

Irreparable Linear Bilevel Problems with Rotations

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ABSTRACT. Classic linear bilevel optimization assumes that the upper-level decision impacts only the right-hand side of the lower-level constraints, which guarantees a solution under mild conditions. We study existence of solutions for linear bilevel problems with rotations (LBPR, for short), i.e., problems for which the upper-level decision modifies the left-hand side coefficients of the linear follower constraints. Under a rather basic compactness and continuity assumption, if the LBPR is such that the follower is feasible for every leader decision, then relaxing all follower inequalities by any positive quantity guarantees a solution for the corresponding perturbed LBPR. Furthermore, we show that the problem’s domain can be partitioned into regions where the follower’s feasible-set mapping is continuous. Finally, we construct examples of LBPR with persistent non-attainment (irreparable) under prescribed perturbations of selected constraint coefficients, additive leader objectives, and nonnegative scalar right-hand-side relaxations. The simultaneous perturbation result keeps the leader domain, the fixed follower constraints, and the follower objective unchanged.

1. INTRODUCTION

Bilevel optimization is a rather young field of mathematical optimization in which a certain set of variables needs to be an optimal solution to another, nested optimization problem (Dempe 2002; Dempe et al. 2015). This allows us to model hierarchical decision-making processes but also leads to highly challenging optimization problems. Despite their computational hardness, even linear bilevel optimization problems are still the subject of current research. For instance, recent work addresses calmness of the solution-set mapping under right-hand-side perturbations (Lefebvre et al. 2026) and the structural role of coupling constraints has also been studied in Henke et al. (2025, 2026).

Usual linear bilevel problems are of the form

$$\min_{x,y} \quad c^\top x + d^\top y \tag{1a}$$

$$\text{s.t.} \quad Px + Qy \leq r, \tag{1b}$$

$$y \in \operatorname{argmin}_{y'} \{e^\top y' : Ay' \leq b - Bx\}. \tag{1c}$$

In particular, this means that the upper-level variable x parameterizes the right-hand side of the lower-level feasible set. Models in which x parameterizes the left-hand side of the lower-level constraints arise, in particular, in the area of counterfactual explanations (CEs) as a tool for explainable artificial intelligence; see, e.g., Kurtz et al. (2026) and Lefebvre and Schmidt (2026). In this case, the bilevel

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problem is given by

$$\min_{x,y} c^\top x + d^\top y \quad (2a)$$

$$\text{s.t. } Px + Qy \leq r, \quad (2b)$$

$$y \in S(x) := \operatorname{argmin}_{y'} \{e^\top y' : A(x)y' \leq b\} \quad (2c)$$

Parameter-dependent constraint matrices have also been studied in parametric linear programming. Zuidwijk (2005) derives local expressions and critical intervals for the optimal value, while Khalilpour and Karimi (2014) develop an algorithm for left-hand side parametric linear programs. More recently, the preprint by Boveroux et al. (2026) studies the complexity of optimizing the value function over a scalar matrix parameter, explicitly using bilevel formulations. Our focus is on existence and continuity and on explicit examples of persistent non-attainment under prescribed perturbation classes.

Geometrically speaking, we replace translations of lower-level constraints in (1) with rotations in (2). Thus, we call problems of the form (2) linear bilevel problems with rotations. At the level of follower optimizers, parameter-dependent objectives and right-hand sides can also be represented by left-hand side coefficients. Indeed, if we consider a lower-level problem of the form $\min_y \{e(x)^\top y : A(x)y \leq b(x)\}$, we can include two extra variables t, z and equivalently write

$$\min_{t,y,z} t \quad (3a)$$

$$\text{s.t. } e(x)^\top y - t \leq 0 \quad (3b)$$

$$\begin{bmatrix} A(x) & -b(x) \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \leq 0 \quad (3c)$$

$$z = 1. \quad (3d)$$

This construction gives a common formal representation of applications such as explainable AI (Kurtz et al. 2026), toll pricing (Brotcorne et al. 2001), and portfolio optimization (Leal et al. 2020). It does not automatically preserve the assumptions used below: the epigraph variable t is unbounded above in the follower feasible set, and the equality $z = 1$, if represented by two inequalities, prevents the strict Slater condition. The existence assumptions and the prescribed perturbation class must therefore be checked separately in a reformulated model.

For the general analysis below, we restrict attention to bilevel problems without coupling constraints, i.e., $Q = 0$, and write $X = \{x : Px \leq r\}$.

Existence of solutions for the classic linear bilevel problem (1) is well understood under standard feasibility and boundedness assumptions (Beck et al. 2026). For model (2), we investigate how this property can fail and construct examples with persistent non-attainment under specified perturbations. The main issue is given by the fact that the feasibility map $K : X \rightrightarrows \mathbb{R}^q$ given by

$$K(x) := \{y : A(x)y \leq b\} \quad (4)$$

may have some discontinuities that might turn the bilevel problem irreparable; see Definition 4.1.

Our main contributions are the following:

- (i) First, under Assumption 2.1 and nonemptiness of $K(x)$ for every $x \in X$, we show in Section 2 that uniformly relaxing every follower inequality by any positive quantity yields a bilevel problem with a solution. The relaxation supplies Slater feasibility, while boundedness of the relaxed feasible range follows from a uniform recession argument.

- (ii) Secondly, in Section 3 we study the structure of the points of discontinuity of K . In particular, we construct a partition such that the restriction of K to any of its elements is continuous.
- (iii) Third and finally, Section 4 gives examples that are irreparable relative to explicitly prescribed perturbation spaces: continuous changes of selected follower coefficients, small C^1 -additive leader-objective perturbations, and nonnegative scalar relaxations of a selected follower inequality. A final example allows these perturbations simultaneously, with the leader domain, fixed follower constraints, and follower objective held fixed. Each sufficiently small perturbation preserves feasibility and a finite infimum, but the infimum is not attained. These examples are adapted, in Section 5, to fit the setting of counterfactual explanations, which is our main motivation.

We close the paper in Section 6 with some conclusions.

2. EXISTENCE OF SOLUTIONS UNDER SLATER'S CONDITION AND A FIRST COUNTEREXAMPLE

Let us consider the bilevel problem

$$\begin{aligned} \min_{x,y} \quad & F(x, y) \\ \text{s.t.} \quad & x \in X, \ y \in S(x), \end{aligned} \tag{5}$$

where $x \in X \subset \mathbb{R}^p$, $y \in \mathbb{R}^q$, $F : X \times \mathbb{R}^q \rightarrow \mathbb{R}$ is not necessarily linear and $S(x) := \operatorname{argmin}_y \{e^\top y : A(x)y \leq b\}$, $A(x) \in \mathbb{R}^{m \times q}$ for every $x \in X$, and $b \in \mathbb{R}^m$. We consider the feasibility map $K : X \rightrightarrows \mathbb{R}^q$ as in (4), i.e., $K(x) := \{y : A(x)y \leq b\}$, and we denote its domain by $\operatorname{dom} K$. We write $v(x) := \inf_{y \in S(x)} F(x, y)$ for the optimistic value function and γ for the value of the bilevel problem, so that we can write $\gamma := \inf_{x \in X} v(x)$.

If the optimal solution of the lower level is unique for each x we write the optimal response as $y(x)$, viewing y as a function of x , with the identification $S(x) = \{y(x)\}$. In that case we can plug the optimal reaction into the leader's objective and obtain that $v(x) = F(x, y(x))$.

Throughout this work, we will adopt the leader-only convention for a solution of a bilevel problem: we say that a point $x^* \in X$ is a solution of (5) if $v(x^*) = \gamma$, or equivalently, under Assumption 2.1, if there exists $y^* \in S(x^*)$ such that $F(x^*, y^*) = \gamma$.

We make the following assumptions.

Assumption 2.1. We suppose that the following conditions are satisfied:

- (i) The matrix-valued mapping $A : \mathbb{R}^p \rightarrow \mathbb{R}^{m \times q}$ is continuous;
- (ii) The feasibility map has a bounded range, i.e., $K(X)$ is bounded;
- (iii) X is non-empty and compact;
- (iv) $F : X \times \mathbb{R}^q \rightarrow \mathbb{R}$ is lower semicontinuous;
- (v) Problem (5) is feasible.

Under this assumption, $S(x)$ is non-empty and compact for each $x \in \operatorname{dom} K$, and the lower semicontinuity of F ensures that the infimum defining $v(x)$ is attained on $S(x)$.

Remark 2.2. The theoretical questions addressed in this paper are primarily motivated by bilevel problems arising in counterfactual explanations in explainable AI (in short, XAI). We briefly discuss how these problems can be formulated within the framework of Model (5). For computing counterfactual explanations, one is

usually interested in problems of the form

$$\begin{aligned} \min_{x,y} \quad & F(x) \\ \text{s.t.} \quad & x \in X, y \in \Theta, y \in S(x). \end{aligned} \tag{6}$$

The idea is that the leader chooses a parameter $x \in X$ for the follower's problem, which is small in some sense that is measured by F , so that the follower's optimal solution $y \in S(x)$ is in some desired set Θ , which is assumed to be non-empty and closed. Such an x is then called a counterfactual explanation.

