

# Signed Budget Uncertainty for Robust Mixed-Integer Optimization

Dimitris Bertsimas

Sloan School of Management, Massachusetts Institute of Technology, USA, dbertsim@mit.edu

Danique de Moor

Sloan School of Management, Massachusetts Institute of Technology, USA, demoor@mit.edu

Dick den Hertog

Amsterdam Business School, University of Amsterdam, The Netherlands, d.denhertog@uva.nl

Federica Donnini

Department of Information Engineering, University of Florence, Italy, federica.donnini@unifi.it

Classical budget uncertainty is widely used in robust optimization. It bounds the aggregate absolute deviation of uncertain parameters from nominal values and yields tractable robust counterparts. In many applications, however, information concerns aggregate signed deviations. We therefore study signed budget uncertainty, in which bounds are imposed on signed aggregates rather than absolute deviations. For robust mixed-integer optimization problems with binary uncertainty-affected variables, we derive a compact robust counterpart requiring only two additional variables and two additional constraints per uncertainty-affected constraint, independently of the number of uncertain coefficients. This contrasts sharply with the standard robust counterpart, whose size grows linearly with the number of uncertain parameters. We further study laminar signed budget uncertainty to capture hierarchical structures arising from geographical, organizational, or categorical information. We derive a compact robust counterpart whose size scales with the number of aggregate sets rather than the number of uncertain coefficients. For problems with a single uncertainty-affected constraint, we show that the robust problem can alternatively be solved through  $2^K$  nominal problems, where  $K$  is the number of aggregate sets. Experiments demonstrate the computational advantages and show that signed budget uncertainty can produce solutions not obtainable by adjusting the classical budget parameter.

*Key words:* Robust Optimization, Signed Budget Uncertainty, Mixed-Integer Optimization, Laminar Families

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## 1. Introduction

Robust optimization (RO) is a widely used framework for decision-making under uncertainty, particularly when uncertain parameters cannot be accurately described by probability distributions but are known to belong to a prescribed uncertainty set. The specification of the uncertainty set is a central modeling choice, since it directly affects both the conservatism of the resulting solutions and the computational tractability of the robust counterpart (RC).

One of the most influential and widely used uncertainty sets in RO is the budget uncertainty set introduced by Bertsimas and Sim (2004). In this uncertainty set, each uncertain parameter can deviate within an individual interval around its nominal value, while a budget limits the sum of the absolute deviations. This construction provides a tractable alternative to box uncertainty and has been widely used in robust combinatorial optimization, network design, facility location, scheduling, routing, and many other application domains (e.g., Bertsimas and Sim (2003), Bertsimas et al. (2011), Alves Pessoa et al. (2015), Bougeret et al. (2019), Agra et al. (2013)).

In many applications, however, reliable information is available for aggregate signed quantities rather than for absolute deviations. For example, forecasts and confidence intervals may be available for total regional demand or total population. Probabilistic limit results such as the Central Limit Theorem (CLT) also directly provide bounds on aggregate signed deviations.

Motivated by this, we study *signed budget uncertainty sets*. As in classical budget uncertainty, each uncertain parameter is individually bounded, but the aggregate constraint is imposed on the sum of the signed deviations rather than on the sum of their absolute values. This distinction is fundamental. Classical budget uncertainty restricts the sum of the magnitudes of the deviations and therefore prevents many coefficients from simultaneously deviating substantially, regardless of the direction. Signed budget uncertainty instead restricts their aggregate net deviation, allowing positive and negative deviations to offset one another as long as the resulting aggregate remains within the prescribed interval.

Uncertainty information is often available at multiple levels of aggregation. Demand may be estimated at the level of cities, provinces, regions, and countries; and customers may be grouped according to product categories. Multiple signed aggregate bounds also arise naturally in hierarchical forecasting (Athanasopoulos et al. 2024), where forecasts and prediction intervals are produced at several levels of aggregation. These applications motivate uncertainty sets with multiple aggregate constraints organized hierarchically. To model such settings, we consider *laminar signed budget uncertainty sets*, which impose signed aggregate bounds on subsets of uncertain coefficients corresponding to a laminar family of sets. A family of sets is laminar if every pair of sets is either disjoint or one is contained in the other. The signed budget uncertainty set with a single aggregate bound is recovered as the special case in which the laminar family consists of a single set containing all uncertain coefficients.

In many applications, the number of uncertain coefficients can be extremely large while the number of aggregate sets remains relatively small. For example, thousands of population points may be grouped into a limited number of municipalities, provinces, and regions. In such settings, robust formulations whose size scales with the number of uncertain coefficients quickly become

impractical, whereas formulations whose size depends only on the aggregation structure remain computationally tractable.

Uncertainty sets based on signed aggregate bounds have previously been considered in specific applications and in robust combinatorial optimization with objective uncertainty. In particular, this work is inspired by Kerkkamp (2014), where a compact RC is derived for a maximum coverage facility location problem with signed budget objective uncertainty. In Stienen et al. (2021) the signed budget uncertainty set is used to model uncertainty in the number of people affected by a disaster in a humanitarian facility location problem. Goerigk and Lendl (2021) study laminar signed budget uncertainty for the special case in which the laminar family is a partition, which they refer to as *locally budgeted uncertainty*. Yaman (2023) studies arbitrary laminar signed budget uncertainty (called *generalized deviation budgeted uncertainty*) and shows that the resulting robust combinatorial optimization problem with objective uncertainty can be solved through  $2^K$  nominal problems, where  $K$  is the number of sets in the laminar family. Laminar signed budget uncertainty sets also appear in robust routing problems (Gounaris et al. 2013, 2016). However, existing tractability results mainly focus on objective uncertainty and specific application domains.

In contrast, in this paper, we consider general mixed-integer optimization problems with binary uncertainty-affected variables and constraint uncertainty. For laminar signed budget uncertainty sets, we derive compact RCs whose additional size depends on the number  $K$  of aggregate sets rather than on the number of uncertain coefficients. For problems with a single uncertainty-affected constraint, we further show that the robust problem can be solved through  $2^K$  nominal problems. This latter approach is particularly attractive for combinatorial optimization problems for which efficient problem-specific algorithms, decomposition methods, or heuristics are available for the nominal formulation, such as routing, covering, and knapsack-type problems.

We further discuss how signed and laminar signed budget uncertainty sets can be constructed from data. In particular, aggregate confidence intervals obtained using the CLT and related asymptotic results directly yield signed aggregate bounds. When the uncertain parameters possess a known hierarchical structure, hierarchical forecasting and forecast-reconciliation methods can be used to construct coherent point forecasts and prediction intervals at different levels of aggregation.

Finally, we evaluate the proposed formulations using robust maximum coverage and knapsack problems. The computational results demonstrate that the compact formulations substantially outperform classical RCs as the number of uncertain coefficients increases and often retain solution times close to those of the nominal formulation. The experiments further show that solutions optimized under signed budget uncertainty are generally not recovered by the classical budget formulations considered in our experiments. For robust knapsack instances under constraint uncertainty, we show that enumerating the  $2^K$  nominal problems is computationally efficient for small  $K$ , but becomes increasingly costly as  $K$  grows.

The main contributions of this paper are the following.

- We study signed budget uncertainty sets in robust mixed-integer optimization, where bounds are imposed on aggregate signed deviations rather than on aggregate absolute deviations. We derive a compact RC that requires only two additional variables and two additional constraints, independently of the number of uncertain coefficients.
- We extend the analysis to laminar signed budget uncertainty sets, which capture hierarchical aggregate information. We derive compact RCs whose additional size depends on the number  $K$  of aggregate sets rather than on the number of uncertain coefficients. For problems with a single uncertainty-affected constraint, we further show that the robust problem can be solved through  $2^K$  nominal problems, extending previous tractability results (Goerigk and Lendl 2021, Yaman 2023) for laminar uncertainty structures to mixed-integer optimization with constraint uncertainty.
- We show how signed and laminar signed budget uncertainty sets can be constructed from data using the CLT and hierarchical forecasting methods, thereby connecting the proposed uncertainty sets to statistical and forecasting information.
- Numerical results on robust maximum coverage instances demonstrate the computational advantages of the compact RCs for signed budget uncertainty. We show that the resulting optimal solutions are not easily recovered under classical budget uncertainty. For robust knapsack instances, comparing the compact RC against enumeration of the  $2^K$  nominal problems reveals which method is preferable for different values of  $K$ .

*Outline* The remainder of the paper is organized as follows. Section 2 studies the computational tractability of the signed budget uncertainty set for robust mixed-integer problems. Section 3 describes laminar signed budget uncertainty sets and studies their computational tractability. Section 4 discusses how laminar signed budget uncertainty sets can be constructed from data. Section 5 reports numerical results, and Section 6 concludes with directions for future research.

*Notation* Calligraphic letters are reserved for sets, whereas finite index sets are denoted using the notation  $[n] = \{1, \dots, n\}$ . We indicate with  $\mathbf{1} \in \mathbb{R}^n$  the vector with  $n$  components all equal to 1.

## 2. Robust Mixed-Integer Optimization with Signed Budget Uncertainty Sets

In this paper, we study robust mixed-integer linear optimization problems of the form

$$\begin{aligned}
 & \min \mathbf{c}^\top \mathbf{x} + \mathbf{d}^\top \mathbf{y} \\
 & \text{s.t. } \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} \leq \boldsymbol{\beta}, \quad \mathbf{A} \in \mathcal{U}, \\
 & \quad \mathbf{x} \in \{0, 1\}^{n_1}, \\
 & \quad \mathbf{y} \in \mathbb{R}^{n_2},
 \end{aligned} \tag{1}$$

with  $\mathbf{c} \in \mathbb{R}^{n_1}$ ,  $\mathbf{d} \in \mathbb{R}^{n_2}$ ,  $\mathbf{A} \in \mathbb{R}^{m \times n_1}$ ,  $\mathbf{B} \in \mathbb{R}^{m \times n_2}$ , and  $\boldsymbol{\beta} \in \mathbb{R}^m$ . We consider the case in which matrix  $\mathbf{A}$  is uncertain and known to reside in an uncertainty set  $\mathcal{U}$ , since the compact reformulations we present exploit the binary structure of the uncertainty-affected variables  $\mathbf{x}$ . We assume that  $\mathbf{c}$ ,  $\mathbf{d}$ ,  $\mathbf{B}$ , and  $\boldsymbol{\beta}$  are deterministic. In fact, uncertainty in the objective coefficients can be reformulated using standard epigraph techniques so that uncertain parameters appear only in the constraints. Moreover, uncertainty in  $\mathbf{B}$  and  $\boldsymbol{\beta}$  can be handled using standard RO techniques from the literature, e.g., Bertsimas and den Hertog (2022), provided that it is independent of the uncertainty in  $\mathbf{A}$ .

Observe that constraint  $\mathbf{Ax} + \mathbf{By} \leq \boldsymbol{\beta}$ ,  $\mathbf{A} \in \mathcal{U}$ , is equivalent to requiring that constraints  $\mathbf{a}_i^\top \mathbf{x} + \mathbf{b}_i^\top \mathbf{y} \leq \beta_i$ ,  $\mathbf{a}_i \in \mathcal{U}_i$ ,  $i \in [m]$ , be satisfied, where  $\mathcal{U}_i$  is the projection of  $\mathcal{U}$  on the uncertain parameters of constraint  $i$ . Hence, as observed in Ben-Tal et al. (2009), robustness with respect to  $\mathcal{U}$  can be formulated constraint-wise, and without loss of generality we can focus on a single constraint:

$$\mathbf{a}^\top \mathbf{x} + \mathbf{b}^\top \mathbf{y} \leq \beta, \quad \forall \mathbf{a} \in \mathcal{U}. \quad (2)$$

### 2.1. Signed Budget Uncertainty Sets

For each  $j \in [n_1]$ , let  $\bar{a}_j$  denote the nominal value of  $a_j$  and let  $\eta_j \geq 0$  denote its maximum deviation. In addition to these individual bounds, we impose lower and upper bounds on the aggregate  $\sum_{j \in [n_1]} a_j$ . Let  $\bar{a}_\Sigma \in \mathbb{R}$  and  $\eta_\Sigma \in \mathbb{R}_+$  denote the center and half-width of this aggregate interval, respectively. Define

$$a_j^- := \bar{a}_j - \eta_j, \quad a_j^+ := \bar{a}_j + \eta_j, \quad j \in [n_1],$$

and

$$a_\Sigma^- := \bar{a}_\Sigma - \eta_\Sigma, \quad a_\Sigma^+ := \bar{a}_\Sigma + \eta_\Sigma.$$

We consider the signed budget uncertainty set

$$\mathcal{U} = \left\{ \mathbf{a} \mid a_j \in [a_j^-, a_j^+], \ j \in [n_1], \sum_{j \in [n_1]} a_j \in [a_\Sigma^-, a_\Sigma^+] \right\}. \quad (3)$$

This set combines individual *box constraints* on the uncertain coefficients with an *aggregate constraint* that bounds their sum from below and above. It differs from the classical budget uncertainty set from Bertsimas and Sim (2004), where the aggregate constraint is replaced by  $\sum_{j \in [n_1]} \frac{|a_j - \bar{a}_j|}{\eta_j} \leq \Gamma$ , i.e., it bounds the sum of scaled absolute deviations instead of the signed quantities. We assume that  $\mathcal{U} \neq \emptyset$ , and after tightening the aggregate bounds in case they are redundant, we assume that it holds

$$\sum_{j \in [n_1]} a_j^- \leq a_\Sigma^- \leq a_\Sigma^+ \leq \sum_{j \in [n_1]} a_j^+. \quad (4)$$

## 2.2. Standard Robust Counterpart

The traditional RC of constraint (2) can be derived using duality theory and it can be written as

$$\begin{aligned}
a_{\Sigma}^+ \lambda - a_{\Sigma}^- \gamma + \sum_{j \in [n_1]} (a_j^+ \mu_j^+ - a_j^- \mu_j^-) + \mathbf{b}^\top \mathbf{y} &\leq \beta, \\
\lambda - \gamma + \mu_j^+ - \mu_j^- &= x_j, \quad j \in [n_1], \\
\lambda, \gamma &\in \mathbb{R}_+, \\
\mu_j^+, \mu_j^- &\in \mathbb{R}_+, \quad j \in [n_1].
\end{aligned} \tag{5}$$

A derivation of this formulation is provided in Appendix A. In this case, the RC requires  $2 + 2n_1$  extra variables and  $n_1$  additional constraints. Whenever  $n_1$  is large, solving (5) becomes computationally prohibitive. This observation motivates the two alternative solution approaches that we present.

## 2.3. Compact Robust Counterpart

The first approach exploits the binary structure of  $\mathbf{x}$  to obtain a compact RC that can be solved as a single optimization model. We begin by characterizing the worst-case value of  $\mathbf{a}^\top \mathbf{x}$  over the signed budget uncertainty set, generalizing the result of Kerkkamp (2014, Section 3.2.4) to the case of constraint uncertainty. Building on this characterization, we give a compact RC of (2). We show that this formulation can be linearized introducing only two additional variables, one binary and one continuous, and two extra constraints, independently of the number of uncertain coefficients  $n_1$ .

**THEOREM 1.** *Let  $\mathcal{U}$  be the signed budget uncertainty set defined in (3). Then, the compact RC of (2) can be written as*

$$(\mathbf{a}^+)^{\top} \mathbf{x} + \min \left\{ 0, a_{\Sigma}^+ - \sum_{j \in [n_1]} (a_j^+ x_j + a_j^- (1 - x_j)) \right\} + \mathbf{b}^\top \mathbf{y} \leq \beta. \tag{6}$$

*Proof* See Appendix B.1. □

The intuition behind Theorem 1 follows from separating the indices into those for which  $x_j = 1$  and those for which  $x_j = 0$ . Ignoring the aggregate constraint, the worst case sets every coefficient with  $x_j = 1$  to its individual upper bound. If this choice remains feasible for the aggregate upper bound after setting the coefficients with  $x_j = 0$  to their individual lower bounds, it is also optimal over  $\mathcal{U}$ . Otherwise, the aggregate upper bound must be binding. In that case, the contribution of the selected coefficients equals the aggregate upper bound minus the smallest possible contribution of the unselected coefficients. These two cases yield the two expressions in Theorem 1.

The minimum in Theorem 1 admits the following standard big- $M$  linearization with one additional continuous variable, one binary variable, and two additional constraints:

$$\begin{aligned} \sum_{j \in [n_1]} a_j^+ x_j + w + \mathbf{b}^\top \mathbf{y} &\leq \beta, \\ w &\geq a_\Sigma^+ - \sum_{j \in [n_1]} (a_j^+ x_j + a_j^- (1 - x_j)) - M(1 - v), \\ w &\geq -Mv, \\ v &\in \{0, 1\}. \end{aligned}$$

As shown in Appendix B.2, a valid choice for the big- $M$  parameter is  $M = 2 \sum_{j \in [n_1]} \eta_j$ .

