

Robust and Distributionally Robust Risk Budgeting with Mean Absolute Deviation

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Abstract

We propose robust and distributionally robust formulations of the risk budgeting problem under the mean absolute deviation (MAD) risk measure. The proposed models address uncertainty in asset returns and their distribution while allocating portfolio risk according to prescribed risk budgets. We first develop a robust model in which historical return observations are perturbed within a correlated budget uncertainty set. Portfolio deviations are measured relative to the mean of the same perturbed empirical distribution, accounting for the effect of uncertainty on both returns and their mean. We then introduce a distributionally robust model in which the return distribution is allowed to vary within a Wasserstein ambiguity set centered at the empirical distribution, and portfolios are constructed by considering the worst-case MAD over the admissible distributions. Tractable optimization formulations are derived for both models. Numerical experiments using real-world stock market data compare the proposed approaches with nominal risk budgeting, minimum MAD, and standard-deviation risk budgeting models. The results suggest that accounting for uncertainty can improve portfolio stability and out-of-sample performance. In particular, the proposed models achieve the lowest turnover and the highest average out-of-sample expected return, Sharpe ratio, and Calmar ratio, together with the smallest average deviations from the risk contributions.

Keywords: Risk budgeting, Mean absolute deviation, Robust optimization, Distributionally robust optimization.

1 Introduction

Portfolio management has been widely studied in both finance and optimization. A foundational model in this area was introduced by [Markowitz \(1952\)](#), where portfolio selection

is formulated by considering the tradeoff between expected return and variance. This mean-variance framework has become one of the main foundations of modern portfolio optimization. However, its performance is highly dependent on the accuracy of the estimated input parameters. For example, [Merton \(1980\)](#) showed that errors in estimating expected returns can have a strong effect on the resulting portfolio. In addition, [Tütüncü and Koenig \(2004\)](#) showed that portfolios obtained from nominal mean-variance models may be highly concentrated, with positive investment in only a small number of assets and zero weights assigned to many others. These limitations have encouraged researchers to study portfolio construction methods that are less sensitive to estimation errors and can produce more stable allocations. One simple benchmark is to invest equally among all assets, which [DeMiguel et al. \(2009\)](#) showed can outperform the mean-variance model in terms of out-of-sample Sharpe ratio. Another approach is risk budgeting, whose main idea is to allocate the total portfolio risk among the assets according to given risk budgets. In this approach, each asset has a prescribed risk budget, and the portfolio weights are chosen so that the relative risk contribution of each asset matches its corresponding budget. When the total portfolio risk is divided equally among all assets, the resulting portfolio is called the risk parity portfolio.

Many studies have developed optimization models for the risk budgeting problem. Early works mainly focused on standard risk parity formulations, including the risk parity model proposed by [Maillard et al. \(2010\)](#). Later, more general formulations were studied. [Roncalli \(2014a\)](#) proposed least-squares formulations of the risk budgeting problem, while [Costa and Kwon \(2020b\)](#) developed more flexible models that can also allow short selling. [Bayat et al. \(2024a\)](#) also considered risk budgets as decision variables and proposed a bi-level risk budgeting model. Another important line of this literature concerns the choice of risk measure in this approach. Standard deviation was used in [Maillard et al. \(2010\)](#), a mean-standard deviation risk measure was considered in [Bai et al. \(2016\)](#), value-at-risk was studied in [Darolles et al. \(2015\)](#), and conditional value-at-risk was used in [Boudt et al. \(2012\)](#) to control tail risk. These studies show that the risk budgeting idea can be used with different ways of measuring portfolio risk. More recently, more general and non-differentiable risk measures have also been considered. [da Costa et al. \(2023\)](#) approximated risk budgeting portfolios by using simulated return scenarios, [Ararat et al. \(2024\)](#) studied

mean absolute deviation risk parity portfolios, and [Mausser and Romanko \(2018\)](#) developed a CVaR-based formulation under discrete distributions. In parallel with these modeling developments, some studies have focused on the theoretical and computational aspects of risk budgeting. In this direction, [Cetingoz et al. \(2024\)](#) studied the existence, uniqueness, and computation of risk budgeting portfolios for a broad class of risk measures, while [Arnaiz Iglesias et al. \(2026\)](#) proposed deterministic and stochastic mirror descent algorithms for computing risk budgeting portfolios under general positively homogeneous and sub-additive risk measures.

The basic risk budgeting framework has also been extended in several directions to make it more useful in practice. Some studies include a minimum expected return constraint [Gambeta and Kwon \(2020\)](#), while others impose explicit bounds on portfolio weights [Richard and Roncalli \(2019\)](#). Another important practical issue is the number of assets in the portfolio. To address this, cardinality constraints have been studied in risk budgeting models. In this direction, [Anis and Kwon \(2022\)](#) developed a mixed-integer risk parity model, [Kaucic \(2019\)](#) proposed a heuristic method for equity portfolios with cardinality limits, and [Feng and Palomar \(2016\)](#) used a nonconvex penalization approach to include cardinality requirements directly in the optimization problem. Beyond these asset-level constraints, a recent direction is to consider diversification not only at the asset level, but also at the factor level. [Cetingoz and Guéant \(2025\)](#) proposed an asset-factor risk budgeting framework that balances asset risk contributions and factor risk contributions in the same portfolio construction model. Dynamic and data-driven extensions have also been proposed. [Li et al. \(2022\)](#) combined risk parity with model predictive control and rolling reoptimization, while [Pesenti et al. \(2025\)](#) extended the idea to time-consistent dynamic risk measures. Another related direction uses hierarchical structures in portfolio construction, where [Koumou \(2024\)](#) proposed a hierarchical risk budgeting approach based on a filtered similarity matrix generated by hierarchical clustering. Recent studies have also combined risk budgeting and risk-based portfolio construction with machine learning methods. [Uysal et al. \(2024\)](#) developed an end-to-end neural network framework for risk budgeting portfolio optimization, where the network learns risk budgets and uses an optimization layer to obtain the portfolio allocation. In a related direction, [Millea and Edalat \(2022\)](#) used deep reinforcement learning to select among hierarchical risk parity and hierarchical equal risk

contribution models over time.

Overall, this literature shows that risk budgeting is a flexible portfolio construction approach that can be combined with different risk measures, constraints, computational methods, and learning-based tools. However, most of these studies focus on models where the input data is treated as fixed after estimation. In practice, these quantities are estimated from historical data and may be affected by sampling error and distributional changes. This motivates the study of risk budgeting under uncertainty, which is the focus of the next subsection.

1.1 Background

Uncertainty plays a central role in portfolio management. Portfolio models are built using historical data, while their inputs, including returns, covariance matrices, and return distributions, are not known exactly and must be estimated from historical data. These estimates may be affected by statistical errors, especially when data are limited, leading to uncertainty, as noted by [Li and Xiang \(2026\)](#). Therefore, uncertainty in the model inputs and their distributions should be explicitly considered. This issue is especially important in risk budgeting, because risk contributions depend on the chosen risk measure and on the data used to estimate it. To deal with this issue, several studies have considered risk budgeting under uncertainty.

[Costa and Kwon \(2020a\)](#) proposed a robust optimization formulation based on a box uncertainty set for the covariance matrix. [Kapsos et al. \(2018\)](#) introduced another robust risk budgeting model that allows both continuous and discrete uncertainty sets. More recently, [Li and Ye \(2026\)](#) proposed a robust risk parity strategy in which covariance uncertainty is handled through an adaptive perturbation set, with the perturbation radius adjusted using volatility forecasting and market regime information. A data-driven distributionally robust approach to the risk parity problem was proposed by [Costa and Kwon \(2022\)](#). Following this line of research, [Bayat et al. \(2025\)](#) studied a distributionally robust risk parity model with the mean-standard deviation risk measure under ambiguity in the variance distribution, while [Bayat et al. \(2024b\)](#) developed a scenario-based stochastic programming formulation of the risk budgeting problem.

Table 1 summarizes representative studies on risk budgeting under uncertainty. It

classifies the existing literature according to the uncertainty framework, namely stochastic programming (SP), robust optimization (RO), and distributionally robust optimization (DRO), and highlights the corresponding risk measures, sources of uncertainty, and key modeling features. The final row summarizes the models proposed in this paper. As

Table 1: Representative risk budgeting models under uncertainty.

Reference	Risk measure	Source of uncertainty	SP	RO	DRO	Key modeling feature
Li and Ye (2026)	Volatility	Covariance matrix		✓		Adaptive covariance perturbation with factor structure.
Bayat et al. (2025)	Mean-standard deviation	Variance distribution			✓	Variance-distribution ambiguity set.
Bayat et al. (2024b)	Mean-standard deviation	Scenario uncertainty for expected return and covariance	✓			Scenario-based stochastic formulation.
Costa and Kwon (2022)	Variance	Return distribution			✓	Data-driven distributional robustness.
Costa and Kwon (2020a)	Variance	Covariance matrix		✓		Box uncertainty for covariance.
Kapsos et al. (2018)	Variance	Covariance matrix		✓		Continuous and discrete uncertainty sets.
Present study	Mean absolute deviation	Asset returns and return distribution		✓	✓	Correlated budget and Wasserstein uncertainty sets.

shown in Table 1, the existing literature has mainly focused on differentiable risk measures, especially variance, volatility, and mean-standard deviation. These studies show that modeling uncertainty is useful for improving the reliability of risk budgeting portfolios. However, the MAD risk measure has received limited attention in uncertainty-based risk budgeting, leaving a gap in the literature. MAD is non-differentiable, which makes the risk budgeting problem more challenging than with other risk measures. To address this gap, this paper develops an uncertainty-based approach for MAD risk budgeting models.