Under Assumption 2.1, suppose that $(\bar{x}, \bar{y})$ is feasible for (6) and choose

$$R > F(\bar{x}) - \min_{x \in X} F(x).$$

Then the exact penalization

$$F(x, y) = F(x) + R \mathbb{1}_{\mathbb{R}^q \setminus \Theta}(y),$$

where $\mathbb{1}_{\mathbb{R}^q \setminus \Theta}$ is the characteristic function of $\mathbb{R}^q \setminus \Theta$, yields a problem of the form in (5) with the same optimal value and optimal solutions as (6). Indeed, if $y \notin \Theta$, then $F(x, y) \geq \min_{z \in X} F(z) + R > F(\bar{x})$, whereas the penalty is zero for $y \in \Theta$. Moreover, this penalized objective is lower semicontinuous because Θ is closed.

Under Assumption 2.1, we can apply Berge's maximum theorem (Aliprantis and Border 2006) to obtain the following classic result; see, e.g., Theorem 3.3.4 in Aussel and Svensson 2020.

Proposition 2.3. *Suppose that Assumption 2.1 holds, and that K is lower semicontinuous on its (non-empty) domain. Then, Problem (5) admits a solution.*

Proof. Since A is continuous, we have that K is closed. Then, since $K(X)$ is bounded, we get that K is upper semicontinuous. The domain of K is closed in X , since feasible selections along a convergent parameter sequence have a convergent subsequence by bounded range, and K has a closed graph. Thus $\text{dom } K$ is compact, and Berge's maximum theorem applies on this domain. Berge's maximum theorem implies that the set-valued map S is closed. Since $\text{gph } S$ is a subset of the bounded set $X \times K(X)$ then $\text{gph } S$ is compact and so Problem (5), which can be written as $\min\{F(x, y) : (x, y) \in (X \times \mathbb{R}^q) \cap \text{gph } S\}$, admits a solution. $\square$

It is well known that constraint set-valued maps defined by inequalities for parameterized convex functions are lower semicontinuous if Slater's condition is satisfied; see, e.g., Example 5.10 in Rockafellar and Wets 1998. Therefore, the following result is an immediate consequence.

Corollary 2.4. *Suppose that Assumption 2.1 holds and that for every $x \in X$, the lower-level problem of (5) satisfies Slater's constraint qualification, i.e., there exists $y \in \mathbb{R}^q$ such that $A(x)y < b$. Then, Problem (5) admits a solution.*

2.1. A First Pathological Example. We now state a simple example illustrating that the conditions of the last two results are very strong.

Example 2.5. Consider the problem

$$\begin{aligned} \min_{x,y} \quad & x + 2y_1 \\ \text{s.t.} \quad & \begin{cases} x \in [-1, 1], \\ y \text{ solves} \begin{cases} \max_y y_1 + y_2 \\ \text{s.t.} \begin{cases} 0 \leq y_1 \leq 1, \\ y_2 \leq 0, \\ y_2 - xy_1 \geq 0; \end{cases} \end{cases} \end{cases} \end{aligned} \tag{7}$$

see Figure 1.

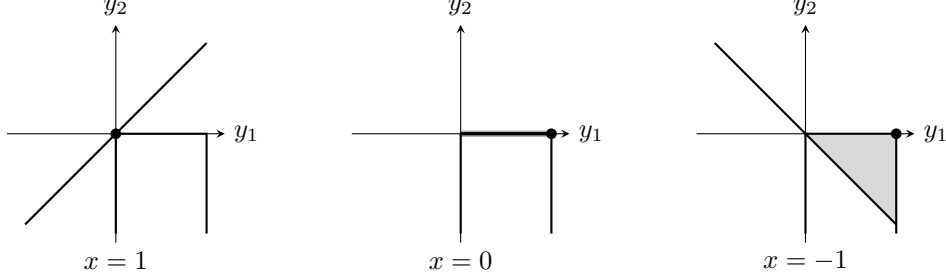


FIGURE 1. Follower's feasible set and response for (7).

Problem (7) fails to have a solution. Indeed, the response of the follower is unique and it is given by

$$y(x) = \begin{cases} (0, 0), & \text{if } x > 0, \\ (1, 0), & \text{if } x \leq 0. \end{cases} \quad (8)$$

Thus, the objective function depends only on x and it is given by

$$v(x) = x + 2y_1(x) = \begin{cases} x, & \text{if } x > 0, \\ x + 2, & \text{if } x \leq 0. \end{cases}$$

Hence, there is no solution since $\inf_{x \in [-1, 1]} v(x) = 0$ but the infimum is never attained. Note that the issue here is at the unique discontinuity $x = 0$, for which the lower-level problem does not satisfy Slater's condition. $\diamond$

Although the bilevel problem with rotations in the previous example fails to have solutions, in this case the issue can be resolved using the following direct corollary.

Corollary 2.6. *Suppose that Assumption 2.1 holds and that $X \subseteq \text{dom } K$. Then for any $\varepsilon > 0$ the perturbed bilevel problem*

$$\begin{aligned} \min_{x, y} \quad & F(x, y) \\ \text{s.t.} \quad & x \in X, \ y \in S_\varepsilon(x) := \operatorname{argmin}_y \{e^\top y : A(x)y \leq b + \varepsilon \mathbf{1}\} \end{aligned} \quad (9)$$

admits an optimal solution. Here, $\mathbf{1} \in \mathbb{R}^m$ denotes the vector whose entries are all one.

Proof. We observe that for any $x \in X$ the lower-level problem of (9) satisfies Slater's condition. Set $K_\varepsilon(x) = \{y : A(x)y \leq b + \varepsilon \mathbf{1}\}$. Since each $K(x)$ is non-empty and bounded, $A(x)d \leq 0$ implies $d = 0$. Hence continuity and compactness yield $\eta := \min_{x \in X, \|d\|=1} \max_{i \in [m]} a_i(x)^\top d > 0$. For every nonzero $y \in K_\varepsilon(x)$, $\eta \|y\| \leq \max_i b_i + \varepsilon$. Thus, $K_\varepsilon(X)$ is bounded. Each $K_\varepsilon(x)$ is non-empty and compact, so the follower optimum exists. Moreover, every $y \in K(x)$ is strictly feasible for the relaxed inequalities. Therefore, the conclusion follows from Corollary 2.4. $\square$

In problem (7), the map K is defined on $X = [-1, 1]$ and has $\text{dom } K = X$, since $(0, 0)$ is feasible for every leader decision. Corollary 2.6 therefore guarantees a solution after uniformly relaxing all four follower inequalities by any $\varepsilon > 0$.

Remark 2.7. Existence for positive relaxations does not imply convergence to a feasible point of the original bilevel problem. For problem (7), the relaxation in

Corollary 2.6 gives the unique follower response $y_2 = \varepsilon$ and

$$y_1 = \begin{cases} 1 + \varepsilon, & x \leq 0, \\ \min\{1 + \varepsilon, 2\varepsilon/x\}, & x > 0. \end{cases}$$

For $0 < \varepsilon < 1/16$, the leader's optimum is

$$x_\varepsilon = 2\sqrt{\varepsilon}, \quad y_\varepsilon = (\sqrt{\varepsilon}, \varepsilon), \quad F(x_\varepsilon, y_\varepsilon) = 4\sqrt{\varepsilon}.$$

Indeed, the branch with $y_1 = 1 + \varepsilon$ has value at least $1 + 2\varepsilon$, while $x + 4\varepsilon/x$ is minimized at $x = 2\sqrt{\varepsilon}$ with value $4\sqrt{\varepsilon} < 1$. Thus $(x_\varepsilon, y_\varepsilon) \rightarrow (0, (0, 0))$, although the original follower solution at $x = 0$ is $(1, 0)$, as indicated in (8).

3. THE STRUCTURE OF DISCONTINUITIES

Based on the example of the preceding section, we aim to study the structure of points of discontinuity of the set-valued map

$$K : \mathbb{R}^p \rightrightarrows \mathbb{R}^q, \quad K(x) = \{y \in \mathbb{R}^q : A(x)y \leq b\}, \quad (10)$$

where $b \in \mathbb{R}^m$ is a fixed vector and $A : \mathbb{R}^p \rightarrow \mathbb{R}^{m \times q}$ is continuous, which yields that K has a closed graph. In what follows we denote by

$$\text{dom } K := \{x \in \mathbb{R}^p : K(x) \neq \emptyset\}$$

the domain of K . We fix a reference set $X \subseteq \mathbb{R}^p$ and we set

$$\tilde{X} := X \cap \text{dom } K.$$

We study the discontinuities of $K|_{\tilde{X}}$. Under Assumption 2.1, this restriction is upper semicontinuous, so its discontinuities are exactly the points where it fails to be lower semicontinuous.

