

# On the Chvátal–Gomory Closure of Integer Quadratic Programs

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## Abstract

The integer quadratic moment hull is the closed convex hull of the pairs  $(z, zz^\top)$  with  $z \in \mathbb{Z}^n$ . We prove that one round of Chvátal–Gomory (CG) cuts applied to its basic semidefinite relaxation recovers this hull exactly if and only if  $n \leq 6$ . Our approach identifies its facets with perfect Delaunay polyhedra and characterizes their recovery through depth, the gap between lattice and continuous minima of a defining quadratic after normalization. A facet inequality is valid for the rank-one CG closure exactly when its depth is less than one. We also analyze the complexity of deciding membership in the CG closure and finding violated inequalities. Every rational point outside the CG closure admits a violated CG inequality whose encoding length is polynomial in the input length. Membership is strongly coNP-complete and exact separation is strongly NP-hard. The same reduction proves strong NP-hardness of quadratic split-inequality separation, resolving a conjecture of Burer and Letchford.

**Keywords.** integer quadratic moment hull; Chvátal–Gomory closure; semidefinite relaxation; perfect Delaunay polyhedron; Erdahl cone; separation complexity

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## 1 Introduction

Integer quadratic optimization is an important class of nonlinear integer programming problems. Consider any quadratic function  $q : \mathbb{R}^n \rightarrow \mathbb{R}$ ,

$$q(z) = z^\top Az + 2b^\top z + \alpha,$$

where  $A \in \mathbb{R}^{n \times n}$  and is symmetric,  $b \in \mathbb{R}^n$ , and  $\alpha \in \mathbb{R}$ . In this work, we focus on optimization of  $q$  over  $z \in \mathbb{Z}^n$ , namely  $\inf_{z \in \mathbb{Z}^n} q(z)$  or  $\sup_{z \in \mathbb{Z}^n} q(z)$ . We impose no convexity or concavity assumption on  $q$ . We use moment lifting to represent each quadratic objective as a linear functional of a matrix containing the first- and second-order terms. The closed convex hull of the lifted integer points then provides a common convex set for both minimization and maximization.

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## 1.1 The moment hull and its relaxations

Throughout, we use  $\mathbb{R}$ ,  $\mathbb{Q}$ , and  $\mathbb{Z}$  to denote the real numbers, rational numbers, and integers, respectively. Let  $\mathbb{S}^n$  denote the space of real symmetric  $n \times n$  matrices for every positive  $n \in \mathbb{Z}$ , and let  $\langle A, B \rangle = \text{tr}(AB)$  denote the trace inner product. For any  $z \in \mathbb{R}^n$ , its lifted moment matrix is defined by

$$W_z := \begin{bmatrix} 1 \\ z \end{bmatrix} \begin{bmatrix} 1 \\ z \end{bmatrix}^\top = \begin{bmatrix} 1 & z^\top \\ z & zz^\top \end{bmatrix}.$$

For the arbitrary quadratic function  $q(z) = z^\top Az + 2b^\top z + \alpha$ , let

$$M_q := \begin{bmatrix} \alpha & b^\top \\ b & A \end{bmatrix},$$

then

$$q(z) = \langle M_q, W_z \rangle.$$

Thus, moment lifting equivalently expresses a quadratic function over  $z \in \mathbb{R}^n$  by a linear function over lifted matrix  $W_z$ .

Specifically, when  $z$  is restricted to  $\mathbb{Z}^n$ , moment lifting naturally gives rise to the set of all lifted matrices associated with  $z \in \mathbb{Z}^n$ . We refer to the closure of its convex hull by *integer quadratic moment hull* and denote it by

$$\text{MIQ}_n := \text{cl conv}\{W_z : z \in \mathbb{Z}^n\}$$

throughout this work. Our notation  $\text{MIQ}_n$  corresponds to the purely integer case  $\text{MIQ}_{n,0}$  of Burer and Letchford [1].

Throughout, we index the rows and columns of an  $(n+1) \times (n+1)$  moment matrix by  $\{0, 1, \dots, n\}$ . We let

$$\mathcal{H}_n := \{W \in \mathbb{S}^{n+1} : W_{00} = 1\},$$

and denote the semidefinite relaxation of  $\text{MIQ}_n$  by

$$\mathcal{P}_n := \{W \in \mathcal{H}_n : W \succeq 0\}.$$

Every lattice moment matrix  $W_z$  is positive semidefinite, so

$$\text{MIQ}_n \subseteq \mathcal{P}_n \subseteq \mathcal{H}_n.$$

Our focus is the *rank-one Chvátal–Gomory (CG) closure*  $\mathcal{C}_n$  of  $\mathcal{P}_n$ , obtained by applying the CG procedure once in the natural integral moment coordinates. To assess the strength of CG rounding, we compare  $\mathcal{C}_n$  with the *split-inequality closure*  $\mathcal{S}_n$ , defined by the quadratic split inequalities introduced by Burer and Letchford [1]. Section 2.4 gives the formal definitions. These sets satisfy

$$\text{MIQ}_n \subseteq \mathcal{C}_n \subseteq \mathcal{S}_n \subsetneq \mathcal{P}_n.$$

This chain gives a baseline comparison. However, we would like to further explore the following research questions. First, how far is  $\mathcal{C}_n$  from fully recovering  $\text{MIQ}_n$ , and how does  $\mathcal{C}_n$  compare with  $\mathcal{S}_n$ ? Second, how hard is exact separation over  $\mathcal{C}_n$  when the dimension is part of the input? We first review the relevant literature.

## 1.2 Related literature

Quadratic optimization with integer decisions arises in wide-ranging applications, such as portfolio selection and facility location [2], unit commitment [3], sparse regression [4], and learning Gaussian Bayesian networks [5]. The breadth of these applications has sustained interest in the geometry of quadratic integer optimization, with convex-hull descriptions and valid inequalities providing a natural bridge from discrete structure to stronger continuous relaxations.

Several earlier works study quadratic optimization by lifting first and second moments. For binary variables, Padberg’s Boolean quadric polytope provides the basic polyhedral model for standard quadratic linearizations [6]. For unconstrained quadratic 0–1 minimization, Boros, Crama, and Hammer gave a polynomial-size linear programming formulation of the first Chvátal closure of the standard linearization relaxation. They also related this closure to odd-cycle inequalities through a signed-graph reformulation [7]. Burer gave an exact formulation over the completely positive cone for a broad class of nonconvex quadratic programs with binary and nonnegative continuous variables [8]. Anstreicher and Burer obtained exact representations of moment convex hulls over several classes of low-dimensional polytopes [9]. Moreover, Burer and Letchford developed a moment-hull framework for mixed-integer quadratic programming [1].

For the purely integer moment hull  $\text{MIQ}_n$ , Burer and Letchford proved that their quadratic split inequalities dominate the positive-semidefinite inequalities [1, Theorem 4] and completely describe the moment hull in dimension two:  $\text{MIQ}_2 = \mathcal{S}_2$  [1, Theorem 8]. In particular, they left open whether split inequalities suffice in any dimension above two. In an attempt to address this, in dimension six, they provided a bounded facet whose defining inequality is not a split inequality and concluded that split inequalities do not completely describe the hull [1, Section 5.3]. The authors also suggested a connection between facets of the integer quadratic moment hull and the structures called Delaunay polytopes. We include relevant background information on them in Section 2.2. Our geometric approach in this work is inspired by this connection. Buchheim and Traversi later recorded that exactness still remained unresolved in dimensions three through five [10, Section 3, discussion following Theorem 4]. They gave an exact separation method for these inequalities and evaluated it computationally. For positive-semidefinite input, the separation reduces to convex quadratic integer minimization. In addition, for input outside the positive-semidefinite cone, their Algorithm 1 finds a violated split inequality in polynomial time [10, Section 4, Theorem 5].

The quadratic split inequalities discussed above use the same lattice disjunctions as classical split cuts for mixed-integer linear programs. Cook, Kannan, and Schrijver introduced split cuts for rational mixed-integer polyhedra and proved that the resulting split closure is polyhedral [11]. Andersen, Cornuéjols, and Li showed that intersection cuts completely describe the split closure, provided infeasible and nonoptimal bases are included [12]. Çezik and Iyengar extended Chvátal–Gomory, lift-and-project, and related cutting-plane constructions to mixed 0–1 conic programs [13].

Our analysis of  $\mathcal{C}_n$  also draws on the broader theory of CG closures. CG procedure builds on the foundational works by Gomory and Chvátal [14, 15]. Based on the result that the elementary closure of a rational polyhedron is again a rational polyhedron by Schrijver [16], Dadush, Dey, and Vielma extended this to strictly convex bodies and their intersections with rational polyhedra [17]. They subsequently proved that the CG closure of every nonempty compact convex set is a rational polytope [18]. On the other hand, they exhibited a strictly convex body in  $\mathbb{R}^3$  whose split closure is not polyhedral [19]. De Meijer and Sotirov systematically developed the CG procedure for integer semidefinite programs and spectrahedra [20]. In contrast, we study a highly structured but unbounded moment spectrahedron.

By exploiting specific constraint and objective structures, constrained mixed- and pure-integer

variants of quadratic optimization have also been studied through strong formulations and valid inequalities. For models in which binary indicators switch continuous variables on or off, perspective cuts and reformulations strengthen convex quadratic relaxations [2, 3]. Perspective strengthening has also demonstrated success combined with outer approximation in mixed-integer convex formulations for learning Gaussian directed acyclic graphs [21]. Moreover, semidefinite methods yield equivalent convex quadratic reformulations for classes of constrained models [22] and exact extended formulations for convex quadratic epigraphs with indicators [23]. Special structures in the quadratic coefficient matrix give rise to further valid inequalities [24, 25]. A different approach uses coordinate-wise optimality conditions to derive optimality-preserving reformulations for convex quadratic programs with indicators [26]. Extensions of submodularity are applied more broadly to structured nonlinear minimization over constrained mixed- and pure-integer feasible sets, yielding epigraph convex-hull descriptions and exact separation algorithms [27, 28]. These results cover special cases of quadratic integer optimization. In our work, we focus on quadratic optimization over the full integer lattice, without imposing structural restrictions on the quadratic objective.

A substantial literature addresses the complexity of elementary closures and cut separation. Eisenbrand proved that deciding whether a given rational point lies outside the elementary closure is NP-complete [29]. Bockmayr and Eisenbrand proved that, in fixed dimension, the elementary closure of a rational polyhedron can be described by polynomially many inequalities and constructed in polynomial time [30]. Cornuéjols and Li compared the elementary closures generated by several classical families of cutting planes [31].

Caprara and Fischetti proved that the separation problem for  $\{0, \frac{1}{2}\}$ -CG cuts is NP-complete [32]. Caprara and Letchford established strong NP-completeness for split-cut separation and related families [33]. Fischetti and Lodi modeled rank-one CG separation as a mixed-integer program and used the resulting separator in a computational study of optimization over the first Chvátal closure [34]. Dash, Günlük, and Lodi proved that the MIR closure coincides with the split closure and that its separation problem is NP-hard [35]. For the quadratic split inequalities considered here, Burer and Letchford conjectured that separation is strongly NP-hard [1, p. 255].

### 1.3 Our contributions

Our results identify the exact dimension threshold for recovery of the integer quadratic moment hull by one round of Chvátal–Gomory cuts. We also settle the open question of whether quadratic split inequalities describe the moment hull in dimensions three through five [1, 10] and prove Burer and Letchford’s conjecture that separating these inequalities is strongly NP-hard [1, p. 255].

For every  $n \geq 1$ , the rank-one CG closure of the basic semidefinite relaxation  $\mathcal{P}_n$  satisfies

$$\mathcal{C}_n = \text{MIQ}_n \iff n \leq 6.$$

The split-inequality closure is exact only through dimension five, so one CG round extends exact recovery by one dimension. The complete comparison is

$$\begin{aligned} 1 \leq n \leq 5 : & \quad \text{MIQ}_n = \mathcal{C}_n = \mathcal{S}_n, \\ n = 6 : & \quad \text{MIQ}_n = \mathcal{C}_n \subsetneq \mathcal{S}_n, \\ n \geq 7 : & \quad \text{MIQ}_n \subsetneq \mathcal{C}_n \subsetneq \mathcal{S}_n. \end{aligned}$$

Moreover,  $\mathcal{S}_n \subsetneq \mathcal{P}_n$  in every dimension.

Our geometric results explain these thresholds. We prove a bijection between the facets of  $\text{MIQ}_n$  and perfect Delaunay polyhedra, making precise the connection suggested by Burer and Letchford [1, Section 5.3]. We associate a depth with each facet: the gap between the lattice and continuous

minima of a primitive integer-valued defining quadratic. We prove that the facet is valid for  $\mathcal{C}_n$  if and only if its depth is less than one. Exact depth calculations for the known types through dimension seven, together with extension of their defining quadratics to higher dimensions, give the comparisons above.

We also establish complexity results for membership and exact separation over  $\mathcal{C}_n$ . Every rational point outside  $\mathcal{C}_n$  admits a violated rank-one CG inequality whose encoding length is polynomial in the input length. When  $n$  is part of the input, nonmembership in  $\mathcal{C}_n$  is strongly NP-complete and membership is strongly coNP-complete, even when the input data are polynomially bounded in  $n$ . The same reduction proves strong NP-hardness of quadratic split-inequality separation, even for input points in  $\mathcal{P}_n$ .

## 1.4 Organization

Section 2 connects quadratic functions with linear inequalities in the moment variables. It introduces the Erdahl cone, consisting of quadratics that are nonnegative at every integer point, and Delaunay polyhedra, the nonempty sets of integer points where a nonzero such quadratic vanishes. It also specifies the coordinates used for CG rounding and defines the CG and split-inequality closures. Section 3 proves the correspondence between the facets of  $\text{MIQ}_n$  and perfect Delaunay polyhedra. Section 4 determines which facet inequalities hold after one round of CG cuts. The key quantity is depth, which measures the gap between the integer and continuous minima of a defining quadratic after choosing a standard scale for its coefficients. We express CG inequalities using integer-valued quadratics, called cut polynomials, and prove that a facet inequality is valid for  $\mathcal{C}_n$  exactly when its depth is less than one. We further provide a rigidity argument to establish that a facet inequality failing this test cannot follow from the other cuts in the same round. Section 5 applies known classifications of perfect Delaunay polyhedra through dimension seven and computes their depths. By extending suitable examples to higher dimensions, we determine exactly when the CG- and split-inequality closures recover the integer quadratic moment hull. In Section 6, we study membership and separation for  $\mathcal{C}_n$ . Specifically, we show that every rational point outside the closure has a short certificate of nonmembership that can be checked efficiently. We then establish the computational difficulty of deciding membership and finding violated inequalities when the dimension is part of the input. Section 7 summarizes the results and discusses directions for further research.

## 2 Quadratic and geometric foundations

This section provides the algebraic and geometric framework used throughout this paper. In Section 2.1, we identify the correspondence between the valid affine inequalities for  $\text{MIQ}_n$  and the elements of the so-called *Erdahl cone*. Section 2.2 introduces Delaunay polyhedra and the notion of perfection, which are later used to describe the facets of  $\text{MIQ}_n$ . Then, in Section 2.3, we discuss the integral moment coordinates and relate them to the integer-valued quadratic polynomials. Lastly, Section 2.4 formally defines the CG closure,  $\mathcal{C}_n$ , and the split-inequality closure,  $\mathcal{S}_n$ . We further prove their containment relation and remark on their invariance under integer-affine changes of variables. We use  $\text{conv}$ ,  $\text{aff}$ , and  $\text{range}$  to denote convex hull, affine hull, and range, respectively.

### 2.1 Quadratic polynomials and the Erdahl cone

Let  $\mathcal{Q}_n$  be the vector space of real polynomials in  $n$  variables of degree at most two. Every  $q \in \mathcal{Q}_n$  has a unique representation

$$q(z) = z^\top A_q z + 2b_q^\top z + \alpha_q,$$

where  $A_q \in \mathbb{S}^n$ ,  $b_q \in \mathbb{R}^n$ , and  $\alpha_q \in \mathbb{R}$ . Equivalently, with

$$M_q := \begin{bmatrix} \alpha_q & b_q^\top \\ b_q & A_q \end{bmatrix} \in \mathbb{S}^{n+1},$$

for  $W = \begin{bmatrix} 1 & x^\top \\ x & X \end{bmatrix} \in \mathcal{H}_n$ , define the linearization

$$L_q(W) := \langle M_q, W \rangle = \langle A_q, X \rangle + 2b_q^\top x + \alpha_q. \quad (2.1)$$

Then  $L_q(W_z) = q(z)$  for every  $z \in \mathbb{R}^n$ . The relevant dimensions follow from the number of independent matrix entries. A symmetric  $(n+1) \times (n+1)$  matrix is determined by its  $n+1$  diagonal entries and its  $\binom{n+1}{2}$  entries above the diagonal. Hence,

$$\dim \mathbb{S}^{n+1} = (n+1) + \binom{n+1}{2} = \binom{n+2}{2}.$$

The constraint  $W_{00} = 1$  removes one degree of freedom, so

$$\dim \mathcal{H}_n = \binom{n+2}{2} - 1.$$

For brevity, we will consistently use

$$d_n := \binom{n+2}{2} - 1 = n + \binom{n+1}{2}$$

to represent this constant. Similarly,  $\dim \mathcal{Q}_n = d_n + 1$ .

The *Erdahl cone* is defined by

$$\text{Erdahl}(n) := \{q \in \mathcal{Q}_n : q(z) \geq 0, \forall z \in \mathbb{Z}^n\}.$$

We call every  $q \in \text{Erdahl}(n)$  a *lattice-nonnegative quadratic* to highlight the property that  $q$  is nonnegative at all integer arguments. We remark that such a quadratic need not be nonnegative on all of  $\mathbb{R}^n$ , nor need it take integer values on  $\mathbb{Z}^n$ .

We first record an elementary property of quadratic polynomials that will be used to identify affine functionals from their values at the lattice moment matrices.

**Lemma 2.1** (Quadratics vanishing on the integer lattice). *If  $p \in \mathcal{Q}_n$  satisfies  $p(z) = 0$  for every  $z \in \mathbb{Z}^n$ , then  $p \equiv 0$ .*

*Proof.* By writing  $p(z) = z^\top A_p z + 2b_p^\top z + \alpha_p$  and evaluating it at  $z = 0$ , we obtain  $\alpha_p = 0$ . Let  $e_i$  be the  $i$ th standard unit vector. Evaluating at  $e_i$  and  $-e_i$  gives

$$(A_p)_{ii} + 2(b_p)_i = 0, \quad (A_p)_{ii} - 2(b_p)_i = 0,$$

so  $(A_p)_{ii} = (b_p)_i = 0$  for every  $i$ . For  $i < j$ , evaluating at  $e_i + e_j$  then gives  $2(A_p)_{ij} = 0$ . Thus  $A_p = 0$ ,  $b_p = 0$ , and  $\alpha_p = 0$ , proving that  $p \equiv 0$ .  $\square$

An immediate consequence is that the lattice moment matrices  $W_z$ ,  $z \in \mathbb{Z}^n$ , affinely span  $\mathcal{H}_n$ . Indeed, every affine functional  $F$  on  $\mathcal{H}_n$  can be written as  $F = L_p$  for some  $p \in \mathcal{Q}_n$ . If  $F(W_z) = 0$  for every  $z \in \mathbb{Z}^n$ , then  $p(z) = 0$  for every  $z \in \mathbb{Z}^n$ , so Lemma 2.1 gives  $p \equiv 0$  and hence  $F \equiv 0$ . Consequently,  $\text{MIQ}_n$  is full-dimensional in  $\mathcal{H}_n$  and has dimension  $d_n$ .

