QUASINORMALITY IN NONLINEAR SECOND-ORDER CONE PROGRAMMING*

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Abstract. We introduce quasinormality and pseudonormality for nonlinear second-order cone programming by exploiting the spectral geometry of the Lorentz cone. The proposed quasinormality distinguishes interior and boundary multipliers and, in the latter case, relates constraint violation to the corresponding spectral direction. It recovers classical quasinormality when all cone blocks are one-dimensional and is strictly weaker than Robinson's condition. Nevertheless, it implies a local error bound and exact penalization, while also guaranteeing boundedness of dual subsequences generated by an augmented Lagrangian method at feasible accumulation points. We also show that the constant rank constraint qualification implies quasinormality for a single conic constraint, whereas this implication fails in general for multiple blocks. We further introduce a relaxed formulation, expressed solely in terms of projections, norms, and distances to the cone boundary, that preserves the local error bound and dual boundedness. Finally, an extension of this condition to infeasible points allows us to establish feasibility of accumulation points satisfying the extended condition.

Key words. Nonlinear second-order cone programming, quasinormality, constraint qualifications, augmented Lagrangian, error bound

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1. Introduction.

Constraint qualifications play a central role in multiplier existence and convergence analysis in nonlinear optimization. Among weaker conditions, quasinormality, introduced by Hestenes [19], excludes nonzero abnormal multipliers compatible with sequences of nearby constraint violations. Pseudonormality [13] implies quasinormality in nonlinear programming. These conditions yield Karush–Kuhn–Tucker (KKT) optimality conditions, exact penalization, local error bounds, and boundedness of augmented Lagrangian multiplier sequences under milder assumptions than classical regularity conditions [13, 9].

In nonlinear second-order cone programming (NSOCP), Robinson's condition [29], a conic counterpart of the Mangasarian–Fromovitz constraint qualification (MFCQ) [25], guarantees a nonempty, closed, bounded multiplier set at local minimizers. Nondegeneracy, analogous to the linear independence constraint qualification (LICQ), ensures multiplier uniqueness and stronger stability properties [15]. Both impose strong geometric requirements that may fail at degenerate points.

Constant-rank conditions offer another approach. Classical nonlinear programming examples include the constant rank constraint qualification (CRCQ) [20], its relaxed version (RCRCQ) [26], and the constant rank of the subspace component constraint qualification (CRSC) [11]. Their extension to NSOCP requires care: the conditions in [32] were shown insufficient for multiplier existence [3]. Subsequent approaches use partial scalar reductions [6], sequential conditions based on spectral directions [7], cone faces [8], and the minimal face [5]. The spectral approach is suited to algorithmic sequences, whereas the facial approach yields second-order optimality conditions for every multiplier. These approaches need not be equivalent.

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Quasinormality and pseudonormality remain less explored in NSOCP because their scalar sign conditions require a replacement reflecting the nonpolyhedral geometry of the Lorentz cone. Recent work on nonlinear semidefinite programming (NSDP) uses spectral quasinormality to establish boundedness of augmented Lagrangian dual sequences without Robinson's condition [30]. Exploiting the simpler spectral structure of the Lorentz cone, we introduce quasinormality and pseudonormality for NSOCP. Our quasinormality distinguishes interior and boundary multipliers, relating constraint violations to the spectral direction determined by each nonzero boundary multiplier. Unlike the quasinormality of Guo, Ye, and Zhang for abstract constraint systems [18], which imposes componentwise sign conditions, our condition is intrinsic to the spectral geometry of the cone and may hold even when theirs fails.

The proposed quasinormality recovers its classical counterpart when all cone blocks are one-dimensional and is strictly weaker than Robinson's condition. It implies a local error bound and exact penalization and guarantees boundedness of augmented Lagrangian dual subsequences at feasible accumulation points. We also prove that CRCQ implies quasinormality for a single conic constraint, but not in general for multiple cone blocks. Finally, we introduce relaxed quasinormality using only projections, norms, and distances to the cone boundary. This condition preserves the local error bound and dual boundedness, while its extension to infeasible points guarantees feasibility of accumulation points satisfying the extended condition.

After fixing the notation, Section 2 reviews preliminaries and optimality conditions. Section 3 introduces quasinormality and pseudonormality, and Section 4 compares them with Robinson's condition, nondegeneracy, CRCQ, and quasinormality for abstract constraint systems. Section 5 establishes a local error bound and exact penalization. Section 6 studies dual boundedness, relaxed quasinormality and its extension to infeasible points, and the resulting optimality and convergence properties.

Notation We work in Euclidean spaces with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. For a set C, we denote by $\operatorname{int} C$, $\operatorname{bd} C$, and $\operatorname{cl} C$ its interior, boundary, and closure, respectively. For a nonempty closed convex set C, Π_C denotes the Euclidean projection onto C, $N_C(x) := \{v : \langle v, z - x\rangle \le 0,\ \forall z \in C\}$, and $\operatorname{dist}(x, C) := \|x - \Pi_C(x)\|$. For a set D, $D^\circ := \{v : \langle v, d\rangle \le 0,\ \forall d \in D\}$, and, for a cone K, $K^* := \{v : \langle v, z\rangle \ge 0,\ \forall z \in K\}$.

2. Preliminaries and the NSOCP problem. In this section, we collect the basic concepts used throughout the paper. For $m \ge 2$, the second-order cone, or Lorentz cone, is $\mathbb{L}_m := \{y = (y_0, \widehat{y}) \in \mathbb{R} \times \mathbb{R}^{m-1} : y_0 \ge \|\widehat{y}\|\}$, and we set $\mathbb{L}_1 := \mathbb{R}_+$. Its interior is $\operatorname{int}(\mathbb{L}_m) = \{y : y_0 > \|\widehat{y}\|\}$, and we denote its positive boundary by $\operatorname{bd}^+(\mathbb{L}_m) := \{y : y_0 = \|\widehat{y}\| > 0\}$. Hence $\operatorname{bd}(\mathbb{L}_m) = \operatorname{bd}^+(\mathbb{L}_m) \cup \{0\}$, with the origin being the vertex of the cone.

In the spectral formulas below, we assume $m \ge 2$. For $y = (y_0, \widehat{y}) \in \mathbb{R}^m$, define $\lambda_-(y) := y_0 - \|\widehat{y}\|$ and $\lambda_+(y) := y_0 + \|\widehat{y}\|$. Then

$$y = \lambda_-(y)u_-(y) + \lambda_+(y)u_+(y),$$

where

$$u_-(y) := \begin{cases} \frac{1}{2}\left(1, -\frac{\widehat{y}}{\|\widehat{y}\|}\right), & \widehat{y} \ne 0, \\ \frac{1}{2}(1, -w), & \widehat{y} = 0, \end{cases} \qquad u_+(y) := \begin{cases} \frac{1}{2}\left(1, \frac{\widehat{y}}{\|\widehat{y}\|}\right), & \widehat{y} \ne 0, \\ \frac{1}{2}(1, w), & \widehat{y} = 0, \end{cases}$$

with arbitrary $w \in \mathbb{R}^{m-1}$ satisfying $\|w\| = 1$. When $\widehat{y} = 0$, the eigenvalues coincide and the vectors $u_-(y)$ and $u_+(y)$ are not uniquely determined. Let $\Gamma_m :=$

$\operatorname{diag}(1,-1,\dots,-1)$. We also define $\det(y) := \lambda_+(y)\lambda_-(y) = y_0^2 - \|\widehat{y}\|^2$. The second-order cone is self-dual, i.e., $\mathbb{L}_m^* = \mathbb{L}_m$.

LEMMA 2.1. *Let $z, u \in \mathbb{L}_m$. Then $\langle z, u\rangle = 0$ if and only if $z = 0$, or $u = 0$, or $z, u \in \operatorname{bd}^+(\mathbb{L}_m)$ and $u = \alpha\Gamma_m z$ for some $\alpha > 0$. In particular, if $z \in \operatorname{bd}^+(\mathbb{L}_m)$, $\langle z, u\rangle = 0$, and $u \neq 0$, then $u \in \operatorname{bd}^+(\mathbb{L}_m)$ and $u_+(u) = u_-(z)$.*

The spectral decomposition also gives the projection onto the second-order cone: $\Pi_{\mathbb{L}_m}(y) = \max\{0, \lambda_+(y)\}u_+(y) + \max\{0, \lambda_-(y)\}u_-(y)$.

LEMMA 2.2. *Let $\Pi := \Pi_{\mathbb{L}_m}$. Then:*

(a) since $\mathbb{L}_m$ is self-dual, Moreau's decomposition gives $y - \Pi(y) = -\Pi(-y) \in -\mathbb{L}_m$ and $\langle \Pi(y), y - \Pi(y)\rangle = 0$;

(b) the function $y \mapsto \frac{1}{2}\operatorname{dist}(y, \mathbb{L}_m)^2$ is continuously differentiable, with gradient $y - \Pi(y)$.

These properties follow from the spectral decomposition of the second-order cone, Moreau's decomposition, and standard results on metric projections onto closed convex sets; see, for instance, [1]. For $m = 1$, we use $\mathbb{L}_1 = \mathbb{R}_+$, $\Pi_{\mathbb{L}_1}(y) = \max\{y, 0\}$, and $\operatorname{bd}^+(\mathbb{L}_1) = \varnothing$. Statements involving spectral vectors are restricted to $m \geq 2$, whereas the projection and complementarity properties remain valid in the scalar case. In componentwise expressions, we set $y_0 = y$ and $\|\widehat{y}\| = 0$ when $m = 1$.

Consider the standard nonlinear second-order cone programming problem

$$\text{(NSOCP)} \qquad \begin{array}{ll} \min & f(x), \\ \text{s.t.} & G_j(x) \in \mathbb{L}_j, \quad j = 1, \dots, p, \end{array}$$

where, throughout the paper, $f : \mathbb{R}^n \to \mathbb{R}$ and $G_j : \mathbb{R}^n \to \mathbb{R}^{m_j}$ are continuously differentiable, and $\mathbb{L}_j := \mathbb{L}_{m_j}$ denotes the cone associated with block j. Notice that the block index in $\mathbb{L}_j$ is distinct from the dimension index in $\mathbb{L}_m$.

