Scenario Tradeoffs in Uncertain Multiobjective Optimization*

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Abstract

Realistic decision problems are inherently multiobjective and uncertain. To manage both of these complexities, robust multiobjective optimization strives to aid the decision maker in finding a decision which is Pareto efficient and is hedged against the worst-case scenario. In this paper, we present a robustness approach which is grounded in the decision maker's preferences by allowing the decision maker to define equivalence classes on the set of scenarios. The upshot of this formulation is the ability to measure the tradeoffs between equivalence classes of scenarios, which allows the decision maker to use multicriteria decision analytic tools on an uncertain multiobjective problem in ways previously not possible. After presenting theoretical results, we demonstrate our method on an example inspired by disaster relief.

Keywords: Optimization under uncertainty, robust optimization, multiobjective optimization, multicriteria decision making/analysis

1 Introduction

Uncertainty abounds in real-world decision problems, which manifests in a myriad of ways, such as uncertainty in the input data or outcome of a selected decision. Besides this, there are conflicting objectives that need to be balanced in the selection of a final decision. Due to these two features of real decision problems, much attention has been devoted to uniting the techniques of optimization under uncertainty, multiobjective optimization, and multicriteria decision making/analysis (MCDM/A).

In this work, we focus on robust multiobjective optimization. The robust optimization approach formulates the inherent uncertainty as a set of "scenarios" and the desired decision is one which minimizes the worst-case scenario. The literature on robust optimization, whether single-objective or multiobjective, is vast. Furthermore, there is a multiplicity of definitions for robustness in the multiobjective case. Here, we present a brief survey of various approaches, but especially focus on works in the multiobjective literature.

Robust optimization was first introduced by Soyster 1973,later developed by Ben-Tal and Nemirovski 1998, and has been extensively studied and used in practice. However, because it hedges against the worst case, it is well known that robust optimal solutions are often too conservative (Bertsimas and Sim 2004). To deal with this, many different approaches have been taken such as probabilistic interpretations of robustness (Xu, Caramanis, and Mannor 2012), bounding uncertainty (Bertsimas and Sim 2004; Poss 2014), using samples from the uncertainty set to build constraints (Ramponi 2018), encoding uncertainty in a single constraint (Roos and den Hertog 2020), and defining cuts in the uncertainty set (Bansal and Zhang 2021). For a survey on the theory and applications of robust optimization, we suggest Gorissen, Yanıkoğlu, and den Hertog 2015.

Due to the utility of robust optimization in the single-objective setting, extensions to multiobjective optimization/multicriteria decision-making have grown in the literature. Early in this literature, Hites et al. 2006 discuss possibilities for applying well-established tools in multicriteria decision analysis (MCDA) to robust decision problems. However, one of the difficulties in extending the robust optimization paradigm to the multiobjective setting is that there are many different interpretations of "robust". An early definition

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is the so-called "component-wise" robust efficiency (Kuroiwa and Lee 2014). This definition defines the "worst-case" as being a component-wise supremum of the vector-valued objective function with respect to the uncertainty set before then minimizing with respect to the decision variables. However, there are many other definitions, such as flimsily robust efficiency, highly robust efficiency, robust efficient paths, etc. Wiecek and Dranichak 2016 provides a good survey of these alternative robustness concepts. Some robustness concepts do not depend on the model uncertainty, but rather uncertainty with respect to a decision maker's preferences (Mavrotas et al. 2015). Closely related to this is the work by Engau 2017, where proper efficiency and criteria tradeoffs are investigated for multiobjective stochastic optimization. Conditions for guaranteeing proper efficiency, that is, efficient solutions with bounded tradeoffs, are shown for countably and uncountably many criteria, respectively.

One of the main definitions of robustness in the multiobjective context uses a set-valued function approach, where the robust counterpart of the uncertain MOP is expressed as a set-valued optimization problem, which essentially generalizes the "min-max" concept of robustness from single-objective optimization. (For an introduction to the theory of set-valued optimization, see Khan, Tammer, and Zălinescu 2015.) This approach has been discussed by Jahn and Ha 2011; Ide and Köbis 2014; Ide, Köbis, et al. 2014; Ide and Schöbel 2016; Eichfelder, Niebling, and Rocktäschel 2020. In particular, Ehrgott, Ide, and Schöbel 2014 shows how to extend the classical scalarization techniques of the weighted-sum and ε-constraint methods to the robust setting, while Bokrantz and Fredriksson 2017 provide necessary and sufficient conditions for the existence of such robust efficient solutions. Wang, Liu, and Chai 2015 also use the set-based notion of robustness but reformulate the problem as a bilevel optimization problem with a set-valued optimization problem in the upper level and a vector optimization problem in the lower level.

The set-based approach, although a useful extension of "min-max" robustness to the multiobjective setting, introduces its own difficulties, especially computationally since optimization ultimately has to occur with set-valued functions, rather than scalar- or vector-valued functions. To address this difficulty, the technique of vectorization has received much attention (Jahn 2015; Eichfelder, Quintana, and Rocktäschel 2022; Eichfelder and Rocktäschel 2023; Eichfelder and Quintana 2024; Eichfelder and Gerlach 2026).

Besides robustness, other techniques for handling uncertainty include fuzzy sets (Ammar 2009; Jana, Sharma, and Mehta 2022), rough sets (Greco, Matarazzo, and Slowinski 2001; Renaud et al. 2007; Atteya 2016), and stochastic multiobjective optimization (Gupta and Hunter 2025).

As can be seen from the preceding discussion, there is a multitude of robustness concepts for multiobjective optimization. To remedy this, there are several works in the literature which provide frameworks for unifying these concepts (Ide and Köbis 2014; Ide, Köbis, et al. 2014; Klamroth et al. 2017; Wei, Chen, and Li 2018).

However, in this work, we follow the perspective developed by Iancu and Trichakis 2014; Botte and Schöbel 2019; Engau and Sigler 2020. First, Iancu and Trichakis 2014 introduce the notion of Pareto robustly optimal (PRO) for single-objective robust optimization. The concept is a reinterpretation of the concept of Pareto efficiency: a solution is considered PRO if it is robust optimal *and* there is no other feasible decision which does at least as well in all possible scenarios, with at least one scenario doing strictly better. This definition has a natural intuition behind it which is readily seen if there are finitely many scenarios in the uncertainty set: instantiate the (scalar-valued) function at each scenario to construct a *vector*-valued function, thereby transforming the original uncertain scalar optimization problem into a *deterministic multiobjective* optimization problem. One result of this transformation is that the original uncertain single-objective optimization problem is now subject to all of the tools of MCDM/A in selecting a robust solution.

Botte and Schöbel 2019 pick up this intuition and extend it in two ways. First, they instantiate a *vector*-valued objective function at each scenario to construct an (admittedly higher-dimensional) new, also deterministic, multiobjective problem. Second, they show how to do this even when the uncertainty set is infinite. Both of these extensions require a generalized version of PRO, which they call "multi-scenario efficiency."

Following this, Engau and Sigler 2020 continue the project of generalizing the idea of a decision which is Pareto efficient with respect to the scenarios by introducing six ordering relations between decisions that encapsulate all the possible relationships between decisions with respect to the possible scenarios. After introducing these ordering relations, they proceed to show generalizations of the weighted-sum method and ε-constraint, which provide such robust Pareto efficient solutions.

However, the drawback of the approaches of Botte and Schöbel 2019 and Engau and Sigler 2020 is that when the uncertainty set is infinite, both methods must work in infinite-dimensional spaces. On one hand, Botte and Schöbel 2019 requires optimization over a space of functionals, which introduces computational complexities due to the infinite-dimensional space. On the other, the weighted-sum method of Engau and Sigler 2020 requires the defining of infinitely many weights, which will generally be meaningless in a decision-making context as the sheer number of weights required to be *a priori* defined cannot represent the decision maker's marginal utility of improvements for each objective. Furthermore, the ε-constraint method also developed in Engau and Sigler 2020 results in a semi-infinite programming problem. Although there are numerous suitable methods in the literature for optimizing semi-infinite programs, this approach again suffers in a decision-making context—how can a decision maker reasonably be expected to provide infinitely many upper bounds on infinitely many objectives?

In this paper, we aim to improve both of these drawbacks by leaning on the original insight that the above works expounded: balancing tradeoffs between *finitely many* scenarios. It turns out that this insight is possible to apply even in the infinite scenario case provided we utilize a key detail in our modeling. Namely, the *decision maker's preferences, experience, and expertise*. The intuition behind our approach is along the following lines. Whatever the decision problem may be, it is likely that this is not the first such decision a DM has faced. She will have seen other cases like it, will have access to case studies or professional standards of practice, and even an intuitive sense of the severity of the scenarios facing her. As such, it is very possible that the DM may consider some scenarios as being essentially *equivalent*. When this is the case, we ask the DM to provide an equivalence relation *on the uncertainty set*, which will effectively serve to "collapse" many scenarios into one. Furthermore, given cognitive limitations, it is very realistic to assume that the DM will provide *finitely* many equivalence classes. Thus, even when there are infinitely many scenarios, the DM's preferences, experiences, and expertise can actually turn the decision problem into one with effectively *finitely* many scenarios.

An early approach that is similar to our paper here is Rockafellar and Wets 1991. In this work, they propose that a *finite* scenario set is partitioned into finitely many disjoint sets, and weights are assigned to these scenarios which may be loosely interpreted as probabilities, even if they are subjective probabilities. The two main differences between Rockafellar and Wets 1991 and this work are that we show how to manage an uncountable set of scenarios and we focus on the multiobjective case.

This work, then, provides the necessary theoretical foundation to transform uncertain multiobjective optimization problems into a single deterministic multiobjective optimization problem which respects the original formulation *and* the DM's own understanding of the problem. This allows us to apply the subproblem tradeoffs from de Castro and Wiecek 2025, which may be readily re-interpreted as providing "scenario tradeoffs". Our approach is distinct from those like Engau 2017 because we measure tradeoffs between (equivalence classes of) scenarios rather than between objectives. This is an important distinction because optimization under uncertainty is greatly affected by the specific scenario (or class of scenarios). A method for measuring the tradeoffs between the scenarios gives the decision maker new insights into how a proposed solution may perform in practice.

The rest of this paper is organized as follows. Section 2 introduces the notation, definitions, and general properties utilized in this work, while Section 3 describes the application of an equivalence relation on the uncertainty set. Section 4 discusses the reformulation of an uncertain MOP using an equivalence relation, which provides the basis for measuring scenario tradeoffs in Section 5. Finally, Section 6 demonstrates the developed theory on an example inspired by disaster relief followed by a conclusion, including a discussion on future research, in Section 7.

