

Reinforcing Old Fences Gives New Facets

and More

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Abstract

The unpublished 1990 technical report reproduced here introduced the t -reinforced k -fence inequalities, the first facets of the acyclic subgraph polytope that are not rank inequalities: each simple k -fence induces $k - 2$ of them, one for each $1 \leq t \leq k - 2$. The report proved this with the spectral method, using a coherent configuration and computer algebra. It went unpublished because referees preferred conventional proofs! Here, we reproduce the report with minor corrections, but otherwise with the same appearance. We then develop the spectral method further. We extend Theorem 4.4 of [Lee and Lee \(1991\)](#) to concordant polytopes that are not balanced, compute the completely reduced form of the fence configuration, and obtain a short proof of the facet result that needs no symbolic computation. We determine the dimension and a maximal irredundant equation system of every face of the acyclic subgraph polytope of a simple fence that shares its symmetry, and show that the symmetric facets are exactly the t -reinforced inequalities. Finally, we show that each block rank of the completely reduced form is the dimension of the span of a few explicit statistics, and use this to determine the symmetric faces of the linear ordering polytope and to treat an unbalanced clique face of the forest polytope.

Keywords: acyclic subgraph polytope; linear ordering polytope; fence inequality; spectral method; coherent configuration.

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1 The Acyclic Subgraph Polytope

Let $G = (N, A)$ be a strict directed graph with *node* set N and *arc* set $A \subseteq N \times N$. Let w_{ij} be the weight of arc (i, j) . For $\tilde{N} \subseteq N$ and $\tilde{A} \subseteq A \cap (\tilde{N} \times \tilde{N})$, the graph $\tilde{G} = (\tilde{N}, \tilde{A})$ is a *subgraph* of G . The graph $\tilde{G}$ is *acyclic* if it contains no directed cycles. For other graph-theoretic terminology, see Bondy and Murty (1976). The (*weighted*) *acyclic subgraph problem* is the problem of finding a maximum-weight acyclic subgraph of $G = (N, A)$, and can be formulated as:

$$\begin{aligned} \mathcal{AC}(G) : \quad & \max \sum_{(i,j) \in A} w_{ij} x_{ij} \\ & \text{s.t.} \\ & \sum_{(i,j) \in C} x_{ij} \leq |C| - 1 && \text{for all directed cycles } C \text{ of } G, \quad (1) \\ & 0 \leq x_{ij} \leq 1 && \text{for all arcs } (i, j) \text{ in } A, \quad (2) \\ & x_{ij} \text{ integer} && \text{for all arcs } (i, j) \text{ in } A. \quad (3) \end{aligned}$$

This problem arises in the study of systems with feedback (see, for example, Ramachandran (1988)); if $\tilde{G} = (\tilde{N}, \tilde{A})$ is an acyclic subgraph of G , then $B = A \setminus \tilde{A}$ is a *feedback arc-set*, and, in applications, it is desirable to find one of small size. Grötschel, Jünger and Reinelt (1985a) discuss the relationship of $\mathcal{AC}(G)$ to several other important problems.

Let $\mathcal{P}_{AC}(G) \subseteq \mathbf{R}^A$ be the convex hull of the feasible solutions of $\mathcal{AC}(G)$. The *dimension* of a polytope $\mathcal{P} \subseteq \mathbf{R}^A$, denoted $\dim(\mathcal{P})$, is one less than the maximum number of affinely independent points in $\mathcal{P}$. It is easy to see that $\mathcal{P}_{AC}(G)$ is *full-dimensional*, that is, $\dim(\mathcal{P}_{AC}(G)) = |A|$. For $a \in \mathbf{R}^A, a \neq 0, a_0 \in \mathbf{R}$, the inequality $a^T x \leq a_0$ is *valid* for $\mathcal{P}$ if $\mathcal{P} \subseteq \{x \in \mathbf{R}^A : a^T x \leq a_0\}$. A valid inequality $a^T x \leq a_0$ represents a *facet* of $\mathcal{P}$ if $\dim(\mathcal{P} \cap \{x \in \mathbf{R}^A : a^T x = a_0\}) = \dim(\mathcal{P}) - 1$.

We are interested in studying the polyhedral structure of $\mathcal{P}_{AC}(G)$ and this paper presents a collection of facets of $\mathcal{P}_{AC}(G)$, which we call *t-reinforced k-fence inequalities*, that are induced by subgraphs of G known as *k-fences*. These facets generalize the *k-fence inequalities* introduced by Grötschel, Jünger and Reinelt (1985a). In many other combinatorial problems defined on graphs, facets of the associated polytope are often induced by subgraphs. The novelty of the facets introduced in this paper is that every *k-fence* induces not one but a family of facets of $\mathcal{P}_{AC}(G)$. In addition, these are the first facets of $\mathcal{P}_{AC}(G)$ to be discovered that are not represented by rank inequalities. (A *rank inequality* for $\mathcal{P}_{AC}(G)$ is an inequality of the form $\sum_{e \in S} x_e \leq r(S)$ where $r(S)$ is the maximum number of arcs in $S \subseteq A$ that comprise the edge-set of an acyclic subgraph of G .)

Grötschel, Jünger and Reinelt (1985a) investigated the acyclic subgraph polytope, determined when the inequalities (1) and (2) represent facets and introduced several other families of facets, all of which are represented by rank inequalities.

For every integer $k \geq 3$, a directed graph is a *simple k-fence* if it is isomorphic to $D = (U \cup W, F_1 \cup F_2)$, where

$$U = \{u_1, u_2, \dots, u_k\} \text{ and } W = \{w_1, w_2, \dots, w_k\} \quad (4)$$

$$\text{with } U \cap W = \emptyset,$$

$$\text{and } F_1 = \{(u_1, w_1), (u_2, w_2), \dots, (u_k, w_k)\}, \quad (5)$$

$$F_2 = \bigcup_{i=1}^k \{(w_i, v) \mid v \in U \setminus \{u_i\}\}. \quad (6)$$

Figure 1 illustrates a simple 3-fence. Following Grötschel, Jünger and Reinelt (1985a), we call the elements of F_1 *pales* and the elements of F_2 *pickets*.

If $G' = (N', A')$ is a directed graph with $(i, j) \in A'$, then the graph $G'' = (N' \cup \{h\}, A' \cup \{(i, h), (h, j)\} \setminus \{(i, j)\})$ is a graph obtained by subdividing arc (i, j) .

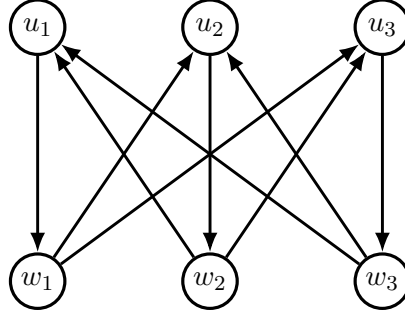


Figure 1: A simple 3-fence.

A k -fence is a directed graph $D' = (V', F'_1 \cup F'_2)$ obtained by repeated subdivision of arcs of a simple k -fence $D = (V, F_1 \cup F_2)$. The *pales* (respectively *pickets*) of D' are the arcs obtained from subdividing the pales (pickets) of D .

If $D' = (V', F'_1 \cup F'_2)$ is a k -fence that is a subgraph of G , then for $1 \leq t \leq k - 2$, we call the inequality

$$t \sum_{(i,j) \in F'_1} x_{ij} + \sum_{(i,j) \in F'_2} x_{ij} \leq t|F'_1| + |F'_2| - tk + \frac{t(t+1)}{2} \quad (7)$$

a t -reinforced k -fence inequality. The k -fence inequalities introduced by Grötschel, Jünger and Reinelt (1985a) are 1-reinforced k -fence inequalities.

The main result in this paper is

Proposition 1 *For $1 \leq t \leq k - 2$, let $D' = (V', F'_1 \cup F'_2)$ be a k -fence that is a subgraph of $G = (N, A)$. Then the t -reinforced k -fence inequality represents a facet of $\mathcal{P}_{AC}(G)$.*

Note that each k -fence induces $k - 2$ facets. The special case of Proposition 1 when $t = 1$ is established in Grötschel, Jünger and Reinelt (1985a). The case when $t = 2$ is related to a result of Leung (1990).

To prove Proposition 1, we first apply spectral methods for determining the

dimension of a polytope (see Lee and Lee (1990)) to show that (7) represents a facet when $G = D' = D$, that is, when the graph G is a simple k -fence and the facet-representing inequality is induced by the same simple k -fence. We then use an ‘extension’ argument to prove the result for arbitrary G . The combination of such arguments is new.

In Section 4, we show that the t -reinforced k -fence inequalities induced by *simple* k -fences also represent facets of the linear ordering polytope. For $t = 1$, this reduces to a result of Grötschel, Jünger and Reinelt (1985b).

2 Just for Fences

In this section, we demonstrate how a mild extension of the spectral method in Lee and Lee (1990) can be used to prove that (7) represents a facet of $\mathcal{P}_{AC}(D')$ when D' is the simple k -fence D .

Proposition 2 *For $1 \leq t \leq k-2$, the inequality (7) represents a facet of $\mathcal{P}_{AC}(D)$ where $D = (U \cup W, F_1 \cup F_2)$ is a simple k -fence.*

Proof. Let $\mathcal{F}_{AC}(D) = \text{conv}\{x \in \mathcal{P}_{AC}(D) : x \text{ satisfies (7) as an equation}\}$. The extreme points of $\mathcal{F}_{AC}(D)$ are defined by:

$$x_e = \begin{cases} 0 & \text{if } e \in F_1 \setminus \{(u_{i_j}, w_{i_j}) : j = 1, \dots, t\} \\ 0 & \text{if } e = (w_{i_j}, u_{i_h}) \text{ for } 1 \leq h < j \leq t, \\ 1 & \text{otherwise,} \end{cases} \quad (8)$$

for every ordered t -tuple $(i_1, i_2, \dots, i_t)$ of distinct integers satisfying $1 \leq i_j \leq k$, for $j = 1, \dots, t$, and by:

$$x_e = \begin{cases} 0 & \text{if } e \in F_1 \setminus \{(u_{i_j}, w_{i_j}) : j = 1, \dots, t+1\} \\ 0 & \text{if } e = (w_{i_j}, u_{i_h}) \text{ for } 1 \leq h < j \leq t+1, \\ 1 & \text{otherwise,} \end{cases} \quad (9)$$

for every ordered $(t+1)$ -tuple $(i_1, i_2, \dots, i_{t+1})$ of distinct integers satisfying $1 \leq i_j \leq k$, for $j = 1, \dots, t+1$. Let the rows of matrix $\mathcal{A}$ be the $\frac{k!}{(k-t)!} + \frac{k!}{(k-t-1)!}$ extreme points of $\mathcal{F}_{AC}(D)$. The *spectrum* of $\mathcal{F}_{AC}(D)$ is the (non-decreasing) list of eigenvalues of $B = \hat{\mathcal{A}}^T \hat{\mathcal{A}}$, where $\hat{\mathcal{A}} = [\mathcal{A} \mid \mathbf{1}]$. The number of non-zero elements in the spectrum is one more than the dimension of $\mathcal{F}_{AC}(D)$.

