

A Generalized Polynomial Lower Bound for the Weighted Completion Time Variance in a Single Machine

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Abstract

This paper studies the single-machine problem of minimizing the weighted completion time variance (WCTV) and proposes a polynomially computable lower-bounding framework that generalizes the benchmark introduced by Nessah and Chu (2010). We first develop a generalized augmented-sequence decomposition that mathematically connects the weighted problem with the classical unweighted variance setting. Using this decomposition, we derive a family of polynomially computable lower bounds for WCTV. The proposed framework exploits three decision parameters: the choice of the first job, the free processing-time parameter assigned to that job through an invariance property of WCTV, and the split level used in the generalized augmented-sequence representation. We show how these parameters can be incorporated into an improved lower-bounding scheme, establish the validity of the resulting bounds, and characterize their optimization structure with respect to the free processing-time parameter and the split level. We also provide an asymptotically motivated approximation for selecting the free processing-time parameter, which can help circumvent numerical stability issues in large-scale applications. We test our approach on publicly available scheduling instances and provide numerical evidence of the effectiveness of the proposed polynomial lower bounds.

Key words: Scheduling; Single machine; Weighted completion time variance; Optimality bounds

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1 Introduction

Completion-time dispersion criteria arise naturally in scheduling environments where regularity, fairness, and temporal balance are as important as aggregate efficiency. This perspective is particularly relevant in just-in-time production systems and service settings, where decision makers seek schedules whose completion times are not excessively heterogeneous (Li et al. 2006, Van de Vonder et al. 2008, Balouka and Cohen 2019, Romero-Silva and Hernández-López 2020, Yu and Han 2021). Variance-based objectives provide a compact way to quantify this form of homogeneity: instead of asking only whether jobs finish early on average, they ask whether completion times remain concentrated around a representative time. Such criteria have generated a substantial literature on single- and multi-machine scheduling (Merten and Muller 1972a, Eilon and Chowdhury 1977, Cai and Cheng 1998, Viswanathkumar and Srinivasan 2003, Li et al. 2010, Srirangacharyulu and Srinivasan 2010, Nasini and Nessah 2021, 2026).

The classical (unweighted) single-machine completion time variance (CTV) minimization problem was introduced by Merten and Muller (1972b), who uncovered rich structural properties. Subsequent contributions have shown that optimal CTV sequences are V-shaped or satisfy closely related dominance conditions (Cai 1995, Eilon and Chowdhury 1977, Mittenthal et al. 1995). Related contributions have identified special positions in optimal sequences: Schrage (1975) has proved that the longest job must be scheduled first, Hall and Kubiak (1991) have characterized the second and last positions, Manna and Prasad (1999) have derived bounds for the position of the smallest job, and Nasini and Nessah (2021) have established additional optimality conditions. These results make the classical (unweighted) CTV minimization problem unusually structured among sequencing problems, and they have been central to the development of analytical lower bounds and exact solution methods.

The WCTV minimization problem is more elusive. In WCTV, each job contributes to the squared deviation according to a positive weight, so that the reference point is the endogenous weighted mean completion time. This weighted centering is useful when jobs have heterogeneous importance, for example in customer-service environments or personnel planning contexts where deviations from the average completion time should be penalized differently across jobs. A related but distinct weighted dispersion problem is studied by Pereira and Vásquez (2017), who consider the weighted mean squared deviation problem with respect to a common due date. In that model, the reference time is exogenous. In WCTV, by contrast, the benchmark is the weighted mean completion time generated by the schedule itself, which produces a different mathematical structure.

It is important to note that the introduction of weights disrupts the symmetry arguments that support the unweighted theory. Consequently, the V-shaped and position-based results available for the homogeneous case do not transfer directly to WCTV. This explains why the weighted problem has received comparatively limited theoretical treatment. To the best of our knowledge, the first and still unique

polynomial-time computable lower bound dedicated to the single-machine WCTV minimization problem is due to Nessah and Chu (2010). Their approach is based on an augmented-instance construction which allows embedding the weighted problem into an unweighted CTV framework. This transformation is appealing because it connects WCTV to the better understood unweighted setting and makes it possible to import lower-bounding arguments from that literature. Although the lower bound of Nessah and Chu (2010) is polynomial and conceptually powerful, Nasini and Nessah (2026) show that it can attain the value zero (the universal lower bound for any variance) in some instances, even when the optimal WCTV value is far from zero. To overcome this limitation, Nasini and Nessah (2026) introduce a head–tail decomposition of the weighted mean squared deviation and a mixed-integer linear programming formulation. Their approach produces near-optimal lower and upper bounds for instances with up to 80 jobs, and the resulting bounds are substantially tighter than the augmented-sequence bound. However, this gain comes at the price of relying on MILP solvers and specialized cutting-plane machinery, a trade-off also encountered in time-indexed formulations for single-machine scheduling (Sousa and Wolsey 1992, Pereira and Vásquez 2017).

In this paper, we generalize the original augmented-sequence idea of Nessah and Chu (2010), with a view to attaining a tighter and polynomial lower bound for the WCTV minimization. The key observation is that converting weights into multiplicities creates a representation in which hidden degrees of freedom can be optimized. In particular, the augmented sequence can be shaped through the choice of the first job, based on an artificial processing time assigned to that first job, and through the split level used to generate the unit-weight pieces. Specifically, for any admissible split level, we express the WCTV of a schedule as the CTV of an associated augmented unweighted sequence plus explicit correction terms. This identity clarifies exactly how the weighted and unweighted objectives differ. We then study the optimization of the resulting bound with respect to the artificial first-job processing time and the split level. The optimization of these parameters fundamentally change the role of the augmented instance, allowing us to preserve polynomial computability while addressing the degenerate cases in which the original bound becomes zero, and while producing systematic improvements on benchmark and large-scale instances.

The numerical results support this theoretical construction. On the instance highlighted by Nasini and Nessah (2026), the original Nessah–Chu bound is equal to zero, the optimal WCTV is 8022.41, and our improved lower bound is 7171.792. On benchmark instances with $n = 10, 20, 40, 80$ provided by Nasini and Nessah (2026), the proposed polynomial bound consistently dominates or matches the original bound of Nessah and Chu (2010); in particular, for one $n = 10$ instance, the bound increases from 1169.43 to 31178.02, close to the MILP lower bound 32398.56. Finally, on larger instances with $n = 200$ and $n = 300$, where the MILP approach can no longer be used, we test the asymptotic optimization of our lower bound which homogeneously improves the baseline bound on all the tested instances.

The remainder of the paper is organized as follows. Section 2 introduces the WCTV problem, notation, and preliminary decomposition results. Section 3 develops the generalized augmented-sequence decomposition and lower-bounding framework. Section 4 studies the optimization of the bound with respect to the artificial processing-time parameter and the split level. Section 5 reports the numerical experiments. Finally, Section 6 concludes the paper. Mathematical proofs are provided in Appendix A.

2 Modeling setup

Let $\mathbb{N}_N = \{1, \dots, N\}$ denote the set of the first N positive integers and consider the set of job indices $\mathcal{N} := \mathbb{N}_n = \{1, \dots, n\}$. Each job $i \in \mathcal{N}$ has a processing time $p_i > 0$ and a weight $w_i > 0$, both integer-valued. Throughout the paper, and consistently with the augmented-sequence construction developed below, we index jobs according to the nondecreasing ratios

$$\frac{p_1}{w_1} \leq \frac{p_2}{w_2} \leq \dots \leq \frac{p_n}{w_n}. \quad (1)$$

A sequence, or schedule, σ is a permutation of $\mathcal{N}$. We denote by $\sigma[h]$ the job scheduled in position h of σ and, to lighten notation, we write $[h]$ in place of $\sigma[h]$ whenever no ambiguity is possible. Thus, the processing time and weight of the job scheduled in position h are written as $p_{\sigma[h]}$ and $w_{\sigma[h]}$, or simply as $p[h]$ and $w[h]$. Let $MS := \sum_{i \in \mathcal{N}} p_i$ and $W := \sum_{i \in \mathcal{N}} w_i$. For any sequence σ , the completion time of job i is denoted by $C_i(\sigma)$, with $C_{\sigma[h]}(\sigma) = \sum_{t=1}^h p_{\sigma[t]}$, and the weighted mean completion time is

$$\overline{C}(\sigma) = \frac{1}{W} \sum_{i=1}^n w_i C_i(\sigma).$$

The WCTV of σ is the minimum value of a weighted mean squared deviation from a common reference time, attained uniquely at $d = \overline{C}(\sigma)$ (Pereira and Vásquez 2017):

$$WCTV(\sigma) = \min_{d \in \mathbb{R}} WMSD(\sigma, d) = \min_{d \in \mathbb{R}} \frac{1}{W} \sum_{i=1}^n w_i (C_i(\sigma) - d)^2 = \frac{1}{W} \sum_{i=1}^n w_i (C_i(\sigma) - \overline{C}(\sigma))^2. \quad (2)$$

The WCTV minimization problem consists of finding a sequence σ that minimizes $WCTV(\sigma)$. To tackle this problem, Nessah and Chu (2010) introduce a polynomially computable lower bound on the minimum WCTV, whereas Nasini and Nessah (2026) show that, in some specific instances, such a bound can attain zero (the trivial bound for a variance). They then introduce a decomposition and a MILP formulation of $WMSD(\sigma, d)$, resulting in near-optimal lower bound. To introduce this decomposition here, let $m := \lfloor \frac{n}{2} \rfloor$ and $\mathcal{K} := \{1, \dots, m\}$. For each $k \in \mathcal{K}$, define the symmetric head–tail jobs $\alpha_k := \sigma[k]$ and $\beta_k := \sigma[n-k+1]$. If n is odd, also set $\alpha_{m+1} = \beta_{m+1} := \sigma[m+1]$.

Proposition 1 (From Nasini and Nessah (2026)). *For any σ and any $d \in \mathbb{R}$, we have*

$$WMSD(\sigma, d) = Q(\sigma) + R(\sigma, d) = \begin{cases} \frac{1}{W} \left(\sum_{t=1}^m Q_t(\sigma) + \sum_{t=1}^m R_t(\sigma, d) \right) & \text{if } n \text{ is even,} \\ \frac{1}{W} \left(\sum_{t=1}^m Q_t(\sigma) + \sum_{t=1}^{m+1} R_t(\sigma, d) \right) & \text{if } n \text{ is odd,} \end{cases}$$

where the terms Q_t ($t = 1, \dots, m$) and R_t ($t = 1, \dots, n - m$) are given by

$$Q_t(\sigma) = \frac{w_{\alpha_t} w_{\beta_t}}{w_{\alpha_t} + w_{\beta_t}} [C_{\beta_t}(\sigma) - C_{\alpha_t}(\sigma)]^2, \quad (3)$$

$$R_t(\sigma, d) = (\bar{w}_{\alpha_t} + \bar{w}_{\beta_t}) \left[d - \frac{\bar{w}_{\beta_t} C_{\beta_t}(\sigma) + \bar{w}_{\alpha_t} C_{\alpha_t}(\sigma)}{\bar{w}_{\beta_t} + \bar{w}_{\alpha_t}} \right]^2, \quad (4)$$

and $(\bar{w}_{\alpha_t}, \bar{w}_{\beta_t}) = (w_{\alpha_t}, w_{\beta_t})$ if $t \leq m$, while $(\bar{w}_{\alpha_t}, \bar{w}_{\beta_t}) = (\frac{w_{\alpha_{m+1}}}{2}, \frac{w_{\alpha_{m+1}}}{2})$ if n is odd and $t = m + 1$. Here, $Q(\sigma) := \frac{1}{W} \sum_{t=1}^m Q_t(\sigma)$ and $R(\sigma, d) := \frac{1}{W} \sum_{t=1}^{n-m} R_t(\sigma, d)$.

Building on this decomposition, Nasini and Nessah (2026) developed a MILP formulation which captures head–tail dispersion independently of the reference time d , while the residual term $R(\sigma, \bar{C}(\sigma))$ captures the effect of the endogenous centering. Although this approach yields near-optimal solutions, its reliance on MILP solvers limits its applicability to large-scale instances. Our goal is to develop a polynomial lower-bounding scheme by returning to the augmented-instance idea of Nessah and Chu (2010), while addressing the cases where this polynomially computable bound yields a zero value. The key idea of Nessah and Chu (2010) (detailed in Section 3) is to convert weights into multiplicities by splitting each job into identical unit-weight pieces, thereby embedding the WCTV problem into an unweighted completion-time-variance framework. This shift has two advantages. First, it allows us to import lower-bounding arguments from the classical unweighted CTV literature. Second, it exposes additional degrees of freedom that were not used in the original construction of Nessah and Chu (2010). The first of these degrees of freedom is justified by the following invariance property.

Lemma 1. *Consider any sequence $\sigma = (\sigma[1], \dots, \sigma[n])$. Then $WCTV(\sigma)$ is independent of the processing time of the job scheduled in the first position, i.e., of $p_{\sigma[1]}$.*

Since changing the processing time of the first job shifts all completion times by the same amount, without changing the centered deviations in (2), we may regard the processing time assigned to the first job as a free positive real parameter. We denote this parameter by p and write $a := \sigma[1]$ for the first job. When $p = p_a$, we recover the original instance. The rest of the paper exploits this invariance through the augmented-sequence construction presented in the next section.

3 Generalized augmented sequence decomposition

Generalizing the idea of Nessah and Chu (2010), this section provides a reformulation of $WCTV(\sigma)$ in terms of an unweighted variance, using an augmented sequence σ_{A_x} obtained by replacing each job i by $k_i = 2xw_i$ unit-weight pieces, where x is a free split level parameter. In this unweighted instance, job i is split into k_i pieces $i_1, \dots, i_{k_i}$ having processing times and weights

$$\tilde{p}_{i_h} = \frac{p_i}{k_i} \quad \text{and} \quad w_{i_h} = 1, \quad h = 1, \dots, k_i. \quad (5)$$

The $2xW = \sum_{i=1}^n k_i$ pieces are re-labeled consecutively as follows:

$$i_h = \begin{cases} h & \text{if } i = 1, \\ h + \sum_{s=1}^{i-1} 2xw_s & \text{if } i > 1, \end{cases} \quad h = 1, \dots, k_i. \quad (6)$$

The augmented sequence σ_{A_x} is constructed from σ by replacing each job $\sigma[t]$ with its block of $k_{\sigma[t]}$ pieces, kept consecutive in σ_{A_x} :

$$\sigma_{A_x} = \left(\underbrace{\frac{p_{\sigma[1]}}{2xw_{\sigma[1]}}, \dots, \frac{p_{\sigma[1]}}{2xw_{\sigma[1]}}}_{2xw_{\sigma[1]} \text{ times}}, \underbrace{\frac{p_{\sigma[2]}}{2xw_{\sigma[2]}}, \dots, \frac{p_{\sigma[2]}}{2xw_{\sigma[2]}}}_{2xw_{\sigma[2]} \text{ times}}, \dots, \underbrace{\frac{p_{\sigma[n]}}{2xw_{\sigma[n]}}, \dots, \frac{p_{\sigma[n]}}{2xw_{\sigma[n]}}}_{2xw_{\sigma[n]} \text{ times}} \right). \quad (7)$$

Let $C_{i_h}(\sigma_{A_x})$ denote the completion time of piece i_h in σ_{A_x} and let

$$\bar{C}(\sigma_{A_x}) = \frac{1}{2xW} \sum_{i=1}^n \sum_{h=1}^{2xw_i} C_{i_h}(\sigma_{A_x}) = \frac{1}{2xW} \sum_{\ell=1}^{2xW} C_{\ell}(\sigma_{A_x})$$

be the (unweighted) mean completion time in the augmented instance. The CTV of σ_{A_x} is then

$$CTV(\sigma_{A_x}) = \frac{1}{2xW} \sum_{i=1}^n \sum_{h=1}^{2xw_i} (C_{i_h}(\sigma_{A_x}) - \bar{C}(\sigma_{A_x}))^2 = \frac{1}{2xW} \sum_{\ell=1}^{2xW} (C_{\ell}(\sigma_{A_x}) - \bar{C}(\sigma_{A_x}))^2.$$

Let p denote the processing time assigned to job a and define

$$\hat{p}_i(a, p) = \begin{cases} p, & i = a, \\ p_i, & i \neq a. \end{cases}$$

Let $\sigma_{A_x}^{(a,p)}$ be the augmented sequence obtained from σ using the processing times $\hat{p}_i(a, p)$, with the pieces of each job kept consecutive. The following theorem establishes a relationship between the weighted variance of a sequence σ and the unweighted variance of the augmented sequence $\sigma_{A_x}^{(a,p)}$.

Theorem 1. Define $\Psi_a(x, p) = \frac{1}{W} \left(\frac{2xw_a - 1}{4x} p + \sum_{i \neq a} \frac{2xw_i - 1}{4x} p_i \right)$ and

$$\begin{cases} A_{a,i}(x, p) = \frac{2xw_i - 1}{4xw_i} \hat{p}_i(a, p) - \Psi_a(x, p), \\ CT_a^{(1)}(x, p) = \Psi_a(x, p)^2 - \frac{1}{W} \left(\frac{(2xw_a - 1)(4xw_a - 1)}{24x^2w_a} p^2 + \sum_{i \neq a} \frac{(2xw_i - 1)(4xw_i - 1)}{24x^2w_i} p_i^2 \right). \end{cases}$$

We have the following decomposition:

$$WCTV(\sigma) = CTV(\sigma_{A_x}^{(a,p)}) + \frac{2}{W} \sum_i w_i A_{a,i}(x, p) \left(C_i^{(a,p)}(\sigma) - \overline{C}^{(a,p)}(\sigma) \right) + CT_a^{(1)}(x, p), \quad (8)$$

where $C_i^{(a,p)}(\sigma)$ and $\overline{C}^{(a,p)}(\sigma)$ denote the completion times and weighted mean under $\hat{p}(a, p)$.

Because $a = \sigma[1]$, Lemma 1 implies that, for every job i , $C_i^{(a,p)}(\sigma) - \overline{C}^{(a,p)}(\sigma) = C_i(\sigma) - \overline{C}(\sigma)$. Based on this characterization, the fundamental idea for constructing a lower bound for the minimization of $WCTV(\sigma)$ is to reduce the problem to an augmented unweighted CTV problem, which allows us to leverage well-established results for CTV through the decomposition in Theorem 1. This idea was originally presented by Nessah and Chu (2010), for the special case $x = \frac{1}{2}$, $a = n$ and $p = p_n$. Let $P_i = \sum_{h=1}^i p_h$ for each $i = 1, \dots, n$, with $P_0 = 0$, and

$$\begin{cases} \theta_j = W - 2 \left\lfloor \frac{W}{2} \right\rfloor + 2(r_1 + \dots + r_{j-1}) - (w_1 + \dots + w_{j-1}), \\ g_j = \frac{2}{3}(r_j - 1)(2r_j - 1) + 2(r_j - 1)(1 + \theta_j) + (1 + \theta_j)^2, \end{cases}$$

with r_j defined recursively by

$$r_1 = \left\lfloor \frac{w_1}{2} \right\rfloor, \quad r_j = \left\lfloor \frac{w_1 + \dots + w_j}{2} \right\rfloor - (r_1 + \dots + r_{j-1}), \quad j = 2, \dots, n.$$

We define

$$CT_a^{(2)}(x, p) := \frac{1}{W} \sum_{i=1}^n w_i A_{a,i}(x, p)^2 = \frac{1}{16x^2 W} \left(\frac{(2xw_a - 1)^2}{w_a} p^2 + \sum_{i \neq a} \frac{(2xw_i - 1)^2}{w_i} p_i^2 \right) - \Psi_a(x, p)^2.$$

Theorem 2. Let σ^* be an optimal solution of the minimization of $WCTV$, and let $a^* = \sigma^*[1]$. For values of a , x , and p for which the following expression is evaluated, define

$$LB_a^{(NC)}(x, p) = \left[\left(\sqrt{\left(LB^{(VS)}(x) + CT_a^{(1)}(x, p) + CT_a^{(2)}(x, p) \right)^+} - \sqrt{CT_a^{(2)}(x, p)} \right)^+ \right]^2, \quad (9)$$

where $(s)^+ = \max(0, s)$. In particular, when $x = \frac{1}{2}$, one has

$$\begin{aligned} LB^{(VS)}\left(\frac{1}{2}\right) &= \frac{1}{2W} \left\{ \Delta^{(VS)} + \sum_{i=1}^n r_i \left[P_{i-1}^2 + 2(r_i + \theta_i) P_{i-1} \frac{p_i}{w_i} + g_i \left(\frac{p_i}{w_i} \right)^2 \right] \right\}, \\ \Delta^{(VS)} &= \begin{cases} 0, & \text{if } w_n \geq 3 \text{ or } (w_n = 1, w_{n-1} \geq 2), \\ \frac{1}{2} \left(\frac{p_n}{w_n} - \frac{p_{n-1}}{w_{n-1}} \right)^2, & \text{if } w_n = 2, \\ \frac{1}{2} \left(\frac{p_{n-1}}{w_{n-1}} - \frac{p_{n-2}}{w_{n-2}} \right)^2, & \text{if } w_n = 1 \text{ and } w_{n-1} = 1. \end{cases} \end{aligned}$$

Then, $WCTV(\sigma^*) \geq LB_{a^*}^{(NC)}(1/2, p_{a^*})$. Moreover, in this specialization $LB_a^{(NC)}(1/2, p_a)$ is independent of a , and hence $WCTV(\sigma^*) \geq LB_n^{(NC)}(1/2, p_n)$.

In Theorem 2, the dependence on a , p , and x is made explicit at the level of the generic augmented decomposition. The term $LB^{(VS)}(x)$ represents the lower bound of Viswanathkumar and Srinivasan (2003) computed on the augmented unweighted instance associated with the split level $x = \frac{1}{2}$, for which the augmented instance contains exactly W unit-weight jobs. However, there is no guarantee that $LB_n^{(NC)}(\frac{1}{2}, p_n) > 0$, as illustrated by Nasini and Nessah (2026). This case is reproduced in Example 1 below.

Example 1. Consider a small WCTV minimization problem with $n = 6$, where the processing times and weights (indexed in nondecreasing ratios p_i/w_i) are $\mathbf{p} = (72, 45, 54, 36, 96, 1200)$ and $\mathbf{w} = (9, 5, 6, 3, 6, 4)$. For this instance, we obtain $LB^{(VS)} = 32645.03$, $CT^{(1)} = -33079.46$, and $CT^{(2)} = 19114.74$. Therefore, $LB^{(NC)} = 0$. The optimal solution, written in the original job labels, is $\sigma^* = (6, 5, 4, 3, 2, 1)$, so that $WCTV(\sigma^*) = 8022.41$.

Building on the three-parameter dependence a , p , and x , Section 3.1 below presents a class of improved lower bounds that aims at circumventing the limitations of the original Nessah–Chu construction, which materializes in cases when $LB^{(VS)}(1/2) \approx CT_n^{(1)}(1/2, p_n)$.