The feasible set of (NSOCP) is denoted by Ω. Given $\bar{x} \in \mathbb{R}^n$, we denote by $DG_j(\bar{x})$ the derivative of $G_j : \mathbb{R}^n \to \mathbb{R}^{m_j}$ at $\bar{x}$, and by $DG_j(\bar{x})^T$ its transpose. For a feasible point $\bar{x}$, define

$$\begin{aligned} I_{\text{int}}(\bar{x}) &:= \{j \in \{1, \dots, p\} : G_j(\bar{x}) \in \operatorname{int}(\mathbb{L}_j)\}, \\ I_B(\bar{x}) &:= \{j \in \{1, \dots, p\} : G_j(\bar{x}) \in \operatorname{bd}^+(\mathbb{L}_j)\}, \\ I_0(\bar{x}) &:= \{j \in \{1, \dots, p\} : G_j(\bar{x}) = 0\}. \end{aligned}$$

We say that $\bar{x}$ satisfies the Karush–Kuhn–Tucker (KKT) conditions if there exist Lagrange multipliers $\mu_j \in \mathbb{L}_j$ such that

$$\nabla f(\bar{x}) - \sum_{j=1}^{p} DG_j(\bar{x})^T \mu_j = 0, \qquad \langle G_j(\bar{x}), \mu_j\rangle = 0, \quad j = 1, \dots, p. \tag{2.1}$$

The tangent cone to Ω at $\bar{x}$ is

$$T_\Omega(\bar{x}) := \{d \in \mathbb{R}^n : \exists\, t_k \to 0^+,\ \exists\, d^k \to d \text{ such that } \bar{x} + t_k d^k \in \Omega \text{ for all } k\},$$

whereas the linearized cone is

$$\begin{aligned} L_\Omega(\bar{x}) &:= \{d \in \mathbb{R}^n : DG_j(\bar{x})d \in \mathcal{T}_{\mathbb{L}_j}(G_j(\bar{x})),\ j = 1, \dots, p\} \\ &= \left\{ d \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} DG_j(\bar{x})d \in \mathbb{L}_j, & j \in I_0(\bar{x}); \\ \langle DG_j(\bar{x})d, \Gamma_{m_j} G_j(\bar{x})\rangle \geq 0, & j \in I_B(\bar{x}). \end{array} \right\}. \end{aligned} \tag{2.2}$$

See [15] for further details on (2.2).

The relationship between the tangent and linearized cones is central to the theory of constraint qualifications. Abadie's constraint qualification requires $T_\Omega(\bar{x}) = L_\Omega(\bar{x})$, whereas Guignard's constraint qualification requires only $T_\Omega(\bar{x})^\circ = L_\Omega(\bar{x})^\circ$. In classical nonlinear programming, where the underlying constraint cone is polyhedral, these geometric conditions are sufficient to guarantee the existence of Lagrange multipliers. In nonpolyhedral conic optimization, however, additional topological properties are required.

Let $G(x) := (G_1(x), \ldots, G_p(x))$ and $\mathbb{L} := \prod_{j=1}^p \mathbb{L}_j$, and define

$$H(\bar{x}) := \{DG(\bar{x})^T y : y \in N_{\mathbb{L}}(G(\bar{x}))\}, \tag{2.3}$$

where $N_{\mathbb{L}}(G(\bar{x})) = \{y \in \mathbb{L}^\circ : \langle G(\bar{x}), y\rangle = 0\}$ is the normal cone to $\mathbb{L}$ at $G(\bar{x})$. Closedness of $H(\bar{x})$ was introduced by Guignard [17] as an additional assumption in multiplier results for optimization problems in Banach spaces. In conic programming, however, equality between the tangent and linearized cones, or even between their polars, does not by itself guarantee the existence of Lagrange multipliers [2]. Moreover, closedness of $H(\bar{x})$ alone is also insufficient, as shown in [3]; see [8] for a more detailed discussion.

These difficulties illustrate why classical geometric conditions may be too restrictive or fail to capture the degeneracy inherent in conic problems, and motivate the study of weaker constraint qualifications that still ensure KKT conditions. In nonlinear programming, quasinormality, introduced by Hestenes [19], is one of the weakest conditions of this type. Unlike constant-rank conditions, which are formulated solely in terms of constraint derivatives, quasinormality is sequential in nature and rules out constraint-violation sequences associated with abnormal multipliers. This provides the motivation for extending quasinormality to the NSOCP setting.

3. Quasinormality and pseudonormality for NSOCP. In this section, we introduce notions of quasinormality and pseudonormality for problem (NSOCP).

Definition 3.1. *Let $\bar{x}$ be a feasible point of problem* (NSOCP). *We say that $\bar{x}$ satisfies* quasinormality *if there do not exist multipliers $\mu_j \in \mathbb{L}_j$, not all zero, and a sequence $\{x^k\}$ converging to $\bar{x}$ such that*

$$\sum_{j=1}^{p} DG_j(\bar{x})^T \mu_j = 0, \tag{3.1}$$

and, for every $j \in \{1, \ldots, p\}$ with $\mu_j \neq 0$, the corresponding cone constraint is violated according to one of the following cases:

(i) if $\mu_j \in \operatorname{int}(\mathbb{L}_j)$, then $G_j(x^k) \in \operatorname{int}(-\mathbb{L}_j)$ for all sufficiently large k;

(ii) if $\mu_j \in \operatorname{bd}^+(\mathbb{L}_j)$, then $G_j(x^k) \notin \mathbb{L}_j$ for all sufficiently large k and

$$u_-(G_j(x^k)) \to u_+(\mu_j).$$

Here, $u_+(\mu_j)$ denotes the spectral vector associated with $\lambda_+(\mu_j)$. If $\widehat{G_j(x^k)} = 0$, condition (ii) requires the existence of choices of the spectral vectors for which the stated convergence holds. For scalar blocks, only condition (i) applies, with $\mu_j > 0$ and $G_j(x^k) < 0$.

Remark 3.2. The conditions in Definition 3.1 imply $\langle G_j(\bar{x}), \mu_j\rangle = 0$ for every j. This is immediate when $\mu_j = 0$. If $\mu_j \in \operatorname{int}(\mathbb{L}_j)$, condition (i), continuity of

G_j, and feasibility of $\bar{x}$ yield $G_j(\bar{x}) \in \mathbb{L}_j \cap (-\mathbb{L}_j) = \{0\}$. If $\mu_j \in \mathrm{bd}^+(\mathbb{L}_j)$, condition (ii) implies $\lambda_-(G_j(x^k)) < 0$. By continuity and feasibility, $\lambda_-(G_j(\bar{x})) = 0$. If $G_j(\bar{x}) = 0$, complementarity is immediate. Otherwise, $G_j(\bar{x}) \in \mathrm{bd}^+(\mathbb{L}_j)$ and continuity of u_- at $G_j(\bar{x})$, together with the alignment condition in (ii), gives $u_-(G_j(\bar{x})) = \lim_{k\to\infty} u_-(G_j(x^k)) = u_+(\mu_j)$. Since $G_j(\bar{x})$ and μ_j are positive multiples of $u_+(G_j(\bar{x}))$ and $u_-(G_j(\bar{x}))$, respectively, the orthogonality of these spectral vectors yields $\langle G_j(\bar{x}), \mu_j\rangle = 0$.

In classical nonlinear programming, quasinormality requires a nearby sequence to violate every constraint associated with a positive multiplier. Definition 3.1 extends this idea to second-order cone constraints by distinguishing the position of μ_j within the cone. If $\mu_j \in \mathrm{int}(\mathbb{L}_j)$, the sequence must satisfy $G_j(x^k) \in \mathrm{int}(-\mathbb{L}_j)$. If $\mu_j \in \mathrm{bd}^+(\mathbb{L}_j)$, only $\lambda_+(\mu_j)$ is positive, and the violation $G_j(x^k) \notin \mathbb{L}_j$ is additionally required to satisfy $u_-(G_j(x^k)) \to u_+(\mu_j)$. Thus, the spectral direction of the violation is matched with the positive spectral direction of the multiplier. No violation condition is imposed when $\mu_j = 0$.

In nonlinear programming, pseudonormality implies quasinormality. We next introduce a counterpart for problem (NSOCP), in which the violation condition is expressed through a multiplier-weighted aggregate. We will show that it also implies the quasinormality of Definition 3.1.

Definition 3.3. *Let $\bar{x}$ be a feasible point of problem* (NSOCP). *We say that $\bar{x}$ satisfies* pseudonormality *if there do not exist multipliers $\mu_j \in \mathbb{L}_j$, not all zero, satisfying* (3.1),

$$\mu_j = 0, \quad j \in I_{\mathrm{int}}(\bar{x}), \qquad \mu_j = \alpha_j \Gamma_{m_j} G_j(\bar{x}), \quad \alpha_j \geq 0, \quad j \in I_B(\bar{x}),$$

and a sequence $x^k \to \bar{x}$ such that $\sum_{j=1}^p \left([\mu_j]_0 [G_j(x^k)]_0 - \|\widehat{\mu_j}\| \, \|\widehat{G_j(x^k)}\| \right) < 0$ for all sufficiently large k.

When all blocks are scalar, the violation condition in Definition 3.3 reduces to $\sum_{j=1}^p \mu_j G_j(x^k) < 0$, recovering the classical Bertsekas–Ozdaglar pseudonormality condition [13].

Remark 3.4. A definition based on the inner product $\langle \mu_j, G_j(x^k)\rangle$ would not preserve the implication pseudonormality $\Rightarrow$ quasinormality. Indeed, consider $G(x) = (1, 1, x) \in \mathbb{L}_3$ at $\bar{x} = 0$, with $DG(\bar{x}) = (0, 0, 1)^T$. The multiplier $\mu = (1, -1, 0)$ satisfies $DG(\bar{x})^T \mu = 0$ and $\langle G(\bar{x}), \mu\rangle = 0$. For $x^k = 1/k$, we have $G(x^k) \notin \mathbb{L}_3$ and $u_-(G(x^k)) \to u_+(\mu)$, so quasinormality fails at $\bar{x}$. On the other hand, $\langle \mu, G(x^k)\rangle = 0$ for every k, while

$$[\mu]_0 [G(x^k)]_0 - \|\widehat{\mu}\| \, \|\widehat{G(x^k)}\| = 1 - \sqrt{1 + \frac{1}{k^2}} < 0.$$

Moreover, every nonzero multiplier satisfying stationarity and complementarity at $\bar{x}$ has the form $t(1, -1, 0)$ with $t > 0$, and its inner product with $G(x)$ vanishes for every x. Hence, replacing the inequality in Definition 3.3 by a negative-inner-product condition would classify $\bar{x}$ as pseudonormal even though quasinormality fails.

Theorem 3.5. *Let $\bar{x}$ be a feasible point of problem* (NSOCP). *If $\bar{x}$ satisfies pseudonormality, then it satisfies quasinormality.*

Proof. Suppose that quasinormality fails at $\bar{x}$. Then there exist multipliers $\mu_j \in \mathbb{L}_j$, not all zero, and a sequence $x^k \to \bar{x}$ satisfying Definition 3.1. By Remark 3.2,

$\langle G_j(\bar{x}), \mu_j \rangle = 0$ for every j. The complementarity characterization of the second-order cone therefore gives

$$\mu_j = 0, \quad j \in I_{\text{int}}(\bar{x}), \qquad \mu_j = \alpha_j \Gamma_{m_j} G_j(\bar{x}), \quad \alpha_j \geq 0, \quad j \in I_B(\bar{x}).$$

Hence, these multipliers satisfy the conditions in Definition 3.3. It remains to verify the violation inequality. Set $s_j^k := [\mu_j]_0 [G_j(x^k)]_0 - \|\widehat{\mu_j}\| \, \|\widehat{G_j(x^k)}\|$. If $\mu_j = 0$, then $s_j^k = 0$. If $\mu_j \in \text{int}(\mathbb{L}_j)$, condition (i) gives $[G_j(x^k)]_0 < 0$ for all sufficiently large k, and therefore $s_j^k < 0$. If $\mu_j \in \text{bd}^+(\mathbb{L}_j)$, then $[\mu_j]_0 = \|\widehat{\mu_j}\| > 0$ and condition (ii) gives

$$s_j^k = [\mu_j]_0 \left([G_j(x^k)]_0 - \|\widehat{G_j(x^k)}\| \right) = [\mu_j]_0 \lambda_-(G_j(x^k)) < 0.$$

For scalar blocks, the corresponding term is $\mu_j G_j(x^k)$, which is also negative whenever $\mu_j > 0$. Since the multipliers are not all zero, $\sum_{j=1}^p s_j^k < 0$ for all sufficiently large k. Thus, pseudonormality also fails at $\bar{x}$. ∎

Example 3.6. The converse of Theorem 3.5 does not hold. Consider $G_1(x) = (x, 0) \in \mathbb{L}_2$ and $G_2(x) = (-x - x^2, 0) \in \mathbb{L}_2$. At $\bar{x} = 0$, both blocks vanish. The multipliers $\mu_1 = \mu_2 = (1, 0)$ satisfy (3.1), and for $x^k = 1/k$,

$$\sum_{j=1}^{2} \left([\mu_j]_0 [G_j(x^k)]_0 - \|\widehat{\mu_j}\| \, \|\widehat{G_j(x^k)}\| \right) = -(x^k)^2 < 0.$$

Hence, pseudonormality fails at $\bar{x}$. On the other hand, any multipliers in $\mathbb{L}_2$, not both zero, satisfying (3.1) must satisfy $\mu_{1,0} = \mu_{2,0} > 0$. Thus both multipliers are nonzero, and failure of quasinormality would require simultaneous violation of both blocks. This is impossible since, for $|x| < 1$, $G_1(x) \notin \mathbb{L}_2$ if and only if $x < 0$, whereas $G_2(x) \notin \mathbb{L}_2$ if and only if $x > 0$. Therefore, $\bar{x}$ is quasinormal.