To this end, set

$$\Sigma := \{x \in \tilde{X} : K|_{\tilde{X}} \text{ is not lsc at } x\}.$$

The set Σ is in fact very small. The following facts are known from Daniilidis and Pang (2011):

- (i) The set $\tilde{X} \setminus \Sigma$ is residual in $\tilde{X}$. In particular, it contains a G_δ -dense subset of $\tilde{X}$; see Daniilidis and Pang (2011, Theorem 2).
- (ii) If X is a semi-algebraic set and A is a semi-algebraic map, then Σ is contained in a set of dimension at most $\dim \tilde{X} - 1$; see Daniilidis and Pang (2011, Theorem 28).

In what follows, for a positive integer $m \in \mathbb{N}$, we write $[m] := \{1, \dots, m\}$. Also, for any matrix $D \in \mathbb{R}^{m \times q}$ and every index set $I \subset [m]$, we denote by D_I the submatrix generated by removing all the rows of D with index in $[m] \setminus I$. We recall that the convex set $K(x)$ satisfies Slater's condition if, for every $i \in [m]$, there exists $y_i \in K(x)$ such that $a_i^\top(x)y_i < b_i$, or equivalently:

$$\exists \hat{y} \in K(x) \quad \text{such that} \quad \forall i \in \{1, \dots, m\} : \quad a_i^\top(x)\hat{y} < b_i. \quad (\mathcal{S})$$

Set

$$\mathcal{U} := \{x \in \tilde{X} : K(x) \text{ satisfies } (\mathcal{S})\}.$$

Notice that $\mathcal{U}$ is relatively open in $\tilde{X}$ and $\Sigma \subset \tilde{X} \setminus \mathcal{U}$. Moreover,

$$x \notin \mathcal{U} \iff \exists i_0 \in [m] \text{ such that } \forall y \in K(x) : \quad a_{i_0}^\top(x)y = b_{i_0}.$$

This motivates us to consider, for $x \in \tilde{X}$, the following set:

$$I(x) := \{i \in [m] : a_i^\top(x)y = b_i \quad \forall y \in K(x)\}.$$

Notice that $I(x) = \emptyset \iff x \in \mathcal{U}$. Further, we define the rank

$$r(x) := \text{rg}(A(x)_{I(x)}),$$

where, according to our notation, $A(x)_{I(x)}$ stands for the $|I(x)| \times q$ submatrix of $A(x)$ obtained by eliminating all rows whose indices do not belong to $I(x)$. By convention, we have $r(x) = 0$ whenever $I(x) = \emptyset$, i.e., whenever $x \in \mathcal{U}$. We further set

$$\mathcal{C} := \{(I, r) : \exists x \in \tilde{X} \text{ such that } I(x) = I, r(x) = r\}.$$

For $(I, r) \in \mathcal{C}$, define

$$\mathcal{X}_{(I, r)} := \{x \in \tilde{X} : I(x) = I, r(x) = r\}.$$

Notice that

$$(I_1, r_1) \neq (I_2, r_2) \implies \mathcal{X}_{(I_1, r_1)} \cap \mathcal{X}_{(I_2, r_2)} = \emptyset,$$

and

$$\tilde{X} = \bigcup_{(I, r) \in \mathcal{C}} \mathcal{X}_{(I, r)}.$$

Notice, in particular, that $\mathcal{U} = \mathcal{X}_{(\emptyset, 0)}$ holds and that this set is possibly empty. Our aim is to show that $K|_{\mathcal{X}_{(I, r)}}$ is continuous for every $(I, r) \in \mathcal{C}$. Before we proceed, let us recall a general fact concerning the description of solutions of a linear system of equations.

Consider a consistent linear system

$$A_* y = b, \tag{11}$$

where A_* is an $(\ell \times q)$ -matrix of rank

$$\text{rg}(A_*) = \text{rg}(B) = r \leq \min\{\ell, q\}, \tag{12}$$

and B is an $(r \times r)$ invertible submatrix of A_* . If $r = 0$, consistency implies $A_* = 0$ and $b = 0$, so the solution set is $\mathbb{R}^q$. In the following formulas, assume $r \geq 1$. After a possible reordering of rows and columns, we may assume that A_* has the form

$$A_* = \begin{bmatrix} B & C \\ & D \end{bmatrix} \quad \text{with } B \text{ being invertible}, \tag{13}$$

where C and D are respectively $r \times (q - r)$ and $(\ell - r) \times q$ submatrices. Then, thanks to (12), the $(r \times q)$ -matrix

$$M = \begin{bmatrix} B & C \end{bmatrix}$$

has the right pseudoinverse $M^\top (MM^\top)^{-1}$ and kernel

$$\ker M = \left\{ \begin{pmatrix} -B^{-1}C\nu \\ \nu \end{pmatrix} : \nu \in \mathbb{R}^{q-r} \right\}.$$

Therefore, the set of solutions of (11) is given by the formula

$$\left\{ M^\top (MM^\top)^{-1} b_{[r]} + \begin{pmatrix} -B^{-1}C\nu \\ \nu \end{pmatrix} : \nu \in \mathbb{R}^{q-r} \right\}, \tag{14}$$

where $b_{[r]} \in \mathbb{R}^r$ is the vector obtained from the first r coordinates of b .

We are now ready to state the main result of this section.

Theorem 3.1. *Let Assumption 2.1 hold. Then, for every $(I, r) \in \mathcal{C}$, the set-valued map $K|_{\mathcal{X}_{(I, r)}}$ is continuous.*

Proof. Each cell restriction has a closed graph and bounded range. Hence, it is upper semicontinuous. It therefore suffices to prove lower semicontinuity. First, we note that for the case $(I, r) = (\emptyset, 0)$ we have that $K|_{\mathcal{U}}$ is lower semicontinuous; see, e.g., Example 5.10 in Rockafellar and Wets (1998). So from now on we assume that $I \neq \emptyset$. If $r = 0$, then, on $\mathcal{X}_{(I, 0)}$, each row indexed by I is identically 0. After deleting these rows, the remaining system satisfies Slater's condition, so $K|_{\mathcal{X}_{(I, 0)}}$ is

lower semicontinuous. Therefore, we may also assume that $r \geq 1$. Let $(x_n)_n \subset \mathcal{X}_{(I,r)}$ and let $\bar{x} \in \mathcal{X}_{(I,r)}$ be such that $x_n \rightarrow \bar{x}$.

Then

$$I(x_n) = I(\bar{x}) = I, \quad r(x_n) = r(\bar{x}) = r \quad \forall n \geq 1.$$

We set $\ell = |I|$. Then, w.l.o.g., after re-ordering the rows, we may assume that for every $i \in \{1, \dots, \ell\}$ and every $y_n \in K(x_n)$, respectively $y \in K(\bar{x})$,

$$a_i^\top(x_n)y_n = b_i, \quad \text{respectively } a_i^\top(\bar{x})y = b_i.$$

Denote by $A_*(x_n)$, respectively $A_*(\bar{x})$, the $(\ell \times q)$ submatrix of $A(x_n)$, respectively of $A(\bar{x})$, obtained by eliminating the last $(m - \ell)$ rows.

Up to reordering, we may assume that $A_*(\bar{x})$ has the form (13), i.e.,

$$A_*(\bar{x}) = \left[\begin{array}{c|c} B(\bar{x}) & C(\bar{x}) \\ \hline & D(\bar{x}) \end{array} \right],$$

where $B(\bar{x})$ is an $(r \times r)$ invertible matrix and

$$\text{rg}(A_*(\bar{x})) = \text{rg}(B(\bar{x})) = r.$$

By continuity of the map

$$x \mapsto B(x),$$

we deduce that $B(x_n)$ is invertible for all $n \geq n_0$. Since $(x_n)_n \subset \mathcal{X}_{(I,r)}$, we get

$$\text{rg}(A_*(x_n)) = \text{rg}(B(x_n)) = r,$$

and $A_*(x_n)$ is also of the form (13) for all $n \geq n_0$. To establish that $K|_{\mathcal{X}_{(I,r)}}$ is lsc at $\bar{x}$, it is well known that it suffices to be able to approximate any $\hat{y} \in \text{rint}K(\bar{x})$ by a sequence $(y_n)_n$ with $y_n \in K(x_n)$; see, e.g., Salas and Svensson 2023. Notice that any such $\hat{y} \in K(\bar{x})$ satisfies

$$a_i^\top(\bar{x})\hat{y} = b_i, \quad i \in [\ell],$$

and

$$a_j^\top(\bar{x})\hat{y} < b_j, \quad j \in [m] \setminus [\ell],$$

Notice that $\hat{y}$ is of the form (14) for some $\hat{v} \in \mathbb{R}^{q-r}$, where

$$M = M(\bar{x}) := \left[\begin{array}{c|c} B(\bar{x}) & C(\bar{x}) \end{array} \right].$$

Set

$$y_n := M(x_n)^\top (M(x_n)M(x_n)^\top)^{-1} b_{[r]} + \begin{pmatrix} -B(x_n)^{-1}C(x_n)\hat{v} \\ \hat{v} \end{pmatrix}.$$