## 2.4. Solution through Two Nominal Problems

Our second approach extends the result from Goerigk and Lendl (2021, Theorem 2) to the case of general mixed-integer RO problems under constraint uncertainty. Instead of constructing a RC, it solves the robust problem through two nominal problems.

We call a problem *nominal* if it has the same structure as (1), with deterministic coefficients and right-hand side. This approach is especially attractive when efficient problem-specific algorithms or highly optimized software are available for the underlying deterministic model.

**COROLLARY 1.** *Let  $\mathcal{U}$  be the signed budget uncertainty set defined in (3) and suppose that Problem (1) contains a single uncertainty-affected constraint. Then it can be solved by solving two nominal problems.*

*Proof* By Theorem 1, the worst-case value in the robust constraint is the minimum of two deterministic expressions. Hence, the robust constraint is satisfied if and only if at least one of the two corresponding deterministic inequalities is satisfied. The robust feasible region is therefore the union of the feasible regions of two nominal problems. Minimizing the same objective function over this union is equivalent to solving both nominal problems and selecting the solution with the smaller objective value.  $\square$

See Appendix B.3 for the explicit expressions of the two nominal problems.

If Problem (1) contains  $m$  uncertainty-affected constraints, then from Corollary 1 it can be solved by solving  $2^m$  nominal problems.

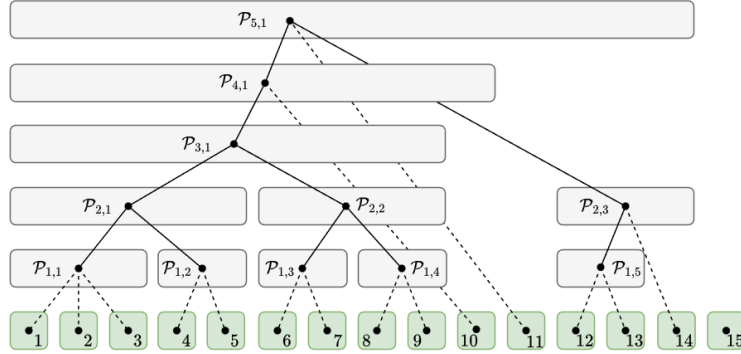
## 3. Robust Mixed-Integer Optimization with Laminar Signed Budget Uncertainty Sets

The signed budget uncertainty set contains a single aggregate bound on the sum of all uncertain parameters. In many applications, however, aggregate bounds are available for several groups at different levels of aggregation, such as cities, provinces, regions, and countries. To represent such

information, we extend the signed budget uncertainty set by imposing aggregate bounds on multiple subsets of uncertain parameters. We restrict attention to subsets that form a *laminar family*: any two subsets are either disjoint or nested. The signed budget uncertainty set is recovered as the special case in which the family consists of the single set  $[n_1]$ .

### 3.1. Laminar Signed Budget Uncertainty Sets

Let  $L \in \mathbb{N}$ , and for each  $\ell \in [L]$ , let  $\{\mathcal{P}_{\ell,1}, \dots, \mathcal{P}_{\ell,K_\ell}\}$  be a collection of  $K_\ell \geq 1$  subsets of  $[n_1]$ , i.e.,  $\mathcal{P}_{\ell,k} \subseteq [n_1]$  for each  $\ell \in [L]$ ,  $k \in [K_\ell]$ . We say that the sets  $\{\mathcal{P}_{\ell,1}, \dots, \mathcal{P}_{\ell,K_\ell}\}$  belong to level  $\ell$ , and we denote with  $K_\ell$  the total number of sets for that level. The full collection is indicated by  $\{\mathcal{P}_{\ell,1}, \dots, \mathcal{P}_{\ell,K_\ell}\}_{\ell \in [L]}$ , and  $K = \sum_{\ell \in [L]} K_\ell$  is the total number of sets across all levels. Following the definition in Yaman (2023), we say that  $\{\mathcal{P}_{\ell,1}, \dots, \mathcal{P}_{\ell,K_\ell}\}_{\ell \in [L]}$  is a laminar family if any two of its elements are either disjoint, or one is contained in the other, i.e., given  $\ell \in [L]$  and  $k_1 \neq k_2$ ,  $k_1, k_2 \in [K_\ell]$ , it holds that  $\mathcal{P}_{\ell,k_1} \cap \mathcal{P}_{\ell,k_2} = \emptyset$ , and given  $\ell_1, \ell_2 \in [L]$ ,  $\ell_1 < \ell_2$ ,  $k_1 \in [K_{\ell_1}]$  and  $k_2 \in [K_{\ell_2}]$ , it holds that  $\mathcal{P}_{\ell_1,k_1} \subset \mathcal{P}_{\ell_2,k_2}$ , or  $\mathcal{P}_{\ell_1,k_1} \cap \mathcal{P}_{\ell_2,k_2} = \emptyset$ . We show an example in Figure 1.



**Figure 1** Example of a laminar family with 5 levels over the set  $\{1, \dots, 15\}$ . Straight lines link pairs of sets  $\mathcal{P}_{\ell,k}$  and  $\mathcal{P}_{\ell+1,k'}$  such that  $\mathcal{P}_{\ell,k} \subset \mathcal{P}_{\ell+1,k'}$ . Dashed lines connect the elements  $j \in \{1, \dots, 15\}$  to the first set  $\mathcal{P}_{\ell,k}$  such that  $j \in \mathcal{P}_{\ell,k}$ .

Starting from a laminar family with  $L$  levels, denoted by  $\{\mathcal{P}_{\ell,1}, \dots, \mathcal{P}_{\ell,K_\ell}\}_{\ell \in [L]}$ , we can define the corresponding laminar signed budget uncertainty set as follows. As before, each parameter  $a_j$ ,  $j \in [n_1]$ , is bounded by its individual lower and upper bounds  $a_j^-$  and  $a_j^+$ . In addition, for each set  $\mathcal{P}_{\ell,k}$ ,  $\ell \in [L]$  and  $k \in [K_\ell]$ , we impose lower and upper bounds on the sum of the parameters indexed by  $\mathcal{P}_{\ell,k}$ . Let  $\bar{a}_{\ell,k} \in \mathbb{R}$  and  $\eta_{\ell,k} \in \mathbb{R}_+$  denote the center and half-width, respectively, of the corresponding aggregate interval, and define

$$a_{\ell,k}^- := \bar{a}_{\ell,k} - \eta_{\ell,k}, \quad a_{\ell,k}^+ := \bar{a}_{\ell,k} + \eta_{\ell,k}.$$

The resulting *laminar signed budget uncertainty set* is defined as

$$\mathcal{U}_L = \left\{ \mathbf{a} \mid a_j \in [a_j^-, a_j^+], j \in [n_1], \sum_{j \in \mathcal{P}_{\ell,k}} a_j \in [a_{\ell,k}^-, a_{\ell,k}^+], \ell \in [L], k \in [K_\ell] \right\}. \quad (7)$$

This set coincides with the *generalized deviation budget uncertainty set* of Yaman (2023).

Throughout this section, we assume that  $\mathcal{U}_L \neq \emptyset$ . Moreover, after tightening aggregate bounds that are redundant with respect to the individual box constraints, we may assume that for all  $\ell \in [L]$  and  $k \in [K_\ell]$  it holds

$$\sum_{j \in \mathcal{P}_{\ell,k}} a_j^- \leq a_{\ell,k}^- \leq a_{\ell,k}^+ \leq \sum_{j \in \mathcal{P}_{\ell,k}} a_j^+. \quad (8)$$

### 3.2. Standard Robust Counterpart

Let  $\{\mathcal{P}_{\ell,1}, \dots, \mathcal{P}_{\ell,K_\ell}\}_{\ell \in [L]}$  be a laminar family of sets with  $L \geq 1$  and  $K = \sum_{\ell \in [L]} K_\ell$ . Let  $\mathcal{U}_L$  be the corresponding laminar signed budget uncertainty set. Then, the traditional RC of constraint (2) can be derived using duality theory (see Appendix A) and can be written as

$$\begin{aligned} & \sum_{\ell \in [L]} \sum_{k \in [K_\ell]} \left( a_{\ell,k}^+ \lambda_{\ell,k} - a_{\ell,k}^- \gamma_{\ell,k} \right) + \sum_{j \in [n_1]} \left( a_j^+ \mu_j^+ - a_j^- \mu_j^- \right) + \mathbf{b}^\top \mathbf{y} \leq \beta, \\ & \sum_{\ell \in [L]} \sum_{\substack{k \in [K_\ell]: \\ j \in \mathcal{P}_{\ell,k}}} (\lambda_{\ell,k} - \gamma_{\ell,k}) + \mu_j^+ - \mu_j^- = x_j, \quad j \in [n_1], \\ & \lambda_{\ell,k}, \gamma_{\ell,k} \in \mathbb{R}_+, \quad \ell \in [L], k \in [K_\ell], \\ & \mu_j^+, \mu_j^- \in \mathbb{R}_+, \quad j \in [n_1]. \end{aligned} \quad (9)$$

The classical RC requires  $2K + 2n_1$  additional variables and  $n_1$  extra constraints. Since typically  $n_1 \gg K$ , (9) can become computationally prohibitive for large  $n_1$ . We therefore develop two approaches paralleling those of Section 2: a compact RC whose size scales with  $K$  rather than  $n_1$ , and an enumeration of nominal problems.

### 3.3. Compact Robust Counterpart

The construction of the compact RC generalizes the worst-case characterization of Theorem 1. For a single aggregate set, the worst-case of  $\mathbf{a}^\top \mathbf{x}$  is the sum of a subset of the individual upper bounds, minus a correction term when the aggregate upper bound is binding. For a laminar family this reasoning applies recursively. Thus, for each set  $\mathcal{P}_{\ell,k}$ , the worst-case contribution depends on those of the lower-level sets it contains, yielding the recursive characterization in Theorem 2.

**THEOREM 2.** *Let  $\{\mathcal{P}_{\ell,1}, \dots, \mathcal{P}_{\ell,K_\ell}\}_{\ell \in [L]}$  be a laminar family of sets. Let  $\mathcal{U}_L$  be as defined in (7) and let  $\mathbf{x} \in \{0,1\}^{n_1}$ . Then, the compact RC of (2) can be written as*

$$(\mathbf{a}^+)^\top \mathbf{x} + \sum_{\ell \in [L]} \sum_{k \in [K_\ell]} w_{\ell,k} + \mathbf{b}^\top \mathbf{y} \leq \beta,$$

where for  $\ell = 1$ ,  $k \in [K_1]$ ,

$$w_{\ell,k} = \min \left\{ 0, a_{\ell,k}^+ - \sum_{j \in \mathcal{P}_{\ell,k}} \left( a_j^- (1 - x_j) + a_j^+ x_j \right) \right\}, \quad (10)$$

and if  $L \geq 2$ , for  $\ell = 1$ ,  $k \in [K_1]$ ,

$$u_{\ell,k} = \max \left\{ 0, a_{\ell,k}^- - \sum_{j \in \mathcal{P}_{\ell,k}} \left( a_j^- (1 - x_j) + a_j^+ x_j \right) \right\}, \quad (11)$$

for  $2 \leq \ell \leq L$  and  $k \in [K_\ell]$ , we define

$$\mathcal{D}_{\ell,k} = \{(\ell', k') : \ell' < \ell, k' \in [K_{\ell'}], \mathcal{P}_{\ell',k'} \subset \mathcal{P}_{\ell,k}\}, \quad (12)$$

and

$$w_{\ell,k} = \min \left\{ 0, a_{\ell,k}^+ - \sum_{j \in \mathcal{P}_{\ell,k}} \left( a_j^- (1 - x_j) + a_j^+ x_j \right) - \sum_{(\ell', k') \in \mathcal{D}_{\ell,k}} (u_{\ell',k'} + w_{\ell',k'}) \right\}, \quad (13)$$

and if  $L \geq 3$ , for  $2 \leq \ell \leq L - 1$ ,  $k \in [K_\ell]$ ,

$$u_{\ell,k} = \max \left\{ 0, a_{\ell,k}^- - \sum_{j \in \mathcal{P}_{\ell,k}} \left( a_j^- (1 - x_j) + a_j^+ x_j \right) - \sum_{(\ell', k') \in \mathcal{D}_{\ell,k}} (u_{\ell',k'} + w_{\ell',k'}) \right\}. \quad (14)$$

*Proof* See Appendix C.5. □

For each  $\ell \in [L]$ ,  $k \in [K_\ell]$ , the term  $w_{\ell,k} \leq 0$  represents a downward correction of the maximal contribution of the selected coefficients ( $x_j = 1$ ) in  $\mathcal{P}_{\ell,k}$  due to the aggregate upper bound, while  $u_{\ell,k} \geq 0$  is an upward correction of the minimal contribution of the unselected coefficients ( $x_j = 0$ ) due to the aggregate lower bound.

The minimum and maximum terms in Theorem 2 can be linearized using standard big- $M$  formulations, as in Section 2. In Appendix C.5, we provide the explicit formulation and a valid choice of the big- $M$  parameters. The resulting linearized RC introduces  $4K - 2K_L$  additional variables and  $4K - 2K_L$  additional constraints. Hence, when  $n_1 \gg K$ , the compact RC can be substantially smaller than (9), as illustrated by the following example.

**EXAMPLE 1.** Consider a robust maximum coverage problem with  $n_1 = 10,000$  uncertain population coefficients aggregated into  $K_1 = 110$  provinces,  $K_2 = 20$  regions, and  $K_3 = 1$  country, yielding  $K = 131$  sets. The standard RC (9) introduces 20,262 additional variables. In contrast, Theorem 2 yields a compact RC with only  $4K - 2K_3 = 522$  additional variables and constraints.

### 3.4. Solution through Nominal Problems

Our second approach generalizes Corollary 1, extending results from Goerigk and Lendl (2021) and Yaman (2023) to mixed-integer RO problems under constraint uncertainty with laminar signed budget uncertainty sets. Rather than constructing a RC, it solves the robust problem through a finite collection of nominal problems.

**COROLLARY 2.** *Let  $\{\mathcal{P}_{\ell,1}, \dots, \mathcal{P}_{\ell,K_\ell}\}_{\ell \in [L]}$  be a laminar family of sets. Let  $\mathcal{U}_L$  be the corresponding laminar signed budget uncertainty set defined in (7) and suppose that Problem (1) contains a single uncertainty-affected constraint. Then it can be solved by solving  $2^K$  nominal problems, where  $K = \sum_{\ell \in [L]} K_\ell$ .*

*Proof* See Appendix C.8. □

The intuition of the proof follows from the recursive worst-case characterization. For each aggregate set, either its upper bound is nonbinding and its contribution is inherited from lower levels, or the upper bound is binding. Fixing one of these two cases for each of the  $K$  aggregate sets yields  $2^K$  deterministic constraints and hence  $2^K$  nominal problems.

This approach is particularly attractive when efficient specialized algorithms are available for the nominal problem. For example, a robust knapsack problem with uncertain weights can be solved through  $2^K$  deterministic knapsack problems, which may be addressed by dynamic programming. Alternatively, Theorem 2 yields a compact RC with  $4K - 2K_L$  additional variables and constraints. We compare these two approaches computationally in Section 5.

## 4. Building Signed Budget Uncertainty Sets from Data

In this section, we provide practical guidelines for constructing (laminar) signed budget uncertainty sets using two approaches. First, following Bandi and Bertsimas (2012), the CLT yields bounds on sums of uncertain parameters and thus a signed budget uncertainty set. Second, when the data has a known aggregation structure, hierarchical forecasting can be used to construct coherent prediction intervals and the corresponding laminar signed budget uncertainty set.

### 4.1. Central Limit Theorem

As shown in Bandi and Bertsimas (2012), asymptotic laws of probability can be used to construct uncertainty sets. Under standard regularity conditions, the CLT states that the suitably normalized sum of the uncertain parameters  $a_j$ ,  $j \in [n_1]$ , converges in distribution to a standard normal random variable as  $n_1$  increases. Although its accuracy depends on the underlying distributions, as a commonly used rule of thumb, the normal approximation is often considered adequate when  $n_1 \geq 30$  (Montgomery and Runger 2011, p. 227).

