A sufficient convergence condition for generalized Benders decomposition with general dual functions

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ABSTRACT

We revisit the framework of generalized Benders decomposition over a compact but non-finite master domain. We show by counterexample that strong general dual functions alone may fail to guarantee convergence. We then define a condition of uniform local strongness and prove that strong general dual functions satisfying this condition guarantee finite ϵ-termination. Finally, we show that this condition is automatically satisfied when the master variables take values in a finite set, thereby recovering the usual finite-domain convergence argument.

1. Introduction

Benders decomposition is a classical method for solving optimization problems with a decomposable structure. In the standard setting in Benders (1962), the subproblem is a linear programming problem, or more generally a convex optimization problem, as in Geoffrion (1972). The original problem is decomposed into a master problem and one or more subproblems. The master problem determines values of the complicating variables and provides a candidate solution and a lower bound; fixing these variables then defines the corresponding parameterized subproblems, whose solutions are used to evaluate the candidate solution and refine the master problem. In this case, strong duality provides valid dual cuts for the master problem. When the subproblem is non-convex, however, strong duality generally does not hold, and classical linear Benders cuts cannot be directly obtained from the dual of the subproblem.

Several extensions of Benders decomposition have been proposed for problems with general subproblems. Hooker and Ottosson (2003) introduced logic-based Benders decomposition (LBBD), where the classical LP dual is replaced by a more general inference dual. Their convergence proof relies on the assumption that the domain of the master variables is finite. Bolusani and Ralphs (2022) extended the LBBD and proposed a generalized Benders decomposition framework based on general dual functions. In this framework, the value function of the subproblem is approximated from below by general dual functions that are *strong* at the previously generated master solutions.

The finite-domain assumption is natural when all master variables are integer and bounded. However, it does not cover the case where some master variables are continuous. In this case, even if every generated general dual function is valid and strong at its generating point, the algorithm may still fail to converge to the true optimal value.

The contributions of this note are as follows:

1. We give a counterexample showing that general dual functions that are strong at their generating points do not guarantee convergence on a compact but non-finite master domain.
2. We introduce a *uniform local strongness at generating points* condition. This condition requires the generated general dual functions to remain uniformly well-behaved near the points where they are generated.
3. We prove that, under this condition, the generalized Benders decomposition framework finitely terminates with an ϵ-optimal solution for any $\epsilon > 0$.
4. We show that this condition is automatically satisfied when the master domain is finite, which recovers the usual finite-domain convergence argument.

This paper is organized as follows. Section 2 introduces the generalized Benders decomposition framework proposed by Bolusani and Ralphs (2022). Section 3 presents a counterexample showing that strong dual functions alone do not guarantee finite convergence to optimality. Section 4 introduces the notion of uniform local strongness at generating points and proves finite ϵ-convergence under this condition. Section 5 concludes the paper.

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2. Framework of generalized Benders decomposition

Generalized Benders decomposition considers optimization problems of the form

$$\begin{aligned} &\text{minimize} && f(x) && (1)\\ &\text{subject to} && x \in X, && (2) \end{aligned}$$

where $X \subseteq \mathbb{R}^n$ is a non-empty set and $f : X \to \mathbb{R} \cup \{+\infty\}$ a lower semi-continuous function. Function $f(x)$ may not have a tractable closed-form expression. For example, $f(x)$ may be the optimal value of a mixed-integer linear programming problem parameterized by x, and $f(x) = +\infty$ corresponds to the case where the parameterized problem is infeasible. The idea of generalized Benders decomposition is to approximate f from below by a sequence of tractable lower-bounding functions.

Throughout this paper, we assume that a finite initial lower bound L_0 is available, i.e.,

$$L_0 \leq f(x), \qquad \forall x \in X.$$

This assumption ensures that the problem is bounded from below.

The following definition is adapted from Bolusani and Ralphs (2022).

Definition 1 (General dual function)**.** *Let $f : X \to \mathbb{R} \cup \{+\infty\}$, and let $\phi : X \times X \to \mathbb{R} \cup \{+\infty\}$. For a given $\hat{x} \in X$, the partial function $\phi(\cdot, \hat{x})$ is called a general dual function of f generated at $\hat{x}$ if*

$$\phi(x, \hat{x}) \leq f(x), \qquad \forall x \in X.$$

It is called strong at $\hat{x}$ if

$$\phi(\hat{x}, \hat{x}) = f(\hat{x}).$$

Moreover, $\phi(\cdot, \cdot)$ is called a general dual function strong at its generating points if, for all $\hat{x} \in X$, its partial function $\phi(\cdot, \hat{x})$ is strong at $\hat{x}$.

