

A Formulation for Power Flow Interdiction with Bounded Big- M Values

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Abstract

Interdiction for the direct current optimal power flow problem is commonly modeled in two stages, where an attacker disables components in the first stage and an operator makes remedial adjustments in the second stage. The resulting bilevel mathematical program is often solved via transformation into a single-level mixed-integer program using duality. A drawback of this approach is that the single-level program often has big- M values that must be chosen large enough to ensure optimality. We present a single-level reformulation along with theoretical bounds on big- M values that allows for interdiction of branches, buses, and generators. Additionally, we provide a general approach to bounding big- M values for combinatorial problems on graphs, leveraging techniques from spectral graph theory. In doing so, we derive several tight lower bounds on the minimal eigenvalues of grounded Laplacians. We show computational results on standard test problems using the theoretical bounds and on real power systems using heuristic bounds. Finally, we demonstrate that our analysis applies to other interdiction problems, such as the decoupled linear power flow problem and a second-order cone formulation.

Keywords: interdiction, bilevel optimization, optimal power flow, big- M

1 Introduction

Modeling interdiction of the power grid is essential for understanding potential impacts from both intentional and natural threats. The interdiction problem seeks to determine a set of components that causes the most disruption to a system if all components in the set simultaneously fail. For a fixed power system, the direct current optimal power flow (DCOPF) formulation is frequently employed for computational tractability as it can be solved with linear programming. The interdiction problem with DCOPF typically has two stages. In the first stage, the attacker/interdictor disables at most k components, where components include generators, buses, and branches. In the second stage, the system operator adjusts the remaining network components to minimize load shed.

The two-stage setting leads to a bilevel max-min formulation, where the attacker disables components to maximize load shed and the operator sets generation levels to minimize load shed. To solve the bilevel formulation, it is common practice to dualize the inner problem, which creates a max-max formulation that can be solved as a single-level mixed-integer program. However, such formulations require McCormick envelopes or disjunctive constraints to handle the product of continuous and binary variables. This results in a formulation requiring dual variable bounds, or equivalently big- M values, and it is difficult to determine big- M values that are large enough to ensure optimality and small enough to allow practical solve times.

We present a novel formulation for the DCOPF interdiction problem that allows for interdiction of generators, buses¹, and branches², and that has theoretical upper bounds on all big- M values. To our knowledge, no such formulation currently exists in the literature. The model allows for overgeneration and bounds on phase angle magnitudes. We also provide a novel, graph-theoretic method for providing valid big- M values for combinatorial optimization problems over graphs. As a byproduct of this analysis, we establish multiple tight lower bounds on the minimal eigenvalues of grounded Laplacians.

We show computational results on three test problems from the MATPOWER power flow package [1] and a real model of the Texas power grid. Using reduced big- M values that violate our bounds, we demonstrate that the formulation remains effective with faster solve times. Finally, we demonstrate that our theoretical results are easily applied to other problems, including interdiction based on the decoupled linear power flow (DLPF) model and a second-order cone formulation for power flow interdiction.

Section 2 gives a brief literature review. Section 3 presents the DCOPF model along with dual variable bounds achieved using direct methods. Section 4 derives the new bounds for grounded Laplacians and shows an alternative approach to bounding DCOPF dual variables. Computational results for the DCOPF model are given in Section 5. Section 6 shows application of both the direct methods and the graph-theoretic methods for bounding variables on other interdiction problems. A conclusion is given in Section 7.

¹We use the term *bus* to refer to any junction node in the model; a physical substation may contain many buses.

²While not shown explicitly in this work, the model can also address interdiction of transformers via branch groupings.

2 Literature Review

General introductions to both interdiction and bilevel optimization can be found in [2, 3]. The survey by Brown et al. discusses approaches for both bilevel (attacker-defender) and trilevel (defender-attacker-defender) interdiction problems with applications to critical infrastructure [4].

For linear bilevel optimization in general, it is natural to ask if a given big- M value is sufficiently large to ensure optimality. It has been shown that this problem is NP-hard [5]. Alternatives to big- M formulations, such as disjunctive programming and special ordered sets, have been developed and implemented in modern solvers [6, 7]. However, they do not yet offer general performance that is competitive with big- M formulations for large problems.

Many studies have developed models for DCOPF interdiction without big- M bounds. Motto et al. present a single-level mixed-integer program allowing for interdiction of generators, branches, buses, and substations [8]. Salmeron et al. develop a generalized form of Benders decomposition to solve a similar problem, but they make limiting assumptions about load shed and do not prove optimality [9]. Schmidt and Thürauf present a lower and upper bounding scheme for networks with potential-based flows, which include the DCOPF branch interdiction problem, but they assume that the network remains connected and they only consider small test problems [10].

There are some theoretical results involving DCOPF interdiction. Johnson and Dey consider the DCOPF interdiction problem where relays are targets [11]. While relays serve as protective devices to shut down components in cases of faults, they can be infiltrated by attackers wishing to disrupt the network. Instead of deriving bounds on dual variables for the full DCOPF model, they prove sufficiency of a relaxed network flow model for uncongested networks. Fattahi et al. consider the related problem of big- M formulations for optimal transmission switching [12]. They show that finding optimal bounds for mixed-integer programs with big- M values is NP-hard and that no constant-factor approximation is possible unless $P=NP$. They also show a bound strengthening method under the assumption that the network remains connected. Assuming that a network remains connected is a problematic assumption for interdiction since often an attacker can be most disruptive by breaking the network into islands.

In work more closely related to this paper, Bienstock and Verma give mixed-integer programming formulations for solving the interdiction problem on power grids in the presence of branch disruptions [13]. They prove bounds on big- M values and their proof techniques inspired many of the approaches that we use in this paper. Our work also extends [14], which develops a formulation for DCOPF allowing the interdiction of branches with bounded big- M values. In addition to including generator and bus interdictions, we allow for nonzero minimum generation bounds as well as bounds on phase angles, which are not included in [14]. We also make no assumptions about network connectivity.

Beyond DCOPF, there are a variety of models offering alternative relaxations and approximations of AC optimal power flow, some of which have been studied in the context of interdiction. These relaxations are based on semidefinite programming (SDP), second-order cone (SOC) programming, and linear programming [15–18]. Kocuk et

al. present a comparison of six different relaxations [19]. Similar to our approach for interdiction, Wu et al. use duality but with an underlying SOC optimal power flow formulation to solve for branch interdictions [20].

3 DCOPF Model

We consider a standard setting for solving the DCOPF problem where a system operator has control of generator power levels and wishes to minimize the sum of load shed and overgeneration. We present the primal formulation where interdiction variables are considered fixed. We then take the dual to allow for simultaneous optimization of interdiction and power flow, allowing the interdiction variables to vary. This creates a bilinear formulation that is linearized to produce the final formulation. To simplify expressions, we omit phase angle bounds from the analysis until the end of the section. Table 1 summarizes all of our notation.

The primal linear program for the optimal power flow problem is shown in (1a) - (1i). The objective function minimizes the sum of load shed and overgeneration. The interdiction variables $(s_b)_{b \in B}$, $(s_l)_{l \in L}$, $(s_g)_{g \in G}$ are equal to one for surviving components and equal to zero for interdicted components. The placement of these variables in the formulation differs from standard models and makes the formulation more amenable to analysis. The interdiction variables are interpreted as fixed parameters here but are allowed to vary later when looking at the dual formulation. Dual variables are shown in parentheses. We assume all reactance values are strictly positive.

(DCOPF-Primal)

$$\min \sum_{b \in B} (r_b + v_b) \quad (1a)$$

$$\text{s.t. } s_l s_{o(l)} s_{d(l)} (\theta_{o(l)} - \theta_{d(l)} - X_l f_l) = 0, \quad \forall l \in L, \quad (\mu_l) \quad (1b)$$

$$\begin{aligned} s_b \sum_{g \in G_b} s_g p_g - \sum_{l|o(l)=b} s_l s_{o(l)} s_{d(l)} f_l \\ + \sum_{l|d(l)=b} s_l s_{o(l)} s_{d(l)} f_l + r_b - v_b = R_b, \quad \forall b \in B, \quad (\lambda_b) \end{aligned} \quad (1c)$$

$$P_g^- \leq p_g \leq P_g^+, \quad \forall g \in G, \quad (\gamma_g^-, \gamma_g^+) \quad (1d)$$

$$-F_l \leq f_l \leq F_l, \quad \forall l \in L, \quad (\phi_l^-, \phi_l^+) \quad (1e)$$

$$r_b \leq R_b, \quad \forall b \in B, \quad (\alpha_b) \quad (1f)$$

$$v_b \leq V_b, \quad \forall b \in B, \quad (\beta_b) \quad (1g)$$

$$r_b, v_b \geq 0, \quad \forall b \in B, \quad (1h)$$

$$\theta_b \text{ free}, \quad \forall b \in B. \quad (1i)$$

Constraint (1b) enforces the appropriate phase angles for a given branch if the branch, source bus, and destination bus are not interdicted. Constraint (1c) ensures flow conservation for each bus. For a given generator, the generator and its corresponding bus must be surviving for the generator to deliver power to the node. For

Table 1 Mathematical Notation

Symbol	Description
Sets	
B	set of buses
G	set of generators
G_b	set of generators at bus b
L	set of branches
Parameters	
$b(g)$	bus connected to generator g
$d(l)$	destination bus of branch l
F_l	flow capacity of branch l
$o(l)$	origin bus of branch l
M_l	upper bound on magnitude of $\bar{\eta}_l$
N_l	upper bound on magnitude of $\bar{\eta}_l - \eta_l$
P_g^-	minimum generation capacity for generator g
P_g^+	maximum generation capacity for generator g
R_b	demand at bus b
T	upper bound on magnitude of phase angle
U_l	upper bound on magnitude of $\bar{\mu}_l$
V_b	overgeneration upper bound at bus b
W_g	upper bound on the magnitudes of $\bar{\lambda}_g$ and $\lambda_{b(g)}$
X_l	reactance of branch l
k	attacker budget
Variables	
a_b	binary variable equal to 1 if bus b is attacked and 0 otherwise
a_g	binary variable equal to 1 if generator g is attacked
a_l	binary variable equal to 1 if branch l is attacked
f_l	power flow through branch l
p_g	power output of generator g
r_b	load shed at bus b
s_l	binary variable equal to 1 if branch l survives and 0 otherwise
s_b	binary variable equal to 1 if bus b survives and 0 otherwise
s_g	binary variable equal to 1 if generator g survives and 0 otherwise
s'_l	binary variable equal to 1 if branch l and adjacent buses survive and 0 otherwise
s'_g	binary variable equal to 1 if generator g and its associated bus survive and 0 otherwise
v_b	overgeneration at bus b
α_b	dual variable for load shed upper bound at bus b
β_b	dual variable for overgeneration upper bound at bus b
γ_g^-	dual variable for generation lower bound at generator g
γ_g^+	dual variable for generation upper bound at generator g
θ_b	phase angle at bus b
λ_b	dual variable for flow conservation at bus b
μ_l	dual variable for phase angle constraint at branch l
$\bar{\mu}_l$	dual variable equal to μ_l if branch l survives and 0 otherwise
ϕ_l^-	dual variable for branch l power flow lower bound
ϕ_l^+	dual variable for branch l power flow upper bound
χ_b^-	dual variable for phase angle lower bound at bus b
χ_b^+	dual variable for phase angle upper bound at bus b

simplified analysis, (1b) allows flow on a branch if it is interdicted, but the interdiction variables in (1c) ensure that any such flow is not realized at bus injection. Similarly, a generator can generate power if it is interdicted in the formulation, but the interdiction variables do not allow the power to be injected at the bus.

We assume that all generators have a nonnegative lower bound on generation, so the upper bound on load shed at a bus is given simply by the sum of the loads at the bus, denoted by R_b . Without loss of generality, we assume for all generators that

$P_g^- < P_g^+$. Likewise, we assume that all loads are nonnegative so the upper bound on overgeneration is given by

$$V_b := \sum_{g \in G_b} P_g^+, \quad \forall b \in B.$$

Treating all interdiction variables s as fixed parameters, the dual linear program is shown in (2a) - (2t). We refer to this as the *dual reformulation*, or DCOPF-DR, since it is a linear programming dual only if the interdiction variables are fixed. The program has been written such that all dual variables for inequality constraints are nonnegative. The interdiction variables can be reinterpreted as variables in the dual as both the dual power flow problem and interdiction problem have maximizing objectives. We assume for the sake of presentation that the attacker/interdictor can attack at most k components as modeled by constraint (2b). The formulation is compatible with other interdiction constraints, such as assigning a unique price to each component.

The attacker only directly selects the attack variables $(a_l)_{l \in L}$, $(a_b)_{b \in B}$, and $(a_g)_{g \in G}$. The surviving variables $(s_l)_{l \in L}$, $(s_b)_{b \in B}$, and $(s_g)_{g \in G}$ are determined as the complements of the attack variables and are included to simplify presentation. Each branch is only functional if its origin bus, destination bus, and the branch itself survive. Hence we define $(s'_l)_{l \in L}$ according to (2c). Similarly, each generator is only functional if the generator's bus and the generator itself survive, so we define $(s'_g)_{g \in G}$ according to (2d). The constraints are nonlinear logical constraints that can be linearized.

(DCOPF-DR)

$$\begin{aligned} \max \quad & - \sum_{l \in L} F_l(\phi_l^+ + \phi_l^-) - \sum_{b \in B} R_b \alpha_b - \sum_{b \in B} V_b \beta_b - \sum_{g \in G} P_g^+ \gamma_g^+ + \sum_{g \in G} P_g^- \gamma_g^- \\ & + \sum_{b \in B} \lambda_b R_b \end{aligned} \quad (2a)$$

$$\text{s.t.} \quad \sum_{l \in L} a_l + \sum_{b \in B} a_b + \sum_{g \in G} a_g \leq k, \quad (2b)$$

$$s'_l = s_l s_{o(l)} s_{d(l)}, \quad \forall l \in L, \quad (2c)$$

$$s'_g = s_{b(g)} s_g, \quad \forall g \in G, \quad (2d)$$

$$s_l = 1 - a_l, \quad \forall l \in L, \quad (2e)$$

$$s_b = 1 - a_b, \quad \forall b \in B, \quad (2f)$$

$$s_g = 1 - a_g, \quad \forall g \in G, \quad (2g)$$

$$a_l, s_l, s'_l \in \{0, 1\}, \quad \forall l \in L, \quad (2h)$$

$$a_b, s_b \in \{0, 1\}, \quad \forall b \in B, \quad (2i)$$

$$a_g, s_g, s'_g \in \{0, 1\}, \quad \forall g \in G, \quad (2j)$$

$$\sum_{l|o(l)=b} s'_l \mu_l - \sum_{l|d(l)=b} s'_l \mu_l = 0, \quad \forall b \in B, \quad (\theta_b) \quad (2k)$$

$$-s'_l X_l \mu_l - s'_l \lambda_{o(l)} + s'_l \lambda_{d(l)} - \phi_l^+ + \phi_l^- = 0, \quad \forall l \in L, \quad (f_l) \quad (2l)$$

$$s'_g \lambda_{b(g)} + \gamma_g^- - \gamma_g^+ = 0, \quad \forall g \in G, \quad (p_g) \quad (2m)$$

$$\lambda_b - \alpha_b \leq 1, \quad \forall b \in B, \quad (r_b) \quad (2n)$$

$$-\lambda_b - \beta_b \leq 1, \quad \forall b \in B, \quad (v_b) \quad (2o)$$

$$\phi_l^+, \phi_l^- \geq 0, \quad \forall l \in L, \quad (2p)$$

$$\alpha_b, \beta_b \geq 0, \quad \forall b \in B, \quad (2q)$$

$$\lambda_b \text{ free}, \quad \forall b \in B, \quad (2r)$$

$$\mu_l \text{ free}, \quad \forall l \in L, \quad (2s)$$

$$\gamma_g^+, \gamma_g^- \geq 0, \quad \forall g \in G. \quad (2t)$$

Allowing the interdiction variables to vary, the dual formulation has multiple bilinear terms. We derive bounds on the continuous variables involved in bilinear terms. The following inequality is required prior to analyzing specific variables.

Lemma 1 Let

$$R' := \sum_{b \in B} \max \left(R_b, \sum_{g \in G_b} P_g^- \right).$$

An optimal solution to (DCOPF-DR) always satisfies

$$\sum_{l \in L} (\phi_l^+ + \phi_l^-) F_l \leq R'.$$

Proof The mathematical program (2a) - (2t) measures the sum of load shed and overgeneration, which must be nonnegative. We can thus restrict variables to the ranges that give a nonnegative objective value, which is necessary for optimality. This gives the condition

$$\begin{aligned} \sum_{l \in L} (\phi_l^+ + \phi_l^-) F_l &\leq - \sum_{b \in B} R_b \alpha_b - \sum_{b \in B} V_b \beta_b - \sum_{g \in G} P_g^+ \gamma_g^+ \\ &\quad + \sum_{g \in G} P_g^- \gamma_g^- + \sum_{b \in B} \lambda_b R_b. \end{aligned} \quad (4a)$$

Let

$$L(g) := -P_g^+ \gamma_g^+ + P_g^- \gamma_g^-, \quad (4b)$$

$$L(b) := -R_b \alpha_b - V_b \beta_b + \sum_{g \in G_b} L(g) + \lambda_b R_b, \quad (4c)$$

so that the right hand side of (4a) is equal to $\sum_{b \in B} L(b)$. Each bus in the optimal solution must have either positive overgeneration, positive load shed, or zero overgeneration and zero load shed.

In the case of positive overgeneration, load shed at a bus must be equal to zero. All surviving generators at the bus must be operating at minimum generation, otherwise overgeneration could be reduced to improve the objective value, so $\gamma_g^- \geq 0$ and $\gamma_g^+ = 0$. For a given bus b with surviving generator $g \in G_b$,

$$L(g) = P_g^- \gamma_g^- = -P_g^- \lambda_b,$$

where we have used $\gamma_g^- = -\lambda_b$ by (2m). If generator g is not surviving then $L(g) = 0$. Since $v_b > 0$, then by complementary slackness, $-\lambda_b - \beta_b = 1$ and $\lambda_b < 0$. Hence

$$\begin{aligned} L(b) &= -V_b\beta_b + \sum_{g \in G_b} L(g) + \lambda_b R_b \\ &\leq -\beta_b \sum_{g \in G_b} P_g^+ + \lambda_b \left(R_b - \sum_{g \in G_b} P_g^- \right) \\ &= \sum_{g \in G_b} P_g^- - R_b + \beta_b \left(-R_b + \sum_{g \in G_b} (P_g^- - P_g^+) \right) \\ &\leq \sum_{g \in G_b} P_g^- - R_b. \end{aligned}$$

For the case of positive load shed at a bus, all surviving generators at the bus must be operating at maximum generation, so $\gamma_g^+ \geq 0$ and $\gamma_g^- = 0$. For a surviving adjacent generator this gives

$$L(g) = -P_g^+ \gamma_g^+ = -P_g^+ \lambda_b = -P_g^+ (1 + \alpha_b).$$

If the generator is not surviving, then $L(g) = 0$, so

$$L(g) \leq 0.$$

Next,

$$\begin{aligned} L(b) &= -R_b\alpha_b + \sum_{g \in G_b} L(g) + \lambda_b R_b \\ &\leq -R_b\alpha_b + (1 + \alpha_b)R_b \\ &= R_b. \end{aligned}$$

For buses with zero load shed and zero overgeneration, there are four subcases. If both $R_b > 0$ and $V_b > 0$, then $\alpha_b = \beta_b = 0$. This gives

$$-1 \leq \lambda_b \leq 1$$

and

$$L(b) \leq \max \left(R_b, \sum_{g \in G_b} P_g^- \right).$$

If $R_b > 0$ and $V_b = 0$, there are no generators at the bus. Since $r_b = 0$, $\alpha_b = 0$ and $\lambda_b \leq 1$. Thus

$$L(b) = R_b\lambda_b \leq R_b.$$

If $V_b > 0$ and $R_b = 0$, then $\beta_b = 0$ implies that $-\lambda_b \leq 1$ and

$$L(b) = \sum_{g \in G_b} L(g).$$

Then if $\lambda_b \geq 0$, $L(g) \leq 0$. On the other hand if $\lambda_b \leq 0$, then

$$L(g) \leq P_g^-,$$

which provides a valid bound for the subcase. Finally if $R_b = 0$ and $V_b = 0$, there are no generators at the bus so

$$L(b) = 0.$$

□

The inequality provided by Lemma 1 is sufficient for deriving upper bounds on $(\phi_l^+, \phi_l^-)_{l \in L}$.

Lemma 2 An optimal solution to (DCOPF-DR) can always be found with values of ϕ^+ and ϕ^- satisfying

$$\begin{aligned}\phi_l^+ &\leq \frac{R'}{F_l}, & \forall l \in L, \\ \phi_l^- &\leq \frac{R'}{F_l}, & \forall l \in L.\end{aligned}$$

Proof We consider the mathematical program for a given $\xi = \phi_l^+$ or $\xi = \phi_l^-$, where constraint (6b) follows from Lemma 1.

$$\max \quad \xi \tag{6a}$$

$$\text{s.t.} \quad \sum_{l \in L} (\phi_l^+ + \phi_l^-) F_l \leq R', \tag{6b}$$

$$\phi_l^+ \geq 0, \quad \phi_l^- \geq 0, \quad \forall l \in L. \tag{6c}$$

The bound follows from direct analysis or by using Lagrange multipliers. $\square$

The dual program (2a) - (2t) has the bilinear terms in constraints (2k), (2l), and (2m). We reformulate (2l) by introducing the variable η_l :

$$s'_l \eta_l = \phi_l^+ - \phi_l^-, \quad \forall l \in L,$$

$$\eta_l = -X_l \mu_l + \lambda_{d(l)} - \lambda_{o(l)}, \quad \forall l \in L.$$

We then define the following variables representing products of continuous and binary variables:

$$\bar{\eta}_l := s'_l \eta_l, \quad \forall l \in L, \tag{7a}$$

$$\bar{\mu}_l := s'_l \mu_l, \quad \forall l \in L, \tag{7b}$$

$$\bar{\lambda}_g := s'_g \lambda_{b(g)}, \quad \forall g \in G.$$

This allows us to write the full linearized dual reformulation (DCOPF-LDR) given by (9a) - (9y).

The linearization for the binary variables s'_l and s'_g is given by (9c) - (9f). Constraints (9m) and (9n) are the McCormick reformulations for $\bar{\eta}_l$. Constraint (9o) is the reformulation for $\bar{\mu}_l$; note that the second constraint in the McCormick reformulation bounding the difference between $\bar{\mu}_l$ and μ_l is not included because the variables μ_l are not present anywhere in the program. Constraints (9q) - (9r) are the reformulations for $\bar{\lambda}_g$.

