# QUASINORMALITY IN NONLINEAR SECOND-ORDER CONE PROGRAMMING*

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**Abstract.** We introduce quasinormality and pseudonormality for nonlinear second-order cone programming by exploiting the spectral geometry of the Lorentz cone. The proposed quasinormality distinguishes interior and boundary multipliers and, in the latter case, relates constraint violation to the corresponding spectral direction. It recovers classical quasinormality when all cone blocks are one-dimensional and is strictly weaker than Robinson's condition. Nevertheless, it implies a local error bound and exact penalization, while also guaranteeing boundedness of dual subsequences generated by an augmented Lagrangian method at feasible accumulation points. We further introduce a relaxed formulation, expressed solely in terms of projections, norms, and distances to the cone boundary, that preserves the local error bound and dual boundedness. Finally, an extension of this condition to infeasible points allows us to establish feasibility of accumulation points satisfying the extended condition.

**Key words.** Nonlinear second-order cone programming, quasinormality, constraint qualifications, augmented Lagrangian, error bound

**MSC codes.** 90C30, 90C46, 65K05

## 1. Introduction.

Constraint qualifications play a central role in nonlinear optimization, both in guaranteeing the existence of Lagrange multipliers and in the convergence analysis of numerical methods. Classical conditions such as the linear independence constraint qualification (LICQ) and the Mangasarian–Fromovitz constraint qualification (MFCQ) provide strong regularity properties, but they may fail at degenerate feasible points. This has motivated the development of weaker conditions that retain important consequences of classical constraint qualifications under milder assumptions.

Among such conditions, quasinormality, introduced by Hestenes [19], occupies a particularly important position. Rather than imposing a direct geometric condition on the linearized feasible set, quasinormality excludes nonzero abnormal multipliers that are compatible with sequences of nearby constraint violations. Pseudonormality, subsequently developed in the multiplier framework of Bertsekas and Ozdaglar [13], is closely related and implies quasinormality in the classical nonlinear programming setting. These conditions have proved useful in establishing Karush–Kuhn–Tucker (KKT) optimality conditions, exact penalization, local error bounds, and boundedness of multiplier sequences generated by augmented Lagrangian methods; see, for instance, [13, 9].

Extending weak constraint qualifications of this type to nonlinear second-order cone programming (NSOCP) is more delicate. The scalar sign structure underlying classical quasinormality is no longer available and must be replaced by information that reflects the geometry of the Lorentz cone. At the same time, degeneracy is common in conic problems, so classical regularity conditions may be unnecessarily restrictive. Robinson's condition, introduced in [29] in an abstract conic setting, can

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be viewed as a conic counterpart of MFCQ [25]. At a local minimizer, it guarantees that the set of Lagrange multipliers is nonempty, closed, and bounded. Nondegeneracy, on the other hand, plays a role analogous to LICQ and yields uniqueness of the Lagrange multipliers together with stronger stability properties; see, for example, [15]. Both conditions, however, impose strong geometric requirements and may fail at degenerate points.

Constant-rank constraint qualifications provide a different approach to conic regularity. In nonlinear programming, classical examples include the constant rank constraint qualification (CRCQ) [20], the relaxed constant rank constraint qualification (RCRCQ) [26], and the constant rank of the subspace component constraint qualification (CRSC) [11]. Their extension to NSOCP has proved considerably more subtle. The conditions proposed in [32], for instance, were subsequently shown in [3] to be insufficient to guarantee the existence of Lagrange multipliers. Later developments include partial reductions to scalar constraints [6], sequential conditions based on spectral directions [7], and a geometric approach based on faces of the cone [8]. The spectral approach is particularly well suited to the analysis of algorithmic sequences, whereas the facial approach yields second-order optimality conditions that hold for every Lagrange multiplier. A constant-rank condition based on the minimal face provides a further development in this direction [5]. These approaches capture different aspects of constant-rank regularity in conic optimization and should not, in general, be regarded as equivalent.

Weak constraint qualifications such as quasinormality and pseudonormality have received comparatively less attention in NSOCP. The main difficulty is already visible at the level of their definitions: the componentwise sign conditions used in nonlinear programming have no intrinsic analogue for a nonpolyhedral cone. For the Lorentz cone, however, the spectral decomposition provides a natural way to replace this scalar structure. In particular, interior and boundary multipliers have different geometric behavior, and a nonzero boundary multiplier determines a distinguished spectral direction that can be related to the direction in which the corresponding cone constraint is violated.

A related spectral viewpoint has recently been used in nonlinear semidefinite programming (NSDP), where quasinormality-type conditions were shown to guarantee boundedness of dual sequences generated by augmented Lagrangian methods without requiring Robinson's condition [30]. The Lorentz cone has a simpler spectral structure than the semidefinite cone, but the same basic issue arises: the notion of constraint violation should be related to an abnormal multiplier in a way that is intrinsic to the cone.

We introduce quasinormality and pseudonormality for NSOCP from this spectral viewpoint. The proposed quasinormality distinguishes interior and boundary multipliers and, in the latter case, relates the violation of the cone constraint to the corresponding spectral direction. It recovers classical quasinormality when all cone blocks are one-dimensional and is strictly weaker than Robinson's condition. Nevertheless, it implies a local error bound and exact penalization and guarantees boundedness of dual subsequences generated by an augmented Lagrangian method at feasible accumulation points.

We also examine the relationship between the proposed condition and constant-rank constraint qualifications for NSOCP. For a single conic constraint, we show that CRCQ implies quasinormality. This implication does not extend, in general, to problems with multiple cone blocks. We further introduce a relaxed version of quasinormality expressed only in terms of projections, norms, and distances to the cone

boundary. This relaxed condition preserves the local error bound and dual boundedness, and its extension to infeasible points guarantees feasibility of accumulation points satisfying the extended condition.

The remainder of the paper is organized as follows. After fixing the notation, Section 2 recalls the basic concepts and optimality conditions used throughout the paper. In Section 3, we introduce quasinormality and pseudonormality for NSOCP. In Section 4, we compare these conditions with Robinson's condition, nondegeneracy, CRCQ, and quasinormality for abstract constraint systems. In Section 5, we establish a local error bound and exact penalization under quasinormality. Finally, in Section 6, we study dual boundedness, introduce relaxed quasinormality and its extension to infeasible points, and derive the corresponding optimality and convergence consequences.

***Notation.*** All spaces are finite-dimensional Euclidean spaces, with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. For a set $C$, we denote by $\operatorname{int} C$, $\operatorname{bd} C$, and $\operatorname{cl} C$ its interior, boundary, and closure, respectively. For a nonempty closed convex set $C$, $\Pi_C$ denotes the Euclidean projection onto $C$,

$$N_C(x) := \{v : \langle v, z - x\rangle \le 0 \text{ for all } z \in C\},$$

and $\operatorname{dist}(x, C) := \|x - \Pi_C(x)\|$. For a set $D$,

$$D^\circ := \{v : \langle v, d\rangle \le 0 \text{ for all } d \in D\},$$

and, for a cone $K$, $K^* := \{v : \langle v, z\rangle \ge 0 \text{ for all } z \in K\}$.

**2. Preliminaries and the NSOCP problem.** In this section, we collect the basic concepts used throughout the paper.

For $m \ge 2$, the second-order cone, or Lorentz cone, is $\mathbb{L}_m := \{y = (y_0, \widehat{y}) \in \mathbb{R} \times \mathbb{R}^{m-1} : y_0 \ge \|\widehat{y}\|\}$, and we set $\mathbb{L}_1 := \mathbb{R}_+$. Its interior is $\operatorname{int}(\mathbb{L}_m) = \{y : y_0 > \|\widehat{y}\|\}$, and we denote its positive boundary by $\operatorname{bd}^+(\mathbb{L}_m) := \{y : y_0 = \|\widehat{y}\| > 0\}$. Hence $\operatorname{bd}(\mathbb{L}_m) = \operatorname{bd}^+(\mathbb{L}_m) \cup \{0\}$, with the origin being the vertex of the cone.

In the spectral formulas below, we assume $m \ge 2$. For $y = (y_0, \widehat{y}) \in \mathbb{R}^m$, define $\lambda_-(y) := y_0 - \|\widehat{y}\|$ and $\lambda_+(y) := y_0 + \|\widehat{y}\|$. Then

$$y = \lambda_-(y)u_-(y) + \lambda_+(y)u_+(y),$$

where

$$u_-(y) := \begin{cases} \frac{1}{2}\left(1, -\frac{\widehat{y}}{\|\widehat{y}\|}\right), & \widehat{y} \ne 0, \\ \frac{1}{2}(1, -w), & \widehat{y} = 0, \end{cases} \qquad u_+(y) := \begin{cases} \frac{1}{2}\left(1, \frac{\widehat{y}}{\|\widehat{y}\|}\right), & \widehat{y} \ne 0, \\ \frac{1}{2}(1, w), & \widehat{y} = 0, \end{cases}$$

with arbitrary $w \in \mathbb{R}^{m-1}$ satisfying $\|w\| = 1$. When $\widehat{y} = 0$, the eigenvalues coincide and the vectors $u_-(y)$ and $u_+(y)$ are not uniquely determined. Let $\Gamma_m := \operatorname{diag}(1, -1, \ldots, -1)$. We also define $\det(y) := \lambda_+(y)\lambda_-(y) = y_0^2 - \|\widehat{y}\|^2$. The second-order cone is self-dual, i.e., $\mathbb{L}_m^* = \mathbb{L}_m$.

LEMMA 2.1. *Let $z, u \in \mathbb{L}_m$. Then $\langle z, u\rangle = 0$ if and only if $z = 0$, or $u = 0$, or $z, u \in \operatorname{bd}^+(\mathbb{L}_m)$ and $u = \alpha\Gamma_m z$ for some $\alpha > 0$. In particular, if $z \in \operatorname{bd}^+(\mathbb{L}_m)$, $\langle z, u\rangle = 0$, and $u \ne 0$, then $u \in \operatorname{bd}^+(\mathbb{L}_m)$ and $u_+(u) = u_-(z)$.*

The spectral decomposition also gives the projection onto the second-order cone:

$$\Pi_{\mathbb{L}_m}(y) = \max\{0, \lambda_+(y)\}u_+(y) + \max\{0, \lambda_-(y)\}u_-(y).$$

LEMMA 2.2. *Let* $\Pi := \Pi_{\mathbb{L}_m}$. *Then:*

*(a) since* $\mathbb{L}_m$ *is self-dual, Moreau's decomposition gives* $y - \Pi(y) = -\Pi(-y) \in -\mathbb{L}_m$ *and* $\langle \Pi(y), y - \Pi(y)\rangle = 0$*;*

*(b) the function* $y \mapsto \frac{1}{2}\operatorname{dist}(y, \mathbb{L}_m)^2$ *is continuously differentiable, with gradient* $y - \Pi(y)$.

These properties follow from the spectral decomposition of the second-order cone, Moreau's decomposition, and standard results on metric projections onto closed convex sets; see, for instance, [1]. For $m = 1$, we use $\mathbb{L}_1 = \mathbb{R}_+$, $\Pi_{\mathbb{L}_1}(y) = \max\{y, 0\}$, and $\mathrm{bd}^+(\mathbb{L}_1) = \varnothing$. Statements involving spectral vectors are restricted to $m \geq 2$, whereas the projection and complementarity properties remain valid in the scalar case. In componentwise expressions, we set $y_0 = y$ and $\|\widehat{y}\| = 0$ when $m = 1$.

We consider the standard nonlinear second-order cone programming problem

$$\text{(NSOCP)} \qquad \begin{array}{ll} \min & f(x), \\ \text{s.t.} & G_j(x) \in \mathbb{L}_j, \quad j = 1, \ldots, p, \end{array}$$

where, throughout the paper, $f : \mathbb{R}^n \to \mathbb{R}$ and $G_j : \mathbb{R}^n \to \mathbb{R}^{m_j}$ are continuously differentiable, and $\mathbb{L}_j := \mathbb{L}_{m_j}$ denotes the cone associated with block $j$. Notice that the block index in $\mathbb{L}_j$ is distinct from the dimension index in $\mathbb{L}_m$.