Let $\hat{E} = F_1 \cup F_2 \cup \{\mathbf{1}\}$. The incidence relations of equivalence classes of pairs of columns of $\hat{\mathcal{A}}$ determine a partition $R = (R_i : i \in T)$ of $\hat{E} \times \hat{E}$, where $T = \{1, 2, \dots, 20\}$. Specifically,

$$\begin{aligned}
R_1 &= \{(\mathbf{1}, \mathbf{1})\} \\
R_2 &= \{(\mathbf{1}, e) : e \in F_1\} \\
R_3 &= \{(\mathbf{1}, f) : f \in F_2\} \\
R_4 &= \{(e, e) : e \in F_1\} \\
R_5 &= \{(e, \mathbf{1}) : e \in F_1\} \\
R_6 &= \{(e_1, e_2) : e_1 \neq e_2, e_1, e_2 \in F_1\} \\
R_7 &= \{(e, f) : e = (u_h, w_h) \in F_1, f = (w_h, u_j) \in F_2 \text{ for some } h \neq j, 1 \leq h, j \leq k\} \\
R_8 &= \{(e, f) : e = (u_h, w_h) \in F_1, f = (w_j, u_h) \in F_2 \text{ for some } h \neq j, 1 \leq h, j \leq k\} \\
R_9 &= \{(e, f) : e = (u_h, w_h) \in F_1, f = (w_i, u_j) \in F_2 \text{ for distinct } h, i, j, \text{ with } 1 \leq h, i, j \leq k\} \\
R_{10} &= \{(f, f) : f \in F_2\} \\
R_{11} &= \{(f, \mathbf{1}) : f \in F_2\} \\
R_{12} &= \{(f, e) : f = (w_j, u_h) \in F_2, e = (u_h, w_h) \in F_1, \text{ for some } h \neq j, 1 \leq h, j \leq k\} \\
R_{13} &= \{(f, e) : f = (w_h, u_j) \in F_2, e = (u_h, w_h) \in F_1, \text{ for some } h \neq j, 1 \leq h, j \leq k\} \\
R_{14} &= \{(f, e) : f = (w_i, u_j) \in F_2, e = (u_h, w_h) \in F_1, \text{ for distinct } h, i, j, \text{ with } 1 \leq h, i, j \leq k\} \\
R_{15} &= \{(f_1, f_2) : f_1 = (w_h, u_i) \in F_2, f_2 = (w_h, u_j) \in F_2 \text{ for distinct } h, i, j, \text{ with } 1 \leq h, i, j \leq k\} \\
R_{16} &= \{(f_1, f_2) : f_1 = (w_i, u_h) \in F_2, f_2 = (w_j, u_h) \in F_2 \text{ for distinct } h, i, j, \text{ with } 1 \leq h, i, j \leq k\} \\
R_{17} &= \{(f_1, f_2) : f_1 = (w_i, u_h) \in F_2, f_2 = (w_h, u_j) \in F_2 \text{ for distinct } h, i, j, \text{ with } 1 \leq h, i, j \leq k\} \\
R_{18} &= \{(f_1, f_2) : f_1, f_2 \in F_2 \text{ with } f_1 \text{ and } f_2 \text{ non-adjacent}\} \\
R_{19} &= \{(f_1, f_2) : f_1 = (w_h, u_j) \in F_2, f_2 = (w_j, u_h) \in F_2 \text{ for some } h \neq j, 1 \leq h, j \leq k\} \\
R_{20} &= \{(f_1, f_2) : f_1 = (w_h, u_i) \in F_2, f_2 = (w_j, u_h) \in F_2 \text{ for distinct } h, i, j, \text{ with } 1 \leq h, i, j \leq k\}
\end{aligned}$$

Let $I = \{1, 4, 10\}$. Note that $\cup_{i \in I} R_i$ is the identity relation $\{(e, e) : e \in \hat{E}\}$.

Let B_i be the $|\hat{E}| \times |\hat{E}|$ incidence matrix of R_i for $i \in T$. For a pair $(e, f) \in R_i$, let τ_i be the inner product of columns e and f of $\hat{\mathcal{A}}$. Elementary counting methods give

$$\begin{aligned}
\tau_1 &= \frac{k!}{(k-t)!} + \frac{k!}{(k-t-1)!} \\
\tau_2 &= \tau_4 = \tau_5 = \tau_1 - (k-t) \frac{(k-1)!}{(k-t-1)!} \\
\tau_3 &= \tau_{10} = \tau_{11} = \tau_1 - \frac{(k-2)!t}{2(k-t)!} ((k-t)(t+1) + (t-1)) \\
\tau_6 &= \tau_1 - 2(k-t) \frac{(k-1)!}{(k-t-1)!} + (k-t-1) \frac{(k-2)!}{(k-t-2)!} \\
\tau_7 &= \tau_8 = \tau_{12} = \tau_{13} = \tau_1 - (k-t) \frac{(k-1)!}{(k-t-1)!} - \frac{(k-2)!t}{2(k-t)!} ((k-t)(t+1) + (t-1)) \\
\tau_9 &= \tau_{14} = \tau_1 - (k-t) \frac{(k-1)!}{(k-t-1)!} - \frac{(k-2)!t}{2(k-t)!} ((k-t)(t+1) + (t-1)) \\
&\quad + \frac{(k-3)!t}{2(k-t-1)!} ((k-t-1)(t+1) + (t-1)) \\
\tau_{15} &= \tau_{16} = \tau_1 - \frac{(k-2)!t}{(k-t)!} ((k-t)(t+1) + (t-1)) + \frac{(k-3)!(t-1)t}{3(k-t)!} ((k-t)(t+1) + t-2) \\
\tau_{17} &= \tau_{20} = \tau_1 - \frac{(k-2)!t}{(k-t)!} ((k-t)(t+1) + (t-1)) + \frac{(k-3)!(t-1)t}{6(k-t)!} ((k-t)(t+1) + t-2) \\
\tau_{18} &= \tau_1 - \frac{(k-2)!t}{(k-t)!} ((k-t)(t+1) + (t-1)) + \frac{(k-4)!t(t-1)(t-2)}{4(k-t)!} ((k-t)(t+1) + t-3) \\
\tau_{19} &= \tau_1 - \frac{(k-2)!t}{(k-t)!} ((k-t)(t+1) + (t-1))
\end{aligned}$$

When e and f are different from $\mathbf{1}$, τ_i counts the extreme points of $\mathcal{F}_{AC}(D)$ that contain both e and f in their support. Note that $B = \sum_{i \in T} \tau_i B_i$. For arbitrary $(e, f) \in R_h$, let p_{ij}^h count the elements g such that $(e, g) \in R_i$ and $(g, f) \in R_j$. (Note that the partition R has the property that p_{ij}^h is independent of the choice of $(e, f) \in R_h$.) The values of p_{ij}^h that are non-zero are recorded in Appendix A. Let P_i be the $|T| \times |T|$ matrix with p_{hi}^j as the entry in row h and column j . Let $P = \sum_{i \in T} \tau_i P_i$. The distinct eigenvalues of P and B are identical (see Higman (1975), or Lee and Lee (1990)), although, generally, their multiplicities are quite different.

The matrix P is a block diagonal matrix with diagonal blocks determined by the partition $(T_l : l \in I)$ of T , defined by

$$T_1 = \{1, 2, 3\},$$

$$T_4 = \{4, 5, 6, 7, 8, 9\}$$

$$\text{and } T_{10} = \{10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}.$$

Let $P^{(l)}$ be the submatrix of P whose rows and columns are indexed from T_l for $l \in I$. Using the symbolic manipulation software of MATHEMATICA, run on a NEXT machine, we determined the eigenvalues of $P^{(l)}$ together with their multiplicities. Let $\sigma = \frac{1}{12} \frac{(k-4)!}{(k-t)!}$ for $k \geq 4$ (for $k = 3$, where $(k-4)!$ is undefined, use the equivalent expressions of Appendix B, which hold for all $k \geq 3$), and let

$$\lambda_0 = 0, \quad \lambda_1 = \sigma r_1, \quad \lambda_2 = \sigma r_2, \quad \lambda_3 = \sigma r'_1, \quad \lambda_4 = \sigma r'_2,$$

$$\begin{aligned} \lambda_5 = & \sigma(-12t + 34kt - 22k^2t + 4k^3t - 24t^2 + 56kt^2 - 34k^2t^2 + 6k^3t^2 \\ & - 24t^3 + 32kt^3 - 14k^2t^3 + 2k^3t^3 - 12t^4 + 10kt^4 - 2k^2t^4), \end{aligned}$$

$$\lambda_6 = \sigma(-6t)(-k+t)(1-k+t)(3-k+2t-kt+t^2),$$

$$\text{and } \lambda_7 = 2\sigma(-3+k)t(2-7k+3k^2+4t-6kt+3k^2t+4t^2-5kt^2+2t^3),$$

where r_1 and r_2 ($r_1 \leq r_2$) are the roots of

$$\begin{aligned} & 5184k^2 - 24192k^3 + 51984k^4 - 67536k^5 + 56880k^6 - 30816k^7 + 10224k^8 - 1872k^9 + 144k^{10} \\ & - 5184kt + 29376k^2t - 70992k^3t + 95328k^4t - 77616k^5t + 39168k^6t - 11952k^7t + 2016k^8t \\ & - 144k^9t - 6480kt^2 + 22464k^2t^2 - 36468k^3t^2 + 36252k^4t^2 - 22716k^5t^2 + 8532k^6t^2 \\ & - 1728k^7t^2 + 144k^8t^2 + 1296t^3 - 6048kt^3 + 15588k^2t^3 - 22500k^3t^3 + 17748k^4t^3 - 7596k^5t^3 \\ & + 1656k^6t^3 - 144k^7t^3 + 2592t^4 - 8208kt^4 + 10440k^2t^4 - 6876k^3t^4 + 2484k^4t^4 - 468k^5t^4 \\ & + 36k^6t^4 + 1296t^5 - 3456kt^5 + 3492k^2t^5 - 1692k^3t^5 + 396k^4t^5 - 36k^5t^5 + (144k - 264k^2 \\ & + 144k^3 + 48k^4 - 132k^5 + 72k^6 - 12k^7 - 72t + 276kt - 336k^2t + 84k^3t + 108k^4t - 72k^5t \\ & + 12k^6t - 90t^2 + 201kt^2 - 264k^2t^2 + 213k^3t^2 - 84k^4t^2 + 12k^5t^2 - 18t^3 + 123kt^3 - 165k^2t^3 \\ & + 78k^3t^3 - 12k^4t^3 + 18t^4 - 33kt^4 + 18k^2t^4 - 3k^3t^4 + 18t^5 - 15kt^5 + 3k^2t^5)x + x^2, \end{aligned}$$

and r'_1 and r'_2 ($r'_1 \leq r'_2$) are the roots of

$$\begin{aligned}
& 1296k^2t - 2808k^3t + 2088k^4t - 648k^5t + 72k^6t - 2592kt^2 + 8208k^2t^2 - 8496k^3t^2 \\
& + 3960k^4t^2 - 864k^5t^2 + 72k^6t^2 + 1296t^3 - 7992kt^3 + 12024k^2t^3 - 7488k^3t^3 \\
& + 2088k^4t^3 - 216k^5t^3 + 2592t^4 - 6912kt^4 + 5688k^2t^4 - 1872k^3t^4 + 216k^4t^4 + 1296t^5 \\
& - 1512kt^5 + 576k^2t^5 - 72k^3t^5 + (72k - 132k^2 + 72k^3 - 12k^4 - 72t + 222kt \\
& - 210k^2t + 84k^3t - 12k^4t - 90t^2 + 156kt^2 - 96k^2t^2 + 18k^3t^2 - 18t^3 + 6kt^3 \\
& + 18k^2t^3 - 6k^3t^3 + 18t^4 - 42kt^4 + 12k^2t^4 + 18t^5 - 6kt^5)x + x^2.
\end{aligned}$$

The multiplicities are given as the first three rows of Table 1.

	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
$P^{(1)}$	1	1	1	0	0	0	0	0
$P^{(4)}$	1	1	1	1	1	1	0	0
$P^{(10)}$	1	1	1	2	2	2	1	1
B	1	1	1	$k-1$	$k-1$	$k-1$	$\frac{k(k-3)}{2}$	$\frac{(k-1)(k-2)}{2}$

Table 1: Multiplicities of the eigenvalues in $P^{(1)}$, $P^{(4)}$, $P^{(10)}$ and B .

To determine the multiplicities of the eigenvalues in B , we use MATHEMATICA, again, to solve the Vandermonde system

$$\sum_{i=0}^7 \mu_i \lambda_i^m = \text{Trace}(B^m) \quad \text{for all } 0 \leq m \leq 7.$$

The multiplicities are recorded as the last row of Table 1.

We use the methodology of Lee and Lee (1990) to determine the necessary traces for the right-hand side of the Vandermonde system. That is, for all $i \in T$, let

$$\begin{aligned}
\theta_i^{(1)} &= \tau_i, \\
\theta_i^{(m)} &= \sum_{h \in T} \sum_{j \in T} \theta_h^{\lfloor \frac{m}{2} \rfloor} \theta_j^{\lceil \frac{m}{2} \rceil} p_{hj}^i, \quad \text{for } m \geq 2.
\end{aligned}$$

Then,

$$\text{Trace}(B^m) = \sum_{i \in I} \theta_i^{(m)} \text{Trace}(B_i) \quad \text{for all } m \geq 1.$$

Note that $\text{Trace}(B_1) = 1$, $\text{Trace}(B_4) = k$, and $\text{Trace}(B_{10}) = k(k-1)$.

The number of non-zero eigenvalues (counting multiplicities) of B is exactly k^2 (see Appendix B), thus proving that the $\dim(\mathcal{F}_{AC}(D)) = k^2 - 1 = \dim(\mathcal{P}_{AC}(D)) - 1$. ■

It is notable that the polytope $\mathcal{P}_{AC}(D)$ is the first polytope to which the spectral method has been applied where the associated matrix $\mathcal{A}$ does not have constant row sums. We are able to apply the method by having R partition $\hat{E} \times \hat{E}$ rather than simply the Cartesian square of the arc-set $F_1 \cup F_2$. This extension of the spectral method addresses an issue raised in Lee (1990).