We deduce by continuity that $y_n \rightarrow \hat{y}$. Moreover, y_n is a solution of (11) for $A_* = A_*(x_n)$, i.e.,

$$a_i^\top(x_n)y_n = b_i \quad \forall i \in [\ell].$$

Moreover, since

$$a_j^\top(\bar{x})\hat{y} < b_j, \quad j \in [m] \setminus [\ell],$$

and

$$a_j(x_n) \rightarrow a_j(\bar{x}), \quad y_n \rightarrow \hat{y},$$

we deduce that, for all sufficiently large n , we have

$$a_j^\top(x_n)y_n < b_j, \quad j \in [m] \setminus [\ell].$$

Thus, $y_n \in K(x_n)$, and the proof is complete. $\square$

Partition vs stratification. Notice that the partition $\{\mathcal{X}_{(I,r)}\}_{(I,r) \in \mathcal{C}}$ does not need to be a stratification. In order to see this, consider first the set-valued map $K : \mathbb{R} \rightrightarrows \mathbb{R}^2$ given by

$$K(x) := \{y \in \mathbb{R}^2 : y_1 \in [0, 1], xy_1 \leq y_2 \leq 0\} \quad \text{for } x \in \mathbb{R}.$$

It is clear that $K(x)$ can be represented in the form of (10) with $b = (0, 1, 0, 0)^\top$ and by considering, for each $x \in \mathbb{R}$, the (4×2) -matrix $A(x)$ with rows

$$\begin{aligned} a_1(x) &= (-1, 0), \\ a_2(x) &= (1, 0), \\ a_3(x) &= (0, 1), \\ a_4(x) &= (x, -1). \end{aligned}$$

We obtain that

$$I(x) = \begin{cases} \{1, 3, 4\}, & x > 0, \\ \{3, 4\}, & x = 0, \\ \emptyset, & x < 0, \end{cases} \quad \text{and} \quad r(x) = \begin{cases} 2, & x > 0, \\ 1, & x = 0, \\ 0, & x < 0. \end{cases}$$

and the corresponding partition consists of the sets $(-\infty, 0)$, $\{0\}$ and $(0, \infty)$, which, in this case, is indeed a stratification of $\mathbb{R}$. However, this construction does not always lead to a stratification. Let us consider the set-valued map $\hat{K}$ defined by $\hat{K}(x) := K(\phi(x))$ with $\phi(x) := \{\max(0, x)\}^2$, with the corresponding matrix-valued function $\hat{A}(x) := A(\phi(x))$. If we write $\hat{I}, \hat{r}$ for the corresponding index set and rank function, we have

$$\hat{I}(x) = \begin{cases} \{1, 3, 4\}, & x > 0, \\ \{3, 4\}, & x \leq 0, \end{cases} \quad \text{and} \quad \hat{r}(x) = \begin{cases} 2, & x > 0, \\ 1, & x \leq 0. \end{cases}$$

The partition is now reduced to the sets

$$\hat{\mathcal{X}}_{(\{3,4\},1)} = (-\infty, 0] \quad \text{and} \quad \hat{\mathcal{X}}_{(\{1,3,4\},2)} = (0, \infty),$$

which is not a stratification of $\mathbb{R}$.

Notice that by introducing the stratum $\{0\}$, we can refine this partition to a stratification, keeping (trivially) the property that K is continuous on each stratum.

Finally, notice that if A and X are definable in the same o-minimal structure, see, e.g., Dries (1998) or Coste (2000), then so are, $\tilde{X}$, the restricted map $K|_{\tilde{X}}$ and the elements of the partition $\mathcal{X}_{(I,r)}$. In particular, the partition $\{\mathcal{X}_{(I,r)}\}$ can always be refined to a stratification. This is due to the following standard result in o-minimal geometry.

Theorem 3.2 (Dries 1998). *Given a finite collection $\mathcal{A} = \{A_1, \dots, A_k\}$ of definable subsets of a definable set $\tilde{X}$, there exists a definable stratification of $\tilde{X}$, which is compatible with the collection $\mathcal{A}$, in the sense that every A_i is a finite union of strata.*

Applying the above result to the partition $\mathcal{A} = \{\mathcal{X}_{(I,r)}\}_{(I,r) \in \mathcal{C}}$, we obtain the desired conclusion: Choose the compatible finite stratification to satisfy the frontier property. Every stratum of dimension $\dim \tilde{X}$ is then relatively open in $\tilde{X}$. By Theorem 3.1, K is continuous on every stratum. Hence Σ , defined relative to $\tilde{X}$, is contained in the union of strata of dimension strictly less than $\dim \tilde{X}$.

4. IRREPARABLE COUNTEREXAMPLES

Considering the results of Section 3, one might hope that the example of Section 2.1 could be in some sense rare. This hope rests on the fact that, for solutions to fail to exist, there must be a coordination between the objective function of the leader, the objective function of the follower, and the discontinuity points of the lower-level feasibility map. Indeed, the objective function of the follower, in the example of Section 2.1, forces a jump at the discontinuity point, which is exactly the unique candidate for a minimizer for the leader. One might therefore expect that small perturbations may break this coordination. However, the residuality result in Section 3 concerns leader parameters for a fixed feasible-set map; it does not establish genericity in a space of problem data.

In fact, despite the rich structure described in Section 3, this section is devoted to presenting variants of the example of Section 2.1 such that the pathologies (nonexistence of solutions) persist under every sufficiently small perturbation in a specified metric space. We call these problems irreparable relative to that space and its topology.

Definition 4.1. *Let us consider a parametric optimization problem*

$$\min_z \{f(z, w) : z \in Z(w)\}, \quad (P(w))$$

where w is a parameter that belongs to a given (possibly functional) metric space W and

$$f : \mathbb{R}^n \times W \rightarrow \mathbb{R} \quad \text{and} \quad Z : W \rightrightarrows \mathbb{R}^n.$$

Given $\bar{w} \in W$, we say that problem $P(\bar{w})$ is irreparable relative to W and its metric if there exists a tolerance $\varepsilon > 0$ such that for all $w \in B(\bar{w}, \varepsilon)$ problem $P(w)$ has no optimal solution.

We are interested in bilevel problems as (2) with specific structure of the perturbations. In this section, $Y \subseteq \mathbb{R}^q$ is a non-empty polytope introduced in order to allow us to perturb a selected subset of the follower's constraints. More precisely, we consider $z = (x, y) \in \mathbb{R}^n$ and perturbed problems of the form

$$\begin{aligned} \min_{x, y} \quad & c^\top x + d^\top y + \rho(x) \\ \text{s.t.} \quad & \begin{cases} x \in X, \\ y \text{ solves } \begin{cases} \min_y e^\top y \\ \text{s.t.} \quad \begin{cases} y \in Y, \\ (A(x) + \alpha(x))y \geq b - \delta \mathbf{1}, \end{cases} \end{cases} \end{cases} \end{aligned} \quad (15)$$

where, unless specified otherwise, the parameter $w = (\rho, \alpha, \delta)$ belongs to

$$W \subseteq \mathcal{C}^1(X; \mathbb{R}) \times \mathcal{C}^1(X; \mathbb{R}^{m \times q}) \times \mathbb{R}_+.$$

The vector $\mathbf{1} \in \mathbb{R}^m$ has all entries equal to one, and the reference parameter is $\bar{w} = (0, 0, 0)$.

We endow W with the distance $d(w, \tilde{w}) = \|w - \tilde{w}\|$ induced by the norm

$$\|(\rho, \alpha, \delta)\| = \max\{\|\rho\|_\infty, \|\nabla \rho\|_\infty, \|\alpha\|_\infty, \|J\alpha\|_\infty, |\delta|\}. \quad (16)$$

For a vector-, matrix-, or tensor-valued function h , we use $\|h\|_\infty := \sup_{x \in X} \max_i |h_i(x)|$, where i ranges over all entries. The matrix $J\alpha$ is the Jacobian after vectorizing the entries of α . On a closed interval or rectangle, $\mathcal{C}^1$ means the restriction of a continuously differentiable function defined on an open neighborhood. When only continuous coefficient perturbations are considered below, their size is measured by $\|\alpha\|_\infty$.

We will start the analysis in Section 4.1 with an irreparable parametric problem considering only the perturbation of the right-hand side of the rotation, i.e., of the form $(0, 0, \delta)$. After that we will show in Section 4.2 an irreparable example under left-hand side perturbations of the rotation of the form $(0, \alpha, 0)$, in Section 4.3 one under perturbations of the leader's objective of the form $(\rho, 0, 0)$, and we end in Section 4.4 with an irreparable example that considers simultaneous perturbations.