The feasible set of (NSOCP) is denoted by $\Omega$. Given $\bar{x} \in \mathbb{R}^n$, we denote by $DG_j(\bar{x})$ the derivative of $G_j : \mathbb{R}^n \to \mathbb{R}^{m_j}$ at $\bar{x}$, and by $DG_j(\bar{x})^T$ its transpose. For a feasible point $\bar{x}$, define

$$\begin{aligned} I_{\mathrm{int}}(\bar{x}) &:= \{j \in \{1, \ldots, p\} : G_j(\bar{x}) \in \mathrm{int}(\mathbb{L}_j)\}, \\ I_B(\bar{x}) &:= \{j \in \{1, \ldots, p\} : G_j(\bar{x}) \in \mathrm{bd}^+(\mathbb{L}_j)\}, \\ I_0(\bar{x}) &:= \{j \in \{1, \ldots, p\} : G_j(\bar{x}) = 0\}. \end{aligned}$$

We say that $\bar{x}$ satisfies the Karush–Kuhn–Tucker (KKT) conditions if there exist Lagrange multipliers $\mu_j \in \mathbb{L}_j$ such that

$$\nabla f(\bar{x}) - \sum_{j=1}^{p} DG_j(\bar{x})^T \mu_j = 0, \qquad \langle G_j(\bar{x}), \mu_j\rangle = 0, \quad j = 1, \ldots, p. \tag{2.1}$$

The tangent cone to $\Omega$ at $\bar{x}$ is

$$T_\Omega(\bar{x}) := \{d \in \mathbb{R}^n : \exists\, t_k \to 0^+,\ \exists\, d^k \to d \text{ such that } \bar{x} + t_k d^k \in \Omega \text{ for all } k\},$$

whereas the linearized cone is

$$\begin{aligned} L_\Omega(\bar{x}) &:= \{d \in \mathbb{R}^n : DG_j(\bar{x})d \in \mathcal{T}_{\mathbb{L}_j}(G_j(\bar{x})),\ j = 1, \ldots, p\} \\ &= \left\{ d \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} DG_j(\bar{x})d \in \mathbb{L}_j, & j \in I_0(\bar{x}); \\ \left\langle DG_j(\bar{x})d, \Gamma_{m_j} G_j(\bar{x})\right\rangle \geq 0, & j \in I_B(\bar{x}). \end{array} \right\}. \end{aligned} \tag{2.2}$$

See [15] for further details on (2.2).

The relationship between the tangent and linearized cones is central to the theory of constraint qualifications. Abadie's constraint qualification requires $T_\Omega(\bar{x}) = L_\Omega(\bar{x})$,

whereas Guignard's constraint qualification requires only $T_\Omega(\bar{x})^\circ = L_\Omega(\bar{x})^\circ$. In classical nonlinear programming, where the underlying constraint cone is polyhedral, these geometric conditions are sufficient to guarantee the existence of Lagrange multipliers. In nonpolyhedral conic optimization, however, additional topological properties are required.

Let $G(x) := (G_1(x), \ldots, G_p(x))$ and $\mathbb{L} := \prod_{j=1}^p \mathbb{L}_j$, and define

$$H(\bar{x}) := \{DG(\bar{x})^T y : y \in N_{\mathbb{L}}(G(\bar{x}))\}, \tag{2.3}$$

where $N_{\mathbb{L}}(G(\bar{x})) = \{y \in \mathbb{L}^\circ : \langle G(\bar{x}), y\rangle = 0\}$ is the normal cone to $\mathbb{L}$ at $G(\bar{x})$. Closedness of $H(\bar{x})$ was introduced by Guignard [17] as an additional assumption in multiplier results for optimization problems in Banach spaces. In conic programming, however, equality between the tangent and linearized cones, or even between their polars, does not by itself guarantee the existence of Lagrange multipliers [2]. Moreover, closedness of $H(\bar{x})$ alone is also insufficient, as shown in [3]; see [8] for a more detailed discussion.

These difficulties illustrate why classical geometric conditions may be too restrictive or fail to capture the degeneracy inherent in conic problems, and motivate the study of weaker constraint qualifications that still ensure KKT conditions. In nonlinear programming, quasinormality, introduced by Hestenes [19], is one of the weakest conditions of this type. Unlike constant-rank conditions, which are formulated solely in terms of constraint derivatives, quasinormality is sequential in nature and rules out constraint-violation sequences associated with abnormal multipliers. This provides the motivation for extending quasinormality to the NSOCP setting.

**3. Quasinormality and pseudonormality for NSOCP.** In this section, we introduce notions of quasinormality and pseudonormality for problem (NSOCP).

DEFINITION 3.1. *Let $\bar{x}$ be a feasible point of problem* (NSOCP). *We say that $\bar{x}$ satisfies* quasinormality *if there do not exist multipliers $\mu_j \in \mathbb{L}_j$, not all zero, and a sequence $\{x^k\}$ converging to $\bar{x}$ such that*

$$\sum_{j=1}^{p} DG_j(\bar{x})^T \mu_j = 0, \tag{3.1}$$

*and, for every $j \in \{1, \ldots, p\}$ with $\mu_j \neq 0$, the corresponding cone constraint is violated according to one of the following cases:*

*(i) if $\mu_j \in \operatorname{int}(\mathbb{L}_j)$, then $G_j(x^k) \in \operatorname{int}(-\mathbb{L}_j)$ for all sufficiently large $k$;*

*(ii) if $\mu_j \in \operatorname{bd}^+(\mathbb{L}_j)$, then $G_j(x^k) \notin \mathbb{L}_j$ for all sufficiently large $k$ and*

$$u_-(G_j(x^k)) \to u_+(\mu_j).$$

*Here, $u_+(\mu_j)$ denotes the spectral vector associated with $\lambda_+(\mu_j)$. If $\widehat{G_j(x^k)} = 0$, condition (ii) requires the existence of choices of the spectral vectors for which the stated convergence holds. For scalar blocks, only condition (i) applies, with $\mu_j > 0$ and $G_j(x^k) < 0$.*

*Remark* 3.2. The conditions in Definition 3.1 imply $\langle G_j(\bar{x}), \mu_j\rangle = 0$ for every $j$. This is immediate when $\mu_j = 0$. If $\mu_j \in \operatorname{int}(\mathbb{L}_j)$, condition (i), continuity of $G_j$, and feasibility of $\bar{x}$ yield $G_j(\bar{x}) \in \mathbb{L}_j \cap (-\mathbb{L}_j) = \{0\}$. If $\mu_j \in \operatorname{bd}^+(\mathbb{L}_j)$, condition (ii) implies $\lambda_-(G_j(x^k)) < 0$. By continuity and feasibility, $\lambda_-(G_j(\bar{x})) =$

0. If $G_j(\bar{x}) = 0$, complementarity is immediate. Otherwise, $G_j(\bar{x}) \in \mathrm{bd}^+(\mathbb{L}_j)$ and continuity of $u_-$ at $G_j(\bar{x})$, together with the alignment condition in (ii), gives $u_-(G_j(\bar{x})) = \lim_{k\to\infty} u_-(G_j(x^k)) = u_+(\mu_j)$. Since $G_j(\bar{x})$ and $\mu_j$ are positive multiples of $u_+(G_j(\bar{x}))$ and $u_-(G_j(\bar{x}))$, respectively, the orthogonality of these spectral vectors yields $\langle G_j(\bar{x}), \mu_j\rangle = 0$.

In classical nonlinear programming, quasinormality requires a nearby sequence to violate every constraint associated with a positive multiplier. Definition 3.1 extends this idea to second-order cone constraints by distinguishing the position of $\mu_j$ within the cone. If $\mu_j \in \mathrm{int}(\mathbb{L}_j)$, the sequence must satisfy $G_j(x^k) \in \mathrm{int}(-\mathbb{L}_j)$. If $\mu_j \in \mathrm{bd}^+(\mathbb{L}_j)$, only $\lambda_+(\mu_j)$ is positive, and the violation $G_j(x^k) \notin \mathbb{L}_j$ is additionally required to satisfy $u_-(G_j(x^k)) \to u_+(\mu_j)$. Thus, the spectral direction of the violation is matched with the positive spectral direction of the multiplier. No violation condition is imposed when $\mu_j = 0$.

In nonlinear programming, pseudonormality implies quasinormality. We next introduce a counterpart for problem (NSOCP), in which the violation condition is expressed through a multiplier-weighted aggregate. We will show that it also implies the quasinormality of Definition 3.1.

Definition 3.3. *Let $\bar{x}$ be a feasible point of problem* (NSOCP). *We say that $\bar{x}$ satisfies* pseudonormality *if there do not exist multipliers $\mu_j \in \mathbb{L}_j$, not all zero, satisfying* (3.1),

$$\mu_j = 0, \quad j \in I_{\mathrm{int}}(\bar{x}), \qquad \mu_j = \alpha_j \Gamma_{m_j} G_j(\bar{x}), \quad \alpha_j \geq 0, \quad j \in I_B(\bar{x}),$$

*and a sequence $x^k \to \bar{x}$ such that*

$$\sum_{j=1}^{p} \left( [\mu_j]_0 [G_j(x^k)]_0 - \|\widehat{\mu_j}\| \, \|\widehat{G_j(x^k)}\| \right) < 0$$

*for all sufficiently large $k$.*

When all blocks are scalar, the violation condition in Definition 3.3 reduces to $\sum_{j=1}^{p} \mu_j G_j(x^k) < 0$, recovering the classical Bertsekas–Ozdaglar pseudonormality condition [13].

*Remark* 3.4. A definition based on the inner product $\langle \mu_j, G_j(x^k)\rangle$ would not preserve the implication pseudonormality $\Rightarrow$ quasinormality. Indeed, consider $G(x) = (1, 1, x) \in \mathbb{L}_3$ at $\bar{x} = 0$, with $DG(\bar{x}) = (0, 0, 1)^T$. The multiplier $\mu = (1, -1, 0)$ satisfies $DG(\bar{x})^T \mu = 0$ and $\langle G(\bar{x}), \mu\rangle = 0$. For $x^k = 1/k$, we have $G(x^k) \notin \mathbb{L}_3$ and $u_-(G(x^k)) \to u_+(\mu)$, so quasinormality fails at $\bar{x}$. On the other hand, $\langle \mu, G(x^k)\rangle = 0$ for every $k$, while

$$[\mu]_0 [G(x^k)]_0 - \|\widehat{\mu}\| \, \|\widehat{G(x^k)}\| = 1 - \sqrt{1 + \frac{1}{k^2}} < 0.$$

Moreover, every nonzero multiplier satisfying stationarity and complementarity at $\bar{x}$ has the form $t(1, -1, 0)$ with $t > 0$, and its inner product with $G(x)$ vanishes for every $x$. Hence, replacing the inequality in Definition 3.3 by a negative-inner-product condition would classify $\bar{x}$ as pseudonormal even though quasinormality fails.