1.2 Main contributions and paper organization

Although risk budgeting under uncertainty has been studied through stochastic programming, robust optimization, and distributionally robust optimization, an important gap still remains. Existing works mainly focus on differentiable risk measures and do not study uncertainty in MAD-based risk budgeting. In this paper, we focus on the risk budgeting problem under the MAD risk measure. However, in practice, future asset returns and their probability distributions are not known exactly and must be estimated from historical data. Therefore, portfolios obtained from nominal MAD models may be sensitive to sampling er-

ror. To address this issue, we develop two uncertainty-aware formulations. The first is a robust optimization framework, in which uncertainty is modeled through perturbations of historical return observations using a correlated budget uncertainty set. The second is a distributionally robust framework, in which the return distribution is allowed to vary within a Wasserstein ambiguity set around the empirical distribution. These two formulations provide different but related approaches to improving the stability of MAD-based risk budgeting portfolios when asset returns and their distributions are uncertain. In general, the main contributions of this paper are as follows:

We propose a robust MAD risk budgeting model based on a correlated budget uncertainty set. In this formulation, uncertainty is modeled through perturbations of historical return observations using the sample covariance structure. Deviations are measured relative to the mean of the same perturbed empirical distribution. A tractable formulation of the proposed model is obtained by using a duality-based approach.

We also develop a distributionally robust risk budgeting model under the mean absolute deviation risk measure. In this model, the true return distribution is not assumed to be known exactly, but is allowed to vary within a Wasserstein ambiguity set centered at the empirical distribution. Then, we derive a tractable reformulation of the proposed distributionally robust model.

Finally, we evaluate the proposed models through numerical experiments using real-world stock market data. The proposed robust and distributionally robust models are compared with nominal risk budgeting, minimum MAD, and standard-deviation risk budgeting benchmark models in order to study their stability and out-of-sample performance. We also assess the out-of-sample quality of risk parity through the mean and maximum absolute deviations of relative risk contributions from the equal-risk target. The purpose of these contributions is to extend the MAD risk budgeting literature by explicitly considering uncertainty.

The remainder of this paper is organized as follows. Section 2 presents the basic risk budgeting framework and introduces the mean absolute deviation risk measure. Section 3 introduces the robust MAD risk budgeting formulation under a correlated budget uncertainty set and provides a tractable formulation. Section 4 develops the distributionally robust risk budgeting model under the MAD risk measure and presents its tractable reformulation based on a Wasserstein ambiguity set. Section 5 reports the numerical results and

compares the proposed models with benchmark portfolio construction methods. Finally, Section 6 concludes the paper and discusses possible direction for future research.

2 Risk Budgeting Approach

We first present the basic risk budgeting framework, which will be used as the main foundation for the uncertainty-based models developed later. Suppose that the investment universe consists of n risky assets indexed by $\mathbb{I} = \{1, 2, \dots, n\}$. Let x_i denote the fraction of the total capital invested in asset $i \in \mathbb{I}$. Since short selling is not allowed, we impose $0 \leq x_i \leq 1$ for all $i \in \mathbb{I}$. We also require the portfolio to be fully invested, that is $\sum_{i \in \mathbb{I}} x_i = 1$. Let $\mathbf{r} = (r_1, \dots, r_n)^\top$ be the random vector of asset returns and let $\mathbf{x} = (x_1, \dots, x_n)^\top$ be the vector of portfolio weights. In risk budgeting, the goal is not only to allocate capital among assets, but also to control how much each asset contributes to the total portfolio risk. Assume that $R(\mathbf{x})$ denotes a risk index associated with the portfolio return $\mathbf{r}^\top \mathbf{x}$ and is positively homogeneous of degree one, meaning that $R(\delta \mathbf{x}) = \delta R(\mathbf{x})$ for all $\delta > 0$. Assume also that R is continuously differentiable at the portfolios under consideration. Under these assumptions, Euler's theorem for homogeneous functions gives the following decomposition:

$$R(\mathbf{x}) = \mathbf{x}^\top \nabla R(\mathbf{x}) = \sum_{i \in \mathbb{I}} x_i \frac{\partial R(\mathbf{x})}{\partial x_i}. \quad (1)$$

This decomposition is the main idea behind risk contributions. The risk contribution of asset i measures its contribution to the total portfolio risk and is defined as $RC_i = x_i \frac{\partial R(\mathbf{x})}{\partial x_i}$. Using (1), the total portfolio risk can therefore be written as the sum of individual risk contributions $R(\mathbf{x}) = \sum_{i \in \mathbb{I}} RC_i(\mathbf{x})$.

Let $\mathbf{b} = (b_1, \dots, b_n)^\top$ be a vector of positive risk budgets satisfying $\sum_{i \in \mathbb{I}} b_i = 1$. The risk budget b_i represents the desired fraction of total risk assigned to asset i . Therefore, a risk budgeting portfolio is a portfolio in which the risk contribution of each asset is equal to its prescribed share of the total risk. Following [Roncalli \(2014b\)](#), this condition can be

written as the following nonlinear system:

$$\begin{cases} RC_i(\mathbf{x}) = b_i R(\mathbf{x}), & \forall i \in \mathbb{I}, \\ \sum_{i \in \mathbb{I}} x_i = 1, \\ x_i \geq 0, & \forall i \in \mathbb{I}. \end{cases} \quad (2)$$

The first condition in (2) means that asset i contributes exactly b_i of the total portfolio risk. In the special case where $b_i = 1/n$ for all $i \in \mathbb{I}$, all assets have the same risk contribution, and the resulting portfolio is called a risk parity portfolio.

2.1 Mean absolute deviation

We now specialize the above risk budgeting framework to the mean absolute deviation risk measure. Let $\mathbb{P}$ be the probability measure associated with the distribution of the random return vector $\mathbf{r}$. For a given portfolio $\mathbf{x}$, the portfolio return is $\mathbf{r}^\top \mathbf{x}$, and its expected return is $\mathbb{E}_{\mathbb{P}}[\mathbf{r}^\top \mathbf{x}]$. The mean absolute deviation of the portfolio is defined as:

$$\text{MAD}(\mathbf{x}) := \mathbb{E}_{\mathbb{P}} \left[\left| \mathbf{r}^\top \mathbf{x} - \mathbb{E}_{\mathbb{P}}[\mathbf{r}^\top \mathbf{x}] \right| \right]. \quad (3)$$

The MAD risk measure was introduced in portfolio optimization by [Konno and Yamazaki \(1991\)](#) as an alternative to variance. It measures the average absolute distance of portfolio returns from their mean. [Konno and Koshizuka \(2005\)](#) reviewed the theoretical properties and computational advantages of the MAD model. An important property of MAD is positive homogeneity. Therefore, MAD satisfies one of the key properties required in the risk budgeting decomposition. However, the portfolio MAD function is not differentiable at every point in general. Following [Ararat et al. \(2024\)](#), its subdifferential at $\mathbf{x}$ can be expressed as:

$$\partial \text{MAD}(\mathbf{x}) = \left\{ \mathbb{E}_{\mathbb{P}} \left[\left(\text{sgn} \left((\mathbf{r} - \boldsymbol{\mu})^\top \mathbf{x} \right) + \theta \mathbf{1}_{\{(\mathbf{r} - \boldsymbol{\mu})^\top \mathbf{x} = 0\}} \right) (\mathbf{r} - \boldsymbol{\mu}) \right] : -1 \leq \theta \leq 1 \right\}, \quad (4)$$

where $\boldsymbol{\mu} := \mathbb{E}_{\mathbb{P}}[\mathbf{r}]$. Hence, $\text{MAD}(\mathbf{x})$ is differentiable at $\mathbf{x}$ when its subdifferential contains only one vector. In contrast, nondifferentiability occurs when the subdifferential contains

multiple elements. This situation is particularly relevant when the return distribution is represented by a discrete empirical distribution. We therefore use subgradients rather than ordinary derivatives in order to define risk contributions in a way that remains valid also at nondifferentiable portfolios. Let $\mathbf{g} = (g_1, \dots, g_n)^\top \in \partial \text{MAD}(\mathbf{x})$ be a subgradient of $\text{MAD}(\mathbf{x})$. Since MAD is convex and positively homogeneous, Euler's theorem implies that, for any such subgradient, we have:

$$\text{MAD}(\mathbf{x}) = \mathbf{x}^\top \mathbf{g} = \sum_{i \in \mathbb{I}} x_i g_i. \quad (5)$$

More generally, let R denote a finite, convex, and positively homogeneous risk measure satisfying $R(\mathbf{x}) > 0$ for every $\mathbf{x} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$. For the nominal MAD case, $R(\mathbf{x}) = \text{MAD}(\mathbf{x})$, and this positivity condition excludes nonzero nonnegative allocations with an almost surely constant portfolio return. For a given subgradient $\mathbf{g} \in \partial R(\mathbf{x})$, the risk contribution of asset i can be written as $RC_i(\mathbf{x}) := x_i g_i$. Using this subgradient-based risk contribution, the risk budgeting framework of [Cesarone et al. \(2020\)](#) can be extended to the following system:

$$\begin{cases} x_i g_i = b_i R(\mathbf{x}), & \forall i \in \mathbb{I}, \\ \sum_{i \in \mathbb{I}} x_i = 1, \\ x_i \geq 0, & \forall i \in \mathbb{I}, \\ \mathbf{g} \in \partial R(\mathbf{x}). \end{cases} \quad (6)$$

Although (6) follows the same risk budgeting idea as (2), solving this system is difficult in general. The main reason is that, for a fixed portfolio $\mathbf{x}$, the subdifferential $\partial R(\mathbf{x})$ may contain more than one subgradient. For this reason, we follow the approach of [Ararat et al. \(2024\)](#) and use an optimization-based characterization of the risk budgeting portfolio. For a given constant $c \in \mathbb{R}$, consider the following model:

$$\begin{aligned} & \underset{\mathbf{y} > 0}{\text{minimize}} && R(\mathbf{y}) \\ & \text{s.t.} && \sum_{i \in \mathbb{I}} b_i \ln y_i \geq c. \end{aligned} \quad (7)$$

Consider $\mathbf{y}^*$ be an optimal solution of problem (7). Under the stated assumptions on R , the corresponding risk budgeting portfolio is obtained by normalizing $\mathbf{y}^*$ as $\mathbf{x}^* = \frac{\mathbf{y}^*}{\mathbf{1}^\top \mathbf{y}^*}$. The resulting portfolio $\mathbf{x}^*$ satisfies the risk budgeting characterization in (6). Problem (7) provides the nominal risk budgeting formulation based on the MAD risk measure (3). To account for uncertainty, the following sections develop two extensions: a robust model based on a correlated budget uncertainty set and a distributionally robust model based on a Wasserstein ambiguity set.