Suppose that the uncertain parameters  $a_j$ ,  $j \in [n_1]$ , are independent random variables with support  $[a_j^-, a_j^+]$ , for which we can estimate the (finite) mean  $\mu_j$  and standard deviation  $\sigma_j$ . Then, we can use the CLT to define a laminar signed budget uncertainty set, even if the data does not have a hierarchical structure and we have no a priori information nor estimates of their sums. Given any subset  $\mathcal{P} \subseteq [n_1]$ , define  $\mu(\mathcal{P}) = \sum_{j \in \mathcal{P}} \mu_j$ , and  $\sigma(\mathcal{P}) = \left( \sum_{j \in \mathcal{P}} \sigma_j^2 \right)^{1/2}$ . Then, under the assumptions of the CLT and for sufficiently large  $|\mathcal{P}|$ , it holds that

$$\frac{\sum_{j \in \mathcal{P}} a_j - \mu(\mathcal{P})}{\sigma(\mathcal{P})} \Rightarrow Z, \quad Z \sim N(0, 1).$$

Since  $\mathbb{P}(|Z| \leq 2) \approx 0.95$  and  $\mathbb{P}(|Z| \leq 3) \approx 0.99$ , choosing  $C \in \{2, 3\}$  gives the approximate bounds

$$\sum_{j \in \mathcal{P}} a_j \in [\mu(\mathcal{P}) - C\sigma(\mathcal{P}), \mu(\mathcal{P}) + C\sigma(\mathcal{P})].$$

Taking  $\mathcal{P} = [n_1]$  yields the signed budget uncertainty set

$$\left\{ \mathbf{a} \mid a_j \in [a_j^-, a_j^+], \ j \in [n_1], \ \sum_{j \in [n_1]} a_j \in [\mu([n_1]) - C\sigma([n_1]), \mu([n_1]) + C\sigma([n_1])] \right\}.$$

More generally, given a laminar family  $\{\mathcal{P}_{\ell,1}, \dots, \mathcal{P}_{\ell,K_\ell}\}_{\ell \in [L]}$  whose sets are sufficiently large, applying the CLT to each  $\mathcal{P}_{\ell,k}$  yields the laminar signed budget uncertainty set where the aggregate bounds for each  $\sum_{j \in \mathcal{P}_{\ell,k}} a_j$  are given by

$$[\mu(\mathcal{P}_{\ell,k}) - C\sigma(\mathcal{P}_{\ell,k}), \mu(\mathcal{P}_{\ell,k}) + C\sigma(\mathcal{P}_{\ell,k})].$$

The CLT belongs to a broader class of weak convergence results that can be used analogously to construct uncertainty sets, including for heavy-tailed distributions (Bandi and Bertsimas 2012). In particular, if the  $a_j$  are i.i.d. with mean  $\mu$  and follow a stable distribution  $Y$  with parameter  $\alpha \in (1, 2)$ , then

$$\frac{\sum_{j \in [n_1]} a_j - n_1 \mu}{n_1^{1/\alpha}} \Rightarrow Y.$$

Consequently, for a constant  $C_{HT}$  determined by the distribution of  $Y$ , the aggregate bounds for  $\sum_{j \in \mathcal{P}_{\ell,k}} a_j$  can be taken as

$$[\mu|\mathcal{P}_{\ell,k}| - C_{HT}|\mathcal{P}_{\ell,k}|^{1/\alpha}, \mu|\mathcal{P}_{\ell,k}| + C_{HT}|\mathcal{P}_{\ell,k}|^{1/\alpha}].$$

These bounds can replace the CLT-based aggregate bounds in the uncertainty set above to obtain a laminar signed budget uncertainty set for heavy-tailed data. The signed budget uncertainty set is recovered as the special case in which the laminar family consists of the single set  $[n_1]$ .

## 4.2. Hierarchical Forecasting

Another possibility is to construct the laminar signed budget uncertainty set from data arising from hierarchical time series. Following Athanasopoulos et al. (2024), a hierarchical time series consists of multiple series linked through aggregation constraints. Let  $\mathbf{u}_t \in \mathbb{R}^{n_1}$  denote the vector of bottom-level observations at time  $t$ , corresponding in our setting to the  $n_1$  individual uncertain parameters. The complete hierarchy can be written as  $\mathbf{z}_t = \mathbf{S}\mathbf{u}_t$ , where  $\mathbf{z}_t \in \mathbb{R}^N$  contains the series at all levels of aggregation and  $\mathbf{S} \in \mathbb{R}^{N \times n_1}$  is the summing matrix describing the aggregation structure. For a laminar family with  $K$  aggregate sets,  $N = n_1 + K$ , and each aggregate series corresponds to the sum of the bottom-level series indexed by one of the sets  $\mathcal{P}_{\ell,k}$ .

Forecasts obtained independently for the different series generally do not satisfy the aggregation constraints. Let  $\hat{\mathbf{z}}_h$  denote a vector of such base forecasts at forecast horizon  $h$ . Forecast reconciliation methods transform these forecasts into coherent forecasts that satisfy the same aggregation constraints as the observed data. These methods can generally be represented as  $\tilde{\mathbf{z}}_h = \mathbf{S}\mathbf{G}\hat{\mathbf{z}}_h$ , where  $\mathbf{G} \in \mathbb{R}^{n_1 \times N}$  maps the base forecasts to the bottom level. One widely used approach is minimum trace (MinT) reconciliation (Wickramasuriya et al. 2019), for which  $\mathbf{G} = (\mathbf{S}^\top \mathbf{W}_h^{-1} \mathbf{S})^{-1} \mathbf{S}^\top \mathbf{W}_h^{-1}$ , where  $\mathbf{W}_h$  denotes the variance-covariance matrix of the base forecast errors. In practice, several approaches for specifying  $\mathbf{W}_h$  have been proposed; see, e.g., Hyndman and Athanasopoulos (2021), Panagiotelis et al. (2023).

The reconciled forecasts and their corresponding prediction intervals can be used to construct the laminar signed budget uncertainty set. In particular, for each aggregate set  $\mathcal{P}_{\ell,k}$ , the reconciled point forecast determines the nominal aggregate value  $\bar{a}_{\ell,k}$ , while the corresponding prediction interval determines the deviation parameter  $\eta_{\ell,k}$ . Hence, hierarchical forecast reconciliation provides aggregate bounds that are coherent across the different levels of the laminar hierarchy. An illustrative example of hierarchical forecast reconciliation and the resulting construction of aggregate bounds is provided in Appendix D.

## 5. Numerical Results

We evaluate the proposed approaches in two sets of numerical experiments. First, using the maximum coverage problem (Church and ReVelle 1974), we compare the computational performance of the standard and compact RCs under signed budget and two-level laminar signed budget uncertainty, and investigate how solutions under signed budget uncertainty differ from those obtained under classical budget ( $\Gamma$ ) uncertainty. Second, using robust knapsack problems with constraint uncertainty, we compare the compact RC with the enumeration of  $2^K$  nominal problems.

All experiments were implemented in Python and solved using Gurobi 12.0.1 on a machine with an Intel Core i5-10600KF processor, 32 GB of RAM, and Ubuntu 24.04.4 LTS. The code is available at [https://github.com/dfede3/sign\\_budget](https://github.com/dfede3/sign_budget).

**Table 1** Results for the robust maximum coverage problem under signed budget uncertainty and laminar signed budget uncertainty. We report average solution times over instances solved to optimality. Superscripts indicate the number of instances, out of five, that reached the time limit, and “\*” indicates that none was solved to optimality.

| $n_1$ | $r$ | $p$ | Nominal             | Signed Budget        |                      | Laminar Signed Budget |                      |
|-------|-----|-----|---------------------|----------------------|----------------------|-----------------------|----------------------|
|       |     |     |                     | $RC_{standard}$      | $RC_{compact}$       | $RC_{standard}$       | $RC_{compact}$       |
| 100   | 0.1 | 5   | 0.01                | 0.04                 | 0.00                 | 0.01                  | 0.04                 |
| 100   | 0.1 | 10  | 0.00                | 0.21                 | 0.03                 | 0.02                  | 0.05                 |
| 100   | 0.2 | 5   | 0.00                | 0.02                 | 0.01                 | 0.04                  | 0.06                 |
| 100   | 0.2 | 10  | 0.00                | 0.00                 | 0.00                 | 0.00                  | 0.00                 |
| 500   | 0.1 | 5   | 0.02                | 80.42                | 0.04                 | 0.63                  | 0.19                 |
| 500   | 0.1 | 10  | 0.02                | *                    | 0.15                 | 6.76                  | 0.29                 |
| 500   | 0.2 | 5   | 0.06                | 104.16               | 0.08                 | 2.97                  | 0.77                 |
| 500   | 0.2 | 10  | 0.43                | 0.76                 | 0.51                 | 0.66                  | 0.64                 |
| 1000  | 0.1 | 5   | 0.08                | *                    | 0.18                 | 14.65                 | 0.49                 |
| 1000  | 0.1 | 10  | 0.08                | *                    | 0.44                 | 706.80                | 1.12                 |
| 1000  | 0.2 | 5   | 0.31                | *                    | 0.79                 | 379.31                | 4.18                 |
| 1000  | 0.2 | 10  | 3.18                | 3.92                 | 3.58                 | 3.53                  | 2.86                 |
| 5000  | 0.1 | 5   | 6.52                | *                    | 21.31                | *                     | 93.60                |
| 5000  | 0.1 | 10  | 7.20                | *                    | 22.04                | *                     | 464.90               |
| 5000  | 0.2 | 5   | 58.38               | *                    | 72.33                | *                     | 655.33               |
| 5000  | 0.2 | 10  | 195.86              | 350.53               | 272.68               | 449.09                | 344.86               |
| 10000 | 0.1 | 5   | 104.39              | *                    | 149.09               | *                     | 150.54               |
| 10000 | 0.1 | 10  | 127.42              | *                    | 176.47               | *                     | 2698.34 <sup>4</sup> |
| 10000 | 0.2 | 5   | 164.11              | *                    | 562.82               | *                     | 2165.94 <sup>4</sup> |
| 10000 | 0.2 | 10  | 786.18 <sup>1</sup> | 1629.41 <sup>2</sup> | 1750.07 <sup>1</sup> | 1300.30 <sup>3</sup>  | 2277.54 <sup>1</sup> |

### 5.1. Robust Maximum Coverage Problem

The maximum coverage problem selects at most  $p$  facility locations to maximize the total population covered within Euclidean distance  $r$ . We generate  $n_1$  candidate facility locations independently and uniformly on the two-dimensional unit square, with demand points at the same locations. Facility  $i \in [n_1]$  covers demand point  $j \in [n_1]$  whenever their Euclidean distance is at most  $r$ . The full formulation is provided in Appendix E.1.

Uncertainty affects the population  $a_j$  at each demand point. For the signed budget uncertainty set,  $\bar{a}_j$  is drawn uniformly from  $\{1, \dots, 14\}$  and  $\eta_j$  from  $\{1, \dots, \min\{\bar{a}_j, 10\}\}$ , while the aggregate population is fixed at its nominal value by setting  $\bar{a}_\Sigma = \sum_{j \in [n_1]} \bar{a}_j$  and  $\eta_\Sigma = 0$ . For the two-level laminar signed budget uncertainty set, the individual bounds are unchanged. At the first level, the unit square is partitioned into four quadrants, with  $\bar{a}_{1,k} = \sum_{j \in \mathcal{P}_{1,k}} \bar{a}_j$  and  $\eta_{1,k} = 0.05\bar{a}_{1,k}$ , for each  $k \in \{1, \dots, 4\}$ . At the second level,  $\mathcal{P}_{2,1} = [n_1]$ , with  $\bar{a}_{2,1} = \sum_{j \in [n_1]} \bar{a}_j$  and  $\eta_{2,1} = 0.01\bar{a}_{2,1}$ .

*Standard versus Compact RCs* We vary  $n_1 \in \{100, 500, 1000, 5000, 10000\}$ ,  $p \in \{5, 10\}$ , and  $r \in \{0.1, 0.2\}$ . For each combination, we generate five random instances and impose a one-hour time limit. Table 1 reports solution times for the nominal problem and for the standard and compact

**Table 2** Results for the robust maximum coverage problem under signed and classical budget uncertainty. For signed budget uncertainty,  $f^*$  and  $t$  denote the optimal objective value and computation time of the compact RC, respectively. For classical budget uncertainty, we first optimize the problem with  $\Gamma = \gamma n_1$  and then evaluate its optimal solution under signed budget uncertainty. We denote the resulting objective value by  $f_\gamma$  and the computation time required to obtain the solution by  $t_\gamma$ .

| $n_1$ | Signed Budget |      | Classical Budget |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |       |       |
|-------|---------------|------|------------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-------|-------|
|       | $f^*$         | $t$  | $f_{0.1}$        | $t_{0.1}$ | $f_{0.2}$ | $t_{0.2}$ | $f_{0.3}$ | $t_{0.3}$ | $f_{0.4}$ | $t_{0.4}$ | $f_{0.5}$ | $t_{0.5}$ | $f_{0.6}$ | $t_{0.6}$ | $f_{0.7}$ | $t_{0.7}$ | $f_{0.8}$ | $t_{0.8}$ | $f_{0.9}$ | $t_{0.9}$ | $f_1$ | $t_1$ |
| 100   | 491           | 0.01 | 491              | 0.01      | 465       | 0.02      | 465       | 0.05      | 462       | 0.04      | 462       | 0.09      | 462       | 0.03      | 462       | 0.03      | 389       | 0.01      | 393       | 0.00      | 393   | 0.00  |
| 250   | 1026          | 0.02 | 1012             | 0.07      | 984       | 0.27      | 984       | 0.57      | 967       | 0.74      | 967       | 0.70      | 967       | 0.61      | 829       | 0.44      | 889       | 0.06      | 887       | 0.02      | 887   | 0.02  |
| 500   | 2222          | 0.07 | 2222             | 0.28      | 2200      | 1.36      | 2183      | 1.73      | 2183      | 6.85      | 2183      | 9.74      | 2019      | 9.32      | 1914      | 3.04      | 1856      | 0.66      | 1854      | 0.08      | 1836  | 0.10  |
| 750   | 2944          | 0.16 | 2936             | 2.08      | 2891      | 5.67      | 2897      | 15.2      | 2812      | 27.4      | 2664      | 35.8      | 2664      | 16.9      | 2290      | 7.28      | 2288      | 2.07      | 2418      | 0.21      | 2424  | 0.20  |
| 1000  | 3638          | 0.29 | 3638             | 5.91      | 3638      | 25.5      | 3638      | 724.3     | 3550      | 615.1     | 3550      | 1382      | 3550      | 195.6     | 3153      | 90.2      | 3103      | 8.24      | 2837      | 3.38      | 2625  | 2.16  |

RCs, denoted by  $\text{RC}_{\text{standard}}$  and  $\text{RC}_{\text{compact}}$ , respectively. We consider both the signed budget and the (two-level) laminar signed budget sets. For the compact formulations, we use Gurobi’s built-in minimum and maximum constraints, which performed better than the explicit big- $M$  linearizations in preliminary tests.

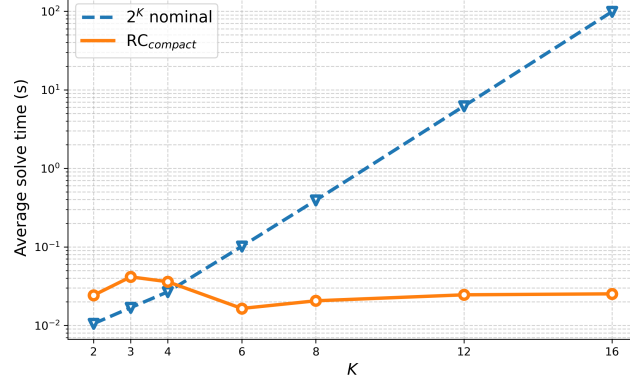
The standard RC quickly becomes computationally challenging, with several instances reaching the time limit for  $n_1 \geq 500$ . Under signed budget uncertainty, the compact RC remains close to the nominal formulation up to  $n_1 = 5000$  and substantially outperforms the standard RC for the largest instances. The two-level laminar structure increases computation time, but the compact RC still solves all instances with  $n_1 \leq 5000$  within the time limit and generally outperforms the standard RC.

*Signed versus Classical Budget Uncertainty* We next investigate whether solutions under signed budget uncertainty can be recovered by tuning the classical budget parameter  $\Gamma$ . We consider the previous maximum coverage instances with  $r = 0.2$ ,  $p = 5$ , and  $n_1 \in \{100, 250, 500, 750, 1000\}$ . We set  $\Gamma = \gamma n_1$ ,  $\gamma \in \{0.1, 0.2, \dots, 1.0\}$ , solve each classical budget instance with a one-hour time limit, and evaluate the resulting solution under signed budget uncertainty.

