem 5.8. *Let $X \subseteq \mathbb{R}^n$ be a nonempty set, let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function with $m \geq 2$, and let $K \subsetneq [m]$ with $|K| \geq 1$. If, for all $k \in K$ there exists a scalar-valued function $\varphi_k\colon \mathbb{R}^{m-|K|} \to \mathbb{R}$ with $f_k(x) = \varphi_k(f_{-K}(x))$ for all $x \in X$, which is monotonically increasing on $f_{-K}(X)$, then f_K is redundant for* $\mathrm{MOP}(f, X)$.

Proof. Since $\mathcal{E}(f_{-K}, X) \subseteq \mathcal{E}(f, X)$ holds by Lemma 5.3, it remains to show that $\mathcal{E}(f, X) \subseteq \mathcal{E}(f_{-K}, X)$. In doing so, we assume that there exists $\bar{x} \in \mathcal{E}(f, X) \setminus \mathcal{E}(f_{-K}, X)$. Then, by definition, there exist $\tilde{x} \in X$ and $k \in K$ such that $f_{-K}(\tilde{x}) \leq f_{-K}(\bar{x})$, $f_{-K}(\tilde{x}) \neq f_{-K}(\bar{x})$ and $f_k(\tilde{x}) > f_k(\bar{x})$. Since $f_k(x) = \varphi_k(f_{-K}(x))$ holds for all $x \in X$ and φ_k is monotonically increasing on $f_{-K}(X)$, the first inequality leads to $f_k(\tilde{x}) = \varphi_k(f_{-K}(\tilde{x})) \leq \varphi_k(f_{-K}(\bar{x})) = f_k(\bar{x})$, which contradicts the strict inequality. □

Theorem 5.8 is also stated in [41, Proposition 5.1] without a proof, but with reference to [35] (in Russian), and under the assumption that the φ_k, $k \in K$ are component-wise monotonically increasing on $f_{-K}(X)$. Based on the cited reference, the term "component-wise monotonically increasing" used there coincides with the monotonicity concept defined in Definition 5.5 (i) and assumed in Theorem 5.8. We have included the corresponding proof for completeness.

As above, we need additional assumptions regarding the Lipschitz continuity of φ_k with $k \in K$, in order to prove a similar result for properly efficient solutions.

Theorem 5.9. *Let $X \subseteq \mathbb{R}^n$ be a nonempty set, $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function with $m \geq 2$, $K \subsetneq [m]$ with $|K| \geq 1$, and Y_{-K} be defined as in* (11)*. If for all $k \in K$ there exists a scalar-valued function $\varphi_k\colon \mathbb{R}^{m-|K|} \to \mathbb{R}$ with $f_k(x) = \varphi_k(f_{-K}(x))$ for all $x \in X$, which is monotonically increasing on Y_{-K} and Lipschitz continuous on an open neighbourhood of Y_{-K}, then f_K is properly redundant for* $\mathrm{MOP}(f, X)$.

Proof. Since $\mathcal{E}_{\mathrm{p}}(f_{-K}, X) \subseteq \mathcal{E}_{\mathrm{p}}(f, X)$ holds by Lemma 5.6, it remains to show that $\mathcal{E}_{\mathrm{p}}(f, X) \subseteq \mathcal{E}_{\mathrm{p}}(f_{-K}, X)$. Hence, let $\bar{x} \in \mathcal{E}_{\mathrm{p}}(f, X)$. Then, $\bar{x} \in \mathcal{E}(f_{-K}, X)$ follows by (1) and Theorem 5.8. Furthermore, by definition, there exists a real number $\bar{M} > 0$ such that for all $i \in [m]$ and $x \in X$ with $f_i(x) < f_i(\bar{x})$ there exists some $j \in [m]_{-\{i\}}$ with

$$f_j(x) > f_j(\bar{x}) \text{ and } \frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leq \bar{M}. \tag{14}$$

Let now $i \in [m]_{-K}$ and $x \in X$ such that $f_i(x) < f_i(\bar{x})$. According to the above there exists $j \in [m]_{-\{i\}}$ with (14). If $j \in [m]_{-K}$, then we are done. Hence, we assume that $j \in K$, and thus

$$\bar{K} := \{j \in K \mid \exists z \in X\colon f_j(z) > f_j(\bar{x})\} \neq \emptyset.$$

Since $f_j(x) = \varphi_j(f_{-K}(x)) > \varphi_j(f_{-K}(\bar{x})) = f_j(\bar{x})$ and φ_j is monotonically increasing on Y_{-K}, there exists $\ell \in [m]_{-K}$ such that $f_\ell(x) > f_\ell(\bar{x})$, and we define

$$\varrho := \max_{\ell \in [m]_{-K}} \{f_\ell(x) - f_\ell(\bar{x})\} > 0.$$

Based on this, let $\tilde{\ell} \in [m]_{-K}$ with $f_{\tilde{\ell}}(x) - f_{\tilde{\ell}}(\bar{x}) = \varrho$. Furthermore, we define the function $w\colon X \to \mathbb{R}^m$ by