The original problem (1)-(2) is then equivalent to the reformulation:

$$\begin{aligned} &\inf_{x,\eta} && \eta && && (3)\\ &\text{subject to} && \eta \geq L_0, && && (4)\\ & && \eta \geq \phi(x, \hat{x}), && \forall \hat{x} \in X && (5)\\ & && x \in X. && && (6) \end{aligned}$$

where ϕ is a general dual function strong at its generating points and L_0 a valid lower bound of the original problem.

This reformulation includes infinitely many constraints. Therefore, we consider only a finite subset of the constraints generated by ϕ in the master problem. The subproblem consists of evaluating the actual value of f and adding new cuts to the master problem.

At iteration k, suppose that k general dual functions $\phi(\cdot, \hat{x}^1), \dots, \phi(\cdot, \hat{x}^k)$ have been generated. The master problem is

$$\begin{aligned} \mathbf{MP}^k \quad &\inf_{x,\eta} && \eta && && (7)\\ &\text{subject to} && \eta \geq L_0, && && (8)\\ & && \eta \geq \phi(x, \hat{x}^i), && i = 1, \dots, k, && (9)\\ & && x \in X. && && (10) \end{aligned}$$

When $k = 0$, no generated Benders cut is present and the master problem only contains the initial lower bound L_0.

The generalized Benders framework can be summarized as Algorithm 1.

This algorithm is known to converge finitely when X is finite; see, e.g., Hooker and Ottosson (2003). However, the finite-domain argument does not directly extend to the case where X is compact but infinite.

Algorithm 1 Generalized Benders framework

Initialize $k \leftarrow 0$, $LB \leftarrow L_0$, and $UB \leftarrow +\infty$.
repeat
  Solve $\mathbf{MP}^k$. Let $\hat{\eta}^k$ be its optimal value.
  if $\hat{\eta}^k = +\infty$ **then**
    Set $x^\star \leftarrow$ None and $LB \leftarrow +\infty$.
    break
  end if
  Let $\hat{x}^{k+1}$ be an optimal solution of $\mathbf{MP}^k$.
  Set $LB \leftarrow \hat{\eta}^k$.
  Evaluate $f(\hat{x}^{k+1})$.
  Generate a general dual function $\phi(\cdot, \hat{x}^{k+1})$ strong at $\hat{x}^{k+1}$, and add the cut

$$\eta \geq \phi(x, \hat{x}^{k+1}).$$

  if $f(\hat{x}^{k+1}) < UB$ **then**
    Set $x^\star \leftarrow \hat{x}^{k+1}$.
    Set $UB \leftarrow f(x^\star)$.
  end if
  Set $k \leftarrow k + 1$.
until $LB = +\infty$ or $UB - LB \leq \epsilon$

3. Counterexample for the framework of GBD

Consider the minimization problem of a one-dimensional concave function:

$$\text{minimize} \quad 1 - \frac{x^2}{2} \tag{11}$$

$$x \in [0, 1] \tag{12}$$

The optimal solution is $x^\star = 1$ with optimal value $f^\star = f(1) = \frac{1}{2}$.

This is a non-convex optimization problem. Not all points in $[0, 1]$ admit a strong dual linear cut. Therefore, we propose a collection of strong general dual functions using norm functions.

$$\phi(x, \hat{x}) = \begin{cases} f(\hat{x}) - \frac{4}{1-2\hat{x}}|x - \hat{x}|, & 0 \leq \hat{x} < \frac{1}{2}, \\ f(\hat{x}) - |x - \hat{x}|, & \frac{1}{2} \leq \hat{x} \leq 1. \end{cases}$$

It is easy to verify that the function $\phi(\cdot, \hat{x})$ is a strong general dual function of f. Indeed, f is 1-Lipschitz on $[0, 1]$. Therefore,

$$f(x) \geq f(\hat{x}) - |x - \hat{x}|, \qquad \forall x, \hat{x} \in [0, 1] \tag{13}$$

$$f(x) \geq f(\hat{x}) - \frac{4}{1 - 2\hat{x}}|x - \hat{x}|, \qquad \forall x \in [0, 1], \forall \hat{x} \in [0, \frac{1}{2}) \tag{14}$$

So we have

$$\phi(x, \hat{x}) \leq f(x), \qquad \forall x \in [0, 1], \ \forall \hat{x} \in [0, 1].$$

Moreover, $\phi(\hat{x}, \hat{x}) = f(\hat{x})$.