2 Preliminaries

We begin by introducing the requisite notation and definitions used in this paper. For a natural number n, we denote the set $\mathbb{N}_n = \{1, \ldots, n\}$. Additionally, we use the following (subsets of) cones to construct the orderings relevant to this work.

$$\mathbb{R}^n_> = \{y \in \mathbb{R}^n \mid y_i > 0,\ \forall i \in \mathbb{N}_n\}$$
$$\mathbb{R}^n_\geq = \{y \in \mathbb{R}^n \setminus \{0\} \mid y_i \geq 0,\ \forall i \in \mathbb{N}_n\}$$
$$\mathbb{R}^n_\geqq = \{y \in \mathbb{R}^n \mid y_i \geq 0,\ \forall i \in \mathbb{N}_n\}$$

Another useful definition for this work is the so-called upper set-less order relation.

Definition 2.1 (Definition 2 in Ide, Köbis, et al. 2014)**.** Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n$ be nonempty sets. We write $\mathcal{A} \leqq^u_{\mathbb{R}^n_\geqq} \mathcal{B}$ whenever

$$\mathcal{A} \subseteq \mathcal{B} - \mathbb{R}^n_\geqq$$

When $\mathcal{A} \leqq^u_{\mathbb{R}^n_\geqq} \mathcal{B}$ we say that $\mathcal{A}$ **dominates** $\mathcal{B}$.

2.1 Deterministic Multicriteria Optimization

We let $\mathcal{X} \subseteq \mathbb{R}^m$ be a nonempty compact set and let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a vector-valued continuous function. A generic *deterministic* multicriteria optimization problem (MOP) is of the following form.

$$\begin{aligned} \min_x \quad & f(x) \\ \text{s.t.} \quad & x \in \mathcal{X} \end{aligned} \tag{MOP}$$

In deterministic multicriteria optimization, optimality is defined using the following ordering.

Definition 2.2. Let $u, v \in \mathbb{R}^n$. We write

- $u < v$ if and only if $u \in v - \mathbb{R}^n_> \iff u_i < v_i\ \forall i \in \mathbb{N}_n$;
- $u \leq v$ if and only if $u \in v - \mathbb{R}^n_\geq \iff u_i \leq v_i\ \forall i \in \mathbb{N}_n$ and $u \neq v$;
- $u \leqq v$ if and only if $u \in v - \mathbb{R}^n_\geqq \iff u_i \leq v_i\ \forall i \in \mathbb{N}_n$

Definition 2.3. Let $x^* \in \mathcal{X}$. We say that x^* is

- a **weakly efficient solution** for (MOP) if there is no $x \in \mathcal{X}$ such that $f(x) < f(x^*)$;
- an **efficient solution** for (MOP) if there is no $x \in \mathcal{X}$ such that $f(x) \leq f(x^*)$;

This definition may be relaxed in the natural way.

Definition 2.4. Let $x^* \in \mathcal{X}$ and $\varepsilon \in \mathbb{R}^n_\geqq$. We say that x^* is

- a **ε-weakly efficient solution** for (MOP) if there is no $x \in \mathcal{X}$ such that $f(x) < f(x^*) - \varepsilon$;
- an **ε-efficient solution** for (MOP) if there is no $x \in \mathcal{X}$ such that $f(x) \leq f(x^*) - \varepsilon$.

The standard approach for finding (weakly) efficient solutions for (MOP) is through the technique of *scalarization*; see, *e.g.*, Ehrgott 2005 for details on classical scalarizations. The specific scalarization relevant for this work is that of achievement scalarizing functions (ASFs), first introduced by Wierzbicki 1977, and their extension to bivariate achievement scalarizing functions (BASFs). More details on (B)ASFs may be found in Wierzbicki 1986; Kaisa Miettinen 1998; de Castro 2026.

Definition 2.5. Let $\sigma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function and let $r \in \mathbb{R}^n$. We say that σ is

- **strictly order preserving** if $y^1 < y^2 \implies \sigma(y^1, r) < \sigma(y^2, r)$ for any $y^1, y^2 \in \mathbb{R}^n$;
- **order preserving** if $y^1 \leqq y^2 \implies \sigma(y^1, r) \leq \sigma(y^2, r)$ for any $y^1, y^2 \in \mathbb{R}^n$;
- **strongly order preserving** if $y^1 \leq y^2 \implies \sigma(y^1, r) < \sigma(y^2, r)$ for any $y^1, y^2 \in \mathbb{R}^n$;

- $\mathcal{C}$**-order representing**, for a solid closed pointed convex cone $\mathcal{C} \subseteq \mathbb{R}^n$, if
$$r - \mathcal{C} = \{y \in \mathbb{R}^n \mid \sigma(y, r) \leq 0\}.$$

If σ is (strictly) order preserving and $\mathcal{C}$-order representing, then σ is an **(bivariate) achievement scalarizing function**.

We may use (B)ASFs to scalarize (MOP).

$$\begin{aligned} \min_{x,r} \quad & \sigma(f(x), r) \\ \text{s.t.} \quad & x \in \mathcal{X} \\ & r \in \mathcal{R} \end{aligned} \tag{σ-MOP}$$

where $\mathcal{R}$ is a nonempty set of reference points. The main properties of (B)ASFs needed in this work are the following.

Theorem 2.1 (Proposition 1 in de Castro 2026)**.** *If $\sigma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a strictly/strongly order preserving and $\mathcal{C}$-order representing (B)ASF then*

- *$\sigma(y, r) < 0$ if and only if $y \in r - \operatorname{int}(\mathcal{C})$;*
- *$\sigma(y, r) = 0$ if and only if $y \in \partial(r - \mathcal{C})$;*
- *$\sigma(y, r) > 0$ if and only if $y \in (r - \mathcal{C})^C$,*

where $\operatorname{int}(\cdot)$, $\partial(\cdot)$, and $(\cdot)^C$ denote the interior, boundary, and complement of a set, respectively.

Theorem 2.2 (Theorem 10 in Wierzbicki 1986, Proposition 4 in de Castro 2026)**.** *Let $\sigma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a (B)ASF and let $\mathcal{R} \subseteq \mathbb{R}^n$ be a nonempty compact set such that $\mathcal{R}$ dominates the Pareto set of $f(\mathcal{X})$.*

1. *If σ is strongly order preserving and $\mathcal{C}$-order representing with $\mathbb{R}^n_{\geqq} \subset \mathcal{C} \subset \mathbb{R}^n_{\delta}$ and $x^* \in \mathcal{X}$ is a δ-properly efficient solution for* (MOP) *such that $f(x^*) \in \mathcal{R}$ then $(x^*, f(x^*))$ is an optimal solution for* (σ-MOP) *with optimal value 0.*
2. *If σ is strictly order preserving and $\mathbb{R}^n_{\geqq}$-order representing and $x^* \in \mathcal{X}$ is a weakly efficient solution for* (MOP) *such that $f(x^*) \in \mathcal{R}$ then $(x^*, f(x^*))$ is an optimal solution for* (σ-MOP)*.*

Remark 2.1. The upshot of Theorem 2.2 is that we can determine if a proposed outcome $f(x^*)$ is indeed Pareto efficient by simply using it as a reference point in a (B)ASF. For more details, including the definition of δ-proper efficiency and the set $\mathbb{R}^p_{\delta}$, see Wierzbicki 1986; de Castro 2026.

2.2 Uncertain Multicriteria Optimization

As discussed in the introduction, many real-world problems are not deterministic. Hence, these types of problems must be modeled by incorporating their inherent uncertainty. Let $\mathcal{U} \subseteq \mathbb{R}^p$ be a nonempty compact set which represents the "scenarios" which may affect the criteria. We thus assume that the objective functions depend on the decisions $x \in \mathcal{X}$ and the scenarios $\xi \in \mathcal{U}$, *i.e.*, $f : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^n$. In general, we need to make a decision *before* a scenario has occurred, which means that we in fact have a class of so-called uncertain multicriteria problems to consider.

$$\left\{ \begin{aligned} \min_{x} \quad & f(x, \xi) \\ \text{s.t.} \quad & x \in \mathcal{X} \end{aligned} \right\}_{\xi \in \mathcal{U}} \tag{UMOP}$$

We let $\mathcal{Y} = \bigcup_{\xi \in \mathcal{U}} \{f(x, \xi) \mid x \in \mathcal{X}\} = f(\mathcal{X} \times \mathcal{U})$ denote the image of f over all decisions $x \in \mathcal{X}$ and all scenarios $\xi \in \mathcal{U}$.

Remark 2.2. It is indeed possible for the feasible set to also depend on the scenario: $\mathcal{X}(\xi)$. However, there are different strategies for doing away with the uncertainty in the constraints. Similarly to Engau and Sigler 2020, we seek decisions which are feasible for all possible scenarios, *i.e.*, $\mathcal{X} = \bigcap_{\xi \in \mathcal{U}} \mathcal{X}(\xi)$. See Wiecek and Dranichak 2016 for a further discussion.

However, if a particular scenario $\xi \in \mathcal{U}$ occurs, we return to a deterministic problem.

$$\begin{aligned} \min_{x} \quad & f(x,\xi) \\ \text{s.t.} \quad & x \in \mathcal{X} \end{aligned} \tag{MOP(ξ)}$$

We let $\mathcal{Y}(\xi) = \{f(x,\xi) \mid x \in \mathcal{X}\}$ denote the image of f over $\mathcal{X}$ for a given $\xi \in \mathcal{U}$.

Optimality for (UMOP) can be understood in various ways and numerous definitions have been proposed in the literature, as discussed in the introduction. There are two particular notions of efficiency from Engau and Sigler 2020 which will be especially relevant for this work.

Definition 2.6 (Definitions 3.1 and 3.4 in Engau and Sigler 2020)**.** Let $x^* \in \mathcal{X}$. We say that x^* is a(n)

- **weakly efficient-under-uncertainty solution** if there is no $x \in \mathcal{X}$ such that $f(x,\xi) < f(x^*,\xi)$ for all $\xi \in \mathcal{U}$;
- **efficient-under-uncertainty solution** if there is no $x \in \mathcal{X}$ such that $f(x,\xi) \leq f(x^*,\xi)$ for all $\xi \in \mathcal{U}$.

We can also relax this definition in the natural way.

Definition 2.7. Let $x^* \in \mathcal{X}$ and let $\varepsilon \in \mathbb{R}^n_{\geqq}$. We say that x^* is a(n)

1. **weakly ε-efficient-under-uncertainty solution** if there is no $x \in \mathcal{X}$ such that $f(x,\xi) < f(x^*,\xi) - \varepsilon$ for all $\xi \in \mathcal{U}$;
2. **ε-efficient-under-uncertainty solution** if there is no $x \in \mathcal{X}$ such that $f(x,\xi) \leq f(x^*,\xi) - \varepsilon$ for all $\xi \in \mathcal{U}$.

Remark 2.3. These definitions are intended to truly extend the notion of Pareto efficiency to the uncertain multicriteria optimization case. The idea is that a feasible solution x^* is (weakly) efficient-under-uncertainty when it is impossible to find another feasible solution which does "better" than x^* across all possible scenarios.