Our NSOCP quasinormality can be viewed as a geometric counterpart of the spectral NSDP condition in [30], which links violations to positive multiplier eigenvalues. Here, interior multipliers require violations in the interior of the negative cone, whereas boundary multipliers require spectral alignment.

4. Relations with other constraint qualifications. In this section, we investigate the relationships among quasinormality, nondegeneracy, Robinson's condition, and constant-rank conditions for (NSOCP).

4.1. Relation with nondegeneracy and Robinson's condition.

Definition 4.1. *[15] Let $\bar{x}$ be a feasible point of problem* (NSOCP). *We say that $\bar{x}$ is* nondegenerate *if the matrix formed by the rows of $DG_j(\bar{x})$, for $j \in I_0(\bar{x})$, together with the row vectors $G_j(\bar{x})^T \Gamma_{m_j} DG_j(\bar{x})$, for $j \in I_B(\bar{x})$, has full row rank.*

Nondegeneracy extends the linear independence constraint qualification (LICQ) from nonlinear programming to NSOCP. When it holds at a local minimizer, the Lagrange multiplier vector exists and is unique.

Definition 4.2. *[15] Let $\bar{x}$ be a feasible point of problem* (NSOCP). *Robinson's condition is said to hold at $\bar{x}$ if there do not exist scalars $\alpha_j \geq 0$, $j \in I_B(\bar{x})$, and multipliers $\mu_j \in \mathbb{L}_j$, $j \in I_0(\bar{x})$, not all zero, such that*

$$\sum_{j \in I_B(\bar{x})} \alpha_j DG_j(\bar{x})^T \Gamma_{m_j} G_j(\bar{x}) + \sum_{j \in I_0(\bar{x})} DG_j(\bar{x})^T \mu_j = 0. \tag{4.1}$$

Nondegeneracy excludes nontrivial vanishing combinations with $\alpha_j \in \mathbb{R}$ and $\mu_j \in \mathbb{R}^{m_j}$, whereas Robinson's condition excludes them only for $\alpha_j \geq 0$ and $\mu_j \in \mathbb{L}_j$. Hence, nondegeneracy implies Robinson's condition. When all blocks are scalar, Robinson's condition reduces to the Mangasarian–Fromovitz constraint qualification (MFCQ).

PROPOSITION 4.3. *Let $\bar{x}$ be feasible for* (NSOCP). *If Robinson's condition holds at $\bar{x}$, then $\bar{x}$ is pseudonormal and, consequently, quasinormal.*

Proof. If pseudonormality failed, the nonzero multipliers in Definition 3.3, after omitting the zero components corresponding to $I_{\text{int}}(\bar{x})$, would satisfy exactly the nontrivial system excluded by Robinson's condition. Hence $\bar{x}$ is pseudonormal. Quasinormality follows from Theorem 3.5. ∎

Example 4.4. Consider $G_1(x) = (x, x) \in \mathbb{L}_2$ and $G_2(x) = (-x, x) \in \mathbb{L}_2$. Since $G_1(x) \in \mathbb{L}_2$ if and only if $x \geq 0$, whereas $G_2(x) \in \mathbb{L}_2$ if and only if $x \leq 0$, the feasible set is $\Omega = \{0\}$. Thus, $\bar{x} = 0$ is the unique feasible point, and both blocks are at the vertex.

Robinson's condition fails at $\bar{x}$. Indeed, $\bar{\mu}_1 = (1, 1)$ and $\bar{\mu}_2 = (1, -1)$ belong to $\mathbb{L}_2$ and satisfy $DG_1(\bar{x})^T\bar{\mu}_1 + DG_2(\bar{x})^T\bar{\mu}_2 = 2 - 2 = 0$.

We now verify quasinormality. Let $\mu_1, \mu_2 \in \mathbb{L}_2$, not both zero, satisfy (3.1). If both are nonzero, failure of quasinormality would require simultaneous violation of both blocks, which is impossible since $G_1(x) \notin \mathbb{L}_2$ if and only if $x < 0$, whereas $G_2(x) \notin \mathbb{L}_2$ if and only if $x > 0$.

If only $\mu_1 \neq 0$, equation (3.1) gives $\mu_{1,0} + \mu_{1,1} = 0$, so $\mu_1 = t(1, -1)$ for some $t > 0$. Violation of the first block requires $x^k < 0$, but then $u_-(G_1(x^k)) = \frac{1}{2}(1, 1) \neq \frac{1}{2}(1, -1) = u_+(\mu_1)$. Similarly, if only $\mu_2 \neq 0$, then $\mu_2 = s(1, 1)$ for some $s > 0$, while violation of the second block requires $x^k > 0$, in which case $u_-(G_2(x^k)) = \frac{1}{2}(1, -1) \neq \frac{1}{2}(1, 1) = u_+(\mu_2)$. Hence no admissible multiplier admits a sequence satisfying Definition 3.1, and $\bar{x}$ is quasinormal.

4.2. Relation with the constant rank constraint qualification. We next recall the constant rank constraint qualification (CRCQ) introduced in [8] for problem (NSOCP). For $j \in I_B(\bar{x})$, define locally around $\bar{x}$, the function $g_j(x) := [G_j(x)]_0 - \|\widehat{G_j(x)}\|$. This function is C^1, and the corresponding cone constraint is locally equivalent to $g_j(x) \geq 0$. In particular,

$$\nabla g_j(\bar{x}) = \frac{1}{[G_j(\bar{x})]_0} DG_j(\bar{x})^T \Gamma_{m_j} G_j(\bar{x}).$$

This identity is used at the reference point; in the constant-rank condition below, the quantity considered in a neighborhood is $\nabla g_j(x)$.

DEFINITION 4.5. *[8] Let $\bar{x}$ be a feasible point of problem* (NSOCP). *We say that the* constant rank constraint qualification *(CRCQ) holds at $\bar{x}$ if there exists a neighborhood V of $\bar{x}$ such that, for every $J_1, J_2 \subseteq I_0(\bar{x})$ and $J_3 \subseteq I_B(\bar{x})$, with $m_j > 1$ for every $j \in J_1$, and every $\eta_j \in \text{bd}^+(\mathbb{L}_{m_j})$, $j \in J_1$, the rank of the family*

$$\bigcup_{\substack{j \in J_1 \\ i=1,\ldots,m_j-1}} \{DG_j(x)^T A_j^i\} \quad \bigcup_{\substack{j \in J_2 \\ i=0,\ldots,m_j-1}} \{\nabla [G_j]_i(x)\} \bigcup_{j \in J_3} \{\nabla g_j(x)\}$$

is constant for all $x \in V$. Here, $A_j \in \mathbb{R}^{m_j \times (m_j-1)}$ is any full-column-rank matrix such that $\text{Im}\, A_j = \{\eta_j\}^\perp$, and A_j^i denotes its ith column. In addition, the set $H(\bar{x})$ defined in (2.3) *is required to be closed.*

The CRCQ of [8] is independent of Robinson's condition and strictly weaker than nondegeneracy. Unlike Robinson's condition, it guarantees a strong second-order optimality condition for every multiplier. For a single cone block active at the vertex, it also ensures local invariance of the Jacobian image, a key property for proving that CRCQ implies quasinormality.

LEMMA 4.6. *Let $\bar{x}$ be a feasible point of an NSOCP problem with a single conic constraint $G(x) \in \mathbb{L}_m$. Suppose that $G(\bar{x}) = 0$ and CRCQ holds at $\bar{x}$, and define $S := \operatorname{Im} DG(\bar{x})$. Then $\operatorname{rank} DG(x) = \dim S$ for all x sufficiently close to $\bar{x}$. Moreover, if $x^k \to \bar{x}$, $G(x^k) \neq 0$, and $G(x^k)/\|G(x^k)\| \to v$, then $v \in S$.*

If, in addition, $\dim S = 1$ and $S = \operatorname{span}\{\eta\}$ for some $\eta \in \operatorname{bd}^+(\mathbb{L}_m)$, then, after possibly shrinking the neighborhood, $\operatorname{Im} DG(x) = S$ and $G(x) \in S$ for every x in that neighborhood.

Proof. Let $r := \dim S = \operatorname{rank} DG(\bar{x})$. Taking the single block in J_2 in Definition 4.5, CRCQ implies, after possibly shrinking the neighborhood, that $\operatorname{rank} DG(x) = r$ for every $x \in V$. If $r = 0$, then $DG(x) = 0$ on V. Taking V convex and using $G(\bar{x}) = 0$, we obtain $G(x) = 0$ throughout V, and the statement concerning normalized secant directions is vacuous. Hence assume $r \geq 1$.

By the constant rank theorem, after shrinking V if necessary, $G(V)$ is locally a C^1 manifold of dimension r through the origin, with $T_{G(V)}(0) = \operatorname{Im} DG(\bar{x}) = S$. Relative to the orthogonal decomposition $\mathbb{R}^m = S \oplus S^\perp$, this manifold can locally be written as the graph of a C^1 mapping $h : U \subset S \to S^\perp$ satisfying $h(0) = 0$ and $Dh(0) = 0$. Thus, for all sufficiently large k, $G(x^k) = s_k + h(s_k)$, where $s_k \in S$ and $s_k \to 0$. Since $h(s_k) = o(\|s_k\|)$ and $S \perp S^\perp$,

$$\frac{\operatorname{dist}(G(x^k), S)}{\|G(x^k)\|} = \frac{\|h(s_k)\|}{\|G(x^k)\|} \leq \frac{\|h(s_k)\|}{\|s_k\|} \longrightarrow 0.$$

Therefore, if $G(x^k)/\|G(x^k)\| \to v$, then $\operatorname{dist}(v, S) = 0$, and hence $v \in S$.