Let us start the generalized Benders decomposition algorithm. The master problem $\mathbf{MP}^k$ is formulated as (7) – (10) with $L_0 = 0$. We suppose that, if $\mathbf{MP}^k$ has more than one optimal solution, the smallest value of x is selected as $\hat{x}^{k+1}$.

Figure 1 illustrates the behavior of the generated cuts. In each figure, the black curve represents the value function $f(x) = 1 - x^2/2$, the dashed horizontal line is the initial lower bound $L_0 = 0$, the thin curves are the generated dual functions, and the thick red curve is the current lower approximation.

The cuts become increasingly concentrated near $x = 1/2$, while the current lower bound remains equal to the initial lower bound $L_0 = 0$. This observation suggests that strongness at the generating points alone is not sufficient to ensure convergence. We now prove this statement formally.

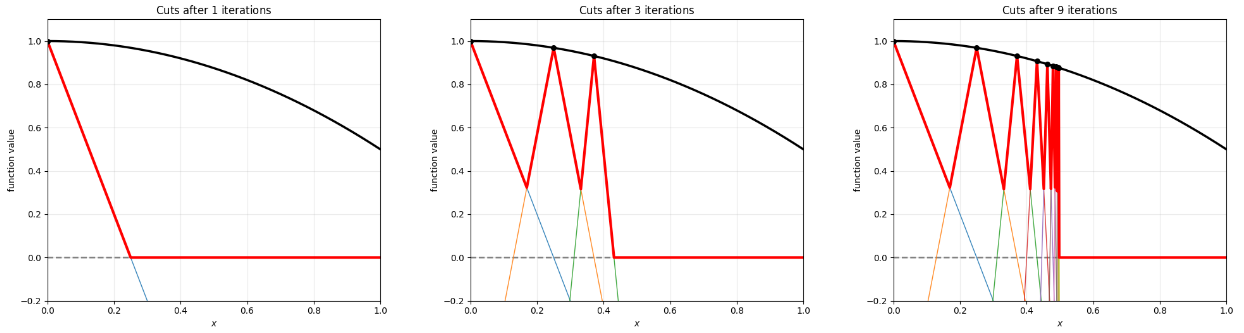

Fig. 1. Generated dual functions and aggregated lower bounds after $n = 1, 3, 9$ iterations.

Lemma 2. *For a given $k \in \mathbb{N}_+$ such that $\hat{x}^k < \frac{1}{2}$, let*

$$S^k = \{x \in [0,1] : \phi(x, \hat{x}^k) > 0\}.$$

Then $S^k \subseteq [0, \frac{1}{4} + \frac{\hat{x}^k}{2}] \subseteq \left[0, \frac{1}{2}\right)$.

Proof. By definition of ϕ, if $x \in S^k$, then $f(\hat{x}^k) - \frac{4}{1-2\hat{x}^k}|x - \hat{x}^k| > 0$. Hence,

$$|x - \hat{x}^k| < \frac{f(\hat{x}^k)(1 - 2\hat{x}^k)}{4}.$$

Since $f(\hat{x}^k) \le 1$, we obtain

$$x \le \hat{x}^k + \frac{1 - 2\hat{x}^k}{4} = \frac{1}{4} + \frac{\hat{x}^k}{2} < \frac{1}{2}.$$

Therefore, $S^k \subseteq [0, \frac{1}{4} + \frac{\hat{x}^k}{2}] \subseteq \left[0, \frac{1}{2}\right)$. □

Proposition 3. *Let $k \ge 0$ be an integer representing the iteration number. With the strong general dual function ϕ defined above, the optimal solution of* ***MP***^{k} *satisfies*

$$\hat{x}^{k+1} < \frac{1}{2}.$$

Proof. We prove this proposition by induction. At the first iteration, the master problem only contains the initial lower bound 0. Hence every point in $[0, 1]$ is optimal for the master problem. By the assumption, the smallest optimal solution is selected, and therefore $\hat{x}^1 = 0 < \frac{1}{2}$. Assume now that

$$\hat{x}^i < \frac{1}{2}, \qquad i = 1, \dots, k.$$

By the lemma, we have

$$S^i \subseteq [0, \frac{1}{4} + \frac{\hat{x}^i}{2}] \subseteq \left[0, \frac{1}{2}\right), \qquad i = 1, \dots, k.$$