From Table 2, solutions optimized under classical budget uncertainty are suboptimal under the signed budget set for most instances. Only for a few instances and carefully chosen values of  $\Gamma$  is the signed budget optimum recovered. This may occur when the aggregate upper bound of the signed budget set is nonbinding at the optimum, i.e., when the minimum in (6) equals 0. The signed budget formulation is also faster in most cases.

## 5.2. Robust Knapsack Problem

We consider the robust knapsack problem with uncertain item weights; the full formulation is provided in Appendix E.2. We compare the compact RC of Theorem 2 with the enumeration of  $2^K$  nominal problems from Corollary 2.



**Figure 2** Average computation times for the compact RC and enumeration of  $2^K$  nominal problems under one-level laminar signed budget uncertainty.

The objective coefficients and nominal weights  $\bar{a}_j$  are drawn uniformly from  $\{1, \dots, 100\}$ , and the capacity is set to the integer approximation of  $0.5 \sum_{j \in [n_1]} \bar{a}_j$ . The deviations  $\eta_j$  are drawn uniformly from  $\{1, \dots, \min\{\bar{a}_j, 20\}\}$ . We use a one-level laminar structure partitioning  $[n_1]$  into equally sized sets, with  $\bar{a}_{1,k} = \sum_{j \in \mathcal{P}_{1,k}} \bar{a}_j$  and  $\eta_{1,k} = 0.05 \bar{a}_{1,k}$ . We set  $n_1 = 192$  and vary  $K \in \{2, 3, 4, 6, 8, 12, 16, 24, 32, 48\}$ . For each  $K$ , we generate five random instances with a one-hour time limit. Figure 2 reports average computation times on a logarithmic scale.

Enumeration is competitive for  $K \leq 4$  but becomes increasingly costly as  $K$  grows; no optimal solution is found within the time limit for  $K \in \{24, 32, 48\}$ . The compact formulation is largely insensitive to  $K$  and solves all instances in less than a second.

## 6. Conclusions

We study signed budget uncertainty for RO problems with binary uncertainty-affected variables. Unlike classical budget uncertainty, which bounds aggregate absolute deviations, signed budget uncertainty imposes bounds on signed deviations and can offer substantial computational advantages. We derive a compact RC that only requires two additional variables and constraints. We then extend the framework to laminar signed budget uncertainty sets, allowing multiple bounds at different levels of aggregation. For this more general setting, we obtain a compact RC whose size grows only with  $K$ , the number of sets in the laminar structure. We also show that the robust problem can alternatively be solved by enumerating  $2^K$  nominal problems, extending existing results from objective uncertainty to constraint uncertainty.

We discuss practical approaches for constructing laminar uncertainty sets using the CLT and hierarchical forecasting. Numerical experiments on robust maximum coverage show that the compact RCs substantially reduce computation time relative to standard RCs and to RCs of classical budget uncertainty, while robust knapsack experiments compare the compact RC with enumeration.

Future research could investigate which structural properties of the optimization problem and aggregation family are sufficient for signed budget uncertainty to admit compact RCs. In this paper, compactness relies on the linear dependence on the uncertain parameters, the binary nature of the uncertainty-affected decision variables, and, in the hierarchical setting, the laminar structure of the aggregate sets. An important direction for future research is to investigate how far these assumptions can be relaxed. In particular, it would be interesting to extend the results to general integer or continuous uncertainty-affected variables, to nonlinear RO problems, and to aggregate sets with overlapping or more general non-laminar structures. Such extensions could lead to a broader characterization of problem classes for which signed aggregate uncertainty remains computationally tractable.

## Appendix A: Proof of the Standard Robust Counterpart

This section derives the standard RC of the signed budget uncertainty set via duality. We present the proof for the more general laminar case; the result for the signed budget set follows immediately as a special case.

Let  $\{\mathcal{P}_{\ell,1}, \dots, \mathcal{P}_{\ell,K_\ell}\}_{\ell \in [L]}$  be a laminar family of sets over  $[n_1]$ , and let  $\mathcal{U}_L$  be the corresponding laminar signed budget uncertainty set. Let us introduce a perturbation variable  $\zeta \in [-1, 1]^{n_1}$ , so that for each  $\mathbf{a} \in \mathcal{U}_L$  we can write

$$a_j = \bar{a}_j + \eta_j \zeta_j, \quad j \in [n_1]. \quad (15)$$

For each set  $\mathcal{P}_{\ell,k}$ , we have  $\sum_{j \in \mathcal{P}_{\ell,k}} a_j = \sum_{j \in \mathcal{P}_{\ell,k}} \bar{a}_j + \sum_{j \in \mathcal{P}_{\ell,k}} \eta_j \zeta_j \in [a_{\ell,k}^-, a_{\ell,k}^+]$ , hence, the aggregation constraints corresponding to  $\mathcal{P}_{\ell,k}$  can be rewritten as

$$\begin{aligned} \sum_{j \in \mathcal{P}_{\ell,k}} \eta_j \zeta_j &\in [a_{\ell,k}^- - \sum_{j \in \mathcal{P}_{\ell,k}} \bar{a}_j, a_{\ell,k}^+ - \sum_{j \in \mathcal{P}_{\ell,k}} \bar{a}_j] \\ &= [\Delta_{\ell,k} - \eta_{\ell,k}, \Delta_{\ell,k} + \eta_{\ell,k}], \end{aligned}$$

where we set  $\Delta_{\ell,k} = \bar{a}_{\ell,k} - \sum_{j \in \mathcal{P}_{\ell,k}} \bar{a}_j$ , for each  $\ell \in [L]$ ,  $k \in [K_\ell]$ .

We define the perturbation set  $\mathcal{Z}_L \subseteq \mathbb{R}^{n_1}$  as follows:

$$\mathcal{Z}_L = \left\{ \zeta \mid -1 \leq \zeta_j \leq 1, j \in [n_1], \sum_{j \in \mathcal{P}_{\ell,k}} \eta_j \zeta_j \in [\Delta_{\ell,k} - \eta_{\ell,k}, \Delta_{\ell,k} + \eta_{\ell,k}], \ell \in [L], k \in [K_\ell] \right\}. \quad (16)$$

The laminar signed budget uncertainty set can now be written as

$$\mathcal{U}_L = \left\{ \mathbf{a} \mid a_j = \bar{a}_j + \eta_j \zeta_j, j \in [n_1], \zeta \in \mathcal{Z}_L \right\}. \quad (17)$$

PROPOSITION 1. Let  $\{\mathcal{P}_{\ell,1}, \dots, \mathcal{P}_{\ell,K_\ell}\}_{\ell \in [L]}$  be a laminar family of sets. The RC of constraint (2) with uncertainty set  $\mathcal{U}_L$  defined by (17) and (16) is given by

$$\begin{aligned}
& \sum_{\ell \in [L]} \sum_{k \in [K_\ell]} (a_{\ell,k}^+ \lambda_{\ell,k} - a_{\ell,k}^- \gamma_{\ell,k}) + \sum_{j \in [n_1]} (a_j^+ \mu_j^+ - a_j^- \mu_j^-) + \mathbf{b}^\top \mathbf{y} \leq \beta, \\
& \sum_{\ell \in [L]} \sum_{\substack{k \in [K_\ell]: \\ j \in \mathcal{P}_{\ell,k}}} (\lambda_{\ell,k} - \gamma_{\ell,k}) + \mu_j^+ - \mu_j^- = x_j, \quad j \in [n_1], \\
& \lambda_{\ell,k}, \gamma_{\ell,k} \in \mathbb{R}_+, \quad \ell \in [L], \quad k \in [K_\ell], \\
& \mu_j^+, \mu_j^- \in \mathbb{R}_+, \quad j \in [n_1].
\end{aligned} \tag{18}$$

*Proof* We can rewrite (2) by using the definition of  $a_j$  in (15) and collecting all the uncertain values:

$$\begin{aligned}
& \sum_{j \in [n_1]} (\bar{a}_j + \eta_j \zeta_j) x_j \leq \beta - \mathbf{b}^\top \mathbf{y}, \quad \forall \zeta \in \mathcal{Z}_L, \\
& \Leftrightarrow \\
& \sum_{j \in [n_1]} \eta_j x_j \zeta_j \leq \beta - \mathbf{b}^\top \mathbf{y} - \sum_{j \in [n_1]} \bar{a}_j x_j, \quad \forall \zeta \in \mathcal{Z}_L,
\end{aligned}$$

which is equivalent to

$$\max_{\zeta \in \mathcal{Z}_L} \sum_{j \in [n_1]} \eta_j x_j \zeta_j \leq \beta - \mathbf{b}^\top \mathbf{y} - \sum_{j \in [n_1]} \bar{a}_j x_j. \tag{19}$$

By recalling the definition of the set  $\mathcal{Z}_L$ , we can rewrite the left-hand side of (19) as

$$\begin{aligned}
& \max \sum_{j \in [n_1]} \eta_j x_j \zeta_j \\
& \text{s.t.} \quad \sum_{j \in \mathcal{P}_{\ell,k}} \eta_j \zeta_j \leq \Delta_{\ell,k} + \eta_{\ell,k}, \quad \ell \in [L], \quad k \in [K_\ell], \\
& \quad \sum_{j \in \mathcal{P}_{\ell,k}} \eta_j \zeta_j \geq \Delta_{\ell,k} - \eta_{\ell,k}, \quad \ell \in [L], \quad k \in [K_\ell], \\
& \quad \zeta_j \leq 1, \quad j \in [n_1], \\
& \quad \zeta_j \geq -1, \quad j \in [n_1], \\
& \quad \zeta_j \in \mathbb{R}, \quad j \in [n_1].
\end{aligned}$$

We now derive its dual, which can be formulated as

$$\begin{aligned}
& \min \sum_{\ell \in [L]} \sum_{k \in [K_\ell]} (\Delta_{\ell,k} + \eta_{\ell,k}) \lambda_{\ell,k} - \sum_{\ell \in [L]} \sum_{k \in [K_\ell]} (\Delta_{\ell,k} - \eta_{\ell,k}) \gamma_{\ell,k} + \sum_{j \in [n_1]} (v_j^+ + v_j^-) \\
& \text{s.t.} \quad \eta_j \left( \sum_{\ell \in [L]} \sum_{\substack{k \in [K_\ell]: \\ j \in \mathcal{P}_{\ell,k}}} \lambda_{\ell,k} \right) - \eta_j \left( \sum_{\ell \in [L]} \sum_{\substack{k \in [K_\ell]: \\ j \in \mathcal{P}_{\ell,k}}} \gamma_{\ell,k} \right) + v_j^+ - v_j^- = \eta_j x_j, \quad j \in [n_1], \\
& \quad \lambda_{\ell,k}, \gamma_{\ell,k} \in \mathbb{R}_+, \quad \ell \in [L], \quad k \in [K_\ell], \\
& \quad v_j^+, v_j^- \in \mathbb{R}_+, \quad j \in [n_1].
\end{aligned}$$

By strong duality, the RC of constraint (2) can be written as

$$\begin{aligned} & \sum_{\ell \in [L]} \sum_{k \in [K_\ell]} (\Delta_{\ell,k} + \eta_{\ell,k}) \lambda_{\ell,k} - \sum_{\ell \in [L]} \sum_{k \in [K_\ell]} (\Delta_{\ell,k} - \eta_{\ell,k}) \gamma_{\ell,k} \\ & + \sum_{j \in [n_1]} (v_j^+ + v_j^-) \leq \beta - \mathbf{b}^\top \mathbf{y} - \sum_{j \in [n_1]} \bar{a}_j x_j, \end{aligned} \quad (20a)$$

$$\eta_j \left( \sum_{\ell \in [L]} \sum_{\substack{k \in [K_\ell]: \\ j \in \mathcal{P}_{\ell,k}}} (\lambda_{\ell,k} - \gamma_{\ell,k}) \right) + v_j^+ - v_j^- = \eta_j x_j, \quad j \in [n_1], \quad (20b)$$

$$\lambda_{\ell,k}, \gamma_{\ell,k} \in \mathbb{R}_+, \quad \ell \in [L], \quad k \in [K_\ell],$$

$$v_j^+, v_j^- \in \mathbb{R}_+, \quad j \in [n_1].$$

If  $\eta_j = 0$  for some  $j \in [n_1]$ , then  $a_j = \bar{a}_j$ , that is, the parameter is not uncertain and we can treat those constraints separately. From now on, we assume  $\eta_j \neq 0$ , and we rescale variables  $v_j^+ = \eta_j \mu_j^+$  and  $v_j^- = \eta_j \mu_j^-$ , so that constraint (20b) is equivalent to

$$\sum_{\ell \in [L]} \sum_{\substack{k \in [K_\ell]: \\ j \in \mathcal{P}_{\ell,k}}} (\lambda_{\ell,k} - \gamma_{\ell,k}) + \mu_j^+ - \mu_j^- = x_j, \quad j \in [n_1], \quad (21)$$

while the inequality in (20a) can be rewritten as

$$\begin{aligned} & \sum_{\ell \in [L]} \sum_{k \in [K_\ell]} (\Delta_{\ell,k} + \eta_{\ell,k}) \lambda_{\ell,k} - \sum_{\ell \in [L]} \sum_{k \in [K_\ell]} (\Delta_{\ell,k} - \eta_{\ell,k}) \gamma_{\ell,k} \\ & + \sum_{j \in [n_1]} (\eta_j \mu_j^+ + \eta_j \mu_j^-) + \sum_{j \in [n_1]} \bar{a}_j x_j \leq \beta - \mathbf{b}^\top \mathbf{y}. \end{aligned} \quad (22)$$

By combining (21) and (22), we obtain

$$\begin{aligned} & \sum_{\ell \in [L]} \sum_{k \in [K_\ell]} (\Delta_{\ell,k} + \eta_{\ell,k}) \lambda_{\ell,k} - \sum_{\ell \in [L]} \sum_{k \in [K_\ell]} (\Delta_{\ell,k} - \eta_{\ell,k}) \gamma_{\ell,k} + \sum_{j \in [n_1]} (\eta_j \mu_j^+ + \eta_j \mu_j^-) \\ & + \sum_{j \in [n_1]} \bar{a}_j \left( \sum_{\ell \in [L]} \sum_{\substack{k \in [K_\ell]: \\ j \in \mathcal{P}_{\ell,k}}} (\lambda_{\ell,k} - \gamma_{\ell,k}) + \mu_j^+ - \mu_j^- \right) \leq \beta - \mathbf{b}^\top \mathbf{y}. \end{aligned} \quad (23)$$

By recalling the definition of  $\Delta_{\ell,k} = \bar{a}_{\ell,k} - \sum_{j \in \mathcal{P}_{\ell,k}} \bar{a}_j$ , and from

$$\sum_{j \in [n_1]} \bar{a}_j \left( \sum_{\ell \in [L]} \sum_{\substack{k \in [K_\ell]: \\ j \in \mathcal{P}_{\ell,k}}} (\lambda_{\ell,k} - \gamma_{\ell,k}) \right) = \sum_{\ell \in [L]} \sum_{k \in [K_\ell]} \left( \sum_{j \in \mathcal{P}_{\ell,k}} \bar{a}_j \right) (\lambda_{\ell,k} - \gamma_{\ell,k}),$$

we can rewrite (23) as

$$\sum_{\ell \in [L]} \sum_{k \in [K_\ell]} (a_{\ell,k}^+ \lambda_{\ell,k} - a_{\ell,k}^- \gamma_{\ell,k}) + \sum_{j \in [n_1]} (a_j^+ \mu_j^+ - a_j^- \mu_j^-) \leq \beta - \mathbf{b}^\top \mathbf{y}.$$

Hence, we conclude that the RC of constraint (2) is equivalent to (18).  $\square$

The signed budget uncertainty set is obtained when the laminar family is such that  $L = 1$ ,  $K_1 = 1$ , and  $\mathcal{P}_{1,1} = [n_1]$ . In this case, the RC of (2) simplifies to

$$\begin{aligned} a_{\Sigma}^+ \lambda - a_{\Sigma}^- \gamma + \sum_{j \in [n_1]} (a_j^+ \mu_j^+ - a_j^- \mu_j^-) + \mathbf{b}^\top \mathbf{y} &\leq \beta, \\ \lambda - \gamma + \mu_j^+ - \mu_j^- &= x_j, \quad j \in [n_1], \\ \lambda, \gamma &\in \mathbb{R}_+, \\ \mu_j^+, \mu_j^- &\in \mathbb{R}_+, \quad j \in [n_1]. \end{aligned}$$

## Appendix B: Proofs for the Signed Budget Uncertainty

We prove the results presented in Section 2. Our approach is based on the observation that the robust constraint (2) is equivalent to

$$\max_{\mathbf{a} \in \mathcal{U}} \mathbf{a}^\top \mathbf{x} + \mathbf{b}^\top \mathbf{y} \leq \beta. \quad (24)$$

Therefore, a key step in deriving compact RCs is to characterize the value of the worst-case expression  $\max_{\mathbf{a} \in \mathcal{U}} \mathbf{a}^\top \mathbf{x}$ . The proof of Theorem 1 follows a similar reasoning as the one used in Kerkkamp (2014, Section 3.2.4) for robust maximum coverage problems with objective uncertainty.