Assume that $r = 1$, $S = \operatorname{span}\{\eta\}$, and $\eta \in \operatorname{bd}^+(\mathbb{L}_m)$. Choose a full-column-rank matrix $A \in \mathbb{R}^{m\times(m-1)}$ such that $\operatorname{Im} A = \{\eta\}^\perp$. Since $\ker DG(\bar{x})^T = S^\perp = \{\eta\}^\perp$, we have $DG(\bar{x})^T A = 0$. Thus, the family associated with J_1 has rank zero at $\bar{x}$. By CRCQ, $DG(x)^T A = 0$ for all x sufficiently close to $\bar{x}$, and therefore $\{\eta\}^\perp \subseteq \ker DG(x)^T$. Since $\operatorname{rank} DG(x) = 1$, both spaces have dimension $m - 1$. Hence $\ker DG(x)^T = \{\eta\}^\perp$, and therefore $\operatorname{Im} DG(x) = S$.

Finally, taking V convex, for every $x \in V$ we have

$$G(x) = \int_0^1 DG\big(\bar{x} + t(x - \bar{x})\big)(x - \bar{x})\, dt.$$

Since the image of every Jacobian along the segment is S, the integrand belongs to S, and therefore $G(x) \in S$. ∎

The next lemma identifies a situation in which the linear image of a Lorentz cone is not closed.

LEMMA 4.7. *Let $B : \mathbb{R}^m \to \mathbb{R}^n$ be linear, with $m \geq 2$, and suppose that*

$$\ker B \cap \operatorname{bd}^+(\mathbb{L}_m) \neq \varnothing, \qquad \ker B \cap \operatorname{int}(\mathbb{L}_m) = \varnothing, \qquad \operatorname{rank} B \geq 2.$$

Then $B(\mathbb{L}_m)$ is not closed.

Proof. Take $0 \neq \mu \in \ker B \cap \operatorname{bd}^+(\mathbb{L}_m)$. After a spatial rotation preserving $\mathbb{L}_m$ and a positive rescaling, we may assume $\mu = (1, 1, 0, \ldots, 0)^T$ and $\Gamma_m \mu = (1, -1, 0, \ldots, 0)^T$.

We first claim that $\ker B \subset (\Gamma_m\mu)^\perp$. Otherwise, there would exist $\xi \in \ker B$ with $\langle \Gamma_m\mu, \xi\rangle \neq 0$. Replacing ξ by $-\xi$ if necessary, assume $\langle \Gamma_m\mu, \xi\rangle > 0$. Since $\nabla \det(\mu) = 2\Gamma_m\mu$,

$$\det(\mu + t\xi) = 2t\langle \Gamma_m\mu, \xi\rangle + O(t^2) > 0$$

for all sufficiently small $t > 0$. The first component of $\mu + t\xi$ remains positive, so $\mu + t\xi \in \operatorname{int}(\mathbb{L}_m) \cap \ker B$, a contradiction. Thus $\Gamma_m\mu \in (\ker B)^\perp$.

Since $\dim(\ker B)^\perp = \operatorname{rank} B \geq 2$, there exists $0 \neq \tau \in (\ker B)^\perp \cap (\Gamma_m\mu)^\perp$. Moreover, $\tau \perp \mu$ because $\mu \in \ker B$, and hence, in the chosen coordinates, $\tau = (0, 0, \widehat{\tau})^T$. For $t > 0$, define $\nu_t := \frac{1}{t}\mu + \tau + \frac{t\|\tau\|^2}{4}\Gamma_m\mu$. A direct computation gives $\det \nu_t = 0$, and its first component is positive; hence $\nu_t \in \operatorname{bd}^+(\mathbb{L}_m)$. Since $B\mu = 0$,

$$B\nu_t = B\tau + \frac{t\|\tau\|^2}{4} B\Gamma_m\mu \longrightarrow B\tau,$$

so $B\tau \in \operatorname{cl} B(\mathbb{L}_m)$.

Suppose that $B\tau = B\eta$ for some $\eta \in \mathbb{L}_m$. Then $\eta - \tau \in \ker B$. Since $\Gamma_m\mu \in (\ker B)^\perp$ and $\tau \perp \Gamma_m\mu$, we obtain $\langle \Gamma_m\mu, \eta\rangle = 0$. In the chosen coordinates this means $\eta_0 = \eta_1$. Since $\eta \in \mathbb{L}_m$, it follows that $\eta \in \mathbb{R}_+\mu$, and therefore $B\eta = 0$. On the other hand, $0 \neq \tau \in (\ker B)^\perp$ implies $B\tau \neq 0$, a contradiction. Thus $B\tau \notin B(\mathbb{L}_m)$, proving that $B(\mathbb{L}_m)$ is not closed. □

The preceding lemmas yield the relationship between CRCQ and quasinormality for a single conic constraint.

THEOREM 4.8. *Let $\bar{x}$ be a feasible point of an NSOCP problem with a single conic constraint $G(x) \in \mathbb{L}_m$. If CRCQ holds at $\bar{x}$, then $\bar{x}$ is quasinormal.*

Proof. Suppose, by contradiction, that quasinormality fails at $\bar{x}$. Then there exist $\mu \in \mathbb{L}_m \setminus \{0\}$ and $x^k \to \bar{x}$ such that $DG(\bar{x})^T\mu = 0$ and the violation conditions of Definition 3.1 hold. By Remark 3.2, $\langle G(\bar{x}), \mu\rangle = 0$. Since $\mu \neq 0$, $G(\bar{x}) \notin \operatorname{int}(\mathbb{L}_m)$, so either $G(\bar{x}) = 0$ or $G(\bar{x}) \in \operatorname{bd}^+(\mathbb{L}_m)$.

Suppose first that $G(\bar{x}) \in \operatorname{bd}^+(\mathbb{L}_m)$. Then $\mu = \alpha\Gamma_m G(\bar{x})$ for some $\alpha > 0$. Let $g(x) := [G(x)]_0 - \|\widehat{G}(x)\|$. We have $\nabla g(\bar{x}) = 0$. Applying CRCQ to the singleton family $\{\nabla g(x)\}$ shows that $\nabla g(x) = 0$ in a neighborhood of $\bar{x}$. Hence $g(x) = g(\bar{x}) = 0$ there. This contradicts $G(x^k) \notin \mathbb{L}_m$, which requires $g(x^k) < 0$.

It remains to consider $G(\bar{x}) = 0$. Set $S := \operatorname{Im} DG(\bar{x})$. Since $DG(\bar{x})^T\mu = 0$, we have $\mu \in S^\perp$.

If $\mu \in \operatorname{int}(\mathbb{L}_m)$, failure of quasinormality gives $G(x^k) \in \operatorname{int}(-\mathbb{L}_m)$. Passing to a subsequence, $G(x^k)/\|G(x^k)\| \to y$ with $\|y\| = 1$. By Lemma 4.6, $y \in S$, and since $-\mathbb{L}_m$ is closed, $y \in -\mathbb{L}_m$. Hence $\langle \mu, y\rangle = 0$. However, $\mu \in \operatorname{int}(\mathbb{L}_m)$ and $0 \neq y \in -\mathbb{L}_m$ imply $\langle \mu, y\rangle < 0$, a contradiction.

Now let $\mu \in \operatorname{bd}^+(\mathbb{L}_m)$ and write $\mu = \mu_0(1, a)$, where $\|a\| = 1$ and $\mu_0 > 0$. Passing again to a subsequence, assume $G(x^k)/\|G(x^k)\| \to y \in S$, with $\|y\| = 1$. We may assume $\widehat{G}(x^k) \neq 0$ for all sufficiently large k; otherwise, a subsequence would satisfy $G(x^k)/\|G(x^k)\| = -e$, yielding $-e \in S$, contrary to $\mu \in S^\perp$. The alignment condition $u_-(G(x^k)) \to u_+(\mu)$ implies that $y = (y_0, -\gamma a)$ for some $\gamma \geq 0$. Since $y \in S \subset \mu^\perp$, we obtain $y_0 = \gamma$, and $\|y\| = 1$ then gives $y = \frac{1}{\sqrt{2}}(1, -a)$. Consequently, $\Gamma_m\mu \in S$.

If $\dim S = 1$, then $S = \operatorname{span}\{\Gamma_m\mu\}$. By the last part of Lemma 4.6, $G(x) \in S$ locally, so $G(x^k) = \alpha_k\Gamma_m\mu$. Since $G(x^k) \notin \mathbb{L}_m$, we must have $\alpha_k < 0$, and therefore $u_-(G(x^k)) = \frac{1}{2}(1, -a) \neq \frac{1}{2}(1, a) = u_+(\mu)$, contradicting the alignment condition. Hence $\dim S \geq 2$.

Let $B := DG(\bar{x})^T$. Then $\mu \in \ker B \cap \mathrm{bd}^+(\mathbb{L}_m)$ and $\Gamma_m\mu \in S = (\ker B)^\perp$. If there existed $\nu \in \ker B \cap \mathrm{int}(\mathbb{L}_m)$, then $\langle \nu, \Gamma_m\mu\rangle > 0$, since $\Gamma_m\mu \in \mathbb{L}_m \setminus \{0\}$. On the other hand, $\nu \in \ker B$ and $\Gamma_m\mu \in (\ker B)^\perp$ imply $\langle \nu, \Gamma_m\mu\rangle = 0$, a contradiction. Hence $\ker B \cap \mathrm{int}(\mathbb{L}_m) = \varnothing$ and $\operatorname{rank} B = \dim S \geq 2$. Lemma 4.7 therefore implies that $B(\mathbb{L}_m)$ is not closed. Since $G(\bar{x}) = 0$,

$$H(\bar{x}) = DG(\bar{x})^T(-\mathbb{L}_m) = -B(\mathbb{L}_m)$$

is also not closed, contradicting CRCQ. Thus $\bar{x}$ is quasinormal. ∎

The converse of Theorem 4.8 fails even for a single conic constraint.

Example 4.9. Consider $G(x_1, x_2) := (x_2, x_1^2, 0) \in \mathbb{L}_3$ and $\bar{x} = (0, 0)$, so that $G(\bar{x}) = 0$ and

$$DG(x) = \begin{pmatrix} 0 & 1 \\ 2x_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $DG(\bar{x})^T\mu = (0, \mu_0)^T$, every $\mu \in \mathbb{L}_3$ with $DG(\bar{x})^T\mu = 0$ satisfies $\mu_0 = 0$, and hence $\mu = 0$. Thus Robinson's condition (Definition 4.2) holds at $\bar{x}$, and $\bar{x}$ is quasinormal by Proposition 4.3. On the other hand, taking $J_2 = \{1\}$ and $J_1 = J_3 = \varnothing$ in Definition 4.5, the relevant family is $\{(0, 1)^T, (2x_1, 0)^T, (0, 0)^T\}$. Its rank is 1 at $\bar{x}$ but 2 whenever $x_1 \neq 0$. Thus CRCQ fails at $\bar{x}$. Hence quasinormality does not imply CRCQ, even for a single conic constraint.

For each block, define $H_j(\bar{x}) := DG_j(\bar{x})^T N_{\mathbb{L}_j}(G_j(\bar{x}))$. Since $N_{\mathbb{L}}(G(\bar{x})) = \prod_{j=1}^p N_{\mathbb{L}_j}(G_j(\bar{x}))$, we have $H(\bar{x}) = \sum_{j=1}^p H_j(\bar{x})$. With multiple cone blocks, closedness of the aggregate set $H(\bar{x})$ does not imply closedness of the individual sets $H_j(\bar{x})$. The next example shows that this distinction allows CRCQ to hold even though quasinormality fails.