For  $q \in \mathcal{Q}_n$ , the validity characterization of Burer and Letchford [1] is

$$L_q(W) \geq 0 \text{ for every } W \in \text{MIQ}_n \iff q \in \text{Erdahl}(n). \quad (2.2)$$

Validity implies  $q(z) = L_q(W_z) \geq 0$  at every integer point. Conversely, if  $q \in \text{Erdahl}(n)$ , then  $L_q$  is nonnegative at every lattice moment matrix and hence on their closed convex hull.

## 2.2 Delaunay polyhedra and perfection

For  $q \in \text{Erdahl}(n)$ , define its *lattice-zero set* by

$$Z(q) := \{z \in \mathbb{Z}^n : q(z) = 0\}.$$

**Definition 2.2** (Delaunay polyhedron). *A nonempty set  $D \subseteq \mathbb{Z}^n$  is a Delaunay polyhedron if  $D = Z(q)$  for some nonzero  $q \in \text{Erdahl}(n)$ .*

Throughout the paper,  $D$  denotes the lattice-point set and  $\text{conv}(D)$  denotes the associated convex set. We call  $D$  bounded when  $\text{conv}(D)$  is bounded, or equivalently when  $D$  is finite. In this case,  $\text{conv}(D)$  is the associated Delaunay polytope. When no confusion can arise, we also call the finite lattice-point set  $D$  a Delaunay polytope.

To formalize when a Delaunay polyhedron uniquely determines its defining quadratic up to scaling, for an arbitrary  $D \subseteq \mathbb{Z}^n$ , let

$$\text{Space}(D) := \{p \in \mathcal{Q}_n : p(z) = 0 \text{ for every } z \in D\}.$$

**Definition 2.3** (Perfect Delaunay polyhedron). *A Delaunay polyhedron  $D$  is perfect if*

$$\dim \text{Space}(D) = 1.$$

If  $D$  is perfect,  $\text{Space}(D) \cap \text{Erdahl}(n)$  is a nonzero ray. We denote a generator of this ray by  $q_D$ , which is unique up to multiplication by a positive scalar.

These notions arise in the theory of inhomogeneous quadratic forms. Erdahl studied the cone of quadratics that are nonnegative on  $\mathbb{Z}^n$ , characterized its extreme rays, and connected one of the extreme-ray classes to perfect Delaunay polyhedra through lattice-zero sets [36]. For further background on lattice geometry and computational methods, see Schürmann [37] and Dutour Sikirić, Schürmann, and Vallentin [38].

In fact, every perfect Delaunay polyhedron is full-dimensional in  $\mathbb{R}^n$ . Suppose otherwise that  $\text{aff}(D) \subsetneq \mathbb{R}^n$ . Then there exists a nonzero affine polynomial  $q \in \mathcal{Q}_n$  that vanishes on  $\text{aff}(D)$ ; that is,  $q(z) = 0$  for every  $z \in \text{aff}(D)$ . To see this, we write  $\text{aff}(D)$  as  $z^0 + V$ , where  $z^0 \in D$  and  $V$  is a linear subspace with  $\dim V \leq n - 1$ . Its orthogonal complement

$$V^\perp := \{u \in \mathbb{R}^n : u^\top v = 0, \forall v \in V\}$$

has dimension  $n - \dim V \geq 1$ . Choose  $u \in V^\perp$  with  $u \neq 0$ , then  $q(z) := u^\top(z - z^0)$  is a nonzero affine polynomial in  $\mathcal{Q}_n$  that vanishes on  $\text{aff}(D)$ . Hence,  $q \in \text{Space}(D)$ . Meanwhile,  $q_D$  cannot be affine because a nonconstant affine polynomial is unbounded below on  $\mathbb{Z}^n$ , and a nonzero constant has no lattice zeros. Therefore,  $q$  and  $q_D$  are linearly independent, contradicting  $\dim \text{Space}(D) = 1$ .

Perfect Delaunay polyhedra may nevertheless be unbounded. Section 3.2 describes their structure and its relation to facet boundedness.

## 2.3 Matrix coordinates and integral moment coordinates

For  $W = \begin{bmatrix} 1 & x^\top \\ x & X \end{bmatrix} \in \mathcal{H}_n$ , we call  $(x, X)$  its *matrix coordinates*, with  $X$  represented by its independent upper-triangular entries.

The CG procedure uses affine functionals that are integer-valued at every lattice moment point. In monomial coordinates, however, an integer-valued quadratic may have fractional coefficients. For example, let

$$\binom{z}{2} := z(z-1)/2$$

for  $z \in \mathbb{R}$ ; here,  $\binom{z}{2}$  is integer-valued on  $z \in \mathbb{Z}$ . We therefore use the Newton basis of  $\mathcal{Q}_n$ :

$$1, \quad z_i, \quad \binom{z_i}{2}, \quad z_i z_j,$$

for  $j \in \{i+1, \dots, n\}$ ,  $i \in \{1, \dots, n\}$ . This basis gives the natural integral coordinates for CG rounding.

The *integral moment coordinates* of  $W$  are

$$\eta(W) = ((x_i)_{i=1}^n; (w_{ii})_{i=1}^n; (w_{ij})_{1 \leq i < j \leq n}) = (x, w) \in \mathbb{R}^{d_n}.$$

Here,  $w$  is given by

$$w_{ij} := \begin{cases} (X_{ii} - x_i)/2, & \forall 1 \leq i = j \leq n, \\ X_{ij}, & \forall 1 \leq i < j \leq n. \end{cases} \quad (2.3)$$

The diagonal coordinates  $w_{11}, \dots, w_{nn}$  precede the off-diagonal coordinates  $w_{ij}$ , whose index pairs  $(i, j)$  with  $i < j$  are ordered lexicographically.

The coordinate lattice for CG rounding is  $\mathbb{Z}^{d_n}$  in the  $(x, w)$  representation. In matrix coordinates, the corresponding affine lattice consists of matrices satisfying  $x_i \in \mathbb{Z}$  and  $X_{ii} - x_i \in 2\mathbb{Z}$  for every  $i$ , and  $X_{ij} \in \mathbb{Z}$  for every  $i < j$ . We denote the inverse coordinate map by  $W : \mathbb{R}^{d_n} \rightarrow \mathcal{H}_n$ . For  $(x, w) \in \mathbb{R}^{d_n}$ , define

$$W(x, w) := \begin{bmatrix} 1 & x^\top \\ x & X(x, w) \end{bmatrix},$$

where  $X(x, w) \in \mathbb{S}^n$  is given by

$$X(x, w)_{ij} := \begin{cases} x_i + 2w_{ii}, & \forall 1 \leq i = j \leq n, \\ w_{ij}, & \forall 1 \leq i < j \leq n, \\ w_{ji}, & \forall 1 \leq j < i \leq n. \end{cases}$$

For any rank-one moment matrix  $W_z$  associated with  $z \in \mathbb{R}^n$ , we define

$$\eta(z) := \eta(W_z) = \left( (z_i)_{i=1}^n; \left( \binom{z_i}{2} \right)_{i=1}^n; (z_i z_j)_{1 \leq i < j \leq n} \right) \quad (2.4)$$

by slight abuse of notation. In particular,  $\eta(z) \in \mathbb{Z}^{d_n}$  for every  $z \in \mathbb{Z}^n$ .

Every  $p \in \mathcal{Q}_n$  has a unique representation in this Newton basis:

$$p(z) = \alpha_p + \sum_{i=1}^n a_i z_i + \sum_{i=1}^n d_i \binom{z_i}{2} + \sum_{1 \leq i < j \leq n} \gamma_{ij} z_i z_j = \alpha_p + c_p^\top \eta(z),$$

where

$$c_p = ((a_i)_{i=1}^n; (d_i)_{i=1}^n; (\gamma_{ij})_{1 \leq i < j \leq n}) \in \mathbb{R}^{d_n}.$$

We call  $c_p$  the *integral moment normal* of  $p$ , and write

$$\text{Int}_2(\mathbb{Z}^n) := \{p \in \mathcal{Q}_n : p(\mathbb{Z}^n) \subseteq \mathbb{Z}\}.$$

The following lemma verifies the correspondence between the two coordinate systems and shows that a quadratic is integer-valued on  $\mathbb{Z}^n$  exactly when its Newton coefficients are integers.

**Lemma 2.4** (Integral moment coordinates and integer-valued quadratics).

(i) The mapping  $\eta : \mathcal{H}_n \rightarrow \mathbb{R}^{d_n}$  given in (2.3) is an affine bijection, and  $\eta(W_z) = \eta(z) \in \mathbb{Z}^{d_n}$  for every  $z \in \mathbb{Z}^n$ .

(ii) For every  $p \in \mathcal{Q}_n$  and every  $W \in \mathcal{H}_n$ ,

$$L_p(W) = \alpha_p + c_p^\top \eta(W). \quad (2.5)$$

(iii) A polynomial  $p \in \mathcal{Q}_n$  satisfies  $p \in \text{Int}_2(\mathbb{Z}^n)$  if and only if  $(\alpha_p, c_p) \in \mathbb{Z} \times \mathbb{Z}^{d_n}$ .

*Proof.* For (i), the formulas above show directly that  $\eta$  and  $W$  are inverse affine maps by construction. Substituting  $x = z$  and  $X = zz^\top$  gives (2.4), whose entries are integral when  $z \in \mathbb{Z}^n$ .

For (ii), the two sides of (2.5) agree at every  $W_z$ . They are affine functions on  $\mathcal{H}_n$ , so the affine-spanning consequence of Lemma 2.1 implies that they agree everywhere.

Lastly, we prove claim (iii). If  $p$  is integer-valued on  $\mathbb{Z}^n$ , its Newton coefficients are recovered by

$$\alpha_p = p(0), \quad a_i = p(e_i) - p(0), \quad d_i = p(2e_i) - 2p(e_i) + p(0),$$

for  $1 \leq i \leq n$ , and for  $1 \leq i < j \leq n$ ,

$$\gamma_{ij} = p(e_i + e_j) - p(e_i) - p(e_j) + p(0).$$

Thus all Newton coefficients are integers. Conversely, every Newton-basis element is integer-valued on  $\mathbb{Z}^n$ , so every integral linear combination of these functions is integer-valued.  $\square$

The Newton representation is convenient for characterizing integer-valued quadratics, while the standard matrix and Hessian representations will be useful for analyzing convexity and continuous minima. We therefore summarize the correspondence between these representations:

$$p(z) = z^\top A_p z + 2b_p^\top z + \alpha_p = \frac{1}{2} z^\top H_p z + \ell_p^\top z + \alpha_p, \quad (2.6)$$

where

$$(A_p)_{ii} = \frac{d_i}{2}, \quad (A_p)_{ij} = \frac{\gamma_{ij}}{2} \ (i < j), \quad (b_p)_i = \frac{a_i}{2} - \frac{d_i}{4}, \quad (2.7)$$

and

$$H_p = 2A_p, \quad (H_p)_{ii} = d_i, \quad (H_p)_{ij} = \gamma_{ij} \ (i < j), \quad (\ell_p)_i = a_i - \frac{d_i}{2}. \quad (2.8)$$

Thus  $H_p$  is integral whenever  $p$  is integer-valued, although  $A_p$ ,  $b_p$ , and  $\ell_p$  need not be integral.

In what follows, under the affine bijection  $\eta$ , we will use the matrices and coordinate vectors interchangeably, while keeping the same notation for sets and functions in either representation. For example, in integral moment coordinates, the semidefinite relaxation is written as

$$\mathcal{P}_n = \{\eta \in \mathbb{R}^{d_n} : W(\eta) \succeq 0\}.$$

## 2.4 The rank-one CG closure and quadratic split inequalities

This section defines the rank-one CG closure  $\mathcal{C}_n$  and the split-inequality closure  $\mathcal{S}_n$ . We establish their basic containment relations with  $\text{MIQ}_n$  and  $\mathcal{P}_n$ , and show that all these four sets are invariant under integer-affine bijections.

For a nonempty closed convex set  $K \subseteq \mathbb{R}^{d_n}$  and  $c \in \mathbb{Z}^{d_n}$ , let

$$\beta_K(c) := \inf\{c^\top \eta : \eta \in K\}.$$

When  $\beta_K(c) > -\infty$ , the associated CG cut is  $c^\top \eta \geq \lceil \beta_K(c) \rceil$ . The CG closure of  $K$  with respect to the integral moment lattice is

$$\text{CG}(K) := \bigcap_{\substack{c \in \mathbb{Z}^{d_n} \\ \beta_K(c) > -\infty}} \left\{ \eta : c^\top \eta \geq \lceil \beta_K(c) \rceil \right\}.$$

We now specialize this construction to the semidefinite relaxation  $\mathcal{P}_n$ . For  $c \in \mathbb{Z}^{d_n}$ , write the integer-valued quadratic  $c^\top \eta(x)$ ,  $x \in \mathbb{R}^n$ , as

$$c^\top \eta(x) = x^\top A_c x + 2b_c^\top x.$$

Then

$$\beta_{\mathcal{P}_n}(c) = \inf_{x \in \mathbb{R}^n} c^\top \eta(x) = \begin{cases} -b_c^\top A_c^\dagger b_c, & A_c \succeq 0 \text{ and } b_c \in \text{range}(A_c), \\ -\infty, & \text{otherwise.} \end{cases} \quad (2.9)$$

Here  $A_c^\dagger$  denotes the Moore–Penrose pseudoinverse. To verify the formula, write  $X = xx^\top + S$  with  $S \succeq 0$ . The linearized objective is

$$c^\top \eta(x) + \langle A_c, S \rangle.$$

If  $A_c \not\succeq 0$ , the objective is unbounded below as  $S$  varies, and  $c^\top \eta(x)$  is unbounded below along a direction of negative curvature. If  $A_c \succeq 0$ , the second term is nonnegative and is minimized at  $S = 0$ . The problem then reduces to minimizing  $c^\top \eta(x)$  over  $\mathbb{R}^n$ . Its value is finite exactly when  $b_c \in \text{range}(A_c)$ , which gives (2.9). In that case,  $x = -A_c^\dagger b_c$  is a minimizer, and the corresponding moment matrix  $W_x$  has matrix rank one.

With this characterization of the relaxation value, we define the *rank-one CG closure* of  $\mathcal{P}_n$  by

$$\mathcal{C}_n := \text{CG}(\mathcal{P}_n) = \bigcap_{\substack{c \in \mathbb{Z}^{d_n} \\ \beta_{\mathcal{P}_n}(c) > -\infty}} \left\{ \eta : c^\top \eta \geq \lceil \beta_{\mathcal{P}_n}(c) \rceil \right\}. \quad (2.10)$$

Here, “rank-one” means that the CG procedure is applied exactly once to  $\mathcal{P}_n$ . This is the elementary CG closure of the spectrahedron  $\mathcal{P}_n$  in the integral coordinates (2.3) (see [20, Definition 1 and Sections 2.1–2.2]).

Although (2.10) is written solely as an intersection of affine halfspaces, the semidefinite constraint defining  $\mathcal{P}_n$  is already implied by this family of CG inequalities. We now verify this fact.

For  $v = (v_0, v') \in \mathbb{Z}^{n+1}$ , consider the integer-valued affine square

$$q_v(z) := (v_0 + v'^\top z)^2.$$

Write  $q_v(z) = v_0^2 + c_v^\top \eta(z)$ , where  $c_v \in \mathbb{Z}^{d_n}$ . Since  $q_v \geq 0$  on  $\mathbb{R}^n$ , (2.9) gives  $\beta_{\mathcal{P}_n}(c_v) \geq -v_0^2$ . The CG inequality for  $c_v$  therefore gives, for every  $\eta \in \mathcal{C}_n$ ,

$$c_v^\top \eta \geq \lceil \beta_{\mathcal{P}_n}(c_v) \rceil \geq -v_0^2,$$

because  $v_0^2$  is an integer. Hence  $v^\top W(\eta)v = v_0^2 + c_v^\top \eta \geq 0$  for every  $\eta \in \mathcal{C}_n$  and every  $v \in \mathbb{Z}^{n+1}$ .

These integral inequalities characterize positive semidefiniteness. Indeed, if  $W \not\geq 0$ , the open set  $\{u \in \mathbb{R}^{n+1} : u^\top W u < 0\}$  contains a rational vector. Scaling that vector gives  $v \in \mathbb{Z}^{n+1}$  with  $v^\top W v < 0$ . Thus,

$$\bigcap_{v \in \mathbb{Z}^{n+1}} \{\eta : v^\top W(\eta)v \geq 0\} = \mathcal{P}_n, \quad \mathcal{C}_n \subseteq \mathcal{P}_n.$$

We next introduce the second family of inequalities considered in this paper, namely the split inequalities, and define the corresponding split-inequality closure. For a primitive  $v \in \mathbb{Z}^n$  and  $s \in \mathbb{Z}$ , define the *primitive split quadratic*

$$\hat{q}_{v,s}(z) := \frac{1}{2}(v^\top z - s)(v^\top z - s - 1). \quad (2.11)$$

Its lattice-zero set is the primitive-split Delaunay polyhedron

$$D_{v,s} := Z(\hat{q}_{v,s}) = \{z \in \mathbb{Z}^n : v^\top z \in \{s, s+1\}\}. \quad (2.12)$$

After an integer-affine change of variables,  $D_{v,s}$  becomes  $\{0, 1\} \times \mathbb{Z}^{n-1}$ , and its convex hull becomes  $[0, 1] \times \mathbb{R}^{n-1}$ .

The *split-inequality closure* is

$$\mathcal{S}_n := \bigcap_{\substack{v \in \mathbb{Z}^n \\ \text{primitive} \\ s \in \mathbb{Z}}} \{W \in \mathcal{H}_n : L_{\hat{q}_{v,s}}(W) \geq 0\}. \quad (2.13)$$

In matrix coordinates, the same inequality is

$$v^\top X v - (2s+1)v^\top x + s(s+1) \geq 0, \quad (2.14)$$

where we omit the positive factor  $1/2$ . These inequalities are the linearizations of the lattice disjunctions  $v^\top z \leq s$  or  $v^\top z \geq s+1$ . Burer and Letchford call inequalities of this form *split inequalities* [1, Section 5.2, Proposition 12]. We call the intersection over primitive directions in (2.13) the *split-inequality closure*.

**Proposition 2.5** (Basic containment relations). *For every  $n \geq 1$ ,*

$$\text{MIQ}_n \subseteq \mathcal{C}_n \subseteq \mathcal{S}_n \subsetneq \mathcal{P}_n. \quad (2.15)$$

*Proof.* Every lattice moment matrix belongs to  $\mathcal{C}_n$ . Indeed, for each admissible  $c$ , the integer  $c^\top \eta(z)$  is at least  $\beta_{\mathcal{P}_n}(c)$  and therefore at least  $\lceil \beta_{\mathcal{P}_n}(c) \rceil$ . Because  $\mathcal{C}_n$  is an intersection of closed halfspaces, it is closed and convex. It therefore contains the closed convex hull of the lattice generators, namely  $\text{MIQ}_n$ .

Every split inequality is a rank-one CG cut. The primitive split quadratic is integer-valued and has continuous minimum  $-1/8$ , as follows by completing the square. Write  $\hat{q}_{v,s}(z) = \alpha + c^\top \eta(z)$ . By Lemma 2.4,  $\alpha \in \mathbb{Z}$  and  $c \in \mathbb{Z}^{d_n}$ . Moreover,

$$\beta_{\mathcal{P}_n}(c) = \min_{y \in \mathbb{R}^n} (\hat{q}_{v,s}(y) - \alpha) = -\alpha - \frac{1}{8},$$

so rounding gives  $c^\top \eta \geq -\alpha$ . This is exactly  $L_{\hat{q}_{v,s}} \geq 0$ . Thus every split inequality occurs in the CG family, giving  $\mathcal{C}_n \subseteq \mathcal{S}_n$ .