4.1. An Irreparable Example under Right-Hand Side Perturbations. The following example shows that right-hand side perturbations (relaxations) of the follower's constraints do not necessarily repair the nonexistence of optimal solutions if the condition $X \subseteq \text{dom } K$ does not hold. See Corollary 2.6 for comparison with the case where the condition $X \subseteq \text{dom } K$ is satisfied. Consider the problem

$$\begin{aligned} \min_{x,y} \quad & x_1 + x_2 + y_1 \\ \text{s.t.} \quad & \begin{cases} x_1 \in [0, 1], \\ x_2 \in [1/2, 1], \\ y \text{ solves } \begin{cases} \min_y & -y_1 \\ \text{s.t.} & \begin{cases} 0 \leq y_1, y_2 \leq 1, \\ x_2 y_2 - x_1 y_1 \geq 1. \end{cases} \end{cases} \end{cases} \end{aligned} \quad (17)$$

Note that x_2 necessarily needs to be 1. Indeed, if the leader chooses $x_2 < 1$, then we can write

$$x_2 y_2 - x_1 y_1 \geq 1 \implies y_2 - \frac{x_1}{x_2} y_1 \geq \frac{1}{x_2} \implies y_2 > 1.$$

The last implication follows using the constraints $x_1 \geq 0$ and $y_1 \geq 0$. Then, $x_2 = 1$ and Problem (17) reduces to a situation similar to the previous one; see Figure 2.

For this example, we also consider the extension to $\mathbb{R}^2$ of the lower-level constraint map, given by $K(x) = \{y \in [0, 1]^2 : x_2 y_2 - x_1 y_1 \geq 1\}$ and its domain is

$$\text{dom } K = \{(x_1, x_2) : x_1 \leq -1 \vee x_2 \geq 1 \vee x_2 - x_1 \geq 1\}.$$

We observe that $X = [0, 1] \times [1/2, 1]$ is clearly not included in $\text{dom } K$. Hence, we cannot apply Corollary 2.6. Moreover, we shall prove in Proposition 4.2 that the problem is irreparable for perturbations of the form $(0, 0, \delta)$, with tolerance $\varepsilon = 1/2$.

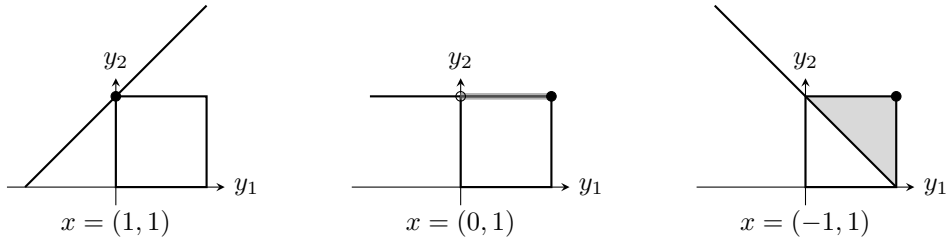


FIGURE 2. Follower's feasible set and optimal response for (17). The last panel illustrates the extended follower map outside the leader domain X .

Proposition 4.2. *For any $\delta \in [0, 1/2)$, the perturbed problem*

$$\begin{aligned} \min_{x,y} \quad & x_1 + x_2 + y_1 \\ \text{s.t.} \quad & \begin{cases} x_1 \in [0, 1], \\ x_2 \in [1/2, 1], \\ y \text{ solves } \begin{cases} \min_y -y_1 \\ \text{s.t.} \begin{cases} 0 \leq y_1, y_2 \leq 1, \\ x_2 y_2 - x_1 y_1 \geq 1 - \delta, \end{cases} \end{cases} \end{cases} \end{aligned} \quad (18)$$

fails to have a solution.

Proof. We will prove that γ , the value of problem (18), satisfies $\gamma = 1 - \delta$ and that this value cannot be attained.

Indeed, let (x_1, x_2) be a feasible point, with a corresponding lower-level reaction (y_1, y_2) . We observe that $x_2 \geq 1 - \delta$, otherwise the lower-level constraints cannot be satisfied as $x_1 y_1 \geq 0$. From this inequality and the non-negativity constraints we see that the upper-level objective function satisfies

$$F(x, y) := x_1 + x_2 + y_1 \geq x_2 \geq 1 - \delta. \quad (19)$$

From (19) we see that $\gamma \geq 1 - \delta$. For the reverse inequality, take $x_1 > 0$ and $x_2 = 1 - \delta$. It is clear that any lower-level reaction (y_1, y_2) has $y_1 = 0$. Therefore,

$$F(x, y) = x_1 + (1 - \delta) + y_1 = x_1 + 1 - \delta$$

and the value converges to $1 - \delta$ as $x_1 \searrow 0$, so that $\gamma \leq 1 - \delta$. Hence, we have proved so far that $\gamma = 1 - \delta$.

To finish the proof we observe that the only way to have an optimal point of the bilevel problem is that all inequalities in (19) become equalities. By the non-negativity constraints, we must have $x_1 = 0$ and $y_1 = 0$. However, given that $x_1 = 0$, it is clear that a lower-level reaction must satisfy $y_1 = 1$, which is a contradiction. Thus, we conclude that problem (18) has no optimal solution. $\square$

4.2. An Irreparable Example under Left-Hand Side Perturbations. Consider the problem

$$\begin{aligned} \min_{x,y} \quad & x + 2y_1 \\ \text{s.t.} \quad & \begin{cases} x \in [-1, 1], \\ y \text{ solves } \begin{cases} \min_y -y_1 \\ \text{s.t.} \begin{cases} 0 \leq y_1 \leq 1, \\ y_2 = 0, \\ y_2 - x y_1 \geq 0. \end{cases} \end{cases} \end{cases} \end{aligned} \quad (20)$$

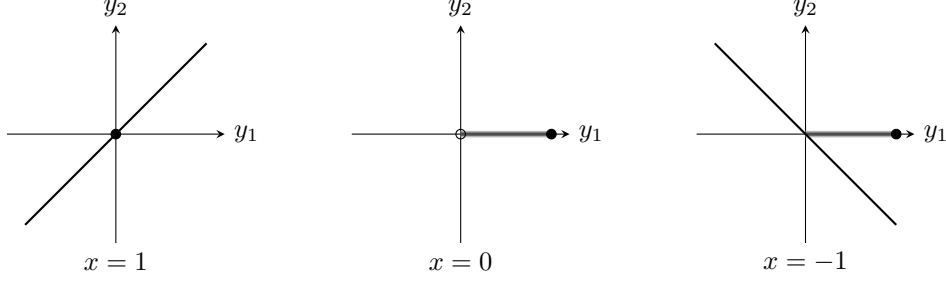
The optimal reaction of the lower-level problem is single-valued and given by $y(x) = (0, 0)$ if $x > 0$, and $y(x) = (1, 0)$ if $x \leq 0$. The problem is illustrated in Figure 3.

Plugging the optimal reaction of the follower into the leader's objective, we have

$$v(x) = F(x, y(x)) = x + 2y_1(x) = \begin{cases} x, & x \in (0, 1], \\ x + 2, & x \in [-1, 0], \end{cases}$$

which is always strictly positive as $x \geq -1$ and the infimum is $\gamma = 0$ by taking the limit at 0^+ . Therefore, Problem (20) has no optimal solution.

We now show that Problem (20) is irreparable for left-hand side perturbations of the form of a continuous function $\alpha : [-1, 1] \rightarrow \mathbb{R}^2$, with tolerance $\varepsilon = 1$.


 FIGURE 3. Feasible set and optimal reaction for $x \in \{1, 0, -1\}$.

Proposition 4.3. *For every continuous function $\alpha : [-1, 1] \rightarrow \mathbb{R}^2$ with $\|\alpha\|_\infty < 1$, the perturbed problem*

$$\begin{aligned} \min_{x,y} \quad & x + 2y_1 \\ \text{s.t.} \quad & \begin{cases} x \in [-1, 1], \\ y \text{ solves } \begin{cases} \min_y -y_1 \\ \text{s.t.} \begin{cases} 0 \leq y_1 \leq 1, \\ y_2 = 0, \\ (1 + \alpha_2(x))y_2 + (-x + \alpha_1(x))y_1 \geq 0, \end{cases} \end{cases} \end{cases} \end{aligned} \quad (21)$$

fails to have a solution.

Proof. Fix $\alpha : [-1, 1] \rightarrow \mathbb{R}^2$ with $\|\alpha\|_\infty < 1$. We note that $1 + \alpha_2(x) > 0$ holds for every $x \in [-1, 1]$. Hence, we can write the perturbed constraint of the lower-level problem as

$$y_2 + \underbrace{\frac{-x + \alpha_1(x)}{1 + \alpha_2(x)}}_{=: \beta(x)} y_1 \geq 0.$$

This yields that the solution of the lower-level problem of Problem (21) is unique and it is given by

$$y(x) = \begin{cases} (0, 0), & \text{if } \beta(x) < 0, \\ (1, 0), & \text{if } \beta(x) \geq 0. \end{cases}$$

Thus, the objective function of Problem (21) is given by

$$v(x) = \begin{cases} x, & \text{if } \beta(x) < 0, \\ x + 2, & \text{if } \beta(x) \geq 0. \end{cases}$$

Clearly, $\beta : [-1, 1] \rightarrow \mathbb{R}$ is continuous. Moreover, since $\|\alpha\|_\infty < 1$, $\beta(-1) > 0$ and $\beta(1) < 0$ holds. Hence, the intermediate value theorem ensures that there is $x \in (-1, 1)$ such that $\beta(x) = 0$.