Example 4.10. The implication in Theorem 4.8 does not extend, in general, to multiple blocks. Let $x = (x_1, x_2) \in \mathbb{R}^2$, $\bar{x} = (0, 0)$, $G_1(x) := (x_1, x_1, x_2) \in \mathbb{L}_3$ and $G_2(x) := (0, x_2, 0) \in \mathbb{L}_3$. We have $G_1(\bar{x}) = G_2(\bar{x}) = 0$, and

$$DG_1(\bar{x})^T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad DG_2(\bar{x})^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Thus $I_B(\bar{x}) = \varnothing$, and both Jacobians are constant. Hence all families appearing in the rank condition of Definition 4.5 have constant rank. On the other hand, $H_1(\bar{x}) = \{(u, v) \in \mathbb{R}^2 : u < 0\} \cup \{(0, 0)\}$ and $H_2(\bar{x}) = \{(0, v) : v \in \mathbb{R}\}$. Although $H_1(\bar{x})$ is not closed, $H(\bar{x}) = H_1(\bar{x}) + H_2(\bar{x}) = \{(u, v) \in \mathbb{R}^2 : u \leq 0\}$ is closed. Therefore, CRCQ holds at $\bar{x}$.

Quasinormality, however, fails. Take $\mu_1 = (1, -1, 0)^T \in \mathrm{bd}^+(\mathbb{L}_3)$ and $\mu_2 = 0$. Then $DG_1(\bar{x})^T\mu_1 = 0$. For $x^k = (1/k, 1/k^2) \to \bar{x}$, we have $G_1(x^k) \notin \mathbb{L}_3$ and

$$u_-(G_1(x^k)) = \frac{1}{2}\begin{pmatrix} 1 \\ -\dfrac{1}{\sqrt{1 + 1/k^2}} \\ -\dfrac{1/k}{\sqrt{1 + 1/k^2}} \end{pmatrix} \longrightarrow \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = u_+(\mu_1).$$

Hence quasinormality fails at $\bar{x}$.

Remark 4.11. Theorem 4.8 highlights a key distinction: in nonlinear programming, scalar blocks ($m_j = 1$) have polyhedral normal cones, whose linear images

remain polyhedral and closed [28]. For nonpolyhedral Lorentz cones, closedness may fail (Lemma 4.7), so closedness of $H(\bar{x})$ requires explicit treatment. This issue also arises when using the Abadie and Guignard conditions to establish the existence of KKT multipliers in nonpolyhedral conic programming [28, 8].

Quasinormality for abstract constraint systems was introduced by Guo, Ye, and Zhang [18] and further studied in [12]. For systems $P(x) \in C$, with P of class C^1 and C closed and convex, their nondirectional quasinormality rules out $0 \neq \zeta \in N_C(P(\bar{x}))$ with $DP(\bar{x})^T\zeta = 0$ for which there exist sequences $x^k \to \bar{x}$, $y^k \to P(\bar{x})$, and $\zeta^k \to \zeta$, with $\zeta^k \in N_C(y^k)$, satisfying $\zeta_i(P_i(x^k) - y_i^k) > 0$ whenever $\zeta_i \neq 0$. Since this notion is formulated for general constraint systems, it is natural to clarify its relationship with the quasinormality proposed here for NSOCP. In the abstract condition, the points $y^k \in C$ are auxiliary and the sign requirement is imposed componentwise. For a nonpolyhedral cone, this condition therefore depends on the chosen coordinates and is not invariant under the automorphisms of $\mathbb{L}_m$, in contrast with Definition 3.1. In our setting, $P = G$, $C = \mathbb{L}$, and $\zeta = -\mu$.

Example 4.12. Definition 3.1 does not imply the abstract condition. Let $G(x) \equiv (1, 1, 0) \in \mathbb{L}_3$ and $\bar{x} = 0$; since the constraint is never violated, quasinormality holds. For $0 < t_k \to 0$, let $x^k = t_k$, $y^k = (1 + t_k, 1 - t_k, 2\sqrt{t_k}) \in \mathrm{bd}^+(\mathbb{L}_3)$, and $\zeta^k = -\Gamma_3 y^k \in N_{\mathbb{L}_3}(y^k)$. Then $\zeta^k \to \zeta = (-1, 1, 0) \neq 0$, $DG(\bar{x})^T\zeta = 0$, and, for every k, $\zeta_0\big([G(x^k)]_0 - y_0^k\big) = t_k > 0$ and $\zeta_1\big([G(x^k)]_1 - y_1^k\big) = t_k > 0$, while $\zeta_2 = 0$ imposes no sign condition. Thus the abstract condition fails, although no cone violation occurs.

Thus, the nondirectional quasinormality of Guo, Ye, and Zhang may fail even when Definition 3.1 holds, already for a single Lorentz-cone constraint. The converse implication remains open: we do not know whether their nondirectional quasinormality, applied to the product-cone formulation $G(x) \in \mathbb{L}$, implies the spectral quasinormality of Definition 3.1.

5. Local error bound and exact penalization under quasinormality. We establish a local error bound under quasinormality and derive exact penalization, following the NLP approach [27, 13], also extended to NSDP in [30]. The key is the relation between constraint violation and projection-induced multipliers. Define $\varphi(x) := \sum_{j=1}^{p} \mathrm{dist}(G_j(x), \mathbb{L}_j)$. For each j, let $R_j(x) := G_j(x) - \Pi_{\mathbb{L}_j}(G_j(x))$. Since $\mathbb{L}_j$ is closed and convex, $R_j(x) \in N_{\mathbb{L}_j}(\Pi_{\mathbb{L}_j}(G_j(x))) \subseteq -\mathbb{L}_j$. Thus, $\varphi(x) = 0$ exactly when $x \in \Omega$.

5.1. Local error bound. The proof of the error bound relies on the relation between the projection residual onto the Lorentz cone and the spectral directions associated with constraint violation. The next result collects the projection properties needed in the proof.

Proposition 5.1. *Let $\mathbb{L}_m \subseteq \mathbb{R}^m$ be the Lorentz cone and, for $w \in \mathbb{R}^m$, define $q(w) := \Pi_{\mathbb{L}_m}(-w) = \Pi_{\mathbb{L}_m}(w) - w$. Then $q(w) \in \mathbb{L}_m$, $q(w) = 0$ if and only if $w \in \mathbb{L}_m$, and $q(w) \in \mathrm{int}(\mathbb{L}_m)$ if and only if $w \in \mathrm{int}(-\mathbb{L}_m)$. Moreover, if $\widehat{q}(w) \neq 0$, then $\widehat{w} \neq 0$ and $u_+(q(w)) = u_-(w)$. Consequently, let $\{w^k\} \subseteq \mathbb{R}^m$, $\tau_k > 0$, and suppose that $\mu^k := \tau_k q(w^k) \to \bar{\mu} \in \mathbb{L}_m \setminus \{0\}$. If $\bar{\mu} \in \mathrm{int}(\mathbb{L}_m)$, then $w^k \in \mathrm{int}(-\mathbb{L}_m)$ for all sufficiently large k. If $\bar{\mu} \in \mathrm{bd}^+(\mathbb{L}_m)$, then, for all sufficiently large k, $w^k \notin \mathbb{L}_m$ and $u_-(w^k) = u_+(\mu^k)$; in particular, $u_-(w^k) \to u_+(\bar{\mu})$.*

Proof. The spectral formula for the projection onto the Lorentz cone gives

$$q(w) = \Pi_{\mathbb{L}_m}(-w) = \begin{cases} -w, & w \in (-\mathbb{L}_m) \setminus \{0\}, \\ -\lambda_-(w)u_-(w), & w \notin \mathbb{L}_m \cup (-\mathbb{L}_m), \\ 0, & w \in \mathbb{L}_m. \end{cases}$$

Hence $q(w) \in \operatorname{int}(\mathbb{L}_m)$ precisely when $w \in \operatorname{int}(-\mathbb{L}_m)$, while in every other case with $q(w) \neq 0$ we have $q(w) \in \operatorname{bd}^+(\mathbb{L}_m)$. If $\widehat{q}(w) \neq 0$, the formula above shows that $q(w)$ is either $-w$ or a positive multiple of $u_-(w)$, and in both cases $u_+(q(w)) = u_-(w)$.

Now suppose that $\mu^k = \tau_k q(w^k) \to \bar{\mu} \neq 0$. If $\bar{\mu} \in \operatorname{int}(\mathbb{L}_m)$, then $\mu^k \in \operatorname{int}(\mathbb{L}_m)$ for all sufficiently large k. Since $\tau_k > 0$, we have $q(w^k) \in \operatorname{int}(\mathbb{L}_m)$ and therefore $w^k \in \operatorname{int}(-\mathbb{L}_m)$.

Suppose instead that $\bar{\mu} \in \operatorname{bd}^+(\mathbb{L}_m)$. Since $\widehat{\bar{\mu}} \neq 0$, we have $\widehat{\mu}^k \neq 0$ for all sufficiently large k. Thus $q(w^k) \neq 0$, so $w^k \notin \mathbb{L}_m$, and $u_-(w^k) = u_+(q(w^k)) = u_+(\mu^k)$. Continuity of u_+ around $\bar{\mu} \in \operatorname{bd}^+(\mathbb{L}_m)$ gives $u_-(w^k) \to u_+(\bar{\mu})$. ∎

THEOREM 5.2. *Let $\bar{x} \in \Omega$ be quasinormal. Then there exist $\delta > 0$ and $M > 0$ such that*

$$\operatorname{dist}(x, \Omega) \leq M \sum_{j=1}^{p} \operatorname{dist}\big(G_j(x), \mathbb{L}_j\big) \qquad \text{whenever } \|x - \bar{x}\| \leq \delta. \tag{5.1}$$

Proof. Define $\psi(x) := \operatorname{dist}(G(x), \mathbb{L})$, so that $\psi(x)^2 = \sum_{j=1}^{p} \operatorname{dist}(G_j(x), \mathbb{L}_j)^2$. Since $\psi(x) \leq \varphi(x)$, it suffices to prove that $\operatorname{dist}(x, \Omega) \leq M\psi(x)$ locally around $\bar{x}$.

Suppose, by contradiction, that no such local estimate holds. Then there exists a sequence $v^k \to \bar{x}$, with $v^k \notin \Omega$, such that, setting $d_k := \operatorname{dist}(v^k, \Omega)$, $d_k > k\psi(v^k)$. Since $\bar{x} \in \Omega$, $0 < d_k \leq \|v^k - \bar{x}\| \to 0$. Set $\varepsilon_k := \psi(v^k) > 0$ and consider the auxiliary problem

$$\min_{\|x - v^k\| \leq d_k/2} \left\{ \psi(x) + \frac{8\varepsilon_k}{d_k^2} \|x - v^k\|^2 \right\}.$$

Let z^k be a minimizer. Such a minimizer exists by continuity and compactness.