3 Robust risk budgeting

In this section, we introduce the first uncertainty-based formulation for the MAD risk budgeting problem. This gives a robust optimization model based on a correlated budget uncertainty set. As discussed below, this uncertainty set also admits an optimal transport interpretation. Let $\mathbf{r}^1, \dots, \mathbf{r}^T$ denote the observed historical return vectors, where $\mathbf{r}^s \in \mathbb{R}^n$ is the return vector in historical observation s . Under the empirical distribution, each historical observation has probability $\hat{p}(\mathbf{r}^s) = \frac{1}{T}$ for all $s = 1, \dots, T$. The sample mean return vector is $\bar{\mathbf{r}} := \frac{1}{T} \sum_{s=1}^T \mathbf{r}^s$. Therefore, under the empirical distribution, the MAD risk measure of a portfolio $\mathbf{y}$ can be written as:

$$\text{MAD}(\mathbf{y}) = \frac{1}{T} \sum_{s=1}^T \left| (\mathbf{r}^s - \bar{\mathbf{r}})^\top \mathbf{y} \right|. \quad (8)$$

Let $\hat{\Sigma} \in \mathbb{S}_+^n$ denote the sample covariance matrix of the asset returns and let $\hat{\Sigma}^{1/2}(\hat{\Sigma}^{1/2})^\top = \hat{\Sigma}$. We assume that the return vector in historical observation $s = 1, \dots, T$ can deviate from its nominal value according to $\tilde{\mathbf{r}}^s = \mathbf{r}^s + \hat{\Sigma}^{1/2} \mathbf{z}^s$, where $\mathbf{z}^s = (z_1^s, \dots, z_n^s)^\top \in \mathbb{R}^n$ is the standardized uncertainty vector associated with historical observation s . To control the amount of uncertainty in each historical observation, we define the following correlated budget uncertainty set directly for the perturbed return observations:

$$\mathcal{U}^\Gamma := \left\{ \tilde{\mathbf{r}} = \left(\mathbf{r}^s + \hat{\Sigma}^{1/2} \mathbf{z}^s \right)_{s=1}^T \mid \|\mathbf{z}^s\|_\infty \leq 1, \quad \|\mathbf{z}^s\|_1 \leq \Gamma_s, \quad s = 1, \dots, T \right\}. \quad (9)$$

Here, $\tilde{\mathbf{r}} = (\tilde{\mathbf{r}}^1, \dots, \tilde{\mathbf{r}}^T)$ denotes the collection of perturbed return observations, and $\Gamma = (\Gamma_1, \dots, \Gamma_T)$ is the vector of uncertainty budgets. For each $s = 1, \dots, T$, the parameter

$\Gamma_s \in [0, n]$ controls the total amount of deviation allowed across standardized uncertainty components in historical observation s . Larger values of Γ_s permit larger total standardized perturbations and increase the level of protection against adverse return deviations. The uncertainty set (9) is also related to type- ∞ optimal transport distributionally robust optimization. In the standardized perturbation space, consider the following transportation cost:

$$\mathbf{c}(\mathbf{z}, \mathbf{z}') = \begin{cases} \|\mathbf{z} - \mathbf{z}'\|_1, & \|\mathbf{z} - \mathbf{z}'\|_\infty \leq 1, \\ +\infty, & \text{otherwise.} \end{cases} \quad (10)$$

For each historical observation s , the restrictions in (9) imply $\mathbf{c}(\mathbf{z}^s, \mathbf{0}) \leq \Gamma_s$. Thus, the ℓ_∞ restriction imposes a hard bound on each standardized perturbation component, while the ℓ_1 restriction bounds the transportation cost of each observation in the standardized space. This provides an observation-wise optimal transport interpretation for the correlated budget uncertainty set.

Considering the uncertainty set (9), we define the worst-case MAD under the correlated budget uncertainty set as:

$$\text{MAD}_\Gamma(\mathbf{y}) := \max_{\tilde{\mathbf{r}} \in \mathcal{U}^\Gamma} \frac{1}{T} \sum_{s=1}^T \left| (\tilde{\mathbf{r}}^s)^\top \mathbf{y} - \mu(\tilde{\mathbf{r}}, \mathbf{y}) \right|, \quad (11)$$

where $\mu(\tilde{\mathbf{r}}, \mathbf{y}) := \frac{1}{T} \sum_{s=1}^T (\tilde{\mathbf{r}}^s)^\top \mathbf{y}$. The function MAD_Γ is nonnegative, convex, and positively homogeneous, since it is the pointwise maximum of empirical MAD functions having these properties. Therefore, under the positivity condition introduced for model (7), the risk budgeting characterization in (6) and (7) can be applied by setting $R(\mathbf{y}) = \text{MAD}_\Gamma(\mathbf{y})$. Then, the robust risk budgeting formulation of model (7) under the correlated budget uncertainty set is defined as:

$$\begin{aligned} & \underset{\mathbf{y} > 0}{\text{minimize}} && \text{MAD}_\Gamma(\mathbf{y}) \\ & \text{s.t.} && \sum_{i \in \mathbb{I}} b_i \ln y_i \geq c. \end{aligned} \quad (12)$$

This model protects the MAD risk budgeting portfolio against adverse perturbations of the historical return observations. In particular, the deviations are measured relative to the mean of the same perturbed empirical distribution rather than relative to the fixed

nominal sample mean. The following proposition gives a tractable reformulation of the constant decision-rule approximation of problem (12).

Proposition 1 *The constant decision-rule approximation of problem (12) can be reformulated as the following tractable optimization model:*

$$\begin{aligned}
& \underset{\mathbf{y} > 0, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\eta}}{\text{minimize}} && \frac{1}{T} \sum_{s=1}^T u^s \\
& \text{s.t.} && u^s \geq (\mathbf{r}^s - \bar{\mathbf{r}})^\top \mathbf{y} + \left(1 - \frac{1}{T}\right) \left(\Gamma_s \lambda^s + \sum_{i \in \mathbb{I}} \eta_i^s \right) + \frac{1}{T} \sum_{\substack{\ell=1 \\ \ell \neq s}}^T \left(\Gamma_\ell \lambda^\ell + \sum_{i \in \mathbb{I}} \eta_i^\ell \right), \quad s = 1, \dots, T, \\
& && u^s \leq -(\mathbf{r}^s - \bar{\mathbf{r}})^\top \mathbf{y} + \left(1 - \frac{1}{T}\right) \left(\Gamma_s \lambda^s + \sum_{i \in \mathbb{I}} \eta_i^s \right) + \frac{1}{T} \sum_{\substack{\ell=1 \\ \ell \neq s}}^T \left(\Gamma_\ell \lambda^\ell + \sum_{i \in \mathbb{I}} \eta_i^\ell \right), \quad s = 1, \dots, T, \\
& && \lambda^\ell + \eta_i^\ell \geq \left[\left(\widehat{\Sigma}^{1/2} \right)^\top \mathbf{y} \right]_i, \quad i \in \mathbb{I}, \quad \ell = 1, \dots, T, \\
& && \lambda^\ell + \eta_i^\ell \leq - \left[\left(\widehat{\Sigma}^{1/2} \right)^\top \mathbf{y} \right]_i, \quad i \in \mathbb{I}, \quad \ell = 1, \dots, T, \\
& && \lambda^\ell \geq 0, \quad \ell = 1, \dots, T, \\
& && \eta_i^\ell \geq 0, \quad i \in \mathbb{I}, \quad \ell = 1, \dots, T, \\
& && \sum_{i \in \mathbb{I}} b_i \ln y_i \geq c.
\end{aligned} \tag{13}$$

Proof: Using constant decision rules for the auxiliary variables u^s , $s = 1, \dots, T$, associated with the absolute value terms, we obtain the following formulation of problem (12):

$$\begin{aligned}
& \underset{\mathbf{y} > 0, \mathbf{u}}{\text{minimize}} && \frac{1}{T} \sum_{s=1}^T u^s \\
& \text{s.t.} && u^s \geq (\tilde{\mathbf{r}}^s)^\top \mathbf{y} - \mu(\tilde{\mathbf{r}}, \mathbf{y}), \quad s = 1, \dots, T, \quad \forall \tilde{\mathbf{r}} \in \mathcal{U}^\Gamma, \\
& && u^s \leq -(\tilde{\mathbf{r}}^s)^\top \mathbf{y} + \mu(\tilde{\mathbf{r}}, \mathbf{y}), \quad s = 1, \dots, T, \quad \forall \tilde{\mathbf{r}} \in \mathcal{U}^\Gamma, \\
& && \sum_{i \in \mathbb{I}} b_i \ln y_i \geq c.
\end{aligned} \tag{14}$$