$$w_t(x) := \max\{f_t(x), f_t(\bar{x})\},\ t \in [m] \text{ for all } x \in X.$$

Then, it holds $w(x) \geq f(x)$ and, since φ_j is monotonically increasing on Y_{-K}, we obtain

$$\varphi_j(f_{-K}(x)) \leq \varphi_j(w_{-K}(x)).$$

Since φ_j is Lipschitz continuous on an open neighbourhood of Y_{-K}, there exists $L_j > 0$ such that for all u and v in this neighbourhood it holds $|\varphi_j(u) - \varphi_j(v)| \leq L_j \|u - v\|_\infty$. This leads to

$$\begin{array}{rcl}
0 & < & f_j(x) - f_j(\bar{x}) \\
 & = & \varphi_j(f_{-K}(x)) - \varphi_j(f_{-K}(\bar{x})) \\
 & \leq & \varphi_j(w_{-K}(x)) - \varphi_j(f_{-K}(\bar{x})) \\
 & \leq & |\varphi_j(w_{-K}(x)) - \varphi_j(f_{-K}(\bar{x}))| \\
 & \leq & L_j \cdot \|w_{-K}(x) - f_{-K}(\bar{x})\|_\infty \\
 & = & L_j \cdot \max_{\ell \in [m]_{-K}} \{w_\ell(x) - f_\ell(\bar{x})\} \\
 & = & L_j \cdot \max_{\ell \in [m]_{-K}} \{\max\{f_\ell(x), f_\ell(\bar{x})\} - f_\ell(\bar{x})\} \\
 & = & L_j \cdot \max_{\ell \in [m]_{-K}} \{\max\{0, f_\ell(x) - f_\ell(\bar{x})\}\} \\
 & = & L_j \cdot \max_{\ell \in [m]_{-K}} \{f_\ell(x) - f_\ell(\bar{x})\} \\
 & = & L_j \cdot \varrho \\
 & = & L_j \cdot (f_{\tilde{\ell}}(x) - f_{\tilde{\ell}}(\bar{x}))
\end{array}$$

and thus $f_{\tilde{\ell}}(x) > f_{\tilde{\ell}}(\bar{x})$. Since $i \in [m]_{-K}$ and $f_i(x) < f_i(\bar{x})$, it follows $\tilde{\ell} \neq i$, that is,

$$\tilde{\ell} \in [m]_{-(K \cup \{i\})}.$$

Hence, we obtain

$$\frac{f_i(\bar{x}) - f_i(x)}{f_{\tilde{\ell}}(x) - f_{\tilde{\ell}}(\bar{x})} \leq L_j \cdot \frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leq L_j \cdot \bar{M}.$$

By choosing

$$M := \max\left\{\bar{M}, \max_{j \in \bar{K}}\{L_j \cdot \bar{M}\}\right\},$$

it follows $\bar{x} \in \mathcal{E}_{\mathrm{p}}(f_{-K}, X)$, and thus $\mathcal{E}_{\mathrm{p}}(f, X) \subseteq \mathcal{E}_{\mathrm{p}}(f_{-K}, X)$. ☐

The following example addresses the requirements for the scalar-valued functions $\varphi_k\colon \mathbb{R}^{m-|K|} \to \mathbb{R}$, $k \in K$, as formulated in the previous results of this section.

Example 5.10. *We consider again the multiobjective optimization problem and the corresponding scalar-valued functions $\varphi, \hat{\varphi}\colon \mathbb{R}^2 \to \mathbb{R}$ introduced in Example 5.4. Then, there exist no $y^1, y^2 \in f_{-\{3\}}(X) = \operatorname{conv}\left((0,1)^\top, (1,0)^\top\right) \subseteq \mathbb{R}^2$ with $y^1 \leq y^2$ and $y^1 \neq y^2$. Thus, φ is, by definition, both monotonically increasing and strictly monotonically increasing on $f_{-\{3\}}(X)$. Furthermore, it is easy to verify that φ is neither monotonically increasing nor strictly monotonically increasing on $Y_{-\{3\}} = [0,1]^2$. That is, while φ satisfies the requirements for the scalar-valued functions in Theorem 5.7 and in Theorem 5.8, it does not satisfy the requirements in Lemma 5.6 and in Theorem 5.9. Consequently, φ can only be applied to ensure that f_3 is weakly redundant and redundant.*

On the other hand, $\hat{\varphi}$ is both monotonically increasing and strictly monotonically increasing on $\mathbb{R}^2$. Since $\hat{\varphi}$ is also Lipschitz continuous on $\mathbb{R}^2$ with Lipschitz constant $L = \sqrt{2}$ (w.r.t. the Euclidean norm), it satisfies the requirements for the scalar-valued functions in the lemma and in all three theorems listed.

The fact that φ does not fulfill the assumptions of Lemma 5.6 and Theorem 5.9 does not mean that the corresponding redundancy properties fail as the formulated conditions are only sufficient. The representation by $\hat{\varphi}$, on the other hand, satisfies all corresponding assumptions and implies that f_3 is even properly redundant. In this sense, the application of the sufficient conditions depends on the chosen representation.