That each inequality (7) induced by a (non-simple) k -fence $D' = (V', F'_1 \cup F'_2)$ represents a facet of $\mathcal{P}_{AC}(D')$ can be shown by applying the following result.

Proposition 3 (*Grötschel, Jünger and Reinelt (1985a)*)

Let $G' = (N', A')$ be a directed graph, and let $a^T x \leq a_0$ be a facet of $\mathcal{P}_{AC}(G')$. Let $G'' = (N'', A'')$ be the graph obtained by subdividing the arc $(i, j) \in A'$ into the arcs (i, h) and (h, j) . Then the inequality $\bar{a}^T x \leq \bar{a}_0$ defines a facet of $\mathcal{P}_{AC}(G'')$, where

$$\bar{a}_e = \begin{cases} a_{(i,j)} & \text{if } e = (i, h) \text{ or } (h, j), \\ a_e & \text{otherwise,} \end{cases}$$

and $\bar{a}_0 = a_0 + a_{(i,j)}.$

3 Extension

In this section, we show that (7) represents a facet for $\mathcal{P}_{AC}(G)$ when D' is a k -fence contained in G . We first examine the case when D' is a simple k -fence.

Proposition 4 Let $G = (N, A)$ be a strict directed graph and let $D' = D = (U \cup W, F_1 \cup F_2)$ be a subgraph of G that is a simple k -fence. Then for $1 \leq t \leq k-2$, (7) represents a facet of $\mathcal{P}_{AC}(G)$.

Proof. Since we have proved that (7) is a facet for $\mathcal{P}_{AC}(D)$, let X_0 be an arbitrary set of k^2 extreme points of $\mathcal{F}_{AC}(D)$ that are affinely independent. Now for any i such that $1 \leq i \leq k$, let $x^{(i)}$ be an extreme point of $\mathcal{F}_{AC}(D)$ with $x_{(u_i, w_i)}^{(i)} = 0$. For any $i \neq j$ with $1 \leq i, j \leq k$, let $x^{(ij)}$ be an extreme point of $\mathcal{F}_{AC}(D)$ as defined by (8) with $i_1 = i$ and $i_t = j$ if $t \geq 2$, and by (9) with $i_1 = i$ and $i_2 = j$ if $t = 1$. Now consider the following $|A|$ vectors in $\mathbf{R}^A$:

- For each $x \in X_0$, define $y \in \mathbf{R}^A$ by

$$y_e = \begin{cases} x_e & \text{if } e \in F_1 \cup F_2, \\ 0 & \text{otherwise,} \end{cases}$$

- For each arc $(i, j) \in A$ with $|\{i, j\} \cap (U \cup W)| \leq 1$, define y by

$$y_e = \begin{cases} x_e & \text{if } e \in F_1 \cup F_2, \\ 1 & \text{if } e = (i, j), \\ 0 & \text{otherwise,} \end{cases}$$

for some $x \in X_0$.

- For each $(w_i, j) \in A$ with $j \in U \cup W$, define y by

$$y_e = \begin{cases} x_e^{(i)} & \text{if } e \in F_1 \cup F_2, \\ 1 & \text{if } e = (w_i, j), \\ 0 & \text{otherwise,} \end{cases}$$

- For each $(u_i, u_j) \in A$ with $i \neq j$ and $1 \leq i, j \leq k$, define y by

$$y_e = \begin{cases} x_e^{(j)} & \text{if } e \in F_1 \cup F_2, \\ 1 & \text{if } e = (u_i, u_j), \\ 0 & \text{otherwise,} \end{cases}$$

- For each $(u_i, w_j) \in A$ with $i \neq j$ and $1 \leq i, j \leq k$, define y by

$$y_e = \begin{cases} x_e^{(ij)} & \text{if } e \in F_1 \cup F_2, \\ 1 & \text{if } e = (u_i, w_j), \\ 0 & \text{otherwise.} \end{cases}$$

These $|A|$ vectors are affinely independent, feasible for $\mathcal{P}_{AC}(G)$, and satisfy (7) with equality. Hence, (7) is a facet for $\mathcal{P}_{AC}(G)$. ■

By similar arguments as in the proof of Proposition 4 (with several more cases to check), Proposition 1 can be established.

4 The Linear Ordering Polytope

A *linear ordering* of an n -element set N is a bijection $\pi : \{1, 2, \dots, n\} \rightarrow N$. Let c_{uv} be the value of having u before v in the linear ordering. The *value* of a linear ordering of N is

$$\left\{ \sum_{(u,v) \in N \times N} c_{uv} : \pi^{-1}(u) < \pi^{-1}(v) \right\}.$$

A directed graph (N, T) is a *tournament* on N if for every pair of nodes $u, v \in N$, exactly one of (u, v) and (v, u) is in T . Given a linear ordering of the nodes N of a directed graph, the arc set $\{(u, v) : \pi^{-1}(u) < \pi^{-1}(v)\}$ forms an acyclic tournament on N ; conversely, an acyclic tournament (N, T) induces a linear ordering of N . Following Grötschel, Jünger and Reinelt (1985b), we let $\langle i_1, i_2, \dots, i_n \rangle$ denote the arc set $\{(i_j, i_k) : 1 \leq j < k \leq n\}$ of the acyclic tournament induced by the linear ordering π where $\pi(j) = i_j$ for $j = 1, 2, \dots, n$.

Let $G_n = (N, A_n)$ be the complete strict directed graph on the n -element node set. Let $\mathcal{T}_n \subseteq 2^{A_n}$ be the collection of the arc sets of acyclic tournaments on N . The *linear ordering polytope* $\mathcal{P}_{LO}^n$ is the convex hull of the incidence vectors of elements of $\mathcal{T}_n$. Grötschel, Jünger and Reinelt (1985b) determine the dimension of $\mathcal{P}_{LO}^n$ and show that

$$x_{ij} + x_{ji} = 1 \quad \forall i, j \in N, \text{ with } i \neq j, \quad (10)$$

is a maximal irredundant equation system for $\mathcal{P}_{LO}^n$. They also show that if the

simple k -fence D is a subgraph of G_n , then the 1-reinforced k -fence inequality (7) for $D' = D$ represents a facet of $\mathcal{P}_{LO}^n$.

In this section, we will show that the t -reinforced k -fence inequality induced by a simple k -fence represents a facet of $\mathcal{P}_{LO}^n$ for $1 \leq t \leq k - 2$.

Proposition 5 *For $1 \leq t \leq k - 2$, let $D' = D = (U \cup W, F_1 \cup F_2)$ be a simple k -fence that is a subgraph of $G_n = (N, A_n)$. Then the t -reinforced k -fence inequality (7) represents a facet of $\mathcal{P}_{LO}^n$.*

The proof of Proposition 5 appeals to the following result.

Proposition 6 *(Grötschel, Jünger and Reinelt (1985b))*

If $a^T x \leq a_0$ defines a facet of $\mathcal{P}_{LO}^n$ for $n \geq 2$, then $\bar{a}^T x \leq a_0$ defines a facet of $\mathcal{P}_{LO}^{n+1}$ where $\bar{a}_{ij} = a_{ij}$ for all $(i, j) \in A_n$ and $\bar{a}_{i,n+1} = \bar{a}_{n+1,i} = 0$ for $i = 1, \dots, n$.

Proof (Proposition 5).

It suffices to prove the proposition for $n = 2k$ and $G_n = G_{2k} = (U \cup W, A_{2k})$ by applying Proposition 6.

Let $\mathcal{F}_{LO} = \text{conv}\{x \in \mathbf{R}^{A_{2k}} : x \in \mathcal{P}_{LO}^{2k} \text{ and } x \text{ satisfies (7) as an equation}\}$. Let $b^T x \leq b_0$ be a valid inequality for $\mathcal{P}_{LO}^{2k}$ such that $\mathcal{P}_{LO}^{2k} \not\subseteq \{x \in \mathbf{R}^{A_{2k}} : b^T x = b_0\}$. We want to show that if $\mathcal{F}_{LO} \subseteq \{x \in \mathcal{P}_{LO}^{2k} : b^T x = b_0\}$, then $b^T x \leq b_0$ is a non-negative multiple of (7) plus a linear combination of equations (10).

By adding a multiple of $x_{ij} + x_{ji} = 1$ to $b^T x \leq b_0$, we can make the coefficient of (i, j) in the resultant inequality zero. Therefore, we can assume, without loss of generality, that

$$\begin{aligned} b_{uv} &= 0 & \forall (v, u) \in F_1 \cup F_2, \\ b_{uv} &= 0 & \forall (u, v) = (u_i, u_j) \text{ or } (w_i, w_j) \text{ with } i < j. \end{aligned}$$

Consider the three tournaments having arc sets

$$\begin{aligned}
& < w_1, w_2, w_3, \dots, w_{k-t}, \\
& \quad u_{k-t+1}, w_{k-t+1}, u_{k-t+2}, w_{k-t+2}, \dots, u_k, w_k, \\
& \quad u_{k-t}, u_{k-t-1}, \dots, u_3, u_2, u_1 >, \\
& < w_2, w_1, w_3, \dots, w_{k-t}, \\
& \quad u_{k-t+1}, w_{k-t+1}, u_{k-t+2}, w_{k-t+2}, \dots, u_k, w_k, \\
& \quad u_{k-t}, u_{k-t-1}, \dots, u_3, u_2, u_1 >, \\
\text{and} \quad < w_1, w_2, w_3, \dots, w_{k-t}, \\
& \quad u_{k-t+1}, w_{k-t+1}, u_{k-t+2}, w_{k-t+2}, \dots, u_k, w_k, \\
& \quad u_{k-t}, u_{k-t-1}, \dots, u_3, u_1, u_2 >.
\end{aligned} \tag{11}$$

It is clear that the associated incidence vectors are in $\mathcal{F}_{LO}$. Substituting these vectors into $b^T x = b_0$ implies that $b_{w_1 w_2} = b_{w_2 w_1} = b_{u_1 u_2} = b_{u_2 u_1} = 0$. By permuting the nodes of U and permuting the nodes of W , we can conclude that

$$b_{uu'} = b_{ww'} = 0 \quad \forall u, u' \in U \text{ and } \forall w, w' \in W.$$

Next, comparing (11) and

$$\begin{aligned}
& < w_1, w_2, \dots, w_{k-t-1}, \\
& \quad u_{k-t}, w_{k-t}, u_{k-t+1}, w_{k-t+1}, \dots, u_k, w_k, \\
& \quad u_{k-t-1}, \dots, u_2, u_1 >,
\end{aligned}$$

we have that

$$\sum_{i=k-t+1}^k b_{w_i u_{k-t}} = b_{u_{k-t} w_{k-t}}.$$

By permuting the nodes of U and W , we can conclude that

$$\begin{aligned}
& b_{u_i w_i} = t\beta_i \quad 1 \leq i \leq k, \\
\text{and} \quad & b_{w_j u_i} = \beta_i \quad 1 \leq i, j \leq k, \ j \neq i,
\end{aligned}$$

for some constants β_i for $1 \leq i \leq k$. Next, comparing

$$\begin{aligned} &< w_1, \dots, w_{k-t-1}, \\ &\quad u_{k-t}, w_{k-t}, u_{k-t+1}, w_{k-t+1}, \dots, u_{k-2}, w_{k-2}, u_{k-1}, w_{k-1}, u_k, w_k, \\ &\quad u_{k-t-1}, \dots, u_1 > \end{aligned}$$

and

$$\begin{aligned} &< w_1, \dots, w_{k-t-1}, \\ &\quad u_{k-t}, w_{k-t}, u_{k-t+1}, w_{k-t+1}, \dots, u_{k-2}, w_{k-2}, u_k, w_k, u_{k-1}, w_{k-1}, \\ &\quad u_{k-t-1}, \dots, u_1 >, \end{aligned}$$

we observe that $b_{w_{k-1}u_k} = b_{w_k u_{k-1}}$, and hence $\beta_i = \beta$ for some constant β for all i with $1 \leq i \leq k$. Hence, $b^T x \leq b_0$ is β times (7). Now consider the incidence vector of

$$\begin{aligned} &< u_1, u_2, \dots, u_{k-t}, \\ &\quad w_k, u_k, w_{k-1}, u_{k-1}, \dots, w_{k-t+1}, u_{k-t+1}, \\ &\quad w_{k-t}, w_{k-t-1}, \dots, w_1 > \end{aligned}$$

which corresponds to the linear ordering that is the reverse of the one associated with (11). Substituting this vector into $b^T x$, we get

$$\beta \left(t(k-t) + \frac{1}{2}t(t-1) \right) = \beta \left(\underbrace{\left[k(k-1) + \frac{1}{2}t(t+1) \right]}_{\text{r.h.s. of (7)}} - \underbrace{\left[(k-t)^2 + k(t-1) + t \right]}_{>0} \right).$$

Since $b^T x \leq b_0$ is valid, we must have $\beta \geq 0$. ■

5 Summary

This paper presents a two-parameter family of facets for the acyclic subgraph polytope and the linear ordering polytope, represented by t -reinforced k -fence inequalities. These facets are induced by subgraphs called k -fences, but each k -fence induces

more than one facet. These are the first facets discovered for these polytopes that are not rank inequalities. The proof that these are facets for the acyclic subgraph problem combines spectral methods introduced by Lee (1990) with direct arguments. The spectral method is particularly suitable in this case, since the family of incidence relations $R = (R_i : i \in T)$ only depends on the subgraph; hence, the parameters p_{ij}^h are independent of the reinforcement parameter t . Thus, to determine the dimension of another face-representing inequality for which R applies, only the τ_i ($i \in T$) need be recomputed. We hope that this method will be useful in the future, not only as an alternative method for identifying facets, but as a tool for identifying other faces induced by k -fences in a systematic manner. A first step in such an analysis would be the calculation of the ‘completely reduced form’ of the algebra of linear combinations of the matrices B_i ($i \in T$) (see Lee and Lee (1990)). This is carried out in Appendix C.