Now, for a fixed portfolio $\mathbf{y}$, define $\mathbf{d}(\mathbf{y}) := \left(\widehat{\Sigma}^{1/2} \right)^\top \mathbf{y}$. Using the definition of the perturbed returns (9), the mean portfolio return is $\mu(\tilde{\mathbf{r}}, \mathbf{y}) = \bar{\mathbf{r}}^\top \mathbf{y} + \frac{1}{T} \sum_{\ell=1}^T (\mathbf{z}^\ell)^\top \mathbf{d}(\mathbf{y})$. Therefore,

$$(\tilde{\mathbf{r}}^s)^\top \mathbf{y} - \mu(\tilde{\mathbf{r}}, \mathbf{y}) = (\mathbf{r}^s - \bar{\mathbf{r}})^\top \mathbf{y} + \left(1 - \frac{1}{T}\right) (\mathbf{z}^s)^\top \mathbf{d}(\mathbf{y}) - \frac{1}{T} \sum_{\substack{\ell=1 \\ \ell \neq s}}^T (\mathbf{z}^\ell)^\top \mathbf{d}(\mathbf{y}).$$

For each $s = 1, \dots, T$, define the entire worst-case perturbation term as:

$$B^s(\mathbf{y}) := \max_{\mathbf{z}^\ell \in \mathbb{R}^n,} \left(1 - \frac{1}{T}\right) (\mathbf{z}^s)^\top \mathbf{d}(\mathbf{y}) - \frac{1}{T} \sum_{\substack{\ell=1 \\ \ell \neq s}}^T (\mathbf{z}^\ell)^\top \mathbf{d}(\mathbf{y})$$

$$s.t. \quad \|\mathbf{z}^\ell\|_\infty \leq 1, \quad \|\mathbf{z}^\ell\|_1 \leq \Gamma_\ell, \quad \ell = 1, \dots, T.$$

For each $\ell = 1, \dots, T$, define $A^\ell(\mathbf{y}) := \max_{\mathbf{z}^\ell \in \mathbb{R}^n} \{(\mathbf{z}^\ell)^\top \mathbf{d}(\mathbf{y}) \mid \|\mathbf{z}^\ell\|_\infty \leq 1, \|\mathbf{z}^\ell\|_1 \leq \Gamma_\ell\}$. Since the uncertainty set for each $\mathbf{z}^\ell$ is symmetric around the origin, the same optimal value is obtained when the objective function is $-(\mathbf{z}^\ell)^\top \mathbf{d}(\mathbf{y})$. Moreover, the uncertainty restrictions are imposed separately for each historical observation. Therefore, the maximization defining $B^s(\mathbf{y})$ is separable, and we obtain $B^s(\mathbf{y}) = \left(1 - \frac{1}{T}\right) A^s(\mathbf{y}) + \frac{1}{T} \sum_{\substack{\ell=1 \\ \ell \neq s}}^T A^\ell(\mathbf{y})$. Hence, the two robust constraints in (14) can be written as:

$$u^s \geq (\mathbf{r}^s - \bar{\mathbf{r}})^\top \mathbf{y} + B^s(\mathbf{y}), \quad s = 1, \dots, T,$$

$$u^s \geq -(\mathbf{r}^s - \bar{\mathbf{r}})^\top \mathbf{y} + B^s(\mathbf{y}), \quad s = 1, \dots, T.$$

It remains to derive a tractable expression for $A^\ell(\mathbf{y})$. Using the symmetry of the uncertainty set, this term can be written as:

$$A^\ell(\mathbf{y}) = \max_{\mathbf{w}^\ell} \sum_{i \in \mathbb{I}} |d_i(\mathbf{y})| w_i^\ell$$

$$s.t. \quad \sum_{i \in \mathbb{I}} w_i^\ell \leq \Gamma_\ell,$$

$$0 \leq w_i^\ell \leq 1, \quad i \in \mathbb{I}.$$
(15)

Problem (15) is feasible and bounded. Therefore, strong duality holds. Let $\lambda^\ell \geq 0$ be the dual variable associated with the budget constraint, and let $\eta_i^\ell \geq 0$ be the dual variable associated with the upper-bound constraint $w_i^\ell \leq 1$. The corresponding dual problem is:

$$A^\ell(\mathbf{y}) = \underset{\lambda^\ell, \boldsymbol{\eta}^\ell}{\text{minimize}} \quad \Gamma_\ell \lambda^\ell + \sum_{i \in \mathbb{I}} \eta_i^\ell$$

$$s.t. \quad \lambda^\ell + \eta_i^\ell \geq |d_i(\mathbf{y})|, \quad i \in \mathbb{I},$$

$$\lambda^\ell \geq 0,$$

$$\eta_i^\ell \geq 0, \quad i \in \mathbb{I}.$$
(16)

To avoid the absolute value $|d_i(\mathbf{y})|$, we equivalently impose the two linear inequalities:

$$\lambda^\ell + \eta_i^\ell \geq d_i(\mathbf{y}), \quad \lambda^\ell + \eta_i^\ell \geq -d_i(\mathbf{y}), \quad i \in \mathbb{I}.$$

By strong duality, the value of $A^\ell(\mathbf{y})$ is therefore represented by $\Gamma_\ell \lambda^\ell + \sum_{i \in \mathbb{I}} \eta_i^\ell$ together with the preceding dual-feasibility constraints. Substituting this dual representation into $B^s(\mathbf{y})$ and then into the robust constraints of (14) gives (13). This completes the proof. $\square$

Based on proposition 1, model (13) provides a tractable formulation of problem (12) under constant decision rules. After solving (13), the final portfolio weights are obtained by normalizing the optimal solution $\mathbf{y}^*$ as $\mathbf{x}^* = \frac{\mathbf{y}^*}{\mathbf{1}^\top \mathbf{y}^*}$. In the next section, we develop the distributionally robust MAD risk budgeting model under a Wasserstein ambiguity set.

4 Distributionally robust risk budgeting

In this section, we develop a distributionally robust formulation of model (7). The nominal MAD risk budgeting model is based on a given probability measure $\mathbb{P}$. However, in practice, this probability measure is not known exactly and is usually estimated from historical data. Therefore, relying only on the empirical distribution may lead to a portfolio that is sensitive to sampling error and distributional misspecification. To address this issue, we allow the return distribution to vary within a Wasserstein ambiguity set and construct the portfolio against the worst-case MAD over the distributions in this set.

We first introduce the Wasserstein ambiguity set under an arbitrary ground norm. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$, and let $\|\cdot\|_*$ denote its dual norm, given by $\|\mathbf{y}\|_* := \max_{\|\mathbf{z}\| \leq 1} \mathbf{z}^\top \mathbf{y}$. Let $\hat{\mathbb{P}}_T := \frac{1}{T} \sum_{s=1}^T \delta_{\mathbf{r}^s}$ denote the empirical distribution of the historical return observations, where $\delta_{\mathbf{r}^s}$ is the point mass at $\mathbf{r}^s$. For two probability distributions $\mathbb{P}, \mathbb{Q} \in \mathcal{P}_1(\mathbb{R}^n)$, the first-order Wasserstein distance induced by the ground norm $\|\cdot\|$ is:

$$W_{1, \|\cdot\|}(\mathbb{P}, \mathbb{Q}) := \inf_{\pi \in \Pi(\mathbb{P}, \mathbb{Q})} \int_{\mathbb{R}^n \times \mathbb{R}^n} \|\xi - \zeta\| \pi(d\xi, d\zeta),$$

where $\Pi(\mathbb{P}, \mathbb{Q})$ denotes the set of probability measures on $\mathbb{R}^n \times \mathbb{R}^n$ with marginals $\mathbb{P}$ and $\mathbb{Q}$. This distance represents the smallest average displacement required to transform one probability distribution into the other. We assume that the true return distribution belongs

to the following Wasserstein ambiguity set centered at $\widehat{\mathbb{P}}_T$:

$$\mathcal{P}_\varepsilon := \left\{ \mathbb{P} \in \mathcal{P}_1(\mathbb{R}^n) \mid W_{1,\|\cdot\|}(\mathbb{P}, \widehat{\mathbb{P}}_T) \leq \varepsilon \right\}, \quad (17)$$

where $\mathcal{P}_1(\mathbb{R}^n)$ denotes the set of probability distributions on $\mathbb{R}^n$ with finite first moment, and $\varepsilon \geq 0$ is the radius of the ambiguity set. A larger value of ε allows greater deviations from the empirical distribution and therefore provides a higher level of protection against distributional uncertainty.