It is well known (see, for instance, [23, Example 5.2]) that affine-linear functions $\varphi\colon \mathbb{R}^\ell \to \mathbb{R}$, defined by $\varphi(x) := a^\top x + \beta$ for all $x \in \mathbb{R}^\ell$, are strictly monotonically increasing on $\mathbb{R}^\ell$ if $a \in \mathbb{R}^\ell_+ \setminus \{0\}$ and $\beta \in \mathbb{R}$, and are monotonically increasing and Lipschitz continuous on $\mathbb{R}^\ell$ if $a \in \mathbb{R}^\ell_+$ and $\beta \in \mathbb{R}$, respectively. Thus, from the previous results in this section, we obtain the following corollary.

Corollary 5.11. *Let $X \subseteq \mathbb{R}^n$ be a nonempty set, let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function with $m \geq 2$, and let $K \subsetneq [m]$ with $|K| \geq 1$. Then it holds:*

(i) If for all $k \in K$ there exist coefficients $a_i^k \in \mathbb{R}_+$, $i \in [m]_{-K}$, not all zero, and $\beta_k \in \mathbb{R}$ such that $f_k(x) = \sum_{i\in[m]_{-K}} a_i^k f_i(x) + \beta_k$ for all $x \in X$, then f_K is weakly redundant for $\mathrm{MOP}(f,X)$.

(ii) If for all $k \in K$ there exist coefficients $a_i^k \in \mathbb{R}_+$, $i \in [m]_{-K}$ and $\beta_k \in \mathbb{R}$ such that $f_k(x) = \sum_{i\in[m]_{-K}} a_i^k f_i(x) + \beta_k$ for all $x \in X$, then f_K is redundant and properly redundant for $\mathrm{MOP}(f,X)$.

The following remark provides some context for the statements in the last corollary.

Remark 5.12. *(i) Under the assumptions of Corollary 5.11 (ii), the subset relation $\mathcal{E}(f_{-K}, X) \subseteq \mathcal{E}(f, X)$ (cf. (10)) holds, as stated in [41, Proposition 5.2] under the weaker condition that for all $k \in K$ there exist $a_i^k \in \mathbb{R}$, $i \in [m]_{-K}$ and $\beta_k \in \mathbb{R}$ such that $f_k = \sum_{i\in[m]_{-K}} a_i^k f_i + \beta_k$.*
(ii) The result for proper redundancy, given in Corollary 5.11 (ii), has already been shown under additional convexity assumptions on f and X in [31, Theorem 4.2].
(iii) If, in Corollary 5.11 (ii), the set K consists of only one element k and it holds $a_i^k = 0$ for all $i \in [m]_{-\{k\}}$, then we obtain $f_k(x) \equiv \beta_k$ for all $x \in X$. Consequently, every function f_k that is constant on X is redundant and properly redundant for $\mathrm{MOP}(f,X)$.

Of course, in addition to affine-linear functions, there are many other possibilities for scalar-valued functions φ that exhibit the corresponding monotonicity and Lipschitz continuity properties.

6 Redundancy and related topics

In the preceding sections, we examined the redundancy of objective functions in multiobjective optimization for various solution concepts, which, as already mentioned, is the primary motivation for this work. At the end of the previous section, sufficient conditions were formulated for all the considered concepts (cf. Theorem 5.7, Theorem 5.8, and Theorem 5.9). In Subsection 6.1 we will show that the sufficient conditions for weak redundancy and redundancy formulated there imply even stronger properties. These properties can be intuitively described in the image space as the preservation of the (corresponding) dominance structure over the entire image set during the transition from f to f_{-K}. For the sake of distinction, we will refer to these concepts as order-redundancy. Note that proper efficiency also depends on quantitative bounds on objective trade-offs. Therefore, we will not consider this solution concept here.

Furthermore, objective redundancy alone does not ensure that a numerical method works in the same way for the original and the reduced problem. Therefore, we briefly examine in the forthcoming Subsection 6.2 additionally in which cases removing redundant functions preserves descent directions and stationary points to stress this difference.

6.1 Order-redundancy

To describe both order-redundancy properties considered here, we use the following two binary relations: For a nonempty set $Z \subseteq \mathbb{R}^n$ and a given function $g\colon \mathbb{R}^n \to \mathbb{R}^\ell$ with $\ell \geq 1$, the binary relations $\prec_{g,Z}$ and $\preceq_{g,Z}$ are defined by

$$\prec_{g,Z} := \{(x^1, x^2) \in Z \times Z \mid g(x^1) < g(x^2)\}$$

and

$$\preceq_{g,Z} := \{(x^1, x^2) \in Z \times Z \mid g(x^1) \leq g(x^2)\}.$$

We would like to point out that, with regard to these notations, we follow [5, 6, 7, 8], where the binary relation $\preceq_{g,Z}$ is already defined and the binary relation $\prec_{g,Z}$ is (at least) briefly mentioned. By using these two binary relations, we are now able to define both order-redundancy concepts we will use in the following.