3 Decomposing the Uncertainty Set

We turn our attention to the primary contribution of this work to the literature of multicriteria decision-making under uncertainty. As Hites et al. 2006 notes, the decision maker is at the center of all such multicriteria modeling and optimization. Numerous decision aid tools have been developed in order to enable the DM to participate in the optimization process. The main reason for this interaction is that the DM's expertise and preferences are essential to the selection of a decision. Traditionally, the DM uses her preferences, expertise, and experience on the *feasible decisions*. In this work, we show how the DM may use her preferences on the *scenarios*.

It is worth being clear on what we mean by a DM's preferences with respect to the scenarios in the uncertainty set $\mathcal{U}$. We do not mean that a DM "prefers" one "disaster" to another, as it were. Rather, we recognize that through experience, a DM can develop preferences in terms of *indifference* between scenarios; whether the specific scenarios ξ^1 or ξ^2 occur does not really affect the decision the DM would choose. These two scenarios are more or less equivalent to her. Mathematically speaking, this means that we may use the DM's preferences on the uncertainty set $\mathcal{U}$ to define an *equivalence relation*.

Definition 3.1. A binary relation $\sim$ on a set $\mathcal{A}$ is an **equivalence relation** if and only if it is

- reflexive: $\forall a \in \mathcal{A},\ a \sim a$;

- symmetric: $\forall a, b \in \mathcal{A},\ a \sim b \implies b \sim a$;
- transitive: $\forall a, b, c \in \mathcal{A},\ a \sim b \land b \sim c \implies a \sim c$.

Let $\sim$ be an equivalence relation on $\mathcal{A}$. For $a \in \mathcal{A}$, the set

$$[a] = \{a' \in \mathcal{A} \mid a' \sim a\}$$

is the **equivalence class** of a. We denote the set of all equivalence classes of $\mathcal{A}$ by $\mathcal{A}/\sim$. For any $a' \in [a]$, we have that $[a'] = [a]$. Any element in an equivalence class used to represent that class is called a **representative**.

There are many tools in the literature for eliciting the preferences of a DM to construct such a relation; see, *e.g.*, Roy 1996. The equivalence classes on $\mathcal{U}$ define a set partition on $\mathcal{U}$, and it is this structure that allows us to measure the tradeoffs of different decisions in different (equivalence classes of) scenarios. In order to do this, however, we must make the following assumption.

Assumption 3.1. The uncertainty set $\mathcal{U}$ with the equivalence relation $\sim$ is partitioned into *finitely* many equivalence classes. That is, $\left|\mathcal{U}/\sim\right| < \infty$.

Remark 3.1. At face value, this may seem to be a restrictive assumption. However, this assumption is actually a generalization of requiring the set of scenarios to be finite. Furthermore, realistically, DMs have finite preferences, or at least preferences which can only reasonably be expressed, explored, and used in a finite way. Indeed, although the uncertainty set is generally considered to be infinite, such as when $\mathcal{U}$ is an interval, a DM may have only a handful of *specific* scenarios in mind which stand as exemplars of the type of scenarios she faces. These exemplars, then, may form the basis for constructing an appropriate equivalence relation, which, being built on finitely many exemplars, results in finitely many equivalence classes.

The equivalence classes on $\mathcal{U}$ allow us to decompose (UMOP) from a single family of uncertain MOPs into a collection of families of uncertain MOPs. Thus, for each $[\xi^*] \in \mathcal{U}/\sim$, we have a subfamily of uncertain MOPs.

$$\left\{\begin{array}{ll} \min\limits_{x} & \begin{bmatrix} f_1(x,\xi) \\ \vdots \\ f_n(x,\xi) \end{bmatrix} \\ \text{s. t.} & x \in \mathcal{X} \end{array}\right\}_{\xi \in [\xi^*]} \tag{UMOP($[\xi^*]$)}$$

Although this may appear to be piling onto a problem which was already difficult, we actually are able to use the equivalence classes to relax the strength of the definition of efficiency-under-uncertainty. Instead of thinking of efficiency-under-uncertainty with respect to *all* possible scenarios, we may restrict our attention to efficiency-under-uncertainty with respect to an *equivalence class* of scenarios. We have the following properties relating (UMOP($[\xi^*]$)) and (UMOP).

Proposition 3.1. *Let $x^* \in \mathcal{X}$, $[\xi^*] \in \mathcal{U}/\sim$, and $\varepsilon \geqq 0$. If x^* is a(n) (weakly) ε-efficient-under-uncertainty solution for* (UMOP($[\xi^*]$)) *then it is a(n) (weakly) ε-efficient-under-uncertainty solution for* (UMOP).

Proof. We show the case when x^* is ε-efficient-under-uncertainty, as the case for weak ε-efficiency-under-uncertainty follows analogously. To that end, let $[\xi^*] \in \mathcal{U}/\sim$ and $\varepsilon \geqq 0$. Let x^* be an ε-efficient-under-uncertainty solution for (UMOP($[\xi^*]$)). Towards a contradiction, suppose x^* is not ε-efficient-under-uncertainty for (UMOP). Then there exists $\bar{x} \in \mathcal{X}$ such that

$$\begin{bmatrix} f_1(\bar{x},\xi) \\ \vdots \\ f_n(\bar{x},\xi) \end{bmatrix} \leq \begin{bmatrix} f_1(x^*,\xi) \\ \vdots \\ f_n(x^*,\xi) \end{bmatrix} - \varepsilon$$

for all $\xi \in \mathcal{U}$. In particular, this means that

$$\begin{bmatrix} f_1(\bar{x},\xi) \\ \vdots \\ f_n(\bar{x},\xi) \end{bmatrix} \leq \begin{bmatrix} f_1(x^*,\xi) \\ \vdots \\ f_n(x^*,\xi) \end{bmatrix} - \varepsilon$$

for all $\xi \in [\xi^*]$. But this contradicts the ε-efficiency-under-uncertainty of x^* for (UMOP($[\xi^*]$)). □

Corollary 3.1. *Let $x^* \in \mathcal{X}$ and $[\xi^i] \in \mathcal{U}/\sim$. If x^* is a(n) (weakly) efficient-under-uncertainty solution for* (UMOP($[\xi^*]$)) *then it is a(n) (weakly) efficient-under-uncertainty solution for* (UMOP).

Proof. Follows directly from the previous proposition by setting $\varepsilon = 0$. □

The upshot of the previous proposition is that efficient-under-uncertainty solutions for an equivalence class in $\mathcal{U}/\sim$ provide efficient-under-uncertainty solutions for (UMOP).

4 Aggregating Equivalence Classes on the Uncertainty Set

So far, we have yet to truly utilize the equivalence classes on $\mathcal{U}$ to assist the DM in working with (UMOP). We turn to this question next.

The motivation for applying an equivalence relation on $\mathcal{U}$ is to reduce the infinite family of MOPs defined by (UMOP) to a single MOP which will enable a decision maker to use MCDM/A tools to select a decision which performs within her preferences across the scenarios. Since each equivalence class, in general, also has infinitely many elements, there is a need to construct new objective functions which still measure the performance of a decision, but also respect the equivalence classes. Thus, we need to define a new function $g : \mathcal{X} \times \mathcal{U}/\sim \to \mathbb{R}^n$ that is well-defined with respect to the equivalence classes. That is, if $\xi^1 \sim \xi^2$ then $g(x,[\xi^1]) = g(x,[\xi^2])$ for all $x \in \mathcal{X}$.

Remark 4.1. This requirement arises from classical results on quotient spaces in point-set topology. When S, T are topological spaces, $\sim$ is an equivalence relation on S, and $g : S \to T$ is a continuous function such that $g(s) = g(s')$ whenever $s \sim s'$, then there exists a continuous function $h : S/\sim \to T$ such that the diagram commutes.

$$\begin{tikzcd} S \arrow[d, "\pi"] \arrow[dr, "g"] & \\ S/\sim \arrow[r, "h"] & T \end{tikzcd}$$

That is, there exists a function $h : S/\sim \to T$ such that $g = h \circ \pi$, where π is the canonical projection of elements in S to $S/\sim$. See Munkres 2000 for more details.

The approach we take here to construct such a function g is based upon the following intuition. We take an equivalence class $[\xi^*] \in \mathcal{U}/\sim$ to mean that a DM views these scenarios as essentially the same. Put another way, regardless of which specific scenario $\xi \in [\xi^*]$ occurs, the DM would evaluate the alternative decisions in the same way and would ultimately make the same decision in any case. In MCDM/A language, each of these scenarios would be equally weighted; in probabilistic language, the DM assumes a uniform probability distribution over a given equivalence class. In this sense, the DM is interested in the *aggregate* performance of a decision over all equivalent scenarios, implying that the objective function that the DM is actually interested in optimizing is the function $g : \mathcal{X} \times \mathcal{U}/\sim \to \mathbb{R}^n$ defined by

$$g(x,[\xi^*]) = (g_1(x,[\xi^*]),\ldots,g_n(x,[\xi^*])) = \begin{bmatrix} \dfrac{1}{M([\xi^*])} \displaystyle\int_{[\xi^*]} f_1(x,\xi)\ d\xi \\ \vdots \\ \dfrac{1}{M([\xi^*])} \displaystyle\int_{[\xi^*]} f_n(x,\xi)\ d\xi \end{bmatrix}$$

where $M([\xi^*])$ is the measure of $[\xi^*]$.

Remark 4.2. With appropriate modifications to the aggregation function g, the results shown in this work may also apply to finitely or countably many scenarios. However, we restrict our attention to the case of uncountably many scenarios.

We make the following assumptions for this construction to be well-defined.

Assumption 4.1. Let (UMOP) be given. The following conditions hold.

1. For each $i \in \mathbb{N}_n$,
   (a) f_i is uniformly continuous on $\mathcal{X} \times \mathcal{U}$;
   (b) f_i is measurable.
2. $\mathcal{X}$ is compact.
3. $\mathcal{U}$ has finite positive measure.
4. Every equivalence class in $\mathcal{U}/\sim$ has finite positive measure.

The function g allows us to define a **subproblem** of (UMOP).

$$\begin{aligned} \min_x \quad & g(x,[\xi^*]) \\ \text{s.t.} \quad & x \in \mathcal{X} \end{aligned} \tag{SP([ξ^*])}$$

The following proposition shows that g is continuous in x.