3.1 A class of improved lower bounds

Throughout this subsection, assume $n \geq 2$ (the case $n = 1$ has zero WCTV), and let $\mathcal{X} := \{x > 0 : 2xw_i \in \mathbb{N} \text{ for all } i \in \mathcal{N}\}$ be the set of admissible split levels. For each $x \in \mathcal{X}$, define

$$k_i(x) := 2xw_i, \quad \tilde{p}_i(x) := \frac{p_i}{k_i(x)} = \frac{p_i}{2xw_i}.$$

For each possible first job $a \in \mathcal{N}$, define $\mathcal{X}_a := \{x \in \mathcal{X} : 2x(2w_a - W) \leq 1\}$. Let Σ_a be the set of schedules whose first job is a and define the restricted problem

$$OPT(a) := \min_{\sigma \in \Sigma_a} WCTV(\sigma).$$

If σ^* is a globally optimal schedule and $a^* := \sigma^*[1]$, then $WCTV(\sigma^*) = OPT(a^*)$. Hence, any lower bound that is valid for every restricted problem $OPT(a)$ immediately yields a valid global lower bound after taking the minimum over $a \in \mathcal{N}$. To construct a class of improved lower bounds, we start from the generalized expression (9) and replace the $LB^{(VS)}(x)$ term by a refined lower bound $\overline{LB}_a^{(VS)}(x, p)$. This leads to the candidate lower bound

$$LB_a^{(NN)}(x, p) := \left[\left(\sqrt{\overline{LB}_a^{(VS)}(x, p) + CT_a^{(1)}(x, p) + CT_a^{(2)}(x, p)} - \sqrt{CT_a^{(2)}(x, p)} \right)^+ \right]^2, \quad (10)$$

where the explicit characterization of $\overline{LB}_a^{(VS)}(x, p)$ is provided in Lemma 2 below. The key distinction here is that the original term $LB^{(VS)}(x)$ in (9) is obtained by Nessah and Chu (2010) by applying the

procedure proposed by Viswanathkumar and Srinivasan (2003) to the augmented unweighted CTV instance with processing times and weights (5) (induced by the split level $x = \frac{1}{2}$). To clarify this construction, let $\mathcal{P}_1(x)$ denote the augmented unweighted problem in which the pieces of every original job must remain consecutive, and let $\mathcal{P}_2(x)$ denote the relaxation obtained by dropping this contiguity requirement. Since every feasible schedule of $\mathcal{P}_1(x)$ is also feasible for $\mathcal{P}_2(x)$, one has

$$\min CTV[\mathcal{P}_2(x)] \leq \min CTV[\mathcal{P}_1(x)].$$

Therefore, any lower bound on $\min CTV[\mathcal{P}_2(x)]$ is a lower bound on $\min CTV[\mathcal{P}_1(x)]$. Recall that, in the augmented instance, the $k_i = 2xw_i$ pieces of job i are labeled by the indices (6). Hence, the processing times of the $2xW$ jobs of $\mathcal{P}_2(x)$ form the multiset

$$\mathcal{Q}_x = \left\{ \underbrace{\frac{p_1}{2xw_1}, \dots, \frac{p_1}{2xw_1}}_{2xw_1 \text{ times}}, \underbrace{\frac{p_2}{2xw_2}, \dots, \frac{p_2}{2xw_2}}_{2xw_2 \text{ times}}, \dots, \underbrace{\frac{p_n}{2xw_n}, \dots, \frac{p_n}{2xw_n}}_{2xw_n \text{ times}} \right\}, \quad (11)$$

where, equivalently, the elements indexed by $i_1, \dots, i_{k_i}$ are exactly the $k_i = 2xw_i$ copies of the value $p_i/(2xw_i)$ associated with original job i . Let $N_x := 2xW$ and let $q_1^{(x)} \leq q_2^{(x)} \leq \dots \leq q_{N_x}^{(x)}$ be the elements of $\mathcal{Q}_x$ sorted in nondecreasing order. Since the original jobs are indexed by the nondecreasing ratios in (1), the sorted list is obtained by concatenating, for $i = 1, \dots, n$, the block of $2xw_i$ identical values $p_i/(2xw_i)$, that is, the block indexed by $i_1, \dots, i_{k_i}$. The Viswanathkumar–Srinivasan procedure provides a lower bound for the unweighted CTV of a list of jobs with sorted processing times. Applied to the present list $q_1^{(x)}, \dots, q_{N_x}^{(x)}$, the pairing argument combines squared completion-time differences with a correction L_x obtained from the extreme pairs. Following Viswanathkumar and Srinivasan (2003), a lower bound for the relaxed problem $\mathcal{P}_2(x)$ takes the form

$$LB^{(VS)}(x) = \frac{2L_x + VS_x}{4xW}, \quad \text{where} \quad L_x = \begin{cases} 0, & \text{if } N_x = 2, \\ \frac{1}{6}(q_2^{(x)} - q_1^{(x)})^2, & \text{if } N_x = 3, \\ \frac{1}{4}(q_{N_x-1}^{(x)} - q_{N_x-2}^{(x)})^2, & \text{if } N_x \geq 4, \end{cases} \quad (12)$$

and the term VS_x is formed by squared cumulative sums of the smallest processing times:

$$VS_x = \begin{cases} \sum_{h=1}^{(N_x-1)/2} (q_1^{(x)} + \dots + q_{2h}^{(x)})^2, & \text{if } N_x \text{ is odd,} \\ \sum_{h=0}^{N_x/2-1} (q_1^{(x)} + \dots + q_{2h+1}^{(x)})^2, & \text{if } N_x \text{ is even.} \end{cases}$$

For the augmented unweighted CTV instance with processing times and weights (5), the sorted list contains repeated values, namely $2xw_i$ copies of $p_i/(2xw_i)$ for each original job i . Therefore, the cumulative sums

entering (12) can be regrouped blockwise by original job. In the special case $x = \frac{1}{2}$, with $W \geq 4$, carrying out this regrouping yields the compact expression used by Nessah and Chu (2010), displayed in Theorem 2.

The refinement developed below keeps the same pairing philosophy, but exploits the fact that, in the restricted problem where the first job is fixed, the first block of the augmented sequence is also fixed. Hence, the pairing construction can be adapted by separating the fixed left block, the matched right tail, and the remaining middle part. Specifically, for each $i = 1, \dots, n-1$, define

$$v_i(a, x) = \begin{cases} 2xw_i, & \text{if } i < a, \\ 2xw_{i+1}, & \text{if } i \geq a, \end{cases} \quad \bar{p}_i(a) = \begin{cases} p_i, & \text{if } i < a, \\ p_{i+1}, & \text{if } i \geq a. \end{cases} \quad (13)$$

After removing job a , the i -th remaining job contributes a block of $v_i(a, x)$ identical pieces to the augmented sequence (7), each of processing time $\bar{p}_i(a)/v_i(a, x)$. Also define $k_a(x) := 2xw_a$. For every $x \in \mathcal{X}$, the sequence

$$\frac{\bar{p}_1(a)}{v_1(a, x)} \leq \frac{\bar{p}_2(a)}{v_2(a, x)} \leq \dots \leq \frac{\bar{p}_{n-1}(a)}{v_{n-1}(a, x)}$$

is nondecreasing, because it is obtained from the original nondecreasing sequence $(p_i/w_i)_{i=1}^n$ after deleting the a -th term and multiplying all ratios by the common positive factor $1/(2x)$. Moreover, among admissible split levels, the following position indexes are well defined precisely for $x \in \mathcal{X}_a$:

$$k_0(a, x) := \max \left\{ k \in \mathbb{N}_{n-1} : k_a(x) \leq 1 + \sum_{h=k}^{n-1} v_h(a, x) \right\}, \quad \text{and} \quad \kappa(a, x) := k_a(x) - 1 - \sum_{h=k_0(a, x)+1}^{n-1} v_h(a, x).$$

By construction, $0 \leq \kappa(a, x) \leq v_{k_0(a, x)}(a, x)$. We also introduce the truncated right-tail and middle multiplicities

$$\bar{v}_i(a, x) = \begin{cases} 0, & \text{if } i < k_0(a, x), \\ \kappa(a, x), & \text{if } i = k_0(a, x), \\ v_i(a, x), & \text{if } i > k_0(a, x), \end{cases} \quad \bar{\bar{v}}_i(a, x) = \begin{cases} v_i(a, x), & \text{if } i < k_0(a, x), \\ v_i(a, x) - \kappa(a, x) + 1, & \text{if } i = k_0(a, x), \\ 0, & \text{if } i > k_0(a, x). \end{cases}$$

The additional unit in the definition of $\bar{\bar{v}}_{k_0}(a, x)$ represents the replacement, in the center-out lower-bounding step, of the last piece of the first block by one additional copy of $\bar{p}_{k_0}(a)/v_{k_0}(a, x)$. This replacement is valid for

$$p \geq \underline{p}_a(x) := k_a(x) \frac{\bar{p}_{k_0(a, x)}(a)}{v_{k_0(a, x)}(a, x)}. \quad (14)$$

This restriction ensures that the last piece of the first block, whose processing time is $p/k_a(x)$, is no smaller than the largest value in the surrogate middle multiset.

When no ambiguity arises, we simply write k_a , k_0 , κ , v_i , $\bar{p}_i$, $\bar{v}_i$, and $\bar{\bar{v}}_i$, in place of $k_a(x)$, $k_0(a, x)$, $\kappa(a, x)$, $v_i(a, x)$, $\bar{p}_i(a)$, $\bar{v}_i(a, x)$, and $\bar{\bar{v}}_i(a, x)$, respectively. These quantities have a simple combinatorial interpretation. After fixing the first job a , the lower-bounding construction uses: (i) a fixed left block of

k_a pieces corresponding to job a , (ii) a right tail consisting of the $k_a - 1$ largest pieces from the remaining jobs, and (iii) a middle part. The actual middle subsequence includes the last piece of the first block. The index $k_0(a, x)$ identifies the truncation block in the right tail, while $\kappa(a, x)$ gives the number of pieces taken from that block. The multiplicities $\bar{v}_i(a, x)$ and $\bar{\bar{v}}_i(a, x)$ describe, respectively, the part of block i used in the right tail and the part retained in the middle block. After fixing the first job a and removing the right-tail pieces encoded by $k_0(a, x)$ and $\kappa(a, x)$, define the multiset of processing times used to lower bound the middle subsequence by

$$\bar{\mathcal{Q}}_{a,x} = \left\{ \underbrace{\frac{\bar{p}_1(a)}{v_1(a,x)}, \dots, \frac{\bar{p}_1(a)}{v_1(a,x)}}_{\bar{\bar{v}}_1(a,x) \text{ times}}, \dots, \underbrace{\frac{\bar{p}_{k_0(a,x)}(a)}{v_{k_0(a,x)}(a,x)}, \dots, \frac{\bar{p}_{k_0(a,x)}(a)}{v_{k_0(a,x)}(a,x)}}_{\bar{\bar{v}}_{k_0(a,x)}(a,x) \text{ times}} \right\}.$$

Let $\bar{q}_1 \leq \dots \leq \bar{q}_{m_x}$ be the elements of $\bar{\mathcal{Q}}_{a,x}$ arranged in nondecreasing order, where, suppressing the dependence of m_x on a , and define

$$m_x := \sum_{h=1}^{k_0(a,x)} \bar{\bar{v}}_h(a, x) = v_{k_0(a,x)}(a, x) - \kappa(a, x) + 1 + \sum_{h=1}^{k_0(a,x)-1} v_h(a, x) = N_x - 2k_a(x) + 2.$$

Building on this subsequence, the Lemma 2 below constructs the refined quantity $\bar{LB}_a^{(VS)}(x, p)$, which is used in place of the baseline term $LB^{(VS)}(x)$ by exploiting the fixed first block. For $i = k_0(a, x), \dots, n-1$, introduce the following quantities, which depend only on the data and on the choice of a and x :

$$\begin{aligned} \bar{\alpha}_i(a, x) &= \left(1 - \sum_{h=i+1}^{n-1} \frac{v_h(a, x)}{k_a(x)}\right)^2 - (\bar{v}_i(a, x) + 1) \frac{1 - \sum_{h=i+1}^{n-1} \frac{v_h(a, x)}{k_a(x)}}{k_a(x)} + \frac{(\bar{v}_i(a, x) + 1)(2\bar{v}_i(a, x) + 1)}{6 k_a(x)^2}, \\ \bar{\beta}_i(a, x) &= 2 \left(\sum_{h=1}^i \bar{p}_h(a) + \frac{\bar{p}_i(a)}{v_i(a, x)} \right) \left(1 - \sum_{h=i+1}^{n-1} \frac{v_h(a, x)}{k_a(x)} \right) \\ &\quad - (\bar{v}_i(a, x) + 1) \left[\left(1 - \sum_{h=i+1}^{n-1} \frac{v_h(a, x)}{k_a(x)} \right) \frac{\bar{p}_i(a)}{v_i(a, x)} + \frac{\sum_{h=1}^i \bar{p}_h(a) + \frac{\bar{p}_i(a)}{v_i(a, x)}}{k_a(x)} \right] \\ &\quad + \frac{(\bar{v}_i(a, x) + 1)(2\bar{v}_i(a, x) + 1)}{3 k_a(x) v_i(a, x)} \bar{p}_i(a), \\ \bar{\gamma}_i(a, x) &= \left(\sum_{h=1}^i \bar{p}_h(a) + \frac{\bar{p}_i(a)}{v_i(a, x)} \right)^2 \\ &\quad - (\bar{v}_i(a, x) + 1) \frac{\left(\sum_{h=1}^i \bar{p}_h(a) + \frac{\bar{p}_i(a)}{v_i(a, x)} \right) \bar{p}_i(a)}{v_i(a, x)} + \frac{(\bar{v}_i(a, x) + 1)(2\bar{v}_i(a, x) + 1)}{6 v_i(a, x)^2} \bar{p}_i(a)^2. \end{aligned}$$

Lemma 2. Fix $a \in \mathcal{N}$, $x \in \mathcal{X}_a$, and $p \in \mathcal{P}_a(x) := [\underline{p}_a(x), +\infty)$. Let $k_a = k_a(x)$, $k_0 = k_0(a, x)$, and $\kappa = \kappa(a, x)$. For $i = k_0, \dots, n-1$, define

$$H_\alpha(a, x) = \sum_{h=k_0}^{n-1} \bar{v}_h(a, x) \bar{\alpha}_h(a, x), \quad H_\beta(a, x) = \sum_{h=k_0}^{n-1} \bar{v}_h(a, x) \bar{\beta}_h(a, x), \quad H_\gamma(a, x) = \sum_{h=k_0}^{n-1} \bar{v}_h(a, x) \bar{\gamma}_h(a, x),$$

where the coefficients $\bar{\alpha}_h(a, x)$, $\bar{\beta}_h(a, x)$ and $\bar{\gamma}_h(a, x)$ are computed with the multiplicities $\bar{v}_h(a, x)$ defined above. Consider

$$\overline{LB}_a^{(VS)}(x, p) = \frac{1}{4xW} [LB_{1,x}(k_0) + H_\alpha(a, x)p^2 + H_\beta(a, x)p + H_\gamma(a, x)], \quad (15)$$

where

$$LB_{1,x}(k_0) = \begin{cases} \sum_{h=1}^{(m_x-1)/2} (\bar{q}_1 + \dots + \bar{q}_{2h})^2, & \text{if } m_x \text{ is odd,} \\ \sum_{h=1}^{m_x/2} (\bar{q}_1 + \dots + \bar{q}_{2h-1})^2, & \text{if } m_x \text{ is even.} \end{cases}$$

Then, for every $\sigma \in \Sigma_a$, where σ_{A_x} is constructed by assigning processing time p to the first job a and keeping the original processing times p_i for $i \neq a$, one has $CTV(\sigma_{A_x}) \geq \overline{LB}_a^{(VS)}(x, p)$.

Having established the refined bound for the augmented CTV problem, we next verify that the square-root terms in (10) are well defined and establish the conditions under which $LB_a^{(NN)}(x, p)$ yields a valid lower bound for WCTV.

Lemma 3. For every admissible split level $x \in \mathcal{X}$, every $a \in \mathcal{N}$, and every $p > 0$, one has $CT_a^{(2)}(x, p) \geq 0$ and $\overline{LB}_a^{(VS)}(x, p) + CT_a^{(1)}(x, p) + CT_a^{(2)}(x, p) \geq 0$.

Theorem 3. Let σ^* be a globally optimal schedule, and let $a^* = \sigma^*[1]$. Let $\mathcal{A} \subseteq \mathcal{N}$ be such that $a^* \in \mathcal{A}$ and $\mathcal{X}_a \neq \emptyset$ for every $a \in \mathcal{A}$. Then

$$WCTV(\sigma^*) \geq \min_{a \in \mathcal{A}} \sup_{x \in \mathcal{X}_a} \sup_{p \geq \underline{p}_a(x)} LB_a^{(NN)}(x, p). \quad (16)$$

By (15), the quantity $\overline{LB}_a^{(VS)}(x, p)$ is a quadratic polynomial in p . The same is true of $CT_a^{(1)}(x, p)$ and $CT_a^{(2)}(x, p)$. It is therefore convenient to isolate the square-root difference appearing in (10), so that for fixed (a, x) the optimization in p reduces to maximizing the difference of two square roots of quadratic polynomials.

Corollary 1. For each fixed $a \in \mathcal{N}$, $x \in \mathcal{X}_a$, and $p \in \mathcal{P}_a(x) = \{p : p \geq \underline{p}_a(x)\}$, we have

$$LB_a^{(NN)}(x, p) = [(g_{a,x}(p))^+]^2 := \left[\left(\sqrt{\alpha_{1,a}(x)p^2 + \alpha_{2,a}(x)p + \alpha_{3,a}(x)} - \sqrt{c_{1,a}(x)p^2 + c_{2,a}(x)p + c_{3,a}(x)} \right)^+ \right]^2.$$

Here, all quantities k_0 , $\bar{v}_h$, $\bar{\alpha}_h$, $\bar{\beta}_h$ and $\bar{\gamma}_h$ are evaluated at (a, x) , and

$$\begin{cases} \alpha_{1,a}(x) = \frac{1}{4xW} \sum_{h=k_0}^{n-1} \bar{v}_h \bar{\alpha}_h - \frac{(2xw_a - 1)(2xw_a + 1)}{48x^2w_aW}, \\ \alpha_{2,a}(x) = \frac{1}{4xW} \sum_{h=k_0}^{n-1} \bar{v}_h \bar{\beta}_h, \\ \alpha_{3,a}(x) = \frac{1}{4xW} \left(\sum_{h=k_0}^{n-1} \bar{v}_h \bar{\gamma}_h + LB_{1,x}(k_0) \right) + \frac{1}{24xW} \sum_{i=1}^{n-1} \frac{1 - v_i(a, x)^2}{v_i(a, x)} \bar{p}_i(a)^2. \\ c_{1,a}(x) = \frac{(2xw_a - 1)^2(W - w_a)}{16x^2w_aW^2}, \\ c_{2,a}(x) = -\frac{2xw_a - 1}{2xW} \Psi_a(x, 0), \\ c_{3,a}(x) = \frac{1}{8xW} \sum_{i=1}^{n-1} \frac{(v_i(a, x) - 1)^2}{v_i(a, x)} \bar{p}_i(a)^2 - \Psi_a(x, 0)^2, \end{cases}$$

with

$$\Psi_a(x, 0) = \frac{1}{W} \sum_{i=1}^{n-1} \frac{v_i(a, x) - 1}{4x} \bar{p}_i(a).$$

By Corollary 1, any interior maximizer $p^* > 0$ at which both radicals are real and differentiable must satisfy the first-order condition $g'_{a,x}(p^*) = 0$, as studied in Section 4.1. By contrast, the dependence on x is only piecewise regular. The source of non-smoothness is combinatorial: the index $k_0(a, x)$, and therefore also $\kappa(a, x)$, may change when x crosses certain thresholds. The next proposition shows that, for fixed a , only two such regimes can occur when $w_a \leq W/2$, that is, in the case where $\mathcal{X}_a = \mathcal{X}$.

Proposition 2. *Assume that $w_i \in \mathbb{N}$ for all $i \in \mathcal{N}$, fix $a \in \mathcal{N}$, and suppose that $w_a \leq W/2$. Let $g := \gcd(w_1, \dots, w_n)$ and $\delta := \frac{1}{2g}$. For $k = 1, \dots, n-1$, define*

$$S_k(a) := \begin{cases} \sum_{h=k}^{a-1} w_h + \sum_{h=a}^{n-1} w_{h+1}, & \text{if } k < a, \\ \sum_{h=k+1}^n w_h, & \text{if } k \geq a, \end{cases} \quad \text{and} \quad S_n(a) := 0.$$

Also define

$$k_a^+ := \max\{k \in \mathbb{N}_{n-1} : S_k(a) \geq w_a\} \quad \text{and} \quad k_a^- := \max\{k \in \mathbb{N}_{n-1} : S_k(a) \geq w_a - g\}. \quad (17)$$

Then, we obtain: $k_0(a, \delta) = k_a^-$, and $k_0(a, x) = k_a^+$ for all $x \in \mathcal{X} \setminus \{\delta\}$. Moreover, $\kappa(a, \delta) = 2\delta(w_a - S_{k_a^-+1}(a)) - 1$, while $\kappa(a, x) = 2x(w_a - S_{k_a^++1}(a)) - 1$, for all $x \in \mathcal{X} \setminus \{\delta\}$.

By Proposition 2, for each fixed $a \in \mathcal{N}$ such that $w_a \leq W/2$, the truncation index $k_0(a, x)$ has at most two possible values on $\mathcal{X}$. Hence the combinatorial part of the construction is completely determined

by two regimes: the minimal admissible split level $x = \delta$, and the set of all larger admissible split levels $x \in \mathcal{X} \setminus \{\delta\}$. In particular, once a is fixed, all quantities whose definition depends on the truncation index, namely $k_0(a, x)$, $\kappa(a, x)$, m_x , $\bar{v}_i(a, x)$, $\bar{\bar{v}}_i(a, x)$ and $LB_{1,x}(k_0)$, are governed by the same structural pattern throughout $\mathcal{X} \setminus \{\delta\}$, while a different pattern may occur only at $x = \delta$. Therefore, for every fixed a with $w_a \leq W/2$, the optimization with respect to x can be decomposed as

$$\sup_{x \in \mathcal{X}} \sup_{p \in \mathcal{P}_a(x)} LB_a^{(NN)}(x, p) = \max \left\{ \sup_{p \in \mathcal{P}_a(x)} LB_a^{(NN)}(\delta, p), \sup_{x \in \mathcal{X} \setminus \{\delta\}} \sup_{p \in \mathcal{P}_a(x)} LB_a^{(NN)}(x, p) \right\}.$$

This two-regime description is exploited in Section 4.2 to organize the optimization over the split level x . Example 2 below illustrates this behavior on a concrete instance.

Example 2. *Let us consider the WCTV minimization problem presented in Example 1. For each possible first job a , define the admissible artificial processing time $p_a^{\text{adm}} := \max \left\{ p_a, \underline{p}_a(\frac{1}{2}) \right\} \in \mathcal{P}_a(\frac{1}{2})$. Table 1 reports, for each choice of a , the values of $k_0(a, 1/2)$, $\underline{p}_a(1/2)$, p_a^{adm} , $\overline{LB}_a^{(VS)}(1/2, p_a^{\text{adm}})$, the two correction terms, and the resulting value of $LB_a^{(NN)}(1/2, p_a^{\text{adm}})$. For the original Nessah–Chu construction, evaluated at the original processing times, one has*

$$LB^{(VS)}(\frac{1}{2}) = 32645.030, \quad CT_a^{(1)}(\frac{1}{2}, p_a) = -33079.460, \quad CT_a^{(2)}(\frac{1}{2}, p_a) = 19114.748,$$

and therefore $LB_n^{(NC)}(1/2, p_n) = 0$.

Table 1: Different values of parameters and lower bounds as functions of the first job a for the re-indexed instance $\mathbf{p} = (72, 45, 54, 36, 96, 1200)$ and $\mathbf{w} = (9, 5, 6, 3, 6, 4)$.

a	1	2	3	4	5	6
$k_0(a, 1/2)$	4	5	4	5	4	5
$\underline{p}_a(1/2)$	144	1500	96	900	72	64
p_a^{adm}	144	1500	96	900	96	1200
$\overline{LB}_a^{(VS)}(1/2, p_a^{\text{adm}})$	83842.288	218190.167	78825.273	112918.121	72732.061	42748.545
$CT_a^{(1)}(1/2, p_a^{\text{adm}})$	-32820.815	-93211.444	-32861.747	-41900.353	-33079.460	-33079.460
$CT_a^{(2)}(1/2, p_a^{\text{adm}})$	18507.012	51998.550	18804.231	22489.822	19114.748	19114.748
$LB_a^{(NN)}(1/2, p_a^{\text{adm}})$	16292.497	37115.858	13774.915	24281.131	10850.073	986.079
$LB^{(VS)}(1/2)$	32645.030					
$LB_n^{(NC)}(1/2, p_n)$	0.000					

From Table 1, we observe that $LB^{(VS)}(\frac{1}{2}) < \overline{LB}_a^{(VS)}(\frac{1}{2}, p_a^{\text{adm}})$, for every $a \in \mathcal{N}$. Moreover, since $p_a^{\text{adm}} \in \mathcal{P}_a(1/2)$ for every a , the resulting valid lower bound is

$$\min_{a \in \mathcal{N}} LB_a^{(NN)}\left(\frac{1}{2}, p_a^{\text{adm}}\right) = 986.079 > LB_n^{(NC)}\left(\frac{1}{2}, p_n\right) = 0.000.$$

Table 2 reports the values of $S_k(a)$, and the indices k_a^+ and k_a^- . By Proposition 2, $g = \gcd(9, 5, 6, 3, 6, 4) = 1$ and $\delta = 1/2$, so that $k_0(a, \delta) = k_a^-$, whereas $k_0(a, x) = k_a^+$ for every $x \in \mathcal{X} \setminus \{\delta\}$.

Table 2: Values of $S_k(a)$, k_a^+ and k_a^- for the re-indexed 6-job instance with $\mathbf{w} = (9, 5, 6, 3, 6, 4)$.

a	$S_1(a)$	$S_2(a)$	$S_3(a)$	$S_4(a)$	$S_5(a)$	k_a^+	k_a^-	$k_0(a, x)$ -regime
1	24	19	13	10	4	4	4	4 for all $x \in \mathcal{X}$
2	28	19	13	10	4	4	5	5 if $x = \frac{1}{2}$; 4 otherwise
3	27	18	13	10	4	4	4	4 for all $x \in \mathcal{X}$
4	30	21	16	10	4	5	5	5 for all $x \in \mathcal{X}$
5	27	18	13	7	4	4	4	4 for all $x \in \mathcal{X}$
6	29	20	15	9	6	5	5	5 for all $x \in \mathcal{X}$

For example, when $a = 2$, one has $k_2^- = 5$ and $k_2^+ = 4$. Therefore,

$$k_0\left(2, \frac{1}{2}\right) = 5, \quad k_0(2, x) = 4 \quad \forall x \in \mathcal{X} \setminus \left\{\frac{1}{2}\right\},$$

so that $x = 1/2$ is the only exceptional point for the truncation index. By contrast, for $a \in \{1, 3, 4, 5, 6\}$, one has $k_a^- = k_a^+$, and hence $k_0(a, x)$ is constant over the whole set $\mathcal{X}$.

4 Enhancing the improved bound

The lower bound $LB_a^{(NN)}(x, p)$ introduced in Section 3.1 depends on three decision parameters: the choice of the first job a , the split level x , and the free processing time p assigned to that first job. The purpose of this section is to study how these parameters can be used to strengthen the bound and, in particular, to avoid the degeneracy that may affect the original $LB^{(NC)}$ bound, as illustrated in Example 1. The mechanism behind this degeneracy is already visible from the fact that the positive part of $LB^{(NC)}$ vanishes whenever $LB^{(VS)} + CT^{(1)} \leq 0$. This situation has a natural interpretation in terms of the splitting construction. In the original Nessah and Chu (2010)'s formulation, each job i is replaced by w_i unit-weight pieces of processing time p_i/w_i . If $w_i = 1$, no internal dispersion is created within the block associated with job i . By contrast, if $w_i > 1$, the completion times of the pieces of job i differ from the completion time of the original job by the internal offsets

$$\frac{w_i - 1}{w_i}p_i, \frac{w_i - 2}{w_i}p_i, \dots, \frac{1}{w_i}p_i, 0.$$

Consequently, a job with a large processing time and weight greater than one may generate a large internal quadratic correction in $CT^{(1)}$. At the same time, the augmented lower bound $LB^{(VS)}$ is computed from the split processing times p_i/w_i , not from the original processing times p_i . Hence a long weighted job may contribute strongly to the negative correction $CT^{(1)}$, while its contribution to the relaxed unweighted lower

bound $LB^{(VS)}$ is diluted by the splitting operation. This explains why instances containing one or a few large processing-time jobs with weights greater than one can lead to $LB^{(NC)} = 0$.

