

A SHARP LOCAL THRESHOLD FOR INTEGER FEASIBILITY IN THREE-WAY TRANSPORTATION POLYTOPES

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Abstract. A three-way transportation polytope consists of nonnegative three-way arrays with prescribed sums along every coordinate line. We show that its geometry and the difficulty of finding its integer points are both controlled by one local quantity: the number of cells on a line that can be positive. The change happens between two and three. The slim construction passes through fractional perfect matchings of hypergraphs and places each hyperedge on a cycle of six cells. If every line has at most two such cells, every nonempty polytope is a product of intervals, and integer feasibility is solvable in polynomial time for integral line sums. If lines of one direction may have three, every nonempty rational polytope is rationally affinely isomorphic to an entire polytope of format $3 \times c \times h$ with line sums in $\{0, 1, 2\}$, and integer feasibility is NP-complete. Hardness persists when the input includes a feasible array that is positive on every cell that can be positive and takes only the values $1/3$ and $2/3$ there. Unless $P = NP$, solving the linear relaxation and knowing its support does not help to round it. With all line sums equal to one, as in the linear relaxation of planar three-index assignment, the threshold becomes three cells in every direction. In this case the vertices have arbitrarily large denominators: every denominator $m \geq 2$ occurs with $O(\log m)$ fractional entries in arrays of order $O(\sqrt{\log m})$, and both bounds are optimal up to constant factors. The slim class with line sums in $\{0, 1, 2\}$ is universal for linear programming.

Key words. three-way transportation polytopes, integer feasibility, three-dimensional matching, planar three-index assignment, universality, line bounds, vertex denominators

MSC codes. 90C10, 52B12, 90C27, 90C60

1. Introduction. Consider a school timetable with N teachers, N classes, and N periods. Each teacher meets each class exactly once, each teacher teaches exactly one lesson in every period, and each class has exactly one lesson in every period. Some combinations of teacher, class, and period are forbidden. If every teacher and class can meet in at most two admissible periods, a timetable can be built by a chain of forced choices. Fixing one lesson determines its neighbors, like a row of falling dominoes (Figure 1). If every requirement may have three admissible options, the picture changes completely. The fractional timetables, in which a lesson may be split across periods, can then form any rational polytope, and deciding whether a genuine timetable exists is NP-complete.

This paper locates that change exactly, for three-way tables with arbitrary prescribed line sums. In each of the three directions, count on every line the cells that are positive in at least one feasible array, and take the largest count; we call the three resulting numbers the line bounds. With unit line sums, as in the timetable, a line bound of two in any one direction forces the feasible set to be a product of intervals. Line bounds $(3, 3, 3)$ already allow every rational polytope as a face. Allowing line sums equal to two lowers the threshold. Line bounds $(2, 2, 2)$ still force a product of intervals, whereas $(3, 2, 2)$ suffices for every rational polytope and for NP-complete integer feasibility, even in slim arrays, in which one index class has size three, that is, arrays of format $3 \times c \times h$. Why should cleaner line sums make the threshold higher? With every line sum equal to one, each direction has as many nonempty lines as the total weight of the array, so the number of cells beyond two per line, counted over all lines, is the same in every direction. A line sum of two lets this excess sit in a single direction. The same threshold separates half-integral vertices from vertices of every

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(a) Two admissible periods				(b) Three admissible periods			
	K_1	K_2	K_3		K_1	K_2	K_3
T_1	1, 2	2, 3 ²	3, 1 ¹	T_1	1	1, 2, 3	1, 2, 3
T_2	2, 3 ²	3, 1 ⁴	1, 2 ³	T_2	1, 2, 3	1, 2, 3	1, 2, 3
T_3	3, 1 ¹	1, 2 ⁴	2, 3 ³	T_3	1, 2, 3	1, 2, 3	1, 2, 3

FIG. 1. Timetables with forbidden combinations, for teachers T_1, T_2, T_3 , classes K_1, K_2, K_3 , and periods 1, 2, 3. Each cell lists the admissible periods for one teacher and one class. (a) With two admissible periods per cell, placing the lesson of T_1 and K_1 in period 1 (boxed) forces every other lesson; small numbers show the order in which lessons are forced, and forced periods are in bold. (b) With all three periods admissible, the same choice leaves the neighboring cells open.

denominator. Table 1 summarizes these thresholds.