Definition 6.1. *Let $X \subseteq \mathbb{R}^n$ be a nonempty set, let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function with $m \geq 2$, and let $K \subsetneq [m]$ with $|K| \geq 1$. Then, f_K is called*

(i) $<$–order-redundant on X w.r.t. f if $\prec_{f_{-K},X} = \prec_{f,X}$, i.e., it holds

$$\{(x^1, x^2) \in X \times X \mid f_{-K}(x^1) < f_{-K}(x^2)\} = \{(x^1, x^2) \in X \times X \mid f(x^1) < f(x^2)\}.$$

(ii) $\leq$–order-redundant on X w.r.t. f if $\preceq_{f_{-K},X} = \preceq_{f,X}$, i.e., it holds

$$\{(x^1, x^2) \in X \times X \mid f_{-K}(x^1) \leq f_{-K}(x^2)\} = \{(x^1, x^2) \in X \times X \mid f(x^1) \leq f(x^2)\}.$$

Based on this, we call a function f_k $<$–order-redundant on X w.r.t. f and $\leq$–order-redundant on X w.r.t. f if the corresponding properties are fulfilled for $K = \{k\}$.

Note that, under the assumptions of Definition 6.1, the subset relations

$$\prec_{f,X} \subseteq \prec_{f_{-K},X}, \ \preceq_{f,X} \subseteq \preceq_{f_{-K},X}, \ \prec_{f,X} \subseteq \preceq_{f,X} \text{ and } \prec_{f_{-K},X} \subseteq \preceq_{f_{-K},X} \tag{15}$$

are fulfilled by definition. The following lemma shows that order-redundancy implies the redundancy considered so far.

Lemma 6.2. *Let $X \subseteq \mathbb{R}^n$ be a nonempty set, let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function with $m \geq 2$, and let $K \subsetneq [m]$ with $|K| \geq 1$. Then it holds:*

(i) If f_K is $<$–order-redundant on X w.r.t. f, then f_K is weakly redundant for $\mathrm{MOP}(f, X)$.

(ii) If f_K is $\leq$–order-redundant on X w.r.t. f, then f_K is redundant for $\mathrm{MOP}(f, X)$.

Proof. We restrict ourselves to the proof of statement (ii). The proof of (i) is similar and even simpler. In doing so, let f_K be $\leq$–order-redundant on X w.r.t. f, let $\bar{x} \in \mathcal{E}(f, X)$, and assume that $\bar{x} \notin \mathcal{E}(f_{-K}, X)$. Then, by definition, there exists $\tilde{x} \in X$ with $(\tilde{x}, \bar{x}) \in \preceq_{f_{-K},X}$ and $f_{-K}(\tilde{x}) \neq f_{-K}(\bar{x})$. Using the $\leq$–order-redundancy of f_K on X w.r.t. f, it follows $(\tilde{x}, \bar{x}) \in \preceq_{f,X}$. As also $f(\tilde{x}) \neq f(\bar{x})$, this contradicts $\bar{x} \in \mathcal{E}(f, X)$. Hence, we obtain $\mathcal{E}(f, X) \subseteq \mathcal{E}(f_{-K}, X)$.

To show the opposite inclusion, let now $\bar{x} \in \mathcal{E}(f_{-K}, X)$ and assume that $\bar{x} \notin \mathcal{E}(f, X)$. Then, by definition, there exists $\tilde{x} \in X$ such that $(\tilde{x}, \bar{x}) \in \preceq_{f,X}$ and $f(\tilde{x}) \neq f(\bar{x})$, which leads, on the one hand, to

$$(\tilde{x}, \bar{x}) \in \preceq_{f_{-K},X} \tag{16}$$

and, on the other hand, to $(\bar{x}, \tilde{x}) \notin \preceq_{f,X}$, which in turn implies $(\bar{x}, \tilde{x}) \notin \preceq_{f_{-K},X}$ by the $\leq$–order-redundancy of f_K on X. Hence, there exists $j \in [m]_{-K}$ such that $f_j(\bar{x}) > f_j(\tilde{x})$ and we obtain $f_{-K}(\tilde{x}) \neq f_{-K}(\bar{x})$, which, together with (16), contradicts $\bar{x} \in \mathcal{E}(f_{-K}, X)$. Thus, it holds $\mathcal{E}(f_{-K}, X) \subseteq \mathcal{E}(f, X)$, and we are done. □

The following example shows, on the other hand, that redundancy does not imply order-redundancy. Therefore, order-redundancy is a stronger property than redundancy.