Let $r_k = \max_{1 \le i \le k}\{\frac{1}{4} + \frac{\hat{x}^i}{2}\}$. Therefore,

$$\bigcup_{i=1}^{k} S^i \subseteq [0, r_k] \subseteq \left[0, \frac{1}{2}\right).$$

Choose any point $\bar{x} \in (r_k, \frac{1}{2})$. Then, $\bar{x} \notin \bigcup_{i=1}^{k} S^i$. For this point, we have

$$\phi(\bar{x}, \hat{x}^i) \le 0, \qquad i = 1, \dots, k.$$

Therefore, the current lower approximation satisfies

$$\inf_{x \in X} \max \left\{0, \phi(x, \hat{x}^1), \dots, \phi(x, \hat{x}^k)\right\} = 0.$$

This implies that $\bar{x}$ is an optimal solution of the master problem $\mathbf{MP}^k$. By the assumption, the next generated point satisfies $\hat{x}^{k+1} \le \bar{x} < \frac{1}{2}$. This completes the induction. □

By this proposition, the generalized Benders decomposition algorithm with this strong general dual function never generates a point in $[\frac{1}{2}, 1]$. Hence, at every finite iteration $k \ge 1$, by the monotonicity of f,

$$f(\hat{x}^k) > f\left(\frac{1}{2}\right) = \frac{7}{8}.$$

Since the upper bound is obtained from the best generated feasible solution, we have $UB_k > \frac{7}{8}$. At the same time, the master lower bound remains equal to 0 at every iteration. Thus the optimality gap does not converge to zero, even though every generated general dual function is valid and strong at its generating point.

4. Uniform local strongness

In the counterexample, we may intuitively attribute the non-convergence to the shrinking "dominant region" of the general Benders cut. Namely, the reverse cone excluded from the epigraph becomes narrower and narrower as the generating point approaches $\frac{1}{2}$. This suggests that strongness at the generating point is not sufficient: the cut should also provide a uniform local approximation around its generating point.

We now define this property formally.

Definition 4 (Uniform local strongness)**.** *Let (X, d) be a metric space, and let*

$$\phi : X \times X \to \mathbb{R} \cup \{+\infty\}$$

be a general dual function strong at its generating points.

We say that ϕ is uniformly locally strong at its generating points *on X if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $\hat{x} \in X$ and every $x \in X$ with $d(x, \hat{x}) < \delta$,*

1. *if $f(x) < +\infty$, then $f(x) - \phi(x, \hat{x}) < \varepsilon$;*
2. *if $f(x) = +\infty$, then $\phi(x, \hat{x}) = +\infty$.*

Theorem 5. *Let (X, d) be a compact metric space and let $f : X \to \mathbb{R} \cup \{+\infty\}$. Suppose that, whenever $\boldsymbol{MP}^k$ has a finite optimal value, it admits an optimal solution $\hat{x}^{k+1} \in X$, and a general dual function $\phi(\cdot, \hat{x}^{k+1})$ strong at $\hat{x}^{k+1}$ is generated.*

If the general dual function ϕ is uniformly locally strong at the generating points on X, then for any tolerance $\varepsilon > 0$, Algorithm 1 either finitely proves that $f \equiv +\infty$ on X, or terminates after finitely many iterations with an ε-optimal solution.

Proof. Fix $\varepsilon > 0$. By uniform local strongness at the generating points, there exists $\delta > 0$ such that, for every $j \ge 1$ and every $x \in X$,

$$d(x, \hat{x}^j) < \delta \quad \Longrightarrow \quad \begin{cases} f(x) - \phi(x, \hat{x}^j) < \varepsilon, & \text{if } f(x) < +\infty, \\ \phi(x, \hat{x}^j) = +\infty, & \text{if } f(x) = +\infty. \end{cases}$$

Since (X, d) is compact, it is totally bounded. Hence, X can be covered by finitely many open balls of radius $\delta/2$. Therefore, the algorithm cannot generate infinitely many pairwise δ-separated points.