For a portfolio $\mathbf{x}$, we define the worst-case MAD risk measure over set (17) as:

$$\text{MAD}_\varepsilon(\mathbf{x}) := \sup_{\mathbb{P} \in \mathcal{P}_\varepsilon} \mathbb{E}_{\mathbb{P}} \left[\left| \mathbf{r}^\top \mathbf{x} - \mathbb{E}_{\mathbb{P}} \left[\mathbf{r}^\top \mathbf{x} \right] \right| \right]. \quad (18)$$

The function MAD_ε is finite, nonnegative, convex, and positively homogeneous. Indeed, for every $\mathbb{P} \in \mathcal{P}_\varepsilon$, the corresponding MAD function has these properties following Ararat et al. (2024), and they are preserved under the pointwise supremum. However, MAD_ε need not be differentiable. Therefore, under the positivity condition introduced for model (7), the risk budgeting characterization in (6) and (7) can be applied by setting $R(\mathbf{x}) = \text{MAD}_\varepsilon(\mathbf{x})$. Accordingly, the risk contributions are defined using $\mathbf{g}^\varepsilon \in \partial \text{MAD}_\varepsilon(\mathbf{x})$. Using this worst-case MAD risk measure, the distributionally robust risk budgeting model can be written as:

$$\begin{aligned} & \underset{\mathbf{y} > 0}{\text{minimize}} && \text{MAD}_\varepsilon(\mathbf{y}) \\ & \text{s.t.} && \sum_{i \in \mathbb{I}} b_i \ln y_i \geq c. \end{aligned} \quad (19)$$

The following proposition gives the main tractable characterization used in the reformulation of model (19).

Proposition 2 *Suppose that the Wasserstein ambiguity set is given by (17), and $\mathbf{y} > 0$ and $\mu \in \mathbb{R}$ are fixed. If $|\bar{\mathbf{r}}^\top \mathbf{y} - \mu| \leq \varepsilon \|\mathbf{y}\|_*$, then*

$$\sup_{\mathbb{P} \in \mathcal{P}_\varepsilon, \mathbb{E}_{\mathbb{P}}[\mathbf{r}^\top \mathbf{y}] = \mu} \mathbb{E}_{\mathbb{P}} \left[\left| \mathbf{r}^\top \mathbf{y} - \mu \right| \right] = \frac{1}{T} \sum_{s=1}^T \left| (\mathbf{r}^s)^\top \mathbf{y} - \mu \right| + \varepsilon \|\mathbf{y}\|_*. \quad (20)$$

If $|\bar{\mathbf{r}}^\top \mathbf{y} - \mu| > \varepsilon \|\mathbf{y}\|_$, then no distribution in $\mathcal{P}_\varepsilon$ satisfies $\mathbb{E}_{\mathbb{P}}[\mathbf{r}^\top \mathbf{y}] = \mu$.*

Proof: To obtain a tractable reformulation of model (19), we introduce an auxiliary scalar $\mu \in \mathbb{R}$. For fixed $\mathbf{y} > 0$ and μ , define:

$$\Psi(\mathbf{y}, \mu) := \sup_{\mathbb{P} \in \mathcal{P}_\varepsilon} \left\{ \mathbb{E}_{\mathbb{P}} \left[\left| \mathbf{r}^\top \mathbf{y} - \mu \right| \right] \mid \mathbb{E}_{\mathbb{P}} \left[\mathbf{r}^\top \mathbf{y} \right] = \mu \right\}. \quad (21)$$

Here, the constraint $\mathbb{E}_{\mathbb{P}}[\mathbf{r}^\top \mathbf{y}] = \mu$ ensures that the deviation is measured around the mean portfolio return under the selected distribution. By the definition of the dual norm, every $\mathbb{P} \in \mathcal{P}_\varepsilon$ satisfies

$$\left| \mathbb{E}_{\mathbb{P}}[\mathbf{r}^\top \mathbf{y}] - \bar{\mathbf{r}}^\top \mathbf{y} \right| \leq \|\mathbf{y}\|_* W_{1, \|\cdot\|} \left(\mathbb{P}, \hat{\mathbb{P}}_T \right) \leq \varepsilon \|\mathbf{y}\|_*.$$

Therefore, the mean constraint is feasible only if $|\bar{\mathbf{r}}^\top \mathbf{y} - \mu| \leq \varepsilon \|\mathbf{y}\|_*$. Conversely, whenever this condition holds, the prescribed mean can be obtained by translating the empirical observations in a direction attaining the dual norm, with transportation cost no greater than ε . Thus, the condition is also sufficient. Applying the general-norm dual reformulation of Chen et al. (2023) to (21) gives:

$$\begin{aligned} \Psi(\mathbf{y}, \mu) = \min_{\gamma_1 \in \mathbb{R}, \gamma_2 \geq 0} \quad & \frac{1}{T} \sum_{s=1}^T \left| (\mathbf{r}^s)^\top \mathbf{y} - \mu \right| + \gamma_1 \left(\bar{\mathbf{r}}^\top \mathbf{y} - \mu \right) + \varepsilon \gamma_2 \\ \text{s.t.} \quad & \|(1 - \gamma_1)\mathbf{y}\|_* \leq \gamma_2, \\ & \|(1 + \gamma_1)\mathbf{y}\|_* \leq \gamma_2. \end{aligned}$$

Since $\max\{|1 - \gamma_1|, |1 + \gamma_1|\} = 1 + |\gamma_1|$, the two dual constraints are equivalent to $\gamma_2 \geq (1 + |\gamma_1|)\|\mathbf{y}\|_*$. Under the feasible-mean condition, we have:

$$\gamma_1 \left(\bar{\mathbf{r}}^\top \mathbf{y} - \mu \right) + \varepsilon \gamma_2 \geq \varepsilon \|\mathbf{y}\|_* + |\gamma_1| \left(\varepsilon \|\mathbf{y}\|_* - \left| \bar{\mathbf{r}}^\top \mathbf{y} - \mu \right| \right) \geq \varepsilon \|\mathbf{y}\|_*.$$

Equality is obtained in the dual problem by choosing $\gamma_1 = 0$ and $\gamma_2 = \|\mathbf{y}\|_*$. Hence $\Psi(\mathbf{y}, \mu) = \frac{1}{T} \sum_{s=1}^T |(\mathbf{r}^s)^\top \mathbf{y} - \mu| + \varepsilon \|\mathbf{y}\|_*$, which proves (20). This completes the proof. $\square$

Proposition 2 shows that, under a general ground norm, the Wasserstein uncertainty adds the robust correction term $\varepsilon \|\mathbf{y}\|_*$ to the empirical absolute deviation about μ . It also gives the interval in which the mean parameter μ must lie. The following proposition gives a tractable reformulation of model (19) under the ℓ_∞ ground norm.

Proposition 3 Suppose that the ground norm in (17) is the ℓ_∞ -norm. Model (19) can be

reformulated as the following tractable optimization model:

$$\begin{aligned}
& \underset{\mathbf{y} > 0, \tau, \mathbf{u}, \mathbf{v}}{\text{minimize}} && \tau + \varepsilon \mathbf{1}^\top \mathbf{y} \\
& \text{s.t.} && \tau \geq \frac{1}{T} \sum_{s=1}^T u_s, \\
& && \tau \geq \frac{1}{T} \sum_{s=1}^T v_s, \\
& && u_s \geq (\mathbf{r}^s - \bar{\mathbf{r}})^\top \mathbf{y} + \varepsilon \mathbf{1}^\top \mathbf{y}, \quad s = 1, \dots, T, \\
& && u_s \geq -(\mathbf{r}^s - \bar{\mathbf{r}})^\top \mathbf{y} - \varepsilon \mathbf{1}^\top \mathbf{y}, \quad s = 1, \dots, T, \\
& && v_s \geq (\mathbf{r}^s - \bar{\mathbf{r}})^\top \mathbf{y} - \varepsilon \mathbf{1}^\top \mathbf{y}, \quad s = 1, \dots, T, \\
& && v_s \geq -(\mathbf{r}^s - \bar{\mathbf{r}})^\top \mathbf{y} + \varepsilon \mathbf{1}^\top \mathbf{y}, \quad s = 1, \dots, T, \\
& && \sum_{i \in \mathbb{I}} b_i \ln y_i \geq c.
\end{aligned} \tag{22}$$

Proof: For a fixed $\mathbf{y} > 0$, the definition of $\Psi(\mathbf{y}, \mu)$ gives $\text{MAD}_\varepsilon(\mathbf{y}) = \max_{\mu \in \mathbb{R}} \Psi(\mathbf{y}, \mu)$. The dual norm of the ℓ_∞ -norm is the ℓ_1 -norm. Since $\mathbf{y} > 0$, we have $\|\mathbf{y}\|_* = \|\mathbf{y}\|_1 = \mathbf{1}^\top \mathbf{y}$. Therefore, by Proposition 2, the worst-case MAD can be calculated as:

$$\begin{aligned}
\text{MAD}_\varepsilon(\mathbf{y}) &= \max_{\mu \in \mathbb{R}} \frac{1}{T} \sum_{s=1}^T \left| (\mathbf{r}^s)^\top \mathbf{y} - \mu \right| + \varepsilon \mathbf{1}^\top \mathbf{y} \\
&\text{s.t.} \quad \bar{\mathbf{r}}^\top \mathbf{y} - \varepsilon \mathbf{1}^\top \mathbf{y} \leq \mu \leq \bar{\mathbf{r}}^\top \mathbf{y} + \varepsilon \mathbf{1}^\top \mathbf{y}.
\end{aligned} \tag{23}$$

For a fixed $\mathbf{y}$, the objective function in (23) is convex with respect to μ . Hence, the maximum over the interval $[\bar{\mathbf{r}}^\top \mathbf{y} - \varepsilon \mathbf{1}^\top \mathbf{y}, \bar{\mathbf{r}}^\top \mathbf{y} + \varepsilon \mathbf{1}^\top \mathbf{y}]$ is attained at one of its two endpoints. We introduce auxiliary variables u_s and v_s for the absolute value terms at the lower and upper endpoints, respectively. Each absolute value is represented by its two linear inequalities. We also introduce τ to represent the maximum of the two averages by imposing $\tau \geq \frac{1}{T} \sum_{s=1}^T u_s$, and $\tau \geq \frac{1}{T} \sum_{s=1}^T v_s$. Minimizing $\tau + \varepsilon \mathbf{1}^\top \mathbf{y}$ together with the risk budgeting constraint gives (22). This completes the proof. $\square$

Based on Proposition 3, model (22) gives a tractable distributionally robust reformulation of model (19). After solving (22), the final risk budgeting portfolio is obtained by normalizing the optimal solution $\mathbf{y}^*$. The model developed in this section accounts for uncertainty at the distribution level by using a Wasserstein ambiguity set. Together with the

robust formulation under a correlated budget uncertainty set in Section 3, it provides the two uncertainty-based approaches studied in this paper. In the next section, we evaluate these models numerically and compare their performance with benchmark portfolio models.