The strengthened bound $LB_a^{(NN)}(x, p)$ mitigates this loss of information in two ways. First, for fixed (a, x) , it replaces the term $LB^{(VS)}$ with the refined quantity $\overline{LB}_a^{(VS)}(x, p)$, which exploits the fixed first augmented block in the restricted problem Σ_a (see Lemma 2). Second, it allows the optimization with respect to p (detailed in Section 4.1 below) and with respect to x (detailed in Section 4.2 below).

4.1 Maximizing $LB_a^{(NN)}(x, p)$ with respect to p

By Lemma 1, the parameter p is artificial from the viewpoint of $WCTV(\sigma)$, but it affects both the augmented lower bound and the correction terms appearing in the decomposition. Optimizing over p can then improve the balance between the refined augmented lower bound and the correction terms. For fixed (a, x) , Corollary 1 shows that $LB_a^{(NN)}(x, p)$ is the square of the positive part of a difference of two square roots of quadratic polynomials in p . This reduces the maximization with respect to the free processing-time parameter to the solution of a one-dimensional stationarity equation. For fixed $a \in \mathcal{N}$ and $x \in \mathcal{X}_a$, define

$$\begin{cases} R_{a,x}(p) &:= \alpha_{1,a}(x)p^2 + \alpha_{2,a}(x)p + \alpha_{3,a}(x), \\ S_{a,x}(p) &:= c_{1,a}(x)p^2 + c_{2,a}(x)p + c_{3,a}(x), \\ D_{a,x} &:= \{p \in \mathcal{P}_a(x) : R_{a,x}(p) \geq 0, S_{a,x}(p) \geq 0\}. \end{cases}$$

The exact maximization of $LB_a^{(NN)}(x, p)$ with respect to p is equivalent to the maximization of the square-root difference $g_{a,x}(p)$ over $D_{a,x}$. A direct analytical route is to characterize the interior stationary points of $g_{a,x}$. Any interior maximizer $p^* \in D_{a,x}$ such that $p^* > 0$, $R_{a,x}(p^*) > 0$ and $S_{a,x}(p^*) > 0$ must satisfy the first-order condition $g'_{a,x}(p^*) = 0$. Squaring the resulting equality of derivatives yields a quartic equation, together with a sign constraint ensuring equivalence with the original unsquared condition. Hence, any such interior stationary point satisfies

$$\begin{cases} 0 &= t_{a,x}p^4 + b_{a,x}p^3 + c_{a,x}p^2 + d_{a,x}p + e_{a,x}, \\ &\text{with the condition: } (2\alpha_{1,a}(x)p + \alpha_{2,a}(x))(2c_{1,a}(x)p + c_{2,a}(x)) \geq 0, \end{cases} \quad (18)$$

where the quartic coefficients are

$$\begin{cases} t_{a,x} = 4\alpha_{1,a}(x)c_{1,a}(x)(\alpha_{1,a}(x) - c_{1,a}(x)), \\ b_{a,x} = 4(\alpha_{1,a}(x) - c_{1,a}(x))(\alpha_{1,a}(x)c_{2,a}(x) + \alpha_{2,a}(x)c_{1,a}(x)), \\ c_{a,x} = 4\alpha_{1,a}(x)^2c_{3,a}(x) + 4\alpha_{1,a}(x)\alpha_{2,a}(x)c_{2,a}(x) + \alpha_{2,a}(x)^2c_{1,a}(x) - 4\alpha_{3,a}(x)c_{1,a}(x)^2 \\ \quad - 4\alpha_{2,a}(x)c_{1,a}(x)c_{2,a}(x) - \alpha_{1,a}(x)c_{2,a}(x)^2, \\ e_{a,x} = \alpha_{2,a}(x)^2c_{3,a}(x) - \alpha_{3,a}(x)c_{2,a}(x)^2, \\ d_{a,x} = 4\alpha_{1,a}(x)\alpha_{2,a}(x)c_{3,a}(x) - 4c_{1,a}(x)c_{2,a}(x)\alpha_{3,a}(x) + \alpha_{2,a}(x)c_{2,a}(x)(\alpha_{2,a}(x) - c_{2,a}(x)). \end{cases}$$

We focus on the generic case in which $\alpha_{1,a}(x)c_{1,a}(x)(\alpha_{1,a}(x) - c_{1,a}(x)) \neq 0$, or equivalently $t_{a,x} \neq 0$, so that the quartic in (18) is non-degenerate.¹

Proposition 3. *For each fixed $a \in \mathcal{N}$ and each fixed $x \in \mathcal{X}_a$, every interior stationary point $p^* \in D_{a,x}$ such that $p^* > 0$, $R_{a,x}(p^*) > 0$ and $S_{a,x}(p^*) > 0$ satisfies the system (18).*

To explicitly characterize the candidate interior stationary points, we solve the quartic in (18) via Ferrari's method. After the standard translation $p = z - \frac{b_{a,x}}{4t_{a,x}}$, the quartic reduces to the depressed form $z^4 + A_{a,x}z^2 + B_{a,x}z + C_{a,x} = 0$, where

$$\begin{aligned} A_{a,x} &= \frac{8t_{a,x}c_{a,x} - 3b_{a,x}^2}{8t_{a,x}^2} \\ B_{a,x} &= \frac{8t_{a,x}^2d_{a,x} + b_{a,x}^3 - 4t_{a,x}b_{a,x}c_{a,x}}{8t_{a,x}^3} \\ C_{a,x} &= \frac{256t_{a,x}^3e_{a,x} + 16t_{a,x}b_{a,x}^2c_{a,x} - 3b_{a,x}^4 - 64t_{a,x}^2b_{a,x}d_{a,x}}{256t_{a,x}^4}. \end{aligned}$$

Ferrari's factorization is governed by the resolvent cubic

$$\Delta_{a,x}(\lambda) = 8\lambda^3 - 4A_{a,x}\lambda^2 - 8C_{a,x}\lambda + 4A_{a,x}C_{a,x} - B_{a,x}^2 = 0. \quad (19)$$

Let $\lambda_0(a, x)$ be a real root of (19) such that $2\lambda_0(a, x) - A_{a,x} > 0$, and define

$$\Delta_{1,a,x} = -2\lambda_0(a, x) - A_{a,x} - 2\frac{B_{a,x}}{\sqrt{2\lambda_0(a, x) - A_{a,x}}}, \quad \Delta_{2,a,x} = -2\lambda_0(a, x) - A_{a,x} + 2\frac{B_{a,x}}{\sqrt{2\lambda_0(a, x) - A_{a,x}}}.$$

When the corresponding discriminants are nonnegative, define the real Ferrari roots

$$\mathcal{Z}_{a,x} := \begin{cases} \frac{1}{2}\left(\sqrt{2\lambda_0(a, x) - A_{a,x}} - \sqrt{\Delta_{1,a,x}}\right), & \frac{1}{2}\left(\sqrt{2\lambda_0(a, x) - A_{a,x}} + \sqrt{\Delta_{1,a,x}}\right), & \text{if } \Delta_{1,a,x} \geq 0, \\ \frac{1}{2}\left(-\sqrt{2\lambda_0(a, x) - A_{a,x}} - \sqrt{\Delta_{2,a,x}}\right), & \frac{1}{2}\left(-\sqrt{2\lambda_0(a, x) - A_{a,x}} + \sqrt{\Delta_{2,a,x}}\right), & \text{if } \Delta_{2,a,x} \geq 0. \end{cases}$$

For each $z \in \mathcal{Z}_{a,x}$, set

$$p(z, a, x) = z - \frac{b_{a,x}}{4t_{a,x}}.$$

Finally, define the feasible Ferrari candidate set

$$\mathcal{F}_{a,x} := \left\{ p(z, a, x) : \begin{array}{l} z \in \mathcal{Z}_{a,x}, \quad p(z, a, x) \in \mathcal{P}_a(x), \quad R_{a,x}(p(z, a, x)) > 0, \quad S_{a,x}(p(z, a, x)) > 0, \\ (2\alpha_{1,a}(x)p(z, a, x) + \alpha_{2,a}(x))(2c_{1,a}(x)p(z, a, x) + c_{2,a}(x)) \geq 0 \end{array} \right\}.$$

Theorem 4 (Almost closed-form stationary candidates of $g_{a,x}(p)$). *Assume that $t_{a,x} \neq 0$. Then every interior stationary point of $g_{a,x}$ belongs to the feasible Ferrari candidate set $\mathcal{F}_{a,x}$. Consequently, if a*

¹The degenerate case $t_{a,x} = 0$ corresponds to a lower-degree stationarity equation and can be handled separately by the same selection rule as below.

maximizer of $g_{a,x}$ over $D_{a,x}$ is attained at an interior point $p^* > 0$ with $R_{a,x}(p^*) > 0$ and $S_{a,x}(p^*) > 0$, then one such interior maximizer can be selected as

$$p_{a,x}^* \in \arg \max \{g_{a,x}(p) : p \in \mathcal{F}_{a,x}\}.$$

As a consequence of Theorem 4, for each fixed admissible split level x and each choice of the first job $a \in \mathcal{N}$, let $p_{a,x}^* \in \arg \max_{p \in \mathcal{P}_a(x)} g_{a,x}(p)$ be a maximizer of the square-root difference introduced in Corollary 1, whenever such a maximizer is attained. When this maximizer is attained at an interior feasible point, Theorem 4 shows that it can be selected from a finite set of Ferrari-type candidates. We then define

$$LB_x(a) := LB_a^{(NN)}(x, p_{a,x}^*) = \left[\max\{0, g_{a,x}(p_{a,x}^*)\} \right]^2, \quad LB^{(NN*)}(x) := \min_{a=1,\dots,n} LB_x(a). \quad (20)$$

By Theorem 3, $LB_x(a)$ is a valid lower bound for the restricted problem with first job fixed to a , and therefore $LB^{(NN*)}(x)$ is a valid lower bound for the original WCTV problem at the split level x . If the split-level optimization is restricted to values x that are admissible for every possible first job, the resulting finite-split lower bound is

$$LB^{(NN*)} = \max_{x \in \cap_{a \in \mathcal{N}} \mathcal{X}_a} LB^{(NN*)}(x) = \max_{x \in \cap_{a \in \mathcal{N}} \mathcal{X}_a} \min_{a=1,\dots,n} LB_x(a). \quad (21)$$

The quartic characterization (18) identifies stationary candidates for optimizing p , but computing them can become numerically unstable as the instance size grows. The coefficients of (18) combine products of the coefficients of the two quadratic radicands; differences in their magnitudes can lead to substantial cancellation. Ferrari's formulas may further amplify numerical errors, particularly in near-degenerate cases where the leading coefficient $t_{a,x}$ is small. Moreover, candidate roots must satisfy the sign constraint in (18) to exclude extraneous solutions introduced by squaring the first-order condition. These difficulties motivate the asymptotic approximation developed below, which simplifies the selection of p while retaining the original lower-bound expression. Example 3 illustrates the differences in scale among the coefficients $\alpha_{1,a}(x)$, $\alpha_{2,a}(x)$, $\alpha_{3,a}(x)$, $c_{1,a}(x)$, $c_{2,a}(x)$ and $c_{3,a}(x)$ defining $g_{a,x}(p)$ in Corollary 1, and provides the motivation for this approximation.

Example 3. Consider again the WCTV instance of Example 1, with $\mathbf{p} = (36, 45, 54, 72, 96, 1200)$ and $\mathbf{w} = (3, 5, 6, 9, 6, 4)$. For the fixed split level $x = \frac{1}{2}$, Table 1 reports, for each possible first job a , the corresponding values of k_0 , $LB^{(VS)}$ and $\overline{LB}_a^{(VS)}(1/2, p_a)$. To complement that picture, Table 3 reports the coefficients entering

$$g_{a,x}(p) = \sqrt{\alpha_{1,a}(x)p^2 + \alpha_{2,a}(x)p + \alpha_{3,a}(x)} - \sqrt{c_{1,a}(x)p^2 + c_{2,a}(x)p + c_{3,a}(x)}$$

at the same split level $x = \frac{1}{2}$.

Table 3: Values of the coefficients in Corollary 1 with $x = \frac{1}{2}$.

a	6	5	4	3	2	1
k_0	3	4	3	4	3	4
$\alpha_{1,a}(x)$	0.016	0.006	0.008	0.002	0.008	0.004
$\alpha_{2,a}(x)$	98.168	70.182	79.470	41.424	76.288	13.288
$\alpha_{3,a}(x)$	55066.424	58068.182	57061.091	54862.136	51366.136	7383.833
$c_{1,a}(x)$	0.039	0.021	0.026	0.009	0.026	0.015
$c_{2,a}(x)$	-16.904	-9.179	-11.267	-4.689	-10.785	-2.174
$c_{3,a}(x)$	20128.698	19486.153	19647.868	19271.640	19912.103	152.764

The table shows a marked scale separation: for every a , the quadratic coefficients $\alpha_{1,a}(x)$ and $c_{1,a}(x)$ are much smaller than the corresponding constant coefficients $\alpha_{3,a}(x)$ and $c_{3,a}(x)$. This separation can make the exact quartic characterization delicate in larger instances, because the coefficients of the stationarity equation combine terms of very different magnitudes. In such cases, the quartic may become numerically ill-conditioned even though the underlying lower-bound expression remains well-defined. This motivates the need for an approximation scheme, which keeps the dominant linear and constant terms in the radicands and avoids the direct manipulation of the full quartic equation.

Building on the scale separation displayed in Example 3, it is natural to consider, for fixed (a, x) , the reduced expression obtained by neglecting the quadratic terms in $g_{a,x}(p)$. Let us multiply $g_{a,x}(p)$ by a constant, an operation that preserves the maximizer, and define

$$\tilde{g}_{a,x}(p) := \left(\frac{1}{\alpha_{3,a}(x)c_{3,a}(x)} \right)^2 \left(\sqrt{\alpha_{1,a}(x)p^2 + \alpha_{2,a}(x)p + \alpha_{3,a}(x)} - \sqrt{c_{1,a}(x)p^2 + c_{2,a}(x)p + c_{3,a}(x)} \right).$$

When

$$\frac{\alpha_{1,a}(x)}{\alpha_{3,a}(x)c_{3,a}(x)} \approx 0 \quad \text{and} \quad \frac{c_{1,a}(x)}{\alpha_{3,a}(x)c_{3,a}(x)} \approx 0,$$

we consider the approximation

$$\arg \max_{p \in \mathcal{D}} \{\tilde{g}_{a,x}(p)\} \approx \arg \max_{p \in \mathcal{D}} \left\{ \tilde{\tilde{g}}_{a,x}(p) := \sqrt{\alpha_{2,a}(x)p + \alpha_{3,a}(x)} - \sqrt{c_{2,a}(x)p + c_{3,a}(x)} \right\}, \quad (22)$$

where we used that the common multiplicative factor $(\alpha_{3,a}(x)c_{3,a}(x))^{-2}$ is positive and independent of p , hence does not affect the maximizer. The natural domain is

$$\mathcal{D} := \{p \in \mathcal{P}_a(x) : \alpha_{2,a}(x)p + \alpha_{3,a}(x) \geq 0, \quad c_{2,a}(x)p + c_{3,a}(x) \geq 0\}.$$

Since $\mathcal{D}$ is obtained by intersecting $\mathcal{P}_a(x)$ with two half-lines, it is always an interval of the form $\mathcal{D} = [\tilde{p}_-, \tilde{p}_+]$ or $\mathcal{D} = [\tilde{p}_-, +\infty)$, where

$$\tilde{p}_- = \max \left\{ \underline{p}_a(x), -\frac{\alpha_{3,a}(x)}{\alpha_{2,a}(x)} \text{ if } \alpha_{2,a}(x) > 0, -\frac{c_{3,a}(x)}{c_{2,a}(x)} \text{ if } c_{2,a}(x) > 0 \right\},$$

and, whenever at least one of the slopes is negative, the upper endpoint is

$$\tilde{p}_+ = \min \left\{ -\frac{\alpha_{3,a}(x)}{\alpha_{2,a}(x)} \text{ if } \alpha_{2,a}(x) < 0, -\frac{c_{3,a}(x)}{c_{2,a}(x)} \text{ if } c_{2,a}(x) < 0 \right\}.$$

This observation will be useful below when characterizing the maximizer of the reduced objective $\tilde{g}_{a,x}(p)$. The approximation is motivated by the numerical structure discussed above. When the quadratic coefficients are negligible relative to the dominant terms in the radicands, the full objective $g_{a,x}(p)$ is well approximated, for the purpose of selecting candidate values of p , by the reduced square-root difference $\tilde{g}_{a,x}(p)$. The advantage is that the maximization of $\tilde{g}_{a,x}(p)$ does not require solving a quartic equation: its critical point, when it exists, is obtained from a simple rational expression, and otherwise the maximum is attained at an endpoint of $\mathcal{D}$. The next theorem legitimates this approximation by showing that, along suitable sequences of instances, the quadratic coefficients become asymptotically negligible relative to the dominant terms.

Theorem 5. *Consider a sequence of instances indexed by n , and for each n let $\alpha_{1,a_n}^{(n)}(x)$, $\alpha_{3,a_n}^{(n)}(x)$, $c_{1,a_n}^{(n)}(x)$, and $c_{3,a_n}^{(n)}(x)$ be the coefficients defined in Corollary 1 for the choice $a = a_n$ and the fixed split level x . Assume that the following hold for all sufficiently large n :*

- (i) *the processing times are uniformly bounded in $[p_-, p_+]$, with $0 < p_- \leq p_+ < +\infty$;*
- (ii) *the weights are uniformly bounded in $[w_-, w_+]$, with $w_- > 2$ and $2xw_- > 1$;*
- (iii) *there exist constants $\eta > 0$ and $p_- \leq p^L < p^H \leq p_+$, independent of n , such that at least ηn of the re-indexed processing times $\bar{p}_i^{(n)}(a_n)$, $i = 1, \dots, n-1$, satisfy $\bar{p}_i^{(n)}(a_n) \leq p^L$, at least ηn of them satisfy $\bar{p}_i^{(n)}(a_n) \geq p^H$, and*

$$\frac{2xw_- - 1}{4xw_-} p^H > \frac{2xw_+ - 1}{4xw_+} p^L. \quad (23)$$

Then, as $n \rightarrow \infty$,

$$\frac{\alpha_{1,a_n}^{(n)}(x)}{\alpha_{3,a_n}^{(n)}(x)c_{3,a_n}^{(n)}(x)} \rightarrow 0 \quad \text{and} \quad \frac{c_{1,a_n}^{(n)}(x)}{\alpha_{3,a_n}^{(n)}(x)c_{3,a_n}^{(n)}(x)} \rightarrow 0. \quad (24)$$

Condition (iii) is a non-concentration assumption on the re-indexed processing times $\bar{p}_i(a_n)$. More precisely, it requires that a positive fraction of these values stays uniformly below some level p^L and that another positive fraction stays uniformly above a larger level p^H . Under the uniform bounds in (i)–(ii), this implies that the transformed quantities

$$u_i(x) = \frac{v_i(a_n, x) - 1}{2v_i(a_n, x)} \bar{p}_i(a_n)$$

retain a uniformly positive weighted variance. In turn, this ensures that $c_{3,a_n}^{(n)}(x)$ remains bounded away from zero, while $\alpha_{3,a_n}^{(n)}(x)$ grows at rate n^2 and the coefficients $\alpha_{1,a_n}^{(n)}(x)$ and $c_{1,a_n}^{(n)}(x)$ are of smaller order. Example 4 below illustrates this asymptotic regime numerically.

Example 4. To generate a deterministic sequence of instances satisfying the assumptions of Theorem 5, fix $x = \frac{1}{2}$, set $a_n := 1$, and choose constants ρ, p^L, p^H such that

$$0 < \rho < \frac{p^L}{4} < \frac{p^H}{4}, \quad p^H > \frac{9}{8}p^L.$$

Let $n = 2m + 1$ with $m \in \mathbb{N}_+$, and define $w_1^{(n)} = 3$, $w_i^{(n)} = 4$, for all $i = 2, \dots, n$, and

$$p_1^{(n)} = 3\rho, \quad p_i^{(n)} = \begin{cases} p^L, & i = 2, \dots, m+1, \\ p^H, & i = m+2, \dots, 2m+1. \end{cases} \quad (25)$$

With this construction, the ordering convention $\frac{p_1^{(n)}}{w_1^{(n)}} \leq \dots \leq \frac{p_n^{(n)}}{w_n^{(n)}}$ is satisfied, because

$$\frac{p_1^{(n)}}{w_1^{(n)}} = \rho < \frac{p^L}{4} < \frac{p^H}{4} = \frac{p_i^{(n)}}{w_i^{(n)}} \quad (i \geq 2).$$

Moreover, $W^{(n)} = 3 + 4(2m) = 8m + 3$. For this family of instances, the simplified expressions derived in Appendix B yield

$$LB_{1, \frac{1}{2}}(2m) = \frac{m(32m^2 - 6m + 1)}{12}(p^L)^2 + m^2(2m - 1)p^L p^H + \frac{m(2m - 1)(4m - 1)}{12}(p^H)^2,$$

and

$$\Gamma_m = 2m^2(p^L + p^H)^2 - \frac{m}{2}p^H(p^L + p^H) + \frac{(p^H)^2}{16}.$$

Hence the two quantities appearing in (24) can be written explicitly as

$$\frac{\alpha_{1, a_n}^{(n)}}{\alpha_{3, a_n}^{(n)} c_{3, a_n}^{(n)}} = \frac{1}{\frac{18(8m+3)}{\alpha_{3, a_n}^{(n)} c_{3, a_n}^{(n)}}} \quad \text{and} \quad \frac{c_{1, a_n}^{(n)}}{\alpha_{3, a_n}^{(n)} c_{3, a_n}^{(n)}} = \frac{\frac{8m}{3(8m+3)^2}}{\frac{\alpha_{3, a_n}^{(n)} c_{3, a_n}^{(n)}}{\alpha_{3, a_n}^{(n)} c_{3, a_n}^{(n)}}},$$

where

$$\begin{aligned} \alpha_{3, a_n}^{(n)} &= \frac{\Gamma_m + LB_{1, \frac{1}{2}}(2m)}{2(8m+3)} - \frac{5m}{16(8m+3)}((p^L)^2 + (p^H)^2), \\ c_{3, a_n}^{(n)} &= \frac{9m}{16(8m+3)^2} [4m(p^H - p^L)^2 + 3((p^L)^2 + (p^H)^2)]. \end{aligned}$$

Therefore both ratios converge to zero, in agreement with Theorem 5. Figure 1 illustrates this behavior for the numerical choice $\rho = 1$, $p^L = 8$, and $p^H = 10$, which satisfies the required conditions. Figure 1 shows a rapid convergence of both ratios. This numerical experiment is consistent with the interpretation of Theorem 5: along this sequence, the quadratic coefficients become negligible relative to the dominant terms entering the approximation scheme (22).

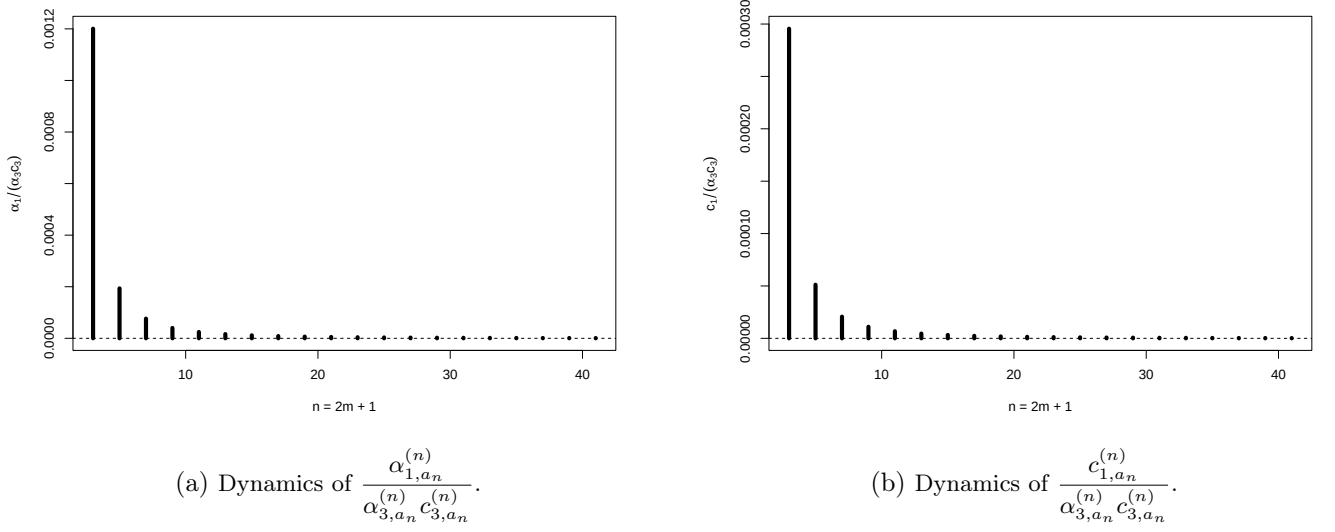


Figure 1: Numerical illustration of Theorem 5 for the deterministic sequence defined in (25), with $\rho = 1$, $p^L = 8$, and $p^H = 10$. Both normalized quadratic coefficients decay rapidly toward zero as the instance size increases.