Suppose first that $f \equiv +\infty$ on X. As long as the master lower bound is finite, a newly generated point $\hat{x}^{i+1}$ must satisfy

$$d(\hat{x}^{i+1}, \hat{x}^j) \geq \delta, \qquad j = 1, \dots, i.$$

Indeed, otherwise uniform local strongness would imply

$$\phi(\hat{x}^{i+1}, \hat{x}^j) = +\infty,$$

while the corresponding cut is already contained in $\mathbf{MP}^i$. Since X cannot contain infinitely many pairwise δ-separated points, the master lower bound becomes $+\infty$ after finitely many iterations. Since all generated dual functions are valid lower bounds of f, this proves that $f \equiv +\infty$ on X.

Suppose now that there exists $\bar{x} \in X$ such that $f(\bar{x}) < +\infty$. By validity of the generated dual functions,

$$\phi(\bar{x}, \hat{x}^j) \leq f(\bar{x}) < +\infty,$$

so every master problem has a finite optimal value. By total boundedness, after finitely many iterations there exist $j \leq i$ such that

$$d(\hat{x}^{i+1}, \hat{x}^j) < \delta.$$

Since the optimal value of $\mathbf{MP}^i$ is finite, we must have $f(\hat{x}^{i+1}) < +\infty$. Otherwise, uniform local strongness would imply

$$\phi(\hat{x}^{i+1}, \hat{x}^j) = +\infty,$$

which contradicts the fact that $\hat{x}^{i+1}$ is an optimal solution of $\mathbf{MP}^i$ with a finite optimal value.

The cut generated at $\hat{x}^j$ is already contained in $\mathbf{MP}^i$, and hence

$$LB = \hat{\eta}^i \geq \phi(\hat{x}^{i+1}, \hat{x}^j).$$

Moreover, $UB \leq f(\hat{x}^{i+1})$. Therefore,

$$UB - LB \leq f(\hat{x}^{i+1}) - \phi(\hat{x}^{i+1}, \hat{x}^j) < \varepsilon.$$

Thus Algorithm 1 terminates after finitely many iterations with an ε-optimal incumbent solution. □

The condition of uniform local strongness at generating points has a simple interpretation. For each generating point $\hat{x}$, the corresponding general dual function must provide a uniformly accurate local representation of f in a neighborhood of $\hat{x}$. More precisely, for points x with finite $f(x)$, the cut value $\phi(x, \hat{x})$ must remain within ϵ of $f(x)$; for points with $f(x) = +\infty$, the cut must also satisfy $\phi(x, \hat{x}) = +\infty$. Hence, for each $\epsilon > 0$, there must exist one neighborhood size $\delta > 0$ that works uniformly for all possible generating points.

However, applying this theorem may be problematic in practice, since verifying its condition requires evaluating the values of f on the neighborhoods of the generating points, which is generally intractable and problem-specific. In the following two subsections, we introduce two special cases where the uniform local strongness can be inferred solely from ϕ, without explicitly evaluating the value of f on any open neighborhood.

4.1. Finite master problem domain

If the master problem domain is finite, i.e. $|X| < \infty$, uniform local strongness holds automatically.

Corollary 6. *If $|X| < \infty$, then any general dual function ϕ which is strong at its generating points is automatically uniformly locally strong at its generating points.*

Proof. We omit the trivial case where $|X| = 1$. Since X is finite, there exists

$$\rho := \min\{d(x, y) : x, y \in X,\ x \neq y\} > 0.$$

For any $\varepsilon > 0$, choose $\delta = \rho/2$. Then, for every $\hat{x} \in X$, the condition $d(x, \hat{x}) < \delta$ implies $x = \hat{x}$.

If $f(\hat{x}) < +\infty$, then

$$f(x) - \phi(x, \hat{x}) = f(\hat{x}) - \phi(\hat{x}, \hat{x}) = 0 < \varepsilon.$$

If $f(\hat{x}) = +\infty$, strongness implies

$$\phi(x, \hat{x}) = \phi(\hat{x}, \hat{x}) = +\infty.$$

Hence ϕ is uniformly locally strong at its generating points. □

This corollary covers the case considered in Hooker and Ottosson (2003) and Bolusani and Ralphs (2022). It also explains why uniform local strongness need not be imposed explicitly in those frameworks.

4.2. Lipschitz function f

If f is Lipschitz continuous, which implies that $f(x) < +\infty$ for all $x \in X$, uniform local strongness can be established by verifying only local properties of ϕ, which is generally more tractable than evaluating f itself.