The following result bounds the magnitude of $\bar{\mu}$. The proof is similar to Lemma 4 in [14] even though it applies to a different model.

Lemma 3 An optimal solution to (DCOPF-LDR) can always be found with values of $\bar{\mu}_l$ satisfying

$$|\bar{\mu}_l| \leq \left(\frac{R'}{\sqrt{X_l}} \right) \max_{t \in L} \left\{ \frac{1}{F_t \sqrt{X_t}} \right\}, \quad \forall l \in L.$$

Proof According to (7b), $\bar{\mu}$ is only nonzero for surviving branches. For a fixed attack A , consider a connected component K of $L \setminus A$, meaning no arcs in K have been destroyed. From (7a), $\eta_l = \bar{\eta}_l$ for all $l \in K$. Rewriting constraints (9j) - (9l) for components restricted to K gives

$$\begin{aligned} N_K \bar{\mu}_K &= 0, \\ -N_K^T \lambda_K - \mathbf{X}_K \bar{\mu}_K &= \phi_K^+ - \phi_K^-, \end{aligned} \quad (8a)$$

where N_K is the node-arc incidence matrix of K , containing +1 entries for origin nodes and -1 entries for destination nodes, and $\mathbf{X}_K$ is the diagonal matrix of $\{X_l : l \in K\}$. Multiplying $-N_K \mathbf{X}_K^{-1}$ to the left-hand side of (8a) gives

$$\begin{aligned} N_K \mathbf{X}_K^{-1} N_K^T \lambda_K &= N_K \mathbf{X}_K^{-1} (\phi_K^- - \phi_K^+) - N_K \bar{\mu}_K \\ &= N_K \mathbf{X}_K^{-1} (\phi_K^- - \phi_K^+). \end{aligned} \quad (8b)$$

We employ a grounding argument for a selected bus b . Let $\hat{\lambda}_K := \lambda_K - \lambda_b \mathbf{1}$. Defining $\tilde{\lambda}_K$ as $\hat{\lambda}_K$ with bus b removed and $\tilde{N}_K$ as N_K with row b removed, we have

$$N_K^T \lambda_K = N_K^T \hat{\lambda}_K = \tilde{N}_K^T \tilde{\lambda}_K, \quad (8c)$$

where we have used that $N_K^T \mathbf{1} = \mathbf{0}$. We then have

$$\tilde{N}_K \mathbf{X}_K^{-1} \tilde{N}_K^T \tilde{\lambda}_K = \tilde{N}_K \mathbf{X}_K^{-1} (\phi_K^- - \phi_K^+). \quad (8d)$$

We compare the solutions in (8d) and (8b):

$$\begin{aligned} \mathbf{X}_K^{-1/2} N_K^T \lambda_K &= \mathbf{X}_K^{-1/2} \tilde{N}_K^T \tilde{\lambda}_K \\ &= \mathbf{X}_K^{-1/2} \tilde{N}_K^T (\tilde{N}_K \mathbf{X}_K^{-1} \tilde{N}_K^T)^{-1} \tilde{N}_K \mathbf{X}_K^{-1} (\phi_K^- - \phi_K^+) \\ &= H \mathbf{X}_K^{-1/2} (\phi_K^- - \phi_K^+) \end{aligned} \quad (8e)$$

where the matrix

$$H := \mathbf{X}_K^{-1/2} \tilde{N}_K^T (\tilde{N}_K \mathbf{X}_K^{-1} \tilde{N}_K^T)^{-1} \tilde{N}_K \mathbf{X}_K^{-1/2}$$

is symmetric and idempotent.

Multiplying $\mathbf{X}_K^{-1/2}$ to the left-hand side of all terms in (8a) gives

$$\begin{aligned} \mathbf{X}_K^{1/2} \bar{\mu}_K &= -\mathbf{X}_K^{-1/2} N_K^T \lambda_K + \mathbf{X}_K^{-1/2} (\phi_K^- - \phi_K^+) \\ &= -H \mathbf{X}_K^{-1/2} (\phi_K^- - \phi_K^+) + \mathbf{X}_K^{-1/2} (\phi_K^- - \phi_K^+) \\ &= (I - H) (\mathbf{X}_K^{-1/2} (\phi_K^- - \phi_K^+)). \end{aligned}$$

Using the fact that if a matrix H is idempotent, $I - H$ is also idempotent,

$$\begin{aligned} \|\mathbf{X}_K^{1/2} \bar{\mu}_K\|_2 &\leq \|I - H\|_2 \|\mathbf{X}_K^{-1/2} (\phi_K^- - \phi_K^+)\|_2 \\ &\leq \|\mathbf{X}_K^{-1/2} (\phi_K^- - \phi_K^+)\|_2. \end{aligned} \quad (8f)$$

Similar to the approach used in Lemma 2, the square of the right-hand side of (8f) is upper-bounded by the optimal value of the mathematical program

$$\max \quad \sum_{l \in K} X_l^{-1} (\phi_l^- - \phi_l^+)^2 \quad (8g)$$

$$\text{s.t.} \quad \sum_{l \in K} (\phi_l^+ + \phi_l^-) F_l \leq R', \quad (8h)$$

$$\phi_l^+ \geq 0, \quad \phi_l^- \geq 0, \quad \forall l \in K. \quad (8i)$$

The optimal objective value of (8g) - (8i) is equal to $\max_{l \in K} \left\{ X_l^{-1} \left(\frac{R'}{F_l} \right)^2 \right\}$ via standard analysis. Using $X_l^{1/2} |\bar{\mu}_l| \leq \|\mathbf{X}_K^{1/2} \bar{\mu}_K\|_2$ for all $l \in K$, we have

$$\begin{aligned} X_l^{1/2} |\bar{\mu}_l| &\leq \max_{l \in K} \left\{ \frac{R'}{F_l \sqrt{X_l}} \right\} \\ &\leq R' \max_{l \in L} \left\{ \frac{1}{F_l \sqrt{X_l}} \right\}. \end{aligned} \quad (8j)$$

The last inequality follows since $K \subseteq L$. The expression (8j) holds for all possible connected components K and hence for all $l \in L$. $\square$

(DCOPF-LDR)

$$\begin{aligned} \max \quad & - \sum_{l \in L} F_l (\phi_l^+ + \phi_l^-) - \sum_{b \in B} R_b \alpha_b - \sum_{b \in B} V_b \beta_b - \sum_{g \in G} P_g^+ \gamma_g^+ + \sum_{g \in G} P_g^- \gamma_g^- \\ & + \sum_{b \in B} \lambda_b R_b \end{aligned} \quad (9a)$$

$$\text{s.t.} \quad \sum_{l \in L} a_l + \sum_{b \in B} a_b + \sum_{g \in G} a_g \leq k, \quad (9b)$$

$$s'_l \leq s_l, s_{o(l)}, s_{d(l)}, \quad \forall l \in L, \quad (9c)$$

$$s'_l \geq s_l + s_{o(l)} + s_{d(l)} - 2, \quad \forall l \in L, \quad (9d)$$

$$s'_g \leq s_{b(g)}, s_g, \quad \forall g \in G, \quad (9e)$$

$$s'_g \geq s_{b(g)} + s_g - 1, \quad \forall g \in G, \quad (9f)$$

$$s_l = 1 - a_l, \quad \forall l \in L, \quad (9g)$$

$$s_b = 1 - a_b, \quad \forall b \in B, \quad (9h)$$

$$s_g = 1 - a_g, \quad \forall g \in G, \quad (9i)$$

$$a_l, s_l, s'_l \in \{0, 1\}, \quad \forall l \in L, \quad (9j)$$

$$a_b, s_b \in \{0, 1\}, \quad \forall b \in B, \quad (9k)$$

$$a_g, s_g, s'_g \in \{0, 1\}, \quad \forall g \in G, \quad (9l)$$

$$\sum_{l|o(l)=b} \bar{\mu}_l - \sum_{l|d(l)=b} \bar{\mu}_l = 0, \quad \forall b \in B, \quad (\theta_b) \quad (9m)$$

$$\bar{\eta}_l = \phi_l^+ - \phi_l^-, \quad \forall l \in L, \quad (9n)$$

$$\eta_l = -X_l \bar{\mu}_l + \lambda_{d(l)} - \lambda_{o(l)}, \quad \forall l \in L, \quad (9l)$$

$$-M_l s'_l \leq \bar{\eta}_l \leq M_l s'_l, \quad \forall l \in L, \quad (9m)$$

$$-N_l(1 - s'_l) \leq \bar{\eta}_l - \eta_l \leq N_l(1 - s'_l), \quad \forall l \in L, \quad (9n)$$

$$-U_l s'_l \leq \bar{\mu}_l \leq U_l s'_l, \quad \forall l \in L, \quad (9o)$$

$$\bar{\lambda}_g + \gamma_g^- - \gamma_g^+ = 0, \quad \forall g \in G, \quad (p_g) \quad (9p)$$

$$-W_g s'_g \leq \bar{\lambda}_g \leq W_g s'_g, \quad \forall g \in G, \quad (9q)$$

$$-W_g(1 - s'_g) \leq \bar{\lambda}_g - \lambda_{b(g)} \leq W_g(1 - s'_g), \quad \forall g \in G, \quad (9r)$$

$$\lambda_b - \alpha_b \leq 1, \quad \forall b \in B, \quad (r_b) \quad (9s)$$

$$-\lambda_b - \beta_b \leq 1, \quad \forall b \in B, \quad (v_b) \quad (9t)$$

$$\phi_l^+, \phi_l^- \geq 0, \quad \forall l \in L, \quad (9u)$$

$$\alpha_b, \beta_b \geq 0 \quad \forall b \in B, \quad (9v)$$

$$\lambda_b \text{ free}, \quad \forall b \in B, \quad (9w)$$

$$\bar{\mu}_l \text{ free}, \quad \forall l \in L, \quad (9x)$$

$$\gamma_g^+, \gamma_g^- \geq 0, \quad \forall g \in G. \quad (9y)$$

The final result is an upper bound on the magnitude of λ_b . The proof technique relies on finding the largest possible sum of edge weights $\lambda_{o(l)} - \lambda_{d(l)}$ along a path in the graph. In this sense, it is analogous to the proof of Lemma 5 in [14] that upper bounds the largest possible sum of edge weights $\theta_{o(l)} - \theta_{d(l)}$ along a path for a primal DCOPF formulation. This construction means the bound is dependent on the graph size; this is likely inevitable since the λ_b variables are expressed as differences in the dual. Lemma 4 establishes that not all λ_b values can be arbitrarily large for connected buses, Lemma 5 bounds the values for disconnected buses, and Lemma 6 gives the final bound.

Lemma 4 For any connected component consisting of buses $B' \subseteq B$, an optimal solution to (DCOPF-LDR) exists with variables $\lambda_{B'}$ satisfying at least one of the two conditions:

- (i) There exists $b^- \in B'$ such that $\lambda_{b^-} \leq 0$ and $b^+ \in B'$ such that $\lambda_{b^+} \geq 0$.
- (ii) There exists $b_1 \in B'$ such that $|\lambda_{b_1}| \leq 1$.

Proof If the first condition is not satisfied, then it must be true that $\lambda_{B'} > 0$ or $\lambda_{B'} < 0$. We consider for each bus $b \in B'$ the portion of the objective function defined by (4b) and (4c). Note that $\lambda_{B'}$ can be shifted by a constant value without impacting feasibility of the constraint (2l).

Starting with the case of $\lambda_{B'} > 0$, if $V_b > 0$, then by complementary slackness, $\lambda_b > -\beta_b - 1$ implies that $v_b = 0$ and hence $\beta_b = 0$. Thus

$$L(b) = -R_b \alpha_b + \sum_{g \in G_b} L(g) + \lambda_b R_b.$$

Since $\lambda_b > 0$, it is not possible to have any connected generators operating at minimum generation. Let G_b^+ be the set of connected generators operating at maximum generation and

let

$$Q_b^+ := \sum_{g \in G_b^+} P_g^+.$$

Then by (2m),

$$L(b) = -R_b \alpha_b - Q_b^+ \lambda_b + \lambda_b R_b.$$

For a given λ_b , the optimal value for α_b according to the objective function and (2n) is given by

$$\alpha_b = \max(0, \lambda_b - 1),$$

so

$$L(b) = -R_b \max(0, \lambda_b - 1) - Q_b^+ \lambda_b + \lambda_b R_b.$$

Since $L(b)$ is non-increasing in λ_b for $\lambda_b \geq 1$, any optimal solution with $\lambda_{B'} > 1$ can be adjusted by reducing $\lambda_{B'}$ by a constant amount until $\min_{b \in B'} \lambda_b = 1$ to obtain an equivalent optimal solution.

For the case of $\lambda_{B'} < 0$, we again consider $L(b)$. If $R_b > 0$, then by complementary slackness $r_b = 0$ and $\alpha_b = 0$, so

$$L(b) = -V_b \beta_b + \sum_{g \in G_b} L(g) + \lambda_b R_b.$$

Let G_b^- be the set of connected generators operating at minimum generation and let

$$Q_b^- := \sum_{g \in G_b^-} P_g^-.$$

We have

$$L(b) = -V_b \beta_b - Q_b^- \lambda_b + \lambda_b R_b.$$

By inspection, the optimal value for β_b given λ_b is

$$\beta_b = \max(-1 - \lambda_b, 0),$$

and thus

$$L(b) = -V_b \max(-1 - \lambda_b, 0) - Q_b^- \lambda_b + \lambda_b R_b.$$

For $\lambda_b \leq -1$, $L(b)$ is non-decreasing in λ_b , as $Q_b^- \leq V_b$. Thus any optimal solution with $\lambda_{B'} < -1$ can be adjusted by increasing $\lambda_{B'}$ by a constant amount until $\max_{b \in B'} \lambda_b = -1$. $\square$

Lemma 5 For any disconnected bus $b \in B$, an optimal solution to (DCOPF-LDR) exists satisfying

$$|\lambda_b| \leq 1.$$

Proof The arguments are similar to the previous proof. Consider an optimal solution to (DCOPF-LDR). If $\lambda_b > 1$, then

$$L(b) = -R_b \alpha_b + \lambda_b R_b.$$

For a given λ_b , the optimal value for α_b according to the objective function and (2n) is given by

$$\alpha_b = \max(0, \lambda_b - 1),$$

so

$$L(b) = -R_b \max(0, \lambda_b - 1) + \lambda_b R_b.$$

Since $L(b)$ is non-increasing in λ_b for $\lambda_b \geq 1$, the solution can be adjusted by reducing λ_b to a value of one without impacting the objective value.

If $\lambda_b < -1$, then

$$L(b) = -V_b \beta_b + \lambda_b R_b.$$

The optimal value for β_b given λ_b is

$$\beta_b = \max(-1 - \lambda_b, 0),$$

and thus

$$L(b) = -V_b \max(-1 - \lambda_b, 0) + \lambda_b R_b.$$

Note that $L(b)$ is non-decreasing in λ_b . Thus the solution can be adjusted by setting $\lambda_b = -1$ without impacting the objective value. $\square$

Lemma 6 An optimal solution to (DCOPF-LDR) can always be found with values of λ_b satisfying

$$|\lambda_b| \leq 1 + R' \max_{l \in L} \left\{ \frac{1}{F_l \sqrt{X_l}} \right\} \left(\sum_{l \in L} X_l \right)^{\frac{1}{2}}, \quad \forall b \in B.$$

Proof The result holds for disconnected buses by the previous lemma. For connected buses, from (8e), we have

$$\mathbf{X}_K^{-1/2} N_K^T \lambda_K = H \mathbf{X}_K^{-1/2} (\phi_K^- - \phi_K^+).$$

Then,

$$\begin{aligned} \|\mathbf{X}_K^{-1/2} N_K^T \lambda_K\|_2 &= \|H \mathbf{X}_K^{-1/2} (\phi_K^- - \phi_K^+)\|_2 \\ &\leq \|\mathbf{X}_K^{-1/2} (\phi_K^- - \phi_K^+)\|_2 \\ &\leq R' \max_{l \in K} \left\{ \frac{1}{F_l \sqrt{X_l}} \right\}, \end{aligned} \tag{12a}$$

where we have used that H is symmetric and idempotent, the inequality from (8f), and the result from Lemma 3. For all $l \in K$, let $d_l := \lambda_{o(l)} - \lambda_{d(l)}$. From Lemma 4, if the first condition holds, then $\|\lambda_K\|_\infty$ is upper bounded by $\sum_{l \in K} |d_l|$. If the second condition holds, $\|\lambda_K\|_\infty$ is bounded by $1 + \sum_{l \in K} |d_l|$. We have that $\sum_{l \in K} |d_l|$ is bounded by the mathematical program

$$\begin{aligned} \max \quad & \sum_{l \in K} |d_l| \\ \text{s.t.} \quad & \sum_{l \in K} \frac{d_l^2}{X_l} \leq \tilde{R}^2, \end{aligned}$$

where $\tilde{R} := R' \max_{l \in L} \left\{ \frac{1}{F_l \sqrt{X_l}} \right\}$ and the constraint represents (12a). Using Lagrange multipliers, it can be shown that the value of the program is equal to $\tilde{R} \left(\sum_{l \in K} X_l \right)^{\frac{1}{2}}$. The result follows since $K \subseteq L$. $\square$

Bounds for the big- M values M_l , N_l , U_l , and W_g follow easily from the above results. Since $|\bar{\eta}_l| \leq \max(|\phi_l^+|, |\phi_l^-|)$, the values for M_l are given by Lemma 2. For N_l , bounds on the difference $\bar{\eta}_l - \eta_l$ are given by Lemma 7 below. Lemma 3 gives the bounds for U_l . Finally, from constraints (9q) - (9r), it holds that $|\bar{\lambda}_g| \leq |\lambda_{b(g)}|$; thus Lemma 6 provides the bounds for W_g .

Lemma 7 An optimal solution to (DCOPF-LDR) can always be found with values of $\bar{\eta}_l - \eta_l$ satisfying

$$|\bar{\eta}_l - \eta_l| \leq 2 + R' \max_{t \in L} \left\{ \frac{1}{F_t \sqrt{X_t}} \right\} \left(\sqrt{X_l} + 2 \left(\sum_{l \in L} X_l \right)^{\frac{1}{2}} \right), \quad \forall l \in L.$$

Proof We begin by bounding the magnitude of η_l . By the relation (9l), Lemma 3, and Lemma 6, we have

$$|\eta_l| \leq 2 + R' \max_{t \in L} \left\{ \frac{1}{F_t \sqrt{X_t}} \right\} \left(\sqrt{X_l} + 2 \left(\sum_{l \in L} X_l \right)^{\frac{1}{2}} \right). \quad (13)$$

If $s'_l = 0$, then $\bar{\eta}_l = 0$ and $\bar{\eta}_l - \eta_l = -\eta_l$. Alternatively, if $s'_l = 1$, then $\bar{\eta}_l - \eta_l = 0$. $\square$

To incorporate phase angle bounds into the preceding analysis, we assume all phase angles have an absolute value of at most T . This adds to the primal problem

$$-T \leq \theta_b \leq T, \quad \forall b \in B, \quad (\chi_b^-, \chi_b^+) \quad (14a)$$

where χ^+, χ^- are the dual variables. The dual objective function becomes

$$\begin{aligned} \max \quad & - \sum_{l \in L} F_l(\phi_l^+ + \phi_l^-) - \sum_{b \in B} R_b \alpha_b - \sum_{b \in B} V_b \beta_b - \sum_{g \in G} P_g^+ \gamma_g^+ + \sum_{g \in G} P_g^- \gamma_g^- \\ & + \sum_{b \in B} \lambda_b R_b - T \sum_{b \in B} (\chi_b^+ + \chi_b^-) \end{aligned}$$

and constraint (2k) is replaced with

$$\sum_{l|o(l)=b} s'_l \mu_l - \sum_{l|d(l)=b} s'_l \mu_l - \chi_b^+ + \chi_b^- = 0, \quad \forall b \in B, \quad (\theta_b) \quad (14b)$$

where

$$\chi_b^+, \chi_b^- \geq 0, \quad \forall b \in B.$$

The following result replaces Lemma 1 to account for phase angle bounds, where R' has the same definition. Lemma 9 bounds the new dual variables, and Lemma 10 replaces Lemma 3. Bounds on λ_b and $\bar{\eta}_l - \eta_l$ can be determined in a fashion similar to the proofs of Lemma 6 and Lemma 7 with the appropriate bound on $\bar{\mu}_l$ from Lemma 10.