Example 6.3. (i) *We consider again the multiobjective optimization problem defined in Example 3.2. Here, as mentioned above, f_3 is weakly redundant for* $\mathrm{MOP}(f, X)$. *Furthermore, let $x^1 := (\frac{1}{2}, 1)^\top$ and $x^2 := (\frac{3}{4}, 1)^\top$. Then it holds $f(x^1) = (\frac{1}{4}, \frac{3}{4}, 1)^\top$ and $f(x^2) = (\frac{9}{16}, \frac{13}{16}, 1)^\top$, which leads to $(x^1, x^2) \in \prec_{f_{-\{3\}},X}$ and $(x^1, x^2) \notin \prec_{f,X}$. Consequently, $\prec_{f_{-\{3\}},X} \nsubseteq \prec_{f,X}$ and f_3 is not $<$–order-redundant on X w.r.t. f.*
(ii) *We consider now the linear multiobjective optimization problem introduced in Example 3.1 (ii). Here, as mentioned above, f_3 is (properly) redundant for* $\mathrm{MOP}(f, X)$. *Let now $x^1 := (0, \frac{1}{2})^\top$ and $x^2 := (1, 0)^\top$. Then it holds $f(x^1) = (0, \frac{1}{2}, \frac{1}{2})^\top$ and $f(x^2) = (1, 1, 0)^\top$, and we obtain $(x^1, x^2) \in \preceq_{f_{-\{3\}},X}$ and $(x^1, x^2) \notin \preceq_{f,X}$. Thus, $\preceq_{f_{-\{3\}},X} \nsubseteq \preceq_{f,X}$ and f_3 is not $\leq$–order-redundant on X w.r.t. f.*

The following result shows that the sufficient conditions for weak redundancy (see Theorem 5.7) and redundancy (see Theorem 5.8) also imply the corresponding order-redundancy properties. The proofs of the two statements of Lemma 6.4 use the first two subset relations of (15). Since the remaining parts of the proofs can be taken almost unchanged from the corresponding proofs of the relevant theorems in Section 5, they have been omitted here.

Lemma 6.4. *Let $X \subseteq \mathbb{R}^n$ be a nonempty set, let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function with $m \geq 2$, and let $K \subsetneq [m]$ with $|K| \geq 1$. Then it holds:*

(i) If for all $k \in K$ there exists a scalar-valued function $\varphi_k\colon \mathbb{R}^{m-|K|} \to \mathbb{R}$ with $f_k(x) = \varphi_k(f_{-K}(x))$ for all $x \in X$, which is strictly monotonically increasing on $f_{-K}(X)$, then f_K is $<$–order-redundant on X w.r.t. f.

(ii) If for all $k \in K$ there exists a scalar-valued function $\varphi_k\colon \mathbb{R}^{m-|K|} \to \mathbb{R}$ with $f_k(x) = \varphi_k(f_{-K}(x))$ for all $x \in X$, which is monotonically increasing on $f_{-K}(X)$, then f_K is $\leq$–order-redundant on X w.r.t. f.

Moreover, it turns out that for the statement in Lemma 6.4 (*ii*), the converse also holds. To verify this, we need the following result.

Lemma 6.5. *Let $X \subseteq \mathbb{R}^n$ be a nonempty set, let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function with $m \geq 2$, and let $K \subsetneq [m]$ with $|K| \geq 1$. If f_K is $\leq$–order-redundant on X w.r.t. f, then for all $x^1, x^2 \in X$, the following implication holds:*

$$f_{-K}(x^1) = f_{-K}(x^2) \;\Rightarrow\; f_K(x^1) = f_K(x^2).$$

Proof. Let $x^1, x^2 \in X$ with $f_{-K}(x^1) = f_{-K}(x^2)$. Then, by definition, both $(x^1, x^2) \in \preceq_{f_{-K},X}$ and $(x^2, x^1) \in \preceq_{f_{-K},X}$ are fulfilled. Using the $\leq$–order-redundancy of f_K on X w.r.t. f it follows $(x^1, x^2) \in \preceq_{f,X}$ and $(x^2, x^1) \in \preceq_{f,X}$. Thus, again by definition, we obtain $f(x^1) = f(x^2)$, and consequently, in particular, $f_K(x^1) = f_K(x^2)$. □

This yields the aforementioned equivalence result in the case of $\leq$–order-redundancy.

Theorem 6.6. *Let $X \subseteq \mathbb{R}^n$ be a nonempty set, let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function with $m \geq 2$, and let $K \subsetneq [m]$ with $|K| \geq 1$. Then f_K is $\leq$–order-redundant on X w.r.t. f if and only if, for all $k \in K$, there exists a scalar-valued function $\varphi_k\colon \mathbb{R}^{m-|K|} \to \mathbb{R}$ with $f_k(x) = \varphi_k(f_{-K}(x))$ for all $x \in X$, which is monotonically increasing on $f_{-K}(X)$.*

Proof. By Lemma 6.4 (*ii*), it suffices to show that the $\leq$–order-redundancy of f_K on X w.r.t. f implies the existence of the desired scalar-valued functions φ_k, $k \in K$. Therefore, let

$$(f_{-K})^{-1}(y) := \{x \in X \mid y = f_{-K}(x)\} \text{ for all } y \in f_{-K}(X)$$