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Appendix A

$p_{1,1}^1 = 1$	$p_{5,1}^5 = 1$	$p_{11,1}^{11} = 1$	$p_{1,2}^2 = 1$
$p_{5,2}^4 = 1$	$p_{5,2}^6 = 1$	$p_{11,2}^{12} = 1$	$p_{11,2}^{13} = 1$
$p_{11,2}^{14} = 1$	$p_{1,3}^3 = 1$	$p_{5,3}^7 = 1$	$p_{5,3}^8 = 1$
$p_{5,3}^9 = 1$	$p_{11,3}^{10} = 1$	$p_{11,3}^{15} = 1$	$p_{11,3}^{16} = 1$
$p_{11,3}^{17} = 1$	$p_{11,3}^{18} = 1$	$p_{11,3}^{19} = 1$	$p_{11,3}^{20} = 1$
$p_{2,4}^2 = 1$	$p_{4,4}^4 = 1$	$p_{6,4}^6 = 1$	$p_{12,4}^{12} = 1$
$p_{13,4}^{13} = 1$	$p_{14,4}^{14} = 1$	$p_{2,5}^1 = k$	$p_{4,5}^5 = 1$
$p_{6,5}^5 = k - 1$	$p_{12,5}^{11} = 1$	$p_{13,5}^{11} = 1$	$p_{14,5}^{11} = k - 2$
$p_{2,6}^2 = k - 1$	$p_{4,6}^6 = 1$	$p_{6,6}^4 = k - 1$	$p_{6,6}^6 = k - 2$
$p_{12,6}^{13} = 1$	$p_{12,6}^{14} = 1$	$p_{13,6}^{14} = 1$	$p_{13,6}^{12} = 1$
$p_{14,6}^{12} = k - 2$	$p_{14,6}^{13} = k - 2$	$p_{14,6}^{14} = k - 3$	$p_{2,7}^3 = 1$
$p_{4,7}^7 = 1$	$p_{6,7}^8 = 1$	$p_{6,7}^9 = 1$	$p_{12,7}^{17} = 1$
$p_{12,7}^{19} = 1$	$p_{13,7}^{15} = 1$	$p_{13,7}^{10} = 1$	$p_{14,7}^{16} = 1$
$p_{14,7}^{18} = 1$	$p_{14,7}^{20} = 1$	$p_{2,8}^3 = 1$	$p_{4,8}^8 = 1$
$p_{6,8}^7 = 1$	$p_{6,8}^9 = 1$	$p_{12,8}^{10} = 1$	$p_{12,8}^{16} = 1$
$p_{13,8}^{19} = 1$	$p_{13,8}^{20} = 1$	$p_{14,8}^{15} = 1$	$p_{14,8}^{18} = 1$
$p_{14,8}^{17} = 1$	$p_{2,9}^3 = k - 2$	$p_{4,9}^9 = 1$	$p_{6,9}^7 = k - 2$
$p_{6,9}^8 = k - 2$	$p_{6,9}^9 = k - 3$	$p_{12,9}^{15} = 1$	$p_{13,9}^{17} = 1$
$p_{12,9}^{18} = 1$	$p_{13,9}^{16} = 1$	$p_{12,9}^{20} = 1$	$p_{13,9}^{18} = 1$
$p_{14,9}^{10} = k - 2$	$p_{14,9}^{15} = k - 3$	$p_{14,9}^{16} = k - 3$	$p_{14,9}^{17} = k - 3$
$p_{14,9}^{19} = k - 2$	$p_{14,9}^{18} = k - 4$	$p_{14,9}^{20} = k - 3$	$p_{3,10}^3 = 1$
$p_{7,10}^7 = 1$	$p_{8,10}^8 = 1$	$p_{9,10}^9 = 1$	$p_{10,10}^{10} = 1$
$p_{15,10}^{15} = 1$	$p_{16,10}^{16} = 1$	$p_{17,10}^{17} = 1$	$p_{18,10}^{18} = 1$
$p_{19,10}^{19} = 1$	$p_{20,10}^{20} = 1$	$p_{3,11}^1 = k(k - 1)$	$p_{7,11}^5 = k - 1$
$p_{8,11}^5 = k - 1$	$p_{9,11}^5 = (k - 1)(k - 2)$	$p_{10,11}^{11} = 1$	$p_{15,11}^{11} = k - 2$
$p_{16,11}^{11} = k - 2$	$p_{17,11}^{11} = k - 2$	$p_{18,11}^{11} = (k - 2)(k - 3)$	$p_{19,11}^{11} = 1$
$p_{20,11}^{11} = k - 2$	$p_{3,12}^2 = k - 1$	$p_{8,12}^4 = k - 1$	$p_{7,12}^6 = 1$
$p_{9,12}^6 = k - 2$	$p_{10,12}^{12} = 1$	$p_{15,12}^{14} = 1$	$p_{16,12}^{12} = k - 2$
$p_{17,12}^{14} = 1$	$p_{18,12}^{14} = k - 3$	$p_{19,12}^{13} = 1$	$p_{20,12}^{13} = k - 2$
$p_{3,13}^2 = k - 1$	$p_{7,13}^4 = k - 1$	$p_{8,13}^6 = 1$	$p_{9,13}^6 = k - 2$
$p_{10,13}^{13} = 1$	$p_{15,13}^{13} = k - 2$	$p_{16,13}^{14} = 1$	$p_{17,13}^{12} = k - 2$
$p_{18,13}^{14} = k - 3$	$p_{19,13}^{12} = 1$	$p_{20,13}^{14} = 1$	$p_{3,14}^2 = (k - 2)(k - 1)$

$p_{7,14}^6 = k - 2$	$p_{8,14}^6 = k - 2$	$p_{9,14}^4 = (k - 1)(k - 2)$	$p_{9,14}^6 = (k - 2)(k - 3)$
$p_{10,14}^{14} = 1$	$p_{15,14}^{14} = k - 3$	$p_{16,14}^{14} = k - 3$	$p_{17,14}^{14} = k - 3$
$p_{18,14}^{14} = (k - 3)(k - 4)$	$p_{19,14}^{14} = 1$	$p_{20,14}^{14} = k - 3$	$p_{15,14}^{12} = k - 2$
$p_{16,14}^{13} = k - 2$	$p_{17,14}^{13} = k - 2$	$p_{18,14}^{12} = (k - 2)(k - 3)$	$p_{18,14}^{13} = (k - 2)(k - 3)$
$p_{20,14}^{12} = k - 2$	$p_{3,15}^3 = k - 2$	$p_{7,15}^7 = k - 2$	$p_{8,15}^9 = 1$
$p_{9,15}^8 = k - 2$	$p_{9,15}^9 = k - 3$	$p_{10,15}^{15} = 1$	$p_{15,15}^{15} = k - 3$
$p_{15,15}^{10} = k - 2$	$p_{16,15}^{18} = 1$	$p_{16,15}^{20} = 1$	$p_{17,15}^{17} = k - 3$
$p_{17,15}^{19} = k - 2$	$p_{18,15}^{16} = k - 3$	$p_{18,15}^{18} = k - 4$	$p_{18,15}^{20} = k - 3$
$p_{19,15}^{17} = 1$	$p_{20,15}^{16} = 1$	$p_{20,15}^{18} = 1$	$p_{3,16}^3 = k - 2$
$p_{7,16}^9 = 1$	$p_{8,16}^8 = k - 2$	$p_{9,16}^7 = k - 2$	$p_{9,16}^9 = k - 3$
$p_{10,16}^{16} = 1$	$p_{15,16}^{17} = 1$	$p_{15,16}^{18} = 1$	$p_{16,16}^{10} = k - 2$
$p_{16,16}^{16} = k - 3$	$p_{17,16}^{15} = 1$	$p_{17,16}^{18} = 1$	$p_{18,16}^{15} = k - 3$
$p_{18,16}^{17} = k - 3$	$p_{18,16}^{18} = k - 4$	$p_{19,16}^{20} = 1$	$p_{20,16}^{19} = k - 2$
$p_{20,16}^{20} = k - 3$	$p_{3,17}^3 = k - 2$	$p_{7,17}^9 = 1$	$p_{8,17}^7 = k - 2$
$p_{9,17}^8 = k - 2$	$p_{9,17}^9 = k - 3$	$p_{10,17}^{17} = 1$	$p_{15,17}^{16} = 1$
$p_{15,17}^{18} = 1$	$p_{16,17}^{17} = k - 3$	$p_{16,17}^{19} = k - 2$	$p_{17,17}^{18} = 1$
$p_{17,17}^{20} = 1$	$p_{18,17}^{16} = k - 3$	$p_{18,17}^{18} = k - 4$	$p_{18,17}^{20} = k - 3$
$p_{19,17}^{15} = 1$	$p_{20,17}^{10} = k - 2$	$p_{20,17}^{15} = k - 3$	$p_{3,18}^3 = (k - 2)(k - 3)$
$p_{7,18}^9 = k - 3$	$p_{8,18}^9 = k - 3$	$p_{9,18}^7 = (k - 2)(k - 3)$	$p_{9,18}^8 = (k - 2)(k - 3)$
$p_{9,18}^9 = (k - 3)(k - 4)$	$p_{10,18}^{18} = 1$	$p_{15,18}^{16} = k - 3$	$p_{15,18}^{17} = k - 3$
$p_{15,18}^{18} = k - 4$	$p_{16,18}^{15} = k - 3$	$p_{18,18}^{18} = k - 4$	$p_{16,18}^{20} = k - 3$
$p_{17,18}^{15} = k - 3$	$p_{17,18}^{18} = k - 4$	$p_{17,18}^{20} = k - 3$	$p_{18,18}^{10} = (k - 2)(k - 3)$
$p_{18,18}^{15} = (k - 3)(k - 4)$	$p_{18,18}^{19} = (k - 2)(k - 3)$	$p_{18,18}^{16} = (k - 3)(k - 4)$	$p_{18,18}^{20} = (k - 3)(k - 4)$
$p_{18,18}^{17} = (k - 3)(k - 4)$	$p_{18,18}^{18} = (k - 4)(k - 5)$	$p_{19,18}^{18} = 1$	$p_{20,18}^{17} = k - 3$
$p_{20,18}^{16} = k - 3$	$p_{20,18}^{18} = k - 4$	$p_{3,19}^3 = 1$	$p_{8,19}^7 = 1$
$p_{7,19}^8 = 1$	$p_{9,19}^9 = 1$	$p_{10,19}^{19} = 1$	$p_{15,19}^{20} = 1$
$p_{16,19}^{17} = 1$	$p_{17,19}^{16} = 1$	$p_{18,19}^{18} = 1$	$p_{19,19}^{10} = 1$
$p_{20,19}^{15} = 1$	$p_{3,20}^3 = k - 2$	$p_{9,20}^7 = k - 2$	$p_{7,20}^8 = k - 2$
$p_{8,20}^9 = 1$	$p_{9,20}^9 = k - 3$	$p_{10,20}^{20} = 1$	$p_{15,20}^{19} = k - 2$
$p_{15,20}^{20} = k - 3$	$p_{16,20}^{15} = 1$	$p_{16,20}^{18} = 1$	$p_{17,20}^{10} = k - 2$
$p_{17,20}^{16} = k - 3$	$p_{18,20}^{17} = k - 3$	$p_{18,20}^{15} = k - 3$	$p_{18,20}^{18} = k - 4$
$p_{19,20}^{16} = 1$	$p_{20,20}^{17} = 1$	$p_{20,20}^{18} = 1$	

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Pages 1–18 reproduce Technical Report 90-22 (November 15, 1990) with its original pagination. We have corrected typographical errors there, and we have filled two small gaps. The typographical corrections are spelling and grammar slips, a missing citation letter, a missing absolute value on p. 10, and the indices of the reversed ordering on p. 14. The gaps are the definition of $x^{(ij)}$ for $t = 1$ in the proof of Proposition 4, and the definition of σ on p. 7, which is undefined for $k = 3$. Some of the material was later published, but with proofs that do not use the spectral method, as J. Leung and J. Lee, *Discrete Appl. Math.* **50** (1994) 185–200. The new appendices B–D develop what the spectral method can further do, in the terminology of Lee and Lee (1990).