This example supports the validity of the asymptotic approximation. This example supports the asymptotic approximation scheme (22). Its practical role is to circumvent the numerical instability that may arise in the exact quartic route when the coefficients of the two radicands have very different scales. In large instances, the terms $\alpha_{3,a}(x)$ and $c_{3,a}(x)$ may dominate $\alpha_{1,a}(x)$ and $c_{1,a}(x)$, so that the quartic coefficients in (18) can involve cancellations between large intermediate quantities. By dropping the asymptotically negligible quadratic terms, the reduced problem avoids the depressed-quartic transformation, the resolvent cubic, and the subsequent filtering of spurious roots introduced by squaring the first-order condition. The following corollary provides a closed-form characterization of $\arg \max_{p \in \mathcal{D}} \tilde{g}_{a,x}(p)$.

Corollary 2. *Assume that $\alpha_{2,a}(x) \neq c_{2,a}(x)$ and that $\mathcal{D}$ is nonempty and bounded, and write $\mathcal{D} = [\tilde{p}_-, \tilde{p}_+]$. Then the maximizers of the approximate objective $\tilde{g}_{a,x}(p)$ over $p \in \mathcal{D}$ are given by*

$$\arg \max_{p \in \mathcal{D}} \tilde{g}_{a,x}(p) = \begin{cases} \{p_{\text{crt}}\}, & \text{if } \alpha_{2,a}(x)c_{2,a}(x) > 0, \quad |\alpha_{2,a}(x)| < |c_{2,a}(x)|, \text{ and } p_{\text{crt}} \in \text{int}(\mathcal{D}), \\ \{\tilde{p}_+\}, & \text{if } \alpha_{2,a}(x) \geq 0, \\ \arg \max_{p \in \{\tilde{p}_-, \tilde{p}_+\}} \tilde{g}_{a,x}(p), & \text{otherwise.} \end{cases}$$

Whenever $\alpha_{2,a}(x)c_{2,a}(x)(\alpha_{2,a}(x) - c_{2,a}(x)) \neq 0$ the only possible interior critical point is

$$p_{\text{crt}} = \frac{c_{2,a}(x)^2 \alpha_{3,a}(x) - \alpha_{2,a}(x)^2 c_{3,a}(x)}{\alpha_{2,a}(x)c_{2,a}(x)(\alpha_{2,a}(x) - c_{2,a}(x))}.$$

The approximation scheme of Corollary 2 is justified asymptotically and is primarily designed to avoid the numerical fragility of the exact quartic characterization when the instance size becomes large. Never-

theless, it is useful to examine whether the same rule already provides reliable candidate values of p on small instances. The approximation replaces the exact maximization of the square-root difference $g_{a,x}(p)$ by a reduced problem whose maximizer is available in closed form or is attained at an endpoint of $\mathcal{D}$. Hence, it can be used as a computationally inexpensive proxy for the free processing-time parameter. The following example shows that, even for a 6-job instance, this asymptotically motivated rule can remain close to the exact maximizer in the binding cases, while also clarifying the situations in which the approximation may lose accuracy.

Example 5. Consider again the 6-job instance of Example 1, with $\mathbf{p} = (36, 45, 54, 72, 96, 1200)$ and $\mathbf{w} = (3, 5, 6, 9, 6, 4)$, and fix the split level at $x = 1/2$. Table 4 reports the coefficients $\alpha_{1,a}(x)$, $\alpha_{2,a}(x)$, $\alpha_{3,a}(x)$, $c_{1,a}(x)$, $c_{2,a}(x)$ and $c_{3,a}(x)$ defining the function $g_{a,x}(p)$ whose maximization determines $LB_a^{(NN)}(x, p)$ with respect to p . The table also reports the exact maximizer $p_{a,x}^*$, the approximate maximizer $\tilde{p}_{a,x}^*$ obtained from Corollary 2, and the corresponding values $LB_a^{(NN)}(x, p_{a,x}^*)$ and $LB_a^{(NN)}(x, \tilde{p}_{a,x}^*)$. The columns in Table 4 follow the sorted order by nondecreasing ratio p_i/w_i , displayed as $a = 6, 5, \dots, 1$.

Table 4: Exact and approximate p -optimization with $x = \frac{1}{2}$.

a	6	5	4	3	2	1
k_0	3	4	3	4	3	4
$\alpha_{1,a}(x)$	0.016	0.006	0.008	0.002	0.008	0.004
$\alpha_{2,a}(x)$	98.168	70.182	79.470	41.424	76.288	13.288
$\alpha_{3,a}(x)$	55066.424	58068.182	57061.091	54862.136	51366.136	7383.833
$c_{1,a}(x)$	0.039	0.021	0.026	0.009	0.026	0.015
$c_{2,a}(x)$	-16.904	-9.179	-11.267	-4.689	-10.785	-2.174
$c_{3,a}(x)$	20128.698	19486.153	19647.868	19271.640	19912.103	152.764
$p_{a,x}^*$	1087.781	1500.000	1214.869	900.000	1210.388	126.523
$\tilde{p}_{a,x}^*$	1190.795	2122.891	1743.808	4110.226	1846.256	70.283
$LB_a^{(NN)}(x, p_{a,x}^*)$	42216.809	37115.858	39014.980	24281.131	33710.003	7171.792
$LB_a^{(NN)}(x, \tilde{p}_{a,x}^*)$	42036.648	32274.885	36144.981	12030.695	30033.336	6839.660

For this instance, one has $\alpha_{2,a}(x) > 0$ and $c_{2,a}(x) < 0$ for every $a \in \mathcal{N}$. Hence, by Corollary 2, the reduced objective $\tilde{g}_{a,x}(p)$ is increasing on its approximate feasible domain and its maximizer is the right endpoint of that interval, namely

$$\tilde{p}_{a,x}^* = -\frac{c_{3,a}(x)}{c_{2,a}(x)}.$$

Comparing this surrogate maximizer with the value $p_{a,x}^*$ returned by the exact optimization of the original function $g_{a,x}(p)$, we observe that the approximate maximizer can be very accurate for some first-job choices. At the fixed split level $x = 1/2$, the exact and approximate lower bounds obtained after optimizing over p

and then taking the minimum over the first job are

$$\min_{a \in \mathcal{N}} LB_a^{(NN)}(1/2, p_{a,1/2}^*) = 7171.792 \quad \text{and} \quad \min_{a \in \mathcal{N}} LB_a^{(NN)}(1/2, \tilde{p}_{a,1/2}^*) = 6839.660.$$

4.2 Maximizing $LB_a^{(NN)}(x, p)$ with respect to the split level x

We now turn to the optimization of the improved lower bound with respect to the split level x . The previous subsection showed that, for fixed (a, x) , the dependence of $LB_a^{(NN)}(x, p)$ on the free processing-time parameter p can be handled through the one-dimensional function $g_{a,x}(p)$. We now study how the resulting bound varies with the split level. For fixed $a \in \mathcal{N}$ such that $w_a \leq W/2$, Proposition 2 implies that $\mathcal{X}_a = \mathcal{X}$. Hence, we define

$$\Phi_a := \sup_{x \in \mathcal{X}} \sup_{p \in \mathcal{P}_a(x)} LB_a^{(NN)}(x, p).$$

At first sight, this optimization appears delicate because the bound depends on several combinatorial quantities induced by the augmented construction, in particular $k_0(a, x)$, $\kappa(a, x)$, the truncated multiplicities $\bar{v}_i(a, x)$ and $\bar{\bar{v}}_i(a, x)$, and the middle-term contribution $LB_{1,x}(k_0)$. Proposition 2 shows, however, that this dependence is much simpler as, based on the integrality of weights, the admissible split levels form the arithmetic progression $\mathcal{X} = \{\delta, 2\delta, 3\delta, \dots\}$. Moreover, recalling the indices k_a^+ and k_a^- defined in (17), Proposition 2 gives $k_0(a, \delta) = k_a^-$ and $k_0(a, x) = k_a^+$, for all $x \in \mathcal{X} \setminus \{\delta\}$. Thus, for each fixed first job a , the combinatorial structure of $\overline{LB}_a^{(VS)}(x, p)$ is constant on $\mathcal{X} \setminus \{\delta\}$ and may differ from it only at the minimal admissible split level $x = \delta$. This observation leads to the decomposition

$$\Phi_a = \max \left\{ \sup_{p \in \mathcal{P}_a(\delta)} LB_a^{(NN)}(\delta, p), \sup_{x \in \mathcal{X} \setminus \{\delta\}} \sup_{p \in \mathcal{P}_a(x)} LB_a^{(NN)}(x, p) \right\}. \quad (26)$$

The advantage of (26) is both conceptual and computational. The point $x = \delta$ is the unique split level at which the truncation index may differ from its value on the rest of the admissible grid. For every $x \in \mathcal{X} \setminus \{\delta\}$, the same value $k_0(a, x) = k_a^+$ is used, so that the formula of the bound has a fixed combinatorial structure and only its coefficients vary with x . Because the combinatorial structure is fixed on that set, it is natural to replace the full search by a search over a finite collection of representative split levels. By letting $\mathcal{X}_a^{\text{rep}} \subseteq \mathcal{X} \setminus \{\delta\}$ be such a finite set, one can restrict the search over admissible split levels. A minimal representative choice is the singleton $\mathcal{X}_a^{\text{rep}} = \{2\delta\}$, namely the smallest admissible split level larger than δ . Therefore, any value of $LB_a^{(NN)}(x, p)$ computed at an admissible pair (x, p) remains a valid lower bound for the restricted problem with first job fixed to a .

Example 6. Consider a WCTV minimization problem with $n = 6$ jobs presented in Example 2.1 of Nessah and Chu (2010): $\mathbf{p} = (22, 37, 45, 55, 74, 98)$ and $\mathbf{w} = (1, 3, 5, 6, 9, 6)$. For this instance, $W = 30$, $g = \gcd(9, 5, 6, 3, 6, 1) = 1$, and therefore $\delta = 1/2$. Table 5 reports the values of $S_k(a)$, together with the indices k_a^+ and k_a^- introduced in (17).

Table 5: Values of $S_k(a)$, k_a^+ and k_a^- for the re-indexed instance of Example 2.1 in Nessah and Chu (2010).

a	$S_1(a)$	$S_2(a)$	$S_3(a)$	$S_4(a)$	$S_5(a)$	k_a^+	k_a^-	Structure in x
1	21	16	10	7	1	3	3	same for all $x \in \mathcal{X}$
2	25	16	10	7	1	4	4	same for all $x \in \mathcal{X}$
3	24	15	10	7	1	4	4	same for all $x \in \mathcal{X}$
4	27	18	13	7	1	4	4	same for all $x \in \mathcal{X}$
5	24	15	10	4	1	3	3	same for all $x \in \mathcal{X}$
6	29	20	15	9	6	5	5	same for all $x \in \mathcal{X}$

Since $k_a^- = k_a^+$ for every $a \in \mathcal{N}$, the combinatorial structure identified in Proposition 2 is the same for every admissible split level $x \in \mathcal{X}$. In other words, for this instance the minimal admissible split level $x = \delta = 1/2$ is not exceptional for any choice of the first job. Figure 2 reports the behavior of the optimized bound as a function of the split level x , for each choice $a = 1, \dots, 6$. More precisely, the plotted value is $LB_a^{(NN)}(x, p_a^*(x))$, where $p_a^*(x)$ denotes the value of the free processing-time parameter obtained from the p -optimization procedure described in Section 4.1.

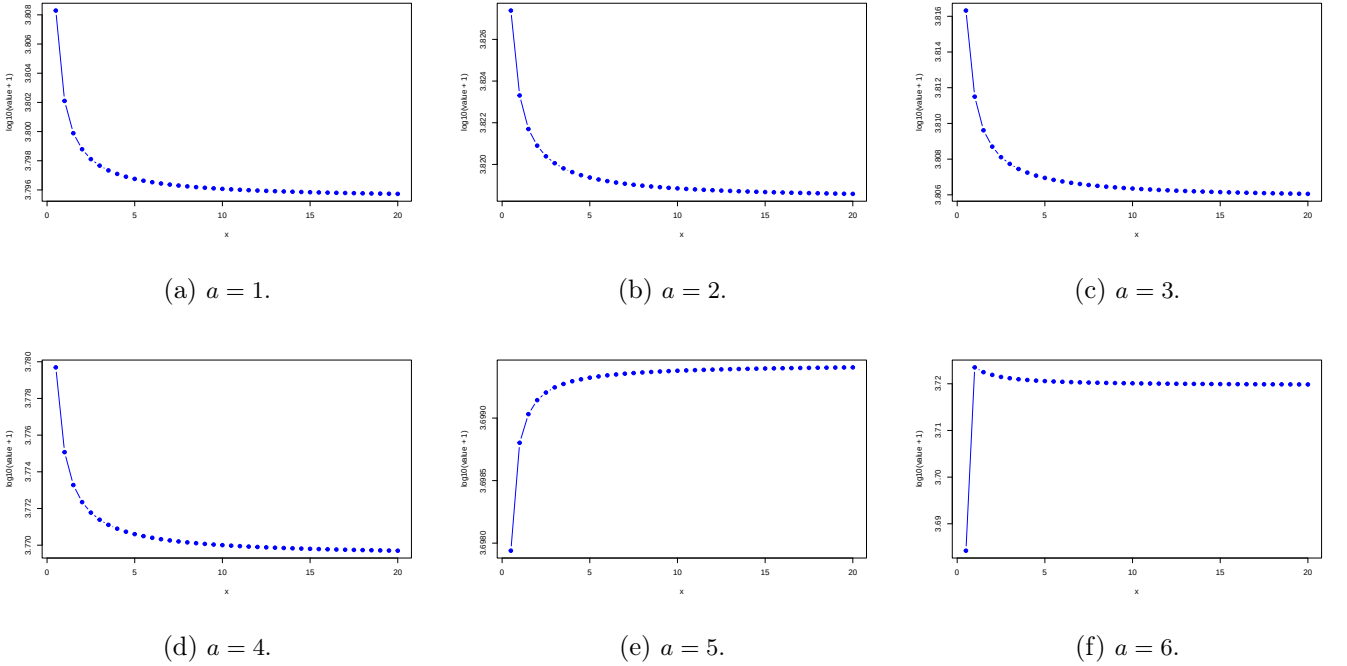


Figure 2: Behavior of $LB_a^{(NN)}(x, p_a^*(x))$ as a function of the split level x , for $a = 1, \dots, 6$. In this example, Proposition 2 implies that the combinatorial structure of the bound is the same for all $x \in \mathcal{X}$, since $k_a^- = k_a^+$ for every a .

In light of Proposition 2, for each fixed a , the only possible structural change could occur at the minimal admissible split level $x = \delta$. In the present example, however, Table 5 shows that no such exceptional case occurs, because $k_a^- = k_a^+$ for every a . Therefore, Figure 2 provides a numerical illustration of a stable

regime in which the truncation structure of the bound is unchanged throughout the admissible set $\mathcal{X}$.

5 Large-scale numerical tests.

We now address large-scale instances in two distinct numerical exercises. First, we compare our proposed polynomial lower bound with the numerical results obtained in our previous MILP-based approach of (Nasini and Nessah 2026) (where $n = 10, 20, 40, 80$). Nasini and Nessah (2026)’s approach relies on the head–tail decomposition of $WMSD(\sigma, d)$ and on a residual lower bound for $R(\sigma, \overline{C}(\sigma))$, which was then embedded in a mixed-integer linear programming formulation solved by a specialized cutting-plane algorithm. By contrast, the present approach remains polynomial: it strengthens the augmented-instance lower bound of Nessah and Chu (2010) by optimizing over the first job a , the free first-job processing time p , and, when activated, the split level x . Second, we assess the capability of the proposed approach to deal with instances larger than $n = 80$ (the maximum size which could be tackled by the MILP-based approach of (Nasini and Nessah 2026)).

Comparison with Nasini and Nessah (2026)’s approach For this comparison, we use the same eight instances considered by Nasini and Nessah (2026), which have been constructed using the two data generation methods proposed by Nasini and Nessah (2021, 2022, 2024). We call them G0 and G1. They simulate statistically dependent and independent processing times vectors, respectively. We denote by $LB^{(NN*)}$ the value of the new polynomial lower bound when the free parameters p and x are optimized, and by $LB^{(NC)}$ the original lower bound of Nessah and Chu (2010). Table 6 reports the comparison. The column BEST MILP LB is the best lower bound obtained in the MILP-based approach of Nasini and Nessah (2026). The column BEST UB is the best feasible upper bound, obtained as the minimum between the MILP-induced feasible value and a benchmark heuristic upper bound, constructed by pairwise exchange local search.

Table 6: Comparison between the proposed $LB^{(NN*)}$ and the MILP-based bounds of Nasini and Nessah (2026).

Gen	n	BENCHMARK LB	$LB^{(NN*)}$	BEST MILP LB	BEST UB	x^*
G0	10	1169.43	31178.02	32398.56	33758.25	0.5
G0	20	33260.66	33260.68	38919.20	43335.69	0.5
G0	40	812397.85	812397.90	1016506.81	1095914.61	0.5
G0	80	8907536.67	8907536.69	10317375.71	11497341.05	0.5
G1	10	829.96	849.74	1256.12	1423.10	1.0
G1	20	17313.18	17313.19	22373.03	24233.58	0.5
G1	40	375020.88	375020.90	444499.27	486931.85	0.5
G1	80	4799493.76	4799493.77	5693169.43	6296708.84	0.5

The comparison reveals three main facts. First, our novel polynomial bound provide homogeneous improvement over the original lower bound of Nessah and Chu (2010). For instances with $n = 10$, the benchmark lower bound is 1169.43 for G0 and 829.96 for G1, against 31178.02 for G0 and 849.74 for G1, attained by $LB^{(NN*)}$. Second, for most of the remaining benchmark instances, the new polynomial bound coincides numerically with the original lower bound. Third, the MILP-based lower bound remains stronger on the complete set of benchmark instances. This is expected, because the two methods strengthen different mathematical objects. The MILP-based approach optimizes a sequence-dependent relaxation derived from the decomposition $WCTV(\sigma) = Q(\sigma) + R(\sigma, \overline{C}(\sigma))$, resulting in a near optimal value. As documented by Nasini and Nessah (2026), this requires multiple hours of computation in a workstation equipped with a 13th Gen Intel(R) Core(TM) i9-13900K, 128 GB of RAM. By contrast, our proposed approach strengthens the polynomial augmented-sequence lower bound directly, avoiding the MILP formulation, though loosing the full sequence-dependent information contained in the residual term $R(\sigma, \overline{C}(\sigma))$.

Large scale asymptotic approximation instances. The previous comparison was limited to the instance sizes for which the MILP-based approach of Nasini and Nessah (2026) was computationally solvable (namely, up to $n = 80$ jobs). We now move beyond that range and test instances with $n = 200$ and $n = 300$ jobs. The purpose is to assess whether the enhanced bound remains informative and can still improve on the original Nessah–Chu bound when the optimization with respect to the artificial processing-time parameter p is performed through the asymptotic approximation rule of Corollary 2.

For each size $n \in \{200, 300\}$, two instances are tested. The split-level search is performed over the finite grid $\mathcal{X}_a^{\text{rep}} = \{\delta, 2\delta, 2^2\delta, 2^3\delta, 2^5\delta\}$. For each first job a and split level $x \in \mathcal{X}_{20}$, the approximate p -optimization uses the closed-form rule of Corollary 2, together with the admissible lower endpoint $\underline{p}_a(x)$ and the original value p_a as safeguards. We denote the resulting finite-grid approximation by $LB_{\text{ap}}^{(NN)}$. Thus, $LB_{\text{ap}}^{(NN)}$ is a computable member of the lower-bound family, not the full supremum over all admissible split levels and all $p \in [\underline{p}_a(x), +\infty)$. Table 7 reports the improved lower bound obtained by the asymptotic p -optimization procedure. The improvement over $LB^{(NC)}$ is systematic. Across the four large-scale instances, the relative gain ranges from 1.72% to 3.14%, with an average gain of approximately 2.35%. These improvements are obtained without solving any MILP and without using the exact quartic-based maximization in p .

Table 7: Enhanced polynomial lower bound with asymptotic p -optimization on large-scale instances.

Instance	$LB^{(NC)}$	$LB_{\text{ap}}^{(NN)}$	a^*	x^*	p^*
$G2_{200}^{(1)}$	2,325.77	2,385.88	199	32	16,060
$G2_{200}^{(2)}$	2,358.13	2,432.28	195	32	16,420
$G2_{300}^{(1)}$	1,729.79	1,763.34	299	32	22,096
$G2_{300}^{(2)}$	2,131.96	2,168.60	298	32	24,199

In Table 7, a^* denotes the binding first-job position after sorting the jobs by nondecreasing p_i/w_i . In all four instances, the selected split level is $x^* = 32$, the largest value in the grid $\mathcal{X}_5$. This behavior suggests that, on these heterogeneous large-scale instances, the additional freedom provided by both the split level and the artificial first-job processing time remains useful even far beyond the instance sizes accessible to the MILP-based approach. The fact that the best value is attained at the boundary of the tested grid should be interpreted carefully. In fact, every reported value is computed from admissible parameters and is therefore a valid lower bound. The experiment confirms that the proposed augmented-sequence refinement can continue to dominate the baseline Nessah–Chu bound on instances with up to 300 jobs, even when the computationally inexpensive asymptotic rule is used for the optimization in p .

6 Conclusions

This paper generalizes the augmented-sequence approach of Nessah and Chu (2010) to develop a family of polynomially computable lower bounds for single-machine WCTV minimization. The key observation is that changing the processing time of the first scheduled job leaves WCTV unchanged. Treating this processing time as a free parameter, we derive a decomposition that relates WCTV to the CTV of an augmented unweighted sequence through explicit correction terms. Exploiting the fixed first block of this sequence then yields a refined CTV bound, which we convert into a lower bound for the corresponding restricted WCTV problem. Under the stated admissibility conditions, taking the minimum over first-job choices provides a valid global lower bound.

For each fixed first job, the bound can be strengthened by optimizing the artificial processing time p and the split level x . Smooth interior stationary points of the objective in p satisfy a polynomial equation of degree at most four. We complement this characterization with an asymptotically motivated approximation that simplifies the selection of p while retaining the original lower-bound expression. When $w_a \leq W/2$, the truncation index has at most two regimes over the admissible split-level grid, which helps organize the search in x .

On the tested benchmark instances, the computed bounds match or improve the original Nessah–Chu bound, with substantial gains in cases where that bound vanishes. On larger instances with up to 300 jobs, the approximation-based selection of p also produces systematic improvements. These results support the practical value of the augmented-sequence refinement when MILP-based methods become computationally demanding.

Future work could improve the numerical reliability of exact p -optimization through robust global one-dimensional procedures, derive sufficient conditions for screening first-job choices, and refine the split-level search. The generalized decomposition may also provide a basis for lower bounds on other weighted dispersion criteria where heterogeneous weights obscure the ordering properties of the unweighted problem.

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A : Mathematical proofs

Lemma 1

Proof. Let $j = \sigma[1]$. For each position h , define the shifted completion times $C'_{\sigma[h]}(\sigma) := C_{\sigma[h]}(\sigma) - p_j$. Then $C'_{\sigma[1]}(\sigma) = 0$ and, for $h \geq 2$, $C'_{\sigma[h]}(\sigma) = \sum_{t=2}^h p_{\sigma[t]}$, so the vector $C'(\sigma)$ is independent of p_j . Moreover,

$$\overline{C}(\sigma) = \frac{1}{W} \sum_{i=1}^n w_i (C'_i(\sigma) + p_j) = \overline{C}'(\sigma) + p_j.$$

Hence, for every job i , $C_i(\sigma) - \overline{C}(\sigma) = C'_i(\sigma) - \overline{C}'(\sigma)$, which does not depend on p_j . Substituting into (2) completes the proof. $\square$

Theorem 1

Proof. Fix $p > 0$ and define

$$\widehat{p}_i = \widehat{p}_i(a, p) := \begin{cases} p, & \text{if } i = a, \\ p_i, & \text{if } i \neq a. \end{cases}$$

Throughout the proof, completion times computed with the modified processing-time vector $(\widehat{p}_i)_{i=1}^n$ are denoted by

$$C_i^{(p)}(\sigma), \quad \overline{C}^{(p)}(\sigma) = \frac{1}{W} \sum_{i=1}^n w_i C_i^{(p)}(\sigma),$$

and the corresponding weighted completion-time variance is denoted by

$$WCTV^{(p)}(\sigma) = \frac{1}{W} \sum_{i=1}^n w_i \left(C_i^{(p)}(\sigma) - \overline{C}^{(p)}(\sigma) \right)^2.$$

Similarly, the augmented sequence constructed from σ using the modified processing times $(\widehat{p}_i)_{i=1}^n$ is denoted by $\sigma_{A_x}^{(p)}$, and its piece completion times and mean completion time are denoted by

$$C_{i_h}^{(p)}(\sigma_{A_x}^{(p)}), \quad \overline{C}^{(p)}(\sigma_{A_x}^{(p)}) = \frac{1}{2xW} \sum_{i=1}^n \sum_{h=1}^{2xw_i} C_{i_h}^{(p)}(\sigma_{A_x}^{(p)}).$$

Since $a = \sigma[1]$, replacing the original processing time p_a by p shifts every completion time in σ by the same constant $p - p_a$. Hence, for every job i ,

$$C_i^{(p)}(\sigma) - \overline{C}^{(p)}(\sigma) = C_i^{(p_a)}(\sigma) - \overline{C}^{(p_a)}(\sigma),$$

and therefore

$$WCTV^{(p)}(\sigma) = WCTV^{(p_a)}(\sigma) = WCTV(\sigma).$$

Thus it is enough to prove the desired identity using the modified processing-time vector $(\widehat{p}_i)_{i=1}^n$.