and let $y \in f_{-K}(X)$. Then for any $x \in (f_{-K})^{-1}(y)$ we obtain, by Lemma 6.5, the same value $f_K(x)$. This allows us to define for every $k \in K$ the corresponding function $\varphi_k\colon \mathbb{R}^{m-|K|} \to \mathbb{R}$ by $\varphi_k(y) := f_k(x)$ if $y \in f_{-K}(X)$ where $x \in (f_{-K})^{-1}(y)$, and $\varphi_k(y) := 0$ if $y \notin f_{-K}(X)$. Therefore, for all $k \in K$ the functions φ_k are well-defined and $f_k(x) = \varphi_k(f_{-K}(x))$ holds for all $x \in X$ by definition. Thus, all that remains is to show the monotonicity of φ_k for all $k \in K$ on $f_{-K}(X)$. In doing so, let $k \in K$ and $y^1, y^2 \in f_{-K}(X)$ with $y^1 \leq y^2$. Then there exist $x^1, x^2 \in X$ with $f_{-K}(x^1) = y^1 \leq y^2 = f_{-K}(x^2)$. Hence, it holds $(x^1, x^2) \in \preceq_{f_{-K},X}$, and by the $\leq$–order-redundancy of f_K on X w.r.t. f we obtain $(x^1, x^2) \in \preceq_{f,X}$, i.e., $f(x^1) \leq f(x^2)$. In particular, given the definition of φ_k, this leads to $\varphi_k(y^1) = f_k(x^1) \leq f_k(x^2) = \varphi_k(y^2)$, and we are done. □

The proof of Lemma 6.5 cannot be extended to $<$–order-redundancy. The reason for this is that by definition, for $(x^1, x^2) \in X \times X$ with $f_{-K}(x^1) = f_{-K}(x^2)$, no corresponding conclusions regarding the equality of $f_K(x^1)$ and $f_K(x^2)$ can be directly derived, that is, without any further assumptions on X, f or φ_k with $k \in K$. The following example shows that the statement of Theorem 6.6 cannot be directly applied to this concept of redundancy, and that neither of the two considered redundancy concepts implies the other.

Example 6.7. *Let $X = \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}^2$ be given by $f = (f_1, f_2)^\top$ with*

$$f_1(x) := \begin{cases} -x^2, & \text{if } x \leq 0 \\ 0, & \text{if } 0 < x \leq 1, \\ (x-1)^2, & \text{if } 1 < x \end{cases} \quad \text{and } f_2(x) := x$$

for all $x \in \mathbb{R}$.

We first consider the case $K = \{2\}$ and show that f_2 is $<$–order-redundant on X w.r.t. f. In doing so, let $(x^1, x^2) \in \prec_{f_{-\{2\}},X}$, i.e., it holds $f_1(x^1) < f_1(x^2)$. Since f_1 is monotonically increasing on $\mathbb{R}$, it follows that $x^1 < x^2$, and thus, by the definition of f_2, that $f_2(x^1) < f_2(x^2)$. Therefore, it holds $f(x^1) < f(x^2)$, and consequently $(x^1, x^2) \in \prec_{f,X}$, i.e., $\prec_{f_{-\{2\}},X} \subseteq \prec_{f,X}$. The $<$–order-redundancy of f_2 follows now by the first subset relation of (15).

Moreover, it holds $f_1(0) = f_1(1) = 0$, $f_2(0) = 0$, and $f_2(1) = 1$. This shows, first of all, that since $(1, 0) \in \preceq_{f_{-\{2\}},X}$ and $(1, 0) \notin \preceq_{f,X}$, the function f_2 is not $\leq$–order-redundant on X w.r.t. f. On the other hand, since $f_1(0) = f_1(1)$ and $f_2(0) \neq f_2(1)$, it also follows that there is no scalar-valued function $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that $f_2(x) = \varphi(f_1(x))$ for all $x \in X$, and thus certainly no strictly monotonically increasing function with this property.

Let now $K = \{1\}$. Here we show that f_1 is $\leq$–order-redundant on X w.r.t. f, but not $<$–order-redundant. Therefore, let $(x^1, x^2) \in \preceq_{f_{-\{1\}},X}$, i.e., it holds $f_2(x^1) \leq f_2(x^2)$ and thus $x^1 \leq x^2$. Since f_1 is monotonically increasing on $\mathbb{R}$, it follows that also $f_1(x^1) \leq f_1(x^2)$. Hence, it holds $f(x^1) \leq f(x^2)$, and consequently $(x^1, x^2) \in \preceq_{f,X}$, i.e., $\preceq_{f_{-\{1\}},X} \subseteq \preceq_{f,X}$. The $\leq$–order-redundancy of f_1 follows now by the second subset relation of (15). Finally, since $(0, 1) \in \prec_{f_{-\{1\}},X}$ and $(0, 1) \notin \prec_{f,X}$, the function f_1 is not $<$–order-redundant on X w.r.t. f.

The authors in [5, 6, 7, 8] address the $\mathcal{NP}$-hard combinatorial problem of determining a subset K with the largest possible cardinality such that f_K satisfies the $\leq$-order redundancy property. In this context, they also investigate approximate variants of order redundancy and consider both exact and heuristic algorithms. Moreover, in [9], the authors give a similar definition of $\leq$-order redundancy for pairs of objectives. More precisely, they say for two objectives f_i and f_j, $i, j \in [m], i \neq j$, that f_i "supports" f_j if $f_i(x) \leq f_i(y)$ implies $f_j(x) \leq f_j(y)$ for all $x, y \in X$.