5 Numerical results

This section presents numerical results for the proposed robust and distributionally robust risk budgeting models. It is divided into four parts. First, we describe the data selection procedure and the portfolio models used for comparison. The second and third parts evaluate the in-sample and out-of-sample performance, respectively. Finally, we examine the sensitivity of the numerical results to the uncertainty parameters.

5.1 Data selection

The computational study is conducted in order to evaluate the performance of the robust formulation under a correlated budget uncertainty set and the distributionally robust formulation under a Wasserstein ambiguity set. All computations are implemented in **Python**. For evaluation, we use weekly historical return data from three real-world stock markets, namely NASDAQ100, FTSE100, and S&P500, obtained from [Bruni et al. \(2016\)](#). The available datasets start in 2004 for NASDAQ100 and S&P500, and in 2002 for FTSE100. For each market, we consider 250 consecutive weekly observations. The dataset is divided into two parts, namely an in-sample set and an out-of-sample set. In the implementation, the first 80% of the observations are used as in-sample data and the remaining 20% are used as out-of-sample data. To construct the test instances, we do not use all available assets in each market. In practice, building a portfolio from the full universe of assets is often unrealistic, and therefore an asset selection step is needed. For this purpose, we apply principal component analysis to the in-sample market data, following [Greenacre et al. \(2022\)](#). Based on this procedure, we consider instances with 15, 25, 35, 45, and 55 selected assets from each market.

For comparison, we consider five portfolio construction methods. The first one is the robust model (13) under the correlated budget uncertainty set denoted by RM. The second one is the distributionally robust model (22) as DRM. The third one is the nominal risk

budgeting (NM) model (7), where $R(\mathbf{y}) = \text{MAD}(\mathbf{y})$ and the nominal return distribution is represented by the empirical distribution. The fourth model, denoted by MinM, is the minimum MAD portfolio, which selects a long-only, fully invested portfolio with the smallest MAD, computed using the sample mean return vector. The fifth one is problem (7) with the standard-deviation risk measure $\sqrt{\mathbf{y}^\top \hat{\Sigma} \mathbf{y}}$, based on the in-sample covariance matrix $\hat{\Sigma}$, and is denoted by SDM.

After solving each model on the in-sample data, the resulting portfolio is evaluated on both parts of the dataset. Following the budgeted uncertainty framework of Bertsimas and Sim (2004), we choose the budget parameter as $\Gamma_s = \rho n$ for $s = 1, \dots, T$, where T denotes the number of in-sample observations and $\rho \in [0, 1]$ controls the level of protection. In our numerical experiments, we set $\rho = 0.2$. For the DRM model, the ambiguity radius is set to $\varepsilon = 0.5$. The parameter settings used in the RM and DRM models are chosen based on preliminary computational tests to provide a representative level of protection without making the models overly conservative. In Subsection 5.4, we also examine the sensitivity of the numerical results to different values of ρ and ε in order to assess whether the main conclusions depend on these particular parameter choices.

5.2 In-sample evaluation

In this subsection, the in-sample performance of the models is compared based on two financial metrics: turnover and Sharpe ratio. Turnover measures the stability of the portfolio and its sensitivity to changes in the estimated input parameters. To evaluate this stability, we use a rolling re-estimation procedure with four portfolio construction rounds. The in-sample data are divided into four equal-sized consecutive segments. The first portfolio is constructed using the first segment, the second portfolio is obtained using the first two segments, the third portfolio is based on the first three segments, and the fourth portfolio is obtained using the full in-sample dataset. Therefore, as new observations become available, the portfolio is reconstructed and compared with the portfolio obtained in the previous round. If w_κ denotes the portfolio obtained at rolling round κ , then turnover is calculated as $\frac{1}{K-1} \sum_{\kappa=1}^{K-1} \|w_{\kappa+1} - w_\kappa\|$, where $K = 4$ is the number of rolling rounds. Lower turnover indicates greater stability across portfolio updates and can also reduce the need for frequent reallocation and the associated transaction costs.

In addition, the Sharpe ratio is used to assess the quality of portfolio returns by incorporating risk through the standard deviation. For this measure, only the final portfolio obtained using the full in-sample dataset is considered. A higher Sharpe ratio indicates better risk-adjusted performance. The in-sample values of turnover and Sharpe ratio are reported in Table 2. The table is organized by the number of selected assets and market. For each instance, the first five numerical columns report the turnover values and the next five columns report the corresponding Sharpe ratios. The smallest turnover and the largest Sharpe ratio in each row are highlighted in bold. All reported values are rounded to four decimal places. The last row of the table reports the average in-sample performance of the five models over all instances.

Observation 1 (Stability and Sharpe ratio). Table 2 shows that DRM consistently yields the lowest turnover across all asset sizes and all three markets, with turnover rounding to zero in all cases at four decimal places. This indicates that the distributionally robust portfolio is much more stable than the other models with respect to changes in the estimated input parameters. SDM and NM also show relatively low average turnover compared with RM and MinM, with SDM achieving the second-lowest average turnover. In terms of the Sharpe ratio, the best-performing model varies across markets and asset sizes. However, when considering the average results, RM achieves the highest average Sharpe ratio among the five models.

5.3 Out-of-sample evaluation

In this subsection, the out-of-sample performance of the five models is evaluated based on six measures, namely expected return, Sharpe ratio, Calmar ratio, CVaR, Mean absolute deviation, and Max absolute deviation. These measures are computed from the out-of-sample data and are used to assess the performance of the portfolios obtained from the in-sample estimation stage. In the out-of-sample analysis, only the final portfolio obtained from the full in-sample dataset is considered for each model. This portfolio is then applied to the out-of-sample returns, and its performance is evaluated over the entire testing period.

The Calmar ratio is used to compare portfolio performance relative to maximum draw-down and is defined as $\frac{\text{Annualized return}}{\text{Max Drawdown}}$, where the annualized return is computed from the out-of-sample portfolio returns as $(\prod_t (1 + r_{p,t}))^{52/T_{\text{out}}} - 1$, and Max Drawdown is computed

Table 2: In-sample comparison based on turnover and Sharpe ratio.

Market index	Turnover					Sharpe ratio				
	RM	DRM	NM	MinM	SDM	RM	DRM	NM	MinM	SDM
15 assets										
FTSE100	0.0944	0.0000	0.1151	0.2892	0.0540	0.9864	0.8433	0.9154	1.1042	0.9500
NASDAQ100	0.1333	0.0000	0.1313	0.4357	0.1103	0.7446	1.0358	0.7964	0.2201	0.8508
S&P500	0.1335	0.0000	0.0875	0.4055	0.0874	0.1754	0.1819	0.1644	0.0721	0.1658
25 assets										
FTSE100	0.1053	0.0000	0.1011	0.4587	0.0611	1.0762	0.9067	0.9840	1.2199	1.0259
NASDAQ100	0.2295	0.0000	0.1840	0.4169	0.1441	0.5852	0.7567	0.6261	0.1988	0.6374
S&P500	0.1792	0.0000	0.1205	0.5264	0.1212	0.1842	0.1104	0.1138	0.0692	0.1447
35 assets										
FTSE100	0.1020	0.0000	0.1141	0.4056	0.0606	1.1183	0.9279	1.0169	1.4276	1.0616
NASDAQ100	0.2390	0.0000	0.2078	0.4445	0.1645	0.7778	0.8976	0.8032	0.3986	0.8110
S&P500	0.2041	0.0000	0.1234	0.9590	0.1293	0.2055	0.1479	0.1497	0.3310	0.1848
45 assets										
FTSE100	0.1062	0.0000	0.1267	0.5465	0.0729	1.2100	1.0015	1.0987	1.3118	1.1555
NASDAQ100	0.2192	0.0000	0.1873	0.5289	0.1609	0.8742	0.8955	0.8648	0.5533	0.8714
S&P500	0.2046	0.0000	0.1235	1.0412	0.1334	0.2785	0.2404	0.2306	0.0834	0.2582
55 assets										
FTSE100	0.1268	0.0000	0.1481	0.6500	0.0929	1.3846	1.0971	1.2511	1.7771	1.3136
NASDAQ100	0.2323	0.0000	0.1834	0.6071	0.1629	0.8388	0.8896	0.8235	0.6289	0.8517
S&P500	0.2063	0.0000	0.1282	0.9119	0.1354	0.3111	0.2533	0.2626	0.2219	0.2904
Average	0.1677	0.0000	0.1388	0.5751	0.1127	0.7167	0.6790	0.6734	0.6412	0.7048

from the out-of-sample wealth process as $\max_t \left(\frac{M_t - W_t}{M_t} \right)$. Here, $r_{p,t}$ is the portfolio return at time t , W_t is the cumulative wealth at time t , T_{out} is the number of out-of-sample observations, and 52 is used because the data are weekly. Moreover, we have $M_t := \max_{s \leq t} W_s$. Note that the initial wealth values of all portfolios are set to one.