The hardness does not come from failing to see where the solution lives. In Theorem 1.3 the input includes a feasible fractional array that is positive on every cell that can be positive at all and takes only the values $1/3$ and $2/3$. The linear relaxation is therefore solved and its support fully revealed, yet, unless $P = NP$, no method running in polynomial time can turn this information into an integer array or certify that none exists. Irving and Jerrum [15] studied integer feasibility for such tables in statistical data security. There the row, column, and file sums of a table of counts are published, and the question is whether an integer table with those sums exists.

1.1. Notation and the local universality theorem. We now make these notions precise. For finite sets A, B, C , let $P(b)$ be the polytope of nonnegative arrays $(x_{abc})_{(a,b,c) \in A \times B \times C}$ satisfying

$$\sum_c x_{abc} = b_{ab}^{AB}, \quad \sum_b x_{abc} = b_{ac}^{AC}, \quad \sum_a x_{abc} = b_{bc}^{BC}.$$

These equations prescribe the sum of x along every *line*, that is, along every set of cells that share two of their three indices; we call the prescribed numbers the *line sums*. This model is also called a *planar* transportation polytope. The axial model instead prescribes sums over coordinate planes [16, Theorem 2.17].

For $S \subseteq A \times B \times C$, write

$$P_S(b) = \{x \in P(b) : x_f = 0 \text{ for every } f \notin S\}.$$

When nonempty, this is a face of $P(b)$. The *union support* of a polytope Q is

$$\text{supp}(Q) = \{f : x_f > 0 \text{ for some } x \in Q\}.$$

Thus an allowed coordinate need not belong to the union support: the equations for the line sums may force it to zero.

Let $d_{AB}(S)$ be the largest number of triples of S sharing an AB pair, and define $d_{AC}(S)$ and $d_{BC}(S)$ in the same way. We say that S has *line bounds* (p, q, r) if these three numbers are at most p, q, r , respectively. The same terminology for a polytope refers to its union support. Measuring on the union support makes the line bounds a

property of the polytope rather than of its description, and it weakens the hypothesis of Proposition 2.1.

The *tristochastic polytope* Δ_N is the case $A = B = C = [N]$ with every line sum equal to one. Its integer points are the incidence arrays of Latin squares. In particular, it is the polytope of fractional Latin squares, or the linear programming relaxation of planar three-index assignment. In the timetable of the opening paragraph, its integer points are exactly the timetables in which no combination is forbidden.

THEOREM 1.1 (Local dichotomy). *Fix line bounds (p, q, r) and consider the polytopes $P_S(b)$ whose union support has these line bounds.*

1. *General line sums. If $\min\{p, q, r\} \geq 2$ and $\max\{p, q, r\} \geq 3$, every nonempty rational polytope is rationally affinely isomorphic to such a polytope, deciding whether such a polytope with integral line sums contains an integer point is NP-complete, and for every $m \geq 2$ some such polytope with integral line sums has a vertex of denominator m . Otherwise every nonempty such polytope is affinely isomorphic to a product of intervals, integer feasibility is solvable in polynomial time, and every vertex is half-integral when the line sums are integral.*
2. *Unit line sums. If $\min\{p, q, r\} \geq 3$, every nonempty rational polytope is rationally affinely isomorphic to such a face of some Δ_N , deciding whether a specified such face contains an integer point is NP-complete, and for every $m \geq 2$ some such face has a vertex of denominator m . Otherwise every nonempty such face is affinely isomorphic to a product of intervals, integer feasibility is solvable in polynomial time, and every vertex is half-integral.*

THEOREM 1.2 (Explicit representations). *Every nonempty rational polytope, given by rational linear inequalities, is rationally affinely isomorphic to each of the following, and both representations can be computed in polynomial time:*

1. *a face of some Δ_N whose union support has line bounds $(3, 3, 3)$;*
2. *an entire $P(b)$ of format $3 \times c \times h$, with all line sums in $\{0, 1, 2\}$, whose union support has line bounds $(3, 2, 2)$. In the two directions bounded by two, every line sum is zero or one.*

Here an affine isomorphism identifies the entire feasible sets, including their faces and vertices. With bounded line sums these isomorphisms cannot in general preserve integer points (Section 4.2), but they do preserve them when the input is already a literal system, a system of equations $\ell_1 + \dots + \ell_q = 1$ in variables $t_i \in [0, 1]$ and their complements $1 - t_i$ (Section 4.1).

The next theorem states the threshold for integer feasibility.