Lemma 8 With the presence of phase angle bounds (14a), an optimal solution to (DCOPF-LDR) always satisfies

$$\sum_{l \in L} (\phi_l^+ + \phi_l^-) F_l + T \sum_{b \in B} (\chi_b^+ + \chi_b^-) \leq R'.$$

Proof Repeating the proof of Lemma 1, starting with the following inequality, gives the result.

$$\begin{aligned} \sum_{l \in L} (\phi_l^+ + \phi_l^-) F_l + T \sum_{b \in B} (\chi_b^+ + \chi_b^-) &\leq - \sum_{b \in B} R_b \alpha_b - \sum_{b \in B} V_b \beta_b - \sum_{g \in G} P_g^+ \gamma_g^+ \\ &\quad + \sum_{g \in G} P_g^- \gamma_g^- + \sum_{b \in B} \lambda_b R_b. \end{aligned}$$

□

Lemma 9 With the presence of phase angle bounds (14a), an optimal solution to (DCOPF-LDR) can always be found with values of χ^+ and χ^- satisfying

$$\begin{aligned} \chi_b^+ &\leq \frac{R'}{T}, \quad \forall b \in B, \\ \chi_b^- &\leq \frac{R'}{T}, \quad \forall b \in B. \end{aligned}$$

Proof Utilizing Lemma 8, the proof is nearly identical to the proof of Lemma 2. □

Lemma 10 With the presence of phase angle bounds (14a), an optimal solution to (DCOPF-LDR) can always be found with values of $\bar{\mu}_l$ satisfying

$$|\bar{\mu}_l| \leq \left(\frac{R'}{\sqrt{X_l}} \right) \max_{t \in L} \left\{ \frac{1}{F_t \sqrt{X_t}} \right\} + \frac{R'}{T}, \quad \forall l \in L.$$

Proof Rewriting constraints (9j) - (9l), with (14b) in place of (2k), for components restricted to K gives

$$\begin{aligned} N_K \bar{\mu}_K &= \chi_K^+ - \chi_K^-, \\ -N_K^T \lambda_K - \mathbf{X}_K \bar{\mu}_K &= \phi_K^+ - \phi_K^-. \end{aligned} \tag{16a}$$

Multiplying $-N_K \mathbf{X}_K^{-1}$ to the left-hand side of (16a) gives

$$N_K \mathbf{X}_K^{-1} N_K^T \lambda_K = N_K \mathbf{X}_K^{-1} (\phi_K^- - \phi_K^+) - N_K \bar{\mu}_K.$$

Similar to the proof of Lemma 3, removing the same arbitrary row from N_K and λ_K , gives the equivalent system

$$\tilde{N}_K \mathbf{X}_K^{-1} \tilde{N}_K^T \tilde{\lambda}_K = \tilde{N}_K \mathbf{X}_K^{-1} (\phi_K^- - \phi_K^+) - \tilde{N}_K \bar{\mu}_K.$$

The grounding argument gives

$$\begin{aligned} \mathbf{X}_K^{-1/2} N_K^T \lambda_K &= \mathbf{X}_K^{-1/2} \tilde{N}_K^T \tilde{\lambda}_K \\ &= \mathbf{X}_K^{-1/2} \tilde{N}_K^T (\tilde{N}_K \mathbf{X}_K^{-1} \tilde{N}_K^T)^{-1} \tilde{N}_K \mathbf{X}_K^{-1} (\phi_K^- - \phi_K^+) \end{aligned}$$

$$\begin{aligned}
& -\mathbf{X}_K^{-1/2} \tilde{N}_K^T (\tilde{N}_K \mathbf{X}_K^{-1} \tilde{N}_K^T)^{-1} \tilde{N}_K \bar{\mu}_K \\
& = H_K \mathbf{X}_K^{-1/2} (\phi_K^- - \phi_K^+) - \mathbf{X}_K^{1/2} J_K (\tilde{\chi}_K^+ - \tilde{\chi}_K^-)
\end{aligned}$$

where the matrix

$$H_K := \mathbf{X}_K^{-1/2} \tilde{N}_K^T (\tilde{N}_K \mathbf{X}_K^{-1} \tilde{N}_K^T)^{-1} \tilde{N}_K \mathbf{X}_K^{-1/2}$$

is symmetric and idempotent and

$$J_K := \mathbf{X}_K^{-1} \tilde{N}_K^T (\tilde{N}_K \mathbf{X}_K^{-1} \tilde{N}_K^T)^{-1}$$

is the PTDF matrix for component K . Multiplying $\mathbf{X}_K^{-1/2}$ to the left-hand side of all terms in (16a) gives

$$\begin{aligned}
\mathbf{X}_K^{1/2} \bar{\mu}_K &= -\mathbf{X}_K^{-1/2} N_K^T \lambda_K + \mathbf{X}_K^{-1/2} (\phi_K^- - \phi_K^+) \\
&= -H_K \mathbf{X}_K^{-1/2} (\phi_K^- - \phi_K^+) + \mathbf{X}_K^{1/2} J_K (\tilde{\chi}_K^+ - \tilde{\chi}_K^-) + \mathbf{X}_K^{-1/2} (\phi_K^- - \phi_K^+) \\
&= (I - H_K) (\mathbf{X}_K^{-1/2} (\phi_K^- - \phi_K^+)) + \mathbf{X}_K^{1/2} J_K (\tilde{\chi}_K^+ - \tilde{\chi}_K^-).
\end{aligned}$$

Looking at the l th component of $\mathbf{X}_K^{1/2} \bar{\mu}_K$,

$$|X_l^{1/2} \bar{\mu}_l| \leq \left| \left((I - H_K) \mathbf{X}_K^{-1/2} (\phi_K^- - \phi_K^+) \right)_l \right| + \left| \left(\mathbf{X}_K^{1/2} J_K (\tilde{\chi}_K^+ - \tilde{\chi}_K^-) \right)_l \right|. \quad (16b)$$

Note that the first term in (16b) is bounded within the proof of Lemma 3 and we have

$$\begin{aligned}
\left| \left((I - H_K) \mathbf{X}_K^{-1/2} (\phi_K^- - \phi_K^+) \right)_l \right| &\leq \| (I - H_K) \mathbf{X}_K^{-1/2} (\phi_K^- - \phi_K^+) \|_2 \\
&\leq R' \max_{l \in L} \left\{ \frac{1}{F_l \sqrt{X_l}} \right\}.
\end{aligned}$$

For the second term, we have

$$\begin{aligned}
\left| \left(J_K (\tilde{\chi}_K^+ - \tilde{\chi}_K^-) \right)_l \right| &= \left| \sum_{b \in K} (J_K)_{l,b} (\chi_b^+ - \chi_b^-) \right| \\
&\leq \sum_{b \in K} |(J_K)_{l,b}| |\chi_b^+ - \chi_b^-| \\
&\leq \sum_{b \in K} |\chi_b^+ - \chi_b^-| \\
&\leq \sum_{b \in B} (\chi_b^+ + \chi_b^-) \\
&\leq \frac{R'}{T},
\end{aligned} \quad (16c)$$

where (16c) follows since a PTDF matrix on a network with positive reactances has entries with magnitude at most one, and the last inequality follows from Lemma 8. Substituting into (16b) gives

$$\sqrt{X_l} |\bar{\mu}_l| \leq R' \max_{l \in L} \left\{ \frac{1}{F_l \sqrt{X_l}} \right\} + \sqrt{X_l} \left(\frac{R'}{T} \right).$$

□

4 Spectral Bounds for Grounded Laplacians

We now consider another approach for obtaining valid big- M values in the interdiction setting presented above. With this, we aim to provide an alternative to Lemma 6 that generalizes well to other interdiction problems on graphs, as we demonstrate in Section 6 below. This new approach is motivated by the fact that the interdiction problems that we consider in this paper are always defined on top of an underlying graph, from which we may be able to extract useful information. Additionally, the constraints in (DCOPF-DR) involving dual variables that we must bound for our formulation depend *directly* on the topology of connected subgraphs of the underlying graph via their incidence matrices. We demonstrate that there is a natural algebraic manipulation of these constraints that yields valid big- M values, provided that one has access to a lower bound on the minimal eigenvalue of the so-called grounded Laplacian of the underlying graph.

In this section, we develop the necessary spectral results to carry out the method alluded to above. Importantly, we expect that our results may be of independent interest to researchers in spectral graph theory, social network analysis, and other fields of operations research that may encounter grounded Laplacians. In Theorem 1 and Theorem 2, we develop two novel bounds on the minimal eigenvalues of grounded Laplacians, both of which are tight for certain classes of graphs. Although the former bound is more useful in the setting of interdiction (i.e., in the worst-case setting), the latter bound provides better control over the minimal eigenvalues of grounded Laplacians of dense graphs.

We also extend our bounds to *signless* grounded Laplacians; while we do not require spectral bounds for these matrices for DCOPF, such results are needed when we study second-order cone relaxations of ACOPF in Section 6.2 below.

To provide motivation for the results presented in this section, we first present the manner in which an algebraic manipulation of the constraints of (DCOPF-DR) yields a natural dependence on the minimal eigenvalue of a grounded Laplacian. Later, in Section 6, we generalize this technique significantly, illustrating its applicability to two additional interdiction models that are more complex than DCOPF.

The following analysis assumes that there are no bounds on phase angles in the primal DCOPF problem. To begin, we require an intermediate lemma, using the same notation as Lemma 6.

Lemma 11 Given a connected component K of $L \setminus A$ for a fixed attack A , as in Lemma 6, there exists a dual-optimal point $\lambda \in \mathbb{R}^{|B|}$ such that

$$\begin{aligned} \|\tilde{\lambda}_K\|_\infty &\leq R' \|(\tilde{N}_K \tilde{N}_K^\top)^{-1} \tilde{N}_K\|_2 \left(\max_{l \in L} \sqrt{X_l} \right) \left(\max_{l \in L} \left\{ \frac{1}{F_l \sqrt{X_l}} \right\} \right) \\ &= \frac{R'}{\sigma_{\min}(\tilde{N}_K^\top)} \left(\max_{l \in L} \sqrt{X_l} \right) \left(\max_{l \in L} \left\{ \frac{1}{F_l \sqrt{X_l}} \right\} \right), \end{aligned}$$

where $\sigma_{\min}(\tilde{N}_K^\top)$ denotes the minimal singular value of $\tilde{N}_K^\top$.

Proof First, we remark that if K is a connected component, then $\tilde{N}_K^\top$ has full column-rank so that $\sigma_{\min}(\tilde{N}_K^\top) > 0$. Now, recall from (8a) that

$$N_K^\top \lambda_K = -\mathbf{X}_K \bar{\mu}_K + (\phi_K^- - \phi_K^+).$$

By again dropping an arbitrary row of N_K to form a matrix $\tilde{N}_K$ with full row-rank, we can ensure that $\tilde{N}_K^\top$ has a well-defined left inverse, given by its pseudoinverse

$$(\tilde{N}_K^\top)^\dagger := (\tilde{N}_K \tilde{N}_K^\top)^{-1} \tilde{N}_K \in \mathbb{R}^{(n-1) \times m}.$$

Note that from (8e), we have

$$\mathbf{X}_K^{-1/2} \tilde{N}_K^\top \tilde{\lambda}_K = H \mathbf{X}_K^{-1/2} (\phi_K^- - \phi_K^+),$$

where H is symmetric and idempotent, which gives

$$\|\tilde{N}_K^\top \tilde{\lambda}_K\|_2 \leq \left(\max_{l \in L} \sqrt{X_l} \right) \left(\max_{l \in L} \left\{ \frac{R'}{F_l \sqrt{X_l}} \right\} \right).$$

We conclude by noting that

$$\begin{aligned} \|\tilde{\lambda}_K\|_\infty &\leq \|\tilde{\lambda}_K\|_2 = \|(\tilde{N}_K^\top)^\dagger \tilde{N}_K^\top \tilde{\lambda}_K\|_2 \\ &\leq \|(\tilde{N}_K^\top)^\dagger\|_2 \|\tilde{N}_K^\top \tilde{\lambda}_K\|_2 \\ &\leq \|(\tilde{N}_K^\top)^\dagger\|_2 \left(\max_{l \in L} \sqrt{X_l} \right) \left(\max_{l \in L} \left\{ \frac{R'}{F_l \sqrt{X_l}} \right\} \right) \\ &= R' \|(\tilde{N}_K \tilde{N}_K^\top)^{-1} \tilde{N}_K\|_2 \left(\max_{l \in L} \sqrt{X_l} \right) \left(\max_{l \in L} \left\{ \frac{1}{F_l \sqrt{X_l}} \right\} \right). \end{aligned} \quad (18a)$$

The conclusion now follows immediately, as the spectral norm of the Moore–Penrose pseudoinverse of a matrix $A \in \mathbb{R}^{m \times (n-1)}$ with full column-rank is simply given by

$$\|(A^\top A)^{-1} A^\top\|_2 = \|A^\dagger\|_2 = \frac{1}{\sigma_{\min}(A)},$$

where $\sigma_{\min}(A)$ denotes the minimal nonzero singular value of A . $\square$

By combining Lemma 11 and Lemma 4, we obtain the following stronger result. Below, we note that $\tilde{N}_K^\top$ depends on the choice of bus removed in the argument from Lemma 3, and as a result, $\sigma_{\min}(\tilde{N}_K^\top)$ also depends on this choice. To make this dependence explicit, we write $\tilde{N}_b(K)^\top$ to denote the incidence matrix induced by removing bus $b \in B'$, where B' is the set of buses in K .

Lemma 12 Given a connected component K of $L \setminus A$ for a fixed attack A , there exists a dual-optimal point $\lambda \in \mathbb{R}^{|B|}$ such that

$$\|\lambda_K\|_\infty \leq \frac{R'}{\min_{b \in B'} \sigma_{\min}(\tilde{N}_b(K)^\top)} \left(\max_{l \in L} \sqrt{X_l} \right) \left(\max_{l \in L} \left\{ \frac{1}{F_l \sqrt{X_l}} \right\} \right) + 1.$$

Proof As in Lemma 3, we may take

$$\hat{\lambda}_K = \lambda_K - \lambda_b \mathbf{1}$$

for some $b \in B'$. Let $\tilde{\lambda}_K$ be the result of removing bus b from $\hat{\lambda}_K$, with $\tilde{N}_b(K)^\top$ the corresponding incidence matrix. Now, by Lemma 4, we also know that at least one of the following must hold:

- (1) There exists $b^- \in B'$ such that $\lambda_{b^-} \leq 0$ and $b^+ \in B'$ such that $\lambda_{b^+} \geq 0$.
- (2) There exists $b_1 \in B'$ such that $|\lambda_{b_1}| \leq 1$.

If condition (2) holds, then observe that

$$\|\lambda_K - \lambda_{b_1} \mathbf{1}\|_\infty \leq \frac{R'}{\sigma_{\min}(\tilde{N}_{b_1}(K)^\top)} \left(\max_{l \in L} \sqrt{X_l} \right) \left(\max_{l \in L} \left\{ \frac{1}{F_l \sqrt{X_l}} \right\} \right)$$

implies that

$$\|\lambda_K\|_\infty \leq \frac{R'}{\sigma_{\min}(\tilde{N}_{b_1}(K)^\top)} \left(\max_{l \in L} \sqrt{X_l} \right) \left(\max_{l \in L} \left\{ \frac{1}{F_l \sqrt{X_l}} \right\} \right) + 1$$

by triangle inequality. If condition (1) holds, then for all $b \in B'$ such that $\lambda_b \geq 0$, we have that

$$\lambda_b - \lambda_{b^-} \leq \frac{R'}{\sigma_{\min}(\tilde{N}_{b^-}(K)^\top)} \left(\max_{l \in L} \sqrt{X_l} \right) \left(\max_{l \in L} \left\{ \frac{1}{F_l \sqrt{X_l}} \right\} \right),$$

which implies that

$$\lambda_b = |\lambda_b| \leq \frac{R'}{\sigma_{\min}(\tilde{N}_{b^-}(K)^\top)} \left(\max_{l \in L} \sqrt{X_l} \right) \left(\max_{l \in L} \left\{ \frac{1}{F_l \sqrt{X_l}} \right\} \right).$$

On the other hand, for all $b \in B'$ such that $\lambda_b \leq 0$, we have that

$$\begin{aligned} |\lambda_{b^+} - \lambda_b| &= \lambda_{b^+} - \lambda_b \\ &\leq \frac{R'}{\sigma_{\min}(\tilde{N}_{b^+}(K)^\top)} \left(\max_{l \in L} \sqrt{X_l} \right) \left(\max_{l \in L} \left\{ \frac{1}{F_l \sqrt{X_l}} \right\} \right) \end{aligned}$$

so that

$$-\lambda_b = |\lambda_b| \leq \frac{R'}{\sigma_{\min}(\tilde{N}_{b^+}(K)^\top)} \left(\max_{l \in L} \sqrt{X_l} \right) \left(\max_{l \in L} \left\{ \frac{1}{F_l \sqrt{X_l}} \right\} \right).$$

In particular, for *all* $b \in B'$, it follows that

$$|\lambda_b| \leq \frac{R'}{\min_{b \in B'} \sigma_{\min}(\tilde{N}_b(K)^\top)} \left(\max_{l \in L} \sqrt{X_l} \right) \left(\max_{l \in L} \left\{ \frac{1}{F_l \sqrt{X_l}} \right\} \right) + 1$$

as claimed. $\square$

The bound in Lemma 12 still depends on $\sigma_{\min}(\tilde{N}_K^\top)$. This quantity is determined by the topology of the underlying graph and the attacker's choice of interdiction. Thus, to obtain a tractable bound from Lemma 12, a universal lower bound on $\sigma_{\min}(\tilde{N}_K^\top)$ is required. The results that follow provide such a lower bound utilizing tools from spectral graph theory. First, we recall the necessary definitions.

Definition 1 Given an undirected graph $G = (V, E)$, with vertex set V and edge set E , the Laplacian of G is given by $L(G) = D(G) - A(G)$. In turn, fixing a vertex $v \in V$, the grounded Laplacian of G with respect to v , denoted by $L_v(G)$, results from removing the row and column of $L(G)$ corresponding to v .

Both the Laplacian and the grounded Laplacian of an undirected graph have an alternative interpretation in terms of incidence matrices. Specifically, if $G = (V, E)$ has incidence matrix $N(G) \in \mathbb{R}^{|V| \times |E|}$, formed by placing an arbitrary orientation

on the edges of G , then $L(G) = N(G)N(G)^\top$. Similarly, removing a row from $N(G)$, corresponding to a vertex $v \in V$, yields a matrix $N_v(G) \in \mathbb{R}^{(|V|-1) \times |E|}$ such that $L_v(G) = N_v(G)N_v(G)^\top$ [21, Lemma 8.3.2]. Consequently, it is immediate that

$$\sigma_{\min}(N_v(G)) = \sqrt{\nu_{\min}(L_v(G))},$$

where $\nu_{\min}(L_v(G))$ denotes the minimal eigenvalue of $L_v(G)$.

Remark 1 The non-standard notation for eigenvalues in this section is intended to avoid notational clash with the variables in (DCOPF-DR) that we aim to bound.

Unifying notation, the matrix $\tilde{N}_K$ above is precisely $N_v(K)$, where K is a connected component of the graph that remains upon interdiction and $v \in K$ is an unknown vertex. With this in mind, we often write $\tilde{L}$ to denote a grounded Laplacian, with $\tilde{L} = \tilde{N}\tilde{N}^\top$, when the grounding vertex $v \in V$ is arbitrary. At this point, we can state our first bound on the minimal eigenvalue of a grounded Laplacian.

Lemma 13 Suppose that $G = (V, E)$ is a connected graph on $|V| = n \geq 2$ vertices. If $\tilde{L}$ is an arbitrary grounded Laplacian of G , then there exists a connected graph $G' = (V', E')$ on $|V'| = 2n - 1$ vertices such that

$$\nu_{n-1}(\tilde{L}) \geq \nu_{2n-2}(L') > 0,$$

where L' is the Laplacian of G' .

Proof To construct G' , take two disjoint copies $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ of the original graph G . If the vertex removed from G to form the grounded Laplacian $\tilde{L}$ is $v \in V$, then connect G_1 and G_2 at v . As a result, $V' = (V_1 \setminus \{v\}) \cup (V_2 \setminus \{v\}) \cup \{v\}$, and the edge set of G' is simply given by $E' = E_1 \cup E_2$. Consequently, $G' = (V', E')$ is a graph on $2n - 1$ vertices.

Now, let L' denote the Laplacian of G' . For any vector $x \in \mathbb{R}^{n-1} \setminus \{0\}$, we observe that

$$x^\top \tilde{L} x = x^\top \tilde{N} \tilde{N}^\top x = \sum_{(i,j) \in E, i,j \neq v} (x_i - x_j)^2 + \sum_{(i,v) \in E} x_i^2,$$

where $\tilde{N}$ is the result of removing the row corresponding to $v \in V$ from the node-arc incidence matrix N of G . On the other hand, we enumerate the vertices of G' , given by $V' = \{v'_1, \dots, v'_{2n-1}\}$, so that $\{v'_1, \dots, v'_{n-1}\} = V_1 \setminus \{v\}$, $v = v'_n$, and $\{v'_{n+1}, \dots, v'_{2n-1}\} = V_2 \setminus \{v\}$. In turn, define $y \in \mathbb{R}^{2n-1}$ by

$$y_i = \begin{cases} x_i & 1 \leq i \leq n-1, \\ 0 & i = n, \\ -x_{i-n} & n+1 \leq i \leq 2n-1. \end{cases}$$

By construction, we observe that

$$\mathbf{1}^\top y = \sum_{i=1}^{2n-1} y_i = \sum_{i=1}^{n-1} x_i - \sum_{i=1}^{n-1} x_i = 0$$

so that y is orthogonal to the nullspace of L' , spanned by the vector of all ones. Moreover, we can explicitly compute

$$\begin{aligned} y^\top L' y &= \sum_{(i,j) \in E_1, i,j \neq v} (x_i - x_j)^2 + \sum_{(i,j) \in E_2, i,j \neq v} (x_i - x_j)^2 + \sum_{(i,v) \in E_1} x_i^2 + \sum_{(i,v) \in E_2} x_i^2 \\ &= 2x^\top \tilde{L} x, \end{aligned}$$

while it is immediately clear that $y^\top y = 2x^\top x$. Thus,

$$\frac{x^\top \tilde{L} x}{x^\top x} = \frac{y^\top L' y}{y^\top y}.$$

If we define

$$W := \{y \in \mathbb{R}^{2n-1} \setminus \{0\} : y^\top \mathbf{1} = 0\},$$

then

$$\frac{x^\top \tilde{L} x}{x^\top x} = \frac{y^\top L' y}{y^\top y} \geq \inf_{w \in W} \frac{w^\top L' w}{w^\top w} = \nu_{2n-2}(L'),$$

applying the Courant–Fischer min-max theorem in the last equality. Applying the same result to the left-hand side in the previous display, this time taking the infimum over all nonzero $x \in \mathbb{R}^{n-1}$, we find that

$$\nu_{n-1}(\tilde{L}) \geq \nu_{2n-2}(L')$$

as claimed. $\square$

As an immediate corollary to Lemma 13, we obtain the following *worst-case* lower bound on the smallest eigenvalue of a grounded Laplacian.

Theorem 1 If $G = (V, E)$ is a connected graph on $|V| = n \geq 2$ vertices and $\tilde{L}$ is an arbitrary grounded Laplacian of G , then

$$\nu_{n-1}(\tilde{L}) \geq 2 \left(1 - \cos \left(\frac{\pi}{2n-1} \right) \right).$$

Proof From Lemma 13, there exists a connected graph G' on $2n-1$ vertices, with Laplacian L' , such that

$$\nu_{n-1}(\tilde{L}) \geq \nu_{2n-2}(L').$$

Now, $\nu_{2n-2}(L')$ is the algebraic connectivity of G' . Among the set of all connected graphs on $2n-1$ vertices, algebraic connectivity is minimized by the path graph P_{2n-1} on $2n-1$ vertices so that

$$\nu_{n-1}(\tilde{L}) \geq \nu_{2n-2}(L') \geq \nu_{2n-2}(P_{2n-1}) = 2 \left(1 - \cos \left(\frac{\pi}{2n-1} \right) \right),$$

with the expression for $\nu_{2n-2}(P_{2n-1})$ coming from [22]. $\square$

Recalling that $\tilde{L} = \tilde{N} \tilde{N}^\top$, we immediately obtain

$$\sigma_{n-1}(\tilde{N}) \geq \sqrt{2 \left(1 - \cos \left(\frac{\pi}{2n-1} \right) \right)} = \Theta(n^{-1}),$$

taking a Taylor expansion of cosine to obtain the asymptotics of the lower bound on $\sigma_{n-1}(\tilde{N})$. For the purposes of converting the bound in Lemma 12 into an inequality that does not depend on K , the connected component that results from the attacker's choice of interdiction, this result suffices and yields the following big- M values.