Finally, Burer and Letchford [1, Theorem 4] establish  $\mathcal{S}_n \subseteq \mathcal{P}_n$  using all integer split normals. Primitive normals suffice because for  $W = \begin{bmatrix} 1 & x^\top \\ x & X \end{bmatrix} \in \mathcal{S}_n$ , choosing  $s = \lfloor v^\top x \rfloor$  in (2.14) gives  $v^\top (X - x x^\top) v \geq 0$  for every primitive  $v \in \mathbb{Z}^n$ . Every nonzero rational direction is a multiple of such a vector, so continuity gives  $X - x x^\top \succeq 0$ . The Schur complement then gives  $W \succeq 0$ . Strictness follows because  $W_{e_1/2} \in \mathcal{P}_n$ , but (2.14) with  $v = e_1$  and  $s = 0$  evaluates to  $1/4 - 1/2 = -1/4 < 0$ .  $\square$

We conclude by recording the integer-affine symmetry shared by all four sets. Let  $T(z) = Uz + t$ , where  $U \in \text{GL}_n(\mathbb{Z})$  is unimodular (equivalently,  $U \in \mathbb{Z}^{n \times n}$  and  $\det(U) = \pm 1$ ) and  $t \in \mathbb{Z}^n$ , and define

$$R_T := \begin{bmatrix} 1 & 0 \\ t & U \end{bmatrix}, \quad \Phi_T(W) := R_T W R_T^\top.$$

Then  $\Phi_T(W_z) = W_{T(z)}$ .

**Proposition 2.6** (Integer-affine symmetries). *The map  $\Phi_T$  is an affine automorphism of  $\mathcal{H}_n$  and satisfies*

$$\Phi_T(\text{MIQ}_n) = \text{MIQ}_n, \quad \Phi_T(\mathcal{P}_n) = \mathcal{P}_n, \quad \Phi_T(\mathcal{C}_n) = \mathcal{C}_n, \quad \Phi_T(\mathcal{S}_n) = \mathcal{S}_n.$$

For every  $p \in \mathcal{Q}_n$  and every  $W \in \mathcal{H}_n$ ,

$$L_{p \circ T^{-1}}(\Phi_T(W)) = L_p(W). \quad (2.16)$$

Composition by  $T^{-1}$  is an automorphism of  $\text{Int}_2(\mathbb{Z}^n)$ .

*Proof.* Because  $T$  bijects  $\mathbb{Z}^n$ , the identity  $\Phi_T(W_z) = W_{T(z)}$  shows that  $\Phi_T$  permutes the lattice moment matrices. Hence it preserves  $\text{MIQ}_n$ . The matrix  $R_T$  is invertible, and congruence by an invertible matrix preserves positive semidefiniteness. Thus  $\Phi_T$  also preserves  $\mathcal{P}_n$ .

Identity (2.16) holds at every  $W_z$  by the definition of linearization. Both sides are affine on  $\mathcal{H}_n$ , so the affine-spanning consequence of Lemma 2.1 extends the identity to all of  $\mathcal{H}_n$ .

Since  $T$  bijects  $\mathbb{Z}^n$ , a quadratic  $p$  is integer-valued on  $\mathbb{Z}^n$  if and only if  $p \circ T^{-1}$  is. Since  $T$  bijects  $\mathbb{R}^n$ , composition by  $T^{-1}$  preserves the infimum.

We now check the effect on a CG inequality. For  $c \in \mathbb{Z}^{d_n}$ , write

$$c^\top \eta(T^{-1}(z)) = \alpha + c'^\top \eta(z).$$

Lemma 2.4 gives  $\alpha \in \mathbb{Z}$  and  $c' \in \mathbb{Z}^{d_n}$ . Applying the same construction to  $T$  gives the inverse map, so  $c \mapsto c'$  is a bijection of  $\mathbb{Z}^{d_n}$ . Moreover,

$$\beta_{\mathcal{P}_n}(c') = \inf_{z \in \mathbb{R}^n} (c'^\top \eta(T^{-1}(z)) - \alpha) = \beta_{\mathcal{P}_n}(c) - \alpha.$$

Thus  $c$  has a finite relaxation value if and only if  $c'$  does. For these normals,  $\alpha$  is integral, so

$$\lceil \beta_{\mathcal{P}_n}(c') \rceil = \lceil \beta_{\mathcal{P}_n}(c) \rceil - \alpha.$$

Together with (2.16), this implies that  $c^\top \eta(W) \geq \lceil \beta_{\mathcal{P}_n}(c) \rceil$  if and only if  $c'^\top \eta(\Phi_T(W)) \geq \lceil \beta_{\mathcal{P}_n}(c') \rceil$ . Thus  $\Phi_T$  permutes the CG inequalities in (2.10) and preserves  $\mathcal{C}_n$ .

Finally,

$$\hat{q}_{v,s} \circ T^{-1} = \hat{q}_{U^{-\top}v, s+v^\top U^{-1}t}.$$

The new normal is primitive because  $U$  is unimodular, and the new level is an integer. Thus composition permutes the primitive split quadratics, so  $\Phi_T$  preserves  $\mathcal{S}_n$ .  $\square$

### 3 Facets of $\text{MIQ}_n$

This section characterizes the facets of  $\text{MIQ}_n$  and determines when they are bounded. The analysis uses Erdahl's structure theorem for lattice-nonnegative quadratics to classify the facet-defining inequalities and describe the geometry of unbounded perfect Delaunay polyhedra.

The affine-spanning consequence of Lemma 2.1 shows that  $\text{MIQ}_n$  is full-dimensional in  $\mathcal{H}_n$ , with dimension  $d_n$ . A *facet* of  $\text{MIQ}_n$  is a nonempty proper face of affine dimension  $d_n - 1$ . Every facet is exposed. To see this, let  $F$  be a facet, choose  $W \in \text{relint } F$ , and take a supporting hyperplane of  $\text{MIQ}_n$  in  $\mathcal{H}_n$  at  $W$ . The exposed face contains  $F$  and has dimension at most  $d_n - 1$ , so it equals  $F$ . Thus every facet is the equality set of a nonconstant valid affine inequality on  $\mathcal{H}_n$ .

### 3.1 The facet–perfect-Delaunay correspondence

The goal of this subsection is to show that the facets of  $\text{MIQ}_n$  are exactly the inequalities arising from perfect Delaunay polyhedra. We first translate perfection into a dimension condition on the corresponding lifted moment points. We then use Erdahl’s classification of the extreme rays of  $\text{Erdahl}(n)$  to prove the converse: every facet-defining inequality must arise from a perfect Delaunay polyhedron.

**Lemma 3.1.** *For a Delaunay polyhedron  $D \subseteq \mathbb{Z}^n$ , the following statements are equivalent:*

(i)  *$D$  is perfect, that is,  $\dim \text{Space}(D) = 1$ ;*

(ii)

$$\dim \text{span}\{W_z : z \in D\} = d_n;$$

(iii)

$$\dim \text{aff}\{\eta(z) : z \in D\} = d_n - 1.$$

*Proof.* Under the linear isomorphism  $p \mapsto M_p$  between  $\mathcal{Q}_n$  and  $\mathbb{S}^{n+1}$ ,

$$\text{Space}(D) = \{p \in \mathcal{Q}_n : \langle M_p, W_z \rangle = 0 \text{ for every } z \in D\}$$

and hence is the orthogonal complement of  $\text{span}\{W_z : z \in D\}$ . Since  $\dim \mathcal{Q}_n = d_n + 1$ , rank–nullity shows that (i) and (ii) are equivalent.

For every  $z \in D$ , the matrix  $W_z$  satisfies  $(W_z)_{00} = 1$ . Thus, the matrices  $W_z$ , for  $z \in D$ , lie in an affine hyperplane that does not contain the origin. Their linear span has dimension one more than their affine hull. Finally, the affine bijection  $W \mapsto \eta(W)$  preserves affine dimension. This proves the equivalence of (ii) and (iii).  $\square$

We use the following form of Erdahl’s extreme-ray classification [36, Structure Theorem 2.1].

**Theorem 3.2** (Erdahl’s extreme-ray theorem). *Every extreme ray of  $\text{Erdahl}(n)$  has a generator of exactly one of the following types:*

(1) *the constant polynomial 1;*

(2) *an affine square*

$$q(z) = (a^\top z + \beta)^2,$$

*where  $a \in \mathbb{R}^n$  is not collinear with any nonzero vector of  $\mathbb{Z}^n$ ;*

(3) *the defining quadratic  $q_D$  of a perfect Delaunay polyhedron  $D \subseteq \mathbb{Z}^n$ .*

**Theorem 3.3** (Facet–perfect-Delaunay correspondence). *The assignment*

$$D \mapsto F_D := \text{MIQ}_n \cap \{W \in \mathcal{H}_n : L_{q_D}(W) = 0\}$$

*is a bijection between the perfect Delaunay polyhedra in  $\mathbb{Z}^n$  and the facets of  $\text{MIQ}_n$ . Equivalently, a nonconstant valid inequality*

$$\langle A, X \rangle + 2b^\top x + \alpha \geq 0$$

*defines a facet of  $\text{MIQ}_n$  if and only if the quadratic  $q(z) = z^\top A z + 2b^\top z + \alpha$  is a positive multiple of the defining quadratic of a perfect Delaunay polyhedron. Moreover, for every integer-affine bijection  $T$ , the map  $\Phi_T$  satisfies  $\Phi_T(F_D) = F_{T(D)}$ .*

*Proof.* Let  $D$  be perfect. Since  $q_D \in \text{Erdahl}(n)$ , the inequality  $L_{q_D} \geq 0$  is valid for  $\text{MIQ}_n$ . For every  $z \in D$ ,  $L_{q_D}(W_z) = q_D(z) = 0$ , so  $W_z \in F_D$ . By Lemma 3.1, these contact matrices have affine hull of dimension  $d_n - 1$ . Hence  $\dim F_D \geq d_n - 1$ . Moreover,  $q_D \neq 0$ , so  $F_D$  lies in a proper supporting hyperplane of the  $d_n$ -dimensional set  $\text{MIQ}_n$ . Therefore,  $\dim F_D = d_n - 1$ , and  $F_D$  is a facet.

Conversely, let  $F$  be a facet and choose a nonconstant valid inequality  $L_q \geq 0$  exposing it. By (2.2),  $q \in \text{Erdahl}(n)$ . We first show that  $q$  generates an extreme ray. Suppose  $q = q_1 + q_2$  with  $q_1, q_2 \in \text{Erdahl}(n)$ . For every  $W \in F$ , the nonnegative quantities  $L_{q_1}(W)$  and  $L_{q_2}(W)$  sum to  $L_q(W) = 0$ . Hence both functionals vanish on  $F$ . Since  $\text{aff}(F)$  is a hyperplane in the  $d_n$ -dimensional affine space  $\mathcal{H}_n$ , every affine functional on  $\mathcal{H}_n$  that vanishes on  $F$  is a scalar multiple of  $L_q$ . The map  $p \mapsto L_p|_{\mathcal{H}_n}$  is injective by Lemma 2.1, so  $q_i = \lambda_i q$  for some  $\lambda_i \in \mathbb{R}$ . Because  $F$  is a proper face, we may choose  $W^0 \in \text{MIQ}_n \setminus F$ . Then  $L_q(W^0) > 0$ , and validity gives  $0 \leq L_{q_i}(W^0) = \lambda_i L_q(W^0)$ . Hence  $\lambda_i \geq 0$ , so  $q$  generates an extreme ray of  $\text{Erdahl}(n)$ .

The first two types in Theorem 3.2 cannot define a facet. The constant inequality  $1 \geq 0$  has an empty equality set. For an affine square  $q(z) = (a^\top z + \beta)^2$ , put  $v := (\beta, a) \in \mathbb{R}^{n+1}$ . Every matrix in  $\text{MIQ}_n$  is positive semidefinite, so

$$L_q(W) = v^\top W v = 0 \implies W v = 0.$$

For  $v \neq 0$ , consider the linear map  $T_v : \mathbb{S}^{n+1} \rightarrow \mathbb{R}^{n+1}$  given by  $T_v(W) = W v$ . This map is surjective. Indeed, for any  $y \in \mathbb{R}^{n+1}$ , the symmetric matrix

$$W_y := \frac{y v^\top + v y^\top}{\|v\|_2^2} - \frac{v^\top y}{\|v\|_2^4} v v^\top$$

satisfies  $W_y v = y$ . Hence  $\dim \ker T_v = (d_n + 1) - (n + 1) = d_n - n$ . The equality face of  $L_q \geq 0$ , if nonempty, therefore has affine dimension at most  $d_n - n$ . For  $n \geq 2$  this is at most  $d_n - 2$ , whereas a facet has dimension  $d_n - 1$ . When  $n = 1$ , every nonzero  $a \in \mathbb{R}$  is collinear with the integer vector 1, so the affine-square type in Erdahl's theorem does not occur. If  $a = 0$ , the square is constant.

Only the third type in Theorem 3.2 remains. Thus  $q$  is a positive multiple of  $q_D$  for some perfect Delaunay polyhedron  $D$ , and  $F = F_D$ .

It remains to prove uniqueness. Suppose  $F_D = F_{D'}$ . The affine hull of a facet is its unique supporting hyperplane, so  $L_{q_D}$  and  $L_{q_{D'}}$  are proportional on  $\mathcal{H}_n$ . The proportionality factor is positive because both inequalities are valid and nonconstant. Injectivity of the linearization map gives that  $q_D$  and  $q_{D'}$  are positive scalar multiples, and therefore

$$D = Z(q_D) = Z(q_{D'}) = D'.$$

Finally, we prove equivariance. Let  $T(z) = Uz + t$  be an integer-affine bijection. Composition by  $T^{-1}$  maps  $\text{Space}(D)$  isomorphically onto  $\text{Space}(T(D))$ , so perfection is preserved. Proposition 2.6 gives

$$L_{q_D \circ T^{-1}}(\Phi_T(W)) = L_{q_D}(W),$$

and  $q_D \circ T^{-1}$  defines  $T(D)$ . Consequently,  $\Phi_T(F_D) = F_{T(D)}$ , proving equivariance.  $\square$

Given the individual facets classified by the last theorem, the next proposition shows that their halfspaces describe the entire moment hull, even when the hull is nonpolyhedral.

**Proposition 3.4** (Perfect-Delaunay facet inequalities describe the moment hull). *One has*

$$\text{MIQ}_n = \bigcap_{D \text{ perfect}} \{W \in \mathcal{H}_n : L_{q_D}(W) \geq 0\}. \quad (3.1)$$

*Proof.* Every perfect-Delaunay inequality is valid for  $\text{MIQ}_n$ , so the inclusion “ $\subseteq$ ” is immediate. Let  $K$  denote the intersection on the right-hand side of (3.1), and fix  $W \in K$ .

Every primitive-split Delaunay polyhedron is perfect [36]. Therefore, the primitive-split inequalities occur among the halfspaces defining  $K$ , and

$$K \subseteq \mathcal{S}_n \subseteq \mathcal{P}_n$$

by Proposition 2.5.

We next use the extreme rays to recover all inequalities in  $\text{Erdahl}(n)$ . The cone  $\text{Erdahl}(n)$  is closed, being an intersection of closed halfspaces. It is also pointed because if both  $q$  and  $-q$  belong to  $\text{Erdahl}(n)$ , then  $q$  vanishes on  $\mathbb{Z}^n$  and hence is the zero polynomial by Lemma 2.1. Every closed pointed cone in a finite-dimensional space is generated by its extreme rays.

We now check the three types of extreme rays at  $W$ . The constant inequality is vacuous. Every affine-square inequality is nonnegative at  $W$  because  $W \in \mathcal{P}_n$ . Every perfect-Delaunay inequality holds by the definition of  $K$ . Thus  $W$  satisfies the inequality associated with every extreme ray in Theorem 3.2. Conic generation then gives  $L_q(W) \geq 0$  for every  $q \in \text{Erdahl}(n)$ .

Suppose, for contradiction, that  $W \notin \text{MIQ}_n$ . Since  $\text{MIQ}_n$  is closed and convex, the separating-hyperplane theorem within  $\mathcal{H}_n$  gives a nonconstant affine inequality that is valid for  $\text{MIQ}_n$  and strictly violated by  $W$ . By (2.2), the corresponding quadratic belongs to  $\text{Erdahl}(n)$ . This contradicts the preceding paragraph. Therefore,  $W \in \text{MIQ}_n$ , which proves the reverse inclusion.  $\square$

### 3.2 Structure of perfect Delaunay polyhedra

The following coordinate form of Erdahl’s structure theorem [36, Structure Theorem 2.1, p. 392] describes a perfect Delaunay polyhedron as a product of a bounded configuration and a free lattice.

**Theorem 3.5** (Erdahl’s structure theorem). *Let  $D \subseteq \mathbb{Z}^n$  be a perfect Delaunay polyhedron. There exist an integer  $k \in \{1, \dots, n\}$ , an integer-affine bijection  $T(z) = Uz + t$  with  $U \in \text{GL}_n(\mathbb{Z})$  and  $t \in \mathbb{Z}^n$ , and a finite perfect Delaunay polyhedron  $D_0 \subseteq \mathbb{Z}^k$  with  $\text{aff}(D_0) = \mathbb{R}^k$ , such that*

$$T(D) = D_0 \times \mathbb{Z}^{n-k}.$$

Moreover, there is a quadratic  $q_0 \in \text{Erdahl}(k)$  with  $Z(q_0) = D_0$  such that

$$q_D(T^{-1}(u, v)) = q_0(u) = u^\top A_0 u + 2b_0^\top u + \alpha_0, \quad A_0 \succ 0,$$

for every  $(u, v) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ . In particular,  $D$  is bounded if and only if  $k = n$ .

This coordinate form follows by choosing bases of Erdahl’s complementary lattices and rescaling the defining quadratic of the bounded factor.

Since  $A_0 \succ 0$ , the quadratic  $q_0$  has a finite, attained continuous minimum. The same is therefore true of  $q_D$ , even when  $D$  is unbounded. This supplies the continuous minimum used to define depth in Section 4.

**Corollary 3.6** (Bounded facets). *For every perfect Delaunay polyhedron  $D \subseteq \mathbb{Z}^n$ , the corresponding facet  $F_D$  of  $\text{MIQ}_n$  is bounded if and only if  $D$  is finite.*

*Proof.* If  $D$  is unbounded, the matrices  $W_z$  with  $z \in D$  form an unbounded subset of  $F_D$ , so  $F_D$  is unbounded. Suppose now that  $D$  is finite. Theorem 3.5 implies that the quadratic part of  $q_D$  is positive definite. Completing the square, write

$$q_D(z) = (z - c)^\top A(z - c) - r, \quad A \succ 0.$$

For  $W = \begin{bmatrix} 1 & x^\top \\ x & X \end{bmatrix} \in F_D$ , put  $\Sigma = X - xx^\top \succeq 0$ , using  $\text{MIQ}_n \subseteq \mathcal{P}_n$ . Since  $L_{q_D}(W) = 0$ , with  $\lambda = \lambda_{\min}(A) > 0$  we obtain

$$r = (x - c)^\top A(x - c) + \langle A, \Sigma \rangle \geq \lambda(\|x - c\|_2^2 + \text{tr } \Sigma).$$

Thus  $x$  and  $\text{tr } \Sigma$  are bounded. Positive semidefiniteness bounds every entry of  $\Sigma$ , so  $X = xx^\top + \Sigma$  is also bounded. Hence  $F_D$  is bounded.  $\square$

Thus the bounded facets correspond exactly to perfect Delaunay polytopes. We next determine which perfect-Delaunay facet inequalities are recovered by one round of CG cuts.