THEOREM 1.3 (Integer feasibility at line bounds $(3, 2, 2)$). *It is NP-complete to decide whether an entire planar transportation polytope $P(b)$ of format $3 \times c \times h$ contains an integer point, even when all of the following hold:*

1. *every line sum belongs to $\{0, 1, 2\}$;*
2. *the union support has line bounds $(3, 2, 2)$, and line sums in the last two directions belong to $\{0, 1\}$;*
3. *a feasible rational point, positive on the union support and taking only the values $1/3$ and $2/3$ there, is supplied with the input.*

The *denominator* of a rational point x is the least positive integer m for which mx is integral. With unit line sums and at most three positive entries per line, vertex denominators are unbounded, and the next theorem measures how few fractional

TABLE 1

Local thresholds for geometry and integer feasibility. Geometry is up to rational affine isomorphism. The complexity column assumes integral line sums and explicit input. The rows with unit line sums concern specified faces of Δ_N ; the last row concerns an entire $P(b)$ of format $3 \times c \times h$. See Theorem 1.1, Proposition 2.1, Section 3, and Corollary 5.4. Vertex denominators refer to integral line sums.

Line sums	Line bounds	Possible geometry	Integer feasibility	Vertex denominators
Unit	Some direction at most two	Products of intervals	P	1 or 2
Unit	$(3, 3, 3)$ or larger	Every rational polytope as a face	NP-complete	every $m \geq 2$
General	$(2, 2, 2)$, or some direction at most one	Products of intervals	P	1 or 2
$\{0, 1, 2\}$	$(3, 2, 2)$ or larger, up to permutation	Every rational polytope as an entire slim $P(b)$	NP-complete	every $m \geq 2$

entries and how small an array suffice.

THEOREM 1.4 (Prescribed denominators with few fractional entries). *Let $m \geq 2$.*

1. *An explicit construction gives a vertex of Δ_N of exact denominator m , with at most three positive entries on each line, $O(\log m)$ fractional entries, and $N = O(\log m)$.*
2. *Merging vertices of this construction keeps the number of fractional entries and gives $N = O(\sqrt{\log m})$.*
3. *Every vertex of Δ_N of denominator m has at least $\frac{3}{2} \log_2 m$ fractional entries, and $N \geq \sqrt{\frac{1}{2} \log_2 m}$.*

The upper and lower bounds determine both the number of fractional entries and the array order up to constant factors. The order $N = \Theta(\sqrt{\log m})$ matches the minimum number of vertices of a simplicial complex whose first homology has torsion $\mathbb{Z}/m\mathbb{Z}$, determined by Newman [21]. The match is not a coincidence: by Proposition 6.1, the support of every vertex of denominator m is a two-dimensional complex on at most $3N$ vertices whose first homology has an element of order m . Both constructions are deterministic and run in polynomial time: the direct one gives $N = O(\log m)$, and a greedy merging of vertices gives $N = O(\sqrt{\log m})$.

COROLLARY 1.5. *Let $D(N)$ be the largest denominator of a vertex of Δ_N . Then $\log_2 D(N) \leq 2N^2$ for every N , and $\log_2 D(N) = \Theta(N^2)$ as $N \rightarrow \infty$.*

1.2. How the constructions work. The engine of the slim construction is a cycle of six cells (Figure 2). A hyperedge $e = (a, b, c)$ of a tripartite hypergraph occupies six cells of a 3×3 slice, two in each row and each column. Every row and every column has line sum one, so the values alternate between t_e and $1 - t_e$ around the cycle. A line with a fixed slice and a fixed vertex v collects one cell from every hyperedge through v ; for a vertex of degree three, the line sums 2, 0, 1 on these lines turn them into the matching constraint $\sum_{e \ni v} t_e = 1$. These lines have up to three cells and all other lines have two, which is where the bounds $(3, 2, 2)$ come from.

For integer feasibility we start from positive 1-in-3 satisfiability, in which every clause becomes an equation $t_i + t_j + t_k = 1$. The gadgets of Průša and Werner

(a) One hyperedge $e = (a, b, c)$

	a	b	c	
$i = 1 \{a, c\}$	$1 - t_e$	0	t_e	sum 1
$i = 2 \{b, c\}$	0	t_e	$1 - t_e$	sum 1
$i = 3 \{a, b\}$	t_e	$1 - t_e$	0	sum 1
	sum 1	sum 1	sum 1	

(b) Vertex $a \in A$ in e_1, e_2, e_3

	e_1	e_2	e_3	
$i = 1$	$1 - t_{e_1}$	$1 - t_{e_2}$	$1 - t_{e_3}$	sum 2
$i = 2$	0	0	0	sum 0
$i = 3$	t_{e_1}	t_{e_2}	t_{e_3}	sum 1