In addition, $\text{CVaR}(95\%)$ measures the average loss in the worst tail of the return distribution. Let L denote the portfolio loss. Then CVaR at confidence level α is estimated using the empirical tail average $\text{CVaR}_\alpha(L) = \mathbb{E}[L \mid L \geq \text{VaR}_\alpha(L)]$. In this empirical calculation, the VaR threshold is estimated from the observed losses, and the expectation averages losses at or above this threshold. Hence, a higher Calmar ratio indicates better return relative to drawdown risk, while a lower CVaR indicates lower exposure to extreme losses.

To evaluate the out-of-sample quality of risk parity, we also consider Mean absolute deviation and Max absolute deviation. Since equal risk budgets are used, the target relative risk contribution of each asset is $1/n$. These measures are defined as the Mean absolute

deviation, $\frac{1}{n} \sum_{i \in \mathbb{I}} \left| \frac{RC_i(\mathbf{x})}{R(\mathbf{x})} - \frac{1}{n} \right|$, and the Max absolute deviation, $\max_{i \in \mathbb{I}} \left| \frac{RC_i(\mathbf{x})}{R(\mathbf{x})} - \frac{1}{n} \right|$.

For this evaluation, when we set $R(\mathbf{x}) = \text{MAD}(\mathbf{x})$, the risk contribution is calculated as $RC_i(\mathbf{x}) = x_i g_i$, where $\mathbf{g} \in \partial \text{MAD}(\mathbf{x})$ is obtained from (4). Lower values indicate that the out-of-sample relative risk contributions remain closer to the equal risk contribution target.

The out-of-sample results are reported in Tables 3, 4, and 5. The organization of these tables is the same as in Table 2, and they are reported for the same instances. Table 3 reports the out-of-sample values of expected return and Sharpe ratio, while Table 4 reports the corresponding values of Calmar ratio and CVaR(95%). Table 5 reports the corresponding values of Mean absolute deviation and Max absolute deviation. For expected return, Sharpe ratio, and Calmar ratio, the largest value in each row is highlighted in bold, while for CVaR(95%), Mean absolute deviation, and Max absolute deviation, the smallest value in each row is highlighted in bold.

Observation 2 (Return and risk-adjusted performance). Table 3 shows that DRM provides the strongest out-of-sample performance in terms of expected return and Sharpe ratio. This suggests that the distributionally robust model is particularly effective in these markets. For FTSE100, the results are more mixed, and the best values are sometimes obtained by NM or MinM. SDM also remains competitive with RM and NM across several instances. This indicates that the benefit of the uncertainty-based models is less pronounced in this market. Looking at the average results, DRM achieves both the highest expected return and the highest Sharpe ratio among the five models.

Observation 3 (Drawdown and tail-risk performance). The Calmar ratio results in table (4) show that DRM performs strongly in NASDAQ100 and S&P500, which means that it generates better return relative to drawdown risk in these markets. For FTSE100, the best Calmar ratio is not concentrated in one model, which again suggests that the relative advantage of the robust and distributionally robust models depends on the market structure. SDM also remains competitive with RM and NM in terms of both Calmar ratio and CVaR. In terms of CVaR(95%), MinM obtains the lowest average value. However, its weaker average performance in expected return and Sharpe ratio shows that this risk reduction may come at the cost of lower return performance. Overall, DRM provides stronger return-drawdown performance, while MinM is more effective in reducing tail risk. These results show the tradeoff between return-risk performance and tail-risk reduction.

Table 3: Out-of-sample comparison based on expected return and Sharpe ratio.

Market index	Expected return					Sharpe ratio				
	RM	DRM	NM	MinM	SDM	RM	DRM	NM	MinM	SDM
15 assets										
FTSE100	0.0037	0.0037	0.0038	0.0033	0.0037	1.0040	1.0184	1.0206	0.7998	1.0021
NASDAQ100	0.0009	0.0026	0.0014	-0.0005	0.0015	0.1045	0.2581	0.1532	-0.0672	0.1623
S&P500	0.0035	0.0039	0.0034	0.0000	0.0034	0.1985	0.2229	0.1984	-0.0010	0.1964
25 assets										
FTSE100	0.0034	0.0033	0.0034	0.0033	0.0034	0.9352	0.9400	0.9711	0.8517	0.9363
NASDAQ100	0.0015	0.0032	0.0021	-0.0009	0.0021	0.1720	0.3465	0.2369	-0.1360	0.2394
S&P500	0.0036	0.0045	0.0040	0.0000	0.0039	0.2132	0.2621	0.2380	0.0025	0.2305
35 assets										
FTSE100	0.0038	0.0036	0.0038	0.0044	0.0038	1.1804	1.0987	1.1792	1.2833	1.1684
NASDAQ100	0.0011	0.0028	0.0016	-0.0014	0.0017	0.1327	0.3152	0.1917	-0.2131	0.2002
S&P500	0.0030	0.0037	0.0034	0.0003	0.0032	0.1948	0.2303	0.2135	0.0253	0.2053
45 assets										
FTSE100	0.0045	0.0041	0.0044	0.0050	0.0045	1.4781	1.2947	1.4499	1.7480	1.4309
NASDAQ100	0.0015	0.0027	0.0018	-0.0013	0.0018	0.1892	0.3124	0.2120	-0.1963	0.2185
S&P500	0.0021	0.0030	0.0026	-0.0005	0.0024	0.1461	0.1953	0.1733	-0.0416	0.1612
55 assets										
FTSE100	0.0048	0.0044	0.0047	0.0052	0.0047	1.6222	1.4250	1.6034	1.9184	1.5646
NASDAQ100	0.0016	0.0027	0.0019	-0.0010	0.0019	0.2042	0.3138	0.2334	-0.1383	0.2332
S&P500	0.0020	0.0027	0.0023	-0.0004	0.0021	0.1450	0.1830	0.1689	-0.0366	0.1563
Average	0.0027	0.0034	0.0030	0.0010	0.0029	0.5280	0.5611	0.5496	0.3866	0.5404

Observation 4 (Out-of-sample risk parity). Based on table 5, DRM obtains the lowest values of both measures in all FTSE100 instances. For NASDAQ100, DRM achieves the lowest Max absolute deviation in all cases, while SDM generally obtains the lowest Mean absolute deviation. For S&P500, DRM generally provides the lowest Mean absolute deviation, while NM achieves the lowest Max absolute deviation in all cases. These results indicate that the model providing the most balanced risk contributions depends on both the market and the evaluation criterion. Looking at the average results, DRM achieves the lowest values of both measures among the five models, indicating the strongest overall out-of-sample risk-parity performance. NM and SDM also remain competitive, while RM has higher average deviations than both models. MinM obtains the largest values of both measures in all instances.

Observation 5 (Overall out-of-sample performance). Overall, the out-of-sample results in Tables 3, 4, and 5 show that DRM provides the most consistent performance

Table 4: Out-of-sample comparison based on Calmar ratio and CVaR(95%).

Market index	Calmar ratio					CVaR(95%)				
	RM	DRM	NM	MinM	SDM	RM	DRM	NM	MinM	SDM
15 assets										
FTSE100	2.5990	2.9051	2.7000	1.5397	2.6884	0.0635	0.0626	0.0628	0.0684	0.0637
NASDAQ100	-0.1287	0.0014	-0.0905	-0.2502	-0.0803	0.1144	0.1265	0.1183	0.1027	0.1179
S&P500	-0.2585	-0.2357	-0.2554	-0.3953	-0.2557	0.2042	0.2054	0.2024	0.1870	0.2019
25 assets										
FTSE100	2.2805	2.4764	2.4105	1.7472	2.3281	0.0607	0.0615	0.0601	0.0650	0.0609
NASDAQ100	-0.0469	0.1433	0.0184	-0.3029	0.0244	0.1072	0.1155	0.1109	0.1004	0.1097
S&P500	-0.2216	-0.1878	-0.2036	-0.3894	-0.2054	0.1945	0.2020	0.1973	0.1857	0.1952
35 assets										
FTSE100	2.9943	2.9504	2.9980	2.9914	2.9857	0.0535	0.0568	0.0540	0.0537	0.0545
NASDAQ100	-0.0827	0.1225	-0.0233	-0.3470	-0.0112	0.1056	0.1125	0.1081	0.0944	0.1072
S&P500	-0.2060	-0.1902	-0.1941	-0.3173	-0.1983	0.1774	0.1840	0.1788	0.1522	0.1780
45 assets										
FTSE100	3.7389	3.4058	3.7007	4.3896	3.6570	0.0502	0.0549	0.0510	0.0463	0.0520
NASDAQ100	-0.0165	0.1262	0.0032	-0.3373	0.0151	0.1039	0.1104	0.1065	0.0894	0.1055
S&P500	-0.2251	-0.2069	-0.2081	-0.3088	-0.2164	0.1701	0.1784	0.1715	0.1444	0.1711
55 assets										
FTSE100	4.2600	3.8509	4.2067	5.3336	4.1468	0.0489	0.0540	0.0498	0.0430	0.0510
NASDAQ100	0.0023	0.1269	0.0326	-0.2965	0.0344	0.1023	0.1087	0.1048	0.0897	0.1037
S&P500	-0.1995	-0.1975	-0.1875	-0.2990	-0.1971	0.1590	0.1699	0.1621	0.1461	0.1615
Average	0.9660	1.0060	0.9938	0.8505	0.9810	0.1144	0.1202	0.1159	0.1046	0.1156

among the five models. It achieves the highest average expected return, Sharpe ratio, and Calmar ratio, which indicates stronger return performance, risk-adjusted return, and return relative to drawdown risk. It also achieves the lowest average Mean absolute deviation and Max absolute deviation, indicating stronger overall out-of-sample risk-parity performance. RM also remains competitive and improves on the nominal model in several cases, especially in terms of risk-adjusted performance. SDM also provides competitive performance and remains close to RM and NM across several out-of-sample measures. NM provides a useful benchmark for assessing the effect of explicitly accounting for uncertainty. MinM performs well in reducing tail risk, as shown by its lowest average CVaR(95%), but it is less balanced across the return-based and risk-adjusted criteria. Overall, these results suggest that explicitly modeling uncertainty can improve the reliability of MAD-based portfolio construction.