Theorem 3.5. *Let $\bar{x}$ be a feasible point of problem* (NSOCP). *If $\bar{x}$ satisfies pseudonormality, then it satisfies quasinormality.*

*Proof.* Suppose that quasinormality fails at $\bar{x}$. Then there exist multipliers $\mu_j \in \mathbb{L}_j$, not all zero, and a sequence $x^k \to \bar{x}$ satisfying Definition 3.1. By Remark 3.2, $\langle G_j(\bar{x}), \mu_j \rangle = 0$ for every $j$. The complementarity characterization of the second-order cone therefore gives

$$\mu_j = 0, \quad j \in I_{\text{int}}(\bar{x}), \qquad \mu_j = \alpha_j \Gamma_{m_j} G_j(\bar{x}), \quad \alpha_j \geq 0, \quad j \in I_B(\bar{x}).$$

Hence, these multipliers satisfy the multiplier conditions in Definition 3.3. It remains to verify the violation inequality. Set

$$s_j^k := [\mu_j]_0 [G_j(x^k)]_0 - \|\widehat{\mu_j}\| \, \|\widehat{G_j(x^k)}\|.$$

If $\mu_j = 0$, then $s_j^k = 0$. If $\mu_j \in \text{int}(\mathbb{L}_j)$, condition (i) gives $[G_j(x^k)]_0 < 0$ for all sufficiently large $k$, and therefore $s_j^k < 0$. If $\mu_j \in \text{bd}^+(\mathbb{L}_j)$, then $[\mu_j]_0 = \|\widehat{\mu_j}\| > 0$ and condition (ii) gives

$$s_j^k = [\mu_j]_0 \left( [G_j(x^k)]_0 - \|\widehat{G_j(x^k)}\| \right) = [\mu_j]_0 \lambda_-(G_j(x^k)) < 0.$$

For scalar blocks, the corresponding term is $\mu_j G_j(x^k)$, which is also negative whenever $\mu_j > 0$. Since the multipliers are not all zero, $\sum_{j=1}^p s_j^k < 0$ for all sufficiently large $k$. Thus, pseudonormality also fails at $\bar{x}$. ∎

*Example* 3.6. The converse of Theorem 3.5 does not hold. Consider $G_1(x) = (x, 0) \in \mathbb{L}_2$ and $G_2(x) = (-x - x^2, 0) \in \mathbb{L}_2$. At $\bar{x} = 0$, both blocks vanish. The multipliers $\mu_1 = \mu_2 = (1, 0)$ satisfy (3.1), and for $x^k = 1/k$,

$$\sum_{j=1}^{2} \left( [\mu_j]_0 [G_j(x^k)]_0 - \|\widehat{\mu_j}\| \, \|\widehat{G_j(x^k)}\| \right) = -(x^k)^2 < 0.$$

Hence, pseudonormality fails at $\bar{x}$. On the other hand, any multipliers in $\mathbb{L}_2$, not both zero, satisfying (3.1) must satisfy $\mu_{1,0} = \mu_{2,0} > 0$. Thus both multipliers are nonzero, and failure of quasinormality would require simultaneous violation of both blocks. This is impossible since, for $|x| < 1$, $G_1(x) \notin \mathbb{L}_2$ if and only if $x < 0$, whereas $G_2(x) \notin \mathbb{L}_2$ if and only if $x > 0$. Therefore, $\bar{x}$ is quasinormal.

The quasinormality proposed here for NSOCP can be viewed as a geometric counterpart of the spectral quasinormality introduced for nonlinear semidefinite programming in [30]. In the NSDP setting, the violation conditions are associated with the positive eigenvalues of the matrix multiplier. For NSOCP, the corresponding distinction is between interior and boundary multipliers: the former require violation in the interior of the negative cone, whereas the latter require violation with spectral alignment.

**4. Relations with other constraint qualifications.** In this section, we investigate the relationships among quasinormality, nondegeneracy, Robinson's condition, and constant-rank conditions for problem (NSOCP).

### 4.1. Relation with nondegeneracy and Robinson's condition.

DEFINITION 4.1. *[15] Let $\bar{x}$ be a feasible point of problem* (NSOCP). *We say that $\bar{x}$ is* nondegenerate *if the matrix formed by the rows of $DG_j(\bar{x})$, for $j \in I_0(\bar{x})$, together with the row vectors $G_j(\bar{x})^T \Gamma_{m_j} DG_j(\bar{x})$, for $j \in I_B(\bar{x})$, has full row rank.*

Nondegeneracy extends the linear independence constraint qualification (LICQ) from nonlinear programming to NSOCP. When it holds at a local minimizer, the Lagrange multiplier vector exists and is unique.

DEFINITION 4.2. *[15] Let $\bar{x}$ be a feasible point of problem* (NSOCP). *Robinson's condition is said to hold at $\bar{x}$ if there do not exist scalars $\alpha_j \geq 0$, $j \in I_B(\bar{x})$, and multipliers $\mu_j \in \mathbb{L}_j$, $j \in I_0(\bar{x})$, not all zero, such that*

$$\sum_{j \in I_B(\bar{x})} \alpha_j DG_j(\bar{x})^T \Gamma_{m_j} G_j(\bar{x}) + \sum_{j \in I_0(\bar{x})} DG_j(\bar{x})^T \mu_j = 0. \tag{4.1}$$

Nondegeneracy excludes nontrivial vanishing combinations with $\alpha_j \in \mathbb{R}$ and $\mu_j \in \mathbb{R}^{m_j}$, whereas Robinson's condition excludes them only for $\alpha_j \geq 0$ and $\mu_j \in \mathbb{L}_j$. Hence, nondegeneracy implies Robinson's condition. When all blocks are scalar, Robinson's condition reduces to the Mangasarian–Fromovitz constraint qualification (MFCQ).

PROPOSITION 4.3. *Let $\bar{x}$ be feasible for* (NSOCP). *If Robinson's condition holds at $\bar{x}$, then $\bar{x}$ is pseudonormal and, consequently, quasinormal.*

*Proof.* If pseudonormality failed, the nonzero multipliers in Definition 3.3, after omitting the zero components corresponding to $I_{\text{int}}(\bar{x})$, would satisfy exactly the nontrivial system excluded by Robinson's condition. Hence $\bar{x}$ is pseudonormal. Quasinormality follows from Theorem 3.5. ∎

Quasinormality does not imply Robinson's condition.

*Example* 4.4. Consider $G_1(x) = (x, x) \in \mathbb{L}_2$ and $G_2(x) = (-x, x) \in \mathbb{L}_2$. Since $G_1(x) \in \mathbb{L}_2$ if and only if $x \geq 0$, whereas $G_2(x) \in \mathbb{L}_2$ if and only if $x \leq 0$, the feasible set is $\Omega = \{0\}$. Thus, $\bar{x} = 0$ is the unique feasible point, and both blocks are at the vertex.

Robinson's condition fails at $\bar{x}$. Indeed, $\bar{\mu}_1 = (1, 1)$ and $\bar{\mu}_2 = (1, -1)$ belong to $\mathbb{L}_2$ and satisfy $DG_1(\bar{x})^T \bar{\mu}_1 + DG_2(\bar{x})^T \bar{\mu}_2 = 2 - 2 = 0$.

We now verify quasinormality. Let $\mu_1, \mu_2 \in \mathbb{L}_2$, not both zero, satisfy (3.1). If both are nonzero, failure of quasinormality would require simultaneous violation of both blocks, which is impossible since $G_1(x) \notin \mathbb{L}_2$ if and only if $x < 0$, whereas $G_2(x) \notin \mathbb{L}_2$ if and only if $x > 0$.

If only $\mu_1 \neq 0$, equation (3.1) gives $\mu_{1,0} + \mu_{1,1} = 0$, so $\mu_1 = t(1, -1)$ for some $t > 0$. Violation of the first block requires $x^k < 0$, but then $u_-(G_1(x^k)) = \frac{1}{2}(1, 1) \neq \frac{1}{2}(1, -1) = u_+(\mu_1)$. Similarly, if only $\mu_2 \neq 0$, then $\mu_2 = s(1, 1)$ for some $s > 0$, while violation of the second block requires $x^k > 0$, in which case $u_-(G_2(x^k)) = \frac{1}{2}(1, -1) \neq \frac{1}{2}(1, 1) = u_+(\mu_2)$. Hence no admissible multiplier admits a sequence satisfying Definition 3.1, and $\bar{x}$ is quasinormal.

**4.2. Relation with the constant rank constraint qualification.** We next recall the constant rank constraint qualification (CRCQ) introduced in [8] for problem (NSOCP). For $j \in I_B(\bar{x})$, define locally around $\bar{x}$, the function $g_j(x) := [G_j(x)]_0 - \widehat{\|G_j(x)\|}$. This function is $C^1$, and the corresponding cone constraint is locally equivalent to $g_j(x) \geq 0$. In particular,

$$\nabla g_j(\bar{x}) = \frac{1}{[G_j(\bar{x})]_0} DG_j(\bar{x})^T \Gamma_{m_j} G_j(\bar{x}).$$

This identity is used at the reference point; in the constant-rank condition below, the quantity considered in a neighborhood is $\nabla g_j(x)$.

DEFINITION 4.5. *[8] Let $\bar{x}$ be a feasible point of problem* (NSOCP). *We say that the* constant rank constraint qualification *(CRCQ) holds at $\bar{x}$ if there exists a neighborhood $V$ of $\bar{x}$ such that, for every $J_1, J_2 \subseteq I_0(\bar{x})$ and $J_3 \subseteq I_B(\bar{x})$, with $m_j > 1$ for every $j \in J_1$, and every $\eta_j \in \mathrm{bd}^+(\mathbb{L}_{m_j})$, $j \in J_1$, the rank of the family*

$$\bigcup_{\substack{j\in J_1\\ i=1,\dots,m_j-1}} \{DG_j(x)^T A_j^i\} \quad \bigcup_{\substack{j\in J_2\\ i=0,\dots,m_j-1}} \{\nabla [G_j]_i(x)\} \bigcup_{j\in J_3} \{\nabla g_j(x)\}$$

*is constant for all $x \in V$. Here, $A_j \in \mathbb{R}^{m_j\times(m_j-1)}$ is any full-column-rank matrix such that $\operatorname{Im} A_j = \{\eta_j\}^\perp$, and $A_j^i$ denotes its $i$th column. In addition, the set $H(\bar{x})$ defined in* (2.3) *is required to be closed.*

The CRCQ of [8] is independent of Robinson's condition and is strictly weaker than nondegeneracy. It also yields a strong second-order optimality condition that holds for every Lagrange multiplier, a property not guaranteed by Robinson's condition.

For a single conic constraint, CRCQ yields an additional local invariance property of the Jacobian image when the constraint is active at the vertex. This property will be crucial in establishing the implication from CRCQ to quasinormality in the one-block case.

LEMMA 4.6. *Let $\bar{x}$ be a feasible point of an NSOCP problem with a single conic constraint $G(x) \in \mathbb{L}_m$. Suppose that $G(\bar{x}) = 0$ and CRCQ holds at $\bar{x}$, and define $S := \operatorname{Im} DG(\bar{x})$. Then $\operatorname{rank} DG(x) = \dim S$ for all $x$ sufficiently close to $\bar{x}$. Moreover, if $x^k \to \bar{x}$, $G(x^k) \neq 0$, and $G(x^k)/\|G(x^k)\| \to v$, then $v \in S$.*

*If, in addition, $\dim S = 1$ and $S = \operatorname{span}\{\eta\}$ for some $\eta \in \mathrm{bd}^+(\mathbb{L}_m)$, then, after possibly shrinking the neighborhood, $\operatorname{Im} DG(x) = S$ and $G(x) \in S$ for every $x$ in that neighborhood.*

*Proof.* Let $r := \dim S = \operatorname{rank} DG(\bar{x})$. Taking the single block in $J_2$ in Definition 4.5, CRCQ implies, after possibly shrinking the neighborhood, that $\operatorname{rank} DG(x) = r$ for every $x \in V$. If $r = 0$, then $DG(x) = 0$ on $V$. Taking $V$ convex and using $G(\bar{x}) = 0$, we obtain $G(x) = 0$ throughout $V$, and the statement concerning normalized secant directions is vacuous. Hence assume $r \geq 1$.

By the constant rank theorem, after shrinking $V$ if necessary, $G(V)$ is locally a $C^1$ manifold of dimension $r$ through the origin, with $T_{G(V)}(0) = \operatorname{Im} DG(\bar{x}) = S$. Relative to the orthogonal decomposition $\mathbb{R}^m = S \oplus S^\perp$, this manifold can locally be written as the graph of a $C^1$ mapping $h : U \subset S \to S^\perp$ satisfying $h(0) = 0$ and $Dh(0) = 0$. Thus, for all sufficiently large $k$, $G(x^k) = s_k + h(s_k)$, where $s_k \in S$ and $s_k \to 0$. Since $h(s_k) = o(\|s_k\|)$ and $S \perp S^\perp$,

$$\frac{\operatorname{dist}(G(x^k), S)}{\|G(x^k)\|} = \frac{\|h(s_k)\|}{\|G(x^k)\|} \leq \frac{\|h(s_k)\|}{\|s_k\|} \longrightarrow 0.$$

Therefore, if $G(x^k)/\|G(x^k)\| \to v$, then $\operatorname{dist}(v, S) = 0$, and hence $v \in S$.