### B.1. Proof of Theorem 1

Let

$$J_0 = \{j \in [n_1] \mid x_j = 0\}, \quad J_1 = \{j \in [n_1] \mid x_j = 1\}. \quad (25)$$

Since  $\mathbf{x}$  is binary, for every  $\mathbf{a} \in \mathcal{U}$  we have  $\mathbf{a}^\top \mathbf{x} = \sum_{j \in J_1} a_j$ . The individual upper bounds imply

$$\sum_{j \in J_1} a_j \leq \sum_{j \in J_1} a_j^+. \quad (26)$$

Moreover, using the aggregate upper bound and the individual lower bounds for  $j \in J_0$ , we get

$$\sum_{j \in J_1} a_j = \sum_{j \in [n_1]} a_j - \sum_{j \in J_0} a_j \leq a_{\Sigma}^+ - \sum_{j \in J_0} a_j^-. \quad (27)$$

Hence, combining (26) and (27), we obtain

$$\max_{\mathbf{a} \in \mathcal{U}} \mathbf{a}^\top \mathbf{x} \leq \min \left\{ \sum_{j \in J_1} a_j^+, a_{\Sigma}^+ - \sum_{j \in J_0} a_j^- \right\}. \quad (28)$$

It remains to show that this bound is attainable. Suppose first that

$$\sum_{j \in J_1} a_j^+ \leq a_{\Sigma}^+ - \sum_{j \in J_0} a_j^-. \quad (29)$$

Set  $a_j = a_j^+$  for all  $j \in J_1$ . We have

$$\sum_{j \in J_0} a_j^- \leq a_{\Sigma}^+ - \sum_{j \in J_1} a_j^+ \leq \sum_{j \in J_0} a_j^+,$$

where the first inequality follows from (29), and the second one from (4). Then, by the Intermediate Value Theorem (IVT) applied to the continuous map  $t \in [0, 1] \mapsto \sum_{j \in J_0} (a_j^- + 2t\eta_j)$ , there exists  $t^* \in [0, 1]$  such that  $\sum_{j \in J_0} (a_j^- + 2t^*\eta_j) = a_\Sigma^+ - \sum_{j \in J_1} a_j^+$ , i.e., the coefficients  $a_j$ ,  $j \in J_0$ , can be chosen within their individual bounds so that  $\sum_{j \in [n_1]} a_j = a_\Sigma^+$ . The resulting solution is feasible for  $\mathcal{U}$  and attains  $\sum_{j \in J_1} a_j^+$ .

Otherwise, assume that (29) does not hold. Set  $a_j = a_j^-$  for all  $j \in J_0$ . With the same reasoning as in the previous case, by (4) we get that the coefficients  $a_j$ ,  $j \in J_1$ , can be chosen within their individual bounds so that  $\sum_{j \in [n_1]} a_j = a_\Sigma^+$ . This solution is feasible for  $\mathcal{U}$  and attains

$$a_\Sigma^+ - \sum_{j \in J_0} a_j^-.$$

Thus, (28) holds with equality and recalling (25), we have that

$$\max_{\mathbf{a} \in \mathcal{U}} \mathbf{a}^\top \mathbf{x} = \sum_{j \in [n_1]} a_j^+ x_j + \min \left\{ 0, a_\Sigma^+ - \sum_{j \in [n_1]} (a_j^- (1 - x_j) + a_j^+ x_j) \right\}.$$

Substituting this expression in (24) yields (6).  $\square$

## B.2. Big- $M$ Coefficients

A suitable choice of  $M$  in the linearization of the compact RC in Section 2.3 can be obtained by deriving valid bounds on the quantity  $a_\Sigma^+ - \sum_{j \in [n_1]} (a_j^+ x_j + a_j^- (1 - x_j))$ . From (4), we have

$$\sum_{j \in [n_1]} a_j^- \leq a_\Sigma^+ \leq \sum_{j \in [n_1]} a_j^+,$$

and, for any  $\mathbf{x} \in \{0, 1\}^{n_1}$ , it holds that

$$\sum_{j \in [n_1]} a_j^- \leq \sum_{j \in [n_1]} (a_j^+ x_j + a_j^- (1 - x_j)) \leq \sum_{j \in [n_1]} a_j^+.$$

Combining the above inequalities, we obtain

$$-2 \sum_{j \in [n_1]} \eta_j \leq a_\Sigma^+ - \sum_{j \in [n_1]} (a_j^+ x_j + a_j^- (1 - x_j)) \leq 2 \sum_{j \in [n_1]} \eta_j.$$

## B.3. Proof of Corollary 1

From Theorem 1, the robust constraint (2) is equivalent to (6). Since the minimum in (6) can only take two values, the constraint is satisfied if at least one of the following inequalities is satisfied by a pair  $(\mathbf{x}, \mathbf{y})$ :

$$\sum_{j \in [n_1]} a_j^+ x_j + \mathbf{b}^\top \mathbf{y} \leq \beta, \tag{30}$$

or

$$a_\Sigma^+ - \sum_{j \in [n_1]} a_j^- (1 - x_j) + \mathbf{b}^\top \mathbf{y} \leq \beta. \tag{31}$$

Observe that (30) is a nominal constraint of the form

$$\hat{\mathbf{a}}^\top \mathbf{x} + \hat{\mathbf{b}}^\top \mathbf{y} \leq \hat{\beta}, \quad (32)$$

with  $\hat{\mathbf{a}} = \mathbf{a}^+$ ,  $\hat{\mathbf{b}} = \mathbf{b}$ ,  $\hat{\beta} = \beta$ . Rearranging the terms in (31), we obtain a nominal constraint of the form (32) with  $\hat{\mathbf{a}} = \mathbf{a}^-$ ,  $\hat{\mathbf{b}} = \mathbf{b}$ ,  $\hat{\beta} = \beta - \mathbf{a}_\Sigma^+ + \sum_{j \in [n_1]} \mathbf{a}_j^-$ . Therefore, solving the two corresponding nominal problems and selecting the solution with the smaller objective value is equivalent to solving the robust problem (1).  $\square$

### Appendix C: Proofs for the Laminar Signed Budget Uncertainty

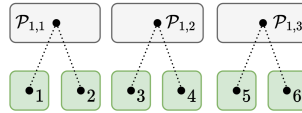
We prove the results from Section 3. For ease of notation and exposition, in this section we consider laminar families such that the sets in each level form a partition of  $[n_1]$ . In Appendix C.7 we show how the proofs can be extended to arbitrary laminar families. In particular, for  $L \geq 2$ , and  $\ell \in [L-1]$ ,  $k \in [K_{\ell+1}]$ , after tightening aggregate bounds that are redundant, we may assume that the following holds:

$$\sum_{\substack{k' \in [K_\ell]: \\ \mathcal{P}_{\ell,k'} \subset \mathcal{P}_{\ell+1,k}}} a_{\ell,k'}^- \leq a_{\ell+1,k}^- \leq a_{\ell+1,k}^+ \leq \sum_{\substack{k' \in [K_\ell]: \\ \mathcal{P}_{\ell,k'} \subset \mathcal{P}_{\ell+1,k}}} a_{\ell,k'}^+. \quad (33)$$

We analyze the maximization problem in (24) for laminar signed budget uncertainty sets  $\mathcal{U}_L$  corresponding to different types of laminar structures: we proceed progressively, starting from simple aggregation structures and gradually increasing the level of hierarchy.

#### C.1. General One-Level Laminar Structure

We extend the analysis in the proof of Theorem 1 to laminar families with one level of aggregation, as illustrated in Figure 3.



**Figure 3** Example of a one-level laminar structure which is a partition of  $\{1, \dots, 6\}$ ,  $\{\mathcal{P}_{1,1}, \mathcal{P}_{1,2}, \mathcal{P}_{1,3}\} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$ .

**LEMMA 1.** Let  $\{\mathcal{P}_{1,1}, \dots, \mathcal{P}_{1,K_1}\}$  be a partition of  $[n_1]$ . Let  $\mathcal{U}_1$  be the corresponding laminar signed budget uncertainty set. Then, for  $\mathbf{x} \in \{0, 1\}^{n_1}$ , it holds

$$\max_{\mathbf{a} \in \mathcal{U}_1} \mathbf{a}^\top \mathbf{x} = \sum_{j \in [n_1]} a_j^+ x_j + \sum_{k \in [K_1]} w_{1,k}, \quad (34)$$

where for each  $k \in [K_1]$ ,  $w_{1,k}$  is defined as in (10).

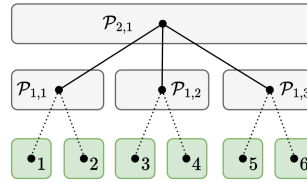
*Proof* Since  $\{\mathcal{P}_{1,1}, \dots, \mathcal{P}_{1,K_1}\}$  is a partition of  $[n_1]$ , and the uncertainty set  $\mathcal{U}_1$  imposes independent signed budget constraints for each set  $\mathcal{P}_{1,k}$ , the maximization problem decomposes across the partition. Applying Theorem 1 to each set  $\mathcal{P}_{1,k}$  we get

$$\begin{aligned} \max_{\mathbf{a} \in \mathcal{U}_1} \mathbf{a}^\top \mathbf{x} &= \sum_{k \in [K_1]} \max_{\mathbf{a} \in \mathcal{U}_1} \sum_{j \in \mathcal{P}_{1,k}} a_j x_j \\ &= \sum_{k \in [K_1]} \left( \sum_{j \in \mathcal{P}_{1,k}} a_j^+ x_j + w_{1,k} \right) \\ &= \sum_{j \in [n_1]} a_j^+ x_j + \sum_{k \in [K_1]} w_{1,k}. \end{aligned}$$

□

## C.2. Two-Level Laminar Structure with One Set in the Second Level

Next, we consider two-level laminar families where the first level partitions  $[n_1]$  and the second level aggregates all first-level sets, as illustrated in Figure 4. This case will provide the basis for our proof by induction for the general  $L$ -level case.



**Figure 4** Example of a two-level laminar structure over  $\{1, \dots, 6\}$ , where  $\mathcal{P}_{2,1} = \{1, \dots, 6\}$ .

The corresponding laminar signed budget uncertainty set is denoted by  $\mathcal{U}_2$ . We denote by  $\mathcal{U}_1$  the one-level uncertainty set obtained by retaining only the first-level aggregate constraints. We want to characterize  $\max_{\mathbf{a} \in \mathcal{U}_2} \mathbf{a}^\top \mathbf{x}$ .

The intuition is that either the second-level upper bound is nonbinding, in which case the worst-case value is the same as over  $\mathcal{U}_1$ , or it is binding and the maximum is obtained by subtracting from this bound the minimum contribution of the coefficients with  $x_j = 0$ . This leads to the following formulation for  $\max_{\mathbf{a} \in \mathcal{U}_2} \mathbf{a}^\top \mathbf{x}$ , derived in Lemma 4:

$$\min \left\{ \max_{\mathbf{a} \in \mathcal{U}_1} \mathbf{a}^\top \mathbf{x}, a_{2,1}^+ - \min_{\mathbf{a} \in \mathcal{U}_1} \mathbf{a}^\top (\mathbf{1} - \mathbf{x}) \right\}.$$

To establish this result, we first derive several auxiliary results. Corollary 3 gives a closed-form expression for  $\min_{\mathbf{a} \in \mathcal{U}_1} \mathbf{a}^\top (\mathbf{1} - \mathbf{x})$  and is a direct consequence of Lemma 1. Lemma 2 shows that there exists a solution in  $\mathcal{U}_1$  that simultaneously maximizes  $\mathbf{a}^\top \mathbf{x}$  and minimizes  $\mathbf{a}^\top (\mathbf{1} - \mathbf{x})$ . Finally, Lemma 3 gives a sufficient condition under which such a solution remains feasible, and hence optimal, after adding the second-level aggregate constraint.

COROLLARY 3. Let  $\{\mathcal{P}_{1,1}, \dots, \mathcal{P}_{1,K_1}\}$  be a partition of  $[n_1]$ . Let  $\mathcal{U}_1$  be the corresponding laminar signed budget uncertainty set. Then, for  $\mathbf{x} \in \{0, 1\}^{n_1}$ , it holds

$$\min_{\mathbf{a} \in \mathcal{U}_1} \mathbf{a}^\top (\mathbf{1} - \mathbf{x}) = \sum_{j \in [n_1]} a_j^- (1 - x_j) + \sum_{k \in [K_1]} u_{1,k}, \quad (35)$$

where, for each  $k \in [K_1]$ ,  $u_{1,k}$  is defined as in (11).

*Proof* Let  $\mathbf{z} = \mathbf{1} - \mathbf{x} \in \{0, 1\}^{n_1}$ . We define  $\tilde{\mathcal{U}}_1 = \{-\mathbf{a} \mid \mathbf{a} \in \mathcal{U}_1\}$ , which by construction is also a one-level laminar signed budget uncertainty set. Then,

$$\min_{\mathbf{a} \in \mathcal{U}_1} \mathbf{a}^\top (\mathbf{1} - \mathbf{x}) = -\max_{\tilde{\mathbf{a}} \in \tilde{\mathcal{U}}_1} \tilde{\mathbf{a}}^\top \mathbf{z}.$$

Applying Lemma 1 to  $\tilde{\mathcal{U}}_1$  and substituting  $\mathbf{z} = \mathbf{1} - \mathbf{x}$  gives

$$\min_{\mathbf{a} \in \mathcal{U}_1} \mathbf{a}^\top (\mathbf{1} - \mathbf{x}) = \sum_{j \in [n_1]} a_j^- (1 - x_j) + \sum_{k \in [K_1]} \max \left\{ 0, a_{1,k}^- - \sum_{j \in \mathcal{P}_{1,k}} (a_j^- x_j + a_j^+ (1 - x_j)) \right\}.$$

□

LEMMA 2. Let  $\mathcal{U}_1$  be the one-level laminar signed budget uncertainty set corresponding to the partition  $\{\mathcal{P}_{1,1}, \dots, \mathcal{P}_{1,K_1}\}$  of  $[n_1]$ . For every  $\mathbf{x} \in \{0, 1\}^{n_1}$ , there exists  $\mathbf{a}^* \in \mathcal{U}_1$  such that

$$\mathbf{a}^* \in \arg \max_{\mathbf{a} \in \mathcal{U}_1} \mathbf{a}^\top \mathbf{x} \cap \arg \min_{\mathbf{a} \in \mathcal{U}_1} \mathbf{a}^\top (\mathbf{1} - \mathbf{x}). \quad (36)$$

*Proof* For each  $k \in [K_1]$ , let

$$J_0(\mathcal{P}_{1,k}) = J_0 \cap \mathcal{P}_{1,k} \quad \text{and} \quad J_1(\mathcal{P}_{1,k}) = J_1 \cap \mathcal{P}_{1,k},$$

where  $J_0$  and  $J_1$  are defined as in (25). Let

$$S_{1,k} = \sum_{j \in J_0(\mathcal{P}_{1,k})} a_j^- + \sum_{j \in J_1(\mathcal{P}_{1,k})} a_j^+.$$

From (10) and (11), we can write

$$\begin{aligned} w_{1,k} &= \min\{0, a_{1,k}^+ - S_{1,k}\}, \\ u_{1,k} &= \max\{0, a_{1,k}^- - S_{1,k}\}. \end{aligned}$$

Since  $a_{1,k}^- \leq a_{1,k}^+$ , if  $w_{1,k} < 0$ , then  $S_{1,k} > a_{1,k}^+$  and hence  $u_{1,k} = 0$ . Similarly, if  $u_{1,k} > 0$ , then  $S_{1,k} < a_{1,k}^-$  and  $w_{1,k} = 0$ . Thus,  $w_{1,k} u_{1,k} = 0$ . We construct  $\mathbf{a}^*$  separately on each set  $\mathcal{P}_{1,k}$ , and since  $w_{1,k} u_{1,k} = 0$  we only consider the following three cases.

1. If  $w_{1,k} = u_{1,k} = 0$ , then  $a_{1,k}^- \leq S_{1,k} \leq a_{1,k}^+$ . We set  $a_j^* = a_j^+$  for  $j \in J_1(\mathcal{P}_{1,k})$ , and  $a_j^* = a_j^-$  for  $j \in J_0(\mathcal{P}_{1,k})$ .