Lemma 14 Given a connected component K of $L \setminus A$ for a fixed attack A , there exists a dual-optimal point $\lambda \in \mathbb{R}^{|B|}$ such that

$$\|\lambda_K\|_\infty \leq R' \left\{ 2 \left(1 - \cos \left(\frac{\pi}{2n-1} \right) \right) \right\}^{-1/2} \left(\max_{l \in L} \sqrt{X_l} \right) \left(\max_{l \in L} \left\{ \frac{1}{F_l \sqrt{X_l}} \right\} \right) + 1.$$

If $R' = O(n)$, as is typically the case from Lemma 1, then we obtain a big- M bound on λ that satisfies $\|\lambda\|_\infty \leq O(n^2)$ from Theorem 1, where n is the number of vertices in the underlying graph.

For the sake of completeness, we present a *different* lower bound on the minimal eigenvalue of a grounded Laplacian that, while not optimal in the worst-case setting, will outperform Theorem 1 for certain dense graphs. Specifically, if the connected component K is known a priori, this alternative bound can be preferable.

Several important results connect the spectra of Laplacians and grounded Laplacians. For instance, the following two forms of Kirchhoff's matrix tree theorem, one of the most fundamental results of spectral graph theory, form an elegant connection between spanning trees of a connected graph and the spectrum of its Laplacian (resp. an arbitrary grounded Laplacian).

Proposition 1 (Theorem 13.2.1, [21]) Let G be a graph on n vertices with Laplacian $L = L(G) \in \mathbb{R}^{n \times n}$. Fix an arbitrary vertex v , and remove the corresponding row and column from L to form the grounded Laplacian $\tilde{L} := L_v(G)$. Then, if $\tau(G)$ denotes the number of spanning trees of G , we have that

$$\tau(G) = \det(\tilde{L}) = \prod_{i=1}^{n-1} \nu_i(\tilde{L}).$$

Note that the above proposition still holds, even if G is disconnected. Indeed, $\tau(G) = 0$ for a disconnected graph, as do both of the characterizations of the number of spanning trees in terms of $\tilde{L}$. We can also characterize $\tau(G)$ *directly* in terms of the spectrum of L .

Proposition 2 (Lemma 13.2.4, [21]) Let G be a graph on n vertices, with the eigenvalues of its Laplacian $L = L(G)$ ordered as $\nu_1(L) \geq \dots \geq \nu_{n-1}(L) \geq \nu_n = 0$. Then, we have that

$$\tau(G) = \frac{1}{n} \prod_{i=1}^{n-1} \nu_i(L).$$

Evidently, combining Proposition 1 and Proposition 2 reveals that, given a connected graph G on n vertices,

$$\prod_{i=1}^{n-1} \nu_i(\tilde{L}) = \frac{1}{n} \prod_{i=1}^{n-1} \nu_i(L),$$

where $\tilde{L}$ is an arbitrary grounded Laplacian of G . To provide a nontrivial bound on the singular values of a grounded Laplacian, we require one more intermediate result from linear algebra: Cauchy's interlacing theorem.

Proposition 3 (Theorem 4.3.17, [23]) Let $B \in \mathbb{R}^{n \times n}$ be a symmetric matrix, with $y \in \mathbb{R}^n$ and $a \in \mathbb{R}$. Taking

$$A = \begin{bmatrix} B & y \\ y^\top & a \end{bmatrix} \in \mathbb{R}^{(n+1) \times (n+1)},$$

we have that

$$\nu_1(A) \geq \nu_1(B) \geq \nu_2(A) \geq \dots \geq \nu_n(A) \geq \nu_n(B) \geq \nu_{n+1}(A),$$

with the eigenvalues of both matrices sorted in decreasing order.

Equipped with the preceding results, we can now state and prove our second spectral bound for grounded Laplacians.

Theorem 2 Let G be a connected graph on n vertices. If N denotes the node-arc incidence matrix of G and $\tilde{N}$ denotes the node-arc incidence matrix with an arbitrary row removed, then

$$\sigma_{n-1}(\tilde{N}) \geq n^{-1/2} \sigma_{n-1}(N).$$

Proof As hinted at above, we combine the results of Proposition 1 and Proposition 2. In particular, if $L = NN^\top$ and $\tilde{L} = \tilde{N}\tilde{N}^\top$ denote the Laplacian and an arbitrary grounded Laplacian of G respectively, we have that

$$\prod_{i=1}^{n-1} \nu_i(\tilde{L}) = \tau(G) = \frac{1}{n} \prod_{i=1}^{n-1} \nu_i(L).$$

Isolating $\nu_{n-1}(\tilde{L}) = \sigma_{n-1}(\tilde{N})^2$, observe that

$$\nu_{n-1}(\tilde{L}) = \left(\frac{1}{n} \prod_{i=1}^{n-2} \frac{\nu_i(L)}{\nu_i(\tilde{L})} \right) \nu_{n-1}(L).$$

Now, recall that $\tilde{L}$ can be equivalently viewed as a principal $(n-1) \times (n-1)$ submatrix of L . Thus, the interlacing result from Proposition 3 applies, revealing that

$$\nu_i(L) \geq \nu_i(\tilde{L}) > 0$$

for all $i = 1, \dots, n-1$. We have that

$$\prod_{i=1}^{n-2} \frac{\nu_i(L)}{\nu_i(\tilde{L})} \geq 1$$

so that

$$\nu_{n-1}(\tilde{L}) \geq \frac{1}{n} \nu_{n-1}(L).$$

Taking the square root of both sides of the above equation yields the result. $\square$

Results analogous to Theorem 2 have arisen in both automatic control and social network analysis; see [24] for instance. In particular, in automatic control, the matrix

$$L_v := \tilde{N}_K \tilde{N}_K^\top$$

is labeled the *grounded* Laplacian of a connected graph, with grounding set of size $|S| = 1$. The minimal eigenvalue of L_v is of particular interest in automatic control systems because it describes the rate of convergence in multi-agent control systems with leader-follower dynamics [25]. To the best of our knowledge, [26] provides the best available bounds on the minimal eigenvalue of L_v .

Proposition 4 (Theorem 1, [26]) Consider a connected graph $G = (V, E)$, and select a grounded vertex $v \in V$. Let ν_{n-1} denote the smallest eigenvalue of the grounded Laplacian $\tilde{L}$ and x the corresponding nonnegative eigenvector, normalized so that $\max_{i=1, \dots, n-1} x_i = 1$. Then, writing $x_{\min} := \min_{i=1, \dots, n} x_i$,

$$\frac{d_v}{n-1} x_{\min} \leq \nu_{n-1} \leq \frac{d_v}{n-1},$$

where d_v is the degree of the grounded vertex v .

The above result relies on the fact that, due to the Perron–Frobenius theorem, the eigenvector of $\tilde{L}$ corresponding to its minimal eigenvalue can be taken to have non-negative components. Furthermore, if removing the grounded vertex and its incident edges does not result in a graph with multiple connected components, then the components of the eigenvector x may be chosen to be positive [26]. Next, [26, Lemma 1] derives a bound on $x_{\min}$, the minimal component of the (ℓ^∞ -normalized) eigenvector corresponding to ν_{n-1} . First, enumerate the vertices of V as $(v_1, \dots, v_n)$, and assume without loss of generality that $v = v_n$. We define, for $i \in \{1, \dots, n-1\}$,

$$\delta_i := \begin{cases} 1 & (v_i, v_n) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Taking $\Delta := \text{diag}(\delta_1, \dots, \delta_{n-1})$, we can write $\tilde{L} = \bar{L} + \Delta$, where $\bar{L}$ is the Laplacian of the graph formed upon removing vertex v and all its incident edges from G . If $\bar{L}$ corresponds to a connected graph, as is the case if the vertex connectivity of G exceeds one, then [26] provides the following:

Proposition 5 (Lemma 1, [26]) Let $\bar{L} \in \mathbb{R}^{(n-1) \times (n-1)}$ be such that $\tilde{L} = \bar{L} + \Delta$, as defined above, and x the eigenvector corresponding to ν_{n-1} , as in Proposition 4. Then,

$$x_{\min} \geq 1 - \frac{2\sqrt{d_v}}{\nu_{n-2}(\bar{L})},$$

where $\nu_{n-2}(\bar{L}) > 0$ is the smallest nonzero eigenvalue of the modified Laplacian $\bar{L}$ and d_v is again the degree of the grounded vertex v .

Here, we remark that for graphs where removing a single vertex results in multiple connected components, the above proposition does not apply. Additionally, the bound on $x_{\min}$ is only useful in the case that $\nu_{n-2}(\bar{L}) > 2\sqrt{d_v}$, which need not be the case even when $\bar{L}$ describes a connected graph. For instance, take L to be the Laplacian of the chordless cycle graph on n vertices. Then, $\bar{L}$ is the Laplacian of the path graph on $n-1$ vertices. In turn, we have that

$$\nu_{n-2}(\bar{L}) = 2 \left(1 - \cos \left(\frac{\pi}{n-1} \right) \right) < 2\sqrt{2} = 2\sqrt{d_v},$$

even though $\bar{L}$ still describes a connected graph, with the above characterization of the algebraic connectivity of the path graph coming from [22]. Thus, in many cases, the result of Proposition 5 provides trivial lower bounds when substituted into Proposition 4.

Given the result of Theorem 2, we return to the bound in Lemma 12. In particular, it remains to establish a tractable bound that holds for any connected component K of a graph after interdiction. Because

$$\sigma_{n_K-1}(\tilde{N}_K) \geq n_K^{-1/2} \sigma_{n_K-1}(N_K),$$

it suffices to establish a lower bound on $\sigma_{n_K-1}(N_K) = \sqrt{\nu_{n_K-1}(L_K)} = \sqrt{\alpha(K)}$, where

$$\alpha(K) := \nu_{n_K-1}(L_K)$$

denotes the algebraic connectivity of K . Here, we distinguish n_K , the number of vertices in a connected component K , from n , which we will use to denote the number of vertices in the underlying graph (prior to interdiction). In turn, we aim to obtain an *optimal* lower bound on $\sigma_{n_K-1}(N_K)$ that only depends on n and other invariants of the underlying graph. To begin, observe that $n_K \leq n$ for any connected component K . On the other hand, because the path graph minimizes algebraic connectivity (see [22]), we have that

$$\alpha(K) \geq 2(1 - \cos(\pi/n_K)) \geq 2(1 - \cos(\pi/n)).$$

Thus, a Taylor expansion of cosine again yields

$$\sigma_{n_K-1}(\tilde{N}_K) \geq n^{-1/2} \sqrt{(\pi^2 n^{-2} + O(n^{-4}))} = \Theta(n^{-3/2}).$$

Converting this into a bound akin to that of Lemma 14, we would obtain $\|\lambda_K\|_\infty \leq O(n^{5/2})$. In the setting of interdiction, it appears difficult to systematically improve this bound without any knowledge of the topology of the connected component K . More specifically, if we could instead find the connected subgraph of a graph that minimizes algebraic connectivity (rather than relying on the coarse lower bound provided by the path graph), then it would also be possible to obtain a better bound in Lemma 14. Unfortunately, this is an NP-hard problem.

Lemma 15 The following decision problem is NP-complete: given a connected graph G on $n \geq 2$ vertices, does there exist a connected subgraph $K \subseteq G$ with $\alpha(K) \leq 2(1 - \cos(\pi/n))$?

Proof We reduce from the Hamiltonian path decision problem: given a graph $G = (V, E)$, does G contain a Hamiltonian path? Recall that a Hamiltonian path is a path that visits every vertex of G exactly once, and the above decision problem is known to be NP-complete for general connected graphs G ; see [27, Section A1.3].

On one hand, suppose that for all connected subgraphs $K \subseteq G$, we have that

$$\alpha(K) > 2(1 - \cos(\pi/n)).$$

Then, G cannot contain a Hamiltonian path P_n . Indeed, if G did contain a Hamiltonian path, then

$$\alpha(P_n) = 2(1 - \cos(\pi/n)) < \alpha(K),$$

a contradiction of the fact that $\alpha(K) > 2(1 - \cos(\pi/n))$ over all connected subgraphs $K \subseteq G$.

On the other hand, suppose that $\alpha(K) \leq 2(1 - \cos(\pi/n))$ for some $K \subseteq G$. If K has $k \leq n$ vertices, then because the path graph minimizes algebraic connectivity over all connected graphs on k vertices (see [22]), we have that

$$\alpha(K) \geq 2(1 - \cos(\pi/k)) \geq 2(1 - \cos(\pi/n)),$$

with equality if and only if $k = n$ due to the monotonicity of the expression $\alpha(P_k)$ in k . Thus, $\alpha(K) = 2(1 - \cos(\pi/n))$, and K has n vertices. From the well-known fact that the path on n vertices *uniquely* minimizes algebraic connectivity over all connected graphs with n vertices (see, for instance, [28, Section 3]), it follows that $K = P_n$. In other words, G has a Hamiltonian path.

We have established that G has a connected subgraph K with $\alpha(K) \leq 2(1 - \cos(\pi/n))$ if and only if G has a Hamiltonian path. This also proves that the decision problem belongs to NP because a Hamiltonian path provides a certificate that can be verified in polynomial time. $\square$

Corollary 1 Given a graph G on n vertices, finding that connected subgraph $K \subseteq G$ that minimizes algebraic connectivity is NP-hard.

We can now comment on the utility of Theorem 1 and Theorem 2, which are tight for different classes of graphs. Indeed, the inequality in Theorem 2 is tight for the complete graph K_n on n vertices, for which

$$\sigma_{n-1}(\tilde{N}) = 1 = n^{-1/2} \sigma_{n-1}(N).$$

On the other hand, for the path graph P_n on n vertices, solving a linear difference equation for the eigenvalues of the grounded Laplacian shows that the bound from Theorem 1 is tight; see Lemma 16 below. Nonetheless, to the best of our knowledge, both Theorem 1 and Theorem 2 provide novel, tight spectral bounds for grounded Laplacians that may be useful beyond the setting of interdiction.

Lemma 16 If P_n is the path graph on n vertices with grounded vertex v an endpoint of the path and $\tilde{L}$ is the corresponding grounded Laplacian, then

$$\nu_{n-1}(\tilde{L}) = 2 \left(1 - \cos \left(\frac{\pi}{2n-1} \right) \right) = \nu_{2n-2}(L'),$$

where L' is the Laplacian of the path graph on $2n-1$ vertices.

Proof We can explicitly compute the eigenvalues and eigenvectors of the grounded Laplacian in this case, observing that the grounded Laplacian is given by

$$\tilde{L} = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & \ddots \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}.$$

In turn, the equation $\tilde{L}x = \lambda x$ can be written as

$$\begin{cases} x_1 - x_2 = \lambda x_1, \\ -x_{k-1} + 2x_k - x_{k+1} = \lambda x_k, & k = 2, \dots, n-1, \\ x_n = 0. \end{cases}$$

First, we solve the homogeneous linear difference equation

$$-x_{k-1} + (2 - \lambda)x_k - x_{k+1} = 0$$

via the ansatz $\lambda = 2(1 - \cos(\theta))$ for some $\theta \in (0, \pi)$. If this is the case, then $2 - \lambda \in (-2, 2)$, and the auxiliary equation $p(r) = r^2 - (2 - \lambda)r + 1$ has two complex roots with magnitude one. As a result, the linear difference equation has a general solution of the form

$$x_k = A \sin(k\theta) + B \cos(k\theta), \quad k = 1, \dots, n$$

for some coefficients $A, B \in \mathbb{R}$. From the condition $x_2 = (1 - \lambda)x_1 = (2 \cos(\theta) - 1)x_1$, we find that

$$\begin{aligned} A \sin(2\theta) + B \cos(2\theta) &= (2 \cos(\theta) - 1)(A \sin(\theta) + B \cos(\theta)) \\ &= 2A \sin(\theta) \cos(\theta) + 2B \cos^2(\theta) - (A \sin(\theta) + B \cos(\theta)). \end{aligned}$$

Applying double-angle identities to the left-hand side, we find that

$$B = A \sin(\theta) + B \cos(\theta).$$

We then find that

$$B = \frac{A \sin(\theta)}{1 - \cos(\theta)} = \frac{2A \sin(\theta/2) \cos(\theta/2)}{1 - (1 - 2 \sin^2(\theta/2))} = A \frac{\cos(\theta/2)}{\sin(\theta/2)}.$$

From the condition $x_n = 0$, we then compute

$$A \sin(n\theta) + B \cos(n\theta) = \sin(n\theta) \sin(\theta/2) + \cos(n\theta) \cos(\theta/2) = 0,$$

assuming that $A \neq 0$. Finally, because $\cos(\theta - \phi) = \cos(\theta) \cos(\phi) + \sin(\theta) \sin(\phi)$, we obtain

$$\cos((n - 1/2)\theta) = 0,$$

from which it follows that

$$\left(\frac{2n-1}{2}\right)\theta = \frac{\pi}{2} + \pi k \implies \theta = \frac{2k+1}{2n-1}\pi$$

for some $k \in \mathbb{Z}$. For $k = 0, \dots, n-2$, this yields $n-1$ distinct eigenvectors of $\tilde{L}$, with corresponding eigenvalues

$$\nu_{n-k-1}(\tilde{L}) = 2 \left(1 - \cos\left(\frac{2k+1}{2n-1}\pi\right)\right).$$

Taking $k = 0$ yields the smallest eigenvalue $\nu_{n-1}(\tilde{L})$ and thus the result. $\square$

We conclude this section by extending some of the above bounds to the setting of *signless* grounded Laplacians. Although these objects do not appear in our study of DCOPF, they will be useful when we extend the methods presented in this section to other power flow models such as the second-order cone relaxations of ACOPF presented in Section 6.2. Therein, we require a bound on the singular values of the *signless* incidence matrix $S \in \mathbb{R}^{|V| \times |E|}$ for a graph $G = (V, E)$, with entries

$$S_{ve} = \begin{cases} 1 & \text{vertex } v \text{ is incident to edge } e, \\ 0 & \text{otherwise.} \end{cases} \quad (23)$$

In turn, S defines a signless Laplacian $Q \in \mathbb{R}^{|V| \times |V|}$ via $Q := SS^\top$. Unlike the standard graph Laplacian, the signless Laplacian is full rank when G is connected and *not* bipartite [29, 30]. The signless incidence matrix S of a graph $G = (V, E)$ is defined in (23) above, and the corresponding signless Laplacian is given by $Q = SS^\top$. As with the standard Laplacian, the signless Laplacian also admits a representation in terms of the degrees $D = \text{diag}(d_1, \dots, d_n)$ and adjacency matrix A of G , with $Q = D + A$.

In the case that G is bipartite, then it is well-known that the Laplacian $L = D - A$ and the signless Laplacian Q are similar via a diagonal matrix R and hence have the same spectrum; see [31, Proposition 1.3.10]. In fact, it is easy to see that this similarity extends to grounded Laplacians and their signless counterparts. In the signless case, the grounded signless Laplacian $\tilde{Q}$ is again a principal submatrix of Q , obtained by removing a row and column indexed by a vertex $v \in V$ from Q .

Lemma 17 If $G = (V, E)$ is a connected bipartite graph with $|V| = n \geq 2$ vertices and $\tilde{L}$ and $\tilde{Q}$ are grounded signed and signless Laplacians respectively, formed using the same grounded vertex $v \in V$, then $\tilde{L}$ and $\tilde{Q}$ are similar and hence have the same spectrum. In particular,

$$\nu_{n-1}(\tilde{Q}) = \nu_{n-1}(\tilde{L}) \geq 2 \left(1 - \cos\left(\frac{\pi}{2n-1}\right)\right).$$

Proof From [31, Proposition 1.3.10], there exists a diagonal matrix $R \in \mathbb{R}^{n \times n}$ such that $L = RQR^{-1}$. Moreover, $\tilde{L} = J^\top LJ$ and $\tilde{Q} = J^\top QJ$, where $J \in \mathbb{R}^{n \times (n-1)}$ is the identity matrix with the column corresponding to the grounded vertex v removed. Observe that $J^\top J = I \in \mathbb{R}^{(n-1) \times (n-1)}$, while $JJ^\top \in \mathbb{R}^{n \times n}$ is a diagonal matrix that commutes with R and R^{-1} , both of which are also diagonal. In turn, we compute

$$\begin{aligned}\tilde{L} &= J^\top LJ = J^\top RQR^{-1}J = J^\top JJ^\top RQR^{-1}JJ^\top J \\ &= J^\top RJJ^\top QJJ^\top R^{-1}J \\ &= \tilde{R}\tilde{Q}\tilde{R}^{-1},\end{aligned}$$

where $\tilde{R} = J^\top RJ$ is diagonal with nonzero diagonal entries by the definition of J . The lower bound on $\nu_{n-1}(\tilde{Q})$ then follows directly from Theorem 1. $\square$

Finally, by eliminating edges from a connected graph to obtain a spanning tree, which is certainly bipartite, we obtain a lower bound on the minimal eigenvalue of the grounded signless Laplacian for *any* connected graph.

Lemma 18 If $G = (V, E)$ is a connected graph on $|V| = n \geq 2$ vertices and $\tilde{Q}$ is an arbitrary grounded, signless Laplacian, then

$$\nu_{n-1}(\tilde{Q}) \geq 2 \left(1 - \cos \left(\frac{\pi}{2n-1} \right) \right).$$

Proof Say that $T = (V, E')$ is a spanning tree of G , with $E' \subseteq E$. Because $\tilde{Q} = \tilde{S}\tilde{S}^\top$, we can write

$$\tilde{Q} = \sum_{e \in E} \tilde{s}_e \tilde{s}_e^\top,$$

where $\tilde{s}_e \in \mathbb{R}^{n-1}$ is the e th column of $\tilde{S}$. The corresponding grounded Laplacian $\tilde{Q}'$ of the spanning tree T is given by

$$\tilde{Q}' = \sum_{e \in E'} \tilde{s}_e \tilde{s}_e^\top \implies \tilde{Q} - \tilde{Q}' = \sum_{e \in E \setminus E'} \tilde{s}_e \tilde{s}_e^\top \succeq 0.$$

Because $\tilde{Q} \succeq \tilde{Q}'$, the variational characterization of the minimal eigenvalue of a matrix implies that $\nu_{n-1}(\tilde{Q}) \geq \nu_{n-1}(\tilde{Q}')$. Finally, T is a spanning tree and hence a bipartite graph on n vertices, so by Lemma 17, the stated lower bound on $\nu_{n-1}(\tilde{Q}')$ and hence $\nu_{n-1}(\tilde{Q})$ follows. $\square$

5 Computational Results

We tested our DCOPF formulation on three test problems from the MATPOWER power flow package ([1]) and a model of the Texas power grid developed by the Electric Reliability Council of Texas (ERCOT). All test problem sizes are shown in Table 2. To match standard practice and improve numerical stability, we normalized all of the power flow, generation, and load values by a factor of 100. We did not include phase angle bounds in these experiments. To match assumptions made in Section 3, negative loads in the data were excluded, and any generators with negative generation bounds

Table 2 Test problem sizes.