6.2 Descent directions and stationary points

Motivated by order-redundancy, the question now arises whether not only the order but also descent directions and stationary points can be preserved when various redundant objectives are removed. For these reasons, we assume in this subsection that the vector-valued function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable and restrict ourselves to the case $X = \mathbb{R}^n$. Considering the constrained case would require assumptions on the structure of the feasible set and would go beyond the scope of this brief discussion. By analogy with Definition 2.1 (ii), we say that $\bar{x} \in \mathbb{R}^n$ is a locally efficient solution of $\mathrm{MOP}(f, \mathbb{R}^n)$ if there exists a neighbourhood $U \subseteq \mathbb{R}^n$ of $\bar{x}$ such that there exists no $x \in U$ with $f(x) \leq f(\bar{x})$ and $f(x) \neq f(\bar{x})$. Based on this we denote the set containing all locally efficient solutions of $\mathrm{MOP}(f, \mathbb{R}^n)$ by $\mathcal{E}^\ell(f, \mathbb{R}^n)$. Then, a necessary condition for a locally efficient solution of $\mathrm{MOP}(f, \mathbb{R}^n)$ is given by the following known result; cf., e.g., [17] and the references therein.

Lemma 6.8. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a continuously differentiable vector-valued function. If $x \in \mathcal{E}^\ell(f, \mathbb{R}^n)$, then there exists no $d \in \mathbb{R}^n \setminus \{0\}$ with*

$$\nabla f_i(x)^\top d < 0 \text{ for all } i \in [m]. \tag{17}$$

Note that if for $x \in \mathbb{R}^n$ there exists no $d \in \mathbb{R}^n \setminus \{0\}$ that satisfies (17), then x is called a *stationary point* of f; see, e.g., [16]. A direction $d \in \mathbb{R}^n \setminus \{0\}$ that satisfies (17) for an $x \in \mathbb{R}^n$ is referred to as a *descent direction* of f at x (see, for instance, [10, Definition 3]). Following this, we define for a given continuously differentiable function $g\colon \mathbb{R}^n \to \mathbb{R}^\ell$ with $\ell \geq 1$ and $x \in \mathbb{R}^n$, the set of descent directions of g at x by

$$\mathcal{D}_<(g, x) := \{d \in \mathbb{R}^n \mid \nabla g_i(x)^\top d < 0 \text{ for all } i \in [\ell]\}.$$

In case of $\mathcal{D}_<(g, x) = \emptyset$ the point x is a stationary point of g. Note that, since $\mathcal{D}_<(f, x) \subseteq \mathcal{D}_<(f_{-K}, x)$ holds for every $x \in \mathbb{R}^n$ by definition, every stationary point of f_{-K} is also a stationary point of f. Furthermore, if $\mathcal{D}_<(f, x) = \mathcal{D}_<(f_{-K}, x)$ is fulfilled for all $x \in \mathbb{R}^n$, then f_{-K} and f have the same stationary points. The following example shows that $<$–order-redundancy generally guarantees neither $\mathcal{D}_<(f, x) = \mathcal{D}_<(f_{-K}, x)$ for all $x \in \mathbb{R}^n$ nor preservation of stationary points.

Example 6.9. *Let $f : \mathbb{R} \to \mathbb{R}^2$ be given by $f_1(x) := x$ and $f_2(x) := x^3$ for all $x \in \mathbb{R}$. Since both f_1 and f_2 are strictly monotonically increasing on $\mathbb{R}$, both functions are also $<$-order-redundant on $\mathbb{R}$ w.r.t. f. Furthermore, it holds $f_1'(0) = 1$ and $f_2'(0) = 0$, which leads to $\mathcal{D}_<(f_{-\{2\}}, 0) = \{d \in \mathbb{R} \mid d < 0\}$ and $\mathcal{D}_<(f, 0) = \emptyset$. Thus, $f_{-\{2\}}$ and f do not have the same set of descent directions at $x = 0$. In particular, $x = 0$ is a stationary point of f, but not of $f_{-\{2\}}$. Hence, $<$-order-redundancy preserves neither common descent directions nor stationary points in general.*

In contrast, the following lemma provides a first sufficient condition regarding the desired property for the descent directions.

Lemma 6.10. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a continuously differentiable vector-valued function with $m \geq 2$ and $K \subsetneq [m]$ with $|K| \geq 1$. Moreover, suppose for all $k \in K$ that there exist $a_i^k \in \mathbb{R}_+$, $i \in [m]_{-K}$, not all zero, and $\beta_k \in \mathbb{R}$ such that*

$$f_k(x) = \sum_{i \in [m]_{-K}} a_i^k f_i(x) + \beta_k \tag{18}$$

is satisfied for all $x \in \mathbb{R}^n$. Then it holds

$$\mathcal{D}_<(f_{-K}, x) = \mathcal{D}_<(f, x)$$

for all $x \in \mathbb{R}^n$. Consequently, f and f_{-K} have the same stationary points.