Appendix B shows that the eigenvalues $\lambda_1, \dots, \lambda_7$ in the proof of Proposition 2 are non-zero, which the proof needs, and gives formulas valid for $k = 3$.

Appendix C carries out the program proposed at the end of Section 5. It gives a version of Theorem 4.4 of Lee and Lee for concordant faces that are not balanced (Proposition 7), and the completely reduced form of the fence configuration (Proposition 8). It gives a proof of Proposition 2 without symbolic computation (Propositions 9–10), and the dimension and a maximal irredundant equation system for every face of $\mathcal{P}_{AC}(D)$ with the symmetry of D ; in particular, the symmetric facets are exactly the t -reinforced inequalities (Proposition 11). Section C.5 treats non-simple fences.

Appendix D demonstrates that each block rank is the dimension of the span of a few explicit statistics (Proposition 15). It applies this to the linear ordering polytope, where it determines all symmetric faces and gives a spectral proof of Proposition 5 (Propositions 16–17), and to an unbalanced clique face of the forest polytope (Proposition 18).

Appendix B

The dimension count at the end of the proof of Proposition 2 requires that each of $\lambda_1, \dots, \lambda_7$ having positive multiplicity in B be non-zero. Then 0 is an eigenvalue of B of multiplicity exactly one (namely λ_0), and B has k^2 non-zero eigenvalues, counting multiplicities. (If two of $\lambda_1, \dots, \lambda_7$ coincide for particular k and t , their multiplicities simply add, which does not affect this count.) We verify this for all $k \geq 3$ and $1 \leq t \leq k - 2$. Throughout, note that $t \leq k - 2$ gives $k - t \geq 2$ and $k - t - 1 \geq 1$.

Removing the factor $k - 3$. Exact symbolic factorization of the polynomials of Section 2 shows that λ_5 and λ_7 contain the factor $k - 3$, and that the coefficients of x and x^0 in the two quadratics defining r_1, r_2 and r'_1, r'_2 are divisible by $k - 3$ and $(k - 3)^2$, respectively. Because $(k - 3)(k - 4)! = (k - 3)!$, let

$$\tilde{\sigma} = \frac{1}{12} \frac{(k - 3)!}{(k - t)!},$$

which is defined and positive for all $k \geq 3$. Then, with $\sigma = \tilde{\sigma}/(k - 3)$ for $k \geq 4$,

$$\begin{aligned} \lambda_1 &= \tilde{\sigma} \tilde{r}_1, & \lambda_2 &= \tilde{\sigma} \tilde{r}_2, & \lambda_3 &= \tilde{\sigma} \tilde{r}'_1, & \lambda_4 &= \tilde{\sigma} \tilde{r}'_2, \\ \lambda_5 &= 2\tilde{\sigma} t(t + 1)(k - 2)(k(t + 2) - t^2 - t - 1), \\ \lambda_6 &= 6\sigma t(k - t)(k - t - 1)((t + 1)(k - t - 1) - 2), \\ \lambda_7 &= 2\tilde{\sigma} t(3k^2t + 3k^2 - 5kt^2 - 6kt - 7k + 2t^3 + 4t^2 + 4t + 2), \end{aligned}$$

where $\tilde{r}_1 \leq \tilde{r}_2$ are the roots of $x^2 + \beta x + \gamma$ and $\tilde{r}'_1 \leq \tilde{r}'_2$ are the roots of $x^2 + \beta' x + \gamma'$, with

$$\begin{aligned} \beta &= 3(k - 2)(-4k^5 + 4k^4t + 4k^4 + 4k^3t^2 - 4k^3t - 4k^2t^3 - 8k^2t^2 - 8k^2t - 8k^2 \\ &\quad - kt^4 + 6kt^3 + 7kt^2 + 12kt + 8k + t^5 + t^4 - t^3 - 5t^2 - 4t), \\ \gamma &= 36(k - t)(k - 1)(k - 2)^2 \left((2k(k - 1) + t(t + 1))^2 + 4k(k - 1 + t^2) \right), \\ \beta' &= 6(k - t)(-2k^2t - 2k^2 - kt^3 + kt^2 + 6kt + 6k + t^4 + t^3 - t^2 - 5t - 4), \\ \gamma' &= 72t(t + 1)(k - 2)(k - t)^2(k - t - 1). \end{aligned}$$

Indeed, $\tilde{r}_i = r_i/(k - 3)$ and $\tilde{r}'_i = r'_i/(k - 3)$, so for $k \geq 4$ these agree with Section

2; all of these identities were obtained by exact symbolic computation. Only λ_6 still involves σ , and for $k = 3$ its multiplicity $k(k - 3)/2$ is zero, so it does not occur. Thus the expressions above, together with Table 1, give the spectrum of B for every $k \geq 3$. For example, for $k = 3$ (so $t = 1$ and $\tilde{\sigma} = 1/24$) they give the eigenvalues 0 ; $(61 \pm \sqrt{3489})/2$; $2 \pm \sqrt{3}$, each twice; and 1 three times (as $\lambda_5 = \lambda_7 = 1$).

Non-vanishing. Each factor below is positive for $k \geq 3$ and $1 \leq t \leq k - 2$, and $\tilde{\sigma} > 0$.

- $\lambda_1 \lambda_2 = \tilde{\sigma}^2 \gamma > 0$: the factors $k - t$, $k - 1$, $(k - 2)^2$ are positive, and the last factor of γ is a square plus $4k(k - 1 + t^2) > 0$. Hence $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$.
- $\lambda_3 \lambda_4 = \tilde{\sigma}^2 \gamma' > 0$, because t , $t + 1$, $k - 2$, $(k - t)^2$ and $k - t - 1$ are all positive.
- $\lambda_5 > 0$, because $k \geq t + 2$ gives $k(t + 2) - t^2 - t - 1 \geq (t + 2)^2 - t^2 - t - 1 = 3t + 3$.
- $\lambda_6 > 0$ whenever it occurs ($k \geq 4$): then $\sigma > 0$, and $(t + 1)(k - t - 1) - 2$ is at least $t + 1 - 2 \geq 1$ if $t \geq 2$, and equals $2(k - 3) \geq 2$ if $t = 1$.
- $\lambda_7 > 0$: substituting $k = t + 1 + m$ with $m = k - t - 1 \geq 1$, the last factor of λ_7 becomes

$$3(t + 1)m^2 + (t^2 + 6t - 1)m + 2(t^2 - 1) \geq 6 + 6 + 0 = 12.$$

Hence $\lambda_1, \dots, \lambda_7$ are non-zero whenever their multiplicities are positive, and B has exactly k^2 non-zero eigenvalues for all $k \geq 3$ and $1 \leq t \leq k - 2$.

Appendix C

This appendix carries out the step proposed at the end of Section 5. We use the terminology of Lee and Lee (1990), which appeared as *Linear Algebra Appl.* **159** (1991) 11–41: the *balance conditions* (their (2.1)), *concordance* (their (3.1)), and the *completely reduced form*, in which the algebra spanned by the B_i is block diagonal with inequivalent blocks $\Delta_j(w)$ of order e_j , each repeated z_j times, acting on orthogonal invariant subspaces $\mathcal{L}_j$ of dimension $e_j z_j$. As there, $y_j(\tau)$ is the number of non-zero eigenvalues of $\Delta_j(\tau)$. We find:

- a form of Theorem 4.4 of Lee and Lee (1990) for polytopes that are concordant but not balanced (Proposition 7),
- the completely reduced form of the configuration of a simple k -fence (Proposition 8),
- a short proof of Proposition 2 that needs no symbolic computation (Propositions 9 and 10), and
- the dimension and a maximal irredundant equation system of every face of $\mathcal{P}_{AC}(D)$ that shares the symmetry of D , with the consequence that the t -reinforced inequalities are exactly the symmetric facets (Table 2, Proposition 11).

C.1 Concordant polytopes without balance

Let $X \subseteq \mathbf{R}^E$ be finite, $\rho = |X|$, $\mathcal{P} = \text{conv}(X)$, and let $R = (R_i : i \in T)$ be a coherent configuration on E with fibers E_i ($i \in \mathcal{J}$), $\kappa_i = |E_i|$ and d^i the incidence vector of E_i . Suppose $\mathcal{P}$ is concordant with respect to R , so that their (3.1) holds with parameters τ_i , and suppose that the column sums $\sum_{x \in X} x_e$ are constant on each fiber, which is their (2.1.ii) and which their (3.1) implies when $X \subseteq \{0, 1\}^E$.

We do *not* assume their (2.1.i). Let $\varpi : \mathbf{R}^E \rightarrow \mathbf{R}^{\mathcal{J}}$, $\varpi(x)_i = \sum_{e \in E_i} x_e$, be the fiber-sum map.

Proposition 7 *Under these hypotheses,*

$$\dim(\mathcal{P}) = \dim(\varpi(\mathcal{P})) + \sum_{j=1}^m z_j y_j(\tau).$$

Proof. Let $\hat{\mathcal{A}} = [\mathcal{A} \mid \mathbf{1}]$, with e_0 the index of the column $\mathbf{1}$, and let $\hat{B} = \hat{\mathcal{A}}^T \hat{\mathcal{A}}$, so that $\text{rank } \hat{B} = \dim(\mathcal{P}) + 1$. In Lee and Lee (1990), $\mathbb{C}^E = \mathcal{L}_0 \oplus \cdots \oplus \mathcal{L}_m$ orthogonally, with $\mathcal{L}_0 = \text{span}\{d^i : i \in \mathcal{J}\}$. Let $y \in \mathcal{L}_j$ with $j \geq 1$. The e_0 -entry of $\hat{B}y$ is $\mathbf{1}^T \mathcal{A}y = \sum_e (\sum_x x_e) y_e$, which vanishes because the column sums are constant on fibers and $y \perp d^i$. Hence $\hat{B}y = B(\tau)y$, and $\{e_{e_0}\} \cup \mathcal{L}_0$ spans an invariant complement $\hat{\mathcal{L}}_0$ of $\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_m$. As $\hat{B}$ is symmetric and positive semidefinite, its rank is the sum of its ranks on these invariant subspaces. On $\mathcal{L}_j$ ($j \geq 1$) the rank is $z_j y_j(\tau)$. On $\hat{\mathcal{L}}_0$ it is the rank of $\hat{\mathcal{A}}$ there, that is, of the matrix with rows $(1, \varpi(x))$, $x \in X$, which is $1 + \dim(\varpi(\mathcal{P}))$. ■

If $\mathcal{P}$ is balanced, then $\varpi(\mathcal{P})$ is the single point $(\mathcal{R}_i)_{i \in \mathcal{J}}$, and Proposition 7 is Theorem 4.4 of Lee and Lee (1990). In general, $\varpi(\mathcal{P})$ is a polytope of dimension at most $|\mathcal{J}|$, so dropping the balance conditions costs one small extra computation. The proof of Proposition 2 instead adjoins $\mathbf{1}$ as a third fiber. The resulting 3×3 block has eigenvalues $\lambda_0, \lambda_1, \lambda_2$, and by Proposition 7 only its rank $1 + \dim(\varpi(\mathcal{P}))$ matters: for (7) it is 2, because ϖ maps the face onto a segment. Thus λ_1 and λ_2 , and part (i) of Appendix B, are not needed.