Step 1: completion times in σ and in the augmented sequence. For each job i , set $k_i := 2xw_i$. Since $x \in \mathcal{X}$, we have $k_i \in \mathbb{N}$ for every i . In the augmented sequence $\sigma_{A_x}^{(p)}$, job i is replaced by the consecutive block of pieces $i_1, \dots, i_{k_i}$, each having processing time $\widehat{p}_i/k_i$. Therefore, for every $h = 1, \dots, k_i$, the completion time of the original job i is the completion time of piece i_h plus the processing time of the pieces of the same block that remain after i_h . Hence

$$C_i^{(p)}(\sigma) = C_{i_h}^{(p)}(\sigma_{A_x}^{(p)}) + (k_i - h) \frac{\widehat{p}_i}{k_i}, \quad h = 1, \dots, k_i. \quad (\text{A.1})$$

Step 2: relation between the two mean completion times. Averaging (A.1) over the pieces of each block gives

$$C_i^{(p)}(\sigma) = \frac{1}{k_i} \sum_{h=1}^{k_i} C_{i_h}^{(p)}(\sigma_{A_x}^{(p)}) + \frac{1}{k_i} \sum_{h=1}^{k_i} (k_i - h) \frac{\widehat{p}_i}{k_i}.$$

Multiplying by w_i , summing over i , and dividing by W , we obtain

$$\overline{C}^{(p)}(\sigma) = \frac{1}{W} \sum_{i=1}^n w_i \frac{1}{k_i} \sum_{h=1}^{k_i} C_{i_h}^{(p)}(\sigma_{A_x}^{(p)}) + \frac{1}{W} \sum_{i=1}^n w_i \frac{1}{k_i} \sum_{h=1}^{k_i} (k_i - h) \frac{\widehat{p}_i}{k_i}.$$

Since $k_i = 2xw_i$, we have $w_i/k_i = 1/(2x)$. Hence the first term is

$$\frac{1}{W} \sum_{i=1}^n w_i \frac{1}{k_i} \sum_{h=1}^{k_i} C_{i_h}^{(p)}(\sigma_{A_x}^{(p)}) = \frac{1}{2xW} \sum_{i=1}^n \sum_{h=1}^{k_i} C_{i_h}^{(p)}(\sigma_{A_x}^{(p)}) = \overline{C}^{(p)}(\sigma_{A_x}^{(p)}).$$

For the second term, using

$$\sum_{h=1}^{k_i} (k_i - h) = \sum_{t=0}^{k_i-1} t = \frac{(k_i - 1)k_i}{2},$$

we get

$$\frac{1}{W} \sum_{i=1}^n w_i \frac{1}{k_i} \sum_{h=1}^{k_i} (k_i - h) \frac{\widehat{p}_i}{k_i} = \frac{1}{W} \sum_{i=1}^n w_i \frac{k_i - 1}{2k_i} \widehat{p}_i = \frac{1}{W} \sum_{i=1}^n \frac{2xw_i - 1}{4x} \widehat{p}_i.$$

Therefore

$$\overline{C}^{(p)}(\sigma) = \overline{C}^{(p)}(\sigma_{A_x}^{(p)}) + \Psi_a(x, p), \quad (\text{A.2})$$

where

$$\Psi_a(x, p) = \frac{1}{W} \left(\frac{2xw_a - 1}{4x} p + \sum_{i \neq a} \frac{2xw_i - 1}{4x} p_i \right).$$

Step 3: expanding the variance. Because $k_i = 2xw_i$, replicating the centered completion time of job i exactly k_i times gives

$$\frac{k_i}{2xW} = \frac{w_i}{W}.$$

Thus

$$WCTV^{(p)}(\sigma) = \frac{1}{W} \sum_{i=1}^n w_i \left(C_i^{(p)}(\sigma) - \overline{C}^{(p)}(\sigma) \right)^2 = \frac{1}{2xW} \sum_{i=1}^n \sum_{h=1}^{k_i} \left(C_{i_h}^{(p)}(\sigma) - \overline{C}^{(p)}(\sigma) \right)^2.$$

Define

$$X_{i,h} := C_{i_h}^{(p)}(\sigma_{A_x}^{(p)}) - \overline{C}^{(p)}(\sigma_{A_x}^{(p)}), \quad Y_{i,h} := (k_i - h) \frac{\widehat{p}_i}{k_i} - \Psi_a(x, p).$$

By (A.1) and (A.2), for every i and every $h = 1, \dots, k_i$,

$$C_i^{(p)}(\sigma) - \overline{C}^{(p)}(\sigma) = X_{i,h} + Y_{i,h}.$$

Consequently,

$$WCTV^{(p)}(\sigma) = \frac{1}{2xW} \sum_{i=1}^n \sum_{h=1}^{k_i} (X_{i,h} + Y_{i,h})^2.$$

Since $X_{i,h} + Y_{i,h} = C_i^{(p)}(\sigma) - \bar{C}^{(p)}(\sigma)$, we have

$$(X_{i,h} + Y_{i,h})^2 = X_{i,h}^2 + 2X_{i,h}Y_{i,h} + Y_{i,h}^2 = X_{i,h}^2 + 2 \left(C_i^{(p)}(\sigma) - \bar{C}^{(p)}(\sigma) \right) Y_{i,h} - Y_{i,h}^2.$$

Therefore,

$$\begin{aligned} WCTV^{(p)}(\sigma) &= CTV(\sigma_{A_x}^{(p)}) + \frac{1}{2xW} \sum_{i=1}^n 2 \left(C_i^{(p)}(\sigma) - \bar{C}^{(p)}(\sigma) \right) \sum_{h=1}^{k_i} Y_{i,h} \\ &\quad - \frac{1}{2xW} \sum_{i=1}^n \sum_{h=1}^{k_i} Y_{i,h}^2. \end{aligned} \tag{A.3}$$

Step 4: simplifying the linear correction term. For each fixed i ,

$$\sum_{h=1}^{k_i} Y_{i,h} = \sum_{h=1}^{k_i} \left((k_i - h) \frac{\hat{p}_i}{k_i} - \Psi_a(x, p) \right) = \frac{k_i - 1}{2} \hat{p}_i - k_i \Psi_a(x, p).$$

Dividing by k_i yields

$$\frac{1}{k_i} \sum_{h=1}^{k_i} Y_{i,h} = \frac{k_i - 1}{2k_i} \hat{p}_i - \Psi_a(x, p) = \frac{2xw_i - 1}{4xw_i} \hat{p}_i - \Psi_a(x, p).$$

Thus, with the notation of the theorem,

$$A_{a,i}(x, p) = \frac{2xw_i - 1}{4xw_i} \hat{p}_i - \Psi_a(x, p),$$

and hence

$$\sum_{h=1}^{k_i} Y_{i,h} = k_i A_{a,i}(x, p) = 2xw_i A_{a,i}(x, p).$$

The linear term in (A.3) therefore becomes

$$\frac{1}{2xW} \sum_{i=1}^n 2 \left(C_i^{(p)}(\sigma) - \bar{C}^{(p)}(\sigma) \right) \sum_{h=1}^{k_i} Y_{i,h} = \frac{2}{W} \sum_{i=1}^n w_i A_{a,i}(x, p) \left(C_i^{(p)}(\sigma) - \bar{C}^{(p)}(\sigma) \right).$$

Step 5: simplifying the quadratic correction term. For each fixed i ,

$$\sum_{h=1}^{k_i} Y_{i,h}^2 = \sum_{h=1}^{k_i} \left((k_i - h) \frac{\hat{p}_i}{k_i} - \Psi_a(x, p) \right)^2.$$

Using

$$\sum_{h=1}^{k_i} (k_i - h) = \frac{(k_i - 1)k_i}{2}, \quad \sum_{h=1}^{k_i} (k_i - h)^2 = \frac{(k_i - 1)k_i(2k_i - 1)}{6},$$

we obtain

$$\sum_{h=1}^{k_i} Y_{i,h}^2 = \frac{(k_i - 1)(2k_i - 1)}{6k_i} \hat{p}_i^2 - (k_i - 1) \hat{p}_i \Psi_a(x, p) + k_i \Psi_a(x, p)^2.$$

Summing over i and dividing by $2xW$ gives

$$\begin{aligned} \frac{1}{2xW} \sum_{i=1}^n \sum_{h=1}^{k_i} Y_{i,h}^2 &= \frac{1}{2xW} \sum_{i=1}^n \frac{(k_i-1)(2k_i-1)}{6k_i} \hat{p}_i^2 \\ &\quad - \frac{\Psi_a(x,p)}{2xW} \sum_{i=1}^n (k_i-1) \hat{p}_i + \frac{\Psi_a(x,p)^2}{2xW} \sum_{i=1}^n k_i. \end{aligned}$$

Now

$$\sum_{i=1}^n k_i = 2xW,$$

and, by the definition of $\Psi_a(x,p)$,

$$\Psi_a(x,p) = \frac{1}{W} \sum_{i=1}^n \frac{k_i-1}{4x} \hat{p}_i = \frac{1}{4xW} \sum_{i=1}^n (k_i-1) \hat{p}_i.$$

Therefore

$$\sum_{i=1}^n (k_i-1) \hat{p}_i = 4xW \Psi_a(x,p).$$

Using these two identities, the last two terms reduce to

$$- \frac{\Psi_a(x,p)}{2xW} \sum_{i=1}^n (k_i-1) \hat{p}_i + \frac{\Psi_a(x,p)^2}{2xW} \sum_{i=1}^n k_i = -\Psi_a(x,p)^2.$$

Moreover, since $k_i = 2xw_i$,

$$\frac{1}{2xW} \frac{(k_i-1)(2k_i-1)}{6k_i} = \frac{1}{W} \frac{(2xw_i-1)(4xw_i-1)}{24x^2w_i}.$$

Hence

$$\frac{1}{2xW} \sum_{i=1}^n \sum_{h=1}^{k_i} Y_{i,h}^2 = \frac{1}{W} \sum_{i=1}^n \frac{(2xw_i-1)(4xw_i-1)}{24x^2w_i} \hat{p}_i^2 - \Psi_a(x,p)^2.$$

Recalling that $\hat{p}_a = p$ and $\hat{p}_i = p_i$ for $i \neq a$, we get

$$\begin{aligned} \frac{1}{2xW} \sum_{i=1}^n \sum_{h=1}^{k_i} Y_{i,h}^2 &= \frac{1}{W} \left(\frac{(2xw_a-1)(4xw_a-1)}{24x^2w_a} p^2 + \sum_{i \neq a} \frac{(2xw_i-1)(4xw_i-1)}{24x^2w_i} p_i^2 \right) - \Psi_a(x,p)^2 \\ &= -CT_a^{(1)}(x,p). \end{aligned}$$

Substituting the results of Steps 4 and 5 into (A.3) yields

$$WCTV^{(p)}(\sigma) = CTV(\sigma_{A_x}^{(p)}) + \frac{2}{W} \sum_{i=1}^n w_i A_{a,i}(x,p) \left(C_i^{(p)}(\sigma) - \overline{C}^{(p)}(\sigma) \right) + CT_a^{(1)}(x,p).$$

Finally, because a is the first job of σ ,

$$WCTV^{(p)}(\sigma) = WCTV(\sigma), \quad C_i^{(p)}(\sigma) - \overline{C}^{(p)}(\sigma) = C_i(\sigma) - \overline{C}(\sigma) \quad \text{for all } i.$$

Dropping the superscript (p) from the augmented sequence notation, as in the statement of the theorem, we obtain exactly (8). This completes the proof. $\square$

Derivation of Theorem 2 (after Nessah and Chu (2010)).

Proof. We follow the proof scheme of Nessah and Chu (2010) and rewrite it in our notation. Since the present theorem recovers the original Nessah–Chu construction, we specialize throughout the proof to $x = \frac{1}{2}$ and to the original processing time of the first job, that is, $p = p_a$ when $a = \sigma[1]$. Since the weights are positive integers, $k_i = 2xw_i = w_i \in \mathbb{N}$, for all $i \in \mathcal{N}$. Thus, job i is replaced by w_i unit-weight pieces, each having processing time p_i/w_i . Moreover, $\hat{p}_i(a, p_a) = p_i$, for all $i \in \mathcal{N}$, and hence

$$\sigma_{A_{1/2}}^{(a, p_a)} = \sigma_{A_{1/2}}, \quad C_i^{(a, p_a)}(\sigma) = C_i(\sigma), \quad \overline{C}^{(a, p_a)}(\sigma) = \overline{C}(\sigma).$$

Therefore, Theorem 1 gives

$$WCTV(\sigma) = CTV(\sigma_{A_{1/2}}) + \frac{2}{W} \sum_{i=1}^n w_i A_{a,i}(\tfrac{1}{2}, p_a) (C_i(\sigma) - \overline{C}(\sigma)) + CT_a^{(1)}(\tfrac{1}{2}, p_a). \quad (\text{A.4})$$

Step 1: lower-bounding the augmented CTV. Recall that $\mathcal{P}_1(\frac{1}{2})$ is the augmented unweighted problem in which the w_i pieces of each original job i must remain consecutive, whereas $\mathcal{P}_2(\frac{1}{2})$ is obtained by dropping this contiguity requirement. Therefore, every feasible sequence of $\mathcal{P}_1(\frac{1}{2})$ is feasible for $\mathcal{P}_2(\frac{1}{2})$. At $x = \frac{1}{2}$, one has $N_{1/2} = W$, and the processing-time multiset of $\mathcal{P}_2(\frac{1}{2})$ is

$$\mathcal{Q}_{1/2} = \left\{ \underbrace{\frac{p_1}{w_1}, \dots, \frac{p_1}{w_1}}_{w_1 \text{ times}}, \dots, \underbrace{\frac{p_n}{w_n}, \dots, \frac{p_n}{w_n}}_{w_n \text{ times}} \right\}.$$

By (1), these processing times are already arranged blockwise in nondecreasing order. Thus, (12) becomes

$$LB^{(VS)}(\tfrac{1}{2}) = \frac{2L_{1/2} + VS_{1/2}}{2W}.$$

The recursive definition of r_j implies

$$\sum_{h=1}^j r_h = \left\lfloor \frac{w_1 + \dots + w_j}{2} \right\rfloor, \quad j = 1, \dots, n.$$

Consequently, the prefix sums entering $VS_{1/2}$ can be grouped according to the original-job blocks as

$$VS_{1/2} = \sum_{j=1}^n \sum_{\ell=1}^{r_j} \left(P_{j-1} + (\theta_j + 2\ell - 1) \frac{p_j}{w_j} \right)^2.$$

Here, if a selected prefix ends exactly at the boundary preceding block j , it may be assigned to block j without changing its cumulative processing time. Since

$$\sum_{\ell=1}^{r_j} (\theta_j + 2\ell - 1) = r_j(r_j + \theta_j)$$

and

$$\sum_{\ell=1}^{r_j} (\theta_j + 2\ell - 1)^2 = r_j g_j,$$

expansion of the preceding expression gives

$$VS_{1/2} = \sum_{j=1}^n r_j \left[P_{j-1}^2 + 2(r_j + \theta_j) P_{j-1} \frac{p_j}{w_j} + g_j \left(\frac{p_j}{w_j} \right)^2 \right].$$

When $W \geq 4$, the correction in (12) satisfies

$$2L_{1/2} = \frac{1}{2} \left(q_{W-1}^{(1/2)} - q_{W-2}^{(1/2)} \right)^2 = \Delta^{(VS)}.$$

Inspection of the multiplicities in the final two blocks of $\mathcal{Q}_{1/2}$ gives precisely the cases for $\Delta^{(VS)}$ displayed in Theorem 2. Hence, for $W \geq 4$, (12) reduces to the compact expression stated in the theorem. If instances with $W = 2$ or $W = 3$ are included, the corresponding small-instance cases in (12) must be used instead; the remainder of the argument is unchanged. The Viswanathkumar–Srinivasan procedure therefore gives

$$\min CTV[\mathcal{P}_2(\tfrac{1}{2})] \geq LB^{(VS)}(\tfrac{1}{2}).$$

Since $\sigma_{A_{1/2}}$ is feasible for $\mathcal{P}_1(\tfrac{1}{2})$, it follows that

$$CTV(\sigma_{A_{1/2}}) \geq \min CTV[\mathcal{P}_1(\tfrac{1}{2})] \geq \min CTV[\mathcal{P}_2(\tfrac{1}{2})] \geq LB^{(VS)}(\tfrac{1}{2}). \quad (\text{A.5})$$

Step 2: lower-bounding the linear term. By the weighted Cauchy–Schwarz inequality,

$$\begin{aligned} & \sum_{i=1}^n w_i A_{a,i}(\tfrac{1}{2}, p_a) (C_i(\sigma) - \overline{C}(\sigma)) \\ & \geq - \left(\sum_{i=1}^n w_i A_{a,i}(\tfrac{1}{2}, p_a)^2 \right)^{1/2} \left(\sum_{i=1}^n w_i (C_i(\sigma) - \overline{C}(\sigma))^2 \right)^{1/2} \\ & = -W \sqrt{WCTV(\sigma) CT_a^{(2)}(\tfrac{1}{2}, p_a)}. \end{aligned} \quad (\text{A.6})$$

The last equality follows from

$$\sum_{i=1}^n w_i A_{a,i}(\tfrac{1}{2}, p_a)^2 = W CT_a^{(2)}(\tfrac{1}{2}, p_a)$$

and

$$\sum_{i=1}^n w_i (C_i(\sigma) - \overline{C}(\sigma))^2 = W WCTV(\sigma).$$

In particular,

$$CT_a^{(2)}(\tfrac{1}{2}, p_a) = \frac{1}{W} \sum_{i=1}^n w_i A_{a,i}(\tfrac{1}{2}, p_a)^2 \geq 0,$$

so the square root in (A.6) is well defined.

Step 3: combining the inequalities and completing the square. Let σ^* be an optimal sequence, let $a^* = \sigma^*[1]$, and set

$$F := WCTV(\sigma^*), \quad D := CT_{a^*}^{(2)}(\tfrac{1}{2}, p_{a^*}).$$

Both F and D are nonnegative. Applying (A.4), (A.5), and (A.6) to σ^* gives

$$F \geq LB^{(VS)}(\tfrac{1}{2}) - 2\sqrt{FD} + CT_{a^*}^{(1)}(\tfrac{1}{2}, p_{a^*}).$$

Rearranging and adding D to both sides yields

$$\left(\sqrt{F} + \sqrt{D}\right)^2 \geq LB^{(VS)}\left(\frac{1}{2}\right) + CT_{a^*}^{(1)}\left(\frac{1}{2}, p_{a^*}\right) + D. \quad (\text{A.7})$$

For any $u \geq 0$ and any $r \in \mathbb{R}$, the inequality $u^2 \geq r$ implies $u \geq \sqrt{r^+}$. Indeed, the assertion is immediate when $r \leq 0$, whereas for $r > 0$ it follows by taking nonnegative square roots. Applying this observation to (A.7) gives

$$\sqrt{F} + \sqrt{D} \geq \sqrt{\left(LB^{(VS)}\left(\frac{1}{2}\right) + CT_{a^*}^{(1)}\left(\frac{1}{2}, p_{a^*}\right) + D\right)^+}.$$

Hence

$$\sqrt{F} \geq \sqrt{\left(LB^{(VS)}\left(\frac{1}{2}\right) + CT_{a^*}^{(1)}\left(\frac{1}{2}, p_{a^*}\right) + D\right)^+} - \sqrt{D}.$$

Since $\sqrt{F} \geq 0$, it follows that

$$\sqrt{F} \geq \left(\sqrt{\left(LB^{(VS)}\left(\frac{1}{2}\right) + CT_{a^*}^{(1)}\left(\frac{1}{2}, p_{a^*}\right) + D\right)^+} - \sqrt{D}\right)^+.$$

Squaring both sides and substituting the definition of D , we obtain

$$\begin{aligned} WCTV(\sigma^*) = F &\geq \left[\left(\sqrt{\left(LB^{(VS)}\left(\frac{1}{2}\right) + CT_{a^*}^{(1)}\left(\frac{1}{2}, p_{a^*}\right) + CT_{a^*}^{(2)}\left(\frac{1}{2}, p_{a^*}\right)\right)^+} - \sqrt{CT_{a^*}^{(2)}\left(\frac{1}{2}, p_{a^*}\right)}\right)^+\right]^2 \\ &= LB_{a^*}^{(NC)}\left(\frac{1}{2}, p_{a^*}\right). \end{aligned}$$

It remains to show that $LB_a^{(NC)}\left(\frac{1}{2}, p_a\right)$ is independent of a . Since $\hat{p}_i(a, p_a) = p_i$ for every $i \in \mathcal{N}$, one has

$$\Psi_a\left(\frac{1}{2}, p_a\right) = \frac{1}{2W} \sum_{i=1}^n (w_i - 1)p_i.$$

Consequently,

$$CT_a^{(1)}\left(\frac{1}{2}, p_a\right) = \left(\frac{1}{2W} \sum_{i=1}^n (w_i - 1)p_i\right)^2 - \frac{1}{W} \sum_{i=1}^n \frac{(w_i - 1)(2w_i - 1)}{6w_i} p_i^2$$

and

$$CT_a^{(2)}\left(\frac{1}{2}, p_a\right) = \frac{1}{4W} \sum_{i=1}^n \frac{(w_i - 1)^2}{w_i} p_i^2 - \left(\frac{1}{2W} \sum_{i=1}^n (w_i - 1)p_i\right)^2.$$

Neither correction term depends on a . Moreover, the augmented multiset $\mathcal{Q}_{1/2}$ is the same for every choice of a , and therefore $LB^{(VS)}\left(\frac{1}{2}\right)$, including its correction $\Delta^{(VS)}$, is also independent of a . It follows that

$$LB_a^{(NC)}\left(\frac{1}{2}, p_a\right) = LB_b^{(NC)}\left(\frac{1}{2}, p_b\right), \quad a, b \in \mathcal{N}.$$

Choosing $a = a^*$ and $b = n$ gives

$$WCTV(\sigma^*) \geq LB_{a^*}^{(NC)}\left(\frac{1}{2}, p_{a^*}\right) = LB_n^{(NC)}\left(\frac{1}{2}, p_n\right),$$

which proves the result. $\square$

Lemma 2

Proof. Fix $\sigma \in \Sigma_a$. To simplify notation, throughout the proof we omit the explicit dependence on (a, x) and write $k_a = k_a(x)$, $k_0 = k_0(a, x)$, $\kappa = \kappa(a, x)$, $v_h = v_h(a, x)$, and $\bar{p}_h = \bar{p}_h(a)$. We also write

$$N_x := 2xW \quad \text{and} \quad M_x := \left\lfloor \frac{N_x}{2} \right\rfloor.$$

Since $x \in \mathcal{X}_a$, one has

$$k_a \leq 1 + \sum_{h=1}^{n-1} v_h = N_x - k_a + 1,$$

and therefore $k_a - 1 \leq M_x$. Thus the first $k_a - 1$ symmetric pairs in the augmented sequence are well defined. Since $\sigma[1] = a$, the first k_a pieces of σ_{A_x} are precisely the pieces of job a , each of processing time p/k_a .

Step 1: Pairing lower bound for $CTV(\sigma_{A_x})$. For any real numbers y, z, μ ,

$$(y - \mu)^2 + (z - \mu)^2 = \frac{(y - z)^2}{2} + 2 \left(\frac{y + z}{2} - \mu \right)^2 \geq \frac{(y - z)^2}{2}.$$

Applying this identity with $\mu = \bar{C}(\sigma_{A_x})$ to the paired positions i and $N_x - i + 1$ and summing for $i = 1, \dots, M_x$ gives

$$CTV(\sigma_{A_x}) \geq \frac{1}{N_x} \sum_{i=1}^{M_x} \frac{(C_{[N_x-i+1]}(\sigma_{A_x}) - C_{[i]}(\sigma_{A_x}))^2}{2},$$

where, if N_x is odd, the nonnegative squared deviation of the unpaired middle completion time is discarded. We split this sum into the contribution of the first $k_a - 1$ pairs and the remaining central contribution:

$$CTV(\sigma_{A_x}) \geq \frac{1}{2xW} [E_{a,x}(p, \sigma) + R_{a,x}(p, \sigma)], \quad (\text{A.8})$$

where

$$E_{a,x}(p, \sigma) := \sum_{i=1}^{k_a-1} \frac{(C_{[N_x-i+1]}(\sigma_{A_x}) - C_{[i]}(\sigma_{A_x}))^2}{2}$$

and

$$R_{a,x}(p, \sigma) := \sum_{i=k_a}^{M_x} \frac{(C_{[N_x-i+1]}(\sigma_{A_x}) - C_{[i]}(\sigma_{A_x}))^2}{2},$$

with the convention that an empty sum is equal to zero.

Step 2: Lower bound for the central contribution. By the definitions of κ and m_x ,

$$\kappa + \sum_{h=k_0+1}^{n-1} v_h = k_a - 1, \quad m_x = \sum_{h=1}^{n-1} v_h - k_a + 2 = N_x - 2k_a + 2 \geq 1.$$

Since the ratios $\bar{p}_h/v_h$ are nondecreasing, the $k_a - 1$ largest pieces among the jobs different from a consist of all pieces from blocks $k_0 + 1, \dots, n - 1$ and κ pieces from block k_0 . Deleting these pieces leaves the $m_x - 1$ smallest pieces among the jobs different from a , namely all v_h pieces from each block $h < k_0$ and $v_{k_0} - \kappa$ pieces from block k_0 . The surrogate multiset $\bar{Q}_{a,x}$ consists of these $m_x - 1$ smallest pieces together with one additional copy of $\bar{p}_{k_0}/v_{k_0}$. This additional value is no smaller than any of the retained values. Consequently, for every $\ell = 1, \dots, m_x - 1$, the values $\bar{q}_1, \dots, \bar{q}_\ell$ are the ℓ smallest processing times among all pieces of jobs different from a .