Let us now define $\bar{x} := \inf \{x \in [-1, 1] : \beta(x) < 0\}$ and note that $\beta(\bar{x}) = 0$. We claim that the value of Problem (21) is $\gamma = \bar{x}$ and that this value is not attained.

Indeed, first note that by construction of $\bar{x}$, there is a sequence $x_n \searrow \bar{x}$ such that $\beta(x_n) < 0$ holds for every $n \in \mathbb{N}$. Therefore, $v(x_n) = x_n$ and $\lim v(x_n) = \bar{x}$. Hence, $\bar{x}$ is an upper bound for the optimal value γ . However, this value is never attained, since for $x > \bar{x}$ we have $v(x) \geq x > \bar{x}$, while for $x \leq \bar{x}$ we have $\beta(x) \geq 0$ and $v(x) = x + 2 \geq -1 + 2 \geq 1 > \bar{x}$. Thus, for all $x \in [-1, 1]$, we have $v(x) > \bar{x} = \gamma = \inf_{x \in [-1, 1]} v(x)$, which finishes the proof. $\square$

Remark 4.4. Note that Proposition 4.3 does not require differentiability of α . In this case, we provide an irreparable problem for perturbations of the form $(0, \alpha, 0)$ with $\alpha \in \mathcal{C}^0(X; \mathbb{R}^{m \times q})$.

4.3. An Irreparable Example under Upper-Level Objective Perturbations. The key point of the example in Section 4.2 is that for small perturbations of the left-hand side of the constraints, we still preserve the fact that the unique candidate for a solution is at a point of discontinuity of the lower-level feasible-set mapping. Before perturbation, that set consisted of the single point $x = 0$. One might hope that if the discontinuity set is fixed and identifiable as in (20), it could be possible to penalize it in the objective function. However, if the penalization is a small $\mathcal{C}^1$ -perturbation (tolerance $\varepsilon = 1/2$), then such a modification will not lead to solutions either, as we show in the next proposition.

Proposition 4.5. *For every $\mathcal{C}^1$ function $\rho : [-1, 1] \rightarrow \mathbb{R}$ with $\|\rho\|_\infty < \frac{1}{2}$ and $\|\rho'\|_\infty < \frac{1}{2}$, the perturbed problem*

$$\begin{aligned} \min_{x,y} \quad & x + 3y_1 + \rho(x) \\ \text{s.t.} \quad & \begin{cases} x \in [-1, 1], \\ y \text{ solves } \begin{cases} \min_y -y_1 \\ \text{s.t.} \begin{cases} 0 \leq y_1 \leq 1, \\ y_2 = 0, \\ y_2 - xy_1 \geq 0, \end{cases} \end{cases} \end{cases} \end{aligned} \quad (22)$$

fails to have solutions.

Proof. Suppose, for a contradiction, that a solution x^* for (22) exists. First, consider the case in which $x^* \leq 0$. Then the optimal reaction of the follower is $y^* = (1, 0)$ and so, using that $\|\rho\|_\infty < 1/2$, we have

$$F(x^*, y^*) = x^* + 3 + \rho(x^*) > -1 + 3 - 1/2 = 3/2.$$

For $x = 1$ we have a strictly better solution as the unique optimal reaction is $y = (0, 0)$ and the upper-level objective is

$$F(1, y) = 1 + \rho(1) \leq 1 + 1/2 = 3/2.$$

From this contradiction it must hold that $x^* > 0$.

Second, consider the case $x^* = 1$. Since $\|\rho\|_\infty < 1/2$, we see that $x = 1/2 - \|\rho\|_\infty > 0$ is a strictly better solution as

$$v(1) = 1 + \rho(1) > 1 - 1/2 = 1/2 \geq \frac{1}{2} - \|\rho\|_\infty + \rho\left(\frac{1}{2} - \|\rho\|_\infty\right) = v(x).$$

Therefore $x^* \in (0, 1)$ and since the optimal reaction for any $x \in (0, 1)$ is $y(x) = (0, 0)$, then x^* must be a solution of

$$\min_{x \in (0, 1)} x + \rho(x).$$

By Fermat's rule this yields that

$$0 = \left. \frac{d}{dx} (x + \rho(x)) \right|_{x=x^*} = 1 + \rho'(x^*) \geq 1 - \|\rho'\|_\infty > 1/2,$$

a contradiction. Since we obtained a contradiction in all cases, we have proved that no optimal solution for (22) exists. $\square$

Note that the bound on $\|\rho'\|$ is essential to the conclusion of Proposition 4.5. To see this, choose $0 < a < 1/3$ and a smooth function $\psi : \mathbb{R} \rightarrow [0, 1]$ with support in $(-1/2, 1/2)$ and $\psi(0) = 1$, and set

$$\rho_a(x) := -2a\psi\left(\frac{x-a}{a}\right).$$

It is clear that $\|\rho_a\|_\infty = 2a \rightarrow 0$ as $a \rightarrow 0$. We note that the leader value at $x = a$ is $v(a) = a + \rho_a(a) = -a < 0$. Outside the interval $[a/2, 3a/2]$, the perturbation vanishes and the leader value is positive, while in this compact interval, the follower response is $(0, 0)$ and the leader objective is continuous. It therefore attains a negative global minimum there. Thus, arbitrarily small smooth perturbations in the uniform norm can repair this example if the derivative is not controlled.

4.4. An Irreparable Example under Simultaneous Perturbations. In this section, we give an example of an irreparable bilevel problem under all the previous types of sufficiently small perturbations together with the $\mathcal{C}^1$ metric specified in (16), for which no optimal solution is attained. The tolerance considered here is $\varepsilon = 1/4$.

Theorem 4.6. *Let $X = [-1, 1] \times [1/2, 3/2]$. Then, for any $\delta \in [0, 1/4)$, and any two continuously differentiable functions $\alpha : X \rightarrow \mathbb{R}^2$ and $\rho : X \rightarrow \mathbb{R}$ with $\|\alpha\|_\infty, \|\rho\|_\infty < \frac{1}{4}$ and $\|J\alpha\|_\infty, \|\nabla\rho\|_\infty < \frac{1}{4}$, the perturbed problem*

$$\begin{aligned} \min_{x,y} \quad & x_1 + x_2 + 4y_1 + \rho(x_1, x_2) \\ \text{s.t.} \quad & \begin{cases} x_1 \in [-1, 1], \\ x_2 \in [1/2, 3/2], \\ y \text{ solves } \begin{cases} \min_y -y_1 \\ \text{s.t. } \begin{cases} 0 \leq y_1, y_2 \leq 1, \\ (x_2 + \alpha_2(x))y_2 - (x_1 + \alpha_1(x))y_1 \geq 1 - \delta, \end{cases} \end{cases} \end{cases} \end{aligned} \quad (23)$$

fails to have a solution.

Proof. Set $\varepsilon = 1/4$. Let us assume that a solution $x^* = (x_1^*, x_2^*)$ exists. Let $S : X \rightrightarrows \mathbb{R}^2$ be the solution mapping of the lower-level problem. We will prove the theorem through several claims, ending with a contradiction.

Claim 1: $\pi_1(\text{dom } S) = [-1, 1]$.

Note first that

$$\text{dom } S \supseteq \{(x_1, x_2) \in X : x_2 + \alpha_2(x_1, x_2) \geq 1 - \delta\}. \quad (24)$$

Then, we have that $\pi_1(\text{dom } S) \supseteq [-1, 1]$, since for any $x_1 \in [-1, 1]$, we can take $x_2 = 3/2$ so that

$$x_2 + \alpha_2(x_1, x_2) \geq \frac{3}{2} - \varepsilon > 1 \geq 1 - \delta,$$

and by (24) we obtain $x_1 \in \pi_1(\text{dom } S)$. Moreover, since the reverse inclusion is trivial we have that $\pi_1(\text{dom } S) = [-1, 1]$.

Claim 2: For all $x_1 \in [-1, 1]$, there exists a unique point $u(x_1) \in (1/2, 3/2)$ such that $u(x_1) + \alpha_2(x_1, u(x_1)) = 1 - \delta$, and u defines a $\mathcal{C}^1$ function with $\|u'\|_\infty \leq 2\varepsilon$.

Indeed, given $x_1 \in [-1, 1]$ we note that the function $w : x_2 \mapsto x_2 + \alpha_2(x_1, x_2)$ satisfies

$$\begin{aligned} w(3/2) &= 3/2 + \alpha_2(x_1, 3/2) \geq 3/2 - \varepsilon > 1 - \delta, \\ w(1/2) &= 1/2 + \alpha_2(x_1, 1/2) \leq 1/2 + \varepsilon < 1 - \delta, \end{aligned}$$

as long as we choose $\varepsilon \leq \frac{1}{4}$. Therefore, as w is continuous we know that there exists a point $x_2 \in (1/2, 3/2)$ satisfying $w(x_2) = 1 - \delta$. The uniqueness follows from the fact that w is strictly increasing as

$$w'(x_2) = 1 + \partial_2 \alpha_2(x_1, x_2) \geq 1 - |\partial_2 \alpha_2(x_1, x_2)| \geq 1 - \varepsilon > 0,$$

where $\partial_i := \frac{\partial}{\partial x_i}$. Thus, our claim about $u(x_1)$ is true.