Assume now that $r = 1$, $S = \operatorname{span}\{\eta\}$, and $\eta \in \mathrm{bd}^+(\mathbb{L}_m)$. Choose a full-column-rank matrix $A \in \mathbb{R}^{m\times(m-1)}$ such that $\operatorname{Im} A = \{\eta\}^\perp$. Since $\ker DG(\bar{x})^T = S^\perp = \{\eta\}^\perp$, we have $DG(\bar{x})^T A = 0$. Thus, the family associated with $J_1$ has rank zero at $\bar{x}$. By CRCQ, $DG(x)^T A = 0$ for all $x$ sufficiently close to $\bar{x}$, and therefore $\{\eta\}^\perp \subseteq \ker DG(x)^T$. Since $\operatorname{rank} DG(x) = 1$, both spaces have dimension $m - 1$, so $\ker DG(x)^T = \{\eta\}^\perp$ and hence $\operatorname{Im} DG(x) = \operatorname{span}\{\eta\} = S$.

Finally, taking $V$ convex, for every $x \in V$ we have

$$G(x) = \int_0^1 DG\big(\bar{x} + t(x - \bar{x})\big)(x - \bar{x})\, dt.$$

Since the image of every Jacobian along the segment is $S$, the integrand belongs to $S$, and therefore $G(x) \in S$. ∎

The next lemma identifies a situation in which the linear image of a Lorentz cone is not closed.

LEMMA 4.7. *Let $B : \mathbb{R}^m \to \mathbb{R}^n$ be linear, with $m \geq 2$, and suppose that*

$$\ker B \cap \mathrm{bd}^+(\mathbb{L}_m) \neq \varnothing, \qquad \ker B \cap \mathrm{int}(\mathbb{L}_m) = \varnothing, \qquad \mathrm{rank}\, B \geq 2.$$

*Then $B(\mathbb{L}_m)$ is not closed.*

*Proof.* Take $0 \neq \mu \in \ker B \cap \mathrm{bd}^+(\mathbb{L}_m)$. After a spatial rotation preserving $\mathbb{L}_m$ and a positive rescaling, we may assume $\mu = (1, 1, 0, \ldots, 0)^T$ and $\Gamma_m \mu = (1, -1, 0, \ldots, 0)^T$.

We first claim that $\ker B \subset (\Gamma_m \mu)^\perp$. Otherwise, there would exist $\xi \in \ker B$ with $\langle \Gamma_m \mu, \xi \rangle \neq 0$. Replacing $\xi$ by $-\xi$ if necessary, assume $\langle \Gamma_m \mu, \xi \rangle > 0$. Since $\nabla \det(\mu) = 2\Gamma_m \mu$,

$$\det(\mu + t\xi) = 2t\langle \Gamma_m \mu, \xi \rangle + O(t^2) > 0$$

for all sufficiently small $t > 0$. The first component of $\mu + t\xi$ remains positive, so $\mu + t\xi \in \mathrm{int}(\mathbb{L}_m) \cap \ker B$, a contradiction. Thus $\Gamma_m \mu \in (\ker B)^\perp$.

Since $\dim(\ker B)^\perp = \mathrm{rank}\, B \geq 2$, there exists $0 \neq \tau \in (\ker B)^\perp \cap (\Gamma_m \mu)^\perp$. Moreover, $\tau \perp \mu$ because $\mu \in \ker B$, and hence, in the chosen coordinates, $\tau = (0, 0, \widehat{\tau})^T$. For $t > 0$, define

$$\nu_t := \frac{1}{t}\mu + \tau + \frac{t\|\tau\|^2}{4}\Gamma_m \mu.$$

A direct computation gives $\det \nu_t = 0$, and its first component is positive; hence $\nu_t \in \mathrm{bd}^+(\mathbb{L}_m)$. Since $B\mu = 0$,

$$B\nu_t = B\tau + \frac{t\|\tau\|^2}{4} B\Gamma_m \mu \longrightarrow B\tau,$$

so $B\tau \in \mathrm{cl}\, B(\mathbb{L}_m)$.

Suppose that $B\tau = B\eta$ for some $\eta \in \mathbb{L}_m$. Then $\eta - \tau \in \ker B$. Since $\Gamma_m \mu \in (\ker B)^\perp$ and $\tau \perp \Gamma_m \mu$, we obtain $\langle \Gamma_m \mu, \eta \rangle = 0$. In the chosen coordinates this means $\eta_0 = \eta_1$. Since $\eta \in \mathbb{L}_m$, it follows that $\eta \in \mathbb{R}_+ \mu$, and therefore $B\eta = 0$. On the other hand, $0 \neq \tau \in (\ker B)^\perp$ implies $B\tau \neq 0$, a contradiction. Thus $B\tau \notin B(\mathbb{L}_m)$, proving that $B(\mathbb{L}_m)$ is not closed. ∎

The preceding lemmas yield the relationship between CRCQ and quasinormality for a single conic constraint.

THEOREM 4.8. *Let $\bar{x}$ be a feasible point of an NSOCP problem with a single conic constraint $G(x) \in \mathbb{L}_m$. If CRCQ holds at $\bar{x}$, then $\bar{x}$ is quasinormal.*

*Proof.* Suppose, by contradiction, that quasinormality fails at $\bar{x}$. Then there exist $\mu \in \mathbb{L}_m \setminus \{0\}$ and $x^k \to \bar{x}$ such that $DG(\bar{x})^T \mu = 0$ and the violation conditions of Definition 3.1 hold. By Remark 3.2, $\langle G(\bar{x}), \mu \rangle = 0$. Since $\mu \neq 0$, $G(\bar{x}) \notin \mathrm{int}(\mathbb{L}_m)$, so either $G(\bar{x}) = 0$ or $G(\bar{x}) \in \mathrm{bd}^+(\mathbb{L}_m)$.

Suppose first that $G(\bar{x}) \in \mathrm{bd}^+(\mathbb{L}_m)$. Then $\mu = \alpha\Gamma_m G(\bar{x})$ for some $\alpha > 0$. Let $g(x) := [G(x)]_0 - \|\widehat{G}(x)\|$. We have $\nabla g(\bar{x}) = 0$. Applying CRCQ to the singleton family $\{\nabla g(x)\}$ shows that $\nabla g(x) = 0$ in a neighborhood of $\bar{x}$. Hence $g(x) = g(\bar{x}) = 0$ there. This contradicts $G(x^k) \notin \mathbb{L}_m$, which requires $g(x^k) < 0$.

It remains to consider $G(\bar{x}) = 0$. Set $S := \operatorname{Im} DG(\bar{x})$. Since $DG(\bar{x})^T\mu = 0$, we have $\mu \in S^\perp$.

If $\mu \in \mathrm{int}(\mathbb{L}_m)$, failure of quasinormality gives $G(x^k) \in \mathrm{int}(-\mathbb{L}_m)$. Passing to a subsequence, $G(x^k)/\|G(x^k)\| \to y$ with $\|y\| = 1$. By Lemma 4.6, $y \in S$, and since $-\mathbb{L}_m$ is closed, $y \in -\mathbb{L}_m$. Hence $\langle \mu, y\rangle = 0$. However, $\mu \in \mathrm{int}(\mathbb{L}_m)$ and $0 \neq y \in -\mathbb{L}_m$ imply $\langle \mu, y\rangle < 0$, a contradiction.

Now let $\mu \in \mathrm{bd}^+(\mathbb{L}_m)$ and write $\mu = \mu_0(1, a)$, where $\|a\| = 1$ and $\mu_0 > 0$. Passing again to a subsequence, assume $G(x^k)/\|G(x^k)\| \to y \in S$, with $\|y\| = 1$. We may assume $\widehat{G}(x^k) \neq 0$ for all sufficiently large $k$; otherwise, a subsequence would satisfy $G(x^k)/\|G(x^k)\| = -e$, yielding $-e \in S$, contrary to $\mu \in S^\perp$. The alignment condition $u_-(G(x^k)) \to u_+(\mu)$ implies that $y = (y_0, -\gamma a)$ for some $\gamma \geq 0$. Since $y \in S \subset \mu^\perp$, we obtain $y_0 = \gamma$, and $\|y\| = 1$ then gives $y = \frac{1}{\sqrt{2}}(1, -a)$. Consequently, $\Gamma_m\mu \in S$.

If $\dim S = 1$, then $S = \mathrm{span}\{\Gamma_m\mu\}$. By the last part of Lemma 4.6, $G(x) \in S$ locally, so $G(x^k) = \alpha_k\Gamma_m\mu$. Since $G(x^k) \notin \mathbb{L}_m$, we must have $\alpha_k < 0$, and therefore $u_-(G(x^k)) = \frac{1}{2}(1, -a) \neq \frac{1}{2}(1, a) = u_+(\mu)$, contradicting the alignment condition. Hence $\dim S \geq 2$.

Let $B := DG(\bar{x})^T$. Then $\mu \in \ker B \cap \mathrm{bd}^+(\mathbb{L}_m)$ and $\Gamma_m\mu \in S = (\ker B)^\perp$. If there existed $\nu \in \ker B \cap \mathrm{int}(\mathbb{L}_m)$, then $\langle \nu, \Gamma_m\mu\rangle > 0$, since $\Gamma_m\mu \in \mathbb{L}_m \setminus \{0\}$. On the other hand, $\nu \in \ker B$ and $\Gamma_m\mu \in (\ker B)^\perp$ imply $\langle \nu, \Gamma_m\mu\rangle = 0$, a contradiction. Hence $\ker B \cap \mathrm{int}(\mathbb{L}_m) = \varnothing$ and $\operatorname{rank} B = \dim S \geq 2$. Lemma 4.7 therefore implies that $B(\mathbb{L}_m)$ is not closed. Since $G(\bar{x}) = 0$,

$$H(\bar{x}) = DG(\bar{x})^T(-\mathbb{L}_m) = -B(\mathbb{L}_m)$$

is also not closed, contradicting CRCQ. Thus $\bar{x}$ is quasinormal. □

The converse of Theorem 4.8 fails even for a single conic constraint.

*Example* 4.9. Consider $G(x_1, x_2) := (x_2, x_1^2, 0) \in \mathbb{L}_3$ and $\bar{x} = (0, 0)$, so that $G(\bar{x}) = 0$ and

$$DG(x) = \begin{pmatrix} 0 & 1 \\ 2x_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $DG(\bar{x})^T\mu = (0, \mu_0)^T$, every $\mu \in \mathbb{L}_3$ with $DG(\bar{x})^T\mu = 0$ satisfies $\mu_0 = 0$, and hence $\mu = 0$. Thus Robinson's condition (Definition 4.2) holds at $\bar{x}$, and $\bar{x}$ is quasinormal by Proposition 4.3. On the other hand, taking $J_2 = \{1\}$ and $J_1 = J_3 = \varnothing$ in Definition 4.5, the relevant family is

$$\{(0, 1)^T, (2x_1, 0)^T, (0, 0)^T\}.$$

Its rank is 1 at $\bar{x}$ but 2 whenever $x_1 \neq 0$. Thus CRCQ fails at $\bar{x}$. Hence quasinormality does not imply CRCQ, even for a single conic constraint.

For each block, define $H_j(\bar{x}) := DG_j(\bar{x})^T N_{\mathbb{L}_j}(G_j(\bar{x}))$. Since

$$N_{\mathbb{L}}(G(\bar{x})) = \prod_{j=1}^{p} N_{\mathbb{L}_j}(G_j(\bar{x})),$$

we have $H(\bar{x}) = \sum_{j=1}^{p} H_j(\bar{x})$. With multiple cone blocks, closedness of the aggregate set $H(\bar{x})$ does not imply closedness of the individual sets $H_j(\bar{x})$. The next example shows that this distinction allows CRCQ to hold even though quasinormality fails.