C.2 The fence configuration

Let D be a simple k -fence with $k \geq 4$, $E = F_1 \cup F_2$, $p_h = (u_h, w_h)$ and $q_{ab} = (w_a, u_b)$. The relations R_i , $i \in T_E = T \setminus \{1, 2, 3, 5, 11\}$, restricted to $E \times E$, are exactly the orbits of S_k (acting on indices) on $E \times E$. They form a coherent

configuration with fibers F_1, F_2 and 15 classes. For every S_k -invariant X , the polytope $\mathcal{P}$ is concordant with respect to it.

For an index h , let $y^{h,1} = e_{p_h}$, $y^{h,\text{out}} = \sum_{j \neq h} e_{q_{hj}}$ (pickets leaving w_h) and $y^{h,\text{in}} = \sum_{j \neq h} e_{q_{jh}}$ (pickets entering u_h), and let $\bar{y}$ denote projection orthogonal to $\mathcal{L}_0 = \text{span}\{d^1, d^2\}$, where d^1 and d^2 are the incidence vectors of F_1 and F_2 . Call a function φ on ordered pairs $a \neq b$ *balanced* if all its row sums $\sum_b \varphi(a, b)$ vanish, and let $f_\varphi = \sum \varphi(a, b) e_{q_{ab}}$.

Proposition 8 *The completely reduced form of the fence configuration has the shape*

j	0	1	2	3
e_j	2	3	1	1
z_j	1	$k-1$	$\frac{k(k-3)}{2}$	$\frac{(k-1)(k-2)}{2}$

with $\sum e_j^2 = 15$ and $\sum e_j z_j = k^2$. Its invariant subspaces have the following bases:

- $\mathcal{L}_0$: $\{d^1, d^2\}$;
- $\mathcal{L}_1$: $\{\bar{y}^{h,1}, \bar{y}^{h,\text{out}}, \bar{y}^{h,\text{in}} : h \neq h_0\}$, for any fixed index h_0 ;
- $\mathcal{L}_2$: $\{f_\varphi\}$ for φ ranging over a basis of the symmetric balanced functions (equivalently, of the kernel of the unsigned incidence matrix of K_k);
- $\mathcal{L}_3$: $\{f_\varphi\}$ for φ ranging over a basis of the antisymmetric balanced functions (equivalently, of the cycle space of K_k).

For $h \neq h_0$, $B(\omega) [\bar{y}^{h,1} \bar{y}^{h,\text{out}} \bar{y}^{h,\text{in}}] = [\bar{y}^{h,1} \bar{y}^{h,\text{out}} \bar{y}^{h,\text{in}}] \Delta_1(\omega)$, and $B(\omega) f_\varphi =$

$\Delta_2(\omega)f_\varphi$ or $\Delta_3(\omega)f_\varphi$ according as φ is symmetric or antisymmetric, where

$$\begin{aligned}\Delta_1(\omega) &= (s_{ij}) \text{ with } s_{11} = \omega_4 - \omega_6, \quad s_{21} = \omega_{13} - \omega_{14}, \quad s_{31} = \omega_{12} - \omega_{14}, \\ s_{12} &= (k-1)\omega_7 - \omega_8 - (k-2)\omega_9, \quad s_{13} = (k-1)\omega_8 - \omega_7 - (k-2)\omega_9, \\ s_{22} &= \omega_{10} + (k-2)\omega_{15} - \omega_{16} - (k-3)\omega_{18} - \omega_{20}, \\ s_{33} &= \omega_{10} + (k-2)\omega_{16} - \omega_{15} - \omega_{17} - (k-3)\omega_{18}, \\ s_{23} &= (k-2)\omega_{20} - \omega_{15} - \omega_{17} - (k-3)\omega_{18} + \omega_{19}, \\ s_{32} &= (k-2)\omega_{17} - \omega_{16} - (k-3)\omega_{18} + \omega_{19} - \omega_{20}, \\ \Delta_2(\omega) &= \omega_{10} - \omega_{15} - \omega_{16} - \omega_{17} + 2\omega_{18} + \omega_{19} - \omega_{20}, \\ \Delta_3(\omega) &= \omega_{10} - \omega_{15} - \omega_{16} + \omega_{17} - \omega_{19} + \omega_{20}.\end{aligned}$$

The shape follows from the decomposition of the permutation modules $\mathbb{C}^{F_1} = S^{(k)} \oplus S^{(k-1,1)}$ and $\mathbb{C}^{F_2} = S^{(k)} \oplus 2S^{(k-1,1)} \oplus S^{(k-2,2)} \oplus S^{(k-2,1,1)}$ of S_k . The blocks follow from Schur's lemma. Each entry is an elementary count, a polynomial in k of degree at most 2; we determined each by exact computation for $4 \leq k \leq 10$, which therefore fixes it. (The block Δ_0 , which is not needed below, is the lower right 2×2 part of the matrix of $B(\omega)$ on $\{e_{e_0}, d^1, d^2\}$.)

The reversal. Reversing every arc of D and exchanging $u_h \leftrightarrow w_h$ fixes each pale, maps $q_{ab} \mapsto q_{ba}$, and maps acyclic subgraphs to acyclic subgraphs. On the classes it exchanges $R_7 \leftrightarrow R_8$, $R_{12} \leftrightarrow R_{13}$, $R_{15} \leftrightarrow R_{16}$ and $R_{17} \leftrightarrow R_{20}$. If ω is invariant under this exchange (as are the τ_i of Section 2, and of every face below), then in the basis $\bar{y}^{h,1}$, $\bar{y}^{h,\text{out}} + \bar{y}^{h,\text{in}}$, $\bar{y}^{h,\text{in}} - \bar{y}^{h,\text{out}}$ the block Δ_1 splits as $\Delta'_1 \oplus \Delta''_1$:

$$\Delta'_1(\omega) = \begin{pmatrix} \omega_4 - \omega_6 & 2(k-2)(\omega_7 - \omega_9) \\ \omega_{12} - \omega_{14} & \omega_{10} + \omega_{19} + (k-4)(\omega_{15} + \omega_{17}) - 2(k-3)\omega_{18} \end{pmatrix},$$

$$\Delta''_1(\omega) = \omega_{10} + (k-2)(\omega_{15} - \omega_{17}) - \omega_{19}.$$

The three basis vectors are mutually orthogonal, so $\Delta'_1(\tau)$ is similar, by a diagonal matrix that depends only on k , to a symmetric positive semidefinite matrix.

What this says about Section 2. The last row of Table 1 lists the z_j (with $\lambda_0, \lambda_1, \lambda_2$ from the block on $\{e_{e_0}, d^1, d^2\}$), so the Vandermonde system is not needed. By an exact symbolic computation with the τ_i of Section 2, the eigenvalues

of $\Delta'_1(\tau)$ are λ_3, λ_4 , and $\lambda_5 = \Delta''_1(\tau)$, $\lambda_6 = \Delta_2(\tau)$ and $\lambda_7 = \Delta_3(\tau)$, which are linear in the τ_i . The zero eigenvalue λ_0 is the equation (7) itself.

C.3 The vertex faces, and a short proof of Proposition 2

For $0 \leq s \leq k$, let X_s be the set of points (8) with $t = s$: the pales p_h with h in an ordered s -tuple S , and every picket except the q_{ab} with $a, b \in S$ and a after b in S . Write $(x)_j = x(x-1) \cdots (x-j+1)$, so that $|X_s| = (k)_s$. Each X_s is invariant under S_k and under the reversal (reverse the tuple).

Proposition 9 *For $0 \leq s \leq k$, with $N = (k)_s$, the parameters of X_s satisfy*

$$\begin{aligned} \Delta'_1(\tau)_{11} &= \frac{N s(k-s)}{k(k-1)}, & \Delta'_1(\tau) \begin{pmatrix} s-1 \\ 1 \end{pmatrix} &= 0, & \Delta''_1(\tau) &= \frac{N(s)_2(s+1)}{6(k)_2}, \\ \Delta_2(\tau) &= \frac{N(s)_2(k-s)_2}{2(k)_4}, & \Delta_3(\tau) &= \frac{N(s)_2(3k-2s-2)}{6(k)_2(k-2)}. \end{aligned}$$

Proof. The first formula is $\tau_4 - \tau_6 = N \frac{s}{k} - N \frac{s(s-1)}{k(k-1)}$. For the second, every $x \in X_s$ satisfies the k equations

$$g_h : \quad (s-1)x_{p_h} + \sum_{j \neq h} (x_{q_{hj}} + x_{q_{jh}}) = 2(k-1) \quad (1 \leq h \leq k),$$

because if h is in position j of S , exactly $(j-1) + (s-j) = s-1$ of the pickets at h are absent, and none are absent if $h \notin S$. So $\mathcal{A}$ annihilates $(s-1)(y^{h,1} - y^{h',1}) + (y^{h,\text{out}} + y^{h,\text{in}}) - (y^{h',\text{out}} + y^{h',\text{in}})$, and hence so does $B(\tau)$. This vector has coordinates $(s-1, 1)$ in the basis of Δ'_1 . In Δ''_1, Δ_2 and Δ_3 the coefficients of the τ_i sum to zero, so we may replace each τ_i by $\tau'_i = |\{x \in X_s : x_f = x_g = 0\}|$ for $(f, g) \in R_i$. A picket q_{ab} is absent exactly when $a, b \in S$ and a follows b , and S is a uniformly random ordered s -tuple, so

$$\tau'_{10} = \frac{N(s)_2}{2(k)_2}, \quad \tau'_{15} = \tau'_{16} = \frac{N(s)_3}{3(k)_3}, \quad \tau'_{17} = \tau'_{20} = \frac{N(s)_3}{6(k)_3}, \quad \tau'_{18} = \frac{N(s)_4}{4(k)_4}, \quad \tau'_{19} = 0.$$

For example, R_{17} prescribes one of the $3!$ orders of three indices, and in R_{19} both

q_{ab} and q_{ba} would have to be absent. Substitution gives the stated values. $\blacksquare$

Proposition 10 *Let $k \geq 4$. The dimensions of the faces of $\mathcal{P}_{AC}(D)$ with point sets X_s ($0 \leq s \leq k$) and $X_t \cup X_{t+1}$ ($0 \leq t \leq k-1$) are as in Table 2. In particular, for $1 \leq t \leq k-2$ the t -reinforced k -fence inequality represents a facet of $\mathcal{P}_{AC}(D)$.*

Proof. By Proposition 7, the dimension of the face is

$$\dim(\varpi) + (k-1)(\text{rk } \Delta'_1 + \text{rk } \Delta''_1) + \frac{k(k-3)}{2} \text{rk } \Delta_2 + \frac{(k-1)(k-2)}{2} \text{rk } \Delta_3,$$

where each of the 1×1 blocks Δ''_1 , Δ_2 , Δ_3 has rank 1 if it is non-zero and rank 0 otherwise.

For X_s , $\varpi(X_s)$ is a point. By Proposition 9, Δ'_1 is singular. It is non-zero for $1 \leq s \leq k-1$; for $s \in \{0, k\}$ its first column vanishes ($\tau_4 = \tau_6$ and $\tau_{12} = \tau_{14}$), and then the kernel vector $(s-1, 1)$ forces it to vanish entirely. Also Δ''_1 and Δ_3 are non-zero exactly when $s \geq 2$, and Δ_2 exactly when $2 \leq s \leq k-2$.

For $X_t \cup X_{t+1}$, ϖ is an edge of dimension 1, and the parameters, hence the blocks, are sums over the two sets. By Proposition 9, every scalar block is a sum of non-negative terms. For $1 \leq t \leq k-2$, the matrix Δ'_1 is (up to the fixed similarity) a sum of two rank-one positive semidefinite matrices with distinct kernels $(t-1, 1)$ and $(t, 1)$, so it has rank 2. Moreover, Δ''_1 , Δ_2 and Δ_3 are positive, because $t+1 \geq 2$, and because one of $t, t+1$ lies in $[2, k-2]$. Hence the dimension is $1 + 3(k-1) + \frac{k(k-3)}{2} + \frac{(k-1)(k-2)}{2} = k^2 - 1$. The cases $t = 0$ and $t = k-1$ are similar. $\blacksquare$

Compared with the proof in Section 2, this argument uses no symbolic computation, no Vandermonde system and none of the eigenvalue formulas. As [Lee and Lee \(1990\)](#) anticipated, the marginal work is calculating a few of the parameters τ_i , and here only for the vertex sets X_s .