For further investigation, we consider the case of 35 selected assets from each market and examine the average cumulative wealth evolution under bootstrap resampling of the

Table 5: Out-of-sample comparison based on Mean and Max absolute deviation.

Market index	Mean absolute deviation					Max absolute deviation				
	RM	DRM	NM	MinM	SDM	RM	DRM	NM	MinM	SDM
15 assets										
FTSE100	0.0400	0.0245	0.0339	0.0940	0.0367	0.1180	0.0447	0.0779	0.7023	0.0932
NASDAQ100	0.0272	0.0213	0.0239	0.0827	0.0214	0.0752	0.0454	0.0644	0.3124	0.0530
S&P500	0.0269	0.0155	0.0222	0.1088	0.0219	0.0522	0.0432	0.0389	0.7231	0.0403
25 assets										
FTSE100	0.0232	0.0142	0.0181	0.0584	0.0199	0.0610	0.0342	0.0420	0.5163	0.0485
NASDAQ100	0.0167	0.0130	0.0138	0.0566	0.0127	0.0544	0.0372	0.0538	0.2536	0.0422
S&P500	0.0139	0.0095	0.0124	0.0695	0.0113	0.0372	0.0264	0.0256	0.7324	0.0297
35 assets										
FTSE100	0.0160	0.0104	0.0118	0.0449	0.0134	0.0421	0.0287	0.0343	0.4934	0.0382
NASDAQ100	0.0120	0.0093	0.0099	0.0424	0.0091	0.0428	0.0295	0.0468	0.2036	0.0353
S&P500	0.0103	0.0071	0.0089	0.0506	0.0081	0.0316	0.0223	0.0212	0.5867	0.0258
45 assets										
FTSE100	0.0127	0.0080	0.0089	0.0351	0.0109	0.0462	0.0246	0.0269	0.4380	0.0472
NASDAQ100	0.0086	0.0071	0.0078	0.0349	0.0069	0.0321	0.0237	0.0370	0.2212	0.0318
S&P500	0.0072	0.0055	0.0059	0.0390	0.0056	0.0231	0.0166	0.0162	0.4887	0.0184
55 assets										
FTSE100	0.0110	0.0067	0.0076	0.0273	0.0089	0.0409	0.0208	0.0231	0.2233	0.0355
NASDAQ100	0.0070	0.0056	0.0060	0.0273	0.0055	0.0308	0.0192	0.0298	0.1966	0.0264
S&P500	0.0057	0.0048	0.0050	0.0323	0.0045	0.0189	0.0153	0.0152	0.3576	0.0162
Average	0.0159	0.0108	0.0131	0.0536	0.0131	0.0471	0.0288	0.0369	0.4300	0.0388

out-of-sample portfolio returns. Figure 1 shows the corresponding results for FTSE100, NASDAQ100, and S&P500. For each of the five models, the portfolio obtained from the in-sample data is applied to the out-of-sample returns. These portfolio returns are then resampled with replacement, and the wealth process for each bootstrap sample is computed from an initial value of 1 by cumulative compounding. Therefore, the plotted curves represent the pointwise averages of the bootstrap wealth paths, rather than the realized chronological wealth paths used to calculate the Calmar ratio.

Observation 6 (Cumulative wealth evolution). Figure 1 provides a visual comparison of the average bootstrap wealth paths of the five models. For NASDAQ100, DRM shows the strongest average cumulative wealth growth and remains above the other models for most of the plotted horizon. SDM also performs strongly and attains the second-highest average terminal wealth. For FTSE100, all models generate increasing average wealth paths and the differences among them are smaller, although MinM attains the highest average terminal

wealth. This suggests that in this market the pure risk-minimizing model performs well in the bootstrap wealth comparison. For S&P500, SDM shows the highest average terminal wealth, while RM also performs strongly and DRM and NM remain competitive. Overall, the figure shows that the benefit of each model depends on the market structure. DRM gives the clearest advantage in NASDAQ100, SDM performs strongly in S&P500, and MinM is more favorable in FTSE100.

Observation 7 (Market dependent performance). The numerical results show that the best-performing model depends on both the market and the evaluation criterion. Overall, DRM provides the most consistent out-of-sample performance across the experiments, particularly for NASDAQ100 and S&P500 in terms of expected return, Sharpe ratio, and Calmar ratio. For FTSE100, the results are more mixed, with MinM performing strongly under several criteria. MinM also obtains the lowest average CVaR, while RM achieves the highest average in-sample Sharpe ratio.

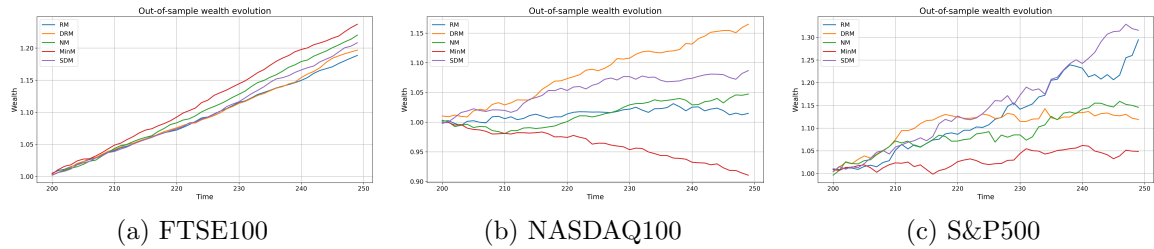


Figure 1: Average bootstrap cumulative wealth based on out-of-sample returns for the case of 35 selected assets.

5.4 Sensitivity Analysis of the Uncertainty Parameters

In this subsection, we investigate the sensitivity of the proposed uncertainty-based risk budgeting models to the choice of their main parameters. For this analysis, we consider the 35-asset instance for each of the FTSE100, NASDAQ100, and S&P500 markets. For RM, we consider $\rho \in \{0, 0.1, 0.2, 0.3, 0.5\}$, while for DRM we consider $\varepsilon \in \{0, 0.25, 0.5, 0.75, 1\}$. For each parameter value, the model is estimated using the same in-sample data and evaluated on the corresponding out-of-sample data. We report the average out-of-sample Sharpe ratio and MAD over the three markets.

Observation 8 (Sensitivity to uncertainty parameters). Table 6 shows that the

Table 6: Sensitivity analysis for the RM and DRM models.

RM			DRM		
ρ	Sharpe ratio	MAD	ε	Sharpe ratio	MAD
0.00	0.5281	0.0507	0.00	0.5281	0.0507
0.10	0.4964	0.0497	0.25	0.5481	0.0524
0.20	0.5026	0.0497	0.50	0.5481	0.0524
0.30	0.5096	0.0497	0.75	0.5481	0.0524
0.50	0.5275	0.0496	1.00	0.5481	0.0524

main results are relatively stable over the considered parameter ranges. For RM, introducing a positive value of ρ slightly reduces the average out-of-sample MAD, which remains within a narrow range across the considered values of ρ . The Sharpe ratio initially decreases for $\rho = 0.1$ and then gradually increases as ρ increases, reaching a value close to the nominal case at $\rho = 0.5$. For DRM, introducing a positive ambiguity radius improves the average out-of-sample Sharpe ratio, while MAD increases only moderately. Moreover, the results remain essentially unchanged across the positive values of ε considered. Overall, the sensitivity analysis indicates that the main conclusions for RM and DRM are not driven by a particular choice of the uncertainty parameters within the considered ranges.

6 Conclusion

In this paper, we studied risk budgeting under the mean absolute deviation (MAD) risk measure in the presence of uncertainty. We first developed a robust model using a correlated budget uncertainty set to perturb historical return observations, with deviations measured relative to the mean of the same perturbed empirical distribution. We then proposed a distributionally robust model based on a Wasserstein ambiguity set centered at the empirical distribution and derived a tractable reformulation. The numerical results showed that explicitly modeling uncertainty can improve the stability and out-of-sample performance of MAD-based portfolios. The distributionally robust model achieved the lowest turnover and the highest average out-of-sample expected return, Sharpe ratio, and Calmar ratio. It also obtained the lowest average Mean absolute deviation and Max absolute deviation of relative risk contributions from the equal-risk target, indicating stronger overall out-of-sample risk-parity performance. The robust model achieved the highest average in-sample

Sharpe ratio, while the minimum MAD model obtained the lowest average CVaR. The sensitivity analysis further showed that the performance of the proposed models remains relatively stable over the considered ranges of their uncertainty parameters. A direction for future research is to extend the proposed framework to dynamic and multi-period settings, where portfolios are updated as new market information becomes available.

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