*Example* 4.10. The implication in Theorem 4.8 does not extend, in general, to multiple blocks. Let $x = (x_1, x_2) \in \mathbb{R}^2$, $\bar{x} = (0, 0)$, $G_1(x) := (x_1, x_1, x_2) \in \mathbb{L}_3$ and $G_2(x) := (0, x_2, 0) \in \mathbb{L}_3$. We have $G_1(\bar{x}) = G_2(\bar{x}) = 0$, and

$$DG_1(\bar{x})^T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad DG_2(\bar{x})^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Thus $I_B(\bar{x}) = \varnothing$, and both Jacobians are constant. Hence all families appearing in the rank condition of Definition 4.5 have constant rank. On the other hand, $H_1(\bar{x}) = \{(u, v) \in \mathbb{R}^2 : u < 0\} \cup \{(0, 0)\}$ and $H_2(\bar{x}) = \{(0, v) : v \in \mathbb{R}\}$. Although $H_1(\bar{x})$ is not closed,

$$H(\bar{x}) = H_1(\bar{x}) + H_2(\bar{x}) = \{(u, v) \in \mathbb{R}^2 : u \leq 0\}$$

is closed. Therefore, CRCQ holds at $\bar{x}$.

Quasinormality, however, fails. Take $\mu_1 = (1, -1, 0)^T \in \mathrm{bd}^+(\mathbb{L}_3)$ and $\mu_2 = 0$. Then $DG_1(\bar{x})^T \mu_1 = 0$. For $x^k = (1/k, 1/k^2) \to \bar{x}$, we have $G_1(x^k) \notin \mathbb{L}_3$ and

$$u_-(G_1(x^k)) = \frac{1}{2} \begin{pmatrix} 1 \\ -\dfrac{1}{\sqrt{1 + 1/k^2}} \\ -\dfrac{1/k}{\sqrt{1 + 1/k^2}} \end{pmatrix} \longrightarrow \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = u_+(\mu_1).$$

Hence quasinormality fails at $\bar{x}$.

*Remark* 4.11. Theorem 4.8 also highlights an important difference between the polyhedral and nonpolyhedral settings. In classical nonlinear programming, every scalar constraint corresponds to a one-dimensional block $m_j = 1$, so the relevant normal cones are polyhedral. Since the linear image of a polyhedral cone is again polyhedral and therefore closed [28], closedness of the normal images causes no additional difficulty.

For nonpolyhedral Lorentz cones, by contrast, a linear image need not be closed, as shown by Lemma 4.7. Thus, closedness of $H(\bar{x})$ becomes an additional issue when passing from nonlinear programming to NSOCP. The same phenomenon arises in connection with the Abadie and Guignard conditions: closedness is automatic in the polyhedral setting, but must be addressed explicitly in nonpolyhedral conic programming when these geometric conditions are used to derive the existence of KKT multipliers; see [28, 8].

Quasinormality for abstract constraint systems was introduced by Guo, Ye, and Zhang [18] and further studied in [12]. For systems $P(x) \in C$, with $P$ of class $C^1$ and $C$ closed and convex, their nondirectional quasinormality rules out $0 \neq \zeta \in N_C(P(\bar{x}))$ with $DP(\bar{x})^T \zeta = 0$ for which there exist sequences $x^k \to \bar{x}$, $y^k \to P(\bar{x})$, and $\zeta^k \to \zeta$, with $\zeta^k \in N_C(y^k)$, satisfying $\zeta_i(P_i(x^k) - y_i^k) > 0$ whenever $\zeta_i \neq 0$. Since this notion is formulated for general constraint systems, it is natural to clarify its relationship with the quasinormality proposed here for NSOCP. In the abstract condition, the points $y^k \in C$ are auxiliary and the sign requirement is imposed componentwise. For a nonpolyhedral cone, this condition therefore depends on the chosen coordinates and

is not invariant under the automorphisms of $\mathbb{L}_m$, in contrast with Definition 3.1. In our setting, $P = G$, $C = \mathbb{L}$, and $\zeta = -\mu$.

*Example* 4.12. Definition 3.1 does not imply the abstract condition. Let $G(x) \equiv (1, 1, 0) \in \mathbb{L}_3$ and $\bar{x} = 0$; since the constraint is never violated, quasinormality holds. For $0 < t_k \to 0$, let $x^k = t_k$, $y^k = (1 + t_k, 1 - t_k, 2\sqrt{t_k}) \in \mathrm{bd}^+(\mathbb{L}_3)$, and $\zeta^k = -\Gamma_3 y^k \in N_{\mathbb{L}_3}(y^k)$. Then $\zeta^k \to \zeta = (-1, 1, 0) \neq 0$, $DG(\bar{x})^T\zeta = 0$, and, for every $k$,

$$\zeta_0\big([G(x^k)]_0 - y_0^k\big) = t_k > 0, \qquad \zeta_1\big([G(x^k)]_1 - y_1^k\big) = t_k > 0,$$

while $\zeta_2 = 0$ imposes no sign condition. Thus the abstract condition fails, although no cone violation occurs.

Thus, the nondirectional quasinormality of Guo, Ye, and Zhang may fail even when Definition 3.1 holds, already for a single Lorentz-cone constraint. The converse implication remains open: we do not know whether their nondirectional quasinormality, applied to the product-cone formulation $G(x) \in \mathbb{L}$, implies the spectral quasinormality of Definition 3.1.

**5. Local error bound and exact penalization under quasinormality.** In this section, we establish a local error bound for the feasible set $\Omega$ of problem (NSOCP) under quasinormality and derive exact penalization as a consequence. This follows the classical nonlinear programming (NLP) route: quasinormality implies a local error bound [27], which in turn yields exact penalization [13]. A similar approach was used for nonlinear semidefinite programming (NSDP) in [30]. For the Lorentz cone, the key ingredient is the relation between the direction of violation of a cone constraint and the multiplier induced by the projection residual.

We measure infeasibility by

$$\varphi(x) := \sum_{j=1}^{p} \mathrm{dist}(G_j(x), \mathbb{L}_j), \tag{5.1}$$

and define the projection residual of block $j$ by $R_j(x) := G_j(x) - \Pi_{\mathbb{L}_j}(G_j(x))$. Since $\mathbb{L}_j$ is closed and convex, $R_j(x) \in N_{\mathbb{L}_j}(\Pi_{\mathbb{L}_j}(G_j(x))) \subseteq -\mathbb{L}_j$ for every $x$. In particular, $\varphi(x) = 0$ if and only if $x \in \Omega$.

**5.1. Local error bound.** The proof of the error bound relies on the relation between the projection residual onto the Lorentz cone and the spectral directions associated with constraint violation. The next result collects the projection properties needed in the proof.

Proposition 5.1. *Let $\mathbb{L}_m \subseteq \mathbb{R}^m$ be the Lorentz cone and, for $w \in \mathbb{R}^m$, define $q(w) := \Pi_{\mathbb{L}_m}(-w) = \Pi_{\mathbb{L}_m}(w) - w$. Then $q(w) \in \mathbb{L}_m$, $q(w) = 0$ if and only if $w \in \mathbb{L}_m$, and $q(w) \in \mathrm{int}(\mathbb{L}_m)$ if and only if $w \in \mathrm{int}(-\mathbb{L}_m)$. Moreover, if $\widehat{q}(w) \neq 0$, then $\widehat{w} \neq 0$ and $u_+(q(w)) = u_-(w)$. Consequently, let $\{w^k\} \subseteq \mathbb{R}^m$, $\tau_k > 0$, and suppose that*

$$\mu^k := \tau_k q(w^k) \to \bar{\mu} \in \mathbb{L}_m \setminus \{0\}.$$

*If $\bar{\mu} \in \mathrm{int}(\mathbb{L}_m)$, then $w^k \in \mathrm{int}(-\mathbb{L}_m)$ for all sufficiently large $k$. If $\bar{\mu} \in \mathrm{bd}^+(\mathbb{L}_m)$, then, for all sufficiently large $k$, $w^k \notin \mathbb{L}_m$ and $u_-(w^k) = u_+(\mu^k)$; in particular, $u_-(w^k) \to u_+(\bar{\mu})$.*

*Proof.* The spectral formula for the projection onto the Lorentz cone gives

$$q(w) = \Pi_{\mathbb{L}_m}(-w) = \begin{cases} -w, & w \in (-\mathbb{L}_m) \setminus \{0\}, \\ -\lambda_-(w)u_-(w), & w \notin \mathbb{L}_m \cup (-\mathbb{L}_m), \\ 0, & w \in \mathbb{L}_m. \end{cases}$$

Hence $q(w) \in \operatorname{int}(\mathbb{L}_m)$ precisely when $w \in \operatorname{int}(-\mathbb{L}_m)$, while in every other case with $q(w) \neq 0$ we have $q(w) \in \operatorname{bd}^+(\mathbb{L}_m)$. If $\widehat{q}(w) \neq 0$, the formula above shows that $q(w)$ is either $-w$ or a positive multiple of $u_-(w)$, and in both cases $u_+(q(w)) = u_-(w)$.

Now suppose that $\mu^k = \tau_k q(w^k) \to \bar{\mu} \neq 0$. If $\bar{\mu} \in \operatorname{int}(\mathbb{L}_m)$, then $\mu^k \in \operatorname{int}(\mathbb{L}_m)$ for all sufficiently large $k$. Since $\tau_k > 0$, we have $q(w^k) \in \operatorname{int}(\mathbb{L}_m)$ and therefore $w^k \in \operatorname{int}(-\mathbb{L}_m)$.

Suppose instead that $\bar{\mu} \in \operatorname{bd}^+(\mathbb{L}_m)$. Since $\widehat{\bar{\mu}} \neq 0$, we have $\widehat{\mu}^k \neq 0$ for all sufficiently large $k$. Thus $q(w^k) \neq 0$, so $w^k \notin \mathbb{L}_m$, and $u_-(w^k) = u_+(q(w^k)) = u_+(\mu^k)$. Continuity of $u_+$ around $\bar{\mu} \in \operatorname{bd}^+(\mathbb{L}_m)$ gives $u_-(w^k) \to u_+(\bar{\mu})$. ∎

THEOREM 5.2. *Let $\bar{x} \in \Omega$ be quasinormal. Then there exist $\delta > 0$ and $M > 0$ such that*

$$\operatorname{dist}(x, \Omega) \leq M \sum_{j=1}^{p} \operatorname{dist}\big(G_j(x), \mathbb{L}_j\big) \qquad \text{whenever } \|x - \bar{x}\| \leq \delta. \tag{5.2}$$

*Proof.* Define

$$\psi(x) := \operatorname{dist}(G(x), \mathbb{L}) = \left( \sum_{j=1}^{p} \operatorname{dist}(G_j(x), \mathbb{L}_j)^2 \right)^{1/2}.$$

Since $\psi(x) \leq \varphi(x)$, it suffices to prove that $\operatorname{dist}(x, \Omega) \leq M\psi(x)$ locally around $\bar{x}$.

Suppose, by contradiction, that no such local estimate holds. Then there exists a sequence $v^k \to \bar{x}$, with $v^k \notin \Omega$, such that, setting $d_k := \operatorname{dist}(v^k, \Omega)$, $d_k > k\psi(v^k)$. Since $\bar{x} \in \Omega$, $0 < d_k \leq \|v^k - \bar{x}\| \to 0$. Set $\varepsilon_k := \psi(v^k) > 0$ and consider the auxiliary problem

$$\min_{\|x - v^k\| \leq d_k/2} \left\{ \psi(x) + \frac{8\varepsilon_k}{d_k^2} \|x - v^k\|^2 \right\}.$$

Let $z^k$ be a minimizer. Such a minimizer exists by continuity and compactness.

By definition of $d_k$, $B(v^k, d_k/2) \cap \Omega = \varnothing$, since $\operatorname{dist}(x, \Omega) \geq d_k - \|x - v^k\| \geq d_k/2 > 0$ on this ball. Hence $\psi(x) > 0$ there. Moreover, $z^k$ lies in the interior of the ball. Indeed, the objective value at $v^k$ is $\varepsilon_k$, whereas on the boundary $\|x - v^k\| = d_k/2$ it is at least $2\varepsilon_k$. Thus $\|z^k - v^k\| < d_k/2$, and consequently $z^k \to \bar{x}$.