FIG. 2. The construction of Lemma 3.1. (a) The six cells of one hyperedge $e = (a, b, c)$ in the slices $i = 1, 2, 3$. The six shaded cells form a cycle in which consecutive cells share a row or a column. Each row and each column has line sum one, so the entries alternate between t_e and $1 - t_e$ around the cycle; the cells marked 0 are excluded by zero line sums. (b) For a vertex $a \in A$ lying in three hyperedges, the line with fixed i and a collects one cell from each hyperedge. Its line sums 2, 0, and 1 encode the matching constraint $t_{e_1} + t_{e_2} + t_{e_3} = 1$. Lines of this kind are the only ones with three cells.

turn such a system into a fractional perfect matching problem: every occurrence of a variable gets its own edges, and a cycle through the occurrences of each variable forces them to carry the same weight (Figure 5). The cycles of six cells then give Theorem 1.3, and the assignment $t_i = 1/3$ gives the fractional witness.

For universality, a normalization step in front writes any rational polytope as a literal system, using halving and addition with complements. The same gadgets, extended to complements and fixed values, then give Theorem 1.2(2).

Unit line sums forbid the line sum two used above. Here each variable becomes a triangulated sphere with triangular holes: interior edges propagate t and $1 - t$ across the sphere, and gluing the holes of different spheres imposes the input equations. A Latin completion then embeds the result as a face of Δ_N .

Merging vertices of the same class leaves the equations unchanged as long as no two edges collide. A greedy coloring finds such a merging and shrinks the arrays to order $O(\sqrt{\log m})$.

Figure 3 shows how the constructions fit together.

Section 2 treats lines with at most two positive cells. Section 3 proves Theorem 1.3, and Section 4 adds the normalization and proves Theorem 1.2(2). Section 5 handles unit line sums, and Section 6 proves Theorem 1.4. Section 7 collects open questions. Appendix A describes the exact certificates.

1.3. Relation to earlier work. Integer feasibility for three-way tables with prescribed line sums is a classical NP-complete problem. Irving and Jerrum [15, Section 4] proved NP-completeness even when every prescribed row, column, and file sum is zero or one. De Loera and Onn [6, Theorem 1.1 and Section 2] proved NP-completeness for slim tables of format $(3, c, h)$, even with line sums encoded in unary. In their construction both the line sums and the line bounds grow with the input: a line sum equal to $n^2 - 1$ occurs, and a feasible point has n^2 positive entries on a single line.

Three-dimensional matching remains NP-complete when every element lies in at most three triples [13, SP1]. Lemma 3.1 places each triple on a cycle of six cells and turns the corresponding fractional matching polytopes into slim transportation

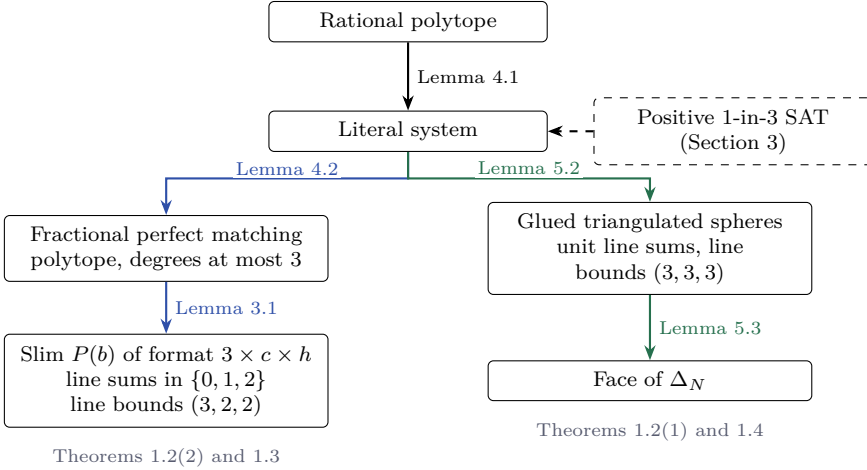


FIG. 3. *The two constructions. Both start from a literal system. The matching route (left) gives the slim representation with line sums in $\{0, 1, 2\}$; the surface route (right) gives faces of Δ_N with unit line sums. The reduction for integer feasibility enters at the literal system through positive 1-in-3 SAT.*

polytopes with line sums in $\{0, 1, 2\}$ and line bounds $(3, 2, 2)$; this already gives items 1 and 2 of Theorem 1.3. The substance of the theorem is item 3. An instance of three-dimensional matching need not have a fractional perfect matching that is positive on every triple, so we start instead from positive 1-in-3 satisfiability, where the assignment $t_i = 1/3$