C.4 All symmetric faces

If an acyclic subgraph uses s pales p_h , $h \in H$, then a picket q_{ab} can lie on a cycle only if $a, b \in H$. Every cycle alternates pales and pickets, so the pickets q_{ab} with $a, b \in H$, read as arcs $a \rightarrow b$, must form an acyclic digraph on H , which has at most $\binom{s}{2}$ arcs. Hence $\varpi(\mathcal{P}_{AC}(D))$ is the polygon

$$\Pi = \left\{ (\xi, \eta) : \xi \geq 0, \eta \geq 0, \xi \leq k, t\xi + \eta \leq k(k-1) + \binom{t+1}{2} \ (0 \leq t \leq k-1) \right\},$$

with vertices $(0,0)$, $(k,0)$ and $V_s = (s, k(k-1) - \binom{s}{2}) = \varpi(X_s)$, and edges on the lines $\xi = 0$, $\eta = 0$, $\xi = k$ and $E_t = [V_t, V_{t+1}]$. An S_k -invariant face is exposed by an invariant inequality (average any exposing inequality over S_k), and such an inequality is constant on F_1 and on F_2 . So the S_k -invariant faces of $\mathcal{P}_{AC}(D)$ are exactly the preimages of the faces of Π . They are listed in Table 2, together with a maximal irredundant system of equations for each, as in Section 5 of [Lee and Lee \(1990\)](#). Because the dimensions are known, it suffices that the listed equations are valid and linearly independent, and that their number is the codimension.

face (preimage of)	dimension	maximal irredundant equation system
V_1	$k-1$	$x_q = 1 \ (q \in F_2); \sum_h x_{p_h} = 1$
$V_s, 2 \leq s \leq k-2$	$k^2 - k - 1$	$\sum_h x_{p_h} = s; \sum_q x_q = k(k-1) - \binom{s}{2}; g_h \ (h \neq h_0)$
V_{k-1}	$\frac{(k-1)(k+2)}{2}$	$\sum_h x_{p_h} = k-1; c_{ab}: x_{q_{ab}} + x_{q_{ba}} + x_{p_a} + x_{p_b} = 3 \ (a < b)$
V_k	$\binom{k}{2}$	$x_{p_h} = 1 \ (\text{all } h); x_{q_{ab}} + x_{q_{ba}} = 1 \ (a < b)$
$E_0: \eta \leq k(k-1)$	k	$x_q = 1 \ (q \in F_2)$
$E_t, 1 \leq t \leq k-2: (7)$	$k^2 - 1$	$(7) \text{ with equality}$
E_{k-1}	$\binom{k+1}{2}$	$c_{ab} \ (a < b)$
$\xi \geq 0$	$k(k-1)$	$x_p = 0 \ (p \in F_1)$
$\eta \geq 0$	k	$x_q = 0 \ (q \in F_2)$
$\xi \leq k$	$k(k-1)$	$x_p = 1 \ (p \in F_1)$

Table 2: The faces of $\mathcal{P}_{AC}(D)$ invariant under S_k ($k \geq 4$), besides the three single points over $(0,0)$, $(k,0)$ and V_0 . Here g_h is as in the proof of Proposition 9, and h_0 is any fixed index.

The dimensions of V_s and E_t are those of Proposition 10. The face $\xi \geq 0$ is

the cube of picket sets, $\eta \geq 0$ the cube of pale sets, and $\xi \leq k$ is affinely isomorphic to the full-dimensional acyclic subgraph polytope of the complete digraph on the k indices. Validity of the equations was shown above (for g_h) or is immediate: on $X_{k-1} \cup X_k$, either all pales are present and exactly one of q_{ab}, q_{ba} is, or only p_m is missing, in which case q_{ab}, q_{ba} are both present when $m \in \{a, b\}$ and exactly one of them is present otherwise. Each system is independent, because each c_{ab} is the only equation involving q_{ab} , and for $s \geq 2$ the pale coefficients separate g_h from $\sum x_{p_h}$. As in Lee and Lee (1990), the equations arise from the invariant subspaces on which the blocks vanish. For instance, for E_{k-1} the $\binom{k}{2}$ equations c_{ab} span the equation (7) from $\mathcal{L}_0$, $k-1$ equations from the kernel of Δ'_1 , and the $\frac{k(k-3)}{2}$ equations of $\mathcal{L}_2$, where $\Delta_2 = 0$.

Proposition 11 *For $k \geq 3$, the facets of $\mathcal{P}_{AC}(D)$ that are invariant under S_k are exactly those represented by the t -reinforced k -fence inequalities (7), $1 \leq t \leq k-2$. Every other S_k -invariant face has dimension at most $k^2 - k$.*

Proof. For $k \geq 4$ this is read off from Table 2 (a facet must lie over an edge of Π , because the preimage of a vertex lies in the preimage of an incident edge). For $k = 3$, the facet $t = 1$ is Proposition 2. The remaining faces have dimension at most $k(k-1)$ by the elementary bounds: the faces $\xi \geq 0$ and $\xi \leq k$ lie in $\{x \in \mathbf{R}^E : x_e = 0 \text{ for all } e \in F_1\}$ and $\{x \in \mathbf{R}^E : x_e = 1 \text{ for all } e \in F_1\}$, E_0 has $k+1$ points, and E_{k-1} satisfies the equations c_{ab} . ■

C.5 Non-simple fences

Subdividing arcs breaks the symmetry of D down to a subgroup, and Proposition 7 still applies, with a coarser configuration. It turns out that no further spectral computation is needed: subdivision changes dimensions additively, by the following two elementary facts. For an arc e subdivided into m pieces, the lift

of an inequality $a^T x \leq a_0$ gives each piece the coefficient a_e and has right-hand side $a_0 + (m - 1)a_e$, as in Proposition 3. Say that a face F *misses* e if some point of F does not contain e .

Proposition 12 *Let $a^T x \leq a_0$ be valid for $\mathcal{P}_{AC}(G)$, with face F , and let $a_e > 0$. Subdivide e into m pieces. Then the lifted inequality is valid for the new graph, and its face F' consists of the points obtained from the points $x \in F$ by taking all pieces if $x_e = 1$, and all pieces but one (in any of m ways) if $x_e = 0$. Hence*

$$\dim(F') = \begin{cases} \dim(F) + m - 1, & \text{if } F \text{ misses } e; \\ \dim(F), & \text{otherwise.} \end{cases}$$

Proof. An acyclic subgraph of the new graph contains all pieces of e only if its contraction contains e , so the sum of the piece variables is at most $x_e + m - 1$, with equality exactly as described; this gives validity and the description of F' , because $a_e > 0$. On F' the pieces satisfy one equation, and if F misses e , then varying the absent piece at such a point gives $m - 1$ new affinely independent directions; otherwise all pieces equal 1 on F' . ■

Proposition 13 *Let h be a node whose only arcs are $e_1 = (i, h)$ and $e_2 = (h, j)$, and let $c^T x \leq c_0$ represent a facet F of $\mathcal{P}_{AC}(G)$. If $c_{e_1} \neq c_{e_2}$, then F is contained in one of $\{x \in \mathbf{R}^A : x_{e_1} = 1\}$, $\{x \in \mathbf{R}^A : x_{e_2} = 1\}$, $\{x \in \mathbf{R}^A : x_{e_1} = 0\}$, $\{x \in \mathbf{R}^A : x_{e_2} = 0\}$. If $c_e < 0$ for any arc e , then $F \subseteq \{x \in \mathbf{R}^A : x_e = 0\}$.*

Proof. Deleting an arc preserves acyclicity, so if $c_e < 0$ no tight point contains e . Let $c_{e_1} > c_{e_2}$; the other case is symmetric. Exchanging e_2 for e_1 preserves acyclicity, because neither set contains the whole path. So no tight point contains e_2 but not e_1 . If some tight point contains neither, then adding e_1 shows $c_{e_1} \leq 0$, so $c_{e_2} < 0$ and $F \subseteq \{x \in \mathbf{R}^A : x_{e_2} = 0\}$. Otherwise every tight point contains e_1 , and $F \subseteq \{x \in \mathbf{R}^A : x_{e_1} = 1\}$. ■

Lifted symmetric faces. Let D' be any k -fence ($k \geq 4$), with M_1 subdivision nodes on pales and M_2 on pickets, so $|A'| = k^2 + M_1 + M_2$. The faces E_t ($1 \leq$

$t \leq k-1$) and V_s ($1 \leq s \leq k$) of Table 2 are exposed by inequalities with positive coefficients on all arcs. For V_s use the sum of the inequalities of E_{s-1} and E_s ; for V_k add $\xi \leq k$ to E_{k-1} . So Proposition 12 applies arc by arc. Every one of these faces misses some pale, except V_k , and misses some picket, except V_1 . Hence their lifts to $\mathcal{P}_{AC}(D')$ have dimensions

lifted face	dimension	lifted face	dimension
V_1	$k-1+M_1$	V_k	$\binom{k}{2}+M_2$
$V_s, 2 \leq s \leq k-2$	$k^2-k-1+M_1+M_2$	$E_t, 1 \leq t \leq k-2$	$ A' - 1$
V_{k-1}	$\frac{(k-1)(k+2)}{2} + M_1 + M_2$	E_{k-1}	$\binom{k+1}{2} + M_1 + M_2$

The entry for E_t is Proposition 1 for D' itself. In passing to D' the codimension grows by M_2 for V_1 and by M_1 for V_k (the new equations are $x = 1$ on every piece of every picket, respectively of every pale), and is otherwise unchanged.

Proposition 14 *Let $k \geq 3$, and let D' be obtained from a simple k -fence by subdividing every pale into a pieces and every picket into b pieces. Then the facets of $\mathcal{P}_{AC}(D')$ that are invariant under S_k are exactly the lifts of the t -reinforced k -fence inequalities, $1 \leq t \leq k-2$.*

Proof. The lifts are facets by Proposition 1. Conversely, let F be an invariant facet. It is not contained in any $\{x \in \mathbf{R}^{A'} : x_e = 0\}$ or $\{x \in \mathbf{R}^{A'} : x_e = 1\}$, because by invariance it would then lie in at least $k \geq 3$ such hyperplanes. So by Proposition 13 its coefficients are constant along each subdivided arc and non-negative, and by invariance they take one value α on all pale pieces and one value β on all picket pieces. If $\beta = 0$, then F lies where all pale pieces are present, which is excluded; likewise if $\alpha = 0$. So $\alpha, \beta > 0$, and F is the lift of the invariant face F_0 of $\mathcal{P}_{AC}(D)$ exposed by $\alpha\xi + \beta\eta \leq c_0 - (a-1)k\alpha - (b-1)k(k-1)\beta$. This inequality is valid, because every point of $\mathcal{P}_{AC}(D)$ lifts to a point of $\mathcal{P}_{AC}(D')$ with the same value. By Proposition 12, $\dim(F_0) \geq \dim(F) - (a-1)k - (b-1)k(k-1) = k^2 - 1$. So F_0 is an invariant facet of $\mathcal{P}_{AC}(D)$, and Proposition 11 completes the proof. ■

Appendix D

This appendix applies the spectral method of Appendix C to two further polytopes: the linear ordering polytope of Section 4, and an unbalanced clique face from the setting of Lee and Lee (1990). Both rest on a convenient way of evaluating the ranks $y_j(\tau)$.

D.1 Block ranks are small spans

Let a finite group Γ act on E , and let $X \subseteq \mathbf{R}^E$ be finite and Γ -invariant. The orbits of Γ on $E \times E$ form a coherent configuration with respect to which $\text{conv}(X)$ is concordant. Its inequivalent blocks Δ_j correspond to the irreducible Γ -modules V_j in $\mathbb{C}^E$: the order e_j is the multiplicity of V_j , and $z_j = \dim V_j$.

Proposition 15 *Let $\iota_1, \dots, \iota_{e_j} : V_j \rightarrow \mathbf{R}^E$ be linearly independent Γ -equivariant maps, and let $0 \neq v \in V_j$. Then*

$$y_j(\tau) = \text{rank } \Delta_j(\tau) = \dim \text{span} \left\{ (\langle \iota_1(v), x \rangle, \dots, \langle \iota_{e_j}(v), x \rangle) : x \in X \right\}.$$

Proof. Changing the ι_r by an invertible matrix changes neither side, so we may take them isometric with orthogonal images. $B = \mathcal{A}^T \mathcal{A} = \sum_{x \in X} x x^T$ commutes with Γ . By Schur's lemma it acts on $\bigoplus_r \iota_r(V_j)$ as $M \otimes I$, where M is equivalent to $\Delta_j(\tau)$ and $M_{rr'} = \langle \iota_r(v), B \iota_{r'}(v) \rangle / \|v\|^2 = \sum_x \langle \iota_r(v), x \rangle \langle \iota_{r'}(v), x \rangle / \|v\|^2$. Thus $M = W^T W / \|v\|^2$, where W has rows $(\langle \iota_r(v), x \rangle)_r$, and $\text{rank } M = \text{rank } W$. ■

With Proposition 7, the dimension of a Γ -invariant face is $\dim(\varpi)$ plus $\sum_j z_j$ times the dimension of the span of at most e_j explicit statistics, and no parameter τ_i needs to be computed. For the fence configuration, the statistics for $\mathcal{L}_1$ are, with $c = e_1 - e_2$, the differences between indices 1 and 2 of the pale indicator, the number of pickets leaving w_h , and the number entering u_h . For $\mathcal{L}_2$ and $\mathcal{L}_3$ they are $\langle f_\varphi, x \rangle$ for one symmetric, respectively antisymmetric, balanced φ . In

this form Proposition 10 reduces to exhibiting a few points, and that is how it is used below.