Test Problem	Buses $ B $	Branches $ L $	Generators $ G $
RTS96	73	110	33
Case300	300	411	69
Case2746wop	2746	3514	514
ERCOT	8412	9971	795

were adjusted to have nonnegative bounds. Negative reactance values in the data were retained, but for calculating bounds as a function of reactance, we used the absolute value of the reactance. For instances containing negative reactances, this substitution is heuristic; the theoretical validity guarantees derived under positive reactances do not directly apply. While we allowed both load shedding and overgeneration, we modified the formulation to only penalize load shed in the primal objective function.

Since big- M formulations are not the only way to represent the product of binary and continuous variables, we considered two alternate encodings for such products: indicator constraints and the generalized disjunctive programming (GDP) hull. For a constraint $y = bx$, where b is binary and x is continuous, an indicator constraint specifies that if $b = 1$, then $y = x$, while an alternative constraint specifies that if $b = 0$ then $y = 0$ [32]. With the GDP hull encoding, the hull reformulation defines separate lifted variables for the two cases $b = 0$ and $b = 1$. The lifted variables are then restricted via linking constraints and bound-scaling constraints in terms of b , allowing for the relaxation $b \in [0, 1]$ during the branch and bound procedure [33]. Indicator constraints perform better with dual variable bounds and the GDP hull requires dual variable bounds, so we employed the same bounds that we used for the big- M values.

For RTS96 and Case300, we used both full big- M values from the bounds presented in Section 3 and reduced big- M values obtained by taking the minimum of each evaluated bound and the corresponding cap in Table 4. The reduced big- M values illustrate that the formulation remains effective with smaller big- M values and runs significantly faster. For the larger cases Case2746wop and ERCOT, optimizing with the full big- M values created numerical issues that rendered the formulation unsolvable, so we were only able to obtain results with the reduced big- M values. The largest evaluated bounds for each variable and test case are shown in Table 3. For RTS96 and Case300, we used the bound on λ_b from Lemma 6 as opposed to the bound from Theorem 2 since it was lower. The caps for all experiments with reduced big- M values are shown in Table 4.

Table 3 Largest big- M value bounds for test cases.

Variable	RTS96	Case300	Case2746wop	ERCOT
λ_b from Lemma 6	1.20E3	4.67E2	1.56E6	2.21E7
λ_b from Theorem 2	3.77E4	1.83E5	4.34E9	7.96E11
ϕ_l^+, ϕ_l^-	4.88E1	1.01E0	3.72E3	6.15E5
$\bar{\mu}_l$	4.35E3	2.2E3	3.08E7	7.39E8
$\eta_l, \bar{\eta}_l - \eta_l$	2.60E3	1.04E3	3.22E6	4.75E7

Table 4 Caps for reduced big- M values.

Variable	Value
λ_b	3
ϕ_l^+, ϕ_l^-	3
$\bar{\mu}_l$	20
$\eta_l, \bar{\eta}_l - \eta_l$	4

The formulation was implemented in Pyomo 6.7 with Python 3.9, using Gurobi 10.0 for optimization ([7, 34, 35]). For the GDP hull experiments, we modeled the linear disjunctions in Pyomo.GDP and generated the hull reformulation using its `gdp.hull` transformation. For all experiments we used a single node on LLNL’s Borax/Boraxo high-performance computing cluster. This cluster is a 64-bit Linux system running TOSS 4 with Intel Xeon Gold architecture and has 2.3 GHz processors with 36 cores per node and 192 GB memory per node. Gurobi parameters were chosen to improve runtime and use the appropriate number of threads for the cluster. The set of parameters differing from the default values is shown in Table 5. We imposed a wall-clock time limit of four days for all experiments.

Table 5 Gurobi parameters.

Parameter	Value
presolve	2
mipfocus	3
heuristics	0.5
cuts	3
threads	36

For all of our experiments, following determination of interdicted components from the mixed-integer programming formulation, we solved a separate linear program for each experiment that calculates load shed resulting from the disabled components. This linear program serves as a validation test as it is not susceptible to numerical issues that can arise in mixed-integer programs with large big- M values, such as trickle flow. Trickle flow occurs when a constraint of the form $y \leq Mb$, where $b \in \{0, 1\}$ and $y \in \mathbb{R}^+$, is intended to fix $y = 0$ when $b = 0$. However, if $M = 1\text{E}5$ and $b = 9.99\text{E}-5$, then with an integrality tolerance of $1\text{E}-4$, y can be as large as 9.99 while b is accepted as zero within the integrality tolerance.

Results for RTS96 with $k = 1$ to $k = 6$ and the full big- M values from bounds are shown in Table 6. The MIP and LP objective values match for $k = 1, 2, 3$ and differ slightly for $k = 4, 5, 6$. The most noticeable discrepancy is with the indicator encoding for $k = 5$, but in all cases the objective value difference is less than 1%. Table 7 shows the results using the reduced big- M values. The smaller big- M values improve numerical stability as all MIP and LP objective values match. All runtimes are reduced to less than five minutes, allowing larger k values to be considered.

Case300 results with full big- M values are shown in Table 8. Since the problem has significantly more buses and branches than RTS96, we were only able to solve to optimality with $k = 1, 2, 3$ for all constraint encodings (along with $k = 4$ for the hull encoding) as we imposed a four day time limit on all experiments. The dagger-marked entries for $k = 4, 5$ indicate the best incumbent solution after four days. There are slight numerical differences in the MIP and LP solutions for $k = 4, 5$. Using the reduced big- M values as shown in Table 9 gives minimal numerical differences and runtimes less than five minutes. For both RTS96 and Case300, there is agreement between the full and reduced big- M LP load shed values for all k tested with both choices of bounds.

Results for Case2746wop and ERCOT are shown in Table 10 and Table 11. The full big- M values were too large to generate numerically stable formulations for these test cases so we cannot claim that the results are optimal. For choosing reduced big- M values, note that if the MIP objective and LP objective values differ significantly for any k , that provides evidence that big- M values are too small and need to be increased. We chose the reduced big- M values by starting with small values and increasing them until there were no discrepancies in MIP and LP objective values. The final values shown in Table 4 were found after only a few experiments; there may be smaller values that still yield the same results as our experiments. While significant differences in MIP and LP objective values often imply that the big- M values are too small, matching objective values do not necessarily imply that the big- M values are sufficiently large. Thus we cannot claim that our selection heuristic is necessarily strong or sufficient.

Across the test cases, the performance of big- M , indicator, and GDP hull formulations is comparable, but the GDP hull formulation is slightly favorable. No formulation was uniformly fastest; for instance, the indicator encoding is faster than the other two encodings on RTS96 with full big- M values for $k = 5, 6$, but it is slower than the GDP hull encoding on Case300 with reduced big- M values for $k = 2, 3, 4, 5$. However, aggregating across all 41 experiments, big- M converged in 37 experiments, indicator in 36, and GDP hull in 38. The average runtimes for the 36 experiments where all three algorithms converged are 5,938.7s for big- M , 7,511.8s for indicator, and 5,713.2s for GDP hull.

The final interdiction solutions in all four test cases are mostly composed of buses and generators as opposed to branches. In all cases for $k = 1$, the solution is a bus interdiction. For $k = 10$, the interdiction for RTS96 is nine buses and one generator, and the interdiction for Case300 is ten buses. Likewise for Case2746wop and ERCOT, the interdictions for $k = 5$ are all buses. This behavior is expected as disabling a bus interrupts all of the branches connecting to it and is often more powerful than disabling a single branch. We conjecture that some optimal interdictions containing generators admit alternative optimal solutions that interdict the corresponding buses instead.

6 Additional Applications

To demonstrate the generality of our results, we apply the proof methods in Section 3 and Section 4 for interdiction under two new models: the decoupled linear power flow (DLPF) formulation from Yang et al. and a second-order cone formulation similar to

Table 6 Experiments for RTS96 with full big- M values from bounds.

k	Big- M		Indicator		Hull		LP Shed
	Shed (MW)	Time (s)	Shed (MW)	Time (s)	Shed (MW)	Time (s)	
1	333.0	4	333.0	6	333.0	6	333.0
2	666.0	14	666.0	16	666.0	19	666.0
3	999.0	105	999.0	79	999.0	108	999.0
4	1317.1	506	1316.8	598	1316.0	371	1316.0
5	1635.2	5334	1636.2	2542	1635.1	6649	1633.0
6	1952.0	33704	1951.2	8046	1951.1	39500	1950.0

Table 7 Experiments for RTS96 with reduced big- M values.

k	Big- M		Indicator		Hull		LP Shed
	Shed (MW)	Time (s)	Shed (MW)	Time (s)	Shed (MW)	Time (s)	
1	333.0	2	333.0	1	333.0	1	333.0
2	666.0	10	666.0	5	666.0	9	666.0
3	999.0	10	999.0	9	999.0	8	999.0
4	1316.0	12	1316.0	17	1316.0	12	1316.0
5	1633.0	18	1633.0	40	1633.0	18	1633.0
6	1950.0	47	1950.0	50	1950.0	38	1950.0
7	2217.0	46	2217.0	71	2217.0	52	2217.0
8	2550.0	50	2550.0	121	2550.0	62	2550.0
9	2883.0	86	2883.0	224	2883.0	89	2883.0
10	3216.0	113	3216.0	281	3216.0	116	3216.0

Table 8 Experiments for Case300 with full big- M values from bounds.

k	Big- M		Indicator		Hull		LP Shed
	Shed (MW)	Time (s)	Shed (MW)	Time (s)	Shed (MW)	Time (s)	
1	1019.2	82	1019.2	92	1019.2	117	1019.2
2	2299.1	624	2299.2	840	2299.1	676	2299.1
3	3318.5	26147	3318.4	25279	3318.8	22352	3318.3
4	4237.3 [†] (52%)	345600	4242.3 [†] (29%)	345600	4238.6	223590	4236.2
5	5517.2 [†] (59%)	345600	5521.9 [†] (48%)	345600	5517.6 [†] (38%)	345600	5516.1

[†]Terminated at time limit of four days with indicated MIP gap.

that of Wu et al. [18, 20]. Since our goal is to demonstrate applicability, we only allow branches to be interdicted in both models. To further simplify algebraic expressions, we do not account for overgeneration, and we assume that all generators have lower bounds on generation equal to zero.

The notation in the following subsections is similar to the DCOPF notation. To distinguish between real and imaginary components, we use the superscripts P and Q throughout. We redefine some variables, such as R' , with the intent of making them specific to each subsection.

Table 9 Experiments for Case300 with reduced big- M values.

k	Big- M		Indicator		Hull		LP Shed
	Shed (MW)	Time (s)	Shed (MW)	Time (s)	Shed (MW)	Time (s)	
1	1019.2	12	1019.2	11	1019.2	4	1019.2
2	2299.1	20	2299.1	21	2299.1	15	2299.1
3	3318.3	30	3318.3	48	3318.3	26	3318.3
4	4236.2	65	4236.2	57	4236.2	51	4236.2
5	5516.1	69	5516.1	147	5516.1	94	5516.1
6	6535.3	75	6535.4	152	6535.3	144	6535.3
7	7389.0	155	7389.0	142	7389.0	125	7389.0
8	8408.2	129	8408.2	155	8408.2	129	8408.2
9	9013.2	159	9013.2	203	9013.2	125	9013.2
10	9631.6	180	9631.7	253	9631.6	171	9631.6

Table 10 Experiments for Case2746wop with reduced big- M values.

k	Big- M		Indicator		Hull		LP Shed
	Shed (MW)	Time (s)	Shed (MW)	Time (s)	Shed (MW)	Time (s)	
1	619.9	209	619.9	281	619.9	205	619.9
2	978.8	862	978.8	1660	978.8	971	978.8
3	1267.8	4351	1267.8	6102	1267.8	4089	1267.8
4	1539.8	20511	1539.8	44334	1539.8	21771	1539.8
5	1806.3	182832	1806.3 [†] (6%)	345600	1806.3	195386	1806.3

[†]Terminated at time limit of four days with indicated MIP gap.

Table 11 Experiments for ERCOT with reduced big- M values.

k	Big- M		Indicator		Hull		LP Shed (%)
	Shed (%)	Time (s)	Shed (%)	Time (s)	Shed (%)	Time (s)	
1	0.9	5590	0.9	5579	0.9	6356	0.9
2	2.9	21017	2.9	30545	2.9	23784	2.9
3	5.4	93444	5.4	142416	5.4	77413	5.4
4	7.4 [†] (19%)	345600	7.4 [†] (61%)	345600	7.4 [†] (20%)	345600	7.4
5	9.9 [†] (84%)	345600	9.9 [†] (97%)	345600	9.9 [†] (94%)	345600	9.9

[†]Terminated at time limit of four days with indicated MIP gap.

6.1 DLPF Model

The main features of the DLPF model are that it models both real and reactive power flow and provides estimates of bus voltage magnitude [18]. This makes it more sophisticated than DCOPF, though it can still be solved for a fixed network as a linear program. The DLPF primal linear program allowing for branch interdictions is shown in (DLPF-Primal). All generation, flow, and load shed variables are complex. The objective function minimizes the sum of combined real and imaginary load shed, where $c \geq 0$ is a fixed constant. We denote the real and imaginary upper bounds on power generation by P_g^P and P_g^Q , respectively. Each bus has a voltage variable V_b constrained by bounds V_b^-, V_b^+ . We assume that there exists a shared voltage W' such

that for all buses, $V_b^- < W' < V_b^+$. The branch parameters B'_l and B''_l are determined based on modeling assumptions. In the simplest case, both B'_l and B''_l are equal to the branch susceptance $B_l = X_l^{-1}$.

The dual reformulation is shown in (DLPF-DR). Its structure is similar to the DCOPF dual but with sets of constraints for real and imaginary components. Basic dual variable bounds are provided in Lemma 19 and Lemma 20. The existence of λ^P and λ^Q values that are close to zero for at least one bus is shown in Lemma 21.

(DLPF-Primal)

$$\min \sum_{b \in B} (r_b^P + cr_b^Q) \quad (24a)$$

$$\text{s.t.} \quad \sum_{g \in G_b} p_g^P - \sum_{l: o(l)=b} s_l f_l^P + \sum_{l: d(l)=b} s_l f_l^P \quad (24b)$$

$$+ r_b^P = R_b^P, \quad \forall b \in B, \quad (\lambda_b^P) \quad (24c)$$

$$\sum_{g \in G_b} p_g^Q - \sum_{l: o(l)=b} s_l f_l^Q + \sum_{l: d(l)=b} s_l f_l^Q \quad (24d)$$

$$+ r_b^Q = R_b^Q, \quad \forall b \in B, \quad (\lambda_b^Q) \quad (24e)$$

$$V_b^- \leq V_b \leq V_b^+, \quad \forall b \in B, \quad (\kappa_b^-, \kappa_b^+) \quad (24f)$$

$$0 \leq p_g^P \leq P_g^P, \quad \forall g \in G, \quad (\gamma_g^P) \quad (24g)$$

$$0 \leq p_g^Q \leq P_g^Q, \quad \forall g \in G, \quad (\gamma_g^Q) \quad (24h)$$

$$0 \leq r_b^P \leq R_b^P, \quad \forall b \in B, \quad (\alpha_b^P) \quad (24i)$$

$$0 \leq r_b^Q \leq R_b^Q, \quad \forall b \in B, \quad (\alpha_b^Q) \quad (24j)$$

$$s_l (f_l^P - B'_l(\theta_{o(l)} - \theta_{d(l)})) = 0, \quad \forall l \in L, \quad (\mu_l^P) \quad (24k)$$

$$s_l (f_l^Q - B''_l(V_{o(l)} - V_{d(l)})) = 0, \quad \forall l \in L, \quad (\mu_l^Q) \quad (24l)$$

$$-F_l \leq f_l^P \leq F_l, \quad \forall l \in L, \quad (\phi_l^{P-}, \phi_l^{P+}) \quad (24m)$$

$$-F_l \leq f_l^Q \leq F_l, \quad \forall l \in L, \quad (\phi_l^{Q-}, \phi_l^{Q+}) \quad (24n)$$

Lemma 19 Let

$$R' := \sum_{b \in B} (R_b^P + cR_b^Q).$$

There always exists an optimal solution to (DLPF-DR) that satisfies

$$\sum_{l \in L} F_l (\phi_l^{P+} + \phi_l^{P-} + \phi_l^{Q+} + \phi_l^{Q-}) \leq R'.$$

Proof The proof is almost identical to the DCOPF case, except we need to show here that

$$\sum_{b \in B} (-\kappa_b^+ V_b^+ + \kappa_b^- V_b^-) \leq 0.$$

It is clear that

$$\sum_{b \in B} (-\kappa_b^+ V_b^+ + \kappa_b^- V_b^-) \leq \sum_{b \in B} W'(-\kappa_b^+ + \kappa_b^-).$$

Summing (27h) over all buses gives

$$\sum_{b \in B} -\kappa_b^+ + \kappa_b^- = \sum_{l \in L} s_l B_l'' \mu_l^Q - \sum_{l \in L} s_l B_l'' \mu_l^Q = 0. \quad (26a)$$

□

(DLPF-DR)

$$\begin{aligned} \max \quad & \sum_{b \in B} \left(R_b^P (\lambda_b^P - \alpha_b^P) + R_b^Q (\lambda_b^Q - \alpha_b^Q) - \kappa_b^+ V_b^+ + \kappa_b^- V_b^- \right) \\ & + \sum_{g \in G} (-P_g^P \gamma_g^P - P_g^Q \gamma_g^Q) \\ & - \sum_{l \in L} F_l (\phi_l^{P+} + \phi_l^{P-} + \phi_l^{Q+} + \phi_l^{Q-}) \end{aligned} \quad (27a)$$

$$\text{s.t.} \quad -\gamma_g^P + \lambda_{b(g)}^P \leq 0, \quad \forall g \in G, \quad (p_g^P) \quad (27b)$$

$$-\gamma_g^Q + \lambda_{b(g)}^Q \leq 0, \quad \forall g \in G, \quad (p_g^Q) \quad (27c)$$

$$-\alpha_b^P + \lambda_b^P \leq 1, \quad \forall b \in B, \quad (r_b^P) \quad (27d)$$

$$-\alpha_b^Q + \lambda_b^Q \leq c, \quad \forall b \in B, \quad (r_b^Q) \quad (27e)$$

$$s_l (-\lambda_{o(l)}^P + \lambda_{d(l)}^P - \mu_l^P) - \phi_l^{P+} + \phi_l^{P-} = 0, \quad \forall l \in L, \quad (f_l^P) \quad (27f)$$

$$s_l (-\lambda_{o(l)}^Q + \lambda_{d(l)}^Q - \mu_l^Q) - \phi_l^{Q+} + \phi_l^{Q-} = 0, \quad \forall l \in L, \quad (f_l^Q) \quad (27g)$$

$$\kappa_b^+ - \kappa_b^- - \sum_{l:o(l)=b} s_l B_l'' \mu_l^Q + \sum_{l:d(l)=b} s_l B_l'' \mu_l^Q = 0, \quad \forall b \in B, \quad (27h)$$

$$\sum_{l:o(l)=b} s_l B_l' \mu_l^P - \sum_{l:d(l)=b} s_l B_l' \mu_l^P = 0, \quad \forall b \in B, \quad (27i)$$

$$\gamma_g^P, \gamma_g^Q \geq 0, \quad \forall g \in G, \quad (27j)$$

$$\alpha_b^P, \alpha_b^Q, \kappa_b^-, \kappa_b^+ \geq 0, \quad \forall b \in B, \quad (27k)$$

$$\lambda_b^P, \lambda_b^Q \text{ free}, \quad \forall b \in B, \quad (27l)$$

$$\phi_l^{P+}, \phi_l^{P-}, \phi_l^{Q+}, \phi_l^{Q-} \geq 0, \quad \forall l \in L, \quad (27m)$$

$$\mu_l^P, \mu_l^Q \text{ free}, \quad \forall l \in L. \quad (27n)$$

Lemma 20 Let

$$V' := \min_{b \in B} \min\{V_b^+ - W', W' - V_b^-\}.$$

There always exists an optimal solution to (DLPF-DR) that satisfies

$$\sum_{b \in B} (\kappa_b^+ + \kappa_b^-) \leq \frac{R'}{V'}.$$

Proof We have

$$\begin{aligned} \sum_{b \in B} (\kappa_b^+ V_b^+ - \kappa_b^- V_b^-) &= \sum_{b \in B} ((V_b^+ - W') \kappa_b^+ + (W' - V_b^-) \kappa_b^-) \\ &\geq V' \sum_{b \in B} (\kappa_b^+ + \kappa_b^-), \end{aligned} \quad (29a)$$

where we have used (26a). Since the dual objective function must be nonnegative at optimality, we have

$$\begin{aligned} \sum_{b \in B} R_b^P (\lambda_b^P - \alpha_b^P) + \sum_{b \in B} R_b^Q (\lambda_b^Q - \alpha_b^Q) + \sum_{b \in B} (-\kappa_b^+ V_b^+ + \kappa_b^- V_b^-) \\ \geq \sum_{g \in G} (P_g^P \gamma_g^P + P_g^Q \gamma_g^Q) + \sum_{l \in L} F_l (\phi_l^{P+} + \phi_l^{P-} + \phi_l^{Q+} + \phi_l^{Q-}) \\ \geq 0. \end{aligned}$$

Then,

$$\begin{aligned} \sum_{b \in B} (\kappa_b^+ V_b^+ - \kappa_b^- V_b^-) &\leq \sum_{b \in B} R_b^P (\lambda_b^P - \alpha_b^P) + \sum_{b \in B} R_b^Q (\lambda_b^Q - \alpha_b^Q) \\ &\leq R'. \end{aligned} \quad (29b)$$

Combining (29b) and (29a) gives the result. $\square$

Lemma 21 For any connected component consisting of buses $B' \subseteq B$, an optimal solution to (DLPF-DR) exists with variables $\lambda_{B'}^P, \lambda_{B'}^Q$, satisfying both conditions:

- (i) There exists $i^* \in B'$ such that $0 \leq \lambda_{i^*}^P \leq 1$.
- (ii) There exists $j^* \in B'$ such that $0 \leq \lambda_{j^*}^Q \leq c$.