Since $z^k \notin \Omega$, we have $G(z^k) \notin \mathbb{L}$. The distance to a closed convex set is differentiable outside the set, so

$$\nabla\psi(z^k) = \frac{1}{\psi(z^k)} \sum_{j=1}^{p} DG_j(z^k)^T R_j(z^k).$$

Because $z^k$ is an interior minimizer of the auxiliary problem,

$$\nabla\psi(z^k) + \frac{16\varepsilon_k}{d_k^2}(z^k - v^k) = 0.$$

Therefore,

$$\|\nabla\psi(z^k)\| < \frac{8\varepsilon_k}{d_k} < \frac{8}{k},$$

and hence $\nabla\psi(z^k) \to 0$. For each $j$, define

$$\mu_j^k := -\frac{R_j(z^k)}{\psi(z^k)} = \frac{\Pi_{\mathbb{L}_j}(G_j(z^k)) - G_j(z^k)}{\psi(z^k)}.$$

By Moreau's decomposition, $\mu_j^k \in \mathbb{L}_j$, and

$$\sum_{j=1}^{p} \|\mu_j^k\|^2 = \frac{\sum_{j=1}^{p} \|R_j(z^k)\|^2}{\psi(z^k)^2} = 1.$$

Passing to a subsequence, assume that $\mu_j^k \to \mu_j \in \mathbb{L}_j$ for every $j$. Then $\sum_{j=1}^{p} \|\mu_j\|^2 = 1$, so the multipliers are not all zero. Moreover,

$$\nabla\psi(z^k) = -\sum_{j=1}^{p} DG_j(z^k)^T \mu_j^k.$$

Passing to the limit yields $\sum_{j=1}^{p} DG_j(\bar{x})^T \mu_j = 0$.

It remains to verify the violation conditions in Definition 3.1. Fix $j$ such that $\mu_j \neq 0$ and let $q_j(w) := \Pi_{\mathbb{L}_j}(w) - w$. Then $\mu_j^k = \tau_k q_j(G_j(z^k))$, where $\tau_k := 1/\psi(z^k) > 0$, and $\mu_j^k \to \mu_j \neq 0$. If $\mu_j \in \operatorname{int}(\mathbb{L}_j)$, Proposition 5.1 gives $G_j(z^k) \in \operatorname{int}(-\mathbb{L}_j)$ for all sufficiently large $k$. If $\mu_j \in \operatorname{bd}^+(\mathbb{L}_j)$, the same proposition gives $G_j(z^k) \notin \mathbb{L}_j$ and $u_-(G_j(z^k)) = u_+(\mu_j^k) \to u_+(\mu_j)$.

Thus, the nonzero multipliers $\mu_j$ and the sequence $z^k \to \bar{x}$ satisfy exactly the conditions excluded by quasinormality, a contradiction. Therefore, there exist $\delta, M > 0$ such that $\operatorname{dist}(x, \Omega) \leq M\psi(x) \leq M\varphi(x)$ whenever $\|x - \bar{x}\| \leq \delta$. ∎

**5.2. Exact penalization.** A direct consequence of the local error bound is exact penalization, which requires only local Lipschitz continuity of $f$. The local minimality conclusion does not require the original minimizer to be strict; strict local minimality is preserved when the penalty parameter is chosen strictly above the threshold obtained below.

THEOREM 5.3. *Let $\bar{x} \in \Omega$ be quasinormal, and let $f$ be locally Lipschitz with a local minimum over $\Omega$ at $\bar{x}$. Then there exists $\bar{c} \geq 0$ such that, for every $c \geq \bar{c}$, $\bar{x}$ is a local minimizer of*

$$P_c(x) := f(x) + c\,\varphi(x) = f(x) + c\sum_{j=1}^{p} \operatorname{dist}\big(G_j(x), \mathbb{L}_j\big)$$

*over $\mathbb{R}^n$. If $\bar{x}$ is a strict local minimizer of the original problem, then it is a strict local minimizer of $P_c$ for every $c > \bar{c}$.*

*Proof.* By Theorem 5.2, there exist $\delta, M > 0$ such that $\operatorname{dist}(x, \Omega) \leq M\varphi(x)$ whenever $\|x - \bar{x}\| \leq \delta$. After decreasing $\delta$ if necessary, assume that $\bar{x}$ minimizes $f$ over $\Omega \cap \{x : \|x - \bar{x}\| \leq 2\delta\}$ and that $f$ is $L_f$-Lipschitz on this larger ball.

For $\|x - \bar{x}\| \leq \delta$, let $\hat{x}$ be a closest point to $x$ in $\Omega$, which exists because $\Omega$ is closed. Since $\bar{x} \in \Omega$, $\|x - \hat{x}\| = \operatorname{dist}(x, \Omega) \leq \|x - \bar{x}\| \leq \delta$, and therefore $\|\hat{x} - \bar{x}\| \leq 2\delta$. Hence $f(\hat{x}) \geq f(\bar{x})$ and, by the error bound, $\|x - \hat{x}\| \leq M\varphi(x)$. Thus

$$P_c(x) \geq f(\hat{x}) - L_f\|x - \hat{x}\| + c\varphi(x) \geq f(\bar{x}) + (c - L_f M)\varphi(x).$$

Taking $\bar{c} := L_f M$, we obtain $P_c(x) \geq f(\bar{x}) = P_c(\bar{x})$ for every $c \geq \bar{c}$, so $\bar{x}$ is a local minimizer of $P_c$.

If the original minimum is strict and $c > \bar{c}$, then the inequality is strict whenever $\varphi(x) > 0$. If $\varphi(x) = 0$, then $x \in \Omega$, and strict local minimality of $f$ on $\Omega$ gives $f(x) > f(\bar{x})$ for every $x \neq \bar{x}$ sufficiently close to $\bar{x}$. Thus $\bar{x}$ is a strict local minimizer of $P_c$. ∎

*Remark* 5.4. When $m_j = 1$ for every $j$, so that $\mathbb{L}_j = \mathbb{R}_+$, the feasible set is described by the scalar inequalities $-G_j(x) \leq 0$, Definition 3.1 reduces to the classical quasinormality of Hestenes, and $\varphi(x) = \sum_j [-G_j(x)]_+$. In this case, Theorems 5.2 and 5.3 recover the classical local error-bound and exact-penalization results; see [27, 13].

**6. Algorithmic consequences.** In this section, we analyze an augmented Lagrangian method with safeguarded multipliers. We first establish dual boundedness under quasinormality. We then introduce relaxed quasinormality and show that this weaker condition still guarantees dual boundedness and the KKT conditions. Finally, an extension to infeasible points is used to establish feasibility of accumulation points.

**6.1. The augmented Lagrangian method.** Several methods have been proposed for this class of nonlinear problems, including exact penalty methods, semismooth Newton methods, SQP-type algorithms, primal-dual interior-point methods, and augmented Lagrangian methods; see [16, 21, 22, 23, 31].

Our purpose is not to introduce a new numerical method, but rather to establish the algorithmic role of the quasinormality condition introduced in Section 3. To this end, we consider a safeguarded scheme based on the Powell–Hestenes–Rockafellar augmented Lagrangian, which is also used in the NSOCP analysis of Liu and Zhang [23]. This framework is consistent with augmented Lagrangian analyses in nonlinear programming [14] and symmetric-cone optimization [24]. We show that quasinormality guarantees boundedness of the multiplier sequence. Consequently, every feasible accumulation point of the primal sequence satisfies the KKT conditions.

Given a penalty parameter $\rho > 0$ and multipliers $\mu_j \in \mathbb{L}_j$, $j = 1, \ldots, p$, we use the following form of the augmented Lagrangian, up to an additive term independent of $x$:

$$\mathcal{L}_\rho(x, \mu) := f(x) + \frac{1}{2\rho} \sum_{j=1}^{p} \left\| \Pi_{\mathbb{L}_j}(\mu_j - \rho G_j(x)) \right\|^2, \tag{6.1}$$

where $\mu = (\mu_1, \ldots, \mu_p) \in \mathbb{L}_1 \times \cdots \times \mathbb{L}_p$. By standard projection properties and the spectral representation of the projection onto the second-order cone [1, 24],

$$\nabla_x \mathcal{L}_\rho(x, \mu) = \nabla f(x) - \sum_{j=1}^{p} DG_j(x)^T \Pi_{\mathbb{L}_j}(\mu_j - \rho G_j(x)). \tag{6.2}$$

We now state the algorithm. As usual in safeguarded augmented Lagrangian schemes, the multipliers used in the subproblems are kept bounded.

**Algorithm 6.1** Augmented Lagrangian method for NSOCP

Let $\tau \in (0,1)$, $\gamma > 1$, $\rho_1 > 0$, and $\widetilde{\mu}^1 \in \mathbb{L}_1 \times \cdots \times \mathbb{L}_p$. Let $\{\varepsilon_k\} \subset \mathbb{R}_+$ satisfy $\varepsilon_k \to 0$. Set $k := 1$ and $\|v^0\| := +\infty$.

**Step 1.** Find an approximately stationary point of the subproblem, i.e., $x^k \in \mathbb{R}^n$ satisfying

$$\left\|\nabla_x \mathcal{L}_{\rho_k}(x^k, \widetilde{\mu}^k)\right\| \le \varepsilon_k.$$

**Step 2.** Define $v^k = (v_1^k, \ldots, v_p^k)$ by

$$v_j^k := \Pi_{\mathbb{L}_j}\left(\frac{\widetilde{\mu}_j^k}{\rho_k} - G_j(x^k)\right) - \frac{\widetilde{\mu}_j^k}{\rho_k}, \qquad j = 1, \ldots, p.$$

If $\|v^k\| \le \tau\|v^{k-1}\|$, set $\rho_{k+1} := \rho_k$; otherwise, set $\rho_{k+1} := \gamma\rho_k$.
**Step 3.** Compute

$$\mu_j^k := \Pi_{\mathbb{L}_j}\left(\widetilde{\mu}_j^k - \rho_k G_j(x^k)\right), \qquad j = 1, \ldots, p, \tag{6.3}$$

and choose $\widetilde{\mu}^{k+1} \in \mathbb{L}_1 \times \cdots \times \mathbb{L}_p$ so that $\{\widetilde{\mu}^k\}$ remains bounded. For example, fix $C > 0$ and take $\widetilde{\mu}^{k+1} = \Pi_{\mathbb{L} \cap \{\|\mu\| \le C\}}(\mu^k)$, where $\mathbb{L} := \prod_j \mathbb{L}_j$.
**Step 4.** Set $k := k + 1$ and return to Step 1.

By Step 1 and the definition of the multipliers in Step 3, the generated sequences satisfy

$$\left\|\nabla f(x^k) - \sum_{j=1}^{p} DG_j(x^k)^T \mu_j^k\right\| \le \varepsilon_k. \tag{6.4}$$

The penalty update rule and approximate stationarity also yield stationarity of a quadratic infeasibility measure without any constraint qualification.

PROPOSITION 6.1. *Let $\bar{x}$ be an accumulation point of a sequence $\{x^k\}$ generated by Algorithm 6.1. Then $\bar{x}$ is a stationary point of the infeasibility minimization problem*

$$\min_{x \in \mathbb{R}^n} \Phi(x) := \sum_{j=1}^{p} \left\|\Pi_{\mathbb{L}_j}(-G_j(x))\right\|^2. \tag{6.5}$$

*Proof.* If $\{\rho_k\}$ is bounded, then the update rule implies that it is eventually constant and $v^k \to 0$. By Moreau's decomposition,

$$G_j(x^k) + v_j^k = \Pi_{\mathbb{L}_j}\left(G_j(x^k) - \frac{\widetilde{\mu}_j^k}{\rho_k}\right) \in \mathbb{L}_j.$$

Hence $\operatorname{dist}(G_j(x^k), \mathbb{L}_j) \le \|v_j^k\| \to 0$. Thus every accumulation point is feasible and therefore stationary for $\Phi$.