D.2 The linear ordering polytope

Let D be a simple k -fence on $N = U \cup W$, $k \geq 4$, and consider $\mathcal{P}_{LO}^{2k}$. By (10) each $x(j, i) = 1 - x(i, j)$, so we may use the coordinates $E' = F_1 \cup F_2 \cup UU \cup WW$, where UU and WW contain all ordered pairs within U and within W ; affine dimensions are unchanged. The group S_k acts on indices, with fibers F_1, F_2, UU, WW , and $\varpi(x) = (\xi, \eta, \binom{k}{2}, \binom{k}{2})$ with $\xi = \xi(x) = \sum_{F_1} x$ and $\eta = \eta(x) = \sum_{F_2} x$. For index h let

$$\begin{aligned} \nu(h) = & (x(u_h, w_h), \#\{b : w_h \prec u_b\}, \#\{a : w_a \prec u_h\}, \\ & \#\{b : u_h \prec u_b\}, \#\{b : w_h \prec w_b\}) \in \mathbf{R}^5, \end{aligned}$$

where $\prec$ is the order. (The in-degree statistics in UU and WW are complements of the out-degrees.) For indices a, b, c , let

$$\psi = (\psi_F, \psi_U, \psi_W), \quad \psi_F = \sum_{\text{cyc}} (x(w_a, u_b) - x(w_b, u_a)),$$

$$\psi_U = \sum_{\text{cyc}} (x(u_a, u_b) - x(u_b, u_a)),$$

with ψ_W analogous. On the copies of $S^{(k-2,2)}$ in UU and WW , every point of $\mathcal{P}_{LO}^{2k}$ gives 0. So Propositions 7 and 15 give, for every S_k -invariant set $\mathcal{O}$ of linear orders,

$$\dim = \dim(\varpi) + (k-1)r_1 + \frac{k(k-3)}{2}r_2 + \frac{(k-1)(k-2)}{2}r_3, \quad (\text{D1})$$

where

- $r_1 \leq 5$ is the dimension of the span of $\nu(h) - \nu(h')$ ($h \neq h'$),
- $r_2 \leq 1$ is the dimension of the span of $\langle f_\varphi, x \rangle$, for symmetric φ on F_2 , and

- $r_3 \leq 3$ is the dimension of the span of ψ .

For $\mathcal{P}_{LO}^{2k}$ itself, (D1) gives $2 + 5(k-1) + \frac{k(k-3)}{2} + \frac{3(k-1)(k-2)}{2} = \binom{2k}{2}$, as it should.

Proposition 16 *Let $0 \leq s \leq k$. A linear order of $U \cup W$ has s pales and the maximum number $k(k-1) - \binom{s}{2}$ of pickets if and only if it has the form*

$$(w_h : h \notin H, \text{ any order}) \quad u_{h_1} w_{h_1} u_{h_2} w_{h_2} \cdots u_{h_s} w_{h_s} \quad (u_h : h \notin H, \text{ any order})$$

for an ordered s -tuple $H = (h_1, \dots, h_s)$. There are $k!(k-s)!$ such orders; call this set Y_s . Its incidence vectors restricted to $F_1 \cup F_2$ form the set X_s of Section C.3, each point repeated $((k-s)!)^2$ times.

Proof. By the bound of Section C.4 the maximum forces every picket q_{ab} with $a \notin H$ or $b \notin H$ to be present, and exactly one of q_{ab}, q_{ba} for $a, b \in H$. So each w_a ($a \notin H$) precedes every u , each u_b ($b \notin H$) follows every w , and the intervals $[u_h, w_h]$, $h \in H$, are pairwise disjoint. ■

Reversing an order is a symmetry of $\mathcal{P}_{LO}^{2k}$ that maps (ξ, η) to $(k-\xi, k(k-1)-\eta)$. So $\varpi(\mathcal{P}_{LO}^{2k})$ is the polygon Π_{LO} with upper vertices $V_s = (s, k(k-1) - \binom{s}{2})$ and lower vertices $\bar{V}_s = (s, \binom{k-s}{2})$, $0 \leq s \leq k$, and with edges

- $E_t = [V_t, V_{t+1}]$, the faces of the t -reinforced inequalities (7);
- $\bar{E}_t = [\bar{V}_t, \bar{V}_{t+1}]$, the images of E_{k-1-t} under reversal; and
- the vertical edges on the lines $\xi = 0$ and $\xi = k$.

The faces $\bar{E}_t$ are the $(k-1-t)$ -reinforced inequalities of the reversed fence $\bar{D}$, which has pales (w_h, u_h) and pickets (u_b, w_a) .

Proposition 17 *Let $k \geq 4$. The S_k -invariant faces of $\mathcal{P}_{LO}^{2k}$ are the preimages of the faces of Π_{LO} , with*

face	(r_1, r_2, r_3)	dimension	face	(r_1, r_2, r_3)	dimension
V_0	$(2, 0, 2)$	$k(k-1)$	E_0	$(3, 0, 2)$	k^2
V_1	$(3, 0, 2)$	$k^2 - 1$	$E_t, 1 \leq t \leq k-2$	$(5, 1, 3)$	$\binom{2k}{2} - 1$
$V_s, 2 \leq s \leq k-2$	$(4, 1, 3)$	$\binom{2k}{2} - k - 1$	E_{k-1}	$(2, 0, 1)$	$\binom{k+1}{2}$
V_{k-1}	$(2, 0, 1)$	$\frac{(k-1)(k+2)}{2}$	$\xi = 0$	$(4, 1, 3)$	$\binom{2k}{2} - k$
V_k	$(1, 0, 1)$	$\binom{k}{2}$			

and $\bar{V}_s \cong V_{k-s}$, $\bar{E}_t \cong E_{k-1-t}$, $\{x \in \mathcal{P}_{LO}^{2k} : \xi(x) = k\} \cong \{x \in \mathcal{P}_{LO}^{2k} : \xi(x) = 0\}$ under reversal. In particular, the S_k -invariant facets of $\mathcal{P}_{LO}^{2k}$ are exactly the t -reinforced k -fence inequalities of D and of $\bar{D}$, $1 \leq t \leq k-2$. By Proposition 6 the former represent facets of $\mathcal{P}_{LO}^n$ for every $n \geq 2k$; this is Proposition 5.

Proof. The first claim is as in Section C.4. For the ranks, Proposition 16 gives the statistics of Y_s explicitly. An index $h \in H$ in position j has $\nu(h) = (1, k-j, k-1-s+j, k-j, s-j)$. An index $h \notin H$ whose w_h is in position α among the w 's outside H , and whose u_h is in position β among the u 's outside H , has $\nu(h) = (0, k-1, k-1, k-s-\beta, k-\alpha)$.

- r_1 for Y_s . Two indices in H contribute the direction $d = (0, -1, 1, -1, -1)$ (if $s \geq 2$). Two indices outside H , whose α and β can be permuted independently, contribute e_4 and e_5 (if $k-s \geq 2$). An index in H against one outside gives a vector $\equiv (1, 0, 1-s, 0, 0)$ modulo e_4, e_5 (if $1 \leq s \leq k-1$). This gives the column r_1 for the V_s .
- r_1 for E_t , $1 \leq t \leq k-2$. Both $s = t$ and $s = t+1$ occur, so $(1, 0, 1-t) - (1, 0, -t)$ yields e_3 , and then d yields e_2 , giving $r_1 = 5$. For E_0 and E_{k-1} the same bookkeeping gives 3 and 2.
- r_3 . If $a, b, c \in H$, then $\psi_F = \psi_U = \psi_W$ is the orientation of the tuple. If $c \notin H$ and $a, b \in H$, all three equal the orientation of (a, b) in H . If at most one of them lies in H , then $\psi_F = 0$ and $\psi_U, \psi_W = \pm 1$ independently. So $r_3 = 3$ when $2 \leq s \leq k-2$ (or both kinds occur), $r_3 = 2$ when $s \leq 1$, and $r_3 = 1$ when $s \geq k-1$.

- r_2 . By Proposition 16 this is the same as for X_s , so by Proposition 9 it is 1 exactly when some $s \in [2, k-2]$ occurs.
- *The face $\xi = 0$.* It contains Y_0 , which gives e_4, e_5 and $r_3 \geq 2$. It contains the reversal of Y_k , which gives d and $\psi_F \neq 0$. It contains $w_1 u_1 w_2 \cdots w_k u_2 \cdots u_k$, which against $w_1 \cdots w_k u_1 \cdots u_k$ gives $(0, 1, -(k-1), 0, 0)$. And it contains $w_1 w_2 u_1 u_2 w_3 \cdots w_k u_3 \cdots u_k$, on which $\langle f_\varphi, x \rangle = 2$ for $\varphi = e_{12} + e_{34} - e_{13} - e_{24}$. Also $r_1 \leq 4$, because $x(u_h, w_h) = 0$.

Finally, $\dim(\varpi)$ is 0 on vertices and 1 on edges, and (D1) gives the table. ■

D.3 An unbalanced clique face

The faces treated by Lee and Lee (1990) are balanced. Proposition 7 removes this restriction, and their blocks for the clique scheme (their Tables 2 and 3) can then be used unchanged. As an example, take the *forest polytope* of K_n , the convex hull of the incidence vectors of forests, which is full-dimensional, and the clique inequality $x(E(K_p)) \leq p-1$. Its tight forests contain a spanning tree of K_p , but the numbers of their edges in the cut E_9 and outside E_{18} vary, so the face is not balanced.

Proposition 18 *For $p \geq 4$ and $n - p \geq 4$, the inequality $x(E(K_p)) \leq p - 1$ represents a facet of the forest polytope of K_n .*

Proof. The group is $S_p \times S_{n-p}$, and Lee and Lee's invariant subspaces $\mathcal{L}_1, \dots, \mathcal{L}_5$ (their Section 5) supply the statistics of Proposition 15. Write $m = n - p$. Here $\dim(\varpi) \leq 2$, because the inside sum is fixed, and it equals 2, because a spanning tree of K_p together with no edge, one cut edge, or one outside edge gives three affinely independent images. For the blocks:

- $\mathcal{L}_3$ ($e_3 = 2$): for inside vertices a, b , the differences of inside degree and of cut degree. A star at a and a star at b give $(\pm(p-2), 0)$, and adding a cut edge at a gives $(\cdot, 1)$; so $y_3 = 2$.
- $\mathcal{L}_1$ ($e_1 = 2$): for outside vertices a, b , the differences of cut degree and of outside degree. A tree plus a cut edge at a gives $(1, 0)$, and a tree plus an edge from a to a third outside vertex gives $(0, 1)$; so $y_1 = 2$.
- $\mathcal{L}_2, \mathcal{L}_5$ ($e = 1$): with $\varphi = e_{ab} + e_{cd} - e_{ac} - e_{bd}$, a tree plus the outside edge ab gives value 1, and a spanning tree of K_p containing ab, bc and cd (for inside a, b, c, d) gives value 2.
- $\mathcal{L}_4$ ($e = 1$): Lee and Lee's vector $t^{(j,k)}$ takes the value 1 on a tree plus the cut edge (j, k) .

Hence, with their z_j ,

$$\dim = 2 + 2(m-1) + \left(\binom{m}{2} - m\right) + 2(p-1) + (p-1)(m-1) + \left(\binom{p}{2} - p\right) = \binom{n}{2} - 1. \quad \blacksquare$$

This facet is classical (it is a closed, connected flat of the graphic matroid). The point is that the spectral method proves it by exhibiting a handful of forests, one for each block, and without any parameter τ_i . Propositions 7 and 15 can be applied in the same way to other unbalanced faces in the clique scheme. The odd-clique faces of the bipartite subgraph polytope would be a natural next test.

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