Proof. Let $x \in \mathbb{R}^n$. Since $\mathcal{D}_<(f,x) \subseteq \mathcal{D}_<(f_{-K},x)$ is trivially fulfilled by definition, we only need to show the opposite subset relation. Therefore, let $d \in \mathcal{D}_<(f_{-K},x)$, i.e., it holds $\nabla f_i(x)^\top d < 0$ for all $i \in [m]_{-K}$, and let $k \in K$. Then, we obtain by differentiating f_k that

$$\nabla f_k(x) = \sum_{i \in [m]_{-K}} a_i^k \nabla f_i(x), \tag{19}$$

and thus

$$\nabla f_k(x)^\top d = \sum_{i \in [m]_{-K}} a_i^k \nabla f_i(x)^\top d < 0.$$

Since $k \in K$ was chosen arbitrarily, we are done. □

Note that the statement of Lemma 6.10 does not hold in general for non-affine-linear functions φ_k. To verify this, see again Example 6.9, where $f_2 = f_1^3$.

In the proof of Lemma 6.10 we can replace assumption (18) by (19), and (19) can even be weakened further: It is enough to assume for all $k \in K$ that there exist maps $a_i^k \colon \mathbb{R}^n \to \mathbb{R}_+$, $i \in [m]_{-K}$ with $a_i^k(x)$, $i \in [m]_{-K}$ not all zero for any $x \in \mathbb{R}^n$ such that

$$\nabla f_k(x) = \sum_{i \in [m]_{-K}} a_i^k(x) \nabla f_i(x) \tag{20}$$

holds for all $x \in \mathbb{R}^n$. This allows us to formulate a similar result to Lemma 6.10 for more general nonlinear functions φ_k.

Theorem 6.11. *Let $f \colon \mathbb{R}^n \to \mathbb{R}^m$ be a continuously differentiable vector-valued function with $m \geq 2$ and $K \subsetneq [m]$ with $|K| \geq 1$. Moreover, suppose for all $k \in K$ that there exists a continuously differentiable function $\varphi_k \colon \mathbb{R}^{m-|K|} \to \mathbb{R}$ such that $f_k(x) = \varphi_k(f_{-K}(x))$ for all $x \in \mathbb{R}^n$ and $\nabla \varphi_k(y) \in \mathbb{R}_+^{m-|K|} \setminus \{0\}$ for all $y \in f_{-K}(\mathbb{R}^n)$. Then it holds*

$$\mathcal{D}_<(f_{-K},x) = \mathcal{D}_<(f,x)$$

for all $x \in \mathbb{R}^n$. Consequently, f and f_{-K} have the same stationary points.

Proof. Let $x \in \mathbb{R}^n$. The multivariable chain rule leads with $[m]_{-K} =: \{i_1, \ldots, i_{m-|K|}\}$ to

$$\nabla f_k(x) = \sum_{j=1}^{m-|K|} \frac{\partial \varphi_k}{\partial y_j}(f_{-K}(x))\, \nabla f_{i_j}(x)$$

for all $k \in K$. By setting $a_{i_j}^k(x) := \frac{\partial \varphi_k}{\partial y_j}(f_{-K}(x))$ for all $j \in [m-|K|]$, we see that (20) is satisfied and thus we can apply the same steps as in the proof of Lemma 6.10. □

Note that in Example 6.9 we have $\varphi_2(y) = y^3$ and thus $\varphi_2'(y) \geq 0$ for all $y \in f_1(\mathbb{R})$. However, due to $0 \in f_1(\mathbb{R})$ and $\varphi_2'(0) = 0$, the assumptions of Theorem 6.11 are not fulfilled. Finally, it is worth noting that if φ_k is strictly monotonically increasing and continuously differentiable on $\mathbb{R}^{m-|K|}$, then it already holds $\nabla \varphi_k(y) \in \mathbb{R}_+^{m-|K|}$ for all $y \in \mathbb{R}^{m-|K|}$.

7 Conclusion and outlook

Our results show that objective redundancy depends essentially on the considered solution concept. In general, weak redundancy, redundancy, and proper redundancy do not imply each other, so an objective function that is redundant in one sense need not to be redundant in another. A notable difference also appears under iterative removal of redundant objectives. For weak redundancy, this leads to a maximal set of weakly redundant objectives, although the result needs not to be necessarily a set with the highest possible cardinality. For (proper) redundancy, the same strategy needs not even to yield a maximal set, since some objectives may be removable only jointly. In addition, we have formulated sufficient conditions for all three concepts of redundancy. Furthermore, preserving the solution sets does not necessarily preserve the order relations between feasible points. In this context, even the stronger concept of order-redundancy, which is also implied under the same sufficient conditions, does not ensure that common descent directions or stationary points remain unchanged after removing various redundant objectives. To address this issue, we also specified sufficient conditions under which both are preserved.

For further research, it would be important to focus on sufficient conditions for each redundancy concept that are easy to verify but do not imply order-redundancy. Finally, it is also of interest to examine more closely the effects of removing redundant objective functions on various solution methods.

Acknowledgments: The authors express their gratitude to Oliver Stein and Felix Neussel for fruitful discussions.

Funding: The work of the first author and the third author is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - Project-ID 528525668.

Declaration of generative AI and AI-assisted technologies
The authors used ChatGPT (OpenAI) for language editing, translation of a Russian text, literature search, TikZ assistance, and checking some mathematical arguments. It was not used for autonomous content creation or to generate the main scientific results or conclusions.

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