Proof We consider the impact of $\lambda_{B'}^P, \lambda_{B'}^Q$ on the objective function and the constraints (27b)–(27e). Define

$$L^Q(b) := R_b^Q (\lambda_b^Q - \alpha_b^Q) - \sum_{g \in G_b} P_g^Q \gamma_g^Q.$$

Choosing α_b^Q and γ_g^Q optimally as a function of λ_b^Q , we have

$$L^Q(b) = \begin{cases} R_b^Q \lambda_b^Q, & \lambda_b^Q < 0, \\ R_b^Q \lambda_b^Q - (\sum_{g \in G_b} P_g^Q) \lambda_b^Q, & 0 \leq \lambda_b^Q \leq c, \\ R_b^Q c - (\sum_{g \in G_b} P_g^Q) \lambda_b^Q, & \lambda_b^Q \geq c, \end{cases}$$

which is a concave function that is maximized when $\lambda_b^Q \in [0, c]$. Note that constant shifts to λ_b^Q do not impact the remaining constraints. Hence the portion of the objective function $\sum_{b \in B'} L^Q(b)$ has a maximum where for some $j^* \in B'$, $\lambda_{j^*}^Q \in [0, c]$. An analogous argument shows that for some $i^* \in B'$, $\lambda_{i^*}^P \in [0, 1]$. $\square$

Note that whenever a surviving connected component is a singleton, Lemma 21 immediately gives bounds on $\|\lambda_K^P\|_\infty$ and $\|\lambda_K^Q\|_\infty$. Hence, we assume that $|V(K)| \geq 2$ throughout the following lemmata.

The bilinear terms in (DLPF-DR) are handled with substitutions analogous to the DCOPF case:

$$\begin{aligned} s_l \eta_l^P &= \phi_l^{P+} - \phi_l^{P-}, & \forall l \in L, \\ \eta_l^P &= -\mu_l^P + \lambda_{d(l)}^P - \lambda_{o(l)}^P, & \forall l \in L, \\ s_l \eta_l^Q &= \phi_l^{Q+} - \phi_l^{Q-}, & \forall l \in L, \\ \eta_l^Q &= -\mu_l^Q + \lambda_{d(l)}^Q - \lambda_{o(l)}^Q, & \forall l \in L. \end{aligned}$$

The new variables are then

$$\begin{aligned} \bar{\eta}_l^P &:= s_l \eta_l^P, & \forall l \in L, \\ \bar{\mu}_l^P &:= s_l \mu_l^P, & \forall l \in L, \\ \bar{\eta}_l^Q &:= s_l \eta_l^Q, & \forall l \in L, \\ \bar{\mu}_l^Q &:= s_l \mu_l^Q, & \forall l \in L. \end{aligned}$$

Bounding $\bar{\mu}_l^P$ is nearly identical to bounding $\bar{\mu}_l$ in the DCOPF case with no phase angle bounds as constraint (27i) is an expression that only involves the dual variables μ_l^P , or equivalently $\bar{\mu}_l^P$. This is shown in Lemma 22. On the other hand, since constraint (27h) contains the voltage bound dual variables $\kappa_b^+ - \kappa_b^-$, bounding $\bar{\mu}_l^Q$ must be accomplished using an approach similar to the DCOPF case with phase angle bounds, as shown in Lemma 23.

Lemma 22 An optimal solution to (DLPF-DR) can always be found with values of $\bar{\mu}_l^P$ satisfying

$$|\bar{\mu}_l^P| \leq \frac{R'}{\sqrt{B'_l}} \max_{t \in L} \left\{ \frac{\sqrt{B'_t}}{F_t} \right\}, \quad \forall l \in L.$$

Proof Consider a surviving connected component K . Let $\mathbf{B}'_K$ be the diagonal matrix for $\{B'_l : l \in K\}$. We have

$$\begin{aligned} N_K \mathbf{B}'_K \bar{\mu}_K^P &= 0, \\ -N_K^T \lambda_K^P - \bar{\mu}_K^P &= \phi_K^{P+} - \phi_K^{P-}. \end{aligned} \tag{32a}$$

Multiplying $-N_K \mathbf{B}'_K$ to the left-hand side of (32a) gives

$$\begin{aligned} N_K \mathbf{B}'_K N_K^T \lambda_K^P &= N_K \mathbf{B}'_K (\phi_K^{P-} - \phi_K^{P+}) - N_K \mathbf{B}'_K \bar{\mu}_K^P \\ &= N_K \mathbf{B}'_K (\phi_K^{P-} - \phi_K^{P+}). \end{aligned} \tag{32b}$$

Grounding gives the equivalent system

$$\tilde{N}_K \mathbf{B}'_K \tilde{N}_K^T \tilde{\lambda}_K^P = \tilde{N}_K \mathbf{B}'_K (\phi_K^{P-} - \phi_K^{P+}). \tag{32c}$$

We compare the solutions in (32c) and (32b):

$$\mathbf{B}'_K^{1/2} N_K^T \lambda_K^P = \mathbf{B}'_K^{1/2} \tilde{N}_K^T \tilde{\lambda}_K^P$$

$$\begin{aligned}
&= \mathbf{B}'_K^{1/2} \tilde{N}_K^T (\tilde{N}_K \mathbf{B}'_K \tilde{N}_K^T)^{-1} \tilde{N}_K \mathbf{B}'_K (\phi_K^{P-} - \phi_K^{P+}) \\
&= H \mathbf{B}'_K^{1/2} (\phi_K^{P-} - \phi_K^{P+})
\end{aligned} \tag{32d}$$

where the matrix

$$H := \mathbf{B}'_K^{1/2} \tilde{N}_K^T (\tilde{N}_K \mathbf{B}'_K \tilde{N}_K^T)^{-1} \tilde{N}_K \mathbf{B}'_K^{1/2}$$

is symmetric and idempotent.

Multiplying $\mathbf{B}'_K^{1/2}$ to the left-hand side of all terms in (32a) gives

$$\begin{aligned}
\mathbf{B}'_K^{1/2} \bar{\mu}_K^P &= -\mathbf{B}'_K^{1/2} N_K^T \lambda_K^P + \mathbf{B}'_K^{1/2} (\phi_K^{P-} - \phi_K^{P+}) \\
&= -H \mathbf{B}'_K^{1/2} (\phi_K^{P-} - \phi_K^{P+}) + \mathbf{B}'_K^{1/2} (\phi_K^{P-} - \phi_K^{P+}) \\
&= (I - H) (\mathbf{B}'_K^{1/2} (\phi_K^{P-} - \phi_K^{P+})).
\end{aligned}$$

Using the fact that $I - H$ is idempotent,

$$\begin{aligned}
\|\mathbf{B}'_K^{1/2} \bar{\mu}_K^P\|_2 &\leq \|I - H\|_2 \|\mathbf{B}'_K^{1/2} (\phi_K^{P-} - \phi_K^{P+})\|_2 \\
&\leq \|\mathbf{B}'_K^{1/2} (\phi_K^{P-} - \phi_K^{P+})\|_2.
\end{aligned} \tag{32e}$$

The square of the right-hand side of (32e) is upper-bounded by the optimal value of the mathematical program

$$\max \sum_{l \in K} B'_l (\phi_l^{P-} - \phi_l^{P+})^2 \tag{32f}$$

$$\text{s.t.} \quad \sum_{l \in K} (\phi_l^{P+} + \phi_l^{P-}) F_l \leq R', \tag{32g}$$

$$\phi_l^{P+} \geq 0, \quad \phi_l^{P-} \geq 0, \quad \forall l \in K. \tag{32h}$$

The optimal objective value of (32f) - (32h) is equal to $\max_{l \in K} \left\{ B'_l \left(\frac{R'}{F_l} \right)^2 \right\}$. Using $\sqrt{B'_l} |\bar{\mu}_l^P| \leq \|\mathbf{B}'_K^{1/2} \bar{\mu}_K^P\|_2$ for all $l \in K$, we have

$$\begin{aligned}
\sqrt{B'_l} |\bar{\mu}_l^P| &\leq \max_{l \in K} \left\{ \frac{R' \sqrt{B'_l}}{F_l} \right\} \\
&\leq R' \max_{l \in L} \left\{ \frac{\sqrt{B'_l}}{F_l} \right\}.
\end{aligned} \tag{32i}$$

The last inequality follows since $K \subseteq L$. The expression (32i) holds for all possible connected components K and hence for all $l \in L$. $\square$

Lemma 23 An optimal solution to (DLPF-DR) can always be found with values of $\bar{\mu}_l^Q$ satisfying

$$|\bar{\mu}_l^Q| \leq \frac{R'}{\sqrt{B'_l}} \max_{t \in L} \left\{ \frac{\sqrt{B'_t}}{F_t} \right\} + \frac{1}{B'_l} \left(\frac{R'}{V'} \right), \quad \forall l \in L.$$

Proof Consider a surviving connected component K . Let $\mathbf{B}''_K$ be the diagonal matrix for $\{B''_l : l \in K\}$. We have

$$\begin{aligned} N_K \mathbf{B}''_K \bar{\mu}_K^Q &= \kappa_K^+ - \kappa_K^-, \\ -N_K^T \lambda_K^Q - \bar{\mu}_K^Q &= \phi_K^{Q+} - \phi_K^{Q-}. \end{aligned} \quad (33a)$$

Multiplying $-N_K \mathbf{B}''_K$ to the left-hand side of (33a) gives

$$N_K \mathbf{B}''_K N_K^T \lambda_K^Q = N_K \mathbf{B}''_K (\phi_K^{Q-} - \phi_K^{Q+}) - N_K \mathbf{B}''_K \bar{\mu}_K^Q.$$

Grounding and removing reference bus b_0 gives the equivalent system

$$\tilde{N}_K \mathbf{B}''_K \tilde{N}_K^T \tilde{\lambda}_K^Q = \tilde{N}_K \mathbf{B}''_K (\phi_K^{Q-} - \phi_K^{Q+}) - \tilde{N}_K \mathbf{B}''_K \bar{\mu}_K^Q.$$

Then,

$$\begin{aligned} \mathbf{B}''^{1/2}_K N_K^T \lambda_K^Q &= \mathbf{B}''^{1/2}_K \tilde{N}_K^T \tilde{\lambda}_K^Q \\ &= \mathbf{B}''^{1/2}_K \tilde{N}_K^T (\tilde{N}_K \mathbf{B}''_K \tilde{N}_K^T)^{-1} \tilde{N}_K \mathbf{B}''_K (\phi_K^{Q-} - \phi_K^{Q+}) \\ &\quad - \mathbf{B}''^{1/2}_K \tilde{N}_K^T (\tilde{N}_K \mathbf{B}''_K \tilde{N}_K^T)^{-1} \tilde{N}_K \mathbf{B}''_K \bar{\mu}_K^Q \\ &= H_K \mathbf{B}''^{1/2}_K (\phi_K^{Q-} - \phi_K^{Q+}) - \mathbf{B}''^{-1/2}_K J_K (\tilde{\kappa}_K^+ - \tilde{\kappa}_K^-) \end{aligned} \quad (33b)$$

where the matrix

$$H_K := \mathbf{B}''^{1/2}_K \tilde{N}_K^T (\tilde{N}_K \mathbf{B}''_K \tilde{N}_K^T)^{-1} \tilde{N}_K \mathbf{B}''^{1/2}_K$$

is symmetric and idempotent and

$$J_K := \mathbf{B}''_K \tilde{N}_K^T (\tilde{N}_K \mathbf{B}''_K \tilde{N}_K^T)^{-1}$$

is the PTDF matrix for component K . Multiplying $\mathbf{B}''^{1/2}_K$ to the left-hand side of all terms in (33a) gives

$$\begin{aligned} \mathbf{B}''^{1/2}_K \bar{\mu}_K^Q &= -\mathbf{B}''^{1/2}_K N_K^T \lambda_K^Q + \mathbf{B}''^{1/2}_K (\phi_K^{Q-} - \phi_K^{Q+}) \\ &= -H_K \mathbf{B}''^{1/2}_K (\phi_K^{Q-} - \phi_K^{Q+}) + \mathbf{B}''^{-1/2}_K J_K (\tilde{\kappa}_K^+ - \tilde{\kappa}_K^-) + \mathbf{B}''^{1/2}_K (\phi_K^{Q-} - \phi_K^{Q+}) \\ &= (I - H_K) (\mathbf{B}''^{1/2}_K (\phi_K^{Q-} - \phi_K^{Q+})) + \mathbf{B}''^{-1/2}_K J_K (\tilde{\kappa}_K^+ - \tilde{\kappa}_K^-). \end{aligned}$$

Looking at the l th component of $\mathbf{B}''^{1/2}_K \bar{\mu}_K^Q$,

$$|\sqrt{B''_l} \bar{\mu}_l^Q| \leq \left| \left((I - H_K) \mathbf{B}''^{1/2}_K (\phi_K^{Q-} - \phi_K^{Q+}) \right)_l \right| + \left| \left(\mathbf{B}''^{-1/2}_K J_K (\tilde{\kappa}_K^+ - \tilde{\kappa}_K^-) \right)_l \right|. \quad (33c)$$

The first term can be bounded using arguments within the proof of Lemma 22 and we have

$$\begin{aligned} \left| \left((I - H_K) \mathbf{B}''^{1/2}_K (\phi_K^{Q-} - \phi_K^{Q+}) \right)_l \right| &\leq \|(I - H_K) \mathbf{B}''^{1/2}_K (\phi_K^{Q-} - \phi_K^{Q+})\|_2 \\ &\leq R' \max_{l \in L} \left\{ \frac{\sqrt{B''_l}}{F_l} \right\}. \end{aligned}$$

For the second term, we have

$$\begin{aligned} \left| \left(J_K (\tilde{\kappa}_K^+ - \tilde{\kappa}_K^-) \right)_l \right| &= \left| \sum_{b \in K, b \neq b_0} (J_K)_{l,b} (\kappa_b^+ - \kappa_b^-) \right| \\ &\leq \sum_{b \in K, b \neq b_0} |(J_K)_{l,b}| |\kappa_b^+ - \kappa_b^-| \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{b \in K} |\kappa_b^+ - \kappa_b^-| \\
&\leq \sum_{b \in B} (\kappa_b^+ + \kappa_b^-) \\
&\leq \frac{R'}{V'}.
\end{aligned}$$

Substituting into (33c) gives

$$\sqrt{B_l''} |\bar{\mu}_l^Q| \leq R' \max_{l \in L} \left\{ \frac{\sqrt{B_l''}}{F_l} \right\} + \frac{1}{\sqrt{B_l''}} \left(\frac{R'}{V'} \right).$$

□

From Lemma 22 and Lemma 23, we can obtain bounds on λ_b^Q and λ_b^P using the same strategy as in Section 4. In particular, from (32a) (resp. (33a)), we can leverage bounds on $\bar{\mu}_K^P$ (resp. $\bar{\mu}_K^Q$) and a bound on the smallest nonzero singular value of $\tilde{N}_K^\top$. For the sake of notational clarity, we define

$$\rho_n := \sqrt{2 \left(1 - \cos \left(\frac{\pi}{2n-1} \right) \right)}, \quad (34)$$

the quantity that arises from the lower bound on the minimal eigenvalue of a grounded Laplacian of a connected graph on n vertices from Theorem 1.

Lemma 24 Given a connected component K of $L \setminus A$ for a fixed attack A , there exist dual optimal points λ_K^Q and λ_K^P such that

$$\|\lambda_K^P\|_\infty \leq \frac{R'}{\rho_n} \left(\max_{l \in L} \frac{1}{\sqrt{B_l'}} \right) \left(\max_{l \in L} \left\{ \frac{\sqrt{B_l'}}{F_l} \right\} \right) + 1$$

and

$$\|\lambda_K^Q\|_\infty \leq \frac{R'}{\rho_n} \left(\sqrt{n-1} \max_{l \in L} \left\{ \frac{1}{\sqrt{B_l''}} \max_{t \in L} \left\{ \frac{\sqrt{B_t''}}{F_t} \right\} + \frac{1}{B_l'' V'} \right\} + \max_{l \in L} \left\{ \frac{1}{F_l} \right\} \right) + c,$$

where n is the number of vertices in the underlying graph.

Proof We replicate the proof of Lemma 12 in both instances. For λ_K^P , we begin with (32d), which states that

$$\mathbf{B}'^{1/2}_K N_K^T \lambda_K^P = H \mathbf{B}'^{1/2}_K (\phi_K^{P-} - \phi_K^{P+}).$$

We can apply Lemma 21 to find an index $b \in K$ such that, at optimality $|\lambda_b^P| \leq 1$. Upon dropping the column of $N_K^\top$ and entry of $\lambda_K^P - \lambda_b^P \mathbf{1}$ corresponding to the bus $b \in K$ to form $\tilde{N}_K^\top$ and $\tilde{\lambda}_K^P$, respectively, we obtain

$$\mathbf{B}'^{1/2}_K \tilde{N}_K^T \tilde{\lambda}_K^P = H \mathbf{B}'^{1/2}_K (\phi_K^{P-} - \phi_K^{P+}),$$

where $\|\lambda_K^P\|_\infty \leq \|\tilde{\lambda}_K^P\|_\infty + 1$. Taking norms of both sides and leveraging the fact that H is symmetric and idempotent, as well as the bound from Lemma 19, we obtain

$$\|\tilde{N}_K^T \tilde{\lambda}_K^P\|_2 \leq R' \max_{l \in L} \left\{ \frac{1}{\sqrt{B'_l}} \right\} \max_{l \in L} \left\{ \frac{\sqrt{B'_l}}{F_l} \right\}.$$

Now, repeating the exact same computation as in (18a), we find that

$$\|\tilde{\lambda}_K^P\|_\infty \leq \frac{R'}{\sigma_{\min}(\tilde{N}_K^T)} \max_{l \in L} \left\{ \frac{1}{\sqrt{B'_l}} \right\} \max_{l \in L} \left\{ \frac{\sqrt{B'_l}}{F_l} \right\}.$$

From Theorem 1, the minimal singular value of $\tilde{N}_K^T$ is bounded below by

$$\sigma_{\min}(\tilde{N}_K^T) \geq \rho_n$$

where n is the number of nodes in the underlying graph, thus yielding the stated bound on $\|\lambda_K^P\|_\infty$.

To obtain a bound on λ_K^Q , we instead utilize (33a), which reads

$$-N_K^T \lambda_K^Q = \bar{\mu}_K^Q + \phi_K^{Q+} - \phi_K^{Q-}.$$

Given a connected component K , we can find a spanning tree $T \subseteq K$ and an associated row-selection matrix $P_K \in \mathbb{R}^{|E(T)| \times |E(K)|}$ such that $P_K N_K^T = N_T^T$, where N_T is the incidence matrix of the spanning tree. Multiplying the above system by P_K , we obtain a modified system

$$-N_T^T \lambda_K^Q = P_K(\bar{\mu}_K^Q + \phi_K^{Q+} - \phi_K^{Q-}),$$

where the right-hand side satisfies

$$\begin{aligned} \|P_K(\bar{\mu}_K^Q + \phi_K^{Q+} - \phi_K^{Q-})\|_2 &\leq \|P_K \bar{\mu}_K^Q\|_2 + \|P_K(\phi_K^{Q+} - \phi_K^{Q-})\|_2 \\ &\leq \sqrt{n-1} \|\bar{\mu}_K^Q\|_\infty + \|\phi_K^{Q+} - \phi_K^{Q-}\|_2. \end{aligned}$$

We use the fact that T has at most $n-1$ edges as a spanning tree of a graph on at most n vertices and for $v \in \mathbb{R}^m$, we have that $\|v\|_2 \leq \sqrt{m} \|v\|_\infty$.

Now, Lemma 21 shows that, at optimality, there exists a bus $b \in K$ such that $|\lambda_b^Q| \leq c$. Let $\tilde{\lambda}_K^Q := \lambda_K^Q - \lambda_b^Q \mathbf{1}$. Define $\tilde{\lambda}_K^Q$ by removing the entry corresponding to bus b from $\tilde{\lambda}_K^Q$, and define $\tilde{N}_T$ by removing the corresponding row from N_T . We now obtain a system of the form

$$-\tilde{N}_T^T \tilde{\lambda}_K^Q = P_K(\bar{\mu}_K^Q + \phi_K^{Q+} - \phi_K^{Q-}),$$

where $\|\lambda_K^Q\|_\infty \leq \|\tilde{\lambda}_K^Q\|_\infty + c$. Repeating the exact same reasoning as before, applying Lemma 23 and Lemma 19, we obtain

$$\|\tilde{\lambda}_K^Q\|_\infty \leq \frac{R'}{\sigma_{\min}(\tilde{N}_T^T)} \left(\sqrt{n-1} \max_{l \in L} \left\{ \frac{1}{\sqrt{B''_l}} \max_{t \in L} \left\{ \frac{\sqrt{B''_t}}{F_t} \right\} + \frac{1}{B''_l V'} \right\} + \max_{l \in L} \left\{ \frac{1}{F_l} \right\} \right).$$

For the first term above, we use the bound on $\|\bar{\mu}_K^Q\|_\infty$ from Lemma 23. For the second term, the same reasoning as in the proof of Lemma 22 yields the bound

$$\|\phi_K^{Q-} - \phi_K^{Q+}\|_2 \leq R' \max_{l \in L} \left\{ \frac{1}{F_l} \right\}.$$

Finally, applying the lower bound on $\sigma_{\min}(\tilde{N}_T^T)$ from Theorem 1 gives the desired bound. $\square$

We can also obtain a slightly cleaner, albeit asymptotically equivalent, bound on λ_K^Q as follows.