Lemma 4.1. *Let* (UMOP) *be given and let $\sim$ be an equivalence relation on $\mathcal{U}$, with $[\xi^*] \in \mathcal{U}/\sim$. If the conditions of Assumption 4.1 hold, then $g_i(x,[\xi^*])$ is continuous in x for every $i \in \{1,\dots,n\}$.*

Proof. Let $M(\mathcal{S})$ be the measure of a set $\mathcal{S}$ and let $i \in \mathbb{N}_n$. Suppose that f_i is uniformly continuous on $\mathcal{X}\times\mathcal{U}$. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that for all $(x,\xi),(y,\xi') \in \mathcal{X}\times\mathcal{U}$ that satisfy $\left\|\begin{bmatrix} x \\ \xi \end{bmatrix} - \begin{bmatrix} y \\ \xi' \end{bmatrix}\right\| < \delta$, we have that $|f_i(x,\xi) - f_i(y,\xi')| < \varepsilon$. In particular, observe that for all $\xi \in \mathcal{U}$,

$$|f_i(x,\xi) - f_i(y,\xi)| < \varepsilon$$

whenever $||x-y|| = \left\|\begin{bmatrix} x \\ \xi \end{bmatrix} - \begin{bmatrix} y \\ \xi \end{bmatrix}\right\| < \delta$. Hence,

$$\begin{aligned} |g_i(x,[\xi^*]) - g_i(y,[\xi^*])| &= \frac{1}{M([\xi^*])}\left|\int_{[\xi^*]} f_i(x,\xi) - f_i(y,\xi)\ d\xi\right| \\ &\le \frac{1}{M([\xi^*])}\int_{[\xi^*]} |f_i(x,\xi) - f_i(y,\xi)|\ d\xi \\ &< \frac{1}{M([\xi^*])}\int_{[\xi^*]} \varepsilon\ d\xi \\ &= \varepsilon. \end{aligned}$$

This implies that $g_i(x,[\xi^*])$ is continuous in x. ☐

Proposition 4.1. *Let* (UMOP) *be given and let $\sim$ be an equivalence relation on $\mathcal{U}$, with $[\xi^*] \in \mathcal{U}/\sim$. If the conditions of Assumption 4.1 hold, then* (SP([ξ^*])) *has an efficient solution.*

Proof. This follows directly from the fact that $\mathcal{X}$ is compact and each $g_i(x,[\xi^*]),\ i = 1,\dots,n$, is continuous in x. ☐

Observe that (SP($[\xi^*]$)) is a deterministic MOP. The real utility of Proposition 4.1 is that the equivalence relation $\sim$ actually transforms the family of decision problems as described by (UMOP) into a finite family of deterministic MOPs.

$$\left\{\begin{matrix} \min\limits_{x} & g(x,[\xi]) \\ \text{s.t.} & x \in \mathcal{X} \end{matrix}\right\}_{[\xi]\in \mathcal{U}/\sim} \tag{UMOP($\sim$)}$$

We now proceed to show the relationship between g and f. In what follows, we assume that the conditions of Assumption 4.1 hold.

Proposition 4.2. *Let $x^* \in \mathcal{X}$ and let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \geqq 0$. If x^* is a(n) (weakly) ε-efficient solution for* (SP($[\xi^*]$)) *then x^* is a(n) (weakly) ε-efficient-under-uncertainty solution for* (UMOP($[\xi^*]$)).

Proof. We show that ε-efficiency implies ε-efficiency-under-uncertainty as the weak case follows analogously. To that end, let $\varepsilon \geqq 0$ and let $x^* \in \mathcal{X}$ be an ε-efficient solution for (SP($[\xi^*]$)). Towards a contradiction, suppose it is not an ε-efficient-under-uncertainty solution for (UMOP($[\xi^*]$)). Then there exists an $x \in \mathcal{X}$ such that for all $\xi \in [\xi^*]$,

$$f(x,\xi) \leq f(x^*,\xi) - \varepsilon.$$

In particular, for each $\xi \in [\xi^*]$ there exists $j \in \mathbb{N}_n$ such that

$$\begin{aligned} f_i(x,\xi) &\leq f_i(x^*,\xi) - \varepsilon_i, \qquad i \neq j \\ f_j(x,\xi) &< f_j(x^*,\xi) - \varepsilon_j. \end{aligned}$$

Notably, this means that for all $i \in \mathbb{N}_n$

$$\begin{aligned} \frac{1}{M([\xi^*])}\int_{[\xi^*]} f_i(x,\xi)\ d\xi &\leq \frac{1}{M([\xi^*])}\int_{[\xi^*]} f_i(x^*,\xi) - \varepsilon_i\ d\xi \\ g_i(x,[\xi^*]) &\leq g_i(x^*,[\xi^*]) - \varepsilon_i \end{aligned} \tag{1}$$

Furthermore, we have that

$$\sum_{k=1}^n f_k(x,\xi) < \sum_{k=1}^n (f_k(x^*,\xi) - \varepsilon_k), \qquad \forall \xi \in [\xi^*].$$

But this implies that

$$\begin{aligned} \frac{1}{M([\xi^*])}\int_{[\xi^*]} \sum_{k=1}^n f_k(x,\xi)\ d\xi &< \frac{1}{M([\xi^*])}\int_{[\xi^*]} \sum_{k=1}^n (f_k(x^*,\xi) - \varepsilon_k)\ d\xi \\ \sum_{k=1}^n \frac{1}{M([\xi^*])}\int_{[\xi^*]} f_k(x,\xi)\ d\xi &< \sum_{k=1}^n \frac{1}{M([\xi^*])}\int_{[\xi^*]} f_k(x^*,\xi) - \varepsilon_k\ d\xi \\ \sum_{k=1}^n g_k(x,[\xi^*]) &< \sum_{k=1}^n (g_k(x^*,[\xi^*]) - \varepsilon_k). \end{aligned}$$

The previous strict inequality and (1) means that there is at least one $j' \in \mathbb{N}_n$ such that

$$g_{j'}(x,[\xi^*]) < g_{j'}(x^*,[\xi^*]) - \varepsilon_{j'}.$$

Hence,

$$g(x,[\xi^*]) \leq g(x^*,[\xi^*]) - \varepsilon,$$

which contradicts the fact that x^* is an ε-efficient solution for (SP($[\xi^*]$)). □

Corollary 4.1. *If x^* is a (weakly) efficient solution for* (SP($[\xi^*]$)) *then x^* is a(n) (weakly) efficient-under-uncertainty solution for* (UMOP).

Proof. This follows from Proposition 4.2 and Proposition 3.1. □

The previous proposition justifies our referring to (SP($[\xi^*]$)) as a "subproblem". By collapsing scenarios into a single equivalence class $[\xi^*] \in \mathcal{U}/\sim$, we are able to use (SP($[\xi^*]$)) to still provide efficient-under-uncertainty solutions for (UMOP) within an equivalence class.

5 Decision Making with Equivalent Scenarios

We now turn to the question of decision-making with an equivalence relation on the set of scenarios. We start by letting (UMOP) be given and suppose that we have an equivalence relation $\sim$ such that $\left|\mathcal{U}/\sim\right| = q < \infty$. Since we have finitely many equivalence classes, this means that there are, in essence, only finitely many scenarios to consider. Thus, in principle, a decision maker may solve each (SP($[\xi^*]$)) and search for a decision $x^* \in \mathcal{X}$ which performs to the DM's satisfaction across all subproblems. This amounts to turning (SP($[\xi^*]$)) from a family of multiobjective optimization problems into a *single* multiobjective optimization, which is possible since there are finitely many equivalence classes. Thus, the master problem that the decision maker desires to solve is the following.

$$\begin{aligned} \min_{x} \quad & \begin{bmatrix} g(x, [\xi^1]) \\ \vdots \\ g(x, [\xi^q]) \end{bmatrix} \\ \text{s.t.} \quad & x \in \mathcal{X} \end{aligned} \tag{MP}$$

The problem (MP) amounts to instantiating the aggregated objective functions at each equivalence class in $\mathcal{U}/\sim$, and since it is multiobjective, it has the ability to model the conflict between equivalence classes, should they arise. Finally, observe that based on the definition of g, (MP) has qn objectives.

Proposition 5.1. *Let $x^* \in \mathcal{X}$ and $[\xi^i] \in \mathcal{U}/\sim$.*

1. *If x^* is a weakly efficient solution for* (SP($[\xi^i]$)) *then it is a weakly efficient solution for* (MP).
2. *If x^* is an efficient solution for* (SP($[\xi^i]$)) *and $g(\cdot, [\xi^i])$ is injective in x for each $[\xi^i] \in \mathcal{U}/\sim$, then x^* is an efficient solution for* (MP).

Proof. We show 2. as 1. follows analogously. Let x^* be an efficient solution for (SP($[\xi^i]$)) and g be an injective function in x. Towards a contradiction, suppose x^* is not an efficient solution for (MP). Then there exists $x \in \mathcal{X}$ such that

$$\begin{bmatrix} g(x, [\xi^1]) \\ \vdots \\ g(x, [\xi^q]) \end{bmatrix} \leq \begin{bmatrix} g(x^*, [\xi^1]) \\ \vdots \\ g(x^*, [\xi^q]) \end{bmatrix}$$

and in particular, $g(x, [\xi^i]) \leqq g(x^*, [\xi^i])$ for some $i \in \mathbb{N}_q$. Observe that $g(x, [\xi^i]) \neq g(x^*, [\xi^i])$ since otherwise, the injectivity of g would imply that $x = x^*$, which is not possible. Thus, it must actually be that $g(x, [\xi^i]) \leq g(x^*, [\xi^i])$, contradicting the efficiency of x^* for (SP($[\xi^*]$)). $\square$

The next proposition connects efficient solutions of (MP) and (UMOP).

Proposition 5.2. *Let $x^* \in \mathcal{X}$. If x^* is a (weakly) efficient solution for* (MP) *then it is a (weakly) efficient-under-uncertainty solution for* (UMOP).