By definition of d_k, $B(v^k, d_k/2) \cap \Omega = \varnothing$, since $\operatorname{dist}(x, \Omega) \geq d_k - \|x - v^k\| \geq d_k/2 > 0$ on this ball. Hence $\psi(x) > 0$ there. Moreover, z^k lies in the interior of the ball. Indeed, the objective value at v^k is ε_k, whereas on the boundary $\|x - v^k\| = d_k/2$ it is at least $2\varepsilon_k$. Thus $\|z^k - v^k\| < d_k/2$, and consequently $z^k \to \bar{x}$.

Since $z^k \notin \Omega$, we have $G(z^k) \notin \mathbb{L}$. The distance to a closed convex set is differentiable outside the set, so $\nabla\psi(z^k) = \frac{1}{\psi(z^k)} \sum_{j=1}^{p} DG_j(z^k)^T R_j(z^k)$. Because z^k is an interior minimizer of the auxiliary problem, $\nabla\psi(z^k) + \frac{16\varepsilon_k}{d_k^2}(z^k - v^k) = 0$. Therefore, $\|\nabla\psi(z^k)\| < \frac{8\varepsilon_k}{d_k} < \frac{8}{k}$, and hence $\nabla\psi(z^k) \to 0$. For each j, define

$$\mu_j^k := -\frac{R_j(z^k)}{\psi(z^k)} = \frac{\Pi_{\mathbb{L}_j}(G_j(z^k)) - G_j(z^k)}{\psi(z^k)}.$$

By Moreau's decomposition, $\mu_j^k \in \mathbb{L}_j$, and $\sum_{j=1}^{p} \|\mu_j^k\|^2 = \frac{\sum_{j=1}^{p} \|R_j(z^k)\|^2}{\psi(z^k)^2} = 1$. Passing to a subsequence, assume that $\mu_j^k \to \mu_j \in \mathbb{L}_j$ for every j. Then $\sum_{j=1}^{p} \|\mu_j\|^2 = 1$, so the multipliers are not all zero. Moreover, $\nabla\psi(z^k) = -\sum_{j=1}^{p} DG_j(z^k)^T \mu_j^k$. Passing to the limit yields $\sum_{j=1}^{p} DG_j(\bar{x})^T \mu_j = 0$.

It remains to verify the violation conditions in Definition 3.1. Fix j such that $\mu_j \neq 0$ and let $q_j(w) := \Pi_{\mathbb{L}_j}(w) - w$. Then $\mu_j^k = \tau_k q_j(G_j(z^k))$, where $\tau_k := 1/\psi(z^k) >$

0, and $\mu_j^k \to \mu_j \neq 0$. If $\mu_j \in \operatorname{int}(\mathbb{L}_j)$, Proposition 5.1 gives $G_j(z^k) \in \operatorname{int}(-\mathbb{L}_j)$ for all sufficiently large k. If $\mu_j \in \operatorname{bd}^+(\mathbb{L}_j)$, the same proposition gives $G_j(z^k) \notin \mathbb{L}_j$ and $u_-(G_j(z^k)) = u_+(\mu_j^k) \to u_+(\mu_j)$.

Thus, the nonzero multipliers μ_j and the sequence $z^k \to \bar{x}$ satisfy exactly the conditions excluded by quasinormality, a contradiction. Therefore, there exist $\delta, M > 0$ such that $\operatorname{dist}(x, \Omega) \leq M\psi(x) \leq M\varphi(x)$ whenever $\|x - \bar{x}\| \leq \delta$. ∎

5.2. Exact penalization. A direct consequence of the local error bound is exact penalization, which requires only local Lipschitz continuity of f. The local minimality conclusion does not require the original minimizer to be strict; strict local minimality is preserved when the penalty parameter is chosen strictly above the threshold obtained below.

THEOREM 5.3. *Let $\bar{x} \in \Omega$ be quasinormal, and let f be locally Lipschitz with a local minimum over Ω at $\bar{x}$. Then there exists $\bar{c} \geq 0$ such that, for every $c \geq \bar{c}$, $\bar{x}$ is a local minimizer of*

$$P_c(x) := f(x) + c\,\varphi(x) = f(x) + c\sum_{j=1}^{p} \operatorname{dist}\big(G_j(x), \mathbb{L}_j\big)$$

over $\mathbb{R}^n$. If $\bar{x}$ is a strict local minimizer of the original problem, then it is a strict local minimizer of P_c for every $c > \bar{c}$.

Proof. By Theorem 5.2, there exist $\delta, M > 0$ such that $\operatorname{dist}(x, \Omega) \leq M\varphi(x)$ whenever $\|x - \bar{x}\| \leq \delta$. After decreasing δ if necessary, assume that $\bar{x}$ minimizes f over $\Omega \cap \{x : \|x - \bar{x}\| \leq 2\delta\}$ and that f is L_f-Lipschitz on this larger ball.

For $\|x - \bar{x}\| \leq \delta$, let $\hat{x}$ be a closest point to x in Ω, which exists because Ω is closed. Since $\bar{x} \in \Omega$, $\|x - \hat{x}\| = \operatorname{dist}(x, \Omega) \leq \|x - \bar{x}\| \leq \delta$, and therefore $\|\hat{x} - \bar{x}\| \leq 2\delta$. Hence $f(\hat{x}) \geq f(\bar{x})$ and, by the error bound, $\|x - \hat{x}\| \leq M\varphi(x)$. Thus, $P_c(x) \geq f(\hat{x}) - L_f\|x - \hat{x}\| + c\varphi(x) \geq f(\bar{x}) + (c - L_f M)\varphi(x)$. Taking $\bar{c} := L_f M$, we obtain $P_c(x) \geq f(\bar{x}) = P_c(\bar{x})$ for every $c \geq \bar{c}$, so $\bar{x}$ is a local minimizer of P_c.

If the original minimum is strict and $c > \bar{c}$, then the inequality is strict whenever $\varphi(x) > 0$. If $\varphi(x) = 0$, then $x \in \Omega$, and strict local minimality of f on Ω gives $f(x) > f(\bar{x})$ for every $x \neq \bar{x}$ sufficiently close to $\bar{x}$. Thus $\bar{x}$ is a strict local minimizer of P_c. ∎

Remark 5.4. When $m_j = 1$ for every j, so that $\mathbb{L}_j = \mathbb{R}_+$, the feasible set is described by the scalar inequalities $-G_j(x) \leq 0$, Definition 3.1 reduces to the classical quasinormality of Hestenes, and $\varphi(x) = \sum_j [-G_j(x)]_+$. In this case, Theorems 5.2 and 5.3 recover the classical local error-bound and exact-penalization results; see [27, 13].

6. Algorithmic consequences. In this section, we analyze an augmented Lagrangian method with safeguarded multipliers. We first establish dual boundedness under quasinormality. We then introduce relaxed quasinormality and show that this weaker condition still guarantees dual boundedness and the KKT conditions. Finally, an extension to infeasible points is used to establish feasibility of accumulation points.

6.1. The augmented Lagrangian method. Several methods have been proposed for this class of problems, including penalty, semismooth Newton, SQP, interior-point, and augmented Lagrangian methods; see [16, 21, 22, 31].

To establish the algorithmic role of quasinormality (Section 3), we consider a safeguarded Powell–Hestenes–Rockafellar augmented Lagrangian scheme, as in [23, 14,

24]. We prove that quasinormality ensures bounded multipliers and KKT conditions at every feasible primal accumulation point.

For $\rho > 0$ and $\mu_j \in \mathbb{L}_j$, $j = 1, \ldots, p$, the augmented Lagrangian, up to an additive term independent of x, is

$$\mathcal{L}_\rho(x, \mu) := f(x) + \frac{1}{2\rho} \sum_{j=1}^{p} \left\| \Pi_{\mathbb{L}_j}(\mu_j - \rho G_j(x)) \right\|^2. \tag{6.1}$$

Its gradient is given by [1, 24]

$$\nabla_x \mathcal{L}_\rho(x, \mu) = \nabla f(x) - \sum_{j=1}^{p} DG_j(x)^T \Pi_{\mathbb{L}_j}(\mu_j - \rho G_j(x)). \tag{6.2}$$

We now state the algorithm, which uses bounded safeguarded multipliers in its subproblems.

Algorithm 6.1 Augmented Lagrangian method for NSOCP

Let $\tau \in (0, 1)$, $\gamma > 1$, $\rho_1 > 0$, and $\widetilde{\mu}^1 \in \mathbb{L}_1 \times \cdots \times \mathbb{L}_p$. Let $\{\varepsilon_k\} \subset \mathbb{R}_+$ satisfy $\varepsilon_k \to 0$. Set $k := 1$ and $\|v^0\| := +\infty$.

Step 1. Find an approximately stationary point of the subproblem, i.e., $x^k \in \mathbb{R}^n$ satisfying

$$\left\| \nabla_x \mathcal{L}_{\rho_k}(x^k, \widetilde{\mu}^k) \right\| \leq \varepsilon_k.$$

Step 2. Define $v^k = (v_1^k, \ldots, v_p^k)$ by

$$v_j^k := \Pi_{\mathbb{L}_j}\left(\frac{\widetilde{\mu}_j^k}{\rho_k} - G_j(x^k) \right) - \frac{\widetilde{\mu}_j^k}{\rho_k}, \qquad j = 1, \ldots, p.$$

If $\|v^k\| \leq \tau \|v^{k-1}\|$, set $\rho_{k+1} := \rho_k$; otherwise, set $\rho_{k+1} := \gamma \rho_k$.

Step 3. Compute

$$\mu_j^k := \Pi_{\mathbb{L}_j}\left(\widetilde{\mu}_j^k - \rho_k G_j(x^k) \right), \qquad j = 1, \ldots, p, \tag{6.3}$$

and choose $\widetilde{\mu}^{k+1} \in \mathbb{L}_1 \times \cdots \times \mathbb{L}_p$ so that $\{\widetilde{\mu}^k\}$ remains bounded. For example, fix $C > 0$ and take $\widetilde{\mu}^{k+1} = \Pi_{\mathbb{L} \cap \{\|\mu\| \leq C\}}(\mu^k)$, where $\mathbb{L} := \prod_j \mathbb{L}_j$.

Step 4. Set $k := k + 1$ and return to Step 1.

By Step 1 and the definition of the multipliers in Step 3, the generated sequences satisfy

$$\left\| \nabla f(x^k) - \sum_{j=1}^{p} DG_j(x^k)^T \mu_j^k \right\| \leq \varepsilon_k. \tag{6.4}$$

The penalty update rule and approximate stationarity also yield stationarity of a quadratic infeasibility measure without any constraint qualification.