Definition 7 (Equicontinuity at generating points). *Let (X, d) be a metric space, and let*

$$\phi : X \times X \to \mathbb{R}.$$

We say that ϕ is equicontinuous at generating points *if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $\hat{x} \in X$ and every $x \in X$,*

$$d(x, \hat{x}) < \delta \Longrightarrow |\phi(x, \hat{x}) - \phi(\hat{x}, \hat{x})| < \varepsilon.$$

Theorem 8. *Let f be an L-Lipschitz function on X, with $L > 0$, and let ϕ be a general dual function of f strong at its generating points and equicontinuous at generating points. Then ϕ is uniformly locally strong at its generating points.*

Proof. For any $\varepsilon > 0$, by equicontinuity at generating points, there exists $\delta_1 > 0$ such that

$$d(x, \hat{x}) < \delta_1 \Longrightarrow |\phi(x, \hat{x}) - \phi(\hat{x}, \hat{x})| < \frac{\varepsilon}{2}.$$

Let $\delta_2 = \frac{\varepsilon}{2L}$. Since f is L-Lipschitz,

$$d(x, \hat{x}) < \delta_2 \Longrightarrow |f(x) - f(\hat{x})| < \frac{\varepsilon}{2}.$$

Taking $\delta = \min\{\delta_1, \delta_2\}$, and using the strongness $\phi(\hat{x}, \hat{x}) = f(\hat{x})$, we obtain, whenever $d(x, \hat{x}) < \delta$,

$$\begin{aligned} f(x) - \phi(x, \hat{x}) &= f(x) - f(\hat{x}) + \phi(\hat{x}, \hat{x}) - \phi(x, \hat{x}) \\ &\leq |f(x) - f(\hat{x})| + |\phi(\hat{x}, \hat{x}) - \phi(x, \hat{x})| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence ϕ is uniformly locally strong at its generating points. □

Intuitively, the equicontinuity at generating points means that, if we translate the generating points of all general dual functions to a common position, they should remain uniformly close to each other near this common location.

This covers the case introduced in Ahmed, Cabral and Freitas Paulo da Costa (2022), where reverse norm cuts are used as general Benders cuts.

Remark 1. In the counterexample of Section 3, the function f is $L-$Lipschitz on $[0, 1]$, but the general dual function ϕ is not equicontinuous at the generating points. Indeed, for $\hat{x} < \frac{1}{2}$, the slope of the generated function is $\frac{4}{1-2\hat{x}}$, which tends to $+\infty$ as $\hat{x}$ approaches $\frac{1}{2}$ from the left. Hence no uniform neighborhood of the generating point can be found for all possible $\hat{x} \in [0, 1]$.

If we use the following general dual function instead

$$\phi(x, \hat{x}) = \ f(\hat{x}) - \rho|x - \hat{x}|,$$

where ρ is a constant bigger than the Lipschitz coefficient of f, this function is then equicontinuous at generating points and guarantees the convergence. This becomes the Lipschitz cut proposed in Ahmed et al. (2022).

5. Conclusion

In this article, we revisited the framework of generalized Benders decomposition proposed by Bolusani and Ralphs (2022). We first provided a counterexample showing that strong general dual functions alone do not guarantee convergence when the master domain is compact but infinite. In particular, the generated cuts may be strong at their generating points while the master lower bound still fails to converge to the true optimal value.

We then introduced the notion of uniform local strongness at generating points for general dual functions. We proved that, if the master domain is compact and the generated strong general dual functions satisfy this property, then the generalized Benders framework finitely terminates with an ϵ-optimal solution for any $\epsilon > 0$. We also showed that this additional condition is automatically satisfied when the master variable region is finite, thereby recovering the usual finite-domain convergence argument. Moreover, it is implied when the objective function is Lipschitz continuous and the strong general dual functions are equicontinuous at their generating points.

It should be emphasized that the condition proposed in this article is only sufficient, not necessary. Other types of cuts may also lead to convergence, including non-strong cuts or cuts that are not uniformly locally strong, but their convergence generally requires a problem-specific analysis. For instance, the non-strong scaled cuts studied in van der Laan and Romeijnders (2024) provide one such example.

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Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During the preparation of this work, the author used ChatGPT (OpenAI) to assist with language editing, manuscript preparation, and reviewing mathematical arguments and proofs. After using this tool, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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