Proof. Let x^* be an efficient solution for (MP). Towards a contradiction, suppose x^* is not an efficient-under-uncertainty solution for (UMOP). Then there exists $x \in \mathcal{X}$ such that $f(x, \xi) \leq f(x^*, \xi)$ for all $\xi \in \mathcal{U}$. In particular, for each $[\xi^i] \in \mathcal{U}/\sim$, we have that $f(x, \xi) \leq f(x^*, \xi)$ for all $\xi \in [\xi^i]$. This means that for each $[\xi^i] \in \mathcal{U}/\sim$ and for each $\xi \in [\xi^i]$, there exists $j \in \mathbb{N}_n$ such that

$$\begin{aligned} f_i(x, \xi) &\leq f_i(x^*, \xi), \qquad i \neq j \\ f_j(x, \xi) &< f_j(x^*, \xi). \end{aligned}$$

In general, this means that

$$\begin{gathered} f_k(x,\xi) \leq f_k(x^*,\xi), \qquad \xi \in [\xi^i] \\ \frac{1}{M([\xi^i])}\int_{[\xi^i]} f_k(x,\xi)\ d\xi \leq \frac{1}{M([\xi^i])}\int_{[\xi^i]} f_k(x^*,\xi)\ d\xi \\ g_k(x,[\xi^i]) \leq g_k(x^*,[\xi^i]) \end{gathered}$$

Therefore, $g(x,[\xi^i]) \leqq g(x^*,[\xi^i])$. However, observe that since $f(x,\xi) \leq f(x^*,\xi)$ for all $\xi \in [\xi^i]$,

$$\begin{gathered} \sum_{k=1}^{n} f_k(x,\xi) < \sum_{k=1}^{n} f_k(x^*,\xi) \\ \frac{1}{M([\xi^i])}\int_{[\xi^i]} \sum_{k=1}^{n} f_k(x,\xi) d\xi < \frac{1}{M([\xi^i])}\int_{[\xi^i]} \sum_{k=1}^{n} f_k(x^*,\xi)\ d\xi \\ \sum_{k=1}^{n} g_k(x,[\xi^i]) < \sum_{k=1}^{n} g_k(x^*,[\xi^i]) \end{gathered}$$

Hence, since the above strict inequality holds and $g(x,[\xi^i]) \leqq g(x^*,[\xi^i])$ then there must be a $j' \in \mathbb{N}_n$ such that

$$g_{j'}(x,[\xi^i]) < g_{j'}(x^*,[\xi^i])$$

which implies that

$$g(x,[\xi^i]) \leq g(x^*,[\xi^i]).$$

Since this is true for all $[\xi^i] \in \mathcal{U}/\sim$, we have that

$$\begin{bmatrix} g(x,[\xi^1]) \\ \vdots \\ g(x,[\xi^q]) \end{bmatrix} \leq \begin{bmatrix} g(x^*,[\xi^1]) \\ \vdots \\ g(x^*,[\xi^q]) \end{bmatrix}.$$

But this contradicts the fact that x^* is an efficient solution for (MP). (The case when x^* is a weakly efficient solution for (MP) follows analogously.) $\square$

When qn is sufficiently small, we may directly solve (MP) and use whatever appropriate MCDM/A tools to assist a decision maker in selecting an efficient solution, which by Proposition 5.2 must be efficient-under-uncertainty for (UMOP). However, if qn is too large, the decision-making process can become cognitively intractable, even with suitable MCDM/A tools (Miller 1956). To address this difficulty, we turn to the concept of scenario tradeoffs.

5.1 Scenario Tradeoffs

To summarize the results shown thus far, Proposition 3.1 shows that efficient-under-uncertainty solutions for (UMOP($[\xi^*]$)) are also efficient-under-uncertainty solutions for (UMOP), while Proposition 4.2 shows that efficient solutions for (SP($[\xi^*]$)) are also efficient-under-uncertainty solutions for (UMOP($[\xi^*]$)), implying Corollary 4.1, which says that efficient solutions for (SP($[\xi^*]$)) are also efficient-under-uncertainty for (UMOP). Finally, Propositions 5.1 and 5.2 show that efficient solutions for (SP($[\xi^*]$)) are efficient for (MP), which are also efficient-under-uncertainty for (UMOP). Observe that these results flow "upward"; that is, they show how each reformulation of (UMOP) actually provides efficient-under-uncertainty solutions for (UMOP). However, reversing the direction of these results gives us some deeper insight into the relationships between the reformulations.

Suppose $x^* \in \mathcal{X}$ is an efficient-under-uncertainty solution for (UMOP) or (UMOP($[\xi^*]$)); then there must be $\varepsilon \geqq 0$ such that x^* is an ε-efficient solution for (MP). This may be readily observed from the fact that the image of x^* in the outcome set of (MP) will land it either in the Pareto set or not. If $(g(x^*,[\xi^1]),\dots g(x^*,[\xi^q]))$ is a Pareto point, then we are done. If, on the other hand, it is not, then by the external stability of (MP) (which is guaranteed to hold since the objective functions are continuous and the feasible set is compact), there must be a Pareto point y^* and a direction $d^* \in \mathbb{R}^{nq}_{\geqq}$ such that $(g(x^*,[\xi^1]),\ldots,g(x^*,[\xi^q])) = y^* + d^*$. This means that x^* is a d^*-efficient solution. (This fact also holds when projecting x^* onto a subproblem, (SP($[\xi^*]$)).)

However, each feasible decision $x \in \mathcal{X}$ will have a *different* $\varepsilon \geqq 0$ for which it is ε-efficient in (MP). This would suggest that in a decision-making context, we desire to find a decision x^* whose ε is the *smallest*. To accomplish this, we suggest shifting attention to the subproblems, (SP($[\xi^*]$)). These subproblems are well-defined with respect to the DM's preferences as they arise out of the equivalence classes defined by the DM. To find a decision x^* which performs as well as possible across all subproblems, *i.e.*, equivalence classes of scenarios, amounts to ensuring that x^* is as close as possible to being Pareto efficient in each subproblem. This can be achieved by taking the image of x^* in each subproblem and "pulling" it close to each respective Pareto set.

This idea follows from the concept of subproblem tradeoffs in de Castro and Wiecek 2025. As discussed in Section 4, we saw that an equivalence class $[\xi^*]$ may be used to define a subproblem. This means that precisely the techniques developed in de Castro and Wiecek 2025 are well suited for our problem here as we desire to balance the tradeoffs between (equivalence classes of) scenarios.

To do so, we use a BASF $\sigma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. For each equivalence class $[\xi^i] \in \mathcal{U}/\sim$, let $\mathcal{R}^i \subseteq \mathbb{R}^n$ be a set containing points which correspond to the DM's aspirations for the performance of g in equivalence class $[\xi^i]$. The **scenario tradeoff problem** is formulated as the following deterministic q-objective optimization problem.

$$\begin{aligned}\min_{x,r^1,\ldots,r^q} \quad & \begin{bmatrix}\sigma(g(x,[\xi^1]),r^1)\\ \vdots \\ \sigma(g(x,[\xi^q]),r^q)\end{bmatrix} \\ \text{s.t.} \quad & x \in \mathcal{X} \\ & r^i \in \mathcal{R}^i \qquad i=1,\ldots,q\end{aligned} \tag{STP}$$

The BASF σ plays the role of "pulling" the performance of x as close as possible to the Pareto set of the corresponding equivalence class $[\xi^i]$, $i=1,\ldots,q$. Since we define $\mathcal{R}^i$ to be a set of aspirations, and we conceive of the decision problem as being minimization, we essentially assume that $\mathcal{R}^i$ dominates the Pareto set of (SP($[\xi^*]$)). (This assumption is further discussed below and in de Castro and Wiecek 2025.)

Definition 5.1. Let $x^* \in \mathcal{X}$ and $r^{i*} \in \mathcal{R}^i$ for $i \in \mathbb{N}_q$. If $(x^*, r^{1*},\ldots,r^{q*})$ is a(n)

- weakly efficient solution for (STP), we say that x^* is a **weakly balanced efficient-under-uncertainty solution** for (UMOP);
- efficient solution for (STP), we say that x^* is a **balanced efficient-under-uncertainty solution** for (UMOP).

Observe that the above definition of efficiency-under-uncertainty depends solely upon whether a decision is *efficient* for (STP). As such, this transforms (UMOP) into a deterministic decision problem where a DM may make her final decision using readily available and well-established MCDM/A tools such as NIMBUS (K. Miettinen and M. Mäkelä 1995; Kaisa Miettinen and M. M. Mäkelä 1996), AHP (Bernasconi, Choirat, and Seri 2010), ELECTRE (Figueira, Mousseau, and Roy 2016), to name a few.

5.2 On Constructing $\mathcal{R}^i$

The *a priori* definition of the set of reference points for each scenario gives the decision maker flexibility in finding a (weakly) balanced efficient-under-uncertainty solution. As such, some comments are necessary to cover the various possibilities for the construction of $\mathcal{R}^i$.

First, in order to guarantee properties analogous to those of autonomous coordination in de Castro and Wiecek 2025, we must ensure that $\mathcal{R}^i$ dominates, in the sense of the upper set-less order relation, the Pareto set of (SP($[\xi^i]$)). Second, the decision maker is able to adjust the cardinality of $\mathcal{R}^i$. A direct application of autonomous coordination from de Castro and Wiecek 2025 would require that $\mathcal{R}^i = \mathcal{P}(g(\mathcal{X}, [\xi^i]))$. This results in the following bilevel optimization problem.

$$\begin{aligned} \min_{x, r^1, \ldots, r^q} \quad & \begin{bmatrix} \sigma(g(x, [\xi^1]), r^1) \\ \vdots \\ \sigma(g(x, [\xi^q]), r^q) \end{bmatrix} \\ \text{s.t.} \quad & x \in \mathcal{X} \\ & r^i \in \mathcal{P}(g(\mathcal{X}, [\xi^i])), \quad i = 1, \ldots, q \end{aligned} \tag{BL-STP}$$

Depending on the problem context, this may be a difficult problem as optimization must occur over q Pareto sets. (For a survey on such bilevel optimization methods, see Eichfelder 2020.) On the other hand, de Castro and Wiecek 2025 suggests using discretizations of the Pareto sets of $g(\mathcal{X}, [\xi^i])$ to transform (BL-STP) into a single-level mixed-binary optimization problem.

$$\min_{\substack{x, r^1, \ldots, r^q \\ \beta^1, \ldots, \beta^q}} \quad \begin{bmatrix} \sigma(g(x, [\xi^1]), r^1) \\ \vdots \\ \sigma(g(x, [\xi^q]), r^q) \end{bmatrix} \tag{MB-STP}$$

$$\text{s.t.} \quad x \in \mathcal{X} \tag{2}$$

$$r^i = \sum_{j=1}^{a_i} \beta^i_j p^i_j, \quad i = 1, \ldots, q \tag{3}$$

$$\sum_{j=1}^{a_i} \beta^i_j = 1, \quad i = 1, \ldots, q \tag{4}$$

$$\beta^i \in \{0, 1\}^{a_i}, \quad i = 1, \ldots, q \tag{5}$$

where for $i = 1, \ldots, q$, a_i is the size of the discretization of $\mathcal{P}(g(\mathcal{X}, [\xi^i]))$, and $p^i_1, \ldots, p^i_{a_i}$ are the points in this discretization. Constraint (3) selects the reference point from the discrete representation of $\mathcal{P}(g(\mathcal{X}, [\xi^i]))$, while constraints (4) and (5) ensure that only one reference point is selected.

An even more straightforward formulation occurs when $|\mathcal{R}^i| = 1$. This places the scenario tradeoff problem back into the classical formulation of (B)ASFs since they were originally conceived of as having a single fixed reference point (Wierzbicki 1986; Kaisa Miettinen 1998). Thus, for fixed $\bar{r}^1, \ldots, \bar{r}^q \in \mathbb{R}^n$, the scenario tradeoff problem becomes the following.

$$\begin{aligned} \min_{x} \quad & \begin{bmatrix} \sigma(g(x, [\xi^1]), \bar{r}^1) \\ \vdots \\ \sigma(g(x, [\xi^q]), \bar{r}^q) \end{bmatrix} \\ \text{s.t.} \quad & x \in \mathcal{X} \end{aligned} \tag{FXD-STP}$$

Remark 5.1. There are a few reasons to construct $\mathcal{R}^i$ as a set, as opposed to a single point. First, uncertainty in the problem may very well extend to the decision maker's aspirations—what performance can even be rationally hoped for? By allowing the reference point itself to vary over different values, the scenario tradeoff problem is able to select aspirations which correspond to the best possible performances, which does provide the decision maker with more information about possible bounds on the performance of her decisions. (For more discussion on this point, see de Castro 2026.)