Lemma 25 There exists a dual optimal point λ_K^Q such that

$$\|\lambda_K^Q\|_\infty \leq \frac{R'}{\rho_n} \max_{l \in L} \left\{ \frac{1}{\sqrt{B_l''}} \right\} \left(\max_{l \in L} \left\{ \frac{\sqrt{B_l''}}{F_l} \right\} + \frac{\sqrt{n-1}}{V'} \max_{l \in L} \left\{ \frac{1}{\sqrt{B_l''}} \right\} \right) + c.$$

Proof Beginning from (33b), we multiply by the same row selection matrix P_K as in the proof of the preceding lemma to obtain the system

$$N_T^\top \lambda_K^Q = P_K (\mathbf{B}''_K)^{-1/2} (H_K \mathbf{B}''_K^{1/2} (\phi_K^{Q-} - \phi_K^{Q+}) - \mathbf{B}''_K^{-1/2} J_K (\tilde{\kappa}_K^+ - \tilde{\kappa}_K^-)).$$

As above, $T \subseteq K$ is a spanning tree. Shifting λ_K^Q by $\lambda_b^Q \mathbf{1}$, where $|\lambda_b^Q| \leq c$, to form $\tilde{\lambda}_K^Q$ and $\tilde{N}_T^\top$, we see that

$$\begin{aligned} \|\tilde{N}_T^\top \tilde{\lambda}_K^Q\|_2 &\leq \frac{1}{\left(\min_{l \in L} \sqrt{B_l''}\right)} \|H_K \mathbf{B}''_K^{1/2} (\phi_K^{Q-} - \phi_K^{Q+})\|_2 + \|P_K \mathbf{B}''_K^{-1} J_K (\tilde{\kappa}_K^+ - \tilde{\kappa}_K^-)\|_2 \\ &\leq \frac{1}{\left(\min_{l \in L} \sqrt{B_l''}\right)} \|\mathbf{B}''_K^{1/2} (\phi_K^{Q-} - \phi_K^{Q+})\|_2 + \|P_K \mathbf{B}''_K^{-1} J_K (\tilde{\kappa}_K^+ - \tilde{\kappa}_K^-)\|_2, \end{aligned}$$

noting that $\|P_K v\|_2 \leq \|v\|_2$ for any $v \in \mathbb{R}^{|E(K)|}$.

Now, by the same reasoning as in the proof of Lemma 22, the first term is bounded by

$$\|\mathbf{B}''_K^{1/2} (\phi_K^{Q-} - \phi_K^{Q+})\|_2 \leq R' \max_{l \in L} \left\{ \frac{\sqrt{B_l''}}{F_l} \right\}.$$

On the other hand, it was established in the proof of Lemma 23 that

$$\|J_K (\tilde{\kappa}_K^+ - \tilde{\kappa}_K^-)\|_\infty \leq \frac{R'}{V'},$$

so it follows that

$$\begin{aligned} \|P_K \mathbf{B}''_K^{-1} J_K (\tilde{\kappa}_K^+ - \tilde{\kappa}_K^-)\|_2 &\leq \sqrt{n-1} \|P_K \mathbf{B}''_K^{-1} J_K (\tilde{\kappa}_K^+ - \tilde{\kappa}_K^-)\|_\infty \\ &\leq \sqrt{n-1} \|\mathbf{B}''_K^{-1} J_K (\tilde{\kappa}_K^+ - \tilde{\kappa}_K^-)\|_\infty \\ &\leq \sqrt{n-1} \|J_K (\tilde{\kappa}_K^+ - \tilde{\kappa}_K^-)\|_\infty \max_{l \in L} \left\{ \frac{1}{B_l''} \right\} \\ &\leq \frac{\sqrt{n-1} R'}{V'} \max_{l \in L} \left\{ \frac{1}{B_l''} \right\}. \end{aligned}$$

Hence, the same spectral bound as in the proof of Lemma 24 shows that

$$\|\tilde{\lambda}_K^Q\|_\infty \leq \frac{R'}{\rho_n} \max_{l \in L} \left\{ \frac{1}{\sqrt{B_l''}} \right\} \left(\max_{l \in L} \left\{ \frac{\sqrt{B_l''}}{F_l} \right\} + \frac{\sqrt{n-1}}{V'} \max_{l \in L} \left\{ \frac{1}{\sqrt{B_l''}} \right\} \right),$$

and the fact that we can choose $\tilde{\lambda}_K^Q$ such that $\|\lambda_K^Q\|_\infty \leq \|\tilde{\lambda}_K^Q\|_\infty + c$ gives the result. $\square$

In both Lemma 24 and Lemma 25, the bounds on $\|\lambda_K^Q\|_\infty$ scale as $O(n^{5/2})$. Indeed, $R' = O(n)$, $\rho_n = \Theta(n^{-1})$ upon invoking Theorem 1 instead of Theorem 2 (as a worst-case bound), and we obtained an extra factor of $\sqrt{n-1}$ from the norm-equivalence bound $\|v\|_2 \leq \sqrt{n-1}\|v\|_\infty$ for $v \in \mathbb{R}^{n-1}$. This stands in contrast to the $O(n^2)$ bound on $\|\lambda_K^P\|_\infty$ from Lemma 24.

6.2 SOC Model

The second-order cone formulation for optimal power flow is formulated over the same directed graph structure as the DCOPF formulation, but each branch has separate variables for forward and backwards flow. For a directed edge (i, j) , we denote by f_{ij}^P and f_{ij}^Q the real and imaginary components for forward directed flow. Analogously, we denote by e_{ij}^P and e_{ij}^Q the real and imaginary components for backwards directed flow. Generation and load shed variables are also complex.

The primal formulation given a fixed branch interdiction is shown in (SOC-Primal). The formulation closely resembles the model given in Wu et al. [20]. Lifted variables are used to represent voltage terms. Given underlying complex bus voltages $(V_i)_{i \in B}$, the squared voltage magnitude $|V_i|^2$ for each bus i is denoted by the quantity W_{ii} . The terms W_{ij}^P and W_{ij}^Q represent the real and imaginary components of the product $V_i V_j^*$. Bounds on squared voltage magnitudes are given by (37f). Similar to the DLPF model, we assume there exists a squared voltage W' such that $V_i^{2-} < W' < V_i^{2+}$ for all buses $i \in B$. Note that (37g) enforces the constraint

$$(W_{ij}^P)^2 + (W_{ij}^Q)^2 \leq W_{ii}W_{jj}.$$

We use the convention that branch admittance is defined as $Y_{ij} = G_{ij} - jB_{ij}$ where $G_{ij} \geq 0$ is the conductance, $B_{ij} > 0$ is the susceptance, and $j := \sqrt{-1}$. This overloads the set notation for generators and buses, but the variable meanings should be clear from context.

The SOC dual reformulation is shown in (SOC-DR). While strong duality ensures that optimal primal and dual objective values are equal in the DCOPF and DLPF models, second-order cone formulations generally only guarantee weak duality. Some efforts have been made to establish conditions ensuring a zero duality gap (see, for example, Farivar and Low [36]), but we assume here that the dual formulation only gives a lower bound on the optimal interdiction.

With the intent of bounding λ^P and λ^Q variables, we show how our methods can be used to express these variables in terms of bounds on the dual variables ω^0 , τ^0 , and δ^0 . While we obtain upper bounds on ω^0 and τ^0 , there is no clear approach for bounding δ^0 . The problem can be modified to yield a bound on δ^0 , for instance by penalizing it in the dual objective function, but we proceed simply by expressing final bounds in terms of a bound on δ^0 .

We anticipate that our methods can be applied to the general lossy case where all branches have $G_{ij} > 0$, but we focus on the simplified lossless case here where all branches satisfy $G_{ij} = 0$. Lemma 26 establishes a new R' for this section as well as an expression for bounding ω_{ij}^0 and τ_{ij}^0 .

(SOC-Primal)

$$\min \quad \sum_{b \in B} r_b^P \quad (37a)$$

$$\text{s.t.} \quad \sum_{g \in G_b} p_g^P - \sum_{(b,j) \in L} s_{bj} f_{bj}^P - \sum_{(i,b) \in L} s_{ib} e_{ib}^P \quad (37b)$$

$$+ r_b^P = R_b^P, \quad \forall b \in B, \quad (\lambda_b^P) \quad (37c)$$

$$\sum_{g \in G_b} p_g^Q - \sum_{(b,j) \in L} s_{bj} f_{bj}^Q - \sum_{(i,b) \in L} s_{ib} e_{ib}^Q \quad (37d)$$

$$+ r_b^Q = R_b^Q, \quad \forall b \in B, \quad (\lambda_b^Q) \quad (37e)$$

$$V_i^{2-} \leq W_{ii} \leq V_i^{2+}, \quad \forall i \in B, \quad (\kappa_i^-, \kappa_i^+) \quad (37f)$$

$$\|2W_{ij}^P, 2W_{ij}^Q, W_{ii} - W_{jj}\|_2 \leq W_{ii} + W_{jj}, \quad \forall (i,j) \in L, \quad (\delta_{ij}) \quad (37g)$$

$$0 \leq p_g^P \leq P_g^P, \quad \forall g \in G, \quad (\gamma_g^P) \quad (37h)$$

$$0 \leq p_g^Q \leq P_g^Q, \quad \forall g \in G, \quad (\gamma_g^Q) \quad (37i)$$

$$0 \leq r_i^P \leq R_i^P, \quad \forall i \in B, \quad (\alpha_i^P) \quad (37j)$$

$$0 \leq r_i^Q \leq R_i^Q, \quad \forall i \in B, \quad (\alpha_i^Q) \quad (37k)$$

$$s_{ij} \left(f_{ij}^P - (G_{ij}(W_{ii} - W_{ij}^P) + B_{ij}W_{ij}^Q) \right) = 0, \quad \forall (i,j) \in L, \quad (\pi_{ij}^P) \quad (37l)$$

$$s_{ij} \left(f_{ij}^Q - (B_{ij}(W_{ii} - W_{ij}^P) - G_{ij}W_{ij}^Q) \right) = 0, \quad \forall (i,j) \in L, \quad (\pi_{ij}^Q) \quad (37m)$$

$$s_{ij} \left(e_{ij}^P - (G_{ij}(W_{jj} - W_{ij}^P) - B_{ij}W_{ij}^Q) \right) = 0, \quad \forall (i,j) \in L, \quad (\psi_{ij}^P) \quad (37n)$$

$$s_{ij} \left(e_{ij}^Q - (B_{ij}(W_{jj} - W_{ij}^P) + G_{ij}W_{ij}^Q) \right) = 0, \quad \forall (i,j) \in L, \quad (\psi_{ij}^Q) \quad (37o)$$

$$\|f_{ij}\|_2 \leq F_{ij}, \quad \forall (i,j) \in L, \quad (\omega_{ij}) \quad (37p)$$

$$\|e_{ij}\|_2 \leq F_{ij}, \quad \forall (i,j) \in L, \quad (\tau_{ij}). \quad (37q)$$

(SOC-DR)

$$\begin{aligned} \max \quad & \sum_{i \in B} \left(R_i^P (\lambda_i^P - \alpha_i^P) + R_i^Q (\lambda_i^Q - \alpha_i^Q) - \kappa_i^+ V_i^{2+} + \kappa_i^- V_i^{2-} \right) \\ & + \sum_{g \in G} (-P_g^P \gamma_g^P - P_g^Q \gamma_g^Q) - \sum_{(i,j) \in L} F_{ij} (\omega_{ij}^0 + \tau_{ij}^0) \end{aligned} \quad (38a)$$

$$\text{s.t.} \quad -\gamma_g^P + \lambda_{b(g)}^P \leq 0, \quad \forall g \in G, \quad (p_g^P) \quad (38b)$$

$$-\gamma_g^Q + \lambda_{b(g)}^Q \leq 0, \quad \forall g \in G, \quad (p_g^Q) \quad (38c)$$

$$-\alpha_i^P + \lambda_i^P \leq 1, \quad \forall i \in B, \quad (r_i^P) \quad (38d)$$

$$-\alpha_i^Q + \lambda_i^Q \leq 0, \quad \forall i \in B, \quad (r_i^Q) \quad (38e)$$

$$s_{ij}(\lambda_i^P + \pi_{ij}^P) + \omega_{ij}^P = 0, \quad \forall (i,j) \in L, \quad (f_{ij}^P) \quad (38f)$$

$$s_{ij}(\lambda_i^Q + \pi_{ij}^Q) + \omega_{ij}^Q = 0, \quad \forall (i,j) \in L, \quad (f_{ij}^Q) \quad (38g)$$

$$s_{ij}(\lambda_j^P + \psi_{ij}^P) + \tau_{ij}^P = 0, \quad \forall (i,j) \in L, \quad (e_{ij}^P) \quad (38h)$$

$$s_{ij}(\lambda_j^Q + \psi_{ij}^Q) + \tau_{ij}^Q = 0, \quad \forall (i,j) \in L, \quad (e_{ij}^Q) \quad (38i)$$

$$\begin{aligned} & \kappa_i^+ - \kappa_i^- \\ & + \sum_{j: (i,j) \in L} \left[s_{ij}(-G_{ij}\pi_{ij}^P - B_{ij}\pi_{ij}^Q) - \delta_{ij}^0 - \delta_{ij}^\Delta \right] \\ & + \sum_{k: (k,i) \in L} \left[s_{ki}(-G_{ki}\psi_{ki}^P - B_{ki}\psi_{ki}^Q) - \delta_{ki}^0 + \delta_{ki}^\Delta \right] = 0, \forall i \in B, \quad (W_{ii}) \end{aligned} \quad (38j)$$

$$s_{ij} \left(G_{ij}(\pi_{ij}^P + \psi_{ij}^P) + B_{ij}(\pi_{ij}^Q + \psi_{ij}^Q) \right) - 2\delta_{ij}^P = 0, \quad \forall (i,j) \in L, \quad (W_{ij}^P) \quad (38k)$$

$$s_{ij} \left(-B_{ij}(\pi_{ij}^P - \psi_{ij}^P) + G_{ij}(\pi_{ij}^Q - \psi_{ij}^Q) \right) - 2\delta_{ij}^Q = 0, \quad \forall (i,j) \in L, \quad (W_{ij}^Q) \quad (38l)$$

$$(\omega_{ij}^P)^2 + (\omega_{ij}^Q)^2 \leq (\omega_{ij}^0)^2, \quad \forall (i,j) \in L, \quad (38m)$$

$$(\tau_{ij}^P)^2 + (\tau_{ij}^Q)^2 \leq (\tau_{ij}^0)^2, \quad \forall (i,j) \in L, \quad (38n)$$

$$(\delta_{ij}^P)^2 + (\delta_{ij}^Q)^2 + (\delta_{ij}^\Delta)^2 \leq (\delta_{ij}^0)^2, \quad \forall (i,j) \in L, \quad (38o)$$

$$\gamma_g^P, \gamma_g^Q \geq 0, \quad \forall g \in G, \quad (38p)$$

$$\alpha_i^P, \alpha_i^Q, \kappa_i^-, \kappa_i^+ \geq 0, \quad \forall i \in B, \quad (38q)$$

$$\lambda_i^P, \lambda_i^Q \text{ free}, \quad \forall i \in B, \quad (38r)$$

$$\omega_{ij}^0, \tau_{ij}^0, \delta_{ij}^0 \geq 0, \quad \forall (i,j) \in L, \quad (38s)$$

$$\pi_{ij}^P, \pi_{ij}^Q, \psi_{ij}^P, \psi_{ij}^Q, \delta_{ij}^P, \delta_{ij}^Q, \delta_{ij}^\Delta \text{ free}, \quad \forall (i,j) \in L. \quad (38t)$$

Lemma 26 Let

$$R' := \sum_{b \in B} R_b^P.$$

Every feasible solution to (SOC-DR) with nonnegative objective value satisfies

$$\sum_{(i,j) \in L} F_{ij}(\omega_{ij}^0 + \tau_{ij}^0) \leq R'.$$

Proof The proof is almost identical to the DCOPF case, except we need to show here that

$$\sum_{i \in B} (-\kappa_i^+ V_i^{2+} + \kappa_i^- V_i^{2-}) \leq 0.$$

It is clear that

$$\sum_{i \in B} (-\kappa_i^+ V_i^{2+} + \kappa_i^- V_i^{2-}) \leq \sum_{i \in B} W'(-\kappa_i^+ + \kappa_i^-).$$

Summing (38j) over all buses gives

$$\begin{aligned} \sum_{i \in B} -\kappa_i^+ + \kappa_i^- &= \sum_{(i,j) \in L} s_{ij}(-G_{ij}(\pi_{ij}^P + \psi_{ij}^P) - B_{ij}(\pi_{ij}^Q + \psi_{ij}^Q)) - 2 \sum_{(i,j) \in L} \delta_{ij}^0 \\ &= -2 \sum_{(i,j) \in L} (\delta_{ij}^P + \delta_{ij}^0) \\ &\leq 0, \end{aligned}$$

where we have substituted (38k) and used that $|\delta_{ij}^P| \leq \delta_{ij}^0$. $\square$

Moving towards bounding the λ variables, for surviving connected lines, we have with simple substitution

$$-B_{ij}(-\omega_{ij}^P - \lambda_i^P + \tau_{ij}^P + \lambda_j^P) + G_{ij}(-\omega_{ij}^Q - \lambda_i^Q + \tau_{ij}^Q + \lambda_j^Q) = 2\delta_{ij}^Q$$

and equivalently,

$$-B_{ij}(\lambda_i^P - \lambda_j^P) + G_{ij}(\lambda_i^Q - \lambda_j^Q) = -2\delta_{ij}^Q - B_{ij}(-\omega_{ij}^P + \tau_{ij}^P) + G_{ij}(-\omega_{ij}^Q + \tau_{ij}^Q).$$

Note that we also have

$$G_{ij}(\lambda_i^P + \lambda_j^P) + B_{ij}(\lambda_i^Q + \lambda_j^Q) = -2\delta_{ij}^P - G_{ij}(\omega_{ij}^P + \tau_{ij}^P) - B_{ij}(\omega_{ij}^Q + \tau_{ij}^Q).$$

In matrix form, using the convention that the node-arc incidence matrix N has +1 at node origins and -1 at node destinations, we express both relationships as

$$\begin{aligned} -\mathbf{B}_K N_K^\top \lambda_K^P + \mathbf{G}_K N_K^\top \lambda_K^Q &= -2\delta_K^Q + \mathbf{B}_K(\omega_K^P - \tau_K^P) - \mathbf{G}_K(\omega_K^Q - \tau_K^Q), \\ \mathbf{G}_K S_K^\top \lambda_K^P + \mathbf{B}_K S_K^\top \lambda_K^Q &= -2\delta_K^P - \mathbf{G}_K(\omega_K^P + \tau_K^P) - \mathbf{B}_K(\omega_K^Q + \tau_K^Q). \end{aligned}$$

Note that in the lossless case, $\mathbf{G}_K = 0$, decoupling λ_K^P and λ_K^Q and providing a path towards bounds in terms of δ^0 , ω^0 , and τ^0 .

To begin, we illustrate how our spectral bounds can be extended to signless incidence matrices. In the lossless setting, λ_K^P and λ_K^Q are decoupled, leaving us with the equations

$$\mathbf{B}_K S_K^\top \lambda_K^Q = \mathbf{v}_K, \quad (41a)$$

$$-\mathbf{B}_K N_K^\top \lambda_K^P = \mathbf{w}_K \quad (41b)$$

having defined

$$\begin{aligned} \mathbf{v}_K &:= -2\delta_K^P - \mathbf{G}_K(\omega_K^P + \tau_K^P) - \mathbf{B}_K(\omega_K^Q + \tau_K^Q), \\ \mathbf{w}_K &:= -2\delta_K^Q + \mathbf{B}_K(\omega_K^P - \tau_K^P) - \mathbf{G}_K(\omega_K^Q - \tau_K^Q). \end{aligned}$$

The right-hand sides of both equations are bounded by the following lemma.

Lemma 27 At every feasible solution to (SOC-DR) with nonnegative objective value, any surviving connected component $K \subseteq L$ satisfies

$$\max\{\|\mathbf{v}_K\|_\infty, \|\mathbf{w}_K\|_\infty\} \leq C_\delta := 2 \left(\|\delta^0\|_\infty + \|\mathbf{B}\|_\infty \max_{(i,j) \in L} \left\{ \frac{R'}{F_{ij}} \right\} \right), \quad (42)$$

where δ^0 and $\mathbf{B}$ are defined over all edges $(i, j) \in L$.

Proof From Lemma 26, we have that

$$\max_{(i,j) \in L} \left\{ \omega_{ij}^0, \tau_{ij}^0 \right\} \leq \max_{(i,j) \in L} \left\{ \frac{R'}{F_{ij}} \right\}.$$

In turn, the SOC constraints in the dual program imply that

$$\max_{(i,j) \in L} \left\{ |\omega_{ij}^P|, |\omega_{ij}^Q|, |\tau_{ij}^P|, |\tau_{ij}^Q| \right\} \leq \max_{(i,j) \in L} \left\{ \frac{R'}{F_{ij}} \right\},$$

while $\max_{(i,j) \in L} \{|\delta_{ij}^P|, |\delta_{ij}^Q|\} \leq \|\delta^0\|_\infty$. Thus, for any surviving connected component $K \subseteq L$, we have that

$$\|\mathbf{v}_K\|_\infty \leq 2 \left(\|\delta^0\|_\infty + \|\mathbf{B}\|_\infty \max_{(i,j) \in L} \left\{ \frac{R'}{F_{ij}} \right\} \right).$$

By identical reasoning, we have the exact same bound on $\|\mathbf{w}_K\|_\infty$ for any surviving connected component K . $\square$

We can now use our spectral framework to bound the spread of both λ_K^P and λ_K^Q , although a true bound on the entries of λ_K^P and λ_K^Q remains slightly out of reach. With a result akin to Lemma 21 for the SOC model, our graph-theoretic approach will yield absolute bounds on both λ_K^P and λ_K^Q . More specifically, with the exact same strategy as Section 4 above, we can bound the oscillation $\text{osc}_{b \in K} \lambda_K^P := \max_{b \in K} \lambda_b^P - \min_{b \in K} \lambda_b^P$. Then, with a slightly more nuanced argument that handles the signless incidence

matrix that arises in (41a), we obtain a similar bound on $\|\lambda_K^Q\|_\infty$ in terms of $|\lambda_{\min}^Q| := \min_{b \in K} |\lambda_b^Q|$. Throughout, it is useful to denote by

$$\rho_n := \sqrt{2 \left(1 - \cos \left(\frac{\pi}{2n-1} \right) \right)} \quad (43a)$$

the tight lower bound on $\sigma_{\min}(\tilde{N}_K^\top)$ that Theorem 1 yields. Recall that C_δ is the constant, depending on δ^0 , that appears in (42).