If $\rho_k \to \infty$, along a subsequence such that $x^k \to \bar{x}$,

$$\frac{\mu_j^k}{\rho_k} = \Pi_{\mathbb{L}_j}\left(\frac{\widetilde{\mu}_j^k}{\rho_k} - G_j(x^k)\right) \longrightarrow \Pi_{\mathbb{L}_j}(-G_j(\bar{x})).$$

Dividing (6.4) by $\rho_k$ and passing to the limit gives $\sum_{j=1}^{p} DG_j(\bar{x})^T \Pi_{\mathbb{L}_j}(-G_j(\bar{x})) = 0$. Since $\nabla\Phi(\bar{x}) = -2\sum_j DG_j(\bar{x})^T \Pi_{\mathbb{L}_j}(-G_j(\bar{x}))$, the conclusion follows. ∎

**6.2. Boundedness of multipliers under quasinormality.** The next lemma records a basic projection property used below.

LEMMA 6.2. *Let $z_j^k := \widetilde{\mu}_j^k - \rho_k G_j(x^k)$. Then $\mu_j^k,\ \mu_j^k - z_j^k \in \mathbb{L}_j$,*

$$\langle \mu_j^k, \mu_j^k - z_j^k \rangle = 0, \qquad \text{and} \qquad \langle G_j(x^k), \mu_j^k \rangle = \rho_k^{-1}\big(\langle \widetilde{\mu}_j^k, \mu_j^k \rangle - \|\mu_j^k\|^2\big). \tag{6.6}$$

*Proof.* This is Moreau's decomposition of $z_j^k$ for the self-dual cone $\mathbb{L}_j$. ∎

The next proposition shows that boundedness of the multiplier sequence is the only missing ingredient for passing from approximate stationarity to the KKT conditions.

PROPOSITION 6.3. *Let $\{x^k\}$ and $\{\mu^k\}$ be generated by Algorithm 6.1, and let $\bar{x}$ be a feasible accumulation point of $\{x^k\}$. Suppose that $x^k \to \bar{x}$ along an infinite index set $K \subset \mathbb{N}$ and that $\{\mu^k\}_{k\in K}$ is bounded. Then $\bar{x}$ is a KKT point of* (NSOCP).

*Proof.* After extracting a subsequence, assume that $\mu^k \to \bar{\mu}$ along $K$, where $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_p)$ and $\bar{\mu}_j \in \mathbb{L}_j$. Passing to the limit in (6.4) yields

$$\nabla f(\bar{x}) - \sum_{j=1}^{p} DG_j(\bar{x})^T \bar{\mu}_j = 0.$$

It remains to prove complementarity. By Lemma 6.2,

$$\langle G_j(x^k), \mu_j^k \rangle = \frac{\langle \widetilde{\mu}_j^k, \mu_j^k \rangle - \|\mu_j^k\|^2}{\rho_k}.$$

If $\rho_k \to +\infty$, boundedness of $\{\widetilde{\mu}^k\}$ and $\{\mu^k\}_{k\in K}$ immediately gives $\langle G_j(x^k), \mu_j^k \rangle \to 0$.

If instead $\{\rho_k\}$ is bounded, the update rule implies $v^k \to 0$. Using

$$\mu_j^k = \rho_k \Pi_{\mathbb{L}_j}\left(\frac{\widetilde{\mu}_j^k}{\rho_k} - G_j(x^k)\right),$$

we obtain $v_j^k = (\mu_j^k - \widetilde{\mu}_j^k)/\rho_k$, and hence $\mu_j^k - \widetilde{\mu}_j^k = \rho_k v_j^k \to 0$. The projection identity (6.6) then gives $\langle G_j(x^k), \mu_j^k \rangle \to 0$. Therefore, $\langle G_j(\bar{x}), \bar{\mu}_j \rangle = 0$ for every $j$. ∎

We now show that quasinormality prevents the multiplier sequence generated by the augmented Lagrangian method from becoming unbounded. This is the main point at which the sequential nature of quasinormality becomes algorithmically relevant.

THEOREM 6.4. *Let $\bar{x}$ be a feasible accumulation point of the primal sequence generated by Algorithm 6.1. Suppose that $x^k \to \bar{x}$ along an infinite index set $K \subset \mathbb{N}$ and that $\bar{x}$ is quasinormal. Then the corresponding multiplier sequence $\{\mu^k\}_{k\in K}$ is bounded.*

*Proof.* Suppose, by contradiction, that $\{\mu^k\}_{k\in K}$ is unbounded. After passing to a subsequence, still indexed by $K$, define $M_k := \|\mu^k\| \to +\infty$ and $\alpha_j^k := \mu_j^k / M_k$. Since $\mu_j^k \in \mathbb{L}_j$ and $\|\alpha^k\| = 1$, we may assume $\alpha_j^k \to \alpha_j \in \mathbb{L}_j$ for every $j$, with $\alpha = (\alpha_1, \ldots, \alpha_p) \neq 0$.

Dividing (6.4) by $M_k$ and passing to the limit gives

$$\sum_{j=1}^{p} DG_j(\bar{x})^T \alpha_j = 0. \tag{6.7}$$

By nonexpansiveness and positive homogeneity of the projection,

$$\left\| \frac{\mu_j^k}{M_k} - \frac{\rho_k}{M_k} \Pi_{\mathbb{L}_j}(-G_j(x^k)) \right\| \leq \frac{\|\widetilde{\mu}_j^k\|}{M_k} \longrightarrow 0.$$

Hence $(\rho_k/M_k)\Pi_{\mathbb{L}_j}(-G_j(x^k)) \to \alpha_j$. For every $j$ with $\alpha_j \neq 0$, Proposition 5.1, applied with $\tau_k = \rho_k/M_k$, yields exactly the violation conditions in Definition 3.1. Together with (6.7) and $\|\alpha\| = 1$, this contradicts quasinormality of $\bar{x}$. Therefore, the dual subsequence is bounded. ∎

Combining Proposition 6.3 and Theorem 6.4 gives the main convergence consequence under quasinormality.

COROLLARY 6.5. *Let $\bar{x}$ be a feasible accumulation point of the primal sequence generated by Algorithm 6.1. If $\bar{x}$ satisfies the quasinormality condition of Definition 3.1, then $\bar{x}$ is a KKT point of* (NSOCP).

**6.3. Relaxed quasinormality and dual boundedness.** The preceding analysis suggests weakening quasinormality by requiring secondary violations to be negligible relative to the dominant ones. This idea is related to relaxed quasinormality in nonlinear programming [10].

For $j = 1, \ldots, p$, define

$$q_j(x) := \Pi_{\mathbb{L}_j}(-G_j(x)), \qquad d_j(x) := \|q_j(x)\|, \qquad b_j(x) := \operatorname{dist}(q_j(x), \operatorname{bd} \mathbb{L}_j). \tag{6.8}$$

Thus, $d_j(x) = \operatorname{dist}(G_j(x), \mathbb{L}_j)$ measures the violation of block $j$, whereas $b_j(x)$ measures the distance of the projection residual to the cone boundary. In particular, $b_j(x) > 0$ if and only if $G_j(x) \in \operatorname{int}(-\mathbb{L}_j)$. For a scalar block, $b_j(x) = d_j(x) = \max\{-G_j(x), 0\}$. Given $\mu = (\mu_1, \ldots, \mu_p) \in \prod_j \mathbb{L}_j \setminus \{0\}$, define

$$J_I(\mu) := \{j : \mu_j \in \operatorname{int} \mathbb{L}_j\}, \quad J_B(\mu) := \{j : \mu_j \in \operatorname{bd}^+(\mathbb{L}_j)\}, \quad J_Z(\mu) := \{j : \mu_j = 0\}.$$

DEFINITION 6.6. *Let $\bar{x} \in \Omega$. We say that $\bar{x}$ satisfies* relaxed quasinormality *(RQN) if there do not exist $\mu \in \prod_j \mathbb{L}_j \setminus \{0\}$ and a sequence $x^k \to \bar{x}$ such that*

$$\sum_{j=1}^{p} DG_j(\bar{x})^T \mu_j = 0 \tag{6.9}$$

*and the following conditions hold:*

(i) *For $j \in J_I(\mu)$, $b_j(x^k) > 0$ for all sufficiently large $k$.*

(ii) *For $j \in J_B(\mu)$, $d_j(x^k) > 0$ for all sufficiently large $k$ and*

$$\frac{q_j(x^k)}{d_j(x^k)} \longrightarrow \frac{\mu_j}{\|\mu_j\|}.$$

(iii) *With the convention* $\min \varnothing = +\infty$*, define*

$$w_k := \min\left\{ \min_{j \in J_I(\mu)} b_j(x^k),\ \min_{j \in J_B(\mu)} d_j(x^k) \right\}. \tag{6.10}$$

*Then* $d_j(x^k) = o(w_k)$ *for* $j \in J_Z(\mu)$ *and* $b_j(x^k) = o(w_k)$ *for* $j \in J_B(\mu)$.

*Remark* 6.7. Since $\mu \neq 0$, the set $J_I(\mu) \cup J_B(\mu)$ is nonempty. Conditions (i)–(ii) guarantee that $0 < w_k < \infty$ for all sufficiently large $k$. For blocks with zero multiplier, the entire violation must be $o(w_k)$. For boundary multipliers, the depth of the residual inside the cone must be $o(w_k)$. No directional condition is imposed on residuals associated with interior multipliers.

THEOREM 6.8. *If a feasible point* $\bar{x}$ *is quasinormal, then it satisfies RQN.*

*Proof.* Suppose that RQN fails, with multipliers $\mu \neq 0$ and $x^k \to \bar{x}$. For $j \in J_I(\mu)$, condition (i) gives $G_j(x^k) \in \operatorname{int}(-\mathbb{L}_j)$. For $j \in J_B(\mu)$, apply Proposition 5.1 to

$$\frac{q_j(x^k)}{d_j(x^k)} \longrightarrow \frac{\mu_j}{\|\mu_j\|}.$$

This yields exactly the violation and alignment requirements in Definition 3.1. Thus the same multipliers and sequence contradict quasinormality of $\bar{x}$. ∎

*Remark* 6.9. The implication in Theorem 6.8 is strict. At the point $\bar{x}$ of Example 4.10, quasinormality fails, but RQN holds. Indeed, (6.9) gives $\mu_1 = (a, -a, 0)$, with $a \geq 0$, and $[\mu_2]_1 = 0$. Since $[G_2(x)]_0 = 0$ for every $x$, condition (i) cannot hold for the second block. If $\mu_2 \in \operatorname{bd}^+(\mathbb{L}_3)$, then $\widehat{\mu_2}$ is a nonzero multiple of $(0, 1)$, whereas $\widehat{q_2(x)}$ is always a multiple of $(1, 0)$, so condition (ii) cannot hold either. Hence $\mu_2 = 0$ and $a > 0$. Write $x^k = (s_k, t_k)$ and $r_k := (s_k^2 + t_k^2)^{1/2}$. If $t_k = 0$, the first block is either feasible or has residual direction $(1, 1, 0)$, which is incompatible with $(1, -1, 0)$. For $t_k \neq 0$, the spatial direction of $q_1(x^k)$ is $-(s_k, t_k)/r_k$, so alignment forces $s_k > 0$ and $t_k/s_k \to 0$. Since $w_k = d_1(x^k) = (r_k - s_k)/\sqrt{2}$ and $d_2(x^k) = |t_k|/\sqrt{2}$,

$$\frac{w_k}{d_2(x^k)} = \frac{r_k - s_k}{|t_k|} = \frac{|t_k|}{r_k + s_k} \longrightarrow 0,$$

which contradicts $d_2(x^k) = o(w_k)$. Thus $\bar{x}$ satisfies CRCQ and RQN, but not quasinormality.

The next property contains the information needed for both dual boundedness and the error-bound argument.

LEMMA 6.10. *Let* $x^k \to \bar{x}$, $t_k > 0$, *and suppose that*

$$t_k q_j(x^k) \to \alpha_j \in \mathbb{L}_j, \qquad \alpha = (\alpha_1, \ldots, \alpha_p) \neq 0.$$

*Then conditions (i)–(iii) of Definition 6.6 hold with* $\mu = \alpha$.