Now fix $i \in \{k_a, \dots, M_x\}$. The completion-time difference

$$C_{[N_x-i+1]}(\sigma_{A_x}) - C_{[i]}(\sigma_{A_x})$$

is the sum of the processing times of the $N_x - 2i + 1$ pieces occupying positions $i + 1, \dots, N_x - i + 1$. Since $i \geq k_a$, all these pieces belong to jobs different from a . Moreover,

$$1 \leq N_x - 2i + 1 \leq N_x - 2k_a + 1 = m_x - 1.$$

Therefore,

$$C_{[N_x-i+1]}(\sigma_{A_x}) - C_{[i]}(\sigma_{A_x}) \geq \sum_{j=1}^{N_x-2i+1} \bar{q}_j.$$

Both sides are nonnegative, so this inequality may be squared and summed over the central pairs.

As i ranges from k_a to M_x , the prefix lengths $N_x - 2i + 1$ are $m_x - 1, m_x - 3, \dots$, ending at 2 if m_x is odd and at 1 if m_x is even. Hence,

$$R_{a,x}(p, \sigma) \geq \frac{1}{2} \sum_{i=k_a}^{M_x} \left(\sum_{j=1}^{N_x-2i+1} \bar{q}_j \right)^2 = \frac{1}{2} L_{B_{1,x}}(k_0). \quad (\text{A.9})$$

When $m_x = 1$, both sums are empty, and the same conclusion holds.

Step 3: Lower bound for the extreme contribution. For $h = k_0, \dots, n - 1$, let

$$q_h := \frac{\bar{p}_h}{v_h}, \quad B_h := \sum_{t=1}^h \bar{p}_t + q_h, \quad D_h := 1 - \sum_{t=h+1}^{n-1} \frac{v_t}{k_a}.$$

We bound the extreme completion-time differences from below by bounding the processing time of each right suffix from above. By the definitions of k_0 and κ , the $k_a - 1$ largest pieces among the jobs different from a consist of all pieces from blocks $k_0 + 1, \dots, n - 1$ and κ pieces from block k_0 .

Fix $h \in \{k_0, \dots, n - 1\}$ and $s \in \{1, \dots, \bar{v}_h\}$, where s counts the selected pieces within block h in the ordering from largest to smallest processing times. The corresponding paired index is

$$i = \sum_{t=h+1}^{n-1} v_t + s.$$

As (h, s) ranges over these indices, i takes each value in $\{1, \dots, k_a - 1\}$ exactly once. Since $i \leq k_a - 1$, the left completion time is

$$C_{[i]}(\sigma_{A_x}) = \frac{i}{k_a} p.$$

Moreover, independently of the order of the remaining jobs in σ , the total processing time of the $i - 1$ pieces after position $N_x - i + 1$ is at most the sum of the $i - 1$ largest pieces among the jobs different from a , namely

$$\sum_{t=h+1}^{n-1} \bar{p}_t + (s - 1)q_h.$$

Therefore,

$$C_{[N_x-i+1]}(\sigma_{A_x}) \geq p + \sum_{t=1}^{n-1} \bar{p}_t - \sum_{t=h+1}^{n-1} \bar{p}_t - (s - 1)q_h = p + \sum_{t=1}^h \bar{p}_t - (s - 1)q_h.$$

It follows that

$$C_{[N_x-i+1]}(\sigma_{A_x}) - C_{[i]}(\sigma_{A_x}) \geq \left(D_h - \frac{s}{k_a}\right)p + B_h - sq_h.$$

The right-hand side is nonnegative. Indeed,

$$D_h - \frac{s}{k_a} = 1 - \frac{\sum_{t=h+1}^{n-1} v_t + s}{k_a} = 1 - \frac{i}{k_a} \geq \frac{1}{k_a},$$

because $i \leq k_a - 1$, and

$$B_h - sq_h = \sum_{t=1}^{h-1} \bar{p}_t + (v_h + 1 - s)q_h \geq 0,$$

because $1 \leq s \leq \bar{v}_h \leq v_h$. Hence, after squaring,

$$(C_{[N_x-i+1]}(\sigma_{A_x}) - C_{[i]}(\sigma_{A_x}))^2 \geq \left[\left(D_h - \frac{s}{k_a}\right)p + B_h - sq_h\right]^2.$$

Consequently,

$$E_{a,x}(p, \sigma) \geq \frac{1}{2} \sum_{h=k_0}^{n-1} \sum_{s=1}^{\bar{v}_h} \left[\left(D_h - \frac{s}{k_a}\right)p + B_h - sq_h\right]^2.$$

Expanding the square and using

$$\sum_{s=1}^{\bar{v}_h} s = \frac{\bar{v}_h(\bar{v}_h + 1)}{2}, \quad \sum_{s=1}^{\bar{v}_h} s^2 = \frac{\bar{v}_h(\bar{v}_h + 1)(2\bar{v}_h + 1)}{6},$$

the definitions of $\bar{\alpha}_h$, $\bar{\beta}_h$ and $\bar{\gamma}_h$ give

$$\sum_{s=1}^{\bar{v}_h} \left[\left(D_h - \frac{s}{k_a}\right)p + B_h - sq_h\right]^2 = \bar{v}_h (\bar{\alpha}_h p^2 + \bar{\beta}_h p + \bar{\gamma}_h).$$

Summing over h yields

$$E_{a,x}(p, \sigma) \geq \frac{1}{2} [H_\alpha(a, x)p^2 + H_\beta(a, x)p + H_\gamma(a, x)], \quad (\text{A.10})$$

where $H_\alpha(a, x)$, $H_\beta(a, x)$ and $H_\gamma(a, x)$ are exactly the quantities defined in the statement.

Step 4: Conclusion. Substituting (A.9) and (A.10) into (A.8) gives

$$CTV(\sigma_{A_x}) \geq \frac{1}{2xW} \left[\frac{1}{2} LB_{1,x}(k_0) + \frac{1}{2} H_\alpha(a, x)p^2 + \frac{1}{2} H_\beta(a, x)p + \frac{1}{2} H_\gamma(a, x) \right].$$

Therefore,

$$CTV(\sigma_{A_x}) \geq \frac{1}{4xW} [LB_{1,x}(k_0) + H_\alpha(a, x)p^2 + H_\beta(a, x)p + H_\gamma(a, x)] = \overline{LB}_a^{(VS)}(x, p).$$

Since $\sigma \in \Sigma_a$ was arbitrary, the inequality holds for every schedule in Σ_a , and in particular for every optimal restricted schedule $\sigma_a^* \in \Sigma_a$. $\square$

Lemma 3

Proof. Fix $x \in \mathcal{X}$, $a \in \mathcal{N}$, and $p > 0$. Since $w_i > 0$ for every $i \in \mathcal{N}$, the definition of $CT_a^{(2)}(x, p)$ immediately gives

$$CT_a^{(2)}(x, p) = \frac{1}{W} \sum_{i=1}^n w_i A_{a,i}(x, p)^2 \geq 0.$$

We now prove the second assertion. As is understood whenever $\overline{LB}_a^{(VS)}(x, p)$ is evaluated, the quantities entering its construction are considered on their defining domain. In particular, $x \in \mathcal{X}_a$. To simplify notation, suppress the dependence on (a, x) and write $k := k_a(x)$, $v_h := v_h(a, x)$, and $\bar{p}_h := \bar{p}_h(a)$, for each $h = 1, \dots, n-1$. Using $k = 2xw_a$, $v_h = 2xw_i$ for the corresponding job $i \neq a$, and $N_x = 2xW = k + \sum_{h=1}^{n-1} v_h$, the definitions of $CT_a^{(1)}(x, p)$ and $CT_a^{(2)}(x, p)$ can be written as

$$\begin{aligned} CT_a^{(1)}(x, p) &= \Psi_a(x, p)^2 - \frac{1}{N_x} \left[\frac{(k-1)(2k-1)}{6k} p^2 + \sum_{h=1}^{n-1} \frac{(v_h-1)(2v_h-1)}{6v_h} \bar{p}_h^2 \right], \\ CT_a^{(2)}(x, p) &= \frac{1}{N_x} \left[\frac{(k-1)^2}{4k} p^2 + \sum_{h=1}^{n-1} \frac{(v_h-1)^2}{4v_h} \bar{p}_h^2 \right] - \Psi_a(x, p)^2. \end{aligned}$$

The terms $\Psi_a(x, p)^2$ cancel. Moreover, for every positive integer r ,

$$\frac{(r-1)^2}{4r} - \frac{(r-1)(2r-1)}{6r} = -\frac{r^2-1}{12r}.$$

Consequently,

$$CT_a^{(1)}(x, p) + CT_a^{(2)}(x, p) = -\frac{1}{12N_x} \left[\frac{k^2-1}{k} p^2 + \sum_{h=1}^{n-1} \frac{v_h^2-1}{v_h} \bar{p}_h^2 \right]. \quad (\text{A.11})$$

It remains to derive a corresponding lower bound for $\overline{LB}_a^{(VS)}(x, p)$. Let

$$R := \sum_{h=1}^{n-1} v_h = N_x - k$$

and let $z_1 \leq \dots \leq z_R$ be the processing times of all pieces belonging to jobs different from a , listed in the nondecreasing block order used in the construction:

$$(z_1, \dots, z_R) = \left(\underbrace{\frac{\bar{p}_1}{v_1}, \dots, \frac{\bar{p}_1}{v_1}}_{v_1 \text{ times}}, \dots, \underbrace{\frac{\bar{p}_{n-1}}{v_{n-1}}, \dots, \frac{\bar{p}_{n-1}}{v_{n-1}}}_{v_{n-1} \text{ times}} \right).$$

Define

$$Z_0 := 0, \quad Z_\ell := \sum_{j=1}^{\ell} z_j, \quad \ell = 1, \dots, R.$$

Since

$$m_x = N_x - 2k + 2 = R - k + 2,$$

the first $m_x - 1$ elements of the surrogate multiset $\overline{\mathcal{Q}}_{a,x}$ coincide, as values, with $z_1, \dots, z_{m_x-1}$. Inspection of the two parity cases in the definition of $LB_{1,x}(k_0)$ therefore gives

$$LB_{1,x}(k_0) = \sum_{\substack{1 \leq \ell \leq m_x-1 \\ \ell \equiv m_x-1 \pmod{2}}} Z_\ell^2, \quad (\text{A.12})$$

with the convention that the sum is zero when $m_x = 1$. Now define

$$F_a(x, p) := LB_{1,x}(k_0) + H_\alpha(a, x)p^2 + H_\beta(a, x)p + H_\gamma(a, x).$$

Expanding the definitions of $\bar{\alpha}_h$, $\bar{\beta}_h$, and $\bar{\gamma}_h$ yields

$$\begin{aligned} & H_\alpha(a, x)p^2 + H_\beta(a, x)p + H_\gamma(a, x) \\ &= \sum_{h=k_0}^{n-1} \sum_{s=1}^{\bar{v}_h} \left[\left(1 - \frac{\sum_{t=h+1}^{n-1} v_t + s}{k} \right) p + \sum_{t=1}^h \bar{p}_t + (1-s) \frac{\bar{p}_h}{v_h} \right]^2. \end{aligned}$$

For every pair (h, s) in this double sum, set

$$i := \sum_{t=h+1}^{n-1} v_t + s.$$

As (h, s) ranges over the indicated values, i takes each value in $\{1, \dots, k-1\}$ exactly once. Furthermore,

$$\sum_{t=1}^h \bar{p}_t + (1-s) \frac{\bar{p}_h}{v_h} = Z_{R-i+1}.$$

It follows that

$$H_\alpha(a, x)p^2 + H_\beta(a, x)p + H_\gamma(a, x) = \sum_{i=1}^{k-1} \left(\frac{k-i}{k} p + Z_{R-i+1} \right)^2. \quad (\text{A.13})$$

Because all processing times are positive,

$$\left(\frac{k-i}{k} p + Z_{R-i+1} \right)^2 \geq \left(\frac{k-i}{k} p \right)^2 + Z_{R-i+1}^2.$$

Moreover, as i ranges from 1 to $k-1$, the index $R-i+1$ ranges from R down to

$$R - k + 2 = m_x.$$

Combining (A.12) and (A.13) therefore gives

$$F_a(x, p) \geq \sum_{\substack{1 \leq \ell \leq R \\ \ell \equiv m_x - 1 \pmod{2}}} Z_\ell^2 + \frac{p^2}{k^2} \sum_{j=1}^{k-1} j^2. \quad (\text{A.14})$$

For the first-job contribution, the standard sum-of-squares formula gives

$$\frac{p^2}{k^2} \sum_{j=1}^{k-1} j^2 = \frac{(k-1)(2k-1)}{6k} p^2 \geq \frac{k^2-1}{6k} p^2,$$

where the inequality follows from $(k-1)(2k-1) - (k^2-1) = (k-1)(k-2) \geq 0$ for every positive integer k . For each $h = 1, \dots, n-1$, let

$$b_h := \sum_{t=1}^{h-1} v_t.$$

The pieces of the h -th remaining job occupy positions $b_h + 1, \dots, b_h + v_h$ in the list $(z_\ell)_{\ell=1}^R$. Hence, for $s = 1, \dots, v_h$,

$$Z_{b_h+s} = \sum_{t=1}^{h-1} \bar{p}_t + s \frac{\bar{p}_h}{v_h} \geq s \frac{\bar{p}_h}{v_h}.$$

Within each block, the congruence

$$b_h + s \equiv m_x - 1 \pmod{2}$$

selects either the even or the odd values of s . For every positive integer v ,

$$\min \left\{ \sum_{\substack{1 \leq s \leq v \\ s \text{ even}}} s^2, \sum_{\substack{1 \leq s \leq v \\ s \text{ odd}}} s^2 \right\} = \frac{v(v^2 - 1)}{6}.$$

Therefore,

$$\sum_{\substack{1 \leq \ell \leq R \\ \ell \equiv m_x - 1 \pmod{2}}} Z_\ell^2 \geq \frac{1}{6} \sum_{h=1}^{n-1} \frac{v_h^2 - 1}{v_h} \bar{p}_h^2.$$

Substituting this inequality and the first-job estimate into (A.14) yields

$$F_a(x, p) \geq \frac{1}{6} \left[\frac{k^2 - 1}{k} p^2 + \sum_{h=1}^{n-1} \frac{v_h^2 - 1}{v_h} \bar{p}_h^2 \right]. \quad (\text{A.15})$$

Finally, by (15) and $N_x = 2xW$,

$$\overline{LB}_a^{(VS)}(x, p) = \frac{F_a(x, p)}{4xW} = \frac{F_a(x, p)}{2N_x}.$$

Combining this identity with (A.11) and (A.15), we obtain

$$\overline{LB}_a^{(VS)}(x, p) + CT_a^{(1)}(x, p) + CT_a^{(2)}(x, p) \geq 0.$$

This completes the proof. $\square$

Theorem 3

Proof. Fix $a \in \mathcal{A}$, $x \in \mathcal{X}_a$, and $p \in \mathcal{P}_a(x)$. Let $\sigma_a^* \in \Sigma_a$ be optimal for $OPT(a)$. Since the first job of σ_a^* is a , Lemma 1 implies that $WCTV(\sigma_a^*)$ is independent of the processing time assigned to that first job. Hence, we may evaluate the decomposition of Theorem 1 on σ_a^* after assigning processing time p to job a . In what follows, the augmented sequence is computed with this assigned value p for job a and with the original processing times p_i for all $i \neq a$. If the completion times computed with this modified processing-time vector are denoted by $C_i^{(a,p)}(\sigma_a^*)$ and $\bar{C}^{(a,p)}(\sigma_a^*)$, then Lemma 1 gives

$$C_i^{(a,p)}(\sigma_a^*) - \bar{C}^{(a,p)}(\sigma_a^*) = C_i(\sigma_a^*) - \bar{C}(\sigma_a^*), \quad i = 1, \dots, n,$$

and the value of $WCTV(\sigma_a^*)$ is unchanged. Therefore, Theorem 1 yields

$$WCTV(\sigma_a^*) = CTV((\sigma_a^*)_{A_x}) + \frac{2}{W} \sum_{i=1}^n w_i A_{a,i}(x, p) (C_i(\sigma_a^*) - \bar{C}(\sigma_a^*)) + CT_a^{(1)}(x, p).$$

Since $p \in \mathcal{P}_a(x)$, Lemma 2 applies and gives

$$CTV((\sigma_a^*)_{A_x}) \geq \overline{LB}_a^{(VS)}(x, p).$$

Moreover, by the weighted Cauchy–Schwarz inequality,

$$\sum_{i=1}^n w_i A_{a,i}(x, p) (C_i(\sigma_a^*) - \overline{C}(\sigma_a^*)) \geq - \left(\sum_{i=1}^n w_i A_{a,i}(x, p)^2 \right)^{1/2} \left(\sum_{i=1}^n w_i (C_i(\sigma_a^*) - \overline{C}(\sigma_a^*))^2 \right)^{1/2}.$$

Using the definitions

$$CT_a^{(2)}(x, p) := \frac{1}{W} \sum_{i=1}^n w_i A_{a,i}(x, p)^2$$

and

$$WCTV(\sigma_a^*) = \frac{1}{W} \sum_{i=1}^n w_i (C_i(\sigma_a^*) - \overline{C}(\sigma_a^*))^2,$$

this inequality becomes

$$\sum_{i=1}^n w_i A_{a,i}(x, p) (C_i(\sigma_a^*) - \overline{C}(\sigma_a^*)) \geq -W \sqrt{WCTV(\sigma_a^*) CT_a^{(2)}(x, p)}.$$

Combining this inequality with the decomposition above and with the bound $CTV((\sigma_a^*)_{A_x}) \geq \overline{LB}_a^{(VS)}(x, p)$ yields

$$WCTV(\sigma_a^*) \geq \overline{LB}_a^{(VS)}(x, p) - 2\sqrt{WCTV(\sigma_a^*) CT_a^{(2)}(x, p)} + CT_a^{(1)}(x, p).$$

Rearranging gives

$$WCTV(\sigma_a^*) + 2\sqrt{WCTV(\sigma_a^*) CT_a^{(2)}(x, p)} + CT_a^{(2)}(x, p) \geq \overline{LB}_a^{(VS)}(x, p) + CT_a^{(1)}(x, p) + CT_a^{(2)}(x, p),$$

or equivalently,

$$\left(\sqrt{WCTV(\sigma_a^*)} + \sqrt{CT_a^{(2)}(x, p)} \right)^2 \geq \overline{LB}_a^{(VS)}(x, p) + CT_a^{(1)}(x, p) + CT_a^{(2)}(x, p).$$

By Lemma 3, we have

$$CT_a^{(2)}(x, p) \geq 0$$

and

$$\overline{LB}_a^{(VS)}(x, p) + CT_a^{(1)}(x, p) + CT_a^{(2)}(x, p) \geq 0.$$

Therefore,

$$\sqrt{WCTV(\sigma_a^*)} + \sqrt{CT_a^{(2)}(x, p)} \geq \sqrt{\overline{LB}_a^{(VS)}(x, p) + CT_a^{(1)}(x, p) + CT_a^{(2)}(x, p)}.$$

It follows that

$$\sqrt{WCTV(\sigma_a^*)} \geq \sqrt{\overline{LB}_a^{(VS)}(x, p) + CT_a^{(1)}(x, p) + CT_a^{(2)}(x, p)} - \sqrt{CT_a^{(2)}(x, p)}.$$

Since the left-hand side is nonnegative, this implies

$$\sqrt{WCTV(\sigma_a^*)} \geq \left(\sqrt{\overline{LB}_a^{(VS)}(x, p) + CT_a^{(1)}(x, p) + CT_a^{(2)}(x, p)} - \sqrt{CT_a^{(2)}(x, p)} \right)^+.$$

Squaring both sides, we obtain

$$OPT(a) = WCTV(\sigma_a^*) \geq LB_a^{(NN)}(x, p).$$

Since this holds for every $p \in \mathcal{P}_a(x)$, it follows that

$$OPT(a) \geq \sup_{p \in \mathcal{P}_a(x)} LB_a^{(NN)}(x, p).$$

Because $x \in \mathcal{X}_a$ was arbitrary, we conclude that

$$OPT(a) \geq \sup_{x \in \mathcal{X}_a} \sup_{p \in \mathcal{P}_a(x)} LB_a^{(NN)}(x, p).$$

Now let σ^* be a globally optimal sequence and set $a^* = \sigma^*[1]$. Since $a^* \in \mathcal{A}$, the previous inequality can be applied with $a = a^*$. Hence

$$WCTV(\sigma^*) = OPT(a^*) \geq \sup_{x \in \mathcal{X}_{a^*}} \sup_{p \in \mathcal{P}_{a^*}(x)} LB_{a^*}^{(NN)}(x, p).$$

Moreover, because $a^* \in \mathcal{A}$,

$$\sup_{x \in \mathcal{X}_{a^*}} \sup_{p \in \mathcal{P}_{a^*}(x)} LB_{a^*}^{(NN)}(x, p) \geq \min_{a \in \mathcal{A}} \sup_{x \in \mathcal{X}_a} \sup_{p \in \mathcal{P}_a(x)} LB_a^{(NN)}(x, p).$$

Combining the last two inequalities proves

$$WCTV(\sigma^*) \geq \min_{a \in \mathcal{A}} \sup_{x \in \mathcal{X}_a} \sup_{p \in \mathcal{P}_a(x)} LB_a^{(NN)}(x, p).$$

□

Corollary 1

Proof. Fix $a \in \mathcal{N}$, $x \in \mathcal{X}_a$, and $p \in \mathcal{P}_a(x)$. Throughout the proof, all quantities k_0 , $\bar{v}_h$, $\bar{\alpha}_h$, $\bar{\beta}_h$ and $\bar{\gamma}_h$ are evaluated at (a, x) . Define

$$\Psi_a(x, 0) = \frac{1}{W} \sum_{i=1}^{n-1} \frac{v_i(a, x) - 1}{4x} \bar{p}_i(a), \quad \Psi_a(x, p) = \frac{2xw_a - 1}{4xW} p + \Psi_a(x, 0).$$

Write

$$\begin{aligned} \overline{LB}_a^{(VS)}(x, p) &= a_1(x)p^2 + a_2(x)p + a_3(x), \\ CT_a^{(1)}(x, p) &= b_1(x)p^2 + b_2(x)p + b_3(x), \\ CT_a^{(2)}(x, p) &= c_{1,a}(x)p^2 + c_{2,a}(x)p + c_{3,a}(x), \end{aligned}$$

where, by the corrected expression for $\overline{LB}_a^{(VS)}(x, p)$,

$$\begin{cases} a_1(x) = \frac{1}{4xW} \sum_{h=k_0}^{n-1} \bar{v}_h \bar{\alpha}_h, \\ a_2(x) = \frac{1}{4xW} \sum_{h=k_0}^{n-1} \bar{v}_h \bar{\beta}_h, \\ a_3(x) = \frac{1}{4xW} \left(\sum_{h=k_0}^{n-1} \bar{v}_h \bar{\gamma}_h + LB_{1,x}(k_0) \right), \\ b_1(x) = \left(\frac{2xw_a - 1}{4xW} \right)^2 - \frac{(2xw_a - 1)(4xw_a - 1)}{24x^2w_aW}, \\ b_2(x) = \frac{2xw_a - 1}{2xW} \Psi_a(x, 0), \\ b_3(x) = \Psi_a(x, 0)^2 - \frac{1}{12xW} \sum_{i=1}^{n-1} \frac{(v_i(a, x) - 1)(2v_i(a, x) - 1)}{v_i(a, x)} \bar{p}_i(a)^2, \\ c_{1,a}(x) = \frac{(2xw_a - 1)^2(W - w_a)}{16x^2w_aW^2}, \\ c_{2,a}(x) = -\frac{2xw_a - 1}{2xW} \Psi_a(x, 0), \\ c_{3,a}(x) = \frac{1}{8xW} \sum_{i=1}^{n-1} \frac{(v_i(a, x) - 1)^2}{v_i(a, x)} \bar{p}_i(a)^2 - \Psi_a(x, 0)^2. \end{cases}$$

Set

$$\alpha_{1,a}(x) = a_1(x) + b_1(x) + c_{1,a}(x), \quad \alpha_{2,a}(x) = a_2(x) + b_2(x) + c_{2,a}(x), \quad \alpha_{3,a}(x) = a_3(x) + b_3(x) + c_{3,a}(x).$$

Then

$$\overline{LB}_a^{(VS)}(x, p) + CT_a^{(1)}(x, p) + CT_a^{(2)}(x, p) = \alpha_{1,a}(x)p^2 + \alpha_{2,a}(x)p + \alpha_{3,a}(x),$$

while

$$CT_a^{(2)}(x, p) = c_{1,a}(x)p^2 + c_{2,a}(x)p + c_{3,a}(x).$$

The second quadratic is nonnegative because

$$CT_a^{(2)}(x, p) = \frac{1}{W} \sum_{i=1}^n w_i A_{a,i}(x, p)^2.$$

The first quadratic is the first radicand in the definition of $LB_a^{(NN)}(x, p)$. It is nonnegative because $\overline{LB}_a^{(VS)}(x, p) \geq LB^{(VS)}(x)$ and

$$LB^{(VS)}(x) + CT_a^{(1)}(x, p) + CT_a^{(2)}(x, p) \geq 0$$

by Lemma 3. Hence,

$$LB_a^{(NN)}(x, p) = \left[\left(\sqrt{\alpha_{1,a}(x)p^2 + \alpha_{2,a}(x)p + \alpha_{3,a}(x)} - \sqrt{c_{1,a}(x)p^2 + c_{2,a}(x)p + c_{3,a}(x)} \right)^+ \right]^2.$$

It remains to simplify the coefficients $\alpha_{1,a}(x)$, $\alpha_{2,a}(x)$ and $\alpha_{3,a}(x)$.