Second, if the decision maker does have a good sense of her aspirations, she is able to not be "tied down" to only one. Indeed, comparing aspirations of performance with actual feasible performances can contribute valuable information in the decision-making process. One such example of this is that the optimal solutions found by solvers are necessarily related to the reference point picked. By allowing multiple reference points, there is the opportunity for the solver to present solutions to the decision maker which would otherwise not be found if restricted to a single reference point.

Third, particularly in the case of the reference points themselves being Pareto points for a class of scenarios, the scenario tradeoff problem adopts useful properties, which are discussed later in this work and in de Castro and Wiecek 2025.

With these considerations in mind, we may proceed to examine some properties of the scenario tradeoff problem.

5.3 Properties of the Scenario Tradeoff Problem

For the remainder of this section, we assume that $\mathcal{R}^i \subseteq \mathcal{P}(g(\mathcal{X}, [\xi^i]))$ for each $i \in \mathbb{N}_q$. The next properties of (STP) follow directly from de Castro and Wiecek 2025.

Theorem 5.1 (Proposition 4 in de Castro and Wiecek 2025)**.** *The ideal point of* (STP) *is nonnegative.*

Theorem 5.2 (Lemma 2 in de Castro and Wiecek 2025)**.** *Let σ be a strictly order preserving and $\mathbb{R}^n_{\geqq}$-order representing BASF and let $(x^*, r^{1*}, \ldots, r^{q*})$ be a feasible solution for* (STP)*. For any $i \in \mathbb{N}_q$, $\sigma(g(x^*, [\xi^i]), r^{i*}) = 0$ if and only if $g(x^*, [\xi^i]) = r^{i*}$.*

The next proposition shows that the scenario tradeoff problem does indeed provide efficient solutions for subproblems.

Proposition 5.3. *Let $x^* \in \mathcal{X}$ and let σ be a strictly order preserving and $\mathbb{R}^n_{\geqq}$-order representing BASF. If $\mathcal{R}^i \subseteq \mathcal{P}(g(\mathcal{X}, [\xi^i]))$, $i = 1, \ldots, q$, and $(x^*, r^{1*}, \ldots, r^{q*})$ is an efficient solution for* (STP) *such that $\sigma(g(x^*, [\xi^i]), r^{i*}) = 0$ for some $i = 1, \ldots, q$ then x^* is an efficient solution for* (SP($[\xi^*]$))*.*

Proof. Follows directly from Theorem 5.2. □

Proposition 5.4. *Let $(x^*, r^{1*}, \ldots, r^{q*})$ be a feasible solution for* (STP)*.*

1. *If $(x^*, r^{1*}, \ldots, r^{q*})$ is a weakly efficient solution for* (STP) *and σ is strictly order preserving then x^* is a weakly efficient solution for* (MP)*.*

2. *If $(x^*, r^{1*}, \ldots, r^{q*})$ is an efficient solution for* (STP) *and σ is strongly order preserving then x^* is an efficient solution for* (MP)*.*

Proof. We show the second case. To that end, let $(x^*, r^{1*}, \ldots, r^{q*})$ be an efficient solution for (STP). Towards a contradiction, suppose x^* is not an efficient solution for (MP). Then there exists $x \in \mathcal{X}$ such that

$$\begin{bmatrix} g(x, [\xi^1]) \\ \vdots \\ g(x, [\xi^q]) \end{bmatrix} \leq \begin{bmatrix} g(x^*, [\xi^1]) \\ \vdots \\ g(x^*, [\xi^q]) \end{bmatrix}$$

Thus, $g(x, [\xi^i]) \leqq g(x^*, [\xi^i])$ for all $i \in \mathbb{N}_q$ and there exists a $j \in \mathbb{N}_q$ such that $g(x, [\xi^j]) \leq g(x^*, [\xi^j])$. In either case, since σ is strongly order preserving, we have that

$$\begin{aligned} \sigma(g(x, [\xi^i]), r^{i*}) &\leq \sigma(g(x^*, [\xi^i]), r^{i*}), \quad i \in \mathbb{N}_q \\ \sigma(g(x, [\xi^j]), r^{j*}) &< \sigma(g(x^*, [\xi^j]), r^{j*}), \end{aligned}$$

which is equivalent to

$$\begin{bmatrix} \sigma(g(x, [\xi^1]), r^{1*}) \\ \vdots \\ \sigma(g(x, [\xi^q]), r^{q*}) \end{bmatrix} \leq \begin{bmatrix} \sigma(g(x^*, [\xi^1]), r^{1*}) \\ \vdots \\ \sigma(g(x^*, [\xi^q]), r^{q*}) \end{bmatrix}.$$

But this contradicts the fact that $(x^*, r^{1*}, \ldots, r^{q*})$ is an efficient solution for (STP). (The first case follows analogously.) □

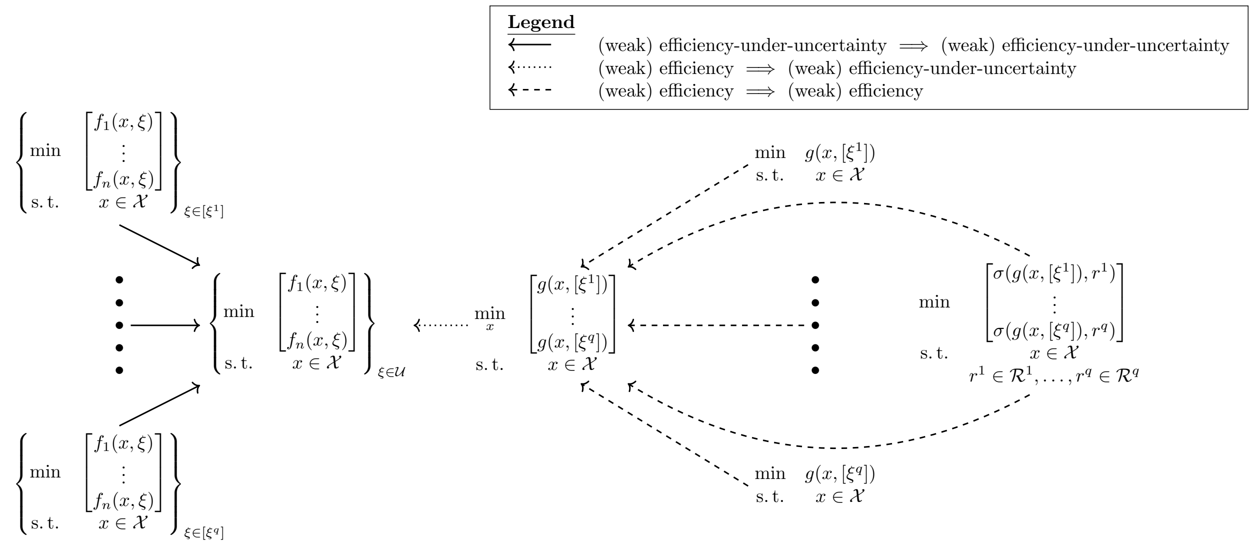


Figure 1: Relationships between formulations and their efficient solutions.

Remark 5.2. Proposition 5.4 shows that when σ is strongly order preserving, we need only solve (STP) to efficiency. This is readily possible with the standard strongly order-preserving (B)ASF

$$\sigma(y, r) = \max_{1 \le i \le n} \{\lambda_i(y_i - r_i)\} + \rho \sum_{i=1}^{n} \lambda_i(y_i - r_i)$$

where $\lambda_1, \dots, \lambda_n, \rho > 0$ are parameters. See Wierzbicki 1986; Kaisa Miettinen 1998; de Castro 2026 for more details.

The main insight provided by (STP) is the ability to measure tradeoffs between equivalence classes of scenarios.

Definition 5.2 (de Castro and Wiecek 2025)**.** Let $(x^*, r^{1*}, \dots, r^{q*})$ be a(n) (weakly) efficient solution for (STP) and let $i, j \in \{1, \dots, q\}$ with $i \neq j$. The **scenario tradeoff** at $(x^*, r^{1*}, \dots, r^{q*})$ is the value

$$\mathcal{ST}_{ij}(x^*, r^{i*}, r^{j*}) = \frac{\sigma(g(x^*, [\xi^i]), r^{i*})}{\sigma(g(x^*, [\xi^j]), r^{j*})}$$

if $\sigma(g(x^*, [\xi^j]), r^{j*}) \neq 0$.

Remark 5.3. When the conditions of Proposition 5.3 are satisfied, there are four possible values for the scenario tradeoff, $\mathcal{ST}_{ij}$, as de Castro and Wiecek 2025 also notes.

1. $\mathcal{ST}_{ij} = 0$: x^* is a(n) (weakly) efficient solution for (SP($[\xi^i]$));
2. $0 < \mathcal{ST}_{ij} < 1$: x^* is "closer" to (weak) efficiency in equivalence class $[\xi^i]$ than in equivalence class $[\xi^j]$;
3. $\mathcal{ST}_{ij} = 1$: x^* is just as "close" to being (weakly) efficient in equivalence class $[\xi^i]$ as in equivalence class $[\xi^j]$;
4. $\mathcal{ST}_{ij} > 1$: x^* is "closer" to being (weakly) efficient in equivalence class $[\xi^j]$ than in equivalence class $[\xi^i]$.

We can also use the standard definition of marginal tradeoffs.

Definition 5.3. Let $(x^*, r^{1*}, \dots, r^{q*}), (\hat{x}, \hat{r}^1, \dots, \hat{r}^q)$ be distinct (weakly) efficient solutions for (STP) and let $i, j \in \{1, \dots, q\}$ with $i \neq j$. The **marginal scenario tradeoff** is the value

$$\Delta\mathcal{ST}_{ij}(x^*, \hat{x}, r^{i*}, \hat{r}^i, r^{j*}, \hat{r}^j) = \left| \frac{\sigma(g(x^*, [\xi^i]), r^{i*}) - \sigma(g(\hat{x}, [\xi^i]), \hat{r}^i)}{\sigma(g(x^*, [\xi^j]), r^{j*}) - \sigma(g(\hat{x}, [\xi^j]), \hat{r}^j)} \right|$$

where $\sigma(g(x^*, [\xi^j]), r^{j*}) - \sigma(g(\hat{x}, [\xi^j]), \hat{r}^j) \neq 0$.

Remark 5.4. We share these two standard definitions of tradeoffs as examples for how a decision maker can measure the tradeoffs between scenarios. However, decision makers can use the tradeoff measurement on (STP) that is relevant for their decision-making procedure.