By the implicit function theorem, u is continuously differentiable with derivative given by

$$u'(x_1) = -\frac{\partial_1 \alpha_2(x_1, u(x_1))}{1 + \partial_2 \alpha_2(x_1, u(x_1))}$$

and we have the bound

$$\|u'\|_\infty \leq \frac{\|\partial_1 \alpha_2\|_\infty}{1 - \|\partial_2 \alpha_2\|_\infty} \leq \frac{\varepsilon}{1 - \varepsilon} \leq 2\varepsilon.$$

Claim 3: $x_1^* + \alpha_1(x^*) > 0$, and $x_2^* + \alpha_2(x^*) \geq 1 - \delta$.

If the first part of the claim were not true, i.e., $x_1^* + \alpha_1(x^*) \leq 0$, then the lower-level optimal response would satisfy $y_1(x^*) = 1$. In this case, if we compare the value of the objective function v of the leader at (x_1^*, x_2^*) and at $(1, u(1))$ using that $\varepsilon < 1/2$ we get

$$\begin{aligned} v(x^*) &= x_1^* + x_2^* + \rho(x^*) + 4 \\ &\geq -1 + 1/2 - \varepsilon + 4 \\ &> 1 + 3/2 + \varepsilon \\ &\geq 1 + u(1) + \rho(1, u(1)) \\ &= v(1, u(1)). \end{aligned} \tag{25}$$

This is a contradiction with the optimality of x^* .

Since we now know that $x_1^* + \alpha_1(x^*) > 0$, we must also have $x_2^* + \alpha_2(x^*) \geq 1 - \delta$, since otherwise the lower-level problem becomes infeasible.

Claim 4: $x_2^* = u(x_1^*)$, or equivalently, $x_2^* + \alpha_2(x^*) = 1 - \delta$.

If the claim is not true then by Claim 3 we have that

$$x_2^* + \alpha_2(x^*) > 1 - \delta.$$

Then, an optimal response of the follower for x^* can be chosen as $(y_1^*, 1)$, where y_1^* is given by

$$y_1^* = \min \left\{ \frac{x_2^* + \alpha_2(x^*) - (1 - \delta)}{x_1^* + \alpha_1(x^*)}, 1 \right\}.$$

Again, from the optimality of (x_1^*, x_2^*) , $y_1^* < 1$ holds, for the same reason as in (25). Thus, for x_2 near enough to x_2^* we have that

$$y_1(x_1^*, x_2) = \frac{x_2 + \alpha_2(x_1^*, x_2) - (1 - \delta)}{x_1^* + \alpha_1(x_1^*, x_2)},$$

which is strictly increasing with respect to x_2 around x_2^* . Indeed, this follows from the fact that it is $\mathcal{C}^1$ and

$$\begin{aligned} \partial_2 y_1(x_1^*, x_2^*) &= \frac{1 + \partial_2 \alpha_2(x^*)}{x_1^* + \alpha_1(x^*)} - \frac{(x_2^* + \alpha_2(x^*) - (1 - \delta)) \partial_2 \alpha_1(x^*)}{(x_1^* + \alpha_1(x^*))^2} \\ &= \frac{1 + \partial_2 \alpha_2(x^*) - y_1^* \partial_2 \alpha_1(x^*)}{x_1^* + \alpha_1(x^*)} \\ &\geq \frac{1 - \varepsilon - y_1^* \varepsilon}{x_1^* + \alpha_1(x^*) + \varepsilon} \\ &\geq \frac{1 - 2\varepsilon}{x_1^* + \alpha_1(x^*) + \varepsilon} > 0. \end{aligned} \tag{26}$$

Note that the mapping $x_2 \mapsto x_2 + \rho(x_1^*, x_2)$ is strictly increasing as well. Then if $x_2^* > 1/2$ there exists $\eta > 0$ small enough such that

$$\begin{aligned} v(x_1^*, x_2^* - \eta) &= x_1^* + x_2^* - \eta + \rho(x_1^*, x_2^* - \eta) + 4y_1(x_1^*, x_2^* - \eta) \\ &< x_1^* + x_2^* + \rho(x^*) + 4y_1^* \\ &= v(x^*), \end{aligned}$$

a contradiction. Then we must have $x_2^* = \frac{1}{2}$. But then $1/2 + \alpha_2(x_1^*, 1/2) < 1 - \delta$ and this is a contradiction with Claim 3.

Reduction to a Problem Without Solution. We can reduce problem (23) to

$$\begin{aligned} \min_{t, y} \quad & t + u(t) + \rho(t, u(t)) + 4y_1 \\ \text{s.t.} \quad & \begin{cases} t \in [-1, 1], \\ y \text{ solves } \begin{cases} \min_y -y_1 \\ \text{s.t. } \begin{cases} 0 \leq y_1, y_2 \leq 1, \\ y_2 - \frac{t + \alpha_1(t, u(t))}{1 - \delta} y_1 \geq 1, \end{cases} \end{cases} \end{cases} \end{aligned} \quad (27)$$

and we know that $t^* = x_1^*$ must be a solution. As in the proof of Proposition 4.3, we can define $\beta(t) = (t + \alpha_1(t, u(t)))/(1 - \delta)$ and get that the response of the follower has the form of $(y_1(t), y_2(t))$ with

$$y_1(t) = \begin{cases} 1, & \text{if } \beta(t) \leq 0, \\ 0, & \text{if } \beta(t) > 0. \end{cases}$$

Clearly, since the perturbation parameters δ and α are small (in norm), we have $\beta(-1) < 0$ and $\beta(1) > 0$, and so there is at least one point $t \in [-1, 1]$ such that $\beta(t) = 0$. Let

$$\bar{t} = \inf\{t \in [-1, 1] : \beta(t) > 0\}.$$

It holds $\beta(\bar{t}) = 0$ and since $x_1^* + \alpha_1(x_1^*, u(x_1^*)) > 0$, we get that $t^* > \bar{t}$. Thus, there is $t \in (\bar{t}, t^*)$ such that $\beta(t) > 0$. However, since $\varepsilon = 1/4$, we have that the mapping $\theta : t \mapsto t + u(t) + \rho(t, u(t))$ is strictly increasing, since

$$\theta'(t) \geq 1 - \|u'\|_\infty - \left| \left\langle \nabla \rho(t, u(t)), \begin{pmatrix} 1 \\ u'(t) \end{pmatrix} \right\rangle \right| \geq 1 - \|u'\|_\infty - 2\|\nabla \rho\|_\infty > 1 - 4\varepsilon = 0.$$

Thus, denoting by ν the objective in Problem (27) and noting that $y_1(t) = 0$ we see that

$$\nu(t^*) \geq \theta(t^*) > \theta(t) = \nu(t),$$

which contradicts the optimality of t^* . We conclude that Problem (23) fails to have solutions, which completes the proof. $\square$

Remark 4.7. In Theorem 4.6, only the upper-level objective and the constraints of the lower level were perturbed. Extending irreparability to perturbations of the upper-level constraints, the follower objective, or an x -dependent right-hand side is a direction for future work. Such extensions require a specified perturbation space and separate bounds and proofs. The formal reformulation (3) alone does not establish them directly: one must also verify preservation of the perturbation class, perturbation size, and correspondence between optimal solutions.

5. IRREPARABLE PROBLEMS IN COUNTERFACTUAL EXPLANATIONS

The results of Proposition 4.3 and Proposition 4.5 are also directly applicable to counterfactual explanations via exact penalization.

Theorem 5.1. *Let $\rho : [-1, 1] \rightarrow \mathbb{R}$ and $\alpha : [-1, 1] \rightarrow \mathbb{R}^2$ be two continuous functions, and consider the perturbed CE problem*

$$\begin{aligned} \min_{x,y} \quad & x + \rho(x) \\ \text{s.t.} \quad & \begin{cases} y \in \Theta := (-\infty, 0] \times \mathbb{R}, \\ x \in [-1, 1], \\ y \text{ solves } \begin{cases} \min_y -y_1 \\ \text{s.t.} \begin{cases} 0 \leq y_1 \leq 1, \\ y_2 = 0, \\ (1 + \alpha_2(x))y_2 + (-x + \alpha_1(x))y_1 \geq 0, \end{cases} \end{cases} \end{cases} \end{aligned} \quad (28)$$

Suppose that $\|\alpha_1\|_\infty, \|\alpha_2\|_\infty < 1$. Then, for any $R > 2 + 2\|\rho\|_\infty$, we have that (28) is equivalent to

$$\begin{aligned} \min_{x,y} \quad & x + \rho(x) + Ry_1 \\ \text{s.t.} \quad & \begin{cases} x \in [-1, 1], \\ y \text{ solves } \begin{cases} \min_y -y_1 \\ \text{s.t.} \begin{cases} 0 \leq y_1 \leq 1, \\ y_2 = 0, \\ (1 + \alpha_2(x))y_2 + (-x + \alpha_1(x))y_1 \geq 0, \end{cases} \end{cases} \end{cases} \end{aligned} \quad (29)$$

In particular, Problem (28) fails to have solutions for $\rho \equiv \alpha \equiv 0$, and also under small values of either α as in Proposition 4.3 or ρ as in Proposition 4.5.