For the quadratic coefficient,

$$\alpha_{1,a}(x) = \frac{1}{4xW} \sum_{h=k_0}^{n-1} \bar{v}_h \bar{\alpha}_h + \left(\frac{2xw_a - 1}{4xW} \right)^2 - \frac{(2xw_a - 1)(4xw_a - 1)}{24x^2w_aW} + \frac{(2xw_a - 1)^2(W - w_a)}{16x^2w_aW^2}.$$

The two positive-square terms combine as

$$\left(\frac{2xw_a - 1}{4xW} \right)^2 + \frac{(2xw_a - 1)^2(W - w_a)}{16x^2w_aW^2} = \frac{(2xw_a - 1)^2}{16x^2w_aW}.$$

Therefore,

$$\alpha_{1,a}(x) = \frac{1}{4xW} \sum_{h=k_0}^{n-1} \bar{v}_h \bar{\alpha}_h + \frac{(2xw_a - 1)^2}{16x^2w_aW} - \frac{(2xw_a - 1)(4xw_a - 1)}{24x^2w_aW}.$$

Factoring out $(2xw_a - 1)/(48x^2w_aW)$ gives

$$\alpha_{1,a}(x) = \frac{1}{4xW} \sum_{h=k_0}^{n-1} \bar{v}_h \bar{\alpha}_h + \frac{(2xw_a - 1)(3(2xw_a - 1) - 2(4xw_a - 1))}{48x^2w_aW},$$

that is,

$$\alpha_{1,a}(x) = \frac{1}{4xW} \sum_{h=k_0}^{n-1} \bar{v}_h \bar{\alpha}_h - \frac{(2xw_a - 1)(2xw_a + 1)}{48x^2w_aW}.$$

For the linear coefficient,

$$\alpha_{2,a}(x) = \frac{1}{4xW} \sum_{h=k_0}^{n-1} \bar{v}_h \bar{\beta}_h + \frac{2xw_a - 1}{2xW} \Psi_a(x, 0) - \frac{2xw_a - 1}{2xW} \Psi_a(x, 0).$$

Thus the two last terms cancel, and

$$\alpha_{2,a}(x) = \frac{1}{4xW} \sum_{h=k_0}^{n-1} \bar{v}_h \bar{\beta}_h.$$

For the constant term,

$$\begin{aligned} \alpha_{3,a}(x) &= \frac{1}{4xW} \left(\sum_{h=k_0}^{n-1} \bar{v}_h \bar{\gamma}_h + LB_{1,x}(k_0) \right) + \Psi_a(x, 0)^2 \\ &\quad - \frac{1}{12xW} \sum_{i=1}^{n-1} \frac{(v_i(a, x) - 1)(2v_i(a, x) - 1)}{v_i(a, x)} \bar{p}_i(a)^2 \\ &\quad + \frac{1}{8xW} \sum_{i=1}^{n-1} \frac{(v_i(a, x) - 1)^2}{v_i(a, x)} \bar{p}_i(a)^2 - \Psi_a(x, 0)^2. \end{aligned}$$

The $\Psi_a(x, 0)^2$ terms cancel. Moreover, for each $v_i = v_i(a, x)$,

$$-\frac{1}{12} \frac{(v_i - 1)(2v_i - 1)}{v_i} + \frac{1}{8} \frac{(v_i - 1)^2}{v_i} = \frac{1 - v_i^2}{24 v_i}.$$

Therefore,

$$\alpha_{3,a}(x) = \frac{1}{4xW} \left(\sum_{h=k_0}^{n-1} \bar{v}_h \bar{\gamma}_h + LB_{1,x}(k_0) \right) + \frac{1}{24xW} \sum_{i=1}^{n-1} \frac{1 - v_i(a, x)^2}{v_i(a, x)} \bar{p}_i(a)^2.$$

This proves the stated formulas. □

Proposition 2

Proof. For each $h = 1, \dots, n-1$, recall that

$$v_h(a, x) = \begin{cases} 2xw_h, & \text{if } h < a, \\ 2xw_{h+1}, & \text{if } h \geq a. \end{cases}$$

Hence, for every $k \in \{1, \dots, n-1\}$,

$$\sum_{h=k}^{n-1} v_h(a, x) = \begin{cases} 2x \left(\sum_{h=k}^{a-1} w_h + \sum_{h=a}^{n-1} w_{h+1} \right), & \text{if } k < a, \\ 2x \sum_{h=k}^{n-1} w_{h+1} = 2x \sum_{h=k+1}^n w_h, & \text{if } k \geq a. \end{cases} \quad (\text{A.16})$$

By the definition of $S_k(a)$, this can be written compactly as

$$\sum_{h=k}^{n-1} v_h(a, x) = 2x S_k(a), \quad k = 1, \dots, n-1. \quad (\text{A.17})$$

We first show that $k_0(a, x)$ is well defined for every $x \in \mathcal{X}$, and that the sets defining k_a^+ and k_a^- are nonempty. For $k = 1$, the tail contains all blocks except the one corresponding to job a , so

$$S_1(a) = W - w_a.$$

Since $w_a \leq W/2$, we have $W - w_a \geq w_a$, hence $S_1(a) \geq w_a \geq w_a - g$. Therefore both sets in (17) are nonempty, so k_a^+ and k_a^- are well defined.

Likewise,

$$\sum_{h=1}^{n-1} v_h(a, x) = 2x(W - w_a).$$

Therefore,

$$k_a(x) \leq 1 + \sum_{h=1}^{n-1} v_h(a, x) \iff 2xw_a \leq 1 + 2x(W - w_a) \iff 2x(2w_a - W) \leq 1.$$

Since $w_a \leq W/2$, the last inequality holds for every $x > 0$. Hence the set defining $k_0(a, x)$ is nonempty, and $k_0(a, x)$ is well defined for every admissible split level $x \in \mathcal{X}$. In particular, under the present assumption $w_a \leq W/2$, one has $\mathcal{X}_a = \mathcal{X}$.

Now let $x \in \mathcal{X}$. Since $w_i/g \in \mathbb{N}$ for all i , the condition $2xg \in \mathbb{N}$ implies $2xw_i \in \mathbb{N}$ for all i . Conversely, if $2xw_i \in \mathbb{N}$ for all i , then $2xg \in \mathbb{N}$ by Bezout's identity applied to $g = \gcd(w_1, \dots, w_n)$. Hence

$$x \in \mathcal{X} \iff 2xg \in \mathbb{N}.$$

Thus there exists an integer $m \geq 1$ such that

$$x = \delta m = \frac{m}{2g}.$$

Since $\delta = \frac{1}{2g}$, one has

$$2x = \frac{m}{g}.$$

Thus,

$$k_a(x) = 2xw_a = m \frac{w_a}{g}.$$

Introduce the normalized integer quantities

$$\widehat{w}_i := \frac{w_i}{g} \in \mathbb{N}, \quad \widehat{S}_k(a) := \frac{S_k(a)}{g} \in \mathbb{N}.$$

Using (A.17), for every $k \in \{1, \dots, n-1\}$,

$$\sum_{h=k}^{n-1} v_h(a, x) = 2x S_k(a) = m \widehat{S}_k(a).$$

Therefore, the defining inequality for $k_0(a, x)$ becomes

$$k_a(x) \leq 1 + \sum_{h=k}^{n-1} v_h(a, x) \iff m\widehat{w}_a \leq 1 + m\widehat{S}_k(a) \iff \widehat{S}_k(a) \geq \widehat{w}_a - \frac{1}{m}.$$

Hence

$$k_0(a, \delta m) = \max \left\{ k \in \{1, \dots, n-1\} : \widehat{S}_k(a) \geq \widehat{w}_a - \frac{1}{m} \right\}. \quad (\text{A.18})$$

We now distinguish two cases.

Case 1: $m = 1$, i.e. $x = \delta$. In this case, (A.18) yields

$$k_0(a, \delta) = \max \left\{ k \in \{1, \dots, n-1\} : \widehat{S}_k(a) \geq \widehat{w}_a - 1 \right\} = k_a^-.$$

Using the definition of $\kappa(a, x)$,

$$\kappa(a, \delta) = k_a(\delta) - 1 - \sum_{h=k_0(a, \delta)+1}^{n-1} v_h(a, \delta),$$

together with $k_a(\delta) = 2\delta w_a$ and (A.17),

$$\sum_{h=k_0(a, \delta)+1}^{n-1} v_h(a, \delta) = 2\delta S_{k_0(a, \delta)+1}(a),$$

we obtain

$$\kappa(a, \delta) = 2\delta w_a - 1 - 2\delta S_{k_0(a, \delta)+1}(a) = 2\delta(w_a - S_{k_a^-+1}(a)) - 1.$$

Case 2: $m \geq 2$, i.e. $x \in \mathcal{X} \setminus \{\delta\}$. If $m \geq 2$, then

$$\widehat{w}_a - 1 < \widehat{w}_a - \frac{1}{m} < \widehat{w}_a.$$

Since each $\widehat{S}_k(a)$ is an integer, the condition

$$\widehat{S}_k(a) \geq \widehat{w}_a - \frac{1}{m}$$

is equivalent to $\widehat{S}_k(a) \geq \widehat{w}_a$. Therefore, by (A.18),

$$k_0(a, \delta m) = \max \left\{ k \in \{1, \dots, n-1\} : \widehat{S}_k(a) \geq \widehat{w}_a \right\} = k_a^+.$$

Hence, for every $x = \delta m \in \mathcal{X} \setminus \{\delta\}$,

$$\kappa(a, x) = k_a(x) - 1 - \sum_{h=k_0(a, x)+1}^{n-1} v_h(a, x) = 2xw_a - 1 - 2x S_{k_a^++1}(a) = 2x(w_a - S_{k_a^++1}(a)) - 1.$$

□

Proposition 3

Proof. Fix $a \in \mathcal{N}$ and $x \in \mathcal{X}_a$, and let p^* be an interior stationary point satisfying $p^* > 0$, $R_{a,x}(p^*) > 0$ and $S_{a,x}(p^*) > 0$. Then $g_{a,x}$ is differentiable at p^* and $g'_{a,x}(p^*) = 0$. Since

$$g'_{a,x}(p) = \frac{2\alpha_{1,a}(x)p + \alpha_{2,a}(x)}{2\sqrt{R_{a,x}(p)}} - \frac{2c_{1,a}(x)p + c_{2,a}(x)}{2\sqrt{S_{a,x}(p)}},$$

we obtain

$$\frac{2\alpha_{1,a}(x)p^* + \alpha_{2,a}(x)}{\sqrt{R_{a,x}(p^*)}} = \frac{2c_{1,a}(x)p^* + c_{2,a}(x)}{\sqrt{S_{a,x}(p^*)}}.$$

This equality implies the sign condition in (18). Squaring the equality and clearing denominators gives

$$(2\alpha_{1,a}(x)p + \alpha_{2,a}(x))^2 S_{a,x}(p) - (2c_{1,a}(x)p + c_{2,a}(x))^2 R_{a,x}(p) = 0.$$

Expanding this expression and collecting powers of p yields precisely the quartic equation in (18), with coefficients $t_{a,x}$, $b_{a,x}$, $c_{a,x}$, $d_{a,x}$ and $e_{a,x}$ as displayed above. $\square$

Theorem 4

Proof. Fix $a \in \mathcal{N}$ and $x \in \mathcal{X}_a$. For simplicity, write

$$\alpha_j = \alpha_{j,a}(x), \quad s_j = c_{j,a}(x), \quad t = t_{a,x}, \quad b = b_{a,x}, \quad c = c_{a,x}, \quad d = d_{a,x}, \quad e = e_{a,x},$$

and

$$R(p) := \alpha_1 p^2 + \alpha_2 p + \alpha_3, \quad S(p) := s_1 p^2 + s_2 p + s_3.$$

By Corollary 1, $g_{a,x}(p) = \sqrt{R(p)} - \sqrt{S(p)}$. Let $p^* \in D_{a,x}$ be an interior stationary point of $g_{a,x}$ such that $p^* > 0$, $R(p^*) > 0$ and $S(p^*) > 0$. Since $g_{a,x}$ is differentiable at p^* , the first-order condition $g'_{a,x}(p^*) = 0$ holds. Differentiating gives

$$g'_{a,x}(p) = \frac{2\alpha_1 p + \alpha_2}{2\sqrt{R(p)}} - \frac{2s_1 p + s_2}{2\sqrt{S(p)}}.$$

Hence $g'_{a,x}(p^*) = 0$ is equivalent to

$$\frac{2\alpha_1 p^* + \alpha_2}{\sqrt{R(p^*)}} = \frac{2s_1 p^* + s_2}{\sqrt{S(p^*)}},$$

which in particular implies the sign constraint

$$(2\alpha_1 p^* + \alpha_2)(2s_1 p^* + s_2) \geq 0. \tag{A.19}$$

Squaring the previous equality and clearing denominators yields

$$(2\alpha_1 p + \alpha_2)^2 S(p) - (2s_1 p + s_2)^2 R(p) = 0.$$

After expansion and collection of powers of p , this becomes exactly the quartic equation in (18):

$$tp^4 + bp^3 + cp^2 + dp + e = 0.$$

Since $t \neq 0$, we may apply the standard depressing substitution

$$p = z - \frac{b}{4t},$$

which transforms the quartic into

$$z^4 + A_{a,x}z^2 + B_{a,x}z + C_{a,x} = 0, \quad (\text{A.20})$$

with

$$A_{a,x} = \frac{8tc - 3b^2}{8t^2}, \quad B_{a,x} = \frac{8t^2d + b^3 - 4tbc}{8t^3}, \quad C_{a,x} = \frac{256t^3e + 16tb^2c - 3b^4 - 64t^2bd}{256t^4}.$$

Ferrari's method consists in writing

$$z^4 + A_{a,x}z^2 + B_{a,x}z + C_{a,x} = (z^2 + \lambda)^2 - \Gamma_\lambda(z),$$

where

$$\Gamma_\lambda(z) := (2\lambda - A_{a,x})z^2 - B_{a,x}z + \lambda^2 - C_{a,x}.$$

The quadratic polynomial $\Gamma_\lambda(z)$ is a perfect square if and only if its discriminant vanishes, namely

$$B_{a,x}^2 - 4(2\lambda - A_{a,x})(\lambda^2 - C_{a,x}) = 0.$$

Expanding this condition yields the resolvent cubic

$$8\lambda^3 - 4A_{a,x}\lambda^2 - 8C_{a,x}\lambda + 4A_{a,x}C_{a,x} - B_{a,x}^2 = 0.$$

Let $\lambda_0(a, x)$ be a real root of this cubic such that $2\lambda_0(a, x) - A_{a,x} > 0$, and set

$$a_{a,x} := \sqrt{2\lambda_0(a, x) - A_{a,x}}.$$

Then

$$\Gamma_{\lambda_0(a,x)}(z) = \left(a_{a,x}z - \frac{B_{a,x}}{2a_{a,x}} \right)^2,$$

and therefore (A.20) factors as

$$z^4 + A_{a,x}z^2 + B_{a,x}z + C_{a,x} = \left(z^2 - a_{a,x}z + \lambda_0(a, x) + \frac{B_{a,x}}{2a_{a,x}} \right) \left(z^2 + a_{a,x}z + \lambda_0(a, x) - \frac{B_{a,x}}{2a_{a,x}} \right).$$

The discriminants of these two quadratic factors are

$$\Delta_{1,a,x} = a_{a,x}^2 - 4 \left(\lambda_0(a, x) + \frac{B_{a,x}}{2a_{a,x}} \right) = -2\lambda_0(a, x) - A_{a,x} - 2\frac{B_{a,x}}{a_{a,x}},$$

and

$$\Delta_{2,a,x} = a_{a,x}^2 - 4 \left(\lambda_0(a, x) - \frac{B_{a,x}}{2a_{a,x}} \right) = -2\lambda_0(a, x) - A_{a,x} + 2\frac{B_{a,x}}{a_{a,x}}.$$

Hence the real roots of the depressed quartic (A.20) are precisely the real elements collected in $\mathcal{Z}_{a,x}$, namely those obtained from the displayed Ferrari formulas for which the corresponding discriminant is nonnegative. Returning to the original variable,

$$p = z - \frac{b}{4t}.$$

Thus every interior stationary point p^* of $g_{a,x}$ belongs to the feasible Ferrari candidate set $\mathcal{F}_{a,x}$ and, by (A.19), satisfies the sign constraint.

Consequently, if a maximizer of $g_{a,x}$ over $D_{a,x}$ is attained at an interior point of $D_{a,x}$, then one such maximizer can be chosen among the feasible Ferrari candidates, so that

$$p_{a,x}^* \in \arg \max \{ g_{a,x}(p) : p \in \mathcal{F}_{a,x} \}.$$

Finally, a real root $\lambda_0(a, x)$ of the resolvent cubic can be obtained in closed form by Cardano's formula, with the usual distinction between the cases of nonnegative and negative cubic discriminant. In the nonnegative-discriminant case, setting

$$\lambda = y + \frac{A_{a,x}}{6}$$

reduces (19) to the depressed cubic

$$y^3 + R_{a,x}^{\text{cub}} y + S_{a,x}^{\text{cub}} = 0,$$

where

$$R_{a,x}^{\text{cub}} = -\frac{A_{a,x}^2 + 12C_{a,x}}{12}, \quad S_{a,x}^{\text{cub}} = -\frac{2A_{a,x}^3 - 72A_{a,x}C_{a,x} + 27B_{a,x}^2}{216}.$$

Define

$$\Delta_{a,x}^{\text{cub}} := \left(\frac{S_{a,x}^{\text{cub}}}{2} \right)^2 + \left(\frac{R_{a,x}^{\text{cub}}}{3} \right)^3.$$

If $\Delta_{a,x}^{\text{cub}} \geq 0$, one real solution is given by

$$y_0(a, x) = \sqrt[3]{-\frac{S_{a,x}^{\text{cub}}}{2} + \sqrt{\Delta_{a,x}^{\text{cub}}}} + \sqrt[3]{-\frac{S_{a,x}^{\text{cub}}}{2} - \sqrt{\Delta_{a,x}^{\text{cub}}}},$$

and therefore

$$\lambda_0(a, x) = y_0(a, x) + \frac{A_{a,x}}{6}.$$

When $\Delta_{a,x}^{\text{cub}} < 0$, the corresponding real roots are obtained from the standard trigonometric form of Cardano's formula, or equivalently by taking the real values generated by the complex Cardano branches. This completes the almost closed-form construction of the Ferrari candidates. $\square$

Theorem 5

Proof. For simplicity of notation, throughout the proof we suppress the superscript (n) and write $a = a_n$, $W = \sum_{i=1}^n w_i$, $k_a = 2xw_a$, $k_0 = k_0(a, x)$, $\kappa = \kappa(a, x)$, together with the corresponding quantities $v_i = v_i(a, x)$, $\bar{p}_i = \bar{p}_i(a)$, $\bar{v}_i = \bar{v}_i(a, x)$ and $\bar{\bar{v}}_i = \bar{\bar{v}}_i(a, x)$. By assumption (ii), $nw_- \leq W \leq nw_+$. In particular, for all sufficiently large n , the quantity W is bounded above and below by positive constant multiples of n . Moreover, since $w_a \leq w_+$ and $W \geq nw_-$, we have $w_a \leq W/2$ for all sufficiently large n . Hence, for all sufficiently large n , $\mathcal{X}_a = \mathcal{X}$, so that the quantities $k_0(a, x)$, $\kappa(a, x)$ and the coefficients in Corollary 1 are well defined for the fixed split level x .

Step 1: the tail indexed by $k_0, \dots, n-1$ has uniformly bounded length. Define

$$L := \max \left\{ 1, \left\lceil \frac{2xw_+ - 1}{2xw_-} \right\rceil \right\}.$$

For all sufficiently large n , one has $n - L \geq 1$. Since $k_a = 2xw_a \leq 2xw_+$ and $v_h \geq 2xw_-$ for every h ,

$$1 + \sum_{h=n-L}^{n-1} v_h \geq 1 + L(2xw_-) \geq 2xw_+ \geq k_a.$$

By the definition of $k_0(a, x)$, this implies $k_0 \geq n - L$. Therefore the set $\{k_0, \dots, n-1\}$ contains at most L indices, where L depends only on x, w_-, w_+ and not on n .

Step 2: bounds for α_1 and c_1 . By Corollary 1,

$$\alpha_1 = \frac{1}{4xW} \sum_{h=k_0}^{n-1} \bar{v}_h \bar{\alpha}_h - \frac{(2xw_a - 1)(2xw_a + 1)}{48x^2w_aW},$$

and

$$c_1 = \frac{(2xw_a - 1)^2(W - w_a)}{16x^2w_aW^2}.$$

Because x is fixed and $w_a \in [w_-, w_+]$, the quantity $k_a = 2xw_a$ is bounded above and below by positive constants independent of n . Moreover, for every $h \geq k_0$,

$$\sum_{t=h+1}^{n-1} v_t \leq \sum_{t=k_0+1}^{n-1} v_t = k_a - 1 - \kappa \leq k_a - 1,$$

so the factor

$$1 - \sum_{t=h+1}^{n-1} \frac{v_t}{k_a}$$

is uniformly bounded. Since also $\bar{v}_h \leq 2xw_+$, it follows from the definition of $\bar{\alpha}_h$ that there exists a constant $C_\alpha > 0$, independent of n , such that

$$|\bar{\alpha}_h| \leq C_\alpha \quad \text{for every } h = k_0, \dots, n-1.$$

Because there are at most L indices in $\{k_0, \dots, n-1\}$ and each $\bar{v}_h \leq 2xw_+$, there exists a constant $C_1 > 0$, independent of n , such that

$$\left| \sum_{h=k_0}^{n-1} \bar{v}_h \bar{\alpha}_h \right| \leq C_1.$$

Using $W \geq nw_-$, we obtain

$$\left| \frac{1}{4xW} \sum_{h=k_0}^{n-1} \bar{v}_h \bar{\alpha}_h \right| \leq \frac{C_1}{4xw_-} \frac{1}{n}.$$

For the second term in α_1 , since $w_a \in [w_-, w_+]$ and $W \geq nw_-$, there exists a constant $C_2 > 0$, independent of n , such that

$$\left| \frac{(2xw_a - 1)(2xw_a + 1)}{48x^2w_aW} \right| \leq \frac{C_2}{n}.$$

Therefore, for all sufficiently large n , there exists $C_3 > 0$ such that

$$|\alpha_1| \leq \frac{C_3}{n}.$$

Similarly, since $w_a \in [w_-, w_+]$ and $W \geq nw_-$, there exists a constant $C_4 > 0$ such that

$$|c_1| = \frac{(2xw_a - 1)^2(W - w_a)}{16x^2w_aW^2} \leq \frac{C_4}{n}$$

for all sufficiently large n .

Step 3: c_3 is bounded away from 0. Define

$$u_i := \frac{v_i - 1}{2v_i} \bar{p}_i, \quad i = 1, \dots, n-1, \quad \text{and} \quad u_0 := 0.$$

Also define

$$\lambda_0 := \frac{w_a}{W}, \quad \lambda_i := \frac{v_i}{2xW}, \quad i = 1, \dots, n-1.$$

Since

$$\sum_{i=1}^{n-1} v_i = \sum_{j \neq a} 2xw_j = 2x(W - w_a),$$

we have

$$\sum_{i=0}^{n-1} \lambda_i = \frac{w_a}{W} + \frac{1}{2xW} \sum_{i=1}^{n-1} v_i = 1.$$

Thus $(\lambda_i)_{i=0}^{n-1}$ is a probability vector. By Corollary 1,

$$c_3 = \frac{1}{8xW} \sum_{i=1}^{n-1} \frac{(v_i - 1)^2}{v_i} \bar{p}_i^2 - \Psi_a(x, 0)^2,$$

with

$$\Psi_a(x, 0) = \frac{1}{W} \sum_{i=1}^{n-1} \frac{v_i - 1}{4x} \bar{p}_i.$$

A direct calculation gives

$$\sum_{i=1}^{n-1} \lambda_i u_i^2 = \frac{1}{8xW} \sum_{i=1}^{n-1} \frac{(v_i - 1)^2}{v_i} \bar{p}_i^2, \quad \sum_{i=1}^{n-1} \lambda_i u_i = \Psi_a(x, 0),$$

and therefore

$$c_3 = \sum_{i=0}^{n-1} \lambda_i u_i^2 - \left(\sum_{i=0}^{n-1} \lambda_i u_i \right)^2 = \frac{1}{2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \lambda_i \lambda_j (u_i - u_j)^2.$$

Now let

$$L_n := \{i \in \{1, \dots, n-1\} : \bar{p}_i \leq p^L\}, \quad H_n := \{i \in \{1, \dots, n-1\} : \bar{p}_i \geq p^H\}.$$

By assumption (iii), both sets L_n and H_n have cardinality at least ηn for all sufficiently large n . Since $2xw_- \leq v_i \leq 2xw_+$ and since the map $z \mapsto \frac{z-1}{2z}$ is increasing and positive on $[2xw_-, 2xw_+]$, it follows that for every $i \in L_n$,

$$u_i = \frac{v_i - 1}{2v_i} \bar{p}_i \leq \frac{2xw_+ - 1}{4xw_+} p^L =: a_*,$$

whereas for every $j \in H_n$,

$$u_j = \frac{v_j - 1}{2v_j} \bar{p}_j \geq \frac{2xw_- - 1}{4xw_-} p^H =: b_*.$$

By (23), one has $b_* > a_*$. Define

$$\lambda(L_n) := \sum_{i \in L_n} \lambda_i, \quad \lambda(H_n) := \sum_{i \in H_n} \lambda_i.$$

Since $\lambda_i = \frac{v_i}{2xW} \geq \frac{w_-}{W}$ for every $i = 1, \dots, n-1$, we obtain

$$\lambda(L_n) \geq \frac{\eta n w_-}{W} \geq \frac{\eta w_-}{w_+}, \quad \lambda(H_n) \geq \frac{\eta n w_-}{W} \geq \frac{\eta w_-}{w_+}.$$

Therefore, for all sufficiently large n , there exists a constant

$$\rho := \frac{\eta w_-}{w_+} > 0$$

such that $\lambda(L_n) \geq \rho$ and $\lambda(H_n) \geq \rho$. Restricting the pairwise representation of c_3 to the two cross-blocks $(i, j) \in L_n \times H_n$ and $(i, j) \in H_n \times L_n$, we obtain

$$c_3 \geq \lambda(L_n)\lambda(H_n) (b_* - a_*)^2 \geq \rho^2(b_* - a_*)^2.$$

Hence, if we define

$$c_*(x) := \rho^2(b_* - a_*)^2 > 0, \tag{A.21}$$

then $c_3 \geq c_*(x)$ for all sufficiently large n .