2. If  $w_{1,k} < 0$ , then  $S_{1,k} > a_{1,k}^+$ . We set  $a_j^* = a_j^-$  for  $j \in J_0(\mathcal{P}_{1,k})$ . Since (8) holds, we have

$$\sum_{j \in J_0(\mathcal{P}_{1,k})} a_j^- + \sum_{j \in J_1(\mathcal{P}_{1,k})} a_j^- \leq a_{1,k}^+ < S_{1,k} = \sum_{j \in J_0(\mathcal{P}_{1,k})} a_j^- + \sum_{j \in J_1(\mathcal{P}_{1,k})} a_j^+$$

and suitably applying the IVT, as in the proof of Theorem 1, we get that the coefficients for  $j \in J_1(\mathcal{P}_{1,k})$  can be chosen within their individual bounds so that  $\sum_{j \in \mathcal{P}_{1,k}} a_j^* = a_{1,k}^+$ .

3. If  $u_{1,k} > 0$ , then  $S_{1,k} < a_{1,k}^-$ . We set  $a_j^* = a_j^+$  for  $j \in J_1(\mathcal{P}_{1,k})$  and choose the coefficients for  $j \in J_0(\mathcal{P}_{1,k})$  within their individual bounds so that  $\sum_{j \in \mathcal{P}_{1,k}} a_j^* = a_{1,k}^-$ .

In the three cases, we can verify that  $\mathbf{a}^* \in \mathcal{U}_1$  satisfies

$$\begin{aligned} \sum_{j \in J_1(\mathcal{P}_{1,k})} a_j^* &= \sum_{j \in J_1(\mathcal{P}_{1,k})} a_j^+ + w_{1,k}, \\ \sum_{j \in J_0(\mathcal{P}_{1,k})} a_j^* &= \sum_{j \in J_0(\mathcal{P}_{1,k})} a_j^- + u_{1,k}. \end{aligned}$$

Recalling the result of Lemma 1 and Corollary 3, and since the sets  $\mathcal{P}_{1,k}$  partition  $[n_1]$ , combining these choices for each  $k \in [K_1]$  gives  $\mathbf{a}^* \in \mathcal{U}_1$  satisfying (36).  $\square$

LEMMA 3. Let  $\mathbf{x} \in \{0, 1\}^{n_1}$ , and let  $w_{1,k}$  and  $u_{1,k}$ ,  $k \in [K_1]$ , be defined as in (10) and (11), respectively. If

$$\sum_{j \in [n_1]} \left( a_j^+ x_j + a_j^- (1 - x_j) \right) + \sum_{k \in [K_1]} (w_{1,k} + u_{1,k}) \leq a_{2,1}^+, \quad (37)$$

then there exists an optimal solution of  $\max_{\mathbf{a} \in \mathcal{U}_1} \mathbf{a}^\top \mathbf{x}$  that is also optimal for  $\max_{\mathbf{a} \in \mathcal{U}_2} \mathbf{a}^\top \mathbf{x}$ .

*Proof* By Lemma 2, there exists  $\mathbf{a}^* \in \mathcal{U}_1$  that satisfies (36). Therefore,

$$\begin{aligned} \sum_{j \in [n_1]} a_j^* &= \max_{\mathbf{a} \in \mathcal{U}_1} \mathbf{a}^\top \mathbf{x} + \min_{\mathbf{a} \in \mathcal{U}_1} \mathbf{a}^\top (\mathbf{1} - \mathbf{x}) \\ &= \sum_{j \in [n_1]} \left( a_j^+ x_j + a_j^- (1 - x_j) \right) + \sum_{k \in [K_1]} (w_{1,k} + u_{1,k}), \end{aligned}$$

where the first equality follows from  $[n_1] = J_1 \cup J_0$  and (36), and the second from Lemma 1 and Corollary 3. Hence, by (37),  $\mathbf{a}^*$  satisfies the second-level upper bound. If  $\mathbf{a}^*$  satisfies the second-level lower bound, then  $\mathbf{a}^* \in \mathcal{U}_2$ . Otherwise, it holds  $\sum_{j \in [n_1]} a_j^* < a_{2,1}^-$ . Let  $\tilde{\mathbf{a}} \in \arg \max_{\mathbf{a} \in \mathcal{U}_1} \mathbf{a}^\top \mathbf{x}$  such that  $\sum_{j \in [n_1]} \tilde{a}_j (1 - x_j)$  is also maximized. By (8) we get  $\sum_{j \in [n_1]} \tilde{a}_j = \sum_{k \in [K_1]} a_{1,k}^+$  and we can write

$$\sum_{j \in [n_1]} a_j^* < a_{2,1}^- \leq a_{2,1}^+ \leq \sum_{k \in [K_1]} a_{1,k}^+ = \sum_{j \in [n_1]} \tilde{a}_j,$$

where the last inequality holds by (33). Then, since  $\mathcal{U}_1$  is convex, by applying the IVT as in the proof of Theorem 1 there exists  $\hat{\mathbf{a}} \in \arg \max_{\mathbf{a} \in \mathcal{U}_1} \mathbf{a}^\top \mathbf{x}$  such that  $\sum_{j \in [n_1]} \hat{a}_j = a_{2,1}^+$ , hence  $\hat{\mathbf{a}} \in \mathcal{U}_2$ . In both cases, there exists an optimal solution for  $\mathcal{U}_1$  that is feasible for  $\mathcal{U}_2$ , and, since  $\mathcal{U}_2 \subseteq \mathcal{U}_1$ , it is also optimal for  $\max_{\mathbf{a} \in \mathcal{U}_2} \mathbf{a}^\top \mathbf{x}$ .  $\square$

LEMMA 4. Let  $L=2$  and  $\{\mathcal{P}_{\ell,1}, \dots, \mathcal{P}_{\ell,K_\ell}\}_{\ell \in [L]}$  be a laminar family such that  $K_2 = 1$  and  $\mathcal{P}_{2,1} = [n_1]$ . Let  $\mathcal{U}_2$  be the corresponding laminar signed budget uncertainty set. For  $\mathbf{x} \in \{0,1\}^{n_1}$ , it holds

$$\max_{\mathbf{a} \in \mathcal{U}_2} \mathbf{a}^\top \mathbf{x} = \min \left\{ \sum_{j \in [n_1]} a_j^+ x_j + \sum_{k \in [K_1]} w_{1,k} a_{2,1}^+ - \sum_{j \in [n_1]} a_j^- (1 - x_j) - \sum_{k \in [K_1]} u_{1,k} \right\},$$

where for  $k \in [K_1]$ ,  $w_{1,k}$  and  $u_{1,k}$  are defined in (10) and (11), respectively.

*Proof* If (37) holds, Lemma 3 implies that an optimal solution over  $\mathcal{U}_1$  remains optimal over  $\mathcal{U}_2$ . Hence, by Lemma 1,  $\max_{\mathbf{a} \in \mathcal{U}_2} \mathbf{a}^\top \mathbf{x}$  takes the value in (34). If (37) does not hold, the second-level upper bound is binding. The worst-case contribution of the coefficients with  $x_j = 1$  is therefore obtained by subtracting the minimum contribution of those with  $x_j = 0$  from  $a_{2,1}^+$ . Using Corollary 3,  $\max_{\mathbf{a} \in \mathcal{U}_2} \mathbf{a}^\top \mathbf{x}$  is equal to

$$a_{2,1}^+ - \sum_{j \in [n_1]} a_j^- (1 - x_j) - \sum_{k \in [K_1]} u_{1,k}.$$

Combining the two cases yields the result.  $\square$

Equivalently, the result can be written as

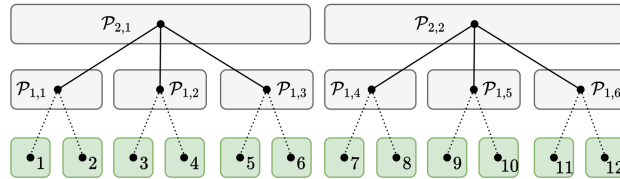
$$\max_{\mathbf{a} \in \mathcal{U}_2} \mathbf{a}^\top \mathbf{x} = \sum_{j \in [n_1]} a_j^+ x_j + \sum_{k \in [K_1]} w_{1,k} + w_{2,1},$$

where  $w_{2,1}$  is defined as

$$\min \left\{ 0, a_{2,1}^+ - \left( \sum_{j \in [n_1]} a_j^- (1 - x_j) + \sum_{k \in [K_1]} u_{1,k} \right) - \left( \sum_{j \in [n_1]} a_j^+ x_j + \sum_{k \in [K_1]} w_{1,k} \right) \right\}.$$

### C.3. General Two-Level Laminar Structure

We generalize Lemma 4 to two-level laminar families in which both levels form partitions of  $[n_1]$ , as illustrated in Figure 5.



**Figure 5** Example of a two-level laminar structure over  $\{1, \dots, 12\}$  where at each level we have a partition.

LEMMA 5. Let  $L = 2$  and  $\{\mathcal{P}_{\ell,1}, \dots, \mathcal{P}_{\ell,K_\ell}\}_{\ell \in [L]}$  be a two-level laminar family such that  $\{\mathcal{P}_{\ell,1}, \dots, \mathcal{P}_{\ell,K_\ell}\}$  is a partition of  $[n_1]$  for each  $\ell \in [L]$ . Let  $\mathcal{U}_2$  be the corresponding laminar signed budget uncertainty set. Then, for  $\mathbf{x} \in \{0,1\}^{n_1}$ ,

$$\max_{\mathbf{a} \in \mathcal{U}_2} \mathbf{a}^\top \mathbf{x} = \sum_{j \in [n_1]} a_j^+ x_j + \sum_{k \in [K_1]} w_{1,k} + \sum_{k \in [K_2]} w_{2,k},$$

where for  $k \in [K_1]$ ,  $w_{1,k}$  and  $u_{1,k}$  are defined in (10) and (11), respectively, and, for  $k \in [K_2]$ ,  $w_{2,k}$  is defined as in (13).

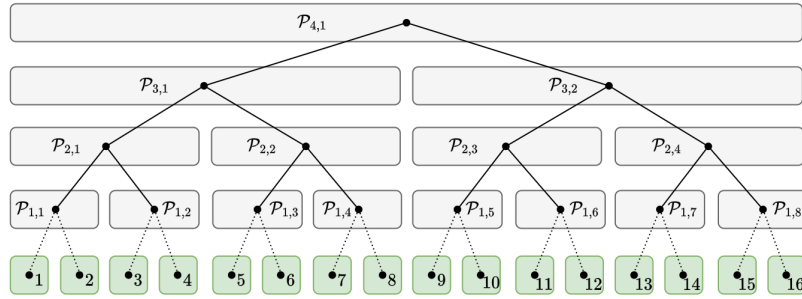
*Proof* Since  $\{\mathcal{P}_{2,1}, \dots, \mathcal{P}_{2,K_2}\}$  is a partition of  $[n_1]$  and the uncertainty set  $\mathcal{U}_2$  imposes independent signed budget constraints across these sets, the maximization problem decomposes across the partition. Hence, Lemma 4 applies to each second-level set separately, and recalling that both levels are partitions of  $[n_1]$ , we get

$$\begin{aligned} \max_{\mathbf{a} \in \mathcal{U}_2} \mathbf{a}^\top \mathbf{x} &= \sum_{k \in [K_2]} \max_{\mathbf{a} \in \mathcal{U}_2} \sum_{j \in \mathcal{P}_{2,k}} a_j x_j \\ &= \sum_{k \in [K_2]} \left( \sum_{j \in \mathcal{P}_{2,k}} a_j^+ x_j + \sum_{\substack{k' \in [K_1]: \\ \mathcal{P}_{1,k'} \subset \mathcal{P}_{2,k}}} w_{1,k'} + w_{2,k} \right) \\ &= \sum_{j \in [n_1]} a_j^+ x_j + \sum_{k \in [K_1]} w_{1,k} + \sum_{k \in [K_2]} w_{2,k}. \end{aligned}$$

□

#### C.4. General $L$ -Level Laminar Structure

We now consider laminar families with  $L$  levels where for each  $\ell \in [L]$ ,  $\{\mathcal{P}_{\ell,1}, \dots, \mathcal{P}_{\ell,K_\ell}\}$  is a partition of  $[n_1]$  (e.g., Figure 6). Let  $\mathcal{U}_L$  be the corresponding laminar signed budget uncertainty set.



**Figure 6** Example of a four-level laminar structure over  $\{1, \dots, 16\}$ . At each level there is a partition of  $\{1, \dots, 16\}$ .

We prove Theorem 2 recalling that the robust constraint (2) is equivalent to (24). We proceed by induction on  $L$ , using the following auxiliary results that extend the 2-level case in Appendix C.2.

For  $L \geq 2$ , let  $\mathcal{U}_{L-1}$  be the laminar signed budget uncertainty set obtained removing from  $\mathcal{U}_L$  the  $L$ -level aggregate constraints. Let  $\mathbf{x} \in \{0, 1\}^{n_1}$ , and let us assume that

$$\max_{\mathbf{a} \in \mathcal{U}_{L-1}} \mathbf{a}^\top \mathbf{x} = \sum_{j \in [n_1]} a_j^+ x_j + \sum_{\ell \in [L-1]} \sum_{k \in [K_\ell]} w_{\ell,k}, \quad (38)$$

where  $w_{\ell,k}$  is defined as in (13). Then, the following results hold as well.

COROLLARY 4. Under the assumptions above, for  $\mathbf{x} \in \{0, 1\}^{n_1}$ , it holds

$$\min_{\mathbf{a} \in \mathcal{U}_{L-1}} \mathbf{a}^\top (\mathbf{1} - \mathbf{x}) = \sum_{j \in [n_1]} a_j^- (1 - x_j) + \sum_{\ell \in [L-1]} \sum_{k \in [K_\ell]} u_{\ell,k}, \quad (39)$$

where for each  $\ell \in [L-1]$ ,  $k \in [K_\ell]$ ,  $u_{\ell,k}$  is defined as (14).

*Proof* The result follows from (38) by applying the same sign change as in the proof of Corollary 3.  $\square$

LEMMA 6. Under the assumptions above, for every  $\mathbf{x} \in \{0, 1\}^{n_1}$ , there exists  $\mathbf{a}^* \in \mathcal{U}_{L-1}$  such that

$$\mathbf{a}^* \in \arg \max_{\mathbf{a} \in \mathcal{U}_{L-1}} \mathbf{a}^\top \mathbf{x} \cap \arg \min_{\mathbf{a} \in \mathcal{U}_{L-1}} \mathbf{a}^\top (\mathbf{1} - \mathbf{x}). \quad (40)$$

*Proof* The proof proceeds by induction on the level  $L' = L - 1 \geq 1$ . For  $L' = 1$  the result holds by Lemma 2. Let  $L' > 1$ , assume that the result holds for  $L' - 1$  and let

$$\tilde{\mathbf{a}} \in \arg \max_{\mathbf{a} \in \mathcal{U}_{L'-1}} \mathbf{a}^\top \mathbf{x} \cap \arg \min_{\mathbf{a} \in \mathcal{U}_{L'-1}} \mathbf{a}^\top (\mathbf{1} - \mathbf{x}).$$

We prove it for  $L'$ . For each  $k \in [K_{L'}]$ , let

$$J_0(\mathcal{P}_{L',k}) = J_0 \cap \mathcal{P}_{L',k}, \quad J_1(\mathcal{P}_{L',k}) = J_1 \cap \mathcal{P}_{L',k},$$

where  $J_0$  and  $J_1$  are defined as in (25). Recalling the definition of  $\mathcal{D}_{L',k}$  in (12), let

$$S_{L',k} = \sum_{j \in J_0(\mathcal{P}_{L',k})} a_j^- + \sum_{j \in J_1(\mathcal{P}_{L',k})} a_j^+ + \sum_{(\ell,k') \in \mathcal{D}_{L',k}} (u_{\ell,k'} + w_{\ell,k'}),$$

From (13) and (14), we can write

$$\begin{aligned} w_{L',k} &= \min\{0, a_{L',k}^+ - S_{L',k}\}, \\ u_{L',k} &= \max\{0, a_{L',k}^- - S_{L',k}\}. \end{aligned}$$

Since  $a_{L',k}^- \leq a_{L',k}^+$ , if  $w_{L',k} < 0$ , then  $S_{L',k} > a_{L',k}^+$  and hence  $u_{L',k} = 0$ . Similarly, if  $u_{L',k} > 0$ , then  $S_{L',k} < a_{L',k}^-$  and  $w_{L',k} = 0$ . Thus,  $w_{L',k} u_{L',k} = 0$ . We construct  $\mathbf{a}^*$  separately on each set  $\mathcal{P}_{L',k}$ , and since  $w_{L',k} u_{L',k} = 0$  we only consider the following three cases.