## 4 Depth and the rank-one Chvátal–Gomory closure

In Section 3, we characterized the facet-defining inequalities of  $\text{MIQ}_n$ . We now explore which of these inequalities can be obtained as rank-one CG cuts from  $\mathcal{P}_n$ . Given that CG rounding depends on positive scaling, we first normalize the defining quadratics so that their linearizations have primitive integer normals in integral moment coordinates. We then introduce the notion of depth for quadratics and perfect Delaunay polyhedra, with which we categorize perfect Delaunay polyhedra as ‘shallow’ or ‘deep’. We show that, when a perfect Delaunay polyhedron is shallow, its associated facet-defining inequality for  $\text{MIQ}_n$  can be obtained in the first round of CG cuts for  $\mathcal{P}_n$ . On the other hand, we prove that the facet-defining inequalities of  $\text{MIQ}_n$  associated with deep perfect Delaunay polyhedra cannot be recovered from the rank-one CG closure of  $\mathcal{P}_n$ .

Determining whether CG rounding from  $\mathcal{P}_n$  recovers a facet-defining inequality requires an integral moment normal on its defining ray. For a nonconstant quadratic  $q \in \mathcal{Q}_n$ , let  $\mathbb{R}_+q := \{tq : t \geq 0\}$  denote the ray generated by  $q$ . We call this ray *rational* if it contains a nonzero quadratic with integer Newton coefficients.

**Lemma 4.1.** *For every perfect Delaunay polyhedron  $D \subseteq \mathbb{Z}^n$ , the ray  $\mathbb{R}_+q_D$  is rational.*

*Proof.* By perfection, the system

$$\alpha + c^\top \eta(z) = 0$$

for  $z \in D$  has a one-dimensional solution space, consisting of the Newton coefficient vectors of the polynomials in  $\text{Space}(D) = \mathbb{R}q_D$ . Since there are only  $d_n + 1$  unknowns, finitely many of these equations define the same solution space. Their coefficients are integers because  $\eta(z)$  is integral for every  $z \in D$ . Gaussian elimination over  $\mathbb{Q}$  therefore gives a nonzero rational solution. Choosing its sign and clearing denominators yields a positive multiple of  $q_D$  with integer Newton coefficients.  $\square$

Rationality allows us to choose an integral representative of the defining ray. We next normalize this representative by requiring its nonconstant Newton coefficient vector to be primitive. A nonzero coefficient vector  $c \in \mathbb{Z}^{d_n}$  is *primitive* if its entries have greatest common divisor (gcd) one. The next lemma shows that this normalization is unique and that the constant coefficient is also integral when the quadratic has a lattice zero.

**Lemma 4.2.** *Suppose that the nonconstant ray  $\mathbb{R}_+q$  is rational and that  $q$  has a lattice zero. Then there is a unique positive scaling*

$$\hat{q}(z) = \alpha + c^\top \eta(z), \quad \alpha \in \mathbb{Z}, \quad c \in \mathbb{Z}^{d_n} \text{ primitive.}$$

*Proof.* Since the ray is rational, choose a positive multiple

$$p(z) = \alpha' + c'^\top \eta(z)$$

of  $q$  with  $\alpha' \in \mathbb{Z}$  and  $c' \in \mathbb{Z}^{d_n}$ . Since  $q$  is nonconstant,  $c' \neq 0$ . Let

$$g := \gcd(c'_1, \dots, c'_{d_n})$$

and define

$$\hat{q} := \frac{p}{g} = \alpha + c^\top \eta(z), \quad c := \frac{c'}{g}.$$

By construction,  $c \in \mathbb{Z}^{d_n}$  is primitive.

It remains to show that  $\alpha = \alpha'/g$  is an integer. Let  $z^0 \in \mathbb{Z}^n$  be a lattice zero of  $q$ . Since  $p$  is a positive multiple of  $q$ ,  $p(z^0) = 0$ , and therefore

$$\alpha' = -c'^\top \eta(z^0).$$

Because every entry of  $c'$  is divisible by  $g$ , the right-hand side is divisible by  $g$ . Hence  $\alpha' / g \in \mathbb{Z}$ , so  $\hat{q}$  is integer-valued by Lemma 2.4.

Finally, this scaling is unique. Indeed, any other positive scaling of  $q$  with a primitive integral nonconstant Newton coefficient vector must have its coefficient vector equal to a positive scalar multiple of  $c$ . Since both vectors are integral and primitive, that scalar must be one.  $\square$

We call  $\hat{q}$  the *primitive integer-valued generator* of the ray. If  $q$  is nonconstant, integer-valued, and has a lattice zero, let  $\Delta(q)$  denote the greatest common divisor of the entries of  $c_q$ . Since  $q$  itself is an integral representative of its defining ray, Lemma 4.2 gives

$$\hat{q} = \frac{q}{\Delta(q)}. \quad (4.1)$$

For a perfect Delaunay polyhedron  $D$ , the normalized inequality  $L_{\hat{q}_D}(W) \geq 0$  defines the same facet of  $\text{MIQ}_n$  as  $L_{q_D}(W) \geq 0$ . The minimum value of  $L_{\hat{q}_D}$  over  $\text{MIQ}_n$  is zero, while by (2.9), its minimum over  $\mathcal{P}_n$  is the continuous minimum of  $\hat{q}_D$ . The notion of depth introduced below quantifies the gap between these two minima.

**Definition 4.3.** Let  $q$  be a nonconstant lattice-nonnegative quadratic such that  $\mathbb{R}_+q$  is rational,  $q$  has at least one lattice-zero, and  $\min_{y \in \mathbb{R}^n} q(y) > -\infty$ . The depth of  $q$  is

$$\delta(q) := -\min_{y \in \mathbb{R}^n} \hat{q}(y) = \min_{z \in \mathbb{Z}^n} \hat{q}(z) - \min_{y \in \mathbb{R}^n} \hat{q}(y) \geq 0. \quad (4.2)$$

This quantity depends only on the positive ray. For a perfect Delaunay polyhedron  $D$ , we write  $\delta(D) := \delta(q_D)$  and call  $D$  shallow when  $\delta(D) < 1$  and deep when  $\delta(D) \geq 1$ .

The lattice minimum in (4.2) is zero because  $q$  is lattice-nonnegative and has a lattice zero. For a perfect Delaunay polyhedron  $D$ , Lemma 4.1 ensures that its defining ray is rational, while the representation in Theorem 3.5 ensures that the continuous minimum of  $q_D$  is finite and attained. Hence  $\delta(D)$  is well defined.

Equation (2.9) gives

$$\min_{W \in \text{MIQ}_n} L_{\hat{q}_D}(W) = 0, \quad \min_{W \in \mathcal{P}_n} L_{\hat{q}_D}(W) = -\delta(D). \quad (4.3)$$

Thus, the depth of  $D$  is the difference between the minimum values of  $L_{\widehat{q}_D}$  over  $\text{MIQ}_n$  and  $\mathcal{P}_n$ . If  $q$  in Definition 4.3 is integer-valued, then (4.1) gives

$$\delta(q) = \frac{-\min_{y \in \mathbb{R}^n} q(y)}{\Delta(q)}.$$

The following example shows how normalization affects CG rounding.

**Example.** The seven-dimensional Gosset polytope  $3_{21}$  is a perfect Delaunay polytope [39]. We use the following defining quadratic on  $\mathbb{R}^7$ , written in the Newton basis:

$$\begin{aligned} q_{G_7}(z) = & 8 \sum_{i=1}^5 \binom{z_i}{2} + 16 \binom{z_6}{2} + 12 \binom{z_7}{2} + 2z_6 \\ & + 6 \sum_{1 \leq i < j \leq 5} z_i z_j + (10z_6 + 8z_7) \sum_{i=1}^5 z_i + 12z_6 z_7. \end{aligned}$$

This polynomial has zero constant term. Its nonconstant Newton coefficients are all even, and the coefficient of  $z_6$  is two. Their greatest common divisor is therefore two, so  $\widehat{q}_{G_7} = q_{G_7}/2$  is the primitive integer-valued generator. The continuous minimum of  $q_{G_7}$  is  $-3/2$ , attained at  $e_7/2$ , where  $e_7$  is the seventh unit vector. Appendix A gives the matrix representation and positive-definiteness check. Thus CG rounding gives

$$\begin{aligned} L_{q_{G_7}}(W) &\geq -\frac{3}{2} & \xrightarrow{\text{CG}} & L_{q_{G_7}}(W) \geq -1, \\ L_{\widehat{q}_{G_7}}(W) &\geq -\frac{3}{4} & \xrightarrow{\text{CG}} & L_{\widehat{q}_{G_7}}(W) \geq 0 \end{aligned}.$$

The two polynomials define the same facet, but rounding with the primitive normal recovers its exact inequality. Using  $q_{G_7}$  directly gives only the weaker bound  $L_{\widehat{q}_{G_7}}(W) \geq -1/2$ . The correctly normalized depth is  $\delta(3_{21}) = 3/4$ .

To use depth for all perfect Delaunay polyhedra, we next examine how it behaves under the structural operations introduced in Section 3.2. The following proposition shows that depth is positive and invariant under integer-affine transformations and products with free lattice factors.

**Proposition 4.4.** *Every perfect Delaunay polyhedron  $D$  has strictly positive depth. In addition, if  $T$  is an integer-affine bijection of  $\mathbb{R}^n$  and  $k \in \mathbb{Z}_{++}$ , then  $T(D)$  and  $D \times \mathbb{Z}^k$  are perfect. Moreover,*

$$\delta(T(D)) = \delta(D) \quad \text{and} \quad \delta(D \times \mathbb{Z}^k) = \delta(D).$$

*Proof.* We first prove  $\delta(D) > 0$  by showing that  $\delta(D) = 0$  would force  $\widehat{q}_D \equiv 0$ . Suppose that  $\delta(D) = 0$ , so  $\min_{\mathbb{R}^n} \widehat{q}_D = 0$ . Write  $\widehat{q}_D(y) = y^\top A y + 2b^\top y + \alpha$ , with symmetric  $A$ , and choose  $z^0 \in D$ . The equality  $\widehat{q}_D(z^0) = 0$  makes  $z^0$  a global minimizer. Thus,  $A \succeq 0$  and  $Az^0 + b = 0$ , which give

$$\widehat{q}_D(y) = (y - z^0)^\top A (y - z^0).$$

For a positive-semidefinite matrix  $A$ , the right-hand side is zero exactly when  $y - z^0 \in \ker A$ . Therefore,

$$\{y \in \mathbb{R}^n : \widehat{q}_D(y) = 0\} = z^0 + \ker A.$$

Every  $z \in D$  satisfies  $\widehat{q}_D(z) = 0$ , so  $D \subseteq z^0 + \ker A$ . The affine subspace  $z^0 + \ker A$  must therefore contain  $\text{aff}(D) = \mathbb{R}^n$ , as established in Section 2.2. It follows that  $\ker A = \mathbb{R}^n$  and  $A = 0$ . The

displayed expression for  $\hat{q}_D$  then gives  $\hat{q}_D \equiv 0$ , contradicting that it is a nonzero generator of the defining ray.

Next, we show that  $T(D)$  is perfect and that  $\delta(T(D)) = \delta(D)$ . By Theorem 3.3,  $T(D)$  is perfect, and its defining ray is generated by  $\hat{q}_D \circ T^{-1}$ . This polynomial is integer-valued and vanishes at the integer point  $T(z^0)$ , where  $z^0 \in D$ .

To compute the depth of  $T(D)$ , we must check that this generator's nonconstant Newton coefficient vector is primitive. Suppose it is not, so some integer  $m > 1$  divides all these coefficients. Since the polynomial vanishes at  $T(z^0)$ , its constant coefficient is also divisible by  $m$ . Thus,  $(\hat{q}_D \circ T^{-1})/m$  is integer-valued. Composing with  $T$  shows that  $\hat{q}_D/m$  is integer-valued. Lemma 2.4 then implies that every entry of its nonconstant Newton coefficient vector  $c_{\hat{q}_D}/m$  is an integer. This means that  $m$  divides every entry of  $c_{\hat{q}_D}$ , contradicting that this vector has gcd one. Hence,  $\hat{q}_D \circ T^{-1}$  is the primitive integer-valued generator of the transformed ray. Because  $T$  is a bijection of  $\mathbb{R}^n$ , the continuous minimum is unchanged. Therefore,  $\delta(T(D)) = \delta(D)$ .

Finally, we argue that  $D \times \mathbb{Z}^k$  is perfect and that  $\delta(D \times \mathbb{Z}^k) = \delta(D)$ . The extension  $(z, u) \mapsto \hat{q}_D(z)$  has lattice-zero set  $D \times \mathbb{Z}^k$ . For  $p \in \text{Space}(D \times \mathbb{Z}^k)$ , write

$$p(z, u) = r(z) + u^\top B u + \ell(z)^\top u, \quad r \in \mathcal{Q}_n, \quad B \in \mathbb{S}^k,$$

where  $\ell : \mathbb{R}^n \rightarrow \mathbb{R}^k$  is affine. For each  $z \in D$ , Lemma 2.1 gives  $p(z, u) = 0$  for all  $u \in \mathbb{R}^k$ , so  $B = 0$ ,  $\ell(z) = 0$ , and  $r(z) = 0$ . The identity  $\text{aff}(D) = \mathbb{R}^n$  gives  $\ell \equiv 0$ , while  $r \in \text{Space}(D) = \mathbb{R}\hat{q}_D$ . Thus,  $\text{Space}(D \times \mathbb{Z}^k)$  consists exactly of the scalar multiples of the extension, which indicates perfection. Its nonconstant Newton coefficient vector is obtained by inserting zero entries into that of  $\hat{q}_D$ , so it remains primitive. Its continuous minimum is unchanged, so  $\delta(D \times \mathbb{Z}^k) = \delta(D)$ .  $\square$

We now determine when CG rounding strengthens the bound  $-\delta(D)$  in (4.3) to the exact bound zero, thereby giving the facet-defining inequality  $L_{\hat{q}_D}(W) \geq 0$  of  $\text{MIQ}_n$ . We express rank-one CG cuts for  $\mathcal{P}_n$  in the form  $L_p(W) \geq 0$ , where  $p$  is an integer-valued quadratic. For a nonzero  $c \in \mathbb{Z}^{d_n}$  with  $\beta_{\mathcal{P}_n}(c) > -\infty$ , we call

$$p(z) := c^\top \eta(z) - \lceil \beta_{\mathcal{P}_n}(c) \rceil \tag{4.4}$$

the *rank-one CG cut polynomial associated with  $c$* . The associated inequality  $L_p(W) \geq 0$  is the rank-one CG cut  $c^\top \eta(W) \geq \lceil \beta_{\mathcal{P}_n}(c) \rceil$  for  $\mathcal{P}_n$ . Equation (2.9) gives

$$\min_{y \in \mathbb{R}^n} p(y) = \beta_{\mathcal{P}_n}(c) - \lceil \beta_{\mathcal{P}_n}(c) \rceil \in (-1, 0]. \tag{4.5}$$

In the next lemma, we discuss exactly when a nonzero integer-valued quadratic is a rank-one CG cut polynomial.

**Lemma 4.5.** *A nonzero polynomial  $p \in \text{Int}_2(\mathbb{Z}^n)$  is a rank-one CG cut if and only if it has a finite continuous minimum satisfying*

$$-1 < \min_{y \in \mathbb{R}^n} p(y) \leq 0. \tag{4.6}$$

*Its normal is uniquely determined by its nonconstant Newton coefficients. Thus,*

$$\mathcal{C}_n = \bigcap_{\substack{p \in \text{Int}_2(\mathbb{Z}^n) \setminus \{0\} \\ -1 < \min_{y \in \mathbb{R}^n} p(y) \leq 0}} \{W \in \mathcal{H}_n : L_p(W) \geq 0\}.$$

*Proof.* Equation (4.5) proves necessity. For the converse, consider  $p(z) = \alpha + c^\top \eta(z)$ , where  $\alpha \in \mathbb{Z}$  and  $c \in \mathbb{Z}^{d_n}$ . A nonzero integer-valued constant cannot satisfy (4.6), so  $c \neq 0$ . If  $m := \min_{y \in \mathbb{R}^n} p(y) \in (-1, 0]$ , then (2.9) gives

$$\beta_{\mathcal{P}_n}(c) = m - \alpha, \quad \lceil \beta_{\mathcal{P}_n}(c) \rceil = -\alpha.$$

Hence,  $p$  has the form (4.4). The Newton-basis representation uniquely determines the coefficient vector  $c$ . We substitute this polynomial representation of the cuts into (2.10) and apply the coordinate identification of Lemma 2.4 to obtain the asserted description of  $\mathcal{C}_n$ . We omit  $c = 0$  because it gives only the trivial inequality.  $\square$

We next determine when a single CG cut along a facet normal recovers the facet inequality.

**Proposition 4.6** (CG rounding along a facet normal). *Let  $D \subseteq \mathbb{Z}^n$  be a perfect Delaunay polyhedron, with primitive integer-valued generator*

$$\widehat{q}_D(z) = \alpha_D + c_D^\top \eta(z), \quad \alpha_D \in \mathbb{Z}, \quad c_D \in \mathbb{Z}^{d_n} \text{ primitive.}$$

*For every integer  $\mu \geq 1$ , the rank-one CG cut obtained from the normal  $\mu c_D$ , after division by  $\mu$ , is*

$$L_{\widehat{q}_D}(W) \geq -\frac{\lfloor \mu \delta(D) \rfloor}{\mu}. \quad (4.7)$$

*This cut is the facet inequality  $L_{\widehat{q}_D}(W) \geq 0$  if and only if  $\mu \delta(D) < 1$ . Equivalently,  $\mu \widehat{q}_D$  is a rank-one CG cut polynomial if and only if  $\mu \delta(D) < 1$ .*

*Proof.* Since

$$\widehat{q}_D(z) = \alpha_D + c_D^\top \eta(z) \quad \text{and} \quad \min_{z \in \mathbb{R}^n} \widehat{q}_D(z) = -\delta(D),$$

we have

$$\beta_{\mathcal{P}_n}(\mu c_D) = -\mu \alpha_D - \mu \delta(D).$$

Because  $\mu \alpha_D$  is integral,

$$\lceil \beta_{\mathcal{P}_n}(\mu c_D) \rceil = -\mu \alpha_D - \lfloor \mu \delta(D) \rfloor.$$

Hence the CG cut associated with the normal  $\mu c_D$  is

$$\mu c_D^\top \eta(W) \geq -\mu \alpha_D - \lfloor \mu \delta(D) \rfloor.$$

Adding  $\mu \alpha_D$  and dividing by  $\mu$  gives

$$L_{\widehat{q}_D}(W) \geq -\frac{\lfloor \mu \delta(D) \rfloor}{\mu}.$$

This is the facet inequality  $L_{\widehat{q}_D}(W) \geq 0$  exactly when  $\lfloor \mu \delta(D) \rfloor = 0$ , or equivalently when  $\mu \delta(D) < 1$ .