Proof. As in the proof of Proposition 4.3, we have that the lower-level problem of Problem (28) admits the unique solution

$$y(x) = \begin{cases} (0, 0), & \text{if } \alpha_1(x) < x, \\ (1, 0), & \text{otherwise.} \end{cases}$$

Set

$$X_0 := \{x \in [-1, 1] : \alpha_1(x) < x\}.$$

We note that $x \in X_0$ if and only if $y_1(x) \leq 0$ so that the constraint $y \in \Theta$ imposes $x \in X_0$. Therefore, since $1 \in X_0$, the CE problem is feasible. The choice of R gives

$$\max_{x \in [-1, 1]} \{x + \rho(x)\} - \min_{x \in [-1, 1]} \{x + \rho(x)\} \leq 2 + 2\|\rho\|_\infty < R.$$

Thus, for every $x \in [-1, 1] \setminus X_0$ and every $z \in X_0$,

$$x + \rho(x) + R > z + \rho(z).$$

On X_0 , both objective functions coincide. Therefore, the two problems have the same optimal value and the same optimal solutions, whether or not the infimum is attained.

Finally, if $\rho \equiv 0$, then X_0 is a non-empty relatively open subset of $[-1, 1]$ that does not contain -1 . Every point of X_0 therefore has a smaller point in X_0 , so the objective x cannot attain its minimum there. If $\alpha \equiv 0$ and ρ satisfies the assumptions of Proposition 4.5, then $X_0 = (0, 1]$ and $1 + \rho'(x) > 0$, so $x + \rho(x)$ has no minimum on X_0 either. This proves the nonexistence assertions. $\square$

Theorem 5.2. *Let $X = [-1, 1] \times [1/2, 3/2]$. Set $\varepsilon = 1/4$. For any $\delta \in [0, \varepsilon)$, and any two continuously differentiable functions $\alpha : X \rightarrow \mathbb{R}^2$ and $\rho : X \rightarrow \mathbb{R}$ with*

$\|\alpha\|_\infty, \|\rho\|_\infty < \varepsilon$ and $\|J\alpha\|_\infty, \|\nabla\rho\|_\infty < \varepsilon$, the perturbed CE problem

$$\begin{aligned} \min_{x,y} \quad & x_1 + x_2 + \rho(x_1, x_2) \\ \text{s.t.} \quad & \begin{cases} x_1 \in [-1, 1], \\ x_2 \in [1/2, 3/2], \\ y \in \Theta := [0, 1/2] \times [0, 1], \\ y \text{ solves } \begin{cases} \min_y -y_1 \\ \text{s.t.} \begin{cases} 0 \leq y_1, y_2 \leq 1, \\ (x_2 + \alpha_2(x_1, x_2))y_2 \\ -(x_1 + \alpha_1(x_1, x_2))y_1 \geq 1 - \delta, \end{cases} \end{cases} \end{cases} \end{aligned} \quad (30)$$

fails to have a solution.

Proof. We take $\varepsilon = 1/4$. As in the proof of Theorem 4.6, there exists a continuously differentiable mapping $u : [-1, 1] \rightarrow (1/2, 3/2)$ such that $u(x_1)$ is the unique value satisfying

$$u(x_1) + \alpha_2(x_1, u(x_1)) = 1 - \delta, \quad \|u'\|_\infty \leq \frac{\varepsilon}{1 - \varepsilon} = \frac{1}{3}.$$

Suppose that (30) admits an optimal solution (x^*, y^*) with $x^* = (x_1^*, x_2^*)$. Necessarily $x_1^* + \alpha_1(x^*) > 0$. Otherwise, increasing y_1 to 1 preserves lower-level feasibility, so every lower-level optimal response has $y_1 = 1$, contradicting the upper-level constraint $y_1 \leq 1/2$. Since $x_2^* + \alpha_2(x^*) > 0$, lower-level feasibility now implies $x_2^* + \alpha_2(x^*) \geq 1 - \delta$. Moreover, the optimal response is $(y_1^*, 1)$, where

$$y_1^* = \frac{x_2^* + \alpha_2(x^*) - (1 - \delta)}{x_1^* + \alpha_1(x^*)} \in [0, 1/2].$$

We claim that $x_2^* + \alpha_2(x^*) = 1 - \delta$. If the inequality were strict, then $x_2^* > u(x_1^*) > 1/2$ and, for x_2 sufficiently close to x_2^* ,

$$y_1(x_1^*, x_2) = \frac{x_2 + \alpha_2(x_1^*, x_2) - (1 - \delta)}{x_1^* + \alpha_1(x_1^*, x_2)}.$$

This function is strictly increasing near x_2^* , since

$$\partial_2 y_1(x_1^*, x_2^*) = \frac{1 + \partial_2 \alpha_2(x^*) - y_1^* \partial_2 \alpha_1(x^*)}{x_1^* + \alpha_1(x^*)} \geq \frac{1 - \frac{3}{2}\varepsilon}{x_1^* + \alpha_1(x^*)} > 0.$$

Thus, for sufficiently small $\eta > 0$, the point $(x_1^*, x_2^* - \eta)$ remains feasible for the CE problem: its lower-level optimal response still satisfies $0 \leq y_1 \leq 1/2$. But $x_2 \mapsto x_2 + \rho(x_1^*, x_2)$ is strictly increasing, because $1 + \partial_2 \rho \geq 1 - \varepsilon > 0$. This contradicts optimality. Hence $x_2^* = u(x_1^*)$.

It follows that $t^* = x_1^*$ must solve the reduced problem

$$\begin{aligned} \min_{t,y} \quad & t + u(t) + \rho(t, u(t)) \\ \text{s.t.} \quad & \begin{cases} t \in [-1, 1], \\ y_1 = 0, \\ y \text{ solves } \begin{cases} \min_y -y_1 \\ \text{s.t.} \begin{cases} 0 \leq y_1, y_2 \leq 1, \\ y_2 - \frac{t + \alpha_1(t, u(t))}{1 - \delta} y_1 \geq 1, \end{cases} \end{cases} \end{cases} \end{aligned} \quad (31)$$

since any feasible point of this reduced problem (31) corresponds to a feasible point of (30) on the curve $x_2 = u(x_1)$ with the same objective value. Indeed, on this curve the optimal first coordinate of the follower is 0 when $t + \alpha_1(t, u(t)) > 0$ and 1

otherwise. Thus, the constraint $y_1 \in [0, 1/2]$ is equivalent to $y_1 = 0$, and the feasible set of parameters in the reduced problem is

$$E := \{t \in [-1, 1] : t + \alpha_1(t, u(t)) > 0\}.$$

This set is relatively open, contains 1, and does not contain -1 . Consequently, for every $t \in E$ there exists $s \in E$ with $s < t$. On the other hand, the reduced objective $\theta(t) := t + u(t) + \rho(t, u(t))$ is strictly increasing, since

$$\theta'(t) \geq 1 - \|u'\|_\infty - \|\nabla \rho\|_\infty(1 + \|u'\|_\infty) \geq 1 - \frac{1}{3} - \frac{1}{4} \left(1 + \frac{1}{3}\right) = \frac{1}{3} > 0.$$

Therefore, θ cannot attain its minimum on E . This contradicts the existence of t^* and completes the proof. $\square$

6. CONCLUSION

We studied linear bilevel problems with rotations, where the leader's decision parameterizes the left-hand side of the follower's constraints. Under Assumption 2.1, if the follower is feasible for every $x \in X$, uniformly relaxing all follower inequalities by any positive quantity yields a bilevel problem with a solution. This is an existence result for the relaxed problems. However, their optimizers need not converge to a bilevel-feasible point of the original problem as Remark 2.7 illustrates. Then, we studied the structure of the points of discontinuity of the follower's feasibility map and constructed a partition of the leader's domain such that the restriction of this map to every element of the partition is continuous. Finally, we constructed examples with persistent non-attainment relative to prescribed perturbation spaces. The leader domain, fixed follower constraints, and follower objective remain unchanged; only selected coefficients, a nonnegative scalar right-hand side relaxation, and an additive leader-objective function are perturbed. The simultaneous result uses a C^1 metric, and the objective example shows why controlling only perturbation values does not suffice.

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GPT-5.6 and GPT-6 Astra assisted with the general revision of the manuscript and proof review; however, all examples in Section 4 and all proofs were independently conceived, developed, and verified by the authors.

DATA AVAILABILITY STATEMENT

No data was used for this research.

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