PROPOSITION 6.1. *Let $\bar{x}$ be an accumulation point of a sequence $\{x^k\}$ generated by Algorithm 6.1. Then $\bar{x}$ is a stationary point of the infeasibility minimization problem*

$$\min_{x \in \mathbb{R}^n} \Phi(x) := \sum_{j=1}^{p} \left\| \Pi_{\mathbb{L}_j}(-G_j(x)) \right\|^2. \tag{6.5}$$

Proof. If $\{\rho_k\}$ is bounded, then the update rule implies that it is eventually constant and $v^k \to 0$. By Moreau's decomposition,

$$G_j(x^k) + v_j^k = \Pi_{\mathbb{L}_j}\left(G_j(x^k) - \frac{\widetilde{\mu}_j^k}{\rho_k}\right) \in \mathbb{L}_j.$$

Hence $\operatorname{dist}(G_j(x^k), \mathbb{L}_j) \le \|v_j^k\| \to 0$. Thus every accumulation point is feasible and therefore stationary for Φ. If $\rho_k \to \infty$, along a subsequence such that $x^k \to \bar{x}$, $\frac{\mu_j^k}{\rho_k} = \Pi_{\mathbb{L}_j}\left(\frac{\widetilde{\mu}_j^k}{\rho_k} - G_j(x^k)\right) \longrightarrow \Pi_{\mathbb{L}_j}(-G_j(\bar{x}))$. Dividing (6.4) by ρ_k and passing to the limit gives $\sum_{j=1}^p DG_j(\bar{x})^T \Pi_{\mathbb{L}_j}(-G_j(\bar{x})) = 0$. Since $\nabla\Phi(\bar{x}) = -2\sum_j DG_j(\bar{x})^T \Pi_{\mathbb{L}_j}(-G_j(\bar{x}))$, the conclusion follows. ∎

6.2. Boundedness of multipliers under quasinormality. The next lemma records a basic projection property used below.

LEMMA 6.2. *Let $z_j^k := \widetilde{\mu}_j^k - \rho_k G_j(x^k)$. Then $\mu_j^k,\ \mu_j^k - z_j^k \in \mathbb{L}_j$,*

$$\langle \mu_j^k, \mu_j^k - z_j^k\rangle = 0, \qquad \text{and} \qquad \langle G_j(x^k), \mu_j^k\rangle = \rho_k^{-1}\big(\langle \widetilde{\mu}_j^k, \mu_j^k\rangle - \|\mu_j^k\|^2\big). \tag{6.6}$$

Proof. This is Moreau's decomposition of z_j^k for the self-dual cone $\mathbb{L}_j$. ∎

The next proposition shows that boundedness of the multiplier sequence is the only missing ingredient for passing from approximate stationarity to the KKT conditions.

PROPOSITION 6.3. *Let $\{x^k\}$ and $\{\mu^k\}$ be generated by Algorithm 6.1, and let $\bar{x}$ be a feasible accumulation point of $\{x^k\}$. Suppose that $x^k \to \bar{x}$ along an infinite index set $K \subset \mathbb{N}$ and that $\{\mu^k\}_{k\in K}$ is bounded. Then $\bar{x}$ is a KKT point of* (NSOCP).

Proof. After extracting a subsequence, assume that $\mu^k \to \bar{\mu}$ along K, where $\bar{\mu} = (\bar{\mu}_1, \dots, \bar{\mu}_p)$ and $\bar{\mu}_j \in \mathbb{L}_j$. Passing to the limit in (6.4) yields $\nabla f(\bar{x}) - \sum_{j=1}^p DG_j(\bar{x})^T \bar{\mu}_j = 0$. It remains to prove complementarity. By Lemma 6.2,

$$\langle G_j(x^k), \mu_j^k\rangle = \frac{\langle \widetilde{\mu}_j^k, \mu_j^k\rangle - \|\mu_j^k\|^2}{\rho_k}.$$

If $\rho_k \to \infty$, boundedness of $\{\widetilde{\mu}^k\}$ and $\{\mu^k\}_{k\in K}$ gives $\langle G_j(x^k), \mu_j^k\rangle \to 0$.

If instead $\{\rho_k\}$ is bounded, the update rule implies $v^k \to 0$. Using $\mu_j^k = \rho_k \Pi_{\mathbb{L}_j}\left(\frac{\widetilde{\mu}_j^k}{\rho_k} - G_j(x^k)\right)$, we obtain $v_j^k = (\mu_j^k - \widetilde{\mu}_j^k)/\rho_k$, and hence $\mu_j^k - \widetilde{\mu}_j^k = \rho_k v_j^k \to 0$. The projection identity (6.6) then gives $\langle G_j(x^k), \mu_j^k\rangle \to 0$. Therefore, $\langle G_j(\bar{x}), \bar{\mu}_j\rangle = 0$ for every j. ∎

We now show that quasinormality prevents the multiplier sequence generated by the augmented Lagrangian method from becoming unbounded. This is the main point at which the sequential nature of quasinormality becomes algorithmically relevant.

THEOREM 6.4. *Let $\bar{x}$ be a feasible accumulation point of the primal sequence generated by Algorithm 6.1. Suppose that $x^k \to \bar{x}$ along an infinite index set $K \subset \mathbb{N}$ and that $\bar{x}$ is quasinormal. Then the corresponding multiplier sequence $\{\mu^k\}_{k\in K}$ is bounded.*

Proof. Suppose, by contradiction, that $\{\mu^k\}_{k\in K}$ is unbounded. After passing to a subsequence, still indexed by K, define $M_k := \|\mu^k\| \to +\infty$ and $\alpha_j^k := \mu_j^k / M_k$.

Since $\mu_j^k \in \mathbb{L}_j$ and $\|\alpha^k\| = 1$, we may assume $\alpha_j^k \to \alpha_j \in \mathbb{L}_j$ for every j, with $\alpha = (\alpha_1, \ldots, \alpha_p) \neq 0$. Dividing (6.4) by M_k and passing to the limit gives

$$\sum_{j=1}^{p} DG_j(\bar{x})^T \alpha_j = 0. \tag{6.7}$$

By nonexpansiveness and positive homogeneity of the projection,

$$\left\| \frac{\mu_j^k}{M_k} - \frac{\rho_k}{M_k} \Pi_{\mathbb{L}_j}(-G_j(x^k)) \right\| \leq \frac{\|\widetilde{\mu}_j^k\|}{M_k} \longrightarrow 0.$$

Hence $(\rho_k/M_k)\Pi_{\mathbb{L}_j}(-G_j(x^k)) \to \alpha_j$. For every j with $\alpha_j \neq 0$, Proposition 5.1, applied with $\tau_k = \rho_k/M_k$, yields exactly the violation conditions in Definition 3.1. Together with (6.7) and $\|\alpha\| = 1$, this contradicts quasinormality of $\bar{x}$. Therefore, the dual subsequence is bounded. ∎

Combining Proposition 6.3 and Theorem 6.4 gives the main convergence consequence under quasinormality.

COROLLARY 6.5. *Let $\bar{x}$ be a feasible accumulation point of the primal sequence generated by Algorithm 6.1. If $\bar{x}$ satisfies the quasinormality condition of Definition 3.1, then $\bar{x}$ is a KKT point of* (NSOCP).

6.3. Relaxed quasinormality and dual boundedness. The preceding analysis suggests weakening quasinormality by requiring secondary violations to be negligible relative to the dominant ones. This idea is related to relaxed quasinormality in nonlinear programming [10]. For $j = 1, \ldots, p$, define

$$q_j(x) := \Pi_{\mathbb{L}_j}(-G_j(x)), \quad d_j(x) := \|q_j(x)\|, \quad b_j(x) := \operatorname{dist}(q_j(x), \operatorname{bd}\mathbb{L}_j). \tag{6.8}$$

Thus, $d_j(x) = \operatorname{dist}(G_j(x), \mathbb{L}_j)$ measures the violation of block j, whereas $b_j(x)$ measures the distance of the projection residual to the cone boundary. In particular, $b_j(x) > 0$ if and only if $G_j(x) \in \operatorname{int}(-\mathbb{L}_j)$. For a scalar block, $b_j(x) = d_j(x) = \max\{-G_j(x), 0\}$. Given $\mu = (\mu_1, \ldots, \mu_p) \in \prod_j \mathbb{L}_j \setminus \{0\}$, define

$$J_I(\mu) := \{j : \mu_j \in \operatorname{int}\mathbb{L}_j\}, J_B(\mu) := \{j : \mu_j \in \operatorname{bd}^+(\mathbb{L}_j)\}, J_Z(\mu) := \{j : \mu_j = 0\}.$$

DEFINITION 6.6. *Let $\bar{x} \in \Omega$. We say that $\bar{x}$ satisfies* relaxed quasinormality *(RQN) if there do not exist $\mu \in \prod_j \mathbb{L}_j \setminus \{0\}$ and a sequence $x^k \to \bar{x}$ such that*

$$\sum_{j=1}^{p} DG_j(\bar{x})^T \mu_j = 0 \tag{6.9}$$

and the following conditions hold:

(i) *For $j \in J_I(\mu)$, $b_j(x^k) > 0$ for all sufficiently large k.*
(ii) *For $j \in J_B(\mu)$, $d_j(x^k) > 0$ for all sufficiently large k and*

$$\frac{q_j(x^k)}{d_j(x^k)} \longrightarrow \frac{\mu_j}{\|\mu_j\|}.$$

(iii) *With the convention* $\min \varnothing = +\infty$*, define*

$$w_k := \min\left\{ \min_{j\in J_I(\mu)} b_j(x^k),\ \min_{j\in J_B(\mu)} d_j(x^k) \right\}. \tag{6.10}$$

Then $d_j(x^k) = o(w_k)$ *for* $j \in J_Z(\mu)$ *and* $b_j(x^k) = o(w_k)$ *for* $j \in J_B(\mu)$.

Remark 6.7. Since $\mu \neq 0$, the set $J_I(\mu) \cup J_B(\mu)$ is nonempty. Conditions (i)–(ii) guarantee that $0 < w_k < \infty$ for all sufficiently large k. For blocks with zero multiplier, the entire violation must be $o(w_k)$. For boundary multipliers, the depth of the residual inside the cone must be $o(w_k)$. No directional condition is imposed on residuals associated with interior multipliers.

THEOREM 6.8. *If a feasible point* $\bar{x}$ *is quasinormal, then it satisfies RQN.*

Proof. Suppose that RQN fails, with multipliers $\mu \neq 0$ and $x^k \to \bar{x}$. For $j \in J_I(\mu)$, condition (i) gives $G_j(x^k) \in \operatorname{int}(-\mathbb{L}_j)$. For $j \in J_B(\mu)$, apply Proposition 5.1 to $\frac{q_j(x^k)}{d_j(x^k)} \longrightarrow \frac{\mu_j}{\|\mu_j\|}$. This yields exactly the violation and alignment requirements in Definition 3.1. Thus the same multipliers and sequence contradict quasinormality of $\bar{x}$. ∎

Remark 6.9. The implication in Theorem 6.8 is strict. At the point $\bar{x}$ of Example 4.10, quasinormality fails, but RQN holds. Indeed, (6.9) gives $\mu_1 = (a, -a, 0)$, with $a \geq 0$, and $[\mu_2]_1 = 0$. Since $[G_2(x)]_0 = 0$ for every x, condition (i) cannot hold for the second block. If $\mu_2 \in \operatorname{bd}^+(\mathbb{L}_3)$, then $\widehat{\mu_2}$ is a nonzero multiple of $(0, 1)$, whereas $\widehat{q_2(x)}$ is always a multiple of $(1, 0)$, so condition (ii) cannot hold either. Hence $\mu_2 = 0$ and $a > 0$. Write $x^k = (s_k, t_k)$ and $r_k := (s_k^2 + t_k^2)^{1/2}$. If $t_k = 0$, the first block is either feasible or has residual direction $(1, 1, 0)$, which is incompatible with $(1, -1, 0)$. For $t_k \neq 0$, the spatial direction of $q_1(x^k)$ is $-(s_k, t_k)/r_k$, so alignment forces $s_k > 0$ and $t_k/s_k \to 0$. Since $w_k = d_1(x^k) = (r_k - s_k)/\sqrt{2}$ and $d_2(x^k) = |t_k|/\sqrt{2}$,

$$\frac{w_k}{d_2(x^k)} = \frac{r_k - s_k}{|t_k|} = \frac{|t_k|}{r_k + s_k} \longrightarrow 0,$$

which contradicts $d_2(x^k) = o(w_k)$. Thus $\bar{x}$ satisfies CRCQ and RQN, but not quasinormality.