Figure 1 provides a summary of the results shown in this work. In particular, a(n) (weakly) efficient solution for (STP) is also a(n) (weakly) efficient solution for (MP). Hence, following the arrows, we see that a(n) (weakly) efficient solution for (STP) must also be a(n) (weakly) efficient-under-uncertainty solution for (UMOP). Indeed, measuring the scenario tradeoffs using (STP) enables the DM to deterministically manage the original uncertain MOP by implementing her preferences in the decision-making model, all the while guaranteeing that her final decision will be efficient-under-uncertainty.

Remark 5.5. It is worth considering the implicit philosophical propositions which undergird the approach presented in this work. The purpose of the original formulation (UMOP) is to model the decision problem as it is, to describe the problem as it occurs in the world, regardless of the individuals who interact with it. In the language of the philosopher Thomas Nagel, this model is specifically formulated from "the view from nowhere" (Nagel 1986). In taking a robust optimization approach, then, it is evidently assumed that $\mathcal{U}$ contains the *actual* worst case in a meaningful way. With a slight abuse of mathematical notions, we take the model to be isomorphic with reality.

But the model leaves out an essential piece of reality—the decision maker's own *subjective view* of the problem. The fact of the matter is that this problem is solved *from somewhere*; there are beliefs about what efficient solutions *should* be optimal. And these sentiments do matter in solving the problem. Unlike mathematical models of physical systems, where solutions are defined, sought, and selected independently of any individual's *sense* about what "should" be optimal or not, mathematical models of decision problems necessarily require the decision maker's input. Of course, optimality for the *model* is independently and abstractly defined as optimality for the *model* arises out of its own mathematical structure. But the rigorous, abstract, mathematical definition of optimality cannot distinguish between optimal solutions.

The preceding argument suggests that although the problem must be modeled from nowhere, it must be solved from somewhere. Now, that "somewhere" may very well be concordant with nowhere: "Avoid the worst case *as the worst case is in reality.*" But it is also possible for "somewhere" to be discordant with nowhere: "Avoid the worst case *as I see the worst case to be.*" Assuming that a decision maker is rational, her subjective view is not totally untethered from reality, but this subjective view can go a long way in *clarifying* the mathematical model.

When a decision maker does have preferences, *the problem she is interested in solving is her subjective view of the objective model*, not merely the objective model. This is where the equivalence relation, the new objectives $g(x, [\xi^1]), \ldots, g(x, [\xi^q])$, the new family of problems (UMOP($\sim$)), and scenario tradeoffs come into the picture. They allow the decision maker's "view from somewhere" to be used in solving the mathematical model which was modeled from the "view from nowhere."

In order to see these results in practice, we consider an example from disaster relief in the next section.

6 Example

We present an example where we consider the problem of formulating a plan for delivering aid in the aftermath of a hurricane. As our focus here is to demonstrate the application of our theoretical developments, we suppose that the damage is inflicted by wind speed alone.

Hurricanes are categorized according to the Saffir-Simpson scale which is divided into five categories (National Hurricane Center 2021). According to this scale, we may consider the uncertainty set to be $\mathcal{U} = [74, 177]$, with units in miles per hour. We define a natural equivalence relation based upon the lower and upper bounds of each category. Namely, for $i = 1, \ldots, 5$, a Category i hurricane with lower wind speed of ℓ_i and upper wind speed of u_i, we say $\xi_1 \sim \xi_2$ if and only if $\ell_i \leq \xi_1, \xi_2 < u_i$ when $i = 1, 2, 3, 4$, or $\xi_1 \sim \xi_2$ if and only if $\ell_5 \leq \xi_1, \xi_2 \leq u_5$. This equivalence relation divides $\mathcal{U}$ into five equivalence classes which match the Saffir-Simpson scale. These equivalence classes are listed in Table 1.

$[\xi^1] = [74, 96)$	$[\xi^2] = [96, 111)$	$[\xi^3] = [111, 130)$
$[\xi^4] = [130, 157)$	$[\xi^5] = [157, 177]$	

Table 1: Equivalence classes for the hurricane example.

Remark 6.1. Although the Saffir-Simpson scale designates no upper bound on a Category 5 hurricane, we provide a reasonable upper bound to ensure that each equivalence class has finite positive measure. This upper bound is chosen based upon the observation that every other equivalence class has a spread of approximately 20 miles per hour.

6.1 Formulation of the Decision Problem

The DM is responsible for constructing a disaster relief plan for four communities connected to a single supply depot. This depot has an inventory of pre-made bundles of goods, denoted by x, and is staffed by disaster response personnel, denoted by y. For ease of analysis, we assume that both x and y are real-valued variables. (We may consider y to measure labor hours as opposed to *individual* disaster-response workers.) The goal is to maximize the utility of the goods and labor delivered to each community. Thus, the problem facing the DM is essentially an uncertain 4-objective assignment problem, which is formulated in (EX-UMOP).

$$\left\{\begin{array}{lll} \max\limits_{x,y} & \begin{bmatrix} U_1(x,y,\xi) \\ U_2(x,y,\xi) \\ U_3(x,y,\xi) \\ U_4(x,y,\xi) \end{bmatrix} & \\ \text{s.t.} & x_1 + x_2 + x_3 + x_4 = 100 & \text{(All goods must be sent out.)} \\ & y_1 + y_2 + y_3 + y_4 = 100 & \text{(All personnel must be sent out.)} \\ & x_1, x_2, x_3, x_4 \geq 100/8 & \text{(At least one-eighth of the available goods must be assigned to each community.)} \\ & y_1, y_2, y_3, y_4 \geq 100/6 & \text{(At least one-sixth of the available labor must be assigned to each community.)} \end{array}\right\}_{\xi \in [74,177]} \tag{EX-UMOP}$$

The objectives in (EX-UMOP) measure the utility of goods and personnel assigned to each respective community. For $i \in \{1, 2, 3, 4\}$, the utility functions are defined piecewise with respect to the scenario.

$$U_1(x,y,\xi) = \begin{cases} \left(1 - \dfrac{\xi}{177}\right)\ln(x_1) + \dfrac{\xi}{177}\ln(y_1), & 74 \le \xi < 130 \\[2ex] \left(1 - \dfrac{\xi+130}{354}\right)\ln(x_1) + \dfrac{\xi+130}{354}\ln(y_1), & 130 \le \xi \le 177 \end{cases}$$

$$U_2(x,y,\xi) = \begin{cases} \dfrac{\xi}{177}\ln(x_2) + \left(1 - \dfrac{\xi}{177}\right)\ln(y_2), & 74 \le \xi < 157 \\[2ex] \dfrac{\xi+157}{354}\ln(x_2) + \left(1 - \dfrac{\xi+157}{354}\right)\ln(y_2), & 157 \le \xi \le 177 \end{cases}$$

$$U_3(x,y,\xi) = \begin{cases} \dfrac{\xi}{177}\ln(x_3) + \left(1 - \dfrac{\xi}{177}\right)\ln(y_3), & 74 \le \xi < 96 \\[2ex] \dfrac{\xi+96}{354}\ln(x_3) + \left(1 - \dfrac{\xi+96}{354}\right)\ln(y_3), & 96 \le \xi \le 177 \end{cases}$$

$$U_4(x,y,\xi) = \begin{cases} \left(1 - \dfrac{\xi}{177}\right)\ln(x_4) + \dfrac{\xi}{177}\ln(y_4), & 74 \le \xi < 111 \\[2ex] \left(1 - \dfrac{\xi+111}{354}\right)\ln(x_4) + \dfrac{\xi+111}{354}\ln(y_4), & 111 \le \xi \le 177 \end{cases}$$

Observe that each utility function $U_1, \dots, U_4$ is in the logarithmic form of a Cobb-Douglas utility function, which depends on the amount of goods and personnel assigned to each community. The coefficients depending on ξ are the parameters of the logarithmic Cobb-Douglas utility function and are piecewise linear.

By using the equivalence classes listed in Table 1, we may calculate the aggregate objective functions, which are listed in Table 2. We let $\overline{U}(x,y,[\xi]) = (\overline{U}_1(x,y,[\xi]), \overline{U}_2(x,y,[\xi]), \overline{U}_3(x,y,[\xi]), \overline{U}_4(x,y,[\xi]))$. These aggregated objective functions enable the DM to formulate the subproblems, which correspond to each equivalence class. For example, the first subproblem is defined using the first equivalence class, namely $[\xi^1] = [74, 96)$.

$$\begin{aligned} \max_{x,y} \quad & \begin{bmatrix} 0.520\ln(x_1) + 0.480\ln(y_1) \\ 0.480\ln(x_2) + 0.520\ln(y_2) \\ 0.480\ln(x_3) + 0.520\ln(y_3) \\ 0.520\ln(x_4) + 0.480\ln(y_4) \end{bmatrix} \\ \text{s.t.} \quad & x_1 + x_2 + x_3 + x_4 = 100 \\ & y_1 + y_2 + y_3 + y_4 = 100 \\ & x_1, x_2, x_3, x_4 \ge 100/8 \\ & y_1, y_2, y_3, y_4 \ge 100/6 \end{aligned} \tag{EX-SP[ξ^1]}$$

However, we focus our attention on the scenario tradeoff problem for this example as it is with the scenario tradeoff problem that our DM ultimately makes her decision. We follow the formulation of (MB-STP) and we therefore use discretizations of the Pareto sets of each subproblem (*i.e.*, each equivalence class), which we denote by $\mathcal{P}[\xi^i] = \{p_1^i, \dots, p_{k_i}^i\}$, for $k_i \ge 1$ and $i = 1,2,3,4,5$. In this example, the Pareto set discretizations each have 35 elements and we use the canonical strictly order preserving and $\mathbb{R}^4_{\geqq}$-order representing BASF defined by

$$\sigma(y,r) = \max_{j=1,2,3,4}\{y_j - r_j\}$$

$$
\overline{U}_1(x,y,[\xi]) = \begin{cases} \dfrac{1}{b-a}\left(\left(b-a-\dfrac{b^2-a^2}{354}\right)\ln(x_1)+\dfrac{b^2-a^2}{354}\ln(y_1)\right), & (a,b)\in\{(74,96),(96,111),\\ & (111,130)\} \\ \dfrac{1}{b-a}\left(\left(\dfrac{112(b-a)}{177}-\dfrac{b^2-a^2}{708}\right)\ln(x_1)+\left(\dfrac{b^2-a^2}{708}+\dfrac{65(b-a)}{177}\right)\ln(y_1)\right), & (a,b)\in\{(130,157),\\ & (157,177)\} \end{cases}
$$