Finally, the corresponding cut polynomial is

$$\mu \widehat{q}_D + \lfloor \mu \delta(D) \rfloor,$$

which equals  $\mu \widehat{q}_D$  exactly under the same condition.  $\square$

Taking  $\mu = 1$  shows that every shallow facet is obtained by CG rounding in its primitive normal. Because  $c_D$  is primitive, every integral normal on its positive ray has the form  $\mu c_D$  with an integer  $\mu \geq 1$ . For a deep perfect Delaunay polyhedron  $D$ , Proposition 4.6 therefore rules out recovering the facet inequality by a single cut in this normal direction. It remains to determine whether rank-one CG cuts with other normals jointly imply this inequality. In general, an infinite family of inequalities can imply an inequality without containing a positive multiple of it. For example,

$$\bigcap_{k \geq 1} \{(x, y) \in \mathbb{R}^2 : kx + y \geq 0\} \subseteq \{(x, y) \in \mathbb{R}^2 : x \geq 0\},$$

because any  $x < 0$  violates  $kx + y \geq 0$  for sufficiently large  $k$ . For the rigidity argument, we first present a coefficient bound for arbitrary real quadratics bounded below by  $-1$ .

**Lemma 4.7** (Newton coefficient bound). *Let  $p(z) = \alpha_p + c_p^\top \eta(z) \in \mathcal{Q}_n$  satisfy  $p(y) \geq -1$  for every  $y \in \mathbb{R}^n$ . For every  $W^0 \in \mathcal{P}_n$  with  $W^0 \succ 0$ ,*

$$\|c_p\|_\infty \leq \frac{3(1 + L_p(W^0))}{\lambda_{\min}(W^0)}.$$

*Proof.* The idea is to represent the globally nonnegative quadratic  $p + 1$  by a positive-semidefinite matrix and then use  $W^0 \succ 0$  to control its entries. Let  $M := M_{p+1}$ , the coefficient matrix of the shifted polynomial  $p + 1$ . For  $t \neq 0$  and  $y \in \mathbb{R}^n$ ,

$$\begin{bmatrix} t \\ y \end{bmatrix}^\top M \begin{bmatrix} t \\ y \end{bmatrix} = t^2(p(y/t) + 1) \geq 0.$$

By continuity, the same inequality holds when  $t = 0$ , and hence  $M \succeq 0$ . Let  $\lambda := \lambda_{\min}(W^0) > 0$ . Since  $W^0 \succeq \lambda I$ ,  $W_{00}^0 = 1$ , and  $M \succeq 0$ ,

$$1 + L_p(W^0) = \langle M, W^0 \rangle \geq \lambda \operatorname{tr}(M).$$

Therefore,

$$\operatorname{tr}(M) \leq \frac{1 + L_p(W^0)}{\lambda}.$$

It remains to relate the entries of  $M$  to the Newton coefficients of  $p$ . Since adding a constant does not change the nonconstant Newton coefficients,  $p$  and  $p + 1$  have the same vector  $c_p$ . Solving the relations in (2.7) for these coefficients gives

$$a_i = 2M_{0i} + M_{ii}, \quad d_i = 2M_{ii}, \quad \gamma_{ij} = 2M_{ij} \quad (i < j).$$

Because  $M \succeq 0$ , every entry of  $M$  has absolute value at most  $\operatorname{tr}(M)$ . Hence

$$|d_i| \leq 2 \operatorname{tr}(M), \quad |\gamma_{ij}| \leq 2 \operatorname{tr}(M), \quad |a_i| \leq 3 \operatorname{tr}(M).$$

Thus,

$$\|c_p\|_\infty \leq 3 \operatorname{tr}(M) \leq \frac{3(1 + L_p(W^0))}{\lambda_{\min}(W^0)}.$$

□

We use this bound to prove the following rigidity theorem. Applied to rank-one CG cuts, it rules out recovering a deep facet inequality through cuts with other normals.

**Theorem 4.8** (Rigidity theorem). *Let  $D \subseteq \mathbb{Z}^n$  be a perfect Delaunay polyhedron, and let  $\mathcal{F} \subseteq \operatorname{Int}_2(\mathbb{Z}^n) \setminus \{0\}$  satisfy  $\inf_{\mathbb{R}^n} p > -1$  for every  $p \in \mathcal{F}$ . Define*

$$\mathcal{K}_{\mathcal{F}} := \{W \in \mathcal{P}_n : L_p(W) \geq 0 \text{ for every } p \in \mathcal{F}\}.$$

*Then  $L_{\hat{q}_D} \geq 0$  is valid for  $\mathcal{K}_{\mathcal{F}}$  if and only if  $\mathcal{F}$  contains  $\mu \hat{q}_D$  for some integer  $\mu \geq 1$ .*

*Proof.* We first prove validity when  $\mathcal{F}$  contains  $\mu \hat{q}_D$  for some integer  $\mu \geq 1$ . Every  $p \in \mathcal{F}$  is lattice-nonnegative because its values on  $\mathbb{Z}^n$  are integers strictly greater than  $-1$ . Thus,  $\operatorname{MIQ}_n \subseteq \mathcal{K}_{\mathcal{F}}$ . If  $\mu \hat{q}_D \in \mathcal{F}$ , every  $W \in \mathcal{K}_{\mathcal{F}}$  satisfies  $\mu L_{\hat{q}_D}(W) \geq 0$ . We divide by  $\mu > 0$  and conclude that  $L_{\hat{q}_D} \geq 0$  is valid for  $\mathcal{K}_{\mathcal{F}}$ .

For the converse, we assume that  $L_{\widehat{q}_D} \geq 0$  is valid for  $\mathcal{K}_{\mathcal{F}}$  and prove that  $\mathcal{F}$  contains a positive integral multiple of  $\widehat{q}_D$ . For a contradiction, suppose that  $\mathcal{F}$  contains no such multiple. We construct a matrix in  $\mathcal{K}_{\mathcal{F}}$  at which  $L_{\widehat{q}_D}$  is negative. Specifically, we will first choose a positive-definite matrix on the facet associated with  $D$  and show that it strictly satisfies every inequality indexed by  $\mathcal{F}$ . We will then use a uniform coefficient estimate to choose a perturbation that remains in  $\mathcal{K}_{\mathcal{F}}$  and makes  $L_{\widehat{q}_D}$  negative.

To construct the starting matrix on the facet associated with  $D$ , we take the average of the matrices  $W_z$  corresponding to selected points  $z \in D$ . By Lemma 3.1, we choose  $z^1, \dots, z^{d_n} \in D$  such that  $W_{z^1}, \dots, W_{z^{d_n}}$  form a basis of  $\text{span}\{W_z : z \in D\}$ , and define

$$\overline{W} := \frac{1}{d_n} \sum_{i=1}^{d_n} W_{z^i}.$$

For every  $p \in \mathcal{Q}_n$ , we have  $p(z) = \langle M_p, W_z \rangle$ . The basis property therefore gives

$$\{p \in \mathcal{Q}_n : p(z^i) = 0 \text{ for } i = 1, \dots, d_n\} = \text{Space}(D) = \mathbb{R}\widehat{q}_D.$$

As a convex combination of the matrices  $W_{z^i}$ , the matrix  $\overline{W}$  belongs to  $\text{MIQ}_n$ . Moreover,  $L_{\widehat{q}_D}(\overline{W}) = 0$  because  $\widehat{q}_D(z^i) = 0$  for every selected point. Thus,  $\overline{W}$  belongs to the facet associated with  $D$ .

We now prove that  $\overline{W} \succ 0$ . For a contradiction, suppose that  $\overline{W}$  is not positive definite. Each  $W_{z^i}$  is positive semidefinite, so  $\overline{W} \succeq 0$ . We can therefore choose a nonzero  $v \in \mathbb{R}^{n+1}$  with

$$0 = v^\top \overline{W} v = \frac{1}{d_n} \sum_{i=1}^{d_n} \left( v^\top \begin{bmatrix} 1 \\ z^i \end{bmatrix} \right)^2.$$

Every summand is nonnegative, so  $v^\top \begin{bmatrix} 1 \\ z^i \end{bmatrix} = 0$  for  $i = 1, \dots, d_n$ . The preceding characterization of  $\text{Space}(D)$  then places the nonzero affine polynomial  $z \mapsto v^\top \begin{bmatrix} 1 \\ z \end{bmatrix}$  in  $\mathbb{R}\widehat{q}_D$ . Because  $\widehat{q}_D$  is not affine, as established in Section 2.2, we reach a contradiction. Hence,  $\overline{W} \succ 0$ .

Next, we establish a positive lower bound on  $L_p(\overline{W})$  that holds for every  $p \in \mathcal{F}$ . Let

$$s(p) := L_p(\overline{W}) = \frac{1}{d_n} \sum_{i=1}^{d_n} p(z^i).$$

The summands are nonnegative integers. Fix  $p \in \mathcal{F}$  and, for a contradiction, suppose that  $s(p) = 0$ . Then  $p(z^i) = 0$  for every  $i$ , so  $p = \mu \widehat{q}_D$  for some nonzero real  $\mu$ . Because  $\widehat{q}_D$  is nonzero and lattice-nonnegative, Lemma 2.1 gives a point  $z \in \mathbb{Z}^n$  with  $\widehat{q}_D(z) > 0$ . The inequality  $p(z) = \mu \widehat{q}_D(z) \geq 0$  and the condition  $\mu \neq 0$  imply  $\mu > 0$ . Moreover,  $c_p = \mu c_D$  is integral and  $c_D$  is primitive, so we obtain  $\mu \in \mathbb{Z}$  from Bézout's identity. We have found a positive integral multiple of  $\widehat{q}_D$  in  $\mathcal{F}$ , which is a contradiction. Thus, at least one summand in the definition of  $s(p)$  is a positive integer, and we obtain

$$s(p) \geq \frac{1}{d_n}, \quad \forall p \in \mathcal{F}. \quad (4.8)$$

To control how the inequalities change under perturbation, apply Lemma 4.7 to each  $p \in \mathcal{F}$  at  $W^0 = \overline{W}$ . With  $\lambda := \lambda_{\min}(\overline{W}) > 0$ , this gives

$$\|c_p\|_\infty \leq \frac{3}{\lambda}(1 + s(p)).$$

Since (4.8) implies  $1 + s(p) \leq (d_n + 1)s(p)$ , we obtain

$$\|c_p\|_\infty \leq C_D s(p), \quad \forall p \in \mathcal{F}, \quad C_D := \frac{3(d_n + 1)}{\lambda}. \quad (4.9)$$

The constant  $C_D$  depends only on the selected points of  $D$ .

Finally, we construct a matrix in  $\mathcal{K}_{\mathcal{F}}$  that violates  $L_{\hat{q}_D} \geq 0$ . We choose  $\varepsilon > 0$  sufficiently small that

$$W^\varepsilon := W(\eta(\overline{W}) - \varepsilon c_D) \succ 0, \quad \varepsilon C_D \|c_D\|_1 \leq \frac{1}{2}.$$

We can make this choice because  $\overline{W} \succ 0$  and  $\eta \mapsto W(\eta)$  is continuous. We use (2.5) and (4.9) to verify that, for every  $p \in \mathcal{F}$ ,

$$\begin{aligned} L_p(W^\varepsilon) &= s(p) - \varepsilon c_p^\top c_D \\ &\geq s(p) - \varepsilon \|c_p\|_\infty \|c_D\|_1 \\ &\geq (1 - \varepsilon C_D \|c_D\|_1) s(p) \geq \frac{1}{2} s(p) > 0. \end{aligned}$$

Thus,  $W^\varepsilon \in \mathcal{K}_{\mathcal{F}}$ , whereas

$$L_{\hat{q}_D}(W^\varepsilon) = -\varepsilon \|c_D\|_2^2 < 0.$$

We assumed that  $L_{\hat{q}_D} \geq 0$  is valid for  $\mathcal{K}_{\mathcal{F}}$ , so we reach a contradiction.  $\square$

We now apply Theorem 4.8 to the family of all rank-one CG cut polynomials to characterize the facet-defining inequalities that are valid for  $\mathcal{C}_n$ .

**Corollary 4.9.** *For every perfect Delaunay polyhedron  $D \subseteq \mathbb{Z}^n$ , the following conditions are equivalent:*

- (i)  $\hat{q}_D$  is a rank-one CG cut polynomial;
- (ii)  $L_{\hat{q}_D} \geq 0$  is valid for  $\mathcal{C}_n$ ;
- (iii)  $\delta(D) < 1$ .

The CG rank of an inequality relative to  $\mathcal{P}_n$  is the minimum number of successive CG rounds needed to recover it, or  $+\infty$  if no finite number suffices. For every deep  $D$ , the facet-defining inequality  $L_{\hat{q}_D}(W) \geq 0$  fails at some point of  $\mathcal{C}_n$ . Therefore, its CG rank is at least two.

*Proof.* With  $\mu = 1$ , Proposition 4.6 gives the equivalence between the first and third conditions. To establish the equivalence with validity for  $\mathcal{C}_n$ , let  $\mathcal{F}$  be the family of all rank-one CG cut polynomials. Their continuous minima belong to  $(-1, 0]$ , so they satisfy the conditions of Theorem 4.8. Lemma 4.5 and the inclusion  $\mathcal{C}_n \subseteq \mathcal{P}_n$  from Section 2.4 give  $\mathcal{K}_{\mathcal{F}} = \mathcal{C}_n$ . By Theorem 4.8, the inequality  $L_{\hat{q}_D} \geq 0$  is valid for  $\mathcal{C}_n$  if and only if  $\mathcal{F}$  contains  $\mu \hat{q}_D$  for some integer  $\mu \geq 1$ . By Proposition 4.6, such a multiple exists if and only if  $\delta(D) < 1$ .  $\square$

We combine Corollary 4.9 with Proposition 3.4 to characterize exactly when  $\mathcal{C}_n = \text{MIQ}_n$ .

**Corollary 4.10.** *For every  $n \geq 1$ ,  $\mathcal{C}_n = \text{MIQ}_n$  if and only if  $\delta(D) < 1$  for every perfect Delaunay polyhedron  $D \subseteq \mathbb{Z}^n$ .*

*Proof.* We first assume that every perfect Delaunay polyhedron in  $\mathbb{Z}^n$  is shallow. By Corollary 4.9, every perfect-Delaunay facet inequality is valid for  $\mathcal{C}_n$ . Proposition 3.4 therefore gives  $\mathcal{C}_n \subseteq \text{MIQ}_n$ . The reverse inclusion holds by (2.15), so  $\mathcal{C}_n = \text{MIQ}_n$ . Conversely, suppose that  $\mathcal{C}_n = \text{MIQ}_n$ . Every facet-defining inequality of  $\text{MIQ}_n$  is then valid for  $\mathcal{C}_n$ . Corollary 4.9 implies that every perfect Delaunay polyhedron in  $\mathbb{Z}^n$  is shallow.  $\square$

In the next section, we will use the classifications of perfect Delaunay polyhedra and their depths to determine the dimensions in which  $\mathcal{C}_n = \text{MIQ}_n$ .

## 5 Exact recovery of $\text{MIQ}_n$

Our results in the previous sections reduce the comparison of  $\text{MIQ}_n$ ,  $\mathcal{C}_n$ , and  $\mathcal{S}_n$  to the geometry of perfect Delaunay polyhedra. By Theorem 3.3, these polyhedra correspond to the facets of  $\text{MIQ}_n$ , while Corollary 4.9 determines whether the associated facet-defining inequality is obtained by one round of CG cuts. We now use the known classification of perfect Delaunay polyhedra in low dimensions to make these results concrete. In particular, we identify the perfect Delaunay polyhedra that occur up to dimension seven, compute their depths, and then use these calculations to determine exactly when  $\mathcal{C}_n$  and  $\mathcal{S}_n$  coincide with  $\text{MIQ}_n$ . Specifically, we prove that  $\text{MIQ}_n = \mathcal{C}_n$  for  $1 \leq n \leq 6$ ,  $\text{MIQ}_n = \mathcal{S}_n$  for  $1 \leq n \leq 5$ , and  $\mathcal{C}_n \subsetneq \mathcal{S}_n$  for every  $n \geq 6$ . We further prove the strict inclusions in higher dimensions using the results established in Section 4.

Throughout this section, we use *type* to refer to an equivalence class of perfect Delaunay polyhedra in a fixed dimension under integer-affine bijections. Perfect Delaunay polytopes have been classified through dimension seven: Erdahl treats dimensions at most five [36, Theorem 5.1 and Corollary 5.2], and Dutour Sikirić treats dimensions six [40, Theorem 2] and seven [41, Theorem 1.1]. For the seven-dimensional polytopes, we use the quadratic-form matrices given by Dutour Sikirić, Erdahl, and Rybnikov [39, Sections 7.1 and 7.2], with the defining quadratics specified in Appendix A. As in Section 2.2, we use the names of these polytopes for their lattice-point sets when discussing integer-affine equivalence, computing depth, or taking products with  $\mathbb{Z}^k$ .

**Theorem 5.1** (Depth classification up to dimension seven). *Up to integer-affine equivalence, the perfect Delaunay polyhedra in  $\mathbb{Z}^n$  for  $1 \leq n \leq 7$ , together with their depths and classification as shallow or deep, are those in Table 1. In particular,  $ER_7$  is the only deep type up to dimension seven, with depth  $43/32$ .*

Table 1: Perfect-Delaunay types and depths up to dimension seven.

| Dimension         | Type                                                 | Depth   | Status  |
|-------------------|------------------------------------------------------|---------|---------|
| $1 \leq n \leq 7$ | $\{0, 1\} \times \mathbb{Z}^{n-1}$ (primitive split) | $1/8$   | shallow |
| 6                 | $2_{21}$                                             | $2/3$   | shallow |
| 7                 | $2_{21} \times \mathbb{Z}$                           | $2/3$   | shallow |
| 7                 | $3_{21}$                                             | $3/4$   | shallow |
| 7                 | $ER_7$                                               | $43/32$ | deep    |

*Proof.* By Theorem 3.5, every perfect Delaunay polyhedron  $D \subseteq \mathbb{Z}^n$  has, after an integer-affine change of coordinates, the form

$$D = D_0 \times \mathbb{Z}^{n-k}, \quad 1 \leq k \leq n,$$

where  $D_0 \subseteq \mathbb{Z}^k$  is the lattice-point set of a  $k$ -dimensional perfect Delaunay polytope. Proposition 4.4 gives  $\delta(D) = \delta(D_0)$ . Thus it suffices to classify these polytopes and compute their depths.

The cited classifications of perfect Delaunay polytopes provide the following types in dimensions up to seven: the type  $\{0, 1\}$  in dimension one, no types in dimensions two to five, the Gosset type  $2_{21}$  in dimension six, and  $3_{21}$  and  $ER_7$  in dimension seven. The products  $\{0, 1\} \times \mathbb{Z}^{n-1}$  represent the primitive-split Delaunay polyhedra  $D_{v,s}$  from (2.12). Up to dimension seven, the only further product with a positive-dimensional integer lattice is  $2_{21} \times \mathbb{Z}$ . These types are summarized in Table 1.

For each integer-valued defining quadratic  $q \in \mathcal{Q}_n$ , the depth is  $-\min_{y \in \mathbb{R}^n} q(y)/\Delta(q)$  by Definition 4.3 and (4.1). Appendix A presents the coefficient and positive-definiteness checks, so we give only the resulting minima and primitive normalizations here.

For the type  $\{0, 1\}$ , take  $v = 1$  and  $s = 0$  in (2.11). Completing the square gives

$$\widehat{q}_{1,0}(z) = \binom{z}{2} = \frac{1}{2} \left( z - \frac{1}{2} \right)^2 - \frac{1}{8}.$$

Its nonconstant Newton coefficient vector is  $(0, 1)$ , so  $\Delta(\widehat{q}_{1,0}) = 1$  and  $\delta(\{0, 1\}) = 1/8$ .