The next property contains the information needed for both dual boundedness and the error-bound argument.

LEMMA 6.10. *Let* $x^k \to \bar{x}$, $t_k > 0$, *and suppose that* $t_k q_j(x^k) \to \alpha_j \in \mathbb{L}_j$ *for* $\alpha = (\alpha_1, \ldots, \alpha_p) \neq 0$. *Then conditions (i)–(iii) of Definition 6.6 hold with* $\mu = \alpha$.

Proof. Continuity of the norm and positive homogeneity of the distance to the boundary of a cone yield $t_k d_j(x^k) \to \|\alpha_j\|$ and $t_k b_j(x^k) \to \operatorname{dist}(\alpha_j, \operatorname{bd}\mathbb{L}_j)$. For $j \in J_I(\alpha)$, the second limit is positive, proving (i). For $j \in J_B(\alpha)$, the first limit is positive and normalization of $t_k q_j(x^k)$ proves (ii). Since there are finitely many blocks,

$$t_k w_k \to \beta := \min\left\{ \min_{j\in J_I(\alpha)} \operatorname{dist}(\alpha_j, \operatorname{bd}\mathbb{L}_j),\ \min_{j\in J_B(\alpha)} \|\alpha_j\| \right\} > 0. \tag{6.11}$$

If $j \in J_Z(\alpha)$, then $t_k d_j(x^k) \to 0$; if $j \in J_B(\alpha)$, then $t_k b_j(x^k) \to 0$. Dividing by (6.11) gives the two dominance relations in (iii). ∎

THEOREM 6.11. *Let $x^k \to \bar{x} \in \Omega$, let $\rho_k > 0$, and let $\{\widetilde{\mu}^k\}$ be bounded. Suppose that $\mu_j^k = \Pi_{\mathbb{L}_j}(\widetilde{\mu}_j^k - \rho_k G_j(x^k))$ and $\nabla f(x^k) - \sum_{j=1}^p DG_j(x^k)^T \mu_j^k \to 0$. If $\bar{x}$ satisfies RQN, then $\{\mu^k\}$ is bounded. In particular, for Algorithm 6.1, this conclusion holds along every primal subsequence converging to a feasible point satisfying RQN, and every accumulation point of the corresponding dual subsequence is a KKT multiplier associated with that point.*

Proof. If the dual sequence were unbounded, after extracting a subsequence we would have $M_k := \|\mu^k\| \to \infty$ and $\mu_j^k/M_k \to \alpha_j$, with $\|\alpha\| = 1$ and $\alpha_j \in \mathbb{L}_j$. Approximate stationarity gives $\sum_{j=1}^p DG_j(\bar{x})^T \alpha_j = 0$. By nonexpansiveness of the projection, $\left\| \frac{\mu_j^k}{M_k} - \frac{\rho_k}{M_k} q_j(x^k) \right\| \le \frac{\|\widetilde{\mu}_j^k\|}{M_k} \to 0$. Thus $(\rho_k/M_k) q_j(x^k) \to \alpha_j$. Lemma 6.10 then yields all conditions in Definition 6.6, contradicting RQN. For sequences generated by Algorithm 6.1, complementarity of every dual limit follows from the proof of Proposition 6.3. ∎

Remark 6.12. The boundedness part of Theorem 6.11 does not use the update rule for ρ_k. This observation allows the result to be applied also to the external-penalty sequences considered below. The algorithmic update rule is needed only to obtain complementarity when the penalty sequence remains bounded.

6.4. Relaxed quasinormality as a constraint qualification. We next use an external-penalty construction that avoids identifying an arbitrary penalty sequence with the particular sequence generated by Algorithm 6.1.

THEOREM 6.13. *Let $\bar{x}$ be a local minimizer of* (NSOCP). *Then, for every sequence $\rho_k \to \infty$, there exist $x^k \to \bar{x}$ and $\mu_j^k = \rho_k q_j(x^k) \in \mathbb{L}_j$ such that*

$$\nabla f(x^k) - \sum_{j=1}^{p} DG_j(x^k)^T \mu_j^k \to 0, \qquad \langle G_j(x^k), \mu_j^k \rangle \to 0, \quad j = 1, \dots, p.$$

Proof. Choose $r > 0$ such that $f(x) \ge f(\bar{x})$ for every $x \in \Omega$ with $\|x - \bar{x}\| \le r$. Let x^k minimize $F_k(x) := f(x) + \frac{1}{2}\|x - \bar{x}\|^2 + \frac{\rho_k}{2} \sum_{j=1}^p d_j(x)^2$ over $\overline{B}_r(\bar{x})$. Since $F_k(x^k) \le F_k(\bar{x}) = f(\bar{x})$ and f is bounded below on the ball, we have $d_j(x^k) \to 0$. Hence every accumulation point x^* is feasible and satisfies $f(x^*) + \frac{1}{2}\|x^* - \bar{x}\|^2 \le f(\bar{x})$. Local minimality of $\bar{x}$ implies $x^* = \bar{x}$, and therefore $x^k \to \bar{x}$. For all sufficiently large k, x^k lies in the interior of the ball, and stationarity of F_k gives

$$\nabla f(x^k) - \sum_{j} DG_j(x^k)^T \rho_k q_j(x^k) = -(x^k - \bar{x}) \to 0.$$

Moreover, $0 \le \frac{\rho_k}{2} \sum_j d_j(x^k)^2 \le f(\bar{x}) - f(x^k) - \frac{1}{2}\|x^k - \bar{x}\|^2 \longrightarrow 0$. By Moreau's identity, $\langle G_j(x^k), q_j(x^k) \rangle = -\|q_j(x^k)\|^2$, which proves asymptotic complementarity. This is the penalty construction underlying sequential optimality conditions for NSOCP; see also [4]. ∎

THEOREM 6.14. *If $\bar{x}$ is a local minimizer of* (NSOCP) *satisfying RQN, then $\bar{x}$ is a KKT point.*

Proof. The sequences provided by Theorem 6.13 satisfy the boundedness assumptions of Theorem 6.11 with $\widetilde{\mu}^k = 0$. Hence $\{\mu^k\}$ is bounded. Extracting a convergent subsequence and passing to the limit in the stationarity and complementarity relations of Theorem 6.13 yields the KKT conditions. ∎

COROLLARY 6.15. *Every local minimizer of* (NSOCP) *satisfying quasinormality or pseudonormality is a KKT point.*

Proof. Combine Theorems 3.5, 6.8, and 6.14. ∎

THEOREM 6.16. *If $\bar{x} \in \Omega$ satisfies RQN, then there exist $\delta, M > 0$ such that* $\operatorname{dist}(x, \Omega) \le M \sum_{j=1}^{p} d_j(x)$ *whenever* $\|x - \bar{x}\| \le \delta$. *Consequently, the exact-penalization conclusions of Theorem 5.3 also hold under RQN.*

Proof. Failure of the error bound would yield, as in Theorem 5.2, $z^k \to \bar{x}$, $\psi(z^k) > 0$, $q_j(z^k)/\psi(z^k) \to \alpha_j \in \mathbb{L}_j$, $\|\alpha\| = 1$, and $\sum_j DG_j(\bar{x})^T \alpha_j = 0$. Lemma 6.10, with $t_k = 1/\psi(z^k)$, contradicts RQN. Exact penalization follows as in Theorem 5.3. ∎

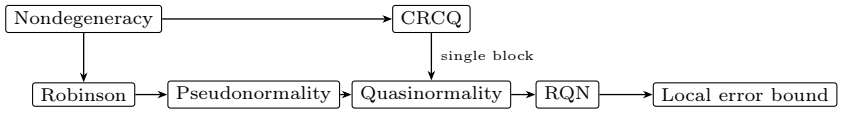


FIGURE 1. *Implications among the conditions studied. The implication from CRCQ to quasinormality holds only for a single cone block.*

6.5. Extended relaxed quasinormality and feasibility. RQN is defined only at feasible points. To derive feasibility from stationarity of the infeasibility measure, we extend the definition without imposing feasibility at the reference point.

DEFINITION 6.17. *A point $\bar{x} \in \mathbb{R}^n$, not necessarily feasible, satisfies* extended relaxed quasinormality *(ERQN) if there do not exist $\mu \in \prod_j \mathbb{L}_j \setminus \{0\}$ and a sequence $x^k \to \bar{x}$ satisfying* (6.9) *and conditions (i)–(iii) of Definition 6.6. Constant sequences are allowed.*

At feasible points, ERQN coincides with RQN. At infeasible points, it rules out, in particular, stationarity of the quadratic infeasibility measure.

THEOREM 6.18. *Let $\bar{x}$ be a stationary point of Φ, defined in* (6.5)*, and suppose that $\bar{x}$ satisfies ERQN. Then $\bar{x}$ is feasible. Consequently, every accumulation point of Algorithm 6.1 satisfying ERQN is feasible and KKT, and the dual subsequence associated with any primal subsequence converging to that point is bounded.*

Proof. Suppose that $\bar{x}$ is infeasible and stationary for Φ. Set $\mu_j := q_j(\bar{x})$. These multipliers are not all zero and satisfy $\sum_j DG_j(\bar{x})^T \mu_j = -\frac{1}{2}\nabla\Phi(\bar{x}) = 0$. For the constant sequence $x^k = \bar{x}$, all conditions in the definition of ERQN are satisfied: residuals in the interior have positive distance from the boundary; for boundary residuals, the normalized residual equals $\mu_j/\|\mu_j\|$; and all secondary terms vanish identically. This contradicts ERQN.

For an accumulation point of Algorithm 6.1, Proposition 6.1 gives stationarity of Φ. Hence the first part yields feasibility. At a feasible point, ERQN reduces to RQN, and Theorem 6.11 gives dual boundedness and the KKT conclusions. ∎

7. Conclusions. Our quasinormality links multiplier position in the Lorentz cone to constraint violation directions. We established its relationships with pseudonormality, Robinson's condition, and CRCQ. CRCQ implies quasinormality for a single cone block, but not generally for multiple blocks, where aggregate closedness may conceal nonclosed individual normal images.

RQN uses only projections, norms, and distances to the cone boundary. It is strictly weaker than quasinormality and ensures local error bounds, exact penaliza-

tion, KKT conditions at local minimizers and feasible augmented Lagrangian accumulation points, and boundedness of the associated dual subsequences. Its extension to infeasible points combines dual analysis with stationarity of the infeasibility measure to yield feasible KKT accumulation points. Dual boundedness is local; extending it to the entire sequence requires suitable assumptions at all primal accumulation points.

Whether CRCQ implies RQN for multiple blocks remains open. Remark 6.9, where both hold but quasinormality fails, is consistent with this implication. We also leave open whether the nondirectional quasinormality of Guo, Ye, and Zhang for the product-cone formulation implies ours; the reverse implication fails.

Declarations.

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