Finally, it is sometimes useful to reduce from a general connected component K to a tree. To do so, take an arbitrary spanning tree $T \subseteq K$. If $E(K)$ and $E(T)$ denote the edge sets of K and T respectively, then there exists a matrix $P_K \in \mathbb{R}^{|E(T)| \times |E(K)|}$ such that $P_K N_K^\top = N_T^\top$ and $P_K S_K^\top = S_T^\top$, where $N_T^\top$ and $S_T^\top$ denote transposes of the standard and signless incidence matrices of the spanning tree T , respectively. Multiplying (41b) and (41a) by $\mathbf{B}_K^{-1}$ and P_K in that order, we obtain

$$S_T^\top \lambda_K^Q = P_K \mathbf{B}_K^{-1} \mathbf{v}_K, \quad (43b)$$

$$N_T^\top \lambda_K^P = -P_K \mathbf{B}_K^{-1} \mathbf{w}_K. \quad (43c)$$

Importantly, because P_K is a row selection matrix, the right-hand sides of (43c) and (43b) remain bounded in norm by Lemma 27.

Lemma 28 Given a connected component K of $L \setminus A$ for a fixed attack A , any dual optimal point satisfies

$$\text{osc}_{b \in K} \lambda_K^P \leq \frac{\sqrt{n-1}}{\rho_n} C_\delta \max_{(i,j) \in L} \left\{ \frac{1}{B_{ij}} \right\},$$

where n is the number of vertices in the underlying graph.

Proof Assuming that $B_{ij} > 0$ for all $(i, j) \in L$, (43c) becomes

$$N_T^\top \lambda_K^P = -P_K \mathbf{B}_K^{-1} \mathbf{w}_K.$$

The right-hand side satisfies

$$\begin{aligned} \|P_K \mathbf{B}_K^{-1} \mathbf{w}_K\|_2 &\leq \max_{(i,j) \in L} \left\{ \frac{1}{B_{ij}} \right\} \sqrt{n-1} \|\mathbf{w}_K\|_\infty \\ &\leq \max_{(i,j) \in L} \left\{ \frac{1}{B_{ij}} \right\} \sqrt{n-1} C_\delta, \end{aligned}$$

leveraging the bound on $\|\mathbf{w}_K\|_\infty$ from Lemma 27 as well as the fact that for $v \in \mathbb{R}^n$, we have that $\|v\|_2 \leq \sqrt{n} \|v\|_\infty$. Importantly, we use the fact that the spanning tree T has at most $n-1$ edges.

On the other hand, picking the index $b \in K$ such that λ_b^P is minimized, recall from Section 4 that $\lambda_K^P - \lambda_b^P \mathbf{1}$ remains a solution to (43c). In particular, if $\tilde{\lambda}_K^P$ is the result of removing the entry of $\lambda_K^P - \lambda_b^P \mathbf{1}$ corresponding to λ_b^P and $\tilde{N}_T$ the result of removing the corresponding row from N_T , then

$$\tilde{N}_T^\top \tilde{\lambda}_K^P = -P_K \mathbf{B}_K^{-1} \mathbf{w}_K, \quad (44)$$

where $\|\tilde{\lambda}_K^P\|_\infty = \text{osc}_{b \in K} \lambda_K^P$ by construction. To conclude, we apply the exact same reasoning as in Section 4, observing that

$$\begin{aligned} \|\tilde{\lambda}_K^P\|_\infty &\leq \|\tilde{\lambda}_K^P\|_2 \leq \frac{1}{\sigma_{\min}(\tilde{N}_T^\top)} \|\tilde{N}_T^\top \tilde{\lambda}_K^P\|_2 \\ &\leq \frac{\sqrt{n-1}}{\rho_n} C_\delta \max_{(i,j) \in L} \left\{ \frac{1}{B_{ij}} \right\} \end{aligned}$$

as claimed. $\square$

With this, we can move towards the more challenging task of bounding the spread of λ_K^Q . In comparison to the previous lemma, such a bound is complicated by the fact that, unlike $N_K^\top$, the nullspace of the signless incidence matrix $S_K^\top$ is only nontrivial if K is bipartite. However, we can sidestep this difficulty with some casework, leveraging a bound from [37] on the minimal eigenvalue of the signless Laplacian $Q_K = S_K S_K^\top$ in the case that K is not bipartite. When K is bipartite, our usual approach, coupled with a bound on the smallest eigenvalue of a grounded signless Laplacian, still suffices with minimal modifications. First, we have the following bound on the minimal eigenvalue of the signless Laplacian when K is not bipartite.

Lemma 29 If $G = (V, E)$ is a connected, non-bipartite graph with $|V| = n \geq 2$ vertices and maximal degree Δ , then if Q denotes the signless Laplacian of G , the minimal eigenvalue of Q satisfies

$$\nu_n(Q) \geq \frac{1}{2\Delta n^2}.$$

Proof We leverage the bound on $\nu_n(Q)$ from [37], which was originally presented in [38, Theorem 3.2], up to an additional factor of two.

First, if $U, W \subset V$, let $e(U) := \{(i, j) \in E : i, j \in U\}$ and $e(U, W) := \{(i, j) \in E : i \in U, j \in W\}$. Similarly, define $\text{cut}(U) := e(U, V \setminus U)$. With this notation, [37] defines the quantity

$$\psi(G) := \min \left\{ \frac{2|e(U)| + 2|e(W)| + |\text{cut}(U \cup W)|}{|U \cup W|} : U, W \subseteq V, U \cap W = \emptyset, U \cup W \neq \emptyset \right\}.$$

In a sense, this quantity measures the “bipartiteness” of a graph, with $\psi(G) = 0$ if and only if G is bipartite. Indeed, $2|e(U)| + 2|e(W)| + |\text{cut}(U \cup W)| = 0$ if and only if U and W form a bipartition, as $\text{cut}(U \cup W) = \emptyset \implies U \cup W = V$, with $e(U) = e(W) = \emptyset$. Thus, in our setting

$$2|e(U)| + 2|e(W)| + |\text{cut}(U \cup W)| \geq 1,$$

while clearly $|U \cup W| \leq n$. Thus, for a non-bipartite graph G ,

$$\psi(G) \geq \frac{1}{n}.$$

Now, as shown in [38, Theorem 3.2] and noted in [37], we have that

$$\nu_n(Q) \geq \frac{\psi(G)^2}{2\Delta},$$

and the desired bound follows immediately from the lower bound on $\psi(G)$ for non-bipartite G . $\square$

With this result, we are equipped to handle both bipartite and non-bipartite components K that result from an attacker's choice of interdiction.

Lemma 30 Suppose that K is a connected component of $L \setminus A$ for a fixed attack A . If K is not bipartite, then any dual optimal point λ_K^Q satisfies

$$\text{osc}_{b \in K} \lambda_K^Q \leq 2n\sqrt{2\Delta m} \max_{(i,j) \in L} \left\{ \frac{1}{B_{ij}} \right\} C_\delta, \quad (45)$$

where C_δ is defined above, Δ is the maximum degree of the underlying graph $G = (V, E)$ containing K , with $|V| = n$ and $|E| = m$.

Proof If K is not bipartite, then $S_K^\top$ has full column rank. From (41a), we know that

$$S_K^\top \lambda_K^Q = \mathbf{B}_K^{-1} \mathbf{v}_K. \quad (46)$$

As in Lemma 28,

$$\|\mathbf{B}_K^{-1} \mathbf{v}_K\|_2 \leq \max_{(i,j) \in L} \left\{ \frac{1}{B_{ij}} \right\} \sqrt{m} C_\delta$$

so that

$$\|S_K^\top \lambda_K^Q\|_2 \leq \max_{(i,j) \in L} \left\{ \frac{1}{B_{ij}} \right\} \sqrt{m} C_\delta.$$

Because $S_K^\top$ has full column rank, we have that

$$\begin{aligned} \|\lambda_K^Q\|_\infty &\leq \|\lambda_K^Q\|_2 = \|(S_K^\top)^\dagger S_K^\top \lambda_K^Q\|_2 \leq \frac{1}{\sigma_{\min}(S_K^\top)} \|S_K^\top \lambda_K^Q\|_2 \\ &\leq \sqrt{2\Delta m} n \max_{(i,j) \in L} \left\{ \frac{1}{B_{ij}} \right\} C_\delta. \end{aligned}$$

In particular, it follows that

$$\text{osc}_{b \in K} \lambda_K^Q \leq 2\sqrt{2\Delta m} n \max_{(i,j) \in L} \left\{ \frac{1}{B_{ij}} \right\} C_\delta.$$

□

If K is bipartite, our path is slightly less direct. On one hand, it *is* possible to use the same strategy to bound the oscillation for buses on the same side of a bipartition of K . To this end, fix a bipartition $U, W \subseteq V$ of K . There exists a vector $\mathbf{u}_K \in \{\pm 1\}^{|K|}$ such that $S_K^\top \mathbf{u}_K = 0$, with $(\mathbf{u}_K)_b = 1$ for $b \in U$ and $(\mathbf{u}_K)_b = -1$ for $b \in W$. If $b \in K$ is the bus at which $|\lambda_b^Q|$ is minimized, we want to bound $\lambda_{b'}^Q - \lambda_b^Q$ for all remaining buses $b' \in K$. Assume, without loss of generality, that $b \in U$. If $b' \in U$, then observe that the corresponding entry of $\lambda_K^Q - \lambda_b^Q \mathbf{u}_K$ is given by $\lambda_{b'}^Q - \lambda_b^Q$, and

$$S_K^\top (\lambda_K^Q - \lambda_b^Q \mathbf{u}_K) = \mathbf{B}_K^{-1} \mathbf{v}_K,$$

and we may moreover reduce to the case of a spanning tree $T \subseteq K$ as above, obtaining

$$S_T^\top (\lambda_K^Q - \lambda_b^Q \mathbf{u}_K) = P_K \mathbf{B}_K^{-1} \mathbf{v}_K. \quad (47)$$

Additionally, the entry of $\lambda_K^Q - \lambda_b^Q \mathbf{u}_K$ corresponding to bus b is zero. Hence, removing the row of S_T corresponding to bus b to form $\tilde{S}_T$ and the corresponding entry of $\lambda_K^Q - \lambda_b^Q \mathbf{u}_K$ to form $\tilde{\lambda}_K^Q$, we obtain a system

$$\tilde{S}_T^\top \tilde{\lambda}_K^Q = P_K \mathbf{B}_K^{-1} \mathbf{v}_K. \quad (48)$$

Taking $\tilde{Q}_T = \tilde{S}_T \tilde{S}_T^\top$, where $\tilde{Q}_T$ denotes the grounded, signless Laplacian corresponding to bus b , Lemma 18 shows that

$$\sigma_{\min}(\tilde{S}_T^\top) \geq \sqrt{2 \left(1 - \cos \left(\frac{\pi}{2n-1} \right) \right)} = \rho_n.$$

Repeating our tried-and-true strategy for obtaining a bound using a pseudoinverse, we have that

$$\|\tilde{\lambda}_K^Q\|_\infty \leq \frac{\sqrt{n-1}}{\sigma_{\min}(\tilde{S}_T^\top)} \max_{(i,j) \in L} \left\{ \frac{1}{B_{ij}} \right\} C_\delta \leq \frac{\sqrt{n-1}}{\rho_n} \max_{(i,j) \in L} \left\{ \frac{1}{B_{ij}} \right\} C_\delta. \quad (49)$$

By the definition of $\tilde{\lambda}_K^Q$, it follows that the difference $\lambda_{b'}^Q - \lambda_b^Q$ satisfies the same bound. However, when $b' \in W = V \setminus U$, across the bipartition from the bus at which $|\lambda_b^Q|$ is minimized, we cannot use this approach. Indeed, shifting by a constant multiple of $\mathbf{u}_K$ yields, at best, a bound on $|\lambda_{b'}^Q + \lambda_b^Q|$ for $b' \in W$.

This can still be used to derive an oscillation type bound that, in the presence of a result analogous to Lemma 21, yields bounds on the entries of λ_K^Q . Specifically, if $b' \in W$, then returning to (47), we observe that $(\lambda_K^Q - \lambda_b^Q \mathbf{u}_K)_b = 0$, while $(\lambda_K^Q - \lambda_b^Q \mathbf{u}_K)_{b'} = \lambda_{b'}^Q + \lambda_b^Q$. Forming $\tilde{S}_T$ and $\tilde{\lambda}_K^Q$ as above, the exact same reasoning as in the case that $b' \in U$ yields

$$\|\tilde{\lambda}_K^Q\|_\infty \leq \frac{\sqrt{n-1}}{\rho_n} \max_{(i,j) \in L} \left\{ \frac{1}{B_{ij}} \right\} C_\delta.$$

However, we now observe that by the reverse triangle inequality

$$|\lambda_{b'}^Q| - |\lambda_b^Q| \leq |\lambda_{b'}^Q + \lambda_b^Q| \leq \frac{\sqrt{n-1}}{\rho_n} \max_{(i,j) \in L} \left\{ \frac{1}{B_{ij}} \right\} C_\delta.$$

Of course, the same inequality also applies to (49), yielding the following result.

Lemma 31 Suppose that K is a connected component of $L \setminus A$ for a fixed attack A . If K is bipartite, then any dual optimal point λ_K^Q satisfies

$$\|\lambda_K^Q\|_\infty \leq \frac{\sqrt{n-1}}{\rho_n} \max_{(i,j) \in L} \left\{ \frac{1}{B_{ij}} \right\} C_\delta + |\lambda_{\min}^Q|,$$

where C_δ is defined above and $|\lambda_{\min}^Q| := \min_{b' \in K} |\lambda_{b'}^Q|$.

7 Conclusion

We have provided a single-level mixed-integer programming formulation for the DCOPF interdiction problem that allows for interdiction of branches, buses, and generators and has bounded big- M values. The model is capable of finding optimal solutions for standard test problems. Using reduced big- M values improves solve times dramatically and allows for analysis of real-world systems. This work establishes a new state-of-the-art standard for DCOPF computational performance and theory.

More generally, we have presented a collection of tools for bounding dual variables using both direct methods and spectral bounds from grounded Laplacians. We established two new tight lower bounds on the minimal eigenvalues of grounded Laplacians. We then showed that both the direct and spectral methods are valuable for finding dual bounds in other interdiction problems, including a second-order cone formulation.

Additional work may investigate how our grounded Laplacian results can be applied further, particularly in second-order cone formulations. For the model we examined, our results can be made more general by considering the lossy case with nonzero conductance. Across all interdiction problems, realized dual variables are often smaller than bounded values, so there are opportunities to improve bounds and find tight examples. It may be possible to establish better bounds from an average-case perspective, for instance, by employing random graphs or randomly generated problem parameters.

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References

- [1] Zimmerman, R.D., Murillo-Sánchez, C.E., Thomas, R.J.: MATPOWER: Steady-state operations, planning, and analysis tools for power systems research and education. *IEEE Transactions on Power Systems* **26**(1), 12–19 (2011) <https://doi.org/10.1109/TPWRS.2010.2051168>
- [2] Wood, R.K.: Bilevel network interdiction models: Formulations and solutions. In: Cochran, J.J., Cox, L.A. Jr., Keskinocak, P., Kharoufeh, J.P., Smith, J.C. (eds.) *Wiley Encyclopedia of Operations Research and Management Science*. John Wiley & Sons, Hoboken, NJ (2011). <https://doi.org/10.1002/9780470400531.eorms0932>
- [3] Dempe, S.: *Foundations of Bilevel Programming*. Kluwer Academic, Boston, MA (2002)

- [4] Brown, G.G., Carlyle, W.M., Wood, R.K.: Optimizing Department of Homeland Security defense investments: Applying defender-attacker (-defender) optimization to terror risk assessment and mitigation. In: Department of Homeland Security Bioterrorism Risk Assessment: A Call for Change. The National Academies Press, Washington, DC (2008). Appendix E. <https://faculty.nps.edu/gbrown/docs/applyingattackerdefenderattackertoterror.pdf>
- [5] Kleinert, T., Labbé, M., Plein, F., Schmidt, M.: There’s no free lunch: On the hardness of choosing a correct big-M in bilevel optimization. *Operations Research* **68**(6), 1716–1721 (2020)
- [6] Kleinert, T., Schmidt, M.: Why there is no need to use a big-M in linear bilevel optimization: A computational study of two ready-to-use approaches. *Computational Management Science* **20**(1), 3 (2023)
- [7] Gurobi Optimization, LLC: Gurobi Optimizer Reference Manual (2023). <https://www.gurobi.com>
- [8] Motto, A.L., Arroyo, J.M., Galiana, F.D.: A mixed-integer LP procedure for the analysis of electric grid security under disruptive threat. *IEEE Transactions on Power Systems* **20**(3), 1357–1365 (2005)
- [9] Salmeron, J., Wood, K., Baldick, R.: Worst-case interdiction analysis of large-scale electric power grids. *IEEE Transactions on Power Systems* **24**(1), 96–104 (2009) <https://doi.org/10.1109/TPWRS.2008.2004825>
- [10] Schmidt, M., Thürauf, J.: An exact method for nonlinear network flow interdiction problems. *SIAM Journal on Optimization* **34**(4), 3623–3652 (2024)
- [11] Johnson, E.S., Dey, S.S.: A scalable lower bound for the worst-case relay attack problem on the transmission grid. *INFORMS Journal on Computing* **34**(4), 2296–2312 (2022)
- [12] Fattahi, S., Lavaei, J., Atamtürk, A.: A bound strengthening method for optimal transmission switching in power systems. *IEEE Transactions on Power Systems* **34**(1), 280–291 (2019) <https://doi.org/10.1109/TPWRS.2018.2867999>
- [13] Bienstock, D., Verma, A.: The $N - k$ problem in power grids: New models, formulations, and numerical experiments. *SIAM Journal on Optimization* **20**(5), 2352–2380 (2010) <https://doi.org/10.1137/08073562X>
- [14] Mastin, A., Baxter, A., Musselman, A., Watson, J.-P.: Best response intersection: An optimal algorithm for interdiction defense. *INFORMS Journal on Optimization* **5**(2), 172–190 (2023)
- [15] Jabr, R.A.: Optimal power flow using an extended conic quadratic formulation. *IEEE Transactions on Power Systems* **23**(3), 1000–1008 (2008)

- [16] Bai, X., Wei, H., Fujisawa, K., Wang, Y.: Semidefinite programming for optimal power flow problems. *International Journal of Electrical Power & Energy Systems* **30**(6–7), 383–392 (2008)
- [17] Coffrin, C., Van Hentenryck, P.: A linear-programming approximation of AC power flows. *INFORMS Journal on Computing* **26**(4), 718–734 (2014)
- [18] Yang, J., Zhang, N., Kang, C., Xia, Q.: A state-independent linear power flow model with accurate estimation of voltage magnitude. *IEEE Transactions on Power Systems* **32**(5), 3607–3617 (2017)
- [19] Kocuk, B., Dey, S.S., Sun, X.A.: Strong SOCP relaxations for the optimal power flow problem. *Operations Research* **64**(6), 1177–1196 (2016)
- [20] Wu, X., Conejo, A.J., Amjady, N.: Robust security constrained ACOPF via conic programming: Identifying the worst contingencies. *IEEE Transactions on Power Systems* **33**(6), 5884–5891 (2018)
- [21] Godsil, C., Royle, G.F.: *Algebraic Graph Theory*. Graduate Texts in Mathematics, vol. 207. Springer, New York, NY (2001). <https://doi.org/10.1007/978-1-4613-0163-9>
- [22] Fiedler, M.: Algebraic connectivity of graphs. *Czechoslovak Mathematical Journal* **23**(2), 298–305 (1973)
- [23] Horn, R.A., Johnson, C.R.: *Matrix Analysis*, 2nd edn. Cambridge University Press, Cambridge; New York (2013)
- [24] Xia, W., Cao, M.: Analysis and applications of spectral properties of grounded Laplacian matrices for directed networks. *Automatica* **80**, 10–16 (2017) <https://doi.org/10.1016/j.automatica.2017.01.009>
- [25] Pirani, M., Shahrivar, E.M., Fidan, B., Sundaram, S.: Robustness of leader–follower networked dynamical systems. *IEEE Transactions on Control of Network Systems* **5**(4), 1752–1763 (2018) <https://doi.org/10.1109/TCNS.2017.2756346>
- [26] Pirani, M., Sundaram, S.: On the smallest eigenvalue of grounded Laplacian matrices. *IEEE Transactions on Automatic Control* **61**(2), 509–514 (2016) <https://doi.org/10.1109/TAC.2015.2444191>
- [27] Garey, M.R., Johnson, D.S.: *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, San Francisco, CA (1979)
- [28] Abreu, N., Justel, C.M., Rojo, O., Trevisan, V.: Ordering trees and graphs with few cycles by algebraic connectivity. *Linear Algebra and its Applications* **458**, 429–453 (2014) <https://doi.org/10.1016/j.laa.2014.06.016>
- [29] Cvetković, D., Rowlinson, P., Simić, S.K.: *Signless Laplacians of finite graphs*.

- Linear Algebra and its Applications **423**(1), 155–171 (2007) <https://doi.org/10.1016/j.laa.2007.01.009>
- [30] Hassani Monfared, K., Mallik, S.: An analog of Matrix Tree Theorem for signless Laplacians. Linear Algebra and its Applications **560**, 43–55 (2019) <https://doi.org/10.1016/j.laa.2018.09.016>
 - [31] Brouwer, A.E., Haemers, W.H.: Spectra of Graphs. Universitext. Springer, New York, NY (2012). <https://doi.org/10.1007/978-1-4614-1939-6>
 - [32] Glover, F.: Improved linear integer programming formulations of nonlinear integer problems. Management Science **22**(4), 455–460 (1975)
 - [33] Balas, E.: Disjunctive programming and a hierarchy of relaxations for discrete optimization problems. SIAM Journal on Algebraic Discrete Methods **6**(3), 466–486 (1985)
 - [34] Bynum, M.L., Hackebeil, G.A., Hart, W.E., Laird, C.D., Nicholson, B.L., Sirola, J.D., Watson, J.-P., Woodruff, D.L.: Pyomo — Optimization Modeling in Python, 3rd edn. Springer Optimization and Its Applications, vol. 67. Springer, Cham, Switzerland (2021). <https://doi.org/10.1007/978-3-030-68928-5>
 - [35] Hart, W.E., Watson, J.-P., Woodruff, D.L.: Pyomo: modeling and solving mathematical programs in Python. Mathematical Programming Computation **3**(3), 219–260 (2011)
 - [36] Farivar, M., Low, S.H.: Branch flow model: Relaxations and convexification—Part I. IEEE Transactions on Power Systems **28**(3), 2554–2564 (2013)
 - [37] Borba, E.M., Schwerdtfeger, U.: Eigenvalue bounds for the signless p -Laplacian. The Electronic Journal of Combinatorics **25**(2) (2018) <https://doi.org/10.37236/6683> . Paper P2.2
 - [38] Desai, M., Rao, V.: A characterization of the smallest eigenvalue of a graph. Journal of Graph Theory **18**(2), 181–194 (1994) <https://doi.org/10.1002/jgt.3190180210>