Burer and Letchford [1, Section 5.3, (10)] exhibited a bounded facet of  $\text{MIQ}_6$  containing 27 extreme points at equality. They observed, without giving a formal proof, that the convex hull of the corresponding  $x$ -vectors is an affine image of the six-dimensional Gosset polytope  $2_{21}$ . For this type, we specialize their lifted facet inequality to obtain the defining quadratic

$$q_{G_6}(z) = \sum_{i=1}^6 (z_i^2 - z_i) + 2z_6 + z_1z_2 + z_2z_3 + z_3z_4 + z_4z_5 + z_5z_1 - z_6 \sum_{i=1}^5 z_i. \quad (5.1)$$

Because the cited inequality is facet-defining for  $\text{MIQ}_6$ , Theorem 3.3 shows that  $Z(q_{G_6})$  is perfect. Appendix A verifies that its quadratic-form matrix is positive definite, so  $Z(q_{G_6})$  is finite. The six-dimensional classification [40, Theorem 2] therefore identifies  $Z(q_{G_6})$  as an integer-affine image of  $2_{21}$ . In addition, Appendix A gives

$$\min_{\mathbb{R}^6} q_{G_6} = -\frac{2}{3}, \quad \Delta(q_{G_6}) = 1, \quad \widehat{q}_{G_6} = q_{G_6}, \quad \delta(2_{21}) = \frac{2}{3}.$$

For  $3_{21}$  and  $ER_7$ , use the defining quadratics  $q_{G_7}$  and  $q_{ER}$  in Appendix A, where we give the explicit matrices and verify that both nonconstant Newton coefficient vectors have greatest common divisor two. Hence,

$$\begin{aligned} \min_{\mathbb{R}^7} q_{G_7} &= -\frac{3}{2}, & \widehat{q}_{G_7} &= \frac{1}{2}q_{G_7}, & \delta(3_{21}) &= \frac{3}{4}, \\ \min_{\mathbb{R}^7} q_{ER} &= -\frac{43}{16}, & \widehat{q}_{ER} &= \frac{1}{2}q_{ER}, & \delta(ER_7) &= \frac{43}{32} > 1. \end{aligned}$$

These calculations and the depth equality above give all values and classifications as shallow or deep in Table 1.  $\square$

The depth calculations now determine the facet-defining inequalities obtained from  $\mathcal{P}_n$  by CG rounding. By Proposition 4.6, rounding the inequalities  $L_{\widehat{q}_{G_6}} \geq -2/3$  over  $\mathcal{P}_6$  and  $L_{\widehat{q}_{G_7}} \geq -3/4$  over  $\mathcal{P}_7$ , using the nonconstant Newton coefficient vectors of these primitive integer-valued generators, gives the corresponding facet-defining inequalities with right-hand side zero. For  $ER_7$ , rounding gives only  $L_{\widehat{q}_{ER}} \geq -1$ . Corollary 4.9 further shows that  $L_{\widehat{q}_{ER}} \geq 0$  is not valid for  $\mathcal{C}_7$ . We next combine these facts with the description of  $\text{MIQ}_n$  in Proposition 3.4 to determine exactly when the two closures describe the integer quadratic moment hull.

The inclusions in (2.15) hold in every dimension. We first establish the equality cases. The classification up to dimension six, together with the description of  $\text{MIQ}_n$  in Proposition 3.4, is sufficient for this step.

**Lemma 5.2.** *For  $1 \leq n \leq 5$ ,  $\text{MIQ}_n = \mathcal{C}_n = \mathcal{S}_n$ . Moreover,  $\text{MIQ}_6 = \mathcal{C}_6$ .*

*Proof.* For  $1 \leq n \leq 5$ , Theorem 5.1 shows that every perfect Delaunay polyhedron is integer-affinely equivalent to  $\{0, 1\} \times \mathbb{Z}^{n-1}$ . These are exactly the primitive-split Delaunay polyhedra  $D_{v,s}$ . Theorem 3.3 therefore identifies every facet-defining inequality of  $\text{MIQ}_n$  as a primitive split inequality, up to positive scaling. Because these inequalities describe  $\text{MIQ}_n$  by Proposition 3.4, the inclusions in (2.15) give  $\text{MIQ}_n = \mathcal{C}_n = \mathcal{S}_n$ .

In dimension six, Theorem 5.1 shows that every perfect Delaunay polyhedron is integer-affinely equivalent to  $\{0, 1\} \times \mathbb{Z}^5$  or to  $2_{21}$ . Both are shallow. Corollary 4.10 therefore gives  $\text{MIQ}_6 = \mathcal{C}_6$ .  $\square$

To prove  $\mathcal{C}_n \subsetneq \mathcal{S}_n$  in higher dimensions, we next exhibit a facet-defining inequality of  $\text{MIQ}_n$  that is valid for  $\mathcal{C}_n$  but not for  $\mathcal{S}_n$ . The inequalities associated with  $2_{21}$  and its products with integer lattices give such examples in every dimension  $n \geq 6$ .

**Lemma 5.3.** *For every  $n \geq 6$ ,  $\mathcal{C}_n \subsetneq \mathcal{S}_n$ .*

*Proof.* Fix  $n \geq 6$  and define

$$\hat{q}_{G_6}^{(n)}(z) := \hat{q}_{G_6}(z_1, \dots, z_6), \quad z \in \mathbb{R}^n.$$

For  $n = 6$ , this is the original primitive integer-valued generator. For  $n > 6$ , its lattice-zero set is  $Z(q_{G_6}) \times \mathbb{Z}^{n-6}$ , which is perfect and has depth  $2/3$  by Proposition 4.4. The nonconstant Newton coefficient vector remains primitive because introducing additional variables only inserts zero coefficients. Thus, in every dimension  $n \geq 6$ , the inequality  $L_{\hat{q}_{G_6}^{(n)}} \geq 0$  is facet-defining for  $\text{MIQ}_n$  and valid for  $\mathcal{C}_n$  by Theorem 3.3 and Corollary 4.9.

To see that this inequality is not valid for  $\mathcal{S}_n$ , apply Theorem 4.8 to the family

$$\mathcal{F} := \{\hat{q}_{v,s} : v \in \mathbb{Z}^n \text{ primitive, } s \in \mathbb{Z}\}.$$

Every polynomial in  $\mathcal{F}$  is integer-valued and has continuous minimum  $-1/8 > -1$ . Moreover,  $\mathcal{K}_{\mathcal{F}} = \mathcal{S}_n$  because  $\mathcal{S}_n \subseteq \mathcal{P}_n$  by Proposition 2.5. The quadratic-form matrix of  $\hat{q}_{G_6}^{(n)}$  has rank six, whereas that of  $\hat{q}_{v,s}$  is  $vv^\top/2$  and has rank one. Hence,  $\mathcal{F}$  contains no positive integral multiple of  $\hat{q}_{G_6}^{(n)}$ . Theorem 4.8 therefore shows that its facet-defining inequality is not valid for  $\mathcal{S}_n$ . Because the same inequality is valid for  $\mathcal{C}_n$ , we obtain  $\mathcal{C}_n \subsetneq \mathcal{S}_n$  for every  $n \geq 6$ .  $\square$

We now show that  $\mathcal{C}_n$  itself fails to equal  $\text{MIQ}_n$  for  $n \geq 7$ , using the facet-defining inequalities associated with  $ER_7$  and its products with integer lattices.

**Lemma 5.4.** *For every  $n \geq 7$ ,  $\text{MIQ}_n \subsetneq \mathcal{C}_n$ .*

*Proof.* Fix  $n \geq 7$  and define

$$\hat{q}_{ER}^{(n)}(z) := \hat{q}_{ER}(z_1, \dots, z_7), \quad z \in \mathbb{R}^n.$$

For  $n = 7$ , its lattice-zero set is the given representative of  $ER_7$  and has depth  $43/32$ . For  $n > 7$ , its lattice-zero set is  $Z(q_{ER}) \times \mathbb{Z}^{n-7}$ , which remains perfect and has the same depth by Proposition 4.4. Thus,  $L_{\hat{q}_{ER}^{(n)}} \geq 0$  is a facet-defining inequality of  $\text{MIQ}_n$  that is not valid for  $\mathcal{C}_n$ , by Theorem 3.3 and Corollary 4.9. This proves the strict inclusion for every  $n \geq 7$ .  $\square$

With these three lemmas, we give the complete comparison of  $\text{MIQ}_n$  with the two closures  $\mathcal{C}_n$  and  $\mathcal{S}_n$ . We also identify the facet-defining inequalities that remain valid for  $\mathcal{C}_7$  with Table 1. In turn, we determine the dimensions in which  $\text{MIQ}_n$  is described by the split inequalities or by the rank-one CG closure of  $\mathcal{P}_n$ .

**Theorem 5.5.** *For every  $n \geq 1$ ,*

$$\begin{aligned} 1 \leq n \leq 5 : & \text{ MIQ}_n = \mathcal{C}_n = \mathcal{S}_n, \\ n = 6 : & \text{ MIQ}_6 = \mathcal{C}_6 \subsetneq \mathcal{S}_6, \\ n \geq 7 : & \text{ MIQ}_n \subsetneq \mathcal{C}_n \subsetneq \mathcal{S}_n. \end{aligned}$$

Moreover,  $\mathcal{S}_n \subsetneq \mathcal{P}_n$  for  $n \geq 1$ . For every perfect Delaunay polyhedron  $D \subseteq \mathbb{Z}^7$ ,  $L_{\widehat{qD}} \geq 0$  is valid for  $\mathcal{C}_7$  if and only if  $D$  is not integer-affinely equivalent to  $ER_7$ .

*Proof.* Lemma 5.2 gives the equality cases. Lemmas 5.3 and 5.4 give the strict inclusions between  $\text{MIQ}_n$ ,  $\mathcal{C}_n$ , and  $\mathcal{S}_n$ . Proposition 2.5 gives  $\mathcal{S}_n \subsetneq \mathcal{P}_n$ . Finally, Corollary 4.9 says that  $L_{\widehat{qD}} \geq 0$  is valid for  $\mathcal{C}_7$  exactly when  $\delta(D) < 1$ . By Theorem 5.1, this holds exactly when  $D$  is not integer-affinely equivalent to  $ER_7$ .  $\square$

In dimension seven, adding the facet inequalities associated with all integer-affine images of  $ER_7$  to  $\mathcal{C}_7$  gives the full moment hull. Let  $T(z) = Uz + t$  range over all integer-affine bijections of  $\mathbb{R}^7$ , where  $U \in \text{GL}_7(\mathbb{Z})$  and  $t \in \mathbb{Z}^7$ . Then

$$\text{MIQ}_7 = \mathcal{C}_7 \cap \bigcap_T \{W \in \mathcal{H}_7 : L_{q_{ER_7 \circ T^{-1}}}(W) \geq 0\}.$$

The added inequalities give all facet inequalities of  $\text{MIQ}_7$  associated with integer-affine images of  $ER_7$ . Every other perfect-Delaunay facet inequality is valid for  $\mathcal{C}_7$  by Theorem 5.5, so Proposition 3.4 gives the equality.

## 6 Complexity of membership and separation

We now study the complexity of membership and exact separation for the rank-one CG closure  $\mathcal{C}_n$ , when the dimension  $n$  is part of the input. Given a rational point  $\bar{\eta} \in \mathbb{Q}^{d_n}$ , we consider the decision problems of determining whether  $\bar{\eta} \notin \mathcal{C}_n$  (CG-VIOLATION) or  $\bar{\eta} \in \mathcal{C}_n$  (CG-MEMBERSHIP). Exact separation means either certifying membership or finding a violated rank-one CG inequality. We first show that every rational point outside  $\mathcal{C}_n$  admits a polynomial-size certificate of nonmembership that can be verified in polynomial time. We then establish strong NP-completeness of CG-VIOLATION and strong NP-hardness of exact separation over both  $\mathcal{C}_n$  and the split-inequality closure  $\mathcal{S}_n$ .

### 6.1 Polynomial-size certificates

Throughout this section, we measure input size by the total binary encoding length of  $(n, \bar{\eta})$ , with rational coordinates represented as reduced fractions.

By Lemma 4.5,  $\bar{\eta} \notin \mathcal{C}_n$  if and only if there exists  $p \in \text{Int}_2(\mathbb{Z}^n)$  such that

$$-1 < \min_{y \in \mathbb{R}^n} p(y) \leq 0, \quad L_p(\bar{\eta}) < 0. \quad (6.1)$$

We call such a polynomial a *cut-polynomial certificate* of nonmembership. We encode  $p$  by its ordered vector of Newton coefficients, as defined in Section 2.3.

Given a proposed certificate  $p(z) = \alpha_p + c_p^\top \eta(z)$ , we can verify its validity in polynomial time. First, we check that its Newton coefficients are integers. Using (2.6), write

$$p(z) = z^\top A_p z + 2b_p^\top z + \alpha_p.$$

The continuous minimum of  $p$  is finite if and only if

$$A_p \succeq 0 \quad \text{and} \quad b_p \in \text{range}(A_p).$$

Both conditions can be checked by exact rational linear algebra. When they hold, we solve  $A_p y = -b_p$  and obtain

$$\min_{z \in \mathbb{R}^n} p(z) = \alpha_p + b_p^\top y,$$

which is independent of the chosen solution. We then check whether this minimum belongs to  $(-1, 0]$  and evaluate

$$L_p(\bar{\eta}) = \alpha_p + c_p^\top \bar{\eta}$$

to determine whether the inequality is violated. All these operations have polynomial bit complexity in the combined encoding lengths of the input and the certificate.

The construction below uses zero extension, by which we mean enlarging a moment matrix by adjoining zero rows and columns. The next lemma shows that this operation preserves membership in the rank-one CG closure.

Fix integers  $1 \leq m < n$ . Using the coordinate bijection in Lemma 2.4, we define the linear map  $\iota : \mathbb{R}^{d_m} \rightarrow \mathbb{R}^{d_n}$  by retaining all integral moment coordinates involving only the first  $m$  variables and setting every coordinate involving one of the last  $n - m$  variables to zero. Equivalently, after ordering the indices so that the new variables come last,

$$W(\iota(\eta)) = \begin{bmatrix} W(\eta) & 0 \\ 0 & 0_{n-m} \end{bmatrix} \quad (\eta \in \mathbb{R}^{d_m}),$$

where  $0_{n-m}$  is the  $(n - m) \times (n - m)$  zero matrix and the off-diagonal blocks are zero matrices of the required sizes. The block formula uniquely determines  $\iota(\eta)$  because  $\eta \mapsto W(\eta)$  is a bijection. The following lemma establishes the CG-closure membership equivalence.

**Lemma 6.1.** *For every  $\eta \in \mathbb{R}^{d_m}$ ,*

$$\eta \in \mathcal{C}_m \iff \iota(\eta) \in \mathcal{C}_n.$$

*Proof.* We first record an evaluation identity for lifting certificates. For  $q \in \mathcal{Q}_m$ , define  $\tilde{q} \in \mathcal{Q}_n$  by

$$\tilde{q}(z) := q(z_1, \dots, z_m), \quad z \in \mathbb{R}^n.$$

With the variables ordered as above, the coefficient matrix of  $\tilde{q}$  is

$$M_{\tilde{q}} = \begin{bmatrix} M_q & 0 \\ 0 & 0_{n-m} \end{bmatrix}.$$

For every  $\eta \in \mathbb{R}^{d_m}$ , the definition of  $\iota$  gives

$$L_{\tilde{q}}(\iota(\eta)) = \langle M_{\tilde{q}}, W(\iota(\eta)) \rangle = \langle M_q, W(\eta) \rangle = L_q(\eta).$$

For the membership equivalence, first suppose that  $\eta \in \mathcal{C}_m$ . Consider any  $c \in \mathbb{Z}^{d_n}$  satisfying  $\beta_{\mathcal{P}_n}(c) > -\infty$ . From this vector, form  $c' \in \mathbb{Z}^{d_m}$  by retaining the coefficients of the nonconstant Newton basis functions involving only the first  $m$  variables. Thus,

$$c'^\top \eta(y) = c^\top \eta((y, 0)) \quad (y \in \mathbb{R}^m),$$

where  $(y, 0) \in \mathbb{R}^n$  has its last  $n - m$  coordinates equal to zero. By (2.9),

$$\begin{aligned}\beta_{\mathcal{P}_m}(c') &= \inf_{y \in \mathbb{R}^m} c'^\top \eta((y, 0)) \\ &\geq \inf_{z \in \mathbb{R}^n} c'^\top \eta(z) = \beta_{\mathcal{P}_n}(c) > -\infty.\end{aligned}$$

Therefore,

$$c^\top \iota(\eta) = c'^\top \eta \geq \lceil \beta_{\mathcal{P}_m}(c') \rceil \geq \lceil \beta_{\mathcal{P}_n}(c) \rceil,$$

where the first inequality is the defining CG inequality for  $c'$ ; when  $c' = 0$ , it is the trivial inequality  $0 \geq 0$ . Because  $c$  was arbitrary,  $\iota(\eta) \in \mathcal{C}_n$ . On the other hand, suppose that  $\iota(\eta) \in \mathcal{C}_n$ . Consider any  $c \in \mathbb{Z}^{d_m}$  satisfying  $\beta_{\mathcal{P}_m}(c) > -\infty$ . Form  $\tilde{c} \in \mathbb{Z}^{d_n}$  by retaining these Newton coefficients and assigning coefficient zero to every basis function involving an additional variable. We thereby obtain a quadratic that is independent of the additional variables, so (2.9) gives

$$\beta_{\mathcal{P}_n}(\tilde{c}) = \beta_{\mathcal{P}_m}(c), \quad \tilde{c}^\top \iota(\eta) = c^\top \eta.$$

Apply the defining CG inequality for  $\tilde{c}$  at  $\iota(\eta)$  and use these identities to obtain  $c^\top \eta \geq \lceil \beta_{\mathcal{P}_m}(c) \rceil$ . Because  $c$  was arbitrary,  $\eta$  satisfies every defining inequality of  $\mathcal{C}_m$ .  $\square$

Now we show that a certificate of polynomial encoding length always exists.

**Theorem 6.2** (Polynomial-size cut-polynomial certificates). *Every rational point  $\bar{\eta} \notin \mathcal{C}_n$  admits a cut-polynomial certificate whose binary encoding length is bounded by a polynomial in the input length of  $(n, \bar{\eta})$ . Consequently, CG-VIOLATION  $\in$  NP.*

*Proof.* Let

$$\bar{W} = \begin{bmatrix} 1 & \bar{x}^\top \\ \bar{x} & \bar{X} \end{bmatrix}, \quad \Sigma := \bar{X} - \bar{x}\bar{x}^\top.$$

By (6.1), every  $\bar{\eta} \notin \mathcal{C}_n$  admits a violated cut-polynomial certificate. We show that such a certificate can be chosen with polynomial binary encoding length by considering three cases.

*Indefinite moment matrix.* Suppose first that  $\bar{W}$  is indefinite. Since  $\bar{W}$  has a negative eigenvalue, scaling and rounding a corresponding eigenvector yields an integer vector  $v = (v_0, v') \in \mathbb{Z}^{n+1}$  of polynomial encoding length such that

$$v^\top \bar{W} v < 0.$$

Appendix B.1 shows the scaling factor and provides the bit-length bound. Necessarily  $v' \neq 0$ , because  $\bar{W}_{00} = 1$ . The affine square

$$p_v(z) := (v_0 + v'^\top z)^2$$

is integer-valued, has continuous minimum zero, and satisfies

$$L_{p_v}(\bar{\eta}) = v^\top \bar{W} v < 0.$$

Its Newton coefficients are sums and pairwise products of entries of  $v$ , so it is a polynomial-size certificate.