*Proof.* Continuity of the norm and positive homogeneity of the distance to the boundary of a cone yield

$$t_k d_j(x^k) \to \|\alpha_j\|, \qquad t_k b_j(x^k) \to \operatorname{dist}(\alpha_j, \operatorname{bd} \mathbb{L}_j).$$

For $j \in J_I(\alpha)$, the second limit is positive, proving (i). For $j \in J_B(\alpha)$, the first limit is positive and normalization of $t_k q_j(x^k)$ proves (ii). Since there are finitely many blocks,

$$t_k w_k \to \beta := \min\left\{ \min_{j\in J_I(\alpha)} \operatorname{dist}(\alpha_j, \operatorname{bd}\mathbb{L}_j),\ \min_{j\in J_B(\alpha)} \|\alpha_j\| \right\} > 0. \tag{6.11}$$

If $j \in J_Z(\alpha)$, then $t_k d_j(x^k) \to 0$; if $j \in J_B(\alpha)$, then $t_k b_j(x^k) \to 0$. Dividing by (6.11) gives the two dominance relations in (iii). ∎

THEOREM 6.11. *Let $x^k \to \bar{x} \in \Omega$, let $\rho_k > 0$, and let $\{\widetilde{\mu}^k\}$ be bounded. Suppose that*

$$\mu_j^k = \Pi_{\mathbb{L}_j}(\widetilde{\mu}_j^k - \rho_k G_j(x^k)), \qquad \nabla f(x^k) - \sum_{j=1}^p DG_j(x^k)^T \mu_j^k \to 0.$$

*If $\bar{x}$ satisfies RQN, then $\{\mu^k\}$ is bounded. In particular, for Algorithm 6.1, this conclusion holds along every primal subsequence converging to a feasible point satisfying RQN, and every accumulation point of the corresponding dual subsequence is a KKT multiplier associated with that point.*

*Proof.* If the dual sequence were unbounded, after extracting a subsequence we would have $M_k := \|\mu^k\| \to \infty$ and $\mu_j^k/M_k \to \alpha_j$, with $\|\alpha\| = 1$ and $\alpha_j \in \mathbb{L}_j$. Approximate stationarity gives $\sum_{j=1}^p DG_j(\bar{x})^T \alpha_j = 0$. By nonexpansiveness of the projection,

$$\left\| \frac{\mu_j^k}{M_k} - \frac{\rho_k}{M_k} q_j(x^k) \right\| \le \frac{\|\widetilde{\mu}_j^k\|}{M_k} \to 0.$$

Thus $(\rho_k/M_k) q_j(x^k) \to \alpha_j$. Lemma 6.10 then yields all conditions in Definition 6.6, contradicting RQN. For sequences generated by Algorithm 6.1, complementarity of every dual limit follows from the proof of Proposition 6.3. ∎

*Remark* 6.12. The boundedness part of Theorem 6.11 does not use the update rule for $\rho_k$. This observation allows the result to be applied also to the external-penalty sequences considered below. The algorithmic update rule is needed only to obtain complementarity when the penalty sequence remains bounded.

**6.4. Relaxed quasinormality as a constraint qualification.** We next use an external-penalty construction that avoids identifying an arbitrary penalty sequence with the particular sequence generated by Algorithm 6.1.

THEOREM 6.13. *Let $\bar{x}$ be a local minimizer of* (NSOCP). *Then, for every sequence $\rho_k \to \infty$, there exist $x^k \to \bar{x}$ and $\mu_j^k = \rho_k q_j(x^k) \in \mathbb{L}_j$ such that*

$$\nabla f(x^k) - \sum_{j=1}^p DG_j(x^k)^T \mu_j^k \to 0, \qquad \langle G_j(x^k), \mu_j^k \rangle \to 0, \quad j = 1, \dots, p.$$

*Proof.* Choose $r > 0$ such that $f(x) \ge f(\bar{x})$ for every $x \in \Omega$ with $\|x - \bar{x}\| \le r$. Let $x^k$ minimize

$$F_k(x) := f(x) + \frac{1}{2}\|x - \bar{x}\|^2 + \frac{\rho_k}{2} \sum_{j=1}^p d_j(x)^2,$$

over the compact set $\{x \in \mathbb{R}^n : \|x - \bar{x}\| \leq r\}$. Since $F_k(x^k) \leq F_k(\bar{x}) = f(\bar{x})$ and $f$ is bounded below on the ball, we have $d_j(x^k) \to 0$. Hence every accumulation point $x^*$ is feasible and satisfies

$$f(x^*) + \frac{1}{2}\|x^* - \bar{x}\|^2 \leq f(\bar{x}).$$

Local minimality of $\bar{x}$ implies $x^* = \bar{x}$, and therefore $x^k \to \bar{x}$.

For all sufficiently large $k$, $x^k$ lies in the interior of the ball, and stationarity of $F_k$ gives

$$\nabla f(x^k) - \sum_j DG_j(x^k)^T \rho_k q_j(x^k) = -(x^k - \bar{x}) \to 0.$$

Moreover,

$$0 \leq \frac{\rho_k}{2} \sum_j d_j(x^k)^2 \leq f(\bar{x}) - f(x^k) - \frac{1}{2}\|x^k - \bar{x}\|^2 \longrightarrow 0.$$

By Moreau's identity, $\langle G_j(x^k), q_j(x^k)\rangle = -\|q_j(x^k)\|^2$, which proves asymptotic complementarity. This is the penalty construction underlying sequential optimality conditions for NSOCP; see also [4]. □

THEOREM 6.14. *If $\bar{x}$ is a local minimizer of* (NSOCP) *satisfying RQN, then $\bar{x}$ is a KKT point.*

*Proof.* The sequences provided by Theorem 6.13 satisfy the boundedness assumptions of Theorem 6.11 with $\widetilde{\mu}^k = 0$. Hence $\{\mu^k\}$ is bounded. Extracting a convergent subsequence and passing to the limit in the stationarity and complementarity relations of Theorem 6.13 yields the KKT conditions. □

COROLLARY 6.15. *Every local minimizer of* (NSOCP) *satisfying quasinormality or pseudonormality is a KKT point.*

*Proof.* Combine Theorems 3.5, 6.8, and 6.14. □

THEOREM 6.16. *If $\bar{x} \in \Omega$ satisfies RQN, then there exist $\delta, M > 0$ such that*

$$\operatorname{dist}(x, \Omega) \leq M \sum_{j=1}^{p} d_j(x) \qquad \text{whenever } \|x - \bar{x}\| \leq \delta.$$

*Consequently, the exact-penalization conclusions of Theorem 5.3 also hold under RQN.*

*Proof.* If the error bound failed, the construction in the proof of Theorem 5.2, up to the point where quasinormality is used, would yield $z^k \to \bar{x}$, $\psi(z^k) > 0$, and

$$\frac{q_j(z^k)}{\psi(z^k)} \longrightarrow \alpha_j \in \mathbb{L}_j, \qquad \|\alpha\| = 1, \qquad \sum_j DG_j(\bar{x})^T \alpha_j = 0.$$

Lemma 6.10, with $t_k = 1/\psi(z^k)$, would imply failure of RQN. Exact penalization then follows from the same argument as in Theorem 5.3. □

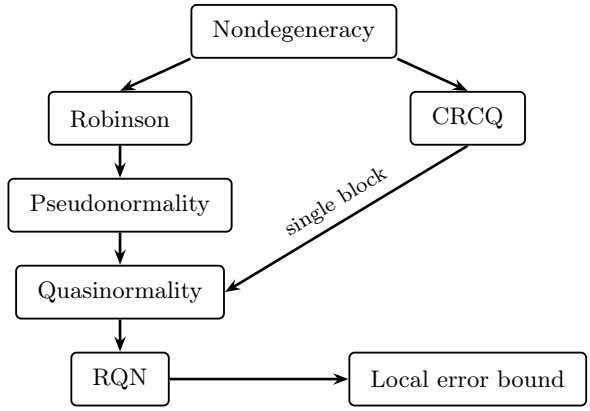


FIGURE 1. *Relationships among the conditions studied. Arrows denote implications. The implication from CRCQ to quasinormality holds for a single cone block and fails in general for multiple blocks. At the point of Remark 6.9, CRCQ and RQN hold, whereas quasinormality fails. Whether CRCQ implies RQN for multiple blocks is left open.*

**6.5. Extended relaxed quasinormality and feasibility.** RQN is defined only at feasible points. To derive feasibility from stationarity of the infeasibility measure, we extend the definition without imposing feasibility at the reference point.

DEFINITION 6.17. *A point $\bar{x} \in \mathbb{R}^n$, not necessarily feasible, satisfies* extended relaxed quasinormality *(ERQN) if there do not exist $\mu \in \prod_j \mathbb{L}_j \setminus \{0\}$ and a sequence $x^k \to \bar{x}$ satisfying* (6.9) *and conditions (i)–(iii) of Definition 6.6. Constant sequences are allowed.*

At feasible points, ERQN coincides with RQN. At infeasible points, it rules out, in particular, stationarity of the quadratic infeasibility measure.

THEOREM 6.18. *Let $\bar{x}$ be a stationary point of $\Phi$, defined in* (6.5)*, and suppose that $\bar{x}$ satisfies ERQN. Then $\bar{x}$ is feasible. Consequently, every accumulation point of Algorithm 6.1 satisfying ERQN is feasible and KKT, and the dual subsequence associated with any primal subsequence converging to that point is bounded.*

*Proof.* Suppose that $\bar{x}$ is infeasible and stationary for $\Phi$. Set $\mu_j := q_j(\bar{x})$. These multipliers are not all zero and satisfy $\sum_j DG_j(\bar{x})^T \mu_j = -\frac{1}{2}\nabla\Phi(\bar{x}) = 0$. For the constant sequence $x^k = \bar{x}$, all conditions in the definition of ERQN are satisfied: residuals in the interior have positive distance from the boundary; for boundary residuals, the normalized residual equals $\mu_j/\|\mu_j\|$; and all secondary terms vanish identically. This contradicts ERQN.

For an accumulation point of Algorithm 6.1, Proposition 6.1 gives stationarity of $\Phi$. Hence the first part yields feasibility. At a feasible point, ERQN reduces to RQN, and Theorem 6.11 gives dual boundedness and the KKT conclusions. ∎

**7. Conclusions.** The proposed notion of quasinormality relates the position of the multiplier within the Lorentz cone to the direction of constraint violation. We established its relationships with pseudonormality, Robinson's condition, and CRCQ, distinguishing the single-block case from the multiple-block setting. In particular, the aggregate closedness required by CRCQ may mask nonclosed individual normal images, which explains why the implication from CRCQ to quasinormality obtained for a single cone block does not extend in general to multiple blocks.

The geometric formulation of RQN is expressed solely in terms of projections,

norms, and distances to the cone boundary. This condition is strictly weaker than quasinormality and guarantees a local error bound and exact penalization. It also guarantees boundedness of dual subsequences associated with feasible accumulation points and yields KKT conditions at local minimizers and at feasible accumulation points of the augmented Lagrangian method. Its extension to infeasible points allows stationarity of the infeasibility measure to be combined with the dual analysis in order to obtain feasible KKT accumulation points. The boundedness results are local in the sense that they hold along primal subsequences converging to a given accumulation point; boundedness of the entire dual sequence requires suitable assumptions at all possible primal accumulation points.

Two questions remain open. First, although CRCQ implies quasinormality for a single cone block and does not imply quasinormality in general for multiple blocks, it remains unknown whether CRCQ implies RQN in the multiple-block setting; the point in Remark 6.9, where CRCQ and RQN hold but quasinormality fails, is consistent with such an implication. Second, for the nondirectional quasinormality of Guo, Ye, and Zhang applied to the product-cone formulation $G(x) \in \mathbb{L}$, we have shown that the quasinormality introduced here does not imply their condition, whereas the converse implication remains unsettled. Clarifying these two relationships is a natural direction for future work.

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