Step 4: lower bound for α_3 . By Corollary 1,

$$\alpha_3 = \frac{1}{4xW} \left(\sum_{h=k_0}^{n-1} \bar{v}_h \bar{\gamma}_h + LB_{1,x}(k_0) \right) + \frac{1}{24xW} \sum_{i=1}^{n-1} \frac{1 - v_i^2}{v_i} \bar{p}_i^2.$$

We begin with $LB_{1,x}(k_0)$. Recall that

$$m_x = v_{k_0} - \kappa + 1 + \sum_{h=1}^{k_0-1} v_h.$$

Since $k_0 \geq n - L$, $0 \leq \kappa \leq v_{k_0} \leq 2xw_+$, and each v_h lies between $2xw_-$ and $2xw_+$, there exist constants $A_1, A_2 > 0$, independent of n , such that, for all sufficiently large n ,

$$A_1 n \leq m_x \leq A_2 n.$$

Also, each expanded term $\bar{q}_j$ entering the definition of $LB_{1,x}(k_0)$ is of the form

$$\bar{q}_j = \frac{\bar{p}_i}{v_i}$$

for some i , and by assumptions (i)–(ii),

$$\bar{q}_j \geq \frac{p_-}{2xw_+} =: q_* > 0.$$

Write $s_r := \bar{q}_1 + \dots + \bar{q}_r$, for all $r = 1, \dots, m_x$. Then $s_r \geq r q_*$ for every r . If m_x is odd, by definition,

$$LB_{1,x}(k_0) = \sum_{h=1}^{(m_x-1)/2} s_{2h}^2 \geq q_*^2 \sum_{h=1}^{(m_x-1)/2} (2h)^2.$$

If m_x is even, by definition,

$$LB_{1,x}(k_0) = \sum_{h=1}^{m_x/2} s_{2h-1}^2 \geq q_*^2 \sum_{h=1}^{m_x/2} (2h-1)^2.$$

Hence there exists a constant $c_{LB} > 0$, independent of n , such that

$$LB_{1,x}(k_0) \geq c_{LB} n^3 \tag{A.22}$$

for all sufficiently large n .

Next, since the set $\{k_0, \dots, n-1\}$ contains at most L indices, it is enough to bound each $\bar{\gamma}_h$ for $h \in \{k_0, \dots, n-1\}$. Define

$$\bar{N}_h := \sum_{t=1}^h \bar{p}_t + \frac{\bar{p}_h}{v_h}.$$

By assumption (i),

$$\bar{N}_h \leq hp_+ + \frac{p_+}{2xw_-} \leq np_+ + \frac{p_+}{2xw_-}.$$

Using the explicit formula for $\bar{\gamma}_h$ and the uniform bounds on v_h , $\bar{v}_h$ and $\bar{p}_h$, there exists a constant $C'_\gamma > 0$, independent of n , such that

$$|\bar{\gamma}_h| \leq C'_\gamma n^2 \quad \text{for every } h = k_0, \dots, n-1.$$

Since there are at most L such indices and $\bar{v}_h \leq 2xw_+$, there exists a constant $C_\gamma > 0$, independent of n , such that

$$\left| \sum_{h=k_0}^{n-1} \bar{v}_h \bar{\gamma}_h \right| \leq C_\gamma n^2$$

for all sufficiently large n . Further, because v_i is uniformly bounded above and below and $\bar{p}_i$ is uniformly bounded, each summand

$$\frac{1 - v_i^2}{v_i} \bar{p}_i^2$$

is bounded in absolute value by a constant independent of n . Since there are $n-1$ such terms and $W \geq nw_-$, there exists $C_0 > 0$, independent of n , such that

$$\left| \frac{1}{24xW} \sum_{i=1}^{n-1} \frac{1 - v_i^2}{v_i} \bar{p}_i^2 \right| \leq C_0$$

for all sufficiently large n .

Combining the previous bounds with (A.22), we obtain

$$\alpha_3 \geq \frac{1}{4xW} (2c_{LB}n^3 - C_\gamma n^2) - C_0.$$

Using $W \leq nw_+$ for the positive term and $W \geq nw_-$ for the negative one, we get

$$\alpha_3 \geq \frac{c_{LB}}{2xw_+} n^2 - \frac{C_\gamma}{4xw_-} n - C_0.$$

Therefore there exists a constant $C_*(x) > 0$, independent of n , such that

$$\alpha_3 \geq C_*(x) n^2 \tag{A.23}$$

for all sufficiently large n .

Step 5: conclusion. Combining (A.21) and (A.23), we obtain

$$\alpha_3 c_3 \geq C_*(x) c_*(x) n^2$$

for all sufficiently large n . Since

$$|\alpha_1| \leq \frac{C_3}{n} \quad \text{and} \quad |c_1| \leq \frac{C_4}{n},$$

it follows that

$$\left| \frac{\alpha_1}{\alpha_3 c_3} \right| \leq \frac{C_3/n}{C_*(x) c_*(x) n^2} = \frac{C_3}{C_*(x) c_*(x)} \frac{1}{n^3} \rightarrow 0,$$

and similarly

$$\left| \frac{c_1}{\alpha_3 c_3} \right| \leq \frac{C_4/n}{C_*(x) c_*(x) n^2} = \frac{C_4}{C_*(x) c_*(x)} \frac{1}{n^3} \rightarrow 0.$$

□

Corollary 2

Proof. For compactness of notation, set

$$A := \alpha_{2,a}(x), \quad B := \alpha_{3,a}(x), \quad C := c_{2,a}(x), \quad E := c_{3,a}(x),$$

and consider

$$f(p) := \tilde{g}_{a,x}(p) = \sqrt{Ap + B} - \sqrt{Cp + E}, \quad p \in \mathcal{D}.$$

By construction, the two radicands are nonnegative on $\mathcal{D}$. Since $\mathcal{D} = [\tilde{p}_-, \tilde{p}_+]$ is compact and f is continuous on $\mathcal{D}$, the maximum of f on $\mathcal{D}$ exists.

We first consider the monotone case $A \geq 0$. Since $C < 0$, the function $p \mapsto \sqrt{Ap + B}$ is nondecreasing on $\mathcal{D}$, while $p \mapsto -\sqrt{Cp + E}$ is also nondecreasing on $\mathcal{D}$. Moreover, because $C < 0$, the second term is strictly increasing on every nondegenerate subinterval on which $Cp + E > 0$. Hence f is nondecreasing on $\mathcal{D}$, and its maximum is attained at the right endpoint:

$$\arg \max_{p \in \mathcal{D}} f(p) = \{\tilde{p}_+\}.$$

In particular, since $C < 0$, if the right endpoint of $\mathcal{D}$ is generated by the constraint $Cp + E \geq 0$, then

$$\tilde{p}_+ = -\frac{E}{C} = -\frac{c_{3,a}(x)}{c_{2,a}(x)}.$$

It remains to analyze the case $A < 0$. Since $C < 0$, we have $AC > 0$. On every interior point of $\mathcal{D}$ at which both radicands are strictly positive, f is differentiable and

$$f'(p) = \frac{A}{2\sqrt{Ap + B}} - \frac{C}{2\sqrt{Cp + E}}.$$

An interior critical point must therefore satisfy

$$\frac{A}{\sqrt{Ap + B}} = \frac{C}{\sqrt{Cp + E}}.$$

Since $AC > 0$, the two sides of this first-order condition have the same sign. Squaring is therefore legitimate and yields

$$A^2(Cp + E) = C^2(Ap + B).$$

Equivalently,

$$AC(A - C)p = C^2B - A^2E.$$

Because $A < 0$, $C < 0$, and $A \neq C$ by assumption, we have $AC(A - C) \neq 0$. Hence there is at most one possible interior critical point, namely

$$p_{\text{crt}} = \frac{C^2B - A^2E}{AC(A - C)}.$$

We now determine when this critical point is a maximum. Suppose that

$$AC > 0, \quad |A| < |C|, \quad p_{\text{crt}} \in \text{int}(\mathcal{D}).$$

Since p_{crt} is an interior point and A and C are nonzero, both radicands are strictly positive at p_{crt} . Define

$$Q(p) := A^2(Cp + E) - C^2(Ap + B).$$

Using the expression for p_{crt} , we obtain

$$Q(p) = AC(A - C)(p - p_{\text{crt}}).$$

Let

$$s := \text{sign}(A) = \text{sign}(C).$$

For every interior point of $\mathcal{D}$ at which the radicands are strictly positive, the sign of $f'(p)$ is the sign of $sQ(p)$. Indeed,

$$f'(p) = \frac{s}{2} \left(\frac{|A|}{\sqrt{Ap+B}} - \frac{|C|}{\sqrt{Cp+E}} \right),$$

and the expression in parentheses is positive if and only if

$$A^2(Cp+E) > C^2(Ap+B),$$

that is, if and only if $Q(p) > 0$. Since $AC > 0$ and $|A| < |C|$, we have

$$sAC(A-C) < 0.$$

Consequently,

$$f'(p) > 0 \quad \text{for } p < p_{\text{crt}}, \quad f'(p) < 0 \quad \text{for } p > p_{\text{crt}},$$

whenever the derivative is defined. By continuity, f increases up to p_{crt} and decreases after p_{crt} on the feasible interval. Hence p_{crt} is the unique global maximizer of f on $\mathcal{D}$, and therefore

$$\arg \max_{p \in \mathcal{D}} f(p) = \{p_{\text{crt}}\}.$$

For completeness, the same conclusion can also be read from the second derivative. At an interior critical point,

$$\frac{|A|}{\sqrt{Ap_{\text{crt}}+B}} = \frac{|C|}{\sqrt{Cp_{\text{crt}}+E}} =: M > 0.$$

Since

$$f''(p) = -\frac{A^2}{4(Ap+B)^{3/2}} + \frac{C^2}{4(Cp+E)^{3/2}},$$

we obtain

$$f''(p_{\text{crt}}) = \frac{M^3}{4} \left(\frac{1}{|C|} - \frac{1}{|A|} \right).$$

Thus

$$f''(p_{\text{crt}}) < 0 \quad \Longleftrightarrow \quad |A| < |C|,$$

which is consistent with the preceding sign-change argument.

We finally consider the remaining cases with $A < 0$. Since $\mathcal{D}$ is compact and f is continuous, a maximizer exists. Suppose that a maximizer is attained at an interior point of $\mathcal{D}$. Because $A < 0$ and $C < 0$, a zero of one of the affine radicands can only occur at an endpoint of the interval determined by the corresponding nonnegativity constraint. Hence both radicands are strictly positive at every interior maximizer, and f is differentiable there. Such an interior maximizer must therefore be an interior critical point. The preceding analysis shows that, outside the case

$$AC > 0, \quad |A| < |C|, \quad p_{\text{crt}} \in \text{int}(\mathcal{D}),$$

there is either no interior critical point, or the only possible interior critical point does not belong to $\text{int}(\mathcal{D})$, or it is not a local maximum. Therefore no remaining case can have a global maximizer strictly inside $\mathcal{D}$. The maximum must then be attained at an endpoint, and

$$\arg \max_{p \in \mathcal{D}} f(p) = \arg \max_{p \in \{\bar{p}_-, \bar{p}_+\}} f(p).$$

Substituting back $A = \alpha_{2,a}(x)$, $B = \alpha_{3,a}(x)$, $C = c_{2,a}(x)$, and $E = c_{3,a}(x)$ gives the stated result. $\square$

B Mathematical simplifications

B.1 Algebraical simplification for $LB_{1,x}(k_0)$.

It is important to note that the direct computation of $LB_{1,x}(k_0)$ requires summing $\mathcal{O}(2xW)$ terms. After simplification, this can be rewritten as a sum of $\mathcal{O}(n)$ terms. We have the reformulation

$$LB_{1,x}(k_0) = \sum_{i=1}^{k_0} r_i \left[P_{i-1}^2 + 2(r_i + \theta_i) P_{i-1} \frac{\bar{p}_i}{v_i} + s_i \left(\frac{\bar{p}_i}{v_i} \right)^2 \right],$$

where $P_j = \sum_{h=1}^j \bar{p}_h$, with $P_0 = 0$, and for each $j = 1, \dots, k_0$,

$$\begin{cases} \theta_j = m_x - 2\tilde{r}_x + 2(r_1 + \dots + r_{j-1}) - (\bar{v}_1 + \dots + \bar{v}_{j-1}), \\ s_j = \frac{2}{3}(r_j - 1)(2r_j - 1) + 2(r_j - 1)(1 + \theta_j) + (1 + \theta_j)^2. \end{cases}$$

The coefficients r_j are defined recursively by

$$r_1 = \left\lfloor \frac{\bar{v}_1}{2} \right\rfloor, \quad r_j = \left\lfloor \frac{\bar{v}_1 + \dots + \bar{v}_j}{2} \right\rfloor - (r_1 + \dots + r_{j-1}), \quad j = 2, \dots, k_0.$$

B.2 Algebraical simplification for Example 4

For the fixed split level $x = \frac{1}{2}$, the generator (25) satisfies the ordering convention

$$\frac{p_1}{w_1} \leq \dots \leq \frac{p_n}{w_n},$$

because

$$\frac{p_1^{(n)}}{w_1^{(n)}} = \rho < \frac{p^L}{4} < \frac{p^H}{4} = \frac{p_i^{(n)}}{w_i^{(n)}} \quad (i \geq 2).$$

Since $a_n = 1$, the reparametrized data of Corollary 1 are particularly simple: for $i = 1, \dots, 2m$, $v_i(a_n, \frac{1}{2}) = 4$, and

$$\bar{p}_i(a_n) = \begin{cases} p^L, & i = 1, \dots, m, \\ p^H, & i = m + 1, \dots, 2m. \end{cases}$$

Moreover,

$$k_a(\frac{1}{2}) = 2 \cdot \frac{1}{2} w_{a_n}^{(n)} = 3, \quad k_0(a_n, \frac{1}{2}) = 2m = n - 1, \quad \kappa(a_n, \frac{1}{2}) = 2.$$

Hence

$$\bar{v}_{2m}(a_n, \frac{1}{2}) = 2, \quad \bar{\bar{v}}_{2m}(a_n, \frac{1}{2}) = 3.$$

The quantity $\Psi_{a_n}(x, 0)$ becomes

$$\Psi_{a_n}(\frac{1}{2}, 0) = \frac{1}{W^{(n)}} \sum_{i=1}^{2m} \frac{v_i(a_n, \frac{1}{2}) - 1}{4(\frac{1}{2})} \bar{p}_i(a_n) = \frac{3m}{2W^{(n)}} (p^L + p^H).$$

For the coefficient $c_{1,a_n}^{(n)}(x)$, Corollary 1 gives

$$c_{1,a_n}^{(n)}(x) = \frac{(2xw_{a_n}^{(n)} - 1)^2 (W^{(n)} - w_{a_n}^{(n)})}{16x^2 w_{a_n}^{(n)} (W^{(n)})^2}.$$

Since $x = \frac{1}{2}$ and $w_{a_n}^{(n)} = 3$, this simplifies to

$$c_{1,a_n}^{(n)}\left(\frac{1}{2}\right) = \frac{8m}{3(8m+3)^2}.$$

For the coefficient $\alpha_{1,a_n}^{(n)}(x)$, Corollary 1 gives

$$\alpha_{1,a_n}^{(n)}(x) = \frac{1}{4xW^{(n)}} \sum_{h=k_0(a_n,x)}^{2m} \bar{v}_h(a_n, x) \bar{\alpha}_h(a_n, x) - \frac{(2xw_{a_n}^{(n)} - 1)(2xw_{a_n}^{(n)} + 1)}{48x^2 w_{a_n}^{(n)} W^{(n)}}.$$

Here only the last index contributes. Since $k_0 = 2m$, $\bar{v}_{2m} = 2$, $k_a = 3$, and

$$\bar{\alpha}_{2m}\left(a_n, \frac{1}{2}\right) = 1 - \frac{\bar{v}_{2m} + 1}{k_a} + \frac{(\bar{v}_{2m} + 1)(2\bar{v}_{2m} + 1)}{6k_a^2} = 1 - \frac{3}{3} + \frac{15}{54} = \frac{5}{18},$$

it follows that

$$\sum_{h=k_0(a_n, \frac{1}{2})}^{2m} \bar{v}_h\left(a_n, \frac{1}{2}\right) \bar{\alpha}_h\left(a_n, \frac{1}{2}\right) = 2 \cdot \frac{5}{18} = \frac{5}{9}.$$

Therefore,

$$\alpha_{1,a_n}^{(n)}\left(\frac{1}{2}\right) = \frac{1}{2W^{(n)}} \cdot \frac{5}{9} - \frac{(3-1)(3+1)}{48(\frac{1}{2})^2 \cdot 3W^{(n)}} = \frac{5}{18W^{(n)}} - \frac{2}{9W^{(n)}} = \frac{1}{18W^{(n)}} = \frac{1}{18(8m+3)}.$$

For the coefficient $c_{3,a_n}^{(n)}(x)$,

$$c_{3,a_n}^{(n)}(x) = \frac{1}{8xW^{(n)}} \sum_{i=1}^{2m} \frac{(v_i(a_n, x) - 1)^2}{v_i(a_n, x)} \bar{p}_i(a_n)^2 - (\Psi_{a_n}(x, 0))^2.$$

Since $v_i(a_n, \frac{1}{2}) = 4$ for all i ,

$$c_{3,a_n}^{(n)}\left(\frac{1}{2}\right) = \frac{9m}{16W^{(n)}} ((p^L)^2 + (p^H)^2) - \frac{9m^2}{4(W^{(n)})^2} (p^L + p^H)^2.$$

Equivalently,

$$c_{3,a_n}^{(n)}\left(\frac{1}{2}\right) = \frac{9m}{16(8m+3)^2} [4m(p^H - p^L)^2 + 3((p^L)^2 + (p^H)^2)].$$

To express $\alpha_{3,a_n}^{(n)}(x)$, first note that

$$LB_{1,x}(k_0) = \sum_{h=1}^{2m} \left(2h \cdot \frac{p^L}{4}\right)^2 + \sum_{r=1}^{2m-1} \left(mp^L + r\frac{p^H}{2}\right)^2,$$

that is,

$$LB_{1,\frac{1}{2}}(2m) = \frac{m(2m+1)(4m+1)}{12} (p^L)^2 + m^2(2m-1)p^L p^H + \frac{m(2m-1)(4m-1)}{12} (p^H)^2 + m^2(2m-1)(p^L)^2.$$

Equivalently,

$$LB_{1, \frac{1}{2}}(2m) = \frac{m(32m^2 - 6m + 1)}{12}(p^L)^2 + m^2(2m - 1)p^L p^H + \frac{m(2m - 1)(4m - 1)}{12}(p^H)^2.$$

Next, define

$$\Gamma_m := \sum_{h=k_0(a_n, \frac{1}{2})}^{2m} \bar{v}_h(a_n, \frac{1}{2}) \bar{\gamma}_h(a_n, \frac{1}{2}).$$

Since again only the last block contributes, one obtains

$$\Gamma_m = 2 \left[\left(m(p^L + p^H) + \frac{p^H}{4} \right)^2 - \frac{3}{4} \left(m(p^L + p^H) + \frac{p^H}{4} \right) p^H + \frac{5}{32} (p^H)^2 \right],$$

which simplifies to

$$\Gamma_m = 2m^2(p^L + p^H)^2 - \frac{m}{2} p^H(p^L + p^H) + \frac{(p^H)^2}{16}.$$

Therefore, using the corrected formula from Corollary 1,

$$\alpha_{3, a_n}^{(n)}(x) = \frac{1}{4xW^{(n)}} (\Gamma_m + LB_{1, x}(k_0)) + \frac{1}{24xW^{(n)}} \sum_{i=1}^{2m} \frac{1 - v_i(a_n, x)^2}{v_i(a_n, x)} \bar{p}_i(a_n)^2.$$

Since $x = \frac{1}{2}$ and $v_i(a_n, \frac{1}{2}) = 4$,

$$\alpha_{3, a_n}^{(n)}\left(\frac{1}{2}\right) = \frac{\Gamma_m + LB_{1, \frac{1}{2}}(2m)}{2(8m + 3)} - \frac{5m}{16(8m + 3)}((p^L)^2 + (p^H)^2).$$

C Supplementary comparison of exact and asymptotic p -optimization

This appendix extends Example 5 by allowing the split level x to vary over a finite grid. We compare exact optimization with respect to the artificial processing-time parameter p with the approximation scheme of Corollary 2, and assess how the choice of p affects the resulting global lower bound. The comparison uses the same 6-job instance, originally given by $\mathbf{p} = (36, 45, 54, 72, 96, 1200)$ and $\mathbf{w} = (3, 5, 6, 9, 6, 4)$. Following the indexing convention in (1), we sort the jobs by nondecreasing ratio p_i/w_i , obtaining

$$(p_1, \dots, p_6) = (72, 45, 54, 36, 96, 1200), \quad (w_1, \dots, w_6) = (9, 5, 6, 3, 6, 4).$$

Throughout this appendix, a denotes the first job according to this sorted indexing. The total weight and total processing time are $W = 33$ and $MS = 1503$. Since $\gcd(w_1, \dots, w_6) = 1$, the minimum admissible split level is $\delta = 1/2$. We restrict the split-level search to

$$\mathcal{X}_{\text{grid}} = \mathcal{X} \cap (0, 20] = \{0.5, 1, 1.5, \dots, 20\}.$$

For this instance, $w_a < W/2$ for every $a \in \mathcal{N}$, so $\mathcal{X}_a = \mathcal{X}$ and the same grid is admissible for every first-job choice.

For each pair (a, x) , the numerical optimization of the exact objective $g_{a,x}(p)$ evaluates the admissible stationary candidates generated by (18), together with the lower endpoint $\underline{p}_a(x)$. The approximation instead selects $\tilde{p}_{a,x}^*$ using Corollary 2. Both procedures evaluate the original lower-bound expression $LB_a^{(NN)}(x, p)$; the approximation concerns the selection of p . For each first job a , define

$$F_a^{\text{ex}} = \max_{x \in \mathcal{X}_{\text{grid}}} \max_{p \in \mathcal{P}_a(x)} LB_a^{(NN)}(x, p) = \max_{x \in \mathcal{X}_{\text{grid}}} LB_a^{(NN)}(x, p_{a,x}^*),$$

$$F_a^{\text{ap}} = \max_{x \in \mathcal{X}_{\text{grid}}} LB_a^{(NN)}(x, \tilde{p}_{a,x}^*).$$

The relative loss from approximation-based selection is

$$\text{Loss}_a(\%) = 100 \frac{F_a^{\text{ex}} - F_a^{\text{ap}}}{F_a^{\text{ex}}}.$$

Table C.1 reports these values and the corresponding choices of x and p for each first job.

Table C.1: Comparison of exact and asymptotic p -optimization over $\mathcal{X}_{\text{grid}}$ for the instance of Example 5. Job indices follow the nondecreasing ratios p_i/w_i .

a	p_a	x_{ex}^*	p_{ex}^*	F_a^{ex}	x_{ap}^*	p_{ap}^*	F_a^{ap}	$\text{Loss}_a(\%)$
1	72	0.5	1087.781	42216.809	0.5	1190.795	42036.648	0.43
2	45	0.5	1500.000	37115.858	0.5	2122.891	32274.885	13.04
3	54	0.5	1214.869	39014.980	0.5	1743.808	36144.981	7.36
4	36	0.5	900.000	24281.131	0.5	4110.226	12030.696	50.45
5	96	0.5	1210.388	33710.003	0.5	1846.256	30033.336	10.91
6	1200	0.5	126.523	7171.792	1.0	116.941	7086.983	1.18

Following Theorem 3, taking the minimum over the first job gives a valid global lower bound for each procedure. On the tested split-level grid, exact p -optimization gives

$$\min_{a \in \mathcal{N}} F_a^{\text{ex}} = 7171.792.$$

The binding choice is $a = 6$, corresponding to the job with original processing time $p_6 = 1200$, with $x_{\text{ex}}^* = 0.5$ and $p_{\text{ex}}^* = 126.523$. Approximation-based selection gives

$$\min_{a \in \mathcal{N}} F_a^{\text{ap}} = 7086.983,$$

with the same binding first job, but with $x_{\text{ap}}^* = 1$ and $p_{\text{ap}}^* = 116.941$. Thus, after optimizing over the tested split levels and taking the minimum over a , the approximation-based lower bound is

$$100 \frac{7171.792 - 7086.983}{7171.792} \approx 1.18\%$$

below the value obtained by exact p -optimization. The larger loss of 50.45% for $a = 4$ occurs because the approximate rule selects the right endpoint of the reduced feasible interval, whereas the exact objective $g_{a,x}(p)$ is maximized much closer to the admissible lower bound. This first-job choice is nonbinding under both procedures. Consequently, the loss in the global lower bound is determined by the binding case $a = 6$, for which the exact and approximation-based values remain close.