*Positive-definite covariance.* Suppose  $\bar{W} \succeq 0$  and  $\Sigma \succ 0$ . Choose any certificate  $p$  from (6.1), and write

$$p(z) = z^\top A z + 2b^\top z + \alpha, \quad A \succeq 0, \quad b \in \text{range}(A).$$

Let  $\min p = -\delta$ , where  $0 \leq \delta < 1$ . A violated certificate cannot have  $\delta = 0$ , because then

$$L_p(\bar{\eta}) = p(\bar{x}) + \langle A, \Sigma \rangle \geq 0.$$

Thus  $0 < \delta < 1$ . Choose  $y$  satisfying  $Ay = -b$ . Completing the square gives

$$L_p(\bar{\eta}) = -\delta + (\bar{x} - y)^\top A(\bar{x} - y) + \langle A, \Sigma \rangle. \quad (6.2)$$

Since both terms following  $-\delta$  are nonnegative, the violation implies

$$\langle A, \Sigma \rangle < 1. \quad (6.3)$$

Let  $\lambda = \lambda_{\min}(\Sigma) > 0$ , then  $\lambda \operatorname{tr}(A) \leq \langle A, \Sigma \rangle$ , and hence

$$\|A\|_2 \leq \operatorname{tr}(A) < \lambda^{-1}.$$

Because  $\Sigma$  is rational and positive definite, the binary length of a valid upper bound on  $\lambda^{-1}$  is polynomial in the input length. The identity (6.2) then bounds  $b$  and  $\alpha$ , and hence all Newton coefficients of  $p$ , by numbers of polynomial binary length. The quantitative estimates are given in Appendix B.2.

*Singular covariance.* It remains to consider  $\bar{W} \succeq 0$  with singular covariance  $\Sigma$ . Let  $r = \operatorname{rank}(\Sigma) < n$ . In this case, the bound  $\langle A, \Sigma \rangle < 1$  still holds, but no longer controls  $A$  along directions in  $\ker(\Sigma)$ . We therefore find an integer basis of  $\ker(\Sigma) \cap \mathbb{Z}^n$  and handle these directions separately.

Choose a positive integer  $D_\Sigma$  that clears the denominators of  $\Sigma$ , and set

$$S = D_\Sigma \Sigma \in \mathbb{Z}^{n \times n}.$$

When  $r > 0$ , since  $\ker(S) = \ker(\Sigma)$ , we can apply the full-row-rank extension of the Kannan–Bachem Hermite-normal-form algorithm [42, Section 2, especially Theorem 2 and p. 504] to  $r$  linearly independent rows of  $S$  to obtain a unimodular matrix

$$Q = [Q_1 \ Z] \in \operatorname{GL}_n(\mathbb{Z}),$$

where  $Q_1 \in \mathbb{Z}^{n \times r}$  consists of the first  $r$  columns of  $Q$ , and the columns of  $Z \in \mathbb{Z}^{n \times (n-r)}$  form a lattice basis of  $\ker(\Sigma) \cap \mathbb{Z}^n$ . When  $r = 0$ , we take  $Q = I_n$ . The matrix  $Q$  has polynomial encoding length; Appendix B.3 provides the verification and the bit bounds.

Suppose first that  $Z^\top \bar{x} \notin \mathbb{Z}^{n-r}$ . Then some column  $v$  of  $Z$  satisfies  $v^\top \bar{x} \notin \mathbb{Z}$ . Since  $Q$  is unimodular,  $v$  is primitive. Let  $s = \lfloor v^\top \bar{x} \rfloor$  and consider the split quadratic

$$\hat{q}_{v,s}(z) = \frac{1}{2}(v^\top z - s)(v^\top z - s - 1).$$

Because  $\Sigma v = 0$ , we have

$$L_{\hat{q}_{v,s}}(\bar{\eta}) = \frac{1}{2}\theta(\theta - 1) < 0, \quad \theta := v^\top \bar{x} - s \in (0, 1).$$

Thus,  $\hat{q}_{v,s}$  is a violated cut-polynomial certificate. Its coefficients have polynomial encoding length because  $v$  and  $s$  do.

Otherwise,  $Z^\top \bar{x} \in \mathbb{Z}^{n-r}$ . In this case, we can eliminate the zero-variance directions by an integer-affine transformation. Let

$$U = Q^{-\top}, \quad t = U \begin{bmatrix} 0 \\ Z^\top \bar{x} \end{bmatrix} \in \mathbb{Z}^n.$$

Under the change of variables  $z = t + Uy$ , the transformed mean is

$$\bar{y} = U^{-1}(\bar{x} - t) = Q^\top \bar{x} - \begin{bmatrix} 0 \\ Z^\top \bar{x} \end{bmatrix} = \begin{bmatrix} Q_1^\top \bar{x} \\ 0 \end{bmatrix}.$$

Similarly, since  $\Sigma Z = 0$ , the transformed covariance is

$$\Sigma' = U^{-1} \Sigma U^{-\top} = Q^\top \Sigma Q = \begin{bmatrix} Q_1^\top \Sigma Q_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Defining  $\bar{y}_r := Q_1^\top \bar{x}$ ,  $\Sigma_r := Q_1^\top \Sigma Q_1$ , we obtain

$$\bar{y} = \begin{bmatrix} \bar{y}_r \\ 0 \end{bmatrix}, \quad \Sigma' = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma_r \succ 0.$$

If  $r = 0$ , the transformed point is  $W_0 \in \mathcal{C}_n$ , contradicting  $\bar{\eta} \notin \mathcal{C}_n$ . Hence  $r \geq 1$ . Since the columns of  $Z$  span  $\ker(\Sigma)$  and  $Q$  is invertible, we have  $\Sigma_r \succ 0$ . Thus, the transformed moment point is the exact zero extension of an  $r$ -dimensional point  $\bar{\eta}_r$  with positive-definite covariance.

By integer-affine invariance and Lemma 6.1,  $\bar{\eta}_r \notin \mathcal{C}_r$ . The positive-definite case therefore gives a polynomial-size cut-polynomial certificate  $p_r$  for  $\bar{\eta}_r$ . Extending  $p_r$  by zero variables and applying the inverse coordinate transformation yields a certificate for  $\bar{\eta}$ . Since the transformation and its inverse have polynomial encoding length, the resulting Newton coefficients also have polynomial encoding length. The details are given in Appendix B.3.

In every case,  $\bar{\eta} \notin \mathcal{C}_n$  admits a polynomial-size cut-polynomial certificate. □

## 6.2 Strong NP-completeness and consequences

Having shown that CG-VIOLATION belongs to NP, we now establish its strong NP-hardness by a reduction from a gap version of the nearest-codeword problem. The reduction produces rational input points with numerator and denominator magnitudes polynomially bounded in  $n$ .

The nearest-codeword problem seeks a vector  $u$  that minimizes the number of violated equations in a linear system over a finite field. We use a gap version that distinguishes systems in which some assignment violates few equations from systems in which every assignment violates many equations. The following hardness result is a consequence of Håstad's inapproximability theorem [43]. The notations  $\mathbb{F}_{11}$  and  $\mathbb{Z}_+$  denote the field of integers modulo 11 and nonnegative integers, respectively.

**Theorem 6.3** (Fixed-field gap nearest codeword; consequence of Håstad [43]). *Given*

$$A \in \mathbb{F}_{11}^{N \times r}, \quad t \in \mathbb{F}_{11}^N, \quad k \in \mathbb{Z}_+,$$

*define*

$$D_H := \min_{u \in \mathbb{F}_{11}^r} \|Au - t\|_0.$$

*It is NP-hard to distinguish instances satisfying*

$$D_H \leq k \quad \text{from those satisfying} \quad D_H > 100k.$$

*Proof.* The proof of Håstad's theorem on MAX-E3-LIN over finite Abelian groups—specifically, its completeness and soundness lemmas—gives the following statement for the additive group of  $\mathbb{F}_{11}$  [43, Theorem 5.9 and Lemmas 5.10–5.11]. For every fixed

$$\rho \in \left(0, \frac{1 - 1/11}{2}\right),$$

it is NP-hard to distinguish a system of  $N_0$  linear equations over  $\mathbb{F}_{11}$  for which some assignment satisfies at least  $(1 - \rho)N_0$  equations from one for which every assignment satisfies at most  $(1/11 + \rho)N_0$  equations.

Set  $\rho := 1/220$  and repeat every equation exactly 220 times. Let  $A \in \mathbb{F}_{11}^{N \times r}$  be the coefficient matrix of the repeated system, let  $t \in \mathbb{F}_{11}^N$  be its right-hand side, and set

$$N := 220N_0, \quad k := N_0.$$

For every assignment  $u$ , the Hamming weight  $\|Au - t\|_0$  is exactly the number of repeated equations violated by  $u$ .

In the YES case, some assignment violates at most  $N_0/220$  of the original equations. After repetition, it therefore violates at most

$$220 \frac{N_0}{220} = N_0 = k$$

equations, and hence  $D_H \leq k$ . In the NO case, every assignment violates at least

$$\left(1 - \frac{1}{11} - \frac{1}{220}\right) N_0 = \frac{199}{220} N_0$$

of the original equations. Consequently,

$$D_H \geq 220 \frac{199}{220} N_0 = 199N_0 = 199k > 100k.$$

If  $A$  is not of full column rank, compute a basis of its column space over  $\mathbb{F}_{11}$  and replace the columns of  $A$  by that basis. This preserves the linear code  $\{Au : u \in \mathbb{F}_{11}^r\}$ , and therefore preserves  $D_H$ , while making  $A$  full column rank. The repetition and column-basis operations are deterministic and polynomial-time, and the construction satisfies  $N = 220k$ .  $\square$

The reduction below uses the instances constructed above, with  $A$  of full column rank and  $N = 220k$ . We next construct an integer least-squares problem whose minimum is a fixed multiple of  $D_H$ . After minimizing over auxiliary integer variables, every violated equation contributes the same amount to the squared error. Choose integer representatives

$$\hat{A} \in \{0, \dots, 10\}^{N \times r}, \quad \hat{t} \in \{0, \dots, 10\}^N.$$

Let  $M := 10N$ , index its rows by pairs  $(j, a)$  with  $j = 1, \dots, N$  and  $a = 1, \dots, 10$ , and put  $m := r + M$ . For  $h = (u, \zeta) \in \mathbb{Z}^r \times \mathbb{Z}^M$ , define  $B \in \mathbb{Z}^{M \times m}$  and  $\tau \in \mathbb{Z}^M$  by

$$(Bh)_{j,a} := a(\hat{A}u)_j + 11\zeta_{j,a}, \quad \tau_{j,a} := a\hat{t}_j.$$

**Lemma 6.4** (Scalar invariance). *For every integer  $d$  with  $11 \nmid d$ ,*

$$\min_{h \in \mathbb{Z}^m} \|Bh - d\tau\|_2^2 = 110D_H.$$

*Proof.* Fix  $d$  and first fix the integer vector  $u$ . For coordinate  $j$ , let  $r_j := (Au - dt)_j \in \mathbb{F}_{11}$ . Minimizing over the independent variables  $\zeta_{j,a}$  gives

$$\sum_{a=1}^{10} \text{dist}(ar_j, 11\mathbb{Z})^2.$$

The contribution is zero when  $r_j = 0$ . When  $r_j \neq 0$ , multiplication by  $r_j$  permutes the nonzero residues modulo 11, whose centered representatives are

$$\pm 1, \pm 2, \pm 3, \pm 4, \pm 5.$$

The contribution is therefore

$$2(1^2 + 2^2 + 3^2 + 4^2 + 5^2) = 110.$$

After minimizing over  $\zeta$ ,

$$\min_{\zeta} \|B(u, \zeta) - d\tau\|_2^2 = 110\|Au - dt\|_0.$$

Changing  $u$  by an element of  $11\mathbb{Z}^r$  can be absorbed into  $\zeta$ , so minimization over integer  $u$  is minimization over  $\mathbb{F}_{11}^r$ . Multiplication by  $d \neq 0$  bijects  $\mathbb{F}_{11}^r$  and preserves Hamming weight, so

$$\min_{u \in \mathbb{F}_{11}^r} \|Au - dt\|_0 = \min_{u \in \mathbb{F}_{11}^r} \|Au - t\|_0 = D_H.$$

□

Define the positive-semidefinite Gram matrix

$$G := \begin{bmatrix} -\tau & B \end{bmatrix}^\top \begin{bmatrix} -\tau & B \end{bmatrix} \succeq 0.$$

Thus, for  $d \in \mathbb{Z}$  and  $h \in \mathbb{Z}^m$ ,

$$\begin{bmatrix} d \\ h \end{bmatrix}^\top G \begin{bmatrix} d \\ h \end{bmatrix} = \|Bh - d\tau\|_2^2. \quad (6.4)$$

Set

$$\varepsilon := \frac{1}{44}, \quad T := 220k, \quad n := m + 1.$$

The first decision coordinate is the anchor, with unit vector  $e_1 \in \mathbb{R}^n$ . Define

$$\bar{x} := \frac{1}{2}e_1, \quad \Sigma := \left(\frac{1}{4} - \varepsilon\right) e_1 e_1^\top + \frac{\varepsilon}{T} G, \quad \bar{X} := \bar{x} \bar{x}^\top + \Sigma. \quad (6.5)$$

Because  $G \succeq 0$  and  $1/4 - \varepsilon = 5/22 > 0$ , the Schur complement shows that the associated rational point  $\bar{\eta}$  belongs to  $\mathcal{P}_n$ .

**Lemma 6.5** (Extraction from an arbitrary violated cut). *If  $\bar{\eta} \notin \mathcal{C}_n$ , then there exist  $d \in \{1, \dots, 7\}$  and  $h \in \mathbb{Z}^m$  such that*

$$\begin{bmatrix} d \\ h \end{bmatrix}^\top G \begin{bmatrix} d \\ h \end{bmatrix} < d^2 T.$$

*Proof.* Choose a cut-polynomial certificate  $p$  from (6.1) and write it in the integral-Hessian form

$$p(z) = \frac{1}{2} z^\top H z + \ell^\top z + \alpha.$$

By (2.8),  $H \in \mathbb{S}^n(\mathbb{Z})$ , and boundedness below implies  $H \succeq 0$ . Since  $p$  is integer-valued and  $\min_{\mathbb{R}^n} p > -1$ ,

$$p(\mathbb{Z}^n) \subseteq \mathbb{Z}_+.$$

Set

$$d := H_{11}, \quad \sigma_p := p(0) + p(e_1) \in \mathbb{Z}_+.$$

The quadratic midpoint identity gives

$$p\left(\frac{1}{2}e_1\right) = \frac{p(0) + p(e_1)}{2} - \frac{d}{8} = \frac{\sigma_p}{2} - \frac{d}{8}. \quad (6.6)$$

Because  $\min p > -1$ , we obtain

$$d < 4\sigma_p + 8, \quad \text{hence} \quad d \leq 4\sigma_p + 7. \quad (6.7)$$

For a quadratic with Hessian  $H$ , evaluation at moments with covariance  $\Sigma$  satisfies

$$L_p(\bar{\eta}) = p(\bar{x}) + \frac{1}{2} \text{tr}(H\Sigma).$$

Substituting (6.5) and (6.6) yields

$$L_p(\bar{\eta}) = \frac{\sigma_p}{2} + \frac{\varepsilon}{2T} (\text{tr}(HG) - dT). \quad (6.8)$$

The matrices  $H$  and  $G$  are positive semidefinite, so  $\text{tr}(HG) \geq 0$ . If  $\sigma_p \geq 1$ , then (6.7) and  $\varepsilon = 1/44$  give

$$L_p(\bar{\eta}) \geq \frac{\sigma_p}{2} - \frac{4\sigma_p + 7}{88} = \frac{40\sigma_p - 7}{88} > 0,$$

contradicting violation. Hence  $\sigma_p = 0$ , and nonnegativity on  $\mathbb{Z}^n$  gives  $p(0) = p(e_1) = 0$ . The bound (6.7) gives  $d \leq 7$ . Positive semidefiniteness gives  $d \geq 0$ , while  $d = 0$  would make (6.8) nonnegative. Hence

$$1 \leq d \leq 7.$$

Let  $h \in \mathbb{Z}^m$  contain the last  $m$  entries of the first column of  $H$ . The Schur complement gives

$$H - \frac{1}{d} \begin{bmatrix} d \\ h \end{bmatrix} \begin{bmatrix} d \\ h \end{bmatrix}^\top \succeq 0.$$

Taking the trace product with  $G \succeq 0$  yields

$$\text{tr}(HG) \geq \frac{1}{d} \begin{bmatrix} d \\ h \end{bmatrix}^\top G \begin{bmatrix} d \\ h \end{bmatrix}.$$

Since  $\sigma_p = 0$  and  $L_p(\bar{\eta}) < 0$ , (6.8) gives  $\text{tr}(HG) < dT$ . Combining the last two inequalities proves the claim.  $\square$

**Theorem 6.6** (Strong NP-completeness and consequences). *The language CG-VIOLATION is strongly NP-complete, even for input points in  $\mathcal{P}_n$ . Consequently, exact separation over  $\mathcal{C}_n$  is strongly NP-hard, and CG-MEMBERSHIP is strongly coNP-complete.*

*Proof.* Membership in NP follows from Theorem 6.2. We prove hardness using the construction above.

Suppose first that  $D_H \leq k$ . Lemma 6.4 with  $d = 1$  gives  $h \in \mathbb{Z}^m$  such that, for  $v = (1, h) \in \mathbb{Z}^n$ ,

$$v^\top G v = 110D_H \leq 110k < T.$$

The vector  $v$  is primitive because its anchor component is one. Consider the primitive split quadratic

$$q_v(z) := \frac{1}{2}(v^\top z)(v^\top z - 1).$$

Since  $q_v$  is integer-valued on  $\mathbb{Z}^n$  and  $\min_{z \in \mathbb{R}^n} q_v(z) = -1/8$ , it is a rank-one CG cut polynomial by Lemma 4.5. Since  $\bar{x} = \frac{1}{2}e_1$  and  $v = (1, h)$ , we have  $v^\top \bar{x} = \frac{1}{2}$ . Using  $\bar{X} = \bar{x}\bar{x}^\top + \Sigma$  and (6.5), we obtain

$$\begin{aligned} L_{q_v}(\bar{\eta}) &= \frac{1}{2}(v^\top \bar{X} v - v^\top \bar{x}) \\ &= \frac{1}{2}((v^\top \bar{x})^2 + v^\top \Sigma v - v^\top \bar{x}) = \frac{1}{2}\left(v^\top \Sigma v - \frac{1}{4}\right) \\ &= \frac{1}{2}\left(\frac{1}{4} - \varepsilon + \frac{\varepsilon}{T}v^\top G v - \frac{1}{4}\right) \\ &= \frac{\varepsilon}{2T}(v^\top G v - T) < 0, \end{aligned}$$

where the last inequality follows from  $v^\top G v < T$ . Thus,  $\bar{\eta} \notin \mathcal{C}_n$ .

Now suppose that  $D_H > 100k$  and, toward a contradiction, that  $\bar{\eta} \notin \mathcal{C}_n$ . Lemma 6.5 gives  $d \in \{1, \dots, 7\}$  and  $h \in \mathbb{Z}^m$  such that

$$\begin{bmatrix} d \\ h \end{bmatrix}^\top G \begin{bmatrix} d \\ h \end{bmatrix} < d^2 T \leq 7^2 \cdot 220k = 10780k.$$

However, since  $1 \leq d \leq 7 < 11$ , Lemma 6.4 and (6.4) imply

$$\begin{bmatrix} d \\ h \end{bmatrix}^\top G \begin{bmatrix} d \\ h \end{bmatrix} \geq 110D_H > 11000k,$$

a contradiction. Hence  $\bar{\eta} \in \mathcal{C}_n$  in the NO case. We have proved

$$D_H \leq k \implies \bar{\eta} \notin \mathcal{C}_n, \quad D_H > 100k \implies \bar{\eta} \in \mathcal{C}_n,$$

which has the required orientation for a deterministic Karp reduction to CG-VIOLATION.