$$
\overline{U}_2(x,y,[\xi]) = \begin{cases} \dfrac{1}{b-a}\left(\left(\dfrac{b^2-a^2}{354}\right)\ln(x_2)+\left(b-a-\dfrac{b^2-a^2}{354}\right)\ln(y_2)\right), & (a,b)\in\{(74,96),(96,111),\\ & (111,130),\\ & (130,157)\} \\ \dfrac{1}{b-a}\left(\left(\dfrac{b^2-a^2}{708}+\dfrac{157(b-a)}{354}\right)\ln(x_2)+\left(\dfrac{197(b-a)}{354}-\dfrac{b^2-a^2}{708}\right)\ln(y_2)\right), & (a,b)\in\{(157,177)\} \end{cases}
$$

$$
\overline{U}_3(x,y,[\xi]) = \begin{cases} \dfrac{1}{b-a}\left(\left(\dfrac{b^2-a^2}{354}\right)\ln(x_3)+\left(b-a-\dfrac{b^2-a^2}{354}\right)\ln(y_3)\right), & (a,b)\in\{(74,96)\} \\ \dfrac{1}{b-a}\left(\left(\dfrac{b^2-a^2}{708}+\dfrac{16(b-a)}{59}\right)\ln(x_3)+\left(\dfrac{43(b-a)}{59}-\dfrac{b^2-a^2}{708}\right)\ln(y_3)\right), & (a,b)\in\{(96,111),(111,130),\\ & (130,157),(157,177)\} \end{cases}
$$

$$
\overline{U}_4(x,y,[\xi]) = \begin{cases} \dfrac{1}{b-a}\left(\left(b-a-\dfrac{b^2-a^2}{354}\right)\ln(x_4)+\dfrac{b^2-a^2}{354}\ln(y_4)\right), & (a,b)\in\{(74,96),(96,111)\} \\ \dfrac{1}{b-a}\left(\left(\dfrac{81(b-a)}{118}-\dfrac{b^2-a^2}{708}\right)\ln(x_4)+\left(\dfrac{b^2-a^2}{708}+\dfrac{37(b-a)}{118}\right)\ln(y_4)\right), & (a,b)\in\{(111,130),(130,157),\\ & (157,177)\} \end{cases}
$$

Table 2: Aggregated objective functions for hurricane example. Observe that (a,b) here refers to ordered pairs defined by the equivalence classes.

The scenario tradeoff problem for this example is the following.

$$\begin{aligned}
\min_{\substack{x,y\\ r^1,\dots,r^q\\ \beta^1,\dots,\beta^5}} \quad & \begin{bmatrix} \sigma(-\overline{U}(x,y,[\xi^1]),-r^1) \\ \sigma(-\overline{U}(x,y,[\xi^2]),-r^2) \\ \sigma(-\overline{U}(x,y,[\xi^3]),-r^3) \\ \sigma(-\overline{U}(x,y,[\xi^4]),-r^4) \\ \sigma(-\overline{U}(x,y,[\xi^5]),-r^5) \end{bmatrix} && \text{(EX-MB-STP)} \\
\text{s.t.} \quad & x_1+x_2+x_3+x_4 = 100 \\
& y_1+y_2+y_3+y_4 = 100 \\
& x_1,x_2,x_3,x_4 \geq 100/8 && \text{(Feasibility.)} \\
& y_1,y_2,y_3,y_4 \geq 100/6 \\
& r^i = \sum_{j=1}^{k_i} \beta^i_j p^i_j, \qquad i=1,2,3,4,5 && \text{(Reference point selection.)} \\
& \sum_{j=1}^{k_i} \beta^i_j = 1, \qquad i=1,2,3,4,5 \\
& \beta^i_j \in \{0,1\}, \qquad j=1,\dots,k_i, \qquad i=1,2,3,4,5
\end{aligned}$$

Remark 6.2. Observe that negative signs are needed in the arguments of the BASFs since the master problem is formulated in terms of a maximization, while the canonical BASF is formulated for a minimization problem.

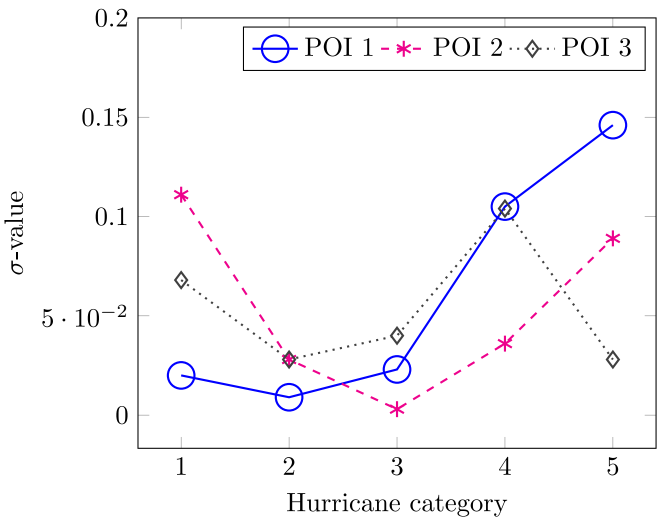
All computational work was completed in `Julia` (Bezanson et al. 2017) using the `JuMP.jl` package (Lubin et al. 2023) with the `Gurobi` solver (Gurobi Optimization, LLC 2026). The discretizations of the Pareto sets of each subproblem were found using the weighted-sum method.

6.2 Marginal Analysis of Three Alternatives

Using a suitable MCDM/A tool, the DM selects three points of interest from the Pareto set of (EX-MB-STP). (These points were also found using the weighted-sum method.) The parallel coordinate plot of these points is found in Figure 2a, while Table 2b lists the ASF values and the allocations of goods and labor to each community corresponding to these three points. Note that the allocations of goods and labor are efficient solutions for (EX-MB-STP) and are thereby weakly efficient-under-uncertainty solutions for (EX-UMOP).

First, we consider POI 1. In subproblem 5, the σ-value is largest, which means that the DM is to expect that POI 1 performs poorly with respect to Category 5 hurricanes, when the wind speeds are between 157 and 177 miles per hour. As for POI 2, sacrifices in performance in subproblem 1 and subproblem 2, as compared to POI 1, result in performance gains in subproblems 3 through 5. Finally, in POI 3, sacrifices in subproblems 1 through 3 also result in improvements in subproblem 5, as compared to POI 1.

This situation is well-suited for a marginal analysis of these three points. Table 3 lists the marginal tradeoffs between the three points of interest with respect to subproblem 1. In particular, Table 3 shows that the gain in performance by choosing POI 3 over POI 1, with respect to Category 5 hurricanes is 245.8% of the loss in performance with respect to Category 1 hurricanes. This means that in comparing POI 1 and POI 3, if the DM is willing to sacrifice performance with respect to Category 1 hurricanes, she stands to gain substantially with respect to Category 5 hurricanes.



(a) Parallel coordinate plot for three Pareto points of (EX-MB-STP), which were identified by the DM.

	σ_1	σ_2	σ_3	σ_4	σ_5
POI 1	0.02	0.009	0.023	0.105	0.146
POI 2	0.111	0.028	0.003	0.036	0.089
POI 3	0.068	0.028	0.04	0.104	0.028
	x_1	x_2	x_3	x_4	-
POI 1	12.5	29.414	27.809	30.278	-
POI 2	16.262	52.417	15.283	16.038	-
POI 3	23.995	31.749	31.755	12.5	-
	y_1	y_2	y_3	y_4	-
POI 1	16.667	19.648	20.672	43.014	-
POI 2	30.925	21.214	16.667	31.195	-
POI 3	50.0	16.667	16.667	16.667	-

(b) Pareto outcomes and efficient solutions corresponding to three points of interest.

Figure 2: Data pertaining to three points of interest identified by the DM by using a suitable MCDM tool on (EX-MB-STP).

$\Delta\mathcal{ST}_{i1}(\text{POI } 1, \text{POI } 2)$	$[\xi^1]$
$[\xi^2]$	0.209
$[\xi^3]$	0.22
$[\xi^4]$	0.758
$[\xi^5]$	0.626

(a) Marginal tradeoffs between POI 1 and POI 2.

$\Delta\mathcal{ST}_{i1}(\text{POI } 1, \text{POI } 3)$	$[\xi^1]$
$[\xi^2]$	0.396
$[\xi^3]$	0.354
$[\xi^4]$	0.021
$[\xi^5]$	2.458

(b) Marginal tradeoffs between POI 1 and POI 3.

$\Delta\mathcal{ST}_{i1}(\text{POI } 2, \text{POI } 3)$	$[\xi^1]$
$[\xi^2]$	0.0
$[\xi^3]$	0.86
$[\xi^4]$	1.581
$[\xi^5]$	1.419

(c) Marginal tradeoffs between POI 2 and POI 3.

Table 3: Marginal tradeoffs between points of interest.

7 Conclusion

The purpose of this paper was to show how a decision maker's preferences, experience, and expertise can be used to manage the uncertainty in robust multiobjective optimization problems. The results shown here open the possibility for further investigations, and we present four open questions.

First, we need an understanding of the relationship between reference points chosen in the aggregated space of $g(x, [\xi^i])$ and the original space of $f(x, \xi^i)$. Since these reference points correspond to a decision maker's aspirations, is there a coherent way to transform reference points between perspectives? *I.e.*, if a reference point is selected with respect to $f(x, \xi^i)$, how can a corresponding reference point with respect to $g(x, [\xi^i])$ be selected and *vice versa*?

Second, we may observe that regardless of the robust efficiency concept used, the inclusion of the decision maker's own preferences, experiences, and expertise in categorizing the possible scenarios can still provide useful insights. As such, there is a need to understand how a robust efficient solution in one equivalence class performs in another equivalence class. Put another way, according to a chosen concept of robust efficiency, under what circumstances is a robust efficient solution preserved between equivalence classes? When does robust efficiency for an equivalence class translate to robust efficiency over the entire uncertainty set? Continuing the analogy of decomposition and coordination from de Castro and Wiecek 2025, it is likely that robustness itself conflicts between equivalence classes; *i.e.*, a robust efficient solution for one equivalence class may very well not be robustly efficient for another. This intuition shows that there is also a need to develop coordination for robustness between equivalence classes of scenarios.

Third, the key mathematical insight for incorporating a decision maker's preferences, experiences, and expertise in handling uncertainty was based upon the notion of a quotient space from topology. We thus needed a function that would ensure that the quotient space diagram would commute, and integration over the equivalence classes is a natural way of building such a function. Future work could include the exploration of other types of functions that can guarantee commutativity and their effects on the decision-making process.

Finally, the requirement of an equivalence relation on the uncertainty set can be restrictive. After all, it requires a decision maker to have well-understood preferences on the set of scenarios, which may be difficult to define even with suitable preference elucidation tools. As such, it would be valuable to fine-tune the approach presented in this work by allowing the required preference structure on the uncertainty set to be relaxed, while still providing much-needed insights into the tradeoffs between scenarios.

Artificial Intelligence Usage Declaration. The author alone developed the idea for this paper. All prose, including the literature review, and proofs in this work were completely and solely written by the author. Artificial Intelligence tools were used only to check for typographical errors and to verify correctness of results. All edits were done completely and solely by the author.

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