1. If  $w_{L',k} = u_{L',k} = 0$ , then  $a_{L',k}^- \leq S_{L',k} \leq a_{L',k}^+$ . Hence, we set  $a_j^* = \tilde{a}_j$  for  $j \in \mathcal{P}_{L',k}$ .
2. If  $w_{L',k} < 0$ , then  $S_{L',k} > a_{L',k}^+$ . Let  $\hat{\mathbf{a}} \in \arg \min_{\mathbf{a} \in \mathcal{U}_{L'-1}} \mathbf{a}^\top (\mathbf{1} - \mathbf{x})$  be such that  $\sum_{j \in [n_1]} \hat{a}_j x_j$  is also minimized. Then, by (33) it holds  $\sum_{j \in \mathcal{P}_{L',k}} \hat{a}_j = \sum_{\substack{k' \in [K_{L'-1}]: \\ \mathcal{P}_{L'-1,k'} \subset \mathcal{P}_{L',k}}} a_{L'-1,k'}^-$  and we can write

$$\sum_{j \in \mathcal{P}_{L',k}} \hat{a}_j \leq a_{L',k}^+ < S_{L',k} = \sum_{j \in \mathcal{P}_{L',k}} \tilde{a}_j.$$

We set  $a_j^* = \tilde{a}_j$  for  $j \in J_0(\mathcal{P}_{L',k})$  and since  $\mathcal{U}_{L'-1}$  is convex, by the IVT we can choose the coefficients for  $j \in J_1(\mathcal{P}_{L',k})$  within the bounds determined by  $\mathcal{U}_{L'-1}$  so that  $\sum_{j \in \mathcal{P}_{L',k}} a_j^* = a_{L',k}^+$ .

3. If  $u_{L',k} > 0$ , then  $S_{L',k} < a_{L',k}^-$ . We set  $a_j^* = \tilde{a}_j$  for  $j \in J_1(\mathcal{P}_{L',k})$  and as in case 2 we choose the coefficients for  $j \in J_0(\mathcal{P}_{L',k})$  within the bounds imposed by  $\mathcal{U}_{L'-1}$  so that  $\sum_{j \in \mathcal{P}_{L',k}} a_j^* = a_{L',k}^-$ .

In the three cases,  $\mathbf{a}^*$  satisfies

$$\begin{aligned} \sum_{j \in J_1(\mathcal{P}_{L',k})} a_j^* &= \sum_{j \in J_1(\mathcal{P}_{L',k})} a_j^+ + \sum_{(\ell,k') \in \mathcal{D}_{L',k}} w_{\ell,k'}, \\ \sum_{j \in J_0(\mathcal{P}_{L',k})} a_j^* &= \sum_{j \in J_0(\mathcal{P}_{L',k})} a_j^- + \sum_{(\ell,k') \in \mathcal{D}_{L',k}} u_{\ell,k'}. \end{aligned}$$

Recalling (38) and (39), and since the sets  $\mathcal{P}_{L',k}$  partition  $[n_1]$ , combining these choices for each  $k \in [K_{L'}]$  gives  $\mathbf{a}^* \in \mathcal{U}_{L'}$  satisfying (40). □

LEMMA 7. Under the assumptions above, let  $K_L = 1$ ,  $\mathcal{P}_{L,1} = [n_1]$  and  $\mathbf{x} \in \{0,1\}^{n_1}$ . If

$$\sum_{j \in [n_1]} (a_j^+ x_j + a_j^- (1 - x_j)) + \sum_{\substack{\ell \in [L-1], \\ k \in [K_\ell]}} (w_{\ell,k} + u_{\ell,k}) \leq a_{L,1}^+, \quad (41)$$

then there exists an optimal solution of  $\max_{\mathbf{a} \in \mathcal{U}_{L-1}} \mathbf{a}^\top \mathbf{x}$  that is also optimal for  $\max_{\mathbf{a} \in \mathcal{U}_L} \mathbf{a}^\top \mathbf{x}$ .

*Proof* Consider the solution  $\mathbf{a}^*$  from Lemma 6. Then,  $\mathbf{a}^* \in \mathcal{U}_{L-1}$  and (40) holds. Hence,

$$\begin{aligned} \sum_{j \in [n_1]} a_j^* &= \max_{\mathbf{a} \in \mathcal{U}_{L-1}} \mathbf{a}^\top \mathbf{x} + \min_{\mathbf{a} \in \mathcal{U}_{L-1}} \mathbf{a}^\top (\mathbf{1} - \mathbf{x}) \\ &= \sum_{j \in [n_1]} (a_j^+ x_j + a_j^- (1 - x_j)) + \sum_{\substack{\ell \in [L-1], \\ k \in [K_\ell]}} (w_{\ell,k} + u_{\ell,k}), \end{aligned}$$

where the last equality follows from (38) and (39). Therefore, by (41),  $\mathbf{a}^*$  satisfies the level- $L$  aggregate upper bound. Following the same reasoning as in the proof of Lemma 3 we show that we can find an optimal solution for  $\mathcal{U}_{L-1}$  that is feasible for  $\mathcal{U}_L$ . Since  $\mathcal{U}_L \subseteq \mathcal{U}_{L-1}$ , such solution is also optimal for  $\max_{\mathbf{a} \in \mathcal{U}_L} \mathbf{a}^\top \mathbf{x}$ . □

### C.5. Proof of Theorem 2

If  $L = 1$ , the result holds by Lemma 1. For  $L = 2$ , the result holds by Lemma 5. Otherwise, we proceed by induction on the level  $L \geq 2$ . Let  $L > 2$  and assume that the result holds for  $L - 1$ . We show that it holds for  $L$ .

First, consider the case  $K_L = 1$ , so that  $\mathcal{P}_{L,1} = [n_1]$  and  $\mathcal{U}_L$  is obtained from  $\mathcal{U}_{L-1}$  by adding the level- $L$  aggregate constraint. If (41) holds, Lemma 7 implies that

$$\max_{\mathbf{a} \in \mathcal{U}_L} \mathbf{a}^\top \mathbf{x} = \max_{\mathbf{a} \in \mathcal{U}_{L-1}} \mathbf{a}^\top \mathbf{x}. \quad (42)$$

Otherwise, the level- $L$  upper aggregate bound is binding and we get

$$\max_{\mathbf{a} \in \mathcal{U}_L} \mathbf{a}^\top \mathbf{x} = a_{L,1}^+ - \min_{\mathbf{a} \in \mathcal{U}_{L-1}} \mathbf{a}^\top (\mathbf{1} - \mathbf{x}). \quad (43)$$

Recalling the induction hypothesis (38) and Corollary 4, we can combine the two cases above and we get that  $\max_{\mathbf{a} \in \mathcal{U}_L} \mathbf{a}^\top \mathbf{x}$  is equivalent to

$$\begin{aligned} & \min \left\{ \sum_{j \in [n_1]} a_j^+ x_j + \sum_{\ell \in [L-1]} \sum_{k \in [K_\ell]} w_{\ell,k}, a_{L,1}^+ - \left( \sum_{j \in [n_1]} a_j^- (1 - x_j) + \sum_{\ell \in [L-1]} \sum_{k \in [K_\ell]} u_{\ell,k} \right) \right\} \\ & = \sum_{j \in [n_1]} a_j^+ x_j + \sum_{\ell \in [L]} \sum_{k \in [K_\ell]} w_{\ell,k}, \end{aligned} \quad (44)$$

where the last equality follows by the definition of  $w_{\ell,k}$  in (13), with  $\ell = L$  and  $k = 1$ .

Now consider  $K_L > 1$ . Since the sets  $\mathcal{P}_{L,k}$ ,  $k \in [K_L]$ , partition  $[n_1]$ , the maximization problem decomposes across these sets. Applying (44) to each  $\mathcal{P}_{L,k}$  gives the result.  $\square$

We can linearize  $w_{\ell,k}$ ,  $\ell \in [L]$ ,  $k \in [K_\ell]$ , and  $u_{\ell,k}$ ,  $\ell \in [L-1]$ ,  $k \in [K_\ell]$ , via standard big- $M$  linearization introducing the corresponding binary variables  $v_{\ell,k}^1$  and  $v_{\ell,k}^2$ . Recalling the definition of  $\mathcal{D}_{\ell,k}$ , note that, for  $\ell = 1$ ,  $\mathcal{D}_{1,k} = \emptyset$  for each  $k \in [K_1]$ . Let

$$S_{\ell,k} = \sum_{j \in \mathcal{P}_{\ell,k}} a_j^- (1 - x_j) + \sum_{j \in \mathcal{P}_{\ell,k}} a_j^+ x_j + \sum_{(\ell',k') \in \mathcal{D}_{\ell,k}} (u_{\ell',k'} + w_{\ell',k'}).$$

This yields the following compact formulation:

$$\begin{aligned} & \sum_{j \in [n_1]} a_j^+ x_j + \sum_{\ell \in [L]} \sum_{k \in [K_\ell]} w_{\ell,k} + \mathbf{b}^\top \mathbf{y} \leq \beta, \\ & w_{\ell,k} \geq a_{\ell,k}^+ - S_{\ell,k} - M_{\ell,k}^1 (1 - v_{\ell,k}^1), & \ell \in [L], k \in [K_\ell], \\ & w_{\ell,k} \geq -M_{\ell,k}^1 v_{\ell,k}^1, & \ell \in [L], k \in [K_\ell], \\ & u_{\ell,k} \leq a_{\ell,k}^- - S_{\ell,k} + M_{\ell,k}^2 (1 - v_{\ell,k}^2), & \ell \in [L-1], k \in [K_\ell], \\ & u_{\ell,k} \leq M_{\ell,k}^2 v_{\ell,k}^2, & \ell \in [L-1], k \in [K_\ell], \\ & v_{\ell,k}^1 \in \{0, 1\}, & \ell \in [L], k \in [K_\ell], \\ & v_{\ell,k}^2 \in \{0, 1\}, & \ell \in [L-1], k \in [K_\ell]. \end{aligned}$$

### C.6. Big- $M$ Coefficients

A suitable choice of the big- $M$  coefficients appearing in the above linearization of the compact RC is obtained by bounding the correction terms  $a_{\ell,k}^\pm - S_{\ell,k}$  that define  $w_{\ell,k}$  and  $u_{\ell,k}$ . We first show that, for every  $\mathbf{x} \in \{0, 1\}^{n_1}$  and every  $\ell \in [L]$ ,  $k \in [K_\ell]$ ,

$$\sum_{j \in \mathcal{P}_{\ell,k}} a_j^- \leq S_{\ell,k} \leq \sum_{j \in \mathcal{P}_{\ell,k}} a_j^+. \quad (45)$$

For  $\ell = 1$ ,  $\mathcal{D}_{1,k} = \emptyset$ , so  $S_{1,k} = \sum_{j \in \mathcal{P}_{1,k}} [a_j^+ x_j + a_j^- (1 - x_j)]$  and (45) holds since each summand lies in  $[a_j^-, a_j^+]$ .

For  $\ell \geq 2$ , observe that by (11)–(14), for every set  $\mathcal{P}_{\ell',k'}$  one has

$$a_{\ell',k'}^- \leq S_{\ell',k'} + w_{\ell',k'} + u_{\ell',k'} \leq a_{\ell',k'}^+,$$

as if  $w_{\ell',k'} < 0$  then  $S_{\ell',k'} > a_{\ell',k'}^+$  and  $u_{\ell',k'} = 0$ , so the middle term equals  $a_{\ell',k'}^+$ . Symmetrically it equals  $a_{\ell',k'}^-$  if  $u_{\ell',k'} > 0$  and otherwise  $w_{\ell',k'} = u_{\ell',k'} = 0$  with  $S_{\ell',k'} \in [a_{\ell',k'}^-, a_{\ell',k'}^+]$ . By (8), the middle term thus lies in  $[\sum_{j \in \mathcal{P}_{\ell',k'}} a_j^-, \sum_{j \in \mathcal{P}_{\ell',k'}} a_j^+]$ . Since the level- $(\ell - 1)$  sets contained in  $\mathcal{P}_{\ell,k}$  partition it, grouping  $\mathcal{D}_{\ell,k}$  by these sets gives

$$S_{\ell,k} = \sum_{k': \mathcal{P}_{\ell-1,k'} \subset \mathcal{P}_{\ell,k}} (S_{\ell-1,k'} + w_{\ell-1,k'} + u_{\ell-1,k'}),$$

and each summand lies in the corresponding child interval. Summing over  $k'$  yields (45).

Combining (8) with (45) then gives, for every  $\ell \in [L]$ ,  $k \in [K_\ell]$ ,

$$-2 \sum_{j \in \mathcal{P}_{\ell,k}} \eta_j \leq a_{\ell,k}^- - S_{\ell,k} \leq a_{\ell,k}^+ - S_{\ell,k} \leq 2 \sum_{j \in \mathcal{P}_{\ell,k}} \eta_j,$$

so a valid choice is  $M_{\ell,k}^1 = M_{\ell,k}^2 = 2 \sum_{j \in \mathcal{P}_{\ell,k}} \eta_j$ .

### C.7. Proofs for General Laminar Families

We show that the results proven in the previous sections hold for general laminar families, and not only for the case in which there is a partition of  $[n_1]$  at each level.

In Lemma 1, given a 1-level laminar family  $\{\mathcal{P}_{1,1}, \dots, \mathcal{P}_{1,K_1}\}$ , we define

$$\mathcal{C}_1 = \cup_{k \in [K_1]} \mathcal{P}_{1,k}, \quad \bar{\mathcal{C}}_1 = [n_1] \setminus \mathcal{C}_1.$$

Then, the uncertainty set  $\mathcal{U}_1$  decomposes across the sets  $\mathcal{C}_1$  and  $\bar{\mathcal{C}}_1$ . From the version of Lemma 1 which holds for partitions of  $[n_1]$ , we get

$$\max_{\mathbf{a} \in \mathcal{U}_1} \sum_{j \in \mathcal{C}_1} a_j x_j = \sum_{j \in \mathcal{C}_1} a_j^+ x_j + \sum_{k \in [K_1]} w_{1,k}.$$

Since for  $j \in \bar{\mathcal{C}}_1$ ,  $\mathcal{U}_1$  only imposes individual bounds on the corresponding  $a_j$ , it holds

$$\max_{\mathbf{a} \in \mathcal{U}_1} \sum_{j \in \bar{\mathcal{C}}_1} a_j x_j = \sum_{j \in \bar{\mathcal{C}}_1} a_j^+ x_j.$$

Hence, we get

$$\begin{aligned} \max_{\mathbf{a} \in \mathcal{U}_1} \mathbf{a}^\top \mathbf{x} &= \max_{\mathbf{a} \in \mathcal{U}_1} \sum_{j \in \mathcal{C}_1} a_j x_j + \max_{\mathbf{a} \in \mathcal{U}_1} \sum_{j \in \bar{\mathcal{C}}_1} a_j x_j \\ &= \sum_{j \in \mathcal{C}_1} a_j^+ x_j + \sum_{k \in [K_1]} w_{1,k} + \sum_{j \in \bar{\mathcal{C}}_1} a_j^+ x_j \\ &= \sum_{j \in [n_1]} a_j^+ x_j + \sum_{k \in [K_1]} w_{1,k}, \end{aligned}$$

successfully generalizing Lemma 1 to 1-level laminar families that are not necessarily partitions of  $[n_1]$ .

This directly implies that Corollary 3 also holds for general 1-level laminar families, since its proof solely relies on the result from Lemma 1.

In Lemma 2, if  $\{\mathcal{P}_{1,1}, \dots, \mathcal{P}_{1,K_1}\}$  is not a partition, to define  $\mathbf{a}^*$  we must consider also the components corresponding to  $j \in \bar{\mathcal{C}}_1$ . By setting  $a_j^* = a_j^+$ , if  $j \in \bar{\mathcal{C}}_1 \cap J_1$  and  $a_j^* = a_j^-$ , if  $j \in \bar{\mathcal{C}}_1 \cap J_0$ , we get the result.