The entries of  $B$  and  $\tau$  are bounded by an absolute constant, so  $G$  has entries  $O(N)$ . Since  $T = N$  and  $n = 1 + r + 10N = \Theta(N)$ , (6.5) and the conversion to integral moment coordinates give polynomial bounds in  $n$  for all numerators and denominators of  $\bar{\eta}$ . The construction is polynomial-time and produces points in  $\mathcal{P}_n$ , proving strong NP-completeness.

Finally, CG-VIOLATION is the decision version of exact separation. Hence exact separation over  $\mathcal{C}_n$  is strongly NP-hard. Since CG-MEMBERSHIP is the complement of the strongly NP-complete language CG-VIOLATION, it is strongly coNP-complete.  $\square$

**Corollary 6.7** (Strong NP-hardness of split-inequality separation). *When  $n$  is part of the input, deciding whether a rational point violates some split inequality is strongly NP-hard. Consequently, exact separation over  $\mathcal{S}_n$  is strongly NP-hard. This resolves the conjecture of Burer and Letchford [1, p. 255].*

*Proof.* Use the reduction in the proof of Theorem 6.6. If  $D_H \leq k$ , the vector  $v = (1, h)$  constructed there is primitive, and the split quadratic  $q_v = \hat{q}_{v,0}$  satisfies  $L_{q_v}(\bar{\eta}) < 0$ . Hence  $\bar{\eta} \notin \mathcal{S}_n$ . If  $D_H > 100k$ , the same proof gives  $\bar{\eta} \in \mathcal{C}_n$ , and therefore  $\bar{\eta} \in \mathcal{S}_n$  by (2.15). The reduction has bounded rational data and produces only points in  $\mathcal{P}_n$ , so the claim follows.  $\square$

## 7 Conclusion

We characterize the facets of  $\text{MIQ}_n$  through perfect Delaunay polyhedra and show that a facet inequality is valid for the rank-one CG closure  $\mathcal{C}_n$  exactly when its depth, measured after primitive integer-valued normalization of the defining quadratic, is less than one. Known classifications and exact depth calculations then establish that one CG round recovers the hull exactly for  $1 \leq n \leq 6$ , while split inequalities do so exactly for  $1 \leq n \leq 5$ , with  $\mathcal{C}_n \subsetneq \mathcal{S}_n$  for every  $n \geq 6$ . For rational input, we prove the existence of polynomial-size certificates of nonmembership in  $\mathcal{C}_n$ . When  $n$  is part of the input, nonmembership is strongly NP-complete and membership is strongly coNP-complete. The same reduction proves strong NP-hardness of quadratic split-inequality separation, resolving Burer and Letchford’s conjecture [1, p. 255].

Future work could determine the CG rank of the  $ER_7$  facet and develop criteria based on depth for recovery under iterated closures. Another direction is to characterize the facets and other maximal proper faces of mixed-integer quadratic moment hulls, including their inheritance from the integer moment hull and the effects of nonnegativity or linear coupling constraints.

## A Low-dimensional generators and exact verification

We record the exact representatives used in Table 1. For a quadratic written as

$$q(z) = (z - c)^\top A(z - c) - r,$$

the continuous minimum is  $-r$ , and its depth is therefore  $r/\Delta(q)$ .

**Primitive splits.** For primitive  $v \in \mathbb{Z}^n$  and  $s \in \mathbb{Z}$ , the quadratic  $\hat{q}_{v,s}$  of (2.11) has  $\Delta(\hat{q}_{v,s}) = 1$  and continuous minimum  $-1/8$ .

**$2_{21}$  in dimension six.** The polynomial in (5.1) satisfies

$$q_{G_6}(z) = \left(z - \frac{1}{3}\mathbf{1}\right)^\top A_{G_6} \left(z - \frac{1}{3}\mathbf{1}\right) - \frac{2}{3},$$

where

$$A_{G_6} = \begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}.$$

Its leading principal minors are

$$1, \quad \frac{3}{4}, \quad \frac{1}{2}, \quad \frac{5}{16}, \quad \frac{1}{8}, \quad \frac{3}{64},$$

so  $A_{G_6} \succ 0$ . The nonconstant Newton coefficients have greatest common divisor one, and hence  $\delta(2_{21}) = 2/3$ .

$3_{21}$  in dimension seven. Let

$$A_7 = \begin{bmatrix} 4 & 3 & 3 & 3 & 3 & 5 & 4 \\ 3 & 4 & 3 & 3 & 3 & 5 & 4 \\ 3 & 3 & 4 & 3 & 3 & 5 & 4 \\ 3 & 3 & 3 & 4 & 3 & 5 & 4 \\ 3 & 3 & 3 & 3 & 4 & 5 & 4 \\ 5 & 5 & 5 & 5 & 5 & 8 & 6 \\ 4 & 4 & 4 & 4 & 4 & 6 & 6 \end{bmatrix}, \quad c_7 = \frac{1}{2}e_7.$$

Then

$$A_7 c_7 = (2, 2, 2, 2, 2, 3, 3)^\top, \quad c_7^\top A_7 c_7 = \frac{3}{2},$$

and

$$q_{G_7}(z) := (z - c_7)^\top A_7 (z - c_7) - \frac{3}{2}$$

has zero constant term. Its Newton coefficients are

$$a = (0, 0, 0, 0, 0, 2, 0), \quad d = (8, 8, 8, 8, 8, 16, 12), \quad \gamma_{ij} = 2(A_7)_{ij}.$$

Their greatest common divisor is two, so

$$\widehat{q}_{G_7} = \frac{1}{2}q_{G_7}, \quad \delta(3_{21}) = \frac{3}{4}.$$

The leading principal minors of  $A_7$  are

$$4, \quad 7, \quad 10, \quad 13, \quad 16, \quad 3, \quad 2,$$

so  $A_7 \succ 0$ .

$ER_7$ . Let

$$A_{ER} = \begin{bmatrix} 8 & 6 & 6 & 6 & 6 & 6 & 9 \\ 6 & 8 & 6 & 6 & 6 & 6 & 9 \\ 6 & 6 & 8 & 6 & 6 & 6 & 9 \\ 6 & 6 & 6 & 8 & 6 & 6 & 9 \\ 6 & 6 & 6 & 6 & 8 & 6 & 9 \\ 6 & 6 & 6 & 6 & 6 & 8 & 9 \\ 9 & 9 & 9 & 9 & 9 & 9 & 13 \end{bmatrix}, \quad c_{ER} = \frac{1}{16}(5, 5, 5, 5, 5, 5, -14)^\top.$$

Then

$$A_{ER} c_{ER} = \left(4, 4, 4, 4, 4, 4, \frac{11}{2}\right)^\top, \quad c_{ER}^\top A_{ER} c_{ER} = \frac{43}{16}.$$

The polynomial

$$q_{ER}(z) := (z - c_{ER})^\top A_{ER} (z - c_{ER}) - \frac{43}{16}$$

has Newton coefficients

$$a = (0, 0, 0, 0, 0, 0, 2), \quad d = (16, 16, 16, 16, 16, 16, 26), \quad \gamma_{ij} \in \{12, 18\}.$$

Their greatest common divisor is two, so

$$\widehat{q}_{ER} = \frac{1}{2}q_{ER}, \quad \delta(ER_7) = \frac{43}{32}.$$

The leading principal minors of  $A_{ER}$  are

$$8, \quad 28, \quad 80, \quad 208, \quad 512, \quad 1216, \quad 256,$$

so  $A_{ER} \succ 0$ .

## B Encoding bounds for separation certificates

This appendix supplies the quantitative bit-length estimates used in the proof of Theorem 6.2. All input rationals are encoded by binary numerators and positive binary denominators.

### B.1 Indefinite moment matrices

Let  $\bar{W} \in \mathbb{Q}^{s \times s}$  be symmetric and indefinite, where  $s = n + 1$ , and let  $L$  denote its binary input length. Clear denominators and write

$$S = D\bar{W} \in \mathbb{Z}^{s \times s}, \quad D > 0.$$

Taking  $D$  to be the product of the input denominators gives  $\log D = O(L)$ . Put

$$M := \max\{1, |S_{ij}| : 1 \leq i, j \leq s\}.$$

Then  $\log M$  is polynomial in  $L$ .

Every eigenvalue  $\mu$  of  $S$  satisfies  $|\mu| \leq sM$ . Indeed, if  $Sx = \mu x$  and  $|x_i| = \|x\|_\infty$ , then

$$|\mu| |x_i| \leq \sum_{j=1}^s |S_{ij}| |x_j| \leq sM |x_i|.$$

Let  $\rho = \text{rank}(S)$  and let  $\mu_1, \dots, \mu_\rho$  be the nonzero eigenvalues. Up to sign, their product is the coefficient of  $t^{s-\rho}$  in the integer polynomial  $\det(tI - S)$ , and hence has absolute value at least one. Therefore every nonzero eigenvalue has magnitude at least

$$\Delta := (sM)^{-(s-1)}.$$

Since  $S$  is indefinite, it has a unit eigenvector  $u$  whose eigenvalue  $\lambda$  satisfies  $\lambda \leq -\Delta$ .

Choose the integer

$$K := 8s^2 M \Delta^{-1} = 8s^2 M (sM)^{s-1}$$

and round  $Ku$  coordinatewise to  $v \in \mathbb{Z}^s$ . With  $y := v/K$ ,

$$\|y - u\|_2 \leq \frac{\sqrt{s}}{2K}.$$

Because  $\|S\|_2 \leq sM$ ,  $\|u\|_2 = 1$ , and  $\|y\|_2 \leq 1 + \sqrt{s}/(2K)$ ,

$$\begin{aligned} |y^\top S y - u^\top S u| &\leq \|S\|_2 \|y - u\|_2 (\|y\|_2 + \|u\|_2) \\ &\leq \frac{3s^{3/2}M}{2K} \leq \frac{3\Delta}{16} < \frac{\Delta}{2}. \end{aligned}$$

Thus

$$y^\top S y < -\frac{\Delta}{2} < 0, \quad v^\top S v = K^2 y^\top S y < 0.$$

Since  $D > 0$ , this is equivalent to  $v^\top \bar{W} v < 0$ .

Finally,

$$\log \Delta^{-1} = (s-1)(\log s + \log M) = \text{poly}(L),$$

so  $\log K = \text{poly}(L)$ . Coordinatewise rounding gives  $|v_i| \leq K+1$ , and hence the binary length of  $v$  is polynomial in  $L$ . The Newton coefficients of  $(v_0 + v'^\top z)^2$  are sums and pairwise products of entries of  $v$ , so their total binary length is polynomial as well.

## B.2 Positive-definite covariance

Let  $\bar{W} \succeq 0$  be rational, let  $\Sigma = \bar{X} - \bar{x}\bar{x}^\top \succ 0$ , and let

$$p(z) = z^\top A z + 2b^\top z + \alpha$$

be any cut-polynomial certificate for  $\bar{\eta}$ . As shown in (6.2)–(6.3), if  $\min p = -\delta$ , then  $0 < \delta < 1$  and

$$\langle A, \Sigma \rangle < 1. \tag{B.1}$$

Write

$$\Sigma = \frac{1}{D_\Sigma} S, \quad D_\Sigma \in \mathbb{Z}_+, \quad S \in \mathbb{Z}^{n \times n} \text{ positive definite,}$$

and let

$$M_\Sigma := \max\{1, |S_{ij}| : 1 \leq i, j \leq n\}.$$

The numbers  $D_\Sigma$  and  $M_\Sigma$  have polynomial binary length in the input size. Since  $\det S$  is a positive integer and  $\lambda_{\max}(S) \leq nM_\Sigma$ ,

$$\lambda_{\min}(\Sigma) = \frac{\lambda_{\min}(S)}{D_\Sigma} \geq \frac{1}{D_\Sigma (nM_\Sigma)^{n-1}}.$$

Define

$$B_\Sigma := D_\Sigma (nM_\Sigma)^{n-1}.$$

The binary length of  $B_\Sigma$  is polynomial in the input length. From (B.1),

$$\|A\|_2 \leq \text{tr}(A) < B_\Sigma. \tag{B.2}$$

Choose  $y$  with  $Ay = -b$ . The master identity gives  $(\bar{x} - y)^\top A(\bar{x} - y) < 1$ , and therefore

$$\|A\bar{x} + b\|_2^2 = (\bar{x} - y)^\top A^2(\bar{x} - y) \leq \|A\|_2 (\bar{x} - y)^\top A(\bar{x} - y) < B_\Sigma.$$

Put

$$X_\infty := \max\{1, |\bar{x}_i| : 1 \leq i \leq n\}.$$

This rational number has polynomial binary length, and  $\|\bar{x}\|_2 \leq \|\bar{x}\|_1 \leq nX_\infty$ . Since  $B_\Sigma \geq 1$ , we have  $\sqrt{B_\Sigma} \leq B_\Sigma$ , and hence

$$\|b\|_\infty \leq \|b\|_2 \leq B_\Sigma(1 + nX_\infty). \tag{B.3}$$

Since  $p$  is integer-valued and its continuous minimum is greater than  $-1$ ,  $p(0) = \alpha \in \mathbb{Z}_+$ . Moreover,  $L_p(\bar{\eta}) < 0$  and  $\langle A, \Sigma \rangle \geq 0$  imply  $p(\bar{x}) < 0$ , whence

$$\begin{aligned} 0 \leq \alpha &< -\bar{x}^\top A \bar{x} - 2b^\top \bar{x} \\ &\leq B_\Sigma n^2 X_\infty^2 + 2B_\Sigma(1 + nX_\infty)nX_\infty. \end{aligned} \tag{B.4}$$

All quantities on the right-hand sides of (B.2)–(B.4) have polynomial binary length. By (2.7), the entries of  $A$  and  $b$  have denominators at most two and four, respectively, while  $\alpha \in \mathbb{Z}$ . The magnitude bounds therefore give polynomial bounds on their binary encoding lengths. The Newton coefficients are recovered by

$$d_i = 2A_{ii}, \quad \gamma_{ij} = 2A_{ij}, \quad a_i = 2b_i + A_{ii},$$

so they also have polynomial total encoding length.

### B.3 Singular covariance

Let  $\Sigma \succeq 0$  be rational and singular. Clear denominators and write  $S = D_\Sigma \Sigma \in \mathbb{Z}^{n \times n}$ . Put  $r := \text{rank}(S)$  and  $k := n - r$ . When  $r > 0$ , select  $r$  linearly independent rows of  $S$  and collect them in  $A_0 \in \mathbb{Z}^{r \times n}$ . The full-row-rank extension of the Kannan–Bachem Hermite-normal-form algorithm [42, Section 2, Theorem 2 and p. 504] computes

$$A_0 Q = \begin{bmatrix} H & 0 \end{bmatrix}, \quad Q \in \text{GL}_n(\mathbb{Z}), \quad H \in \mathbb{Z}^{r \times r} \text{ nonsingular}, \tag{B.5}$$

with polynomial running time and polynomially bounded output bit length. If  $r = 0$ , let  $A_0$  be the empty  $0 \times n$  matrix and take  $Q = I_n$ . In either case, let  $Z$  be the last  $k$  columns of  $Q$ . The selected rows span the row space of  $S$ , so  $\ker_{\mathbb{R}}(A_0) = \ker_{\mathbb{R}}(S) = \ker_{\mathbb{R}}(\Sigma)$ . Moreover, for  $x = Qy \in \mathbb{Z}^n$ , equation (B.5) gives

$$Sx = 0 \iff A_0 x = 0 \iff Hy_{1:r} = 0 \iff y_{1:r} = 0.$$

Thus the columns of  $Z$  form a lattice basis of  $\ker(\Sigma) \cap \mathbb{Z}^n$ . Set

$$R := Z^\top, \quad U := Q^{-\top}.$$

Because  $\Sigma$  is symmetric,  $\text{range}(\Sigma) = \ker(\Sigma)^\perp$ , and the identity  $Q^\top Q^{-\top} = I_n$  yields

$$\ker_{\mathbb{R}} R = \text{range}(\Sigma), \quad RU = \begin{bmatrix} 0 & I_k \end{bmatrix}. \tag{B.6}$$

The matrix  $R$  has polynomial binary length. Since  $Q$  is unimodular, the cofactor formula gives the same conclusion for  $U = Q^{-\top}$ , while  $U^{-1} = Q^\top$  already has polynomial binary length.

Suppose first that  $R\bar{x} \notin \mathbb{Z}^k$ . Choose a row  $v^\top$  of  $R$  with  $v^\top \bar{x} \notin \mathbb{Z}$  and set  $s = \lfloor v^\top \bar{x} \rfloor$ . The vector  $v$  is primitive because it is the transpose of a column of the unimodular matrix  $Q$ , and  $v \in \ker(\Sigma)$ . With  $\theta = v^\top \bar{x} - s \in (0, 1)$ ,

$$L_{q_{v,s}}(\bar{\eta}) = \frac{1}{2} \left[ v^\top \Sigma v + \theta(\theta - 1) \right] = \frac{1}{2} \theta(\theta - 1) < 0.$$

The entries of  $v$  and the integer  $s$  have polynomial binary length, so the split polynomial is a polynomial-size certificate.

Suppose now that  $R\bar{x} \in \mathbb{Z}^k$ . Define

$$z^0 := U \begin{bmatrix} 0 \\ R\bar{x} \end{bmatrix} \in \mathbb{Z}^n. \quad (\text{B.7})$$

Then  $R(z^0 - \bar{x}) = 0$ , and hence  $z^0 - \bar{x} \in \text{range}(\Sigma)$  by (B.6). Apply the integer-affine transformation

$$z = z^0 + Uy.$$

The transformed mean and covariance are

$$\bar{x}' = U^{-1}(\bar{x} - z^0), \quad \Sigma' = U^{-1}\Sigma U^{-\top} = Q^\top \Sigma Q. \quad (\text{B.8})$$

Equation (B.6) shows that the last  $k$  coordinates of  $\bar{x}'$  vanish. Partition  $Q = [Q_1 \ Z]$ . Since  $\Sigma Z = 0$ ,

$$\Sigma' = \begin{bmatrix} Q_1^\top \Sigma Q_1 & 0 \\ 0 & 0 \end{bmatrix} =: \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Congruence preserves rank, so  $\Sigma_r$  is an  $r \times r$  positive-definite matrix. The transformed moment point is therefore the exact zero extension of an  $r$ -dimensional rational point  $\bar{\eta}_r$  with positive-definite covariance. If  $r = 0$ , then  $\bar{x}' = 0$  and the transformed point is the integer generator  $W_0$ , contradicting nonmembership. Hence  $r \geq 1$ .

All entries of  $\bar{\eta}_r$  have polynomial binary length by (B.5)–(B.8). The positive-definite case supplies a certificate  $p_r$  of polynomial size for  $\bar{\eta}_r$ . Extend  $p_r$  so that it is independent of the last  $k$  transformed variables and pull it back by

$$p(z) := p_r\left(\left[U^{-1}(z - z^0)\right]_{1:r}\right). \quad (\text{B.9})$$

Integer-affine invariance shows that  $p$  is integer-valued, has the same continuous minimum as  $p_r$ , and satisfies  $L_p(\bar{\eta}) = L_{p_r}(\bar{\eta}_r) < 0$ . The entries of  $U^{-1}$  and  $z^0$  have polynomial binary length, and expanding the quadratic in (B.9) requires only polynomially many arithmetic operations on polynomial-size numbers. The resulting Newton coefficients therefore have polynomial total encoding length.

## Declaration of AI use

The authors used GPT to assist with reviewing the manuscript, refining its exposition, and preparing the bibliography. They reviewed all AI-assisted material and take full responsibility for the manuscript.

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