Before we generalize the remaining results, note that when we consider a general laminar family condition (33) does not necessarily hold. Instead, for  $L \geq 2$ ,  $\ell \in [L-1]$  and  $k \in [K_{\ell+1}]$ , after tightening any redundant bounds we assume that

$$\min_{\mathbf{a} \in \mathcal{U}_\ell} \sum_{j \in \mathcal{P}_{\ell+1,k}} a_j \leq a_{\ell+1,k}^- \leq a_{\ell+1,k}^+ \leq \max_{\mathbf{a} \in \mathcal{U}_\ell} \sum_{j \in \mathcal{P}_{\ell+1,k}} a_j. \quad (46)$$

Note that, to determine the quantities at the two extremes of (46) we need not solve an optimization problem, since they only depend on the structure of the laminar family.

In Lemma 3, the proof follows as in the main text, until we consider the case in which  $\mathbf{a}^*$  violates the second-level lower bound. Then, we define  $\tilde{\mathbf{a}} \in \arg \max_{\mathbf{a} \in \mathcal{U}_1} \mathbf{a}^\top \mathbf{x}$ , such that  $\sum_{j \in [n_1]} \tilde{a}_j(1-x_j)$  is also maximized. Then, it holds  $\sum_{j \in [n_1]} \tilde{a}_j = \max_{\mathbf{a} \in \mathcal{U}_1} \sum_{j \in [n_1]} a_j$ . Therefore, we have that

$$\sum_{j \in [n_1]} a_j^* < a_{2,1}^- \leq a_{2,1}^+ \leq \max_{\mathbf{a} \in \mathcal{U}_1} \sum_{j \in [n_1]} a_j = \sum_{j \in [n_1]} \tilde{a}_j,$$

where the second inequality follows from (46) for  $\ell = 1$  and  $k = 1$ , since in this case  $\mathcal{P}_{2,1} = [n_1]$ . Then, since  $\mathcal{U}_1$  is convex, by the Intermediate Value Theorem (IVT) there exists  $\hat{\mathbf{a}} \in \arg \max_{\mathcal{U}_1} \mathbf{a}^\top \mathbf{x}$  such that  $\sum_{j \in [n_1]} \hat{a}_j = a_{2,1}^+$ , i.e.,  $\hat{\mathbf{a}} \in \mathcal{U}_2$ . The rest of the proof continues as in the main text.

The proof of Lemma 4 relies on Lemma 1, Lemma 3 and Corollary 3. Hence, having proven that these hold for general laminar families, Lemma 4 does as well.

In Lemma 5, if  $\{\mathcal{P}_{2,1}, \dots, \mathcal{P}_{2,K_2}\}$  is not a partition of  $[n_1]$ , then we define

$$\mathcal{C}_2 = \cup_{k \in [K_2]} \mathcal{P}_{2,k}, \quad \bar{\mathcal{C}}_2 = [n_1] \setminus \mathcal{C}_2.$$

Then, we get

$$\max_{\mathbf{a} \in \mathcal{U}_2} \mathbf{a}^\top \mathbf{x} = \max_{\mathbf{a} \in \mathcal{U}_2} \sum_{j \in \mathcal{C}_2} a_j x_j + \max_{\mathbf{a} \in \mathcal{U}_2} \sum_{j \in \bar{\mathcal{C}}_2} a_j x_j,$$

and we apply Lemma 5 to compute  $\max_{\mathbf{a} \in \mathcal{U}_2} \sum_{j \in \mathcal{C}_2} a_j x_j$ , and the generalization of Lemma 1 to compute the other term, since the second-level bounds of  $\mathcal{U}_2$  do not involve the coefficients corresponding to  $j \in \bar{\mathcal{C}}_2$ , hence

$$\max_{\mathbf{a} \in \mathcal{U}_2} \sum_{j \in \bar{\mathcal{C}}_2} a_j x_j = \max_{\mathbf{a} \in \mathcal{U}_1} \sum_{j \in \bar{\mathcal{C}}_2} a_j x_j,$$

where in this case  $\mathcal{U}_1$  is the one-level laminar signed budget uncertainty set obtained ignoring the second level bounds of  $\mathcal{U}_2$ .

Corollary 4 follows straightforwardly, since it only depends on the induction hypothesis, i.e., (38), and on a sign change.

To generalize Lemma 6, we proceed as in the generalization of Lemma 2. We define

$$\mathcal{C}_{L'} = \cup_{k \in [K_{L'}]} \mathcal{P}_{L',k}, \quad \bar{\mathcal{C}}_{L'} = [n_1] \setminus \mathcal{C}_{L'},$$

and in the definition of  $\mathbf{a}^*$  we must consider also the components corresponding to  $j \in \bar{\mathcal{C}}_{L'}$ . We set  $a_j^* = \tilde{a}_j$ , for each  $j \in \bar{\mathcal{C}}_{L'}$ , then, by suitably replacing (33) with (46), the proof of cases 2 and 3 still holds and we can verify that the result holds as well.

Lemma 7 holds for general laminar families since its proof only depends on the previous results.

In the proof of Theorem 2, the case in which  $K_L > 1$  and  $\{\mathcal{P}_{L,1}, \dots, \mathcal{P}_{L,K_L}\}$  is not a partition of  $[n_1]$  is generalized following the same reasoning as in the generalization of Lemma 1 at the beginning of this section.

### C.8. Proof of Corollary 2

Since Corollary 2 holds for general laminar families, in the following proof, when we refer to results from Appendix C, we intend their extension to general laminar families, which we addressed in Appendix C.7.

*Proof* We proceed by induction on the level  $L \geq 1$ .

For  $L = 1$ , if  $K_1 = 1$  the result holds by Corollary 1. If  $K_1 > 1$ , from Lemma 1 we know that to compute  $\max_{\mathbf{a} \in \mathcal{U}_1} \mathbf{a}^\top \mathbf{x}$  we have  $K_1$  terms  $w_{1,k}$  that can only take two values each. Hence, there are a total of  $2^{K_1}$  possible combinations, since in this case  $K = K_1$ . In particular, when  $\mathbf{w}_1 \in \mathbb{R}^{K_1}$  is fixed, we obtain a nominal constraint of the form (32) where  $\hat{\mathbf{b}} = \mathbf{b}$ , and for  $k \in [K_1]$ , for each  $j \in \mathcal{P}_{1,k}$  we set

$$\hat{a}_j = \begin{cases} a_j^+, & \text{if } w_{1,k} = 0, \\ a_j^-, & \text{otherwise,} \end{cases}$$

and

$$\hat{\beta} = \beta - \sum_{\substack{k \in [K_1]: \\ w_{1,k} \neq 0}} \left( a_{1,k}^+ + \sum_{j \in \mathcal{P}_{1,k}} a_j^- \right).$$

Let  $L > 1$ , and assume that the result holds for  $L - 1$ . We prove that it holds for  $L$ .

First, we consider the case  $K_L = 1$ . From the proof of Theorem 2, we know that in this case  $\max_{\mathbf{a} \in \mathcal{U}_L} \mathbf{a}^\top \mathbf{x}$  can only take two values. In the first case, when the upper bound  $a_{L,1}^+$  is not binding,

(42) holds which by induction hypothesis, can be solved by solving  $2^{K'}$  nominal problems, where  $K' = \sum_{\ell \in [L-1]} K_\ell$ . In the second case, when the upper bound is active, then (43) holds instead, and, as a consequence of the induction hypotheses and of the proof of Corollary 4, the above minimization problem can also be solved via  $2^{K'}$  nominal problems.

Therefore, solving all  $2^{K'} + 2^{K'} = 2^{K'+1}$  nominal problems and taking the best solution is equivalent to solving the original robust problem. From  $K_L = 1$ , we get that  $K' + 1 = K$ , which proves the first part of the result.

The case  $K_L > 1$  can be obtained from  $K_L = 1$  following the same reasoning as in the proof for the case  $K_1 > 1$ .  $\square$

## Appendix D: Hierarchical Forecasting Example

A difficulty of the MinT solution lies in estimating the covariance matrix  $\mathbf{W}_h$ , especially for  $h > 1$ . Many alternative approximations have been proposed, here we only report two (Hyndman and Athanasopoulos 2021):

1.  $\mathbf{W}_h = k_h \text{diag}(\hat{\mathbf{W}}_1)$ , where  $\hat{\mathbf{W}}_1 = \frac{1}{T} \sum_{t=1}^T \boldsymbol{\epsilon}_t \boldsymbol{\epsilon}_t^\top$  in which  $\boldsymbol{\epsilon}_t \in \mathbb{R}^N$  is a vector of residuals of the models that generated the base forecasts (stacked in the same order as the data). This is referred to as the *weighted least squares* (WLS) estimator using variance scaling;
2.  $\mathbf{W}_h = k_h \boldsymbol{\Lambda}$ , where  $\boldsymbol{\Lambda} = \text{diag}(\mathbf{S}\mathbf{1})$ . This is referred to as WLS with *structural scaling*.

The scalar multiple  $k_h$  cancels out when computing  $\mathbf{G}$  so it does not need to be estimated. The second estimator only depends on the structure of the aggregations and not on the actual data, therefore, it is useful for example when the residuals are not available.

EXAMPLE 2. We consider three bottom-level series corresponding to the demand at three different locations, and one aggregate series representing the total demand across all locations. The summing matrix for this setting is

$$\mathbf{S} = \begin{bmatrix} 1 & 1 & 1 \\ \mathbf{I}_3 \end{bmatrix}.$$

The true value of the bottom-level series at the forecast horizon  $h$  is such that  $\mathbf{u}_{T+h} = [10, 10, 10]^\top$ , i.e., the demand is equal to 10 at all three locations, hence

$$\mathbf{z}_{T+h} = \mathbf{S}\mathbf{u}_{T+h} = [30, 10, 10, 10]^\top.$$

Using a regression model, we can obtain independent forecasts for all the bottom level series and the aggregate series. Suppose that the results obtained applying linear regression can be expressed as the following base forecasts vector:

$$\hat{\mathbf{z}}_h = [28, 9, 11, 10]^\top.$$

Clearly, these base forecasts are incoherent, since the sum of the forecasts of the bottom level series is different from the forecast of the aggregate series,  $9 + 11 + 10 = 30 \neq 28$ . Note that, if we compute the mean squared error, we obtain

$$\|z_{T+h} - \hat{z}_h\|^2 = 2^2 + 1 + 1 + 0 = 6.$$

We show how to apply the hierarchical forecasting techniques introduced above to obtain coherent forecasts with a better accuracy.

For example, we choose to use structural scaling. Then, in this case, following the definition we can write

$$\mathbf{W}_h = \text{diag}(\mathbf{S}\mathbf{1}) = \text{diag}(3, 1, 1, 1).$$

Note that, in this approximation of the covariance matrix  $\mathbf{W}_h$  the weight associated to each series is proportional to the number of bottom-level series that contribute to it. In particular, the aggregate series receives weight 3, while each bottom level series weights exactly 1. The MinT reconciliation for the total aggregation is

$$\tilde{z}_h = \mathbf{S}(\mathbf{S}^\top \mathbf{W}_h^{-1} \mathbf{S})^{-1} \mathbf{S}^\top \mathbf{W}_h^{-1} \hat{z}_h. \quad (47)$$

Computing explicitly:

$$\mathbf{S}^\top \mathbf{W}_h^{-1} \mathbf{S} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4/3 & 1/3 & 1/3 \\ 1/3 & 4/3 & 1/3 \\ 1/3 & 1/3 & 4/3 \end{bmatrix},$$

and its inverse

$$(\mathbf{S}^\top \mathbf{W}_h^{-1} \mathbf{S})^{-1} = \frac{1}{6} \begin{bmatrix} 5 & -1 & -1 \\ -1 & 5 & -1 \\ -1 & -1 & 5 \end{bmatrix},$$

we get that the reconciled forecasts are

$$\tilde{z}_h = \mathbf{S}(\mathbf{S}^\top \mathbf{W}_h^{-1} \mathbf{S})^{-1} \mathbf{S}^\top \mathbf{W}_h^{-1} \hat{z}_h = \frac{1}{6} \begin{bmatrix} 3 & 3 & 3 & 3 \\ 1 & 5 & -1 & -1 \\ 1 & -1 & 5 & -1 \\ 1 & -1 & -1 & 5 \end{bmatrix} \begin{bmatrix} 28 \\ 9 \\ 11 \\ 10 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 87 \\ 26 \\ 32 \\ 29 \end{bmatrix} \approx \begin{bmatrix} 29 \\ 8.7 \\ 10.7 \\ 9.7 \end{bmatrix}.$$

These forecasts are now coherent, since we can verify that the total aggregation exactly matches the sum of the bottom-level series ( $\frac{87}{3} = \frac{1}{3}(26 + 32 + 29)$ ). Furthermore, using structural scaling not only enforces hierarchical consistency but it also improves forecast accuracy. In fact, the current mean squared error is

$$\|z_{T+h} - \tilde{z}_h\|^2 = 1 + \frac{16}{9} + \frac{4}{9} + \frac{1}{9} \approx 3.33.$$

For a given approximation of  $\mathbf{W}_h$ , equation (47) provides coherent point forecasts for the hierarchical time series, however, we are interested in computing prediction intervals for the forecast values. Several approaches exist to generate reconciled probabilistic forecasts (Hyndman and Athanasopoulos 2021, Panagiotelis et al. 2023). Here, we only report the following two methods:

1. If the base forecasts are normally distributed, i.e.,  $\hat{\mathbf{z}}_h \sim N(\hat{\boldsymbol{\mu}}_h, \hat{\boldsymbol{\Sigma}}_h)$ , then the reconciled forecasts are also normally distributed and

$$\tilde{\mathbf{z}}_h \sim N(\mathbf{S}\mathbf{G}\hat{\boldsymbol{\mu}}_h, \mathbf{S}\mathbf{G}\hat{\boldsymbol{\Sigma}}_h\mathbf{G}^\top\mathbf{S}^\top).$$

Hence, each component  $\tilde{z}_{h,i}$ ,  $i \in [N]$ , of  $\tilde{\mathbf{z}}_h$  has a normal distribution with mean  $(\mathbf{S}\mathbf{G}\hat{\boldsymbol{\mu}}_h)_i$  and variance  $(\mathbf{S}\mathbf{G}\hat{\boldsymbol{\Sigma}}_h\mathbf{G}^\top\mathbf{S}^\top)_{ii}$ . To generate intervals we can use the same reasoning as explained in Section 4.1.

2. If we cannot assume normality of the base forecasts, intervals can be derived by using bootstrapping. First,  $B$  independent simulated sample paths  $(\hat{\mathbf{z}}_h^1, \dots, \hat{\mathbf{z}}_h^B)$  are generated from the models used to produce the base forecasts. Then, by premultiplying them by  $\mathbf{S}\mathbf{G}$ , we obtain the set  $(\mathbf{S}\mathbf{G}\hat{\mathbf{z}}_h^1, \dots, \mathbf{S}\mathbf{G}\hat{\mathbf{z}}_h^B)$ , from which percentiles can be calculated, in order to construct prediction intervals (see Hyndman and Athanasopoulos (2021) for additional details).

## Appendix E: Numerical Experiment Problem Formulations

### E.1. Maximum Coverage Problem

We consider a set of  $[n_1]$  population points, each of which is also a candidate facility location. For each  $i \in [n_1]$ , the binary variable  $x_i$  indicates whether a facility is opened at location  $i$ . Similarly, for each  $j \in [n_1]$ , the binary variable  $z_j$  indicates whether population point  $j$  is *covered* by at least one opened facility. At most  $p$  facilities can be opened, and  $r$  denotes the coverage radius of each facility. Let  $d_{i,j}$  denote the Euclidean distance between candidate facility  $i$  and population point  $j$ . The objective is to maximize the total covered population, where  $\mathbf{a} \in \mathbb{R}^{n_1}$  denotes the population weight associated with each point. The resulting maximum coverage problem is formulated as follows:

$$\begin{aligned} \max \quad & \mathbf{a}^\top \mathbf{z} \\ \text{s.t.} \quad & \sum_{i \in [n_1]} x_i \leq p, \\ & \sum_{\substack{i \in [n_1]: \\ d_{i,j} \leq r}} x_i \geq z_j, \quad j \in [n_1], \\ & x_i \in \{0, 1\}, \quad i \in [n_1], \\ & z_j \in \{0, 1\}, \quad j \in [n_1]. \end{aligned}$$

### E.2. Knapsack Problem

We consider a set of  $n_1$  items. For each item  $j \in [n_1]$ , let  $c_j$  denote its value and  $a_j$  its weight. The binary variable  $x_j$  indicates whether item  $j$  is selected. The total weight of the selected items cannot

exceed the knapsack capacity  $\beta \in \mathbb{R}$ . The objective is to maximize the value of the selected items subject to this capacity constraint. The problem is formulated as follows.

$$\begin{aligned} \max \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & \mathbf{a}^\top \mathbf{x} \leq \beta, \\ & \mathbf{x} \in \{0, 1\}^{n_1}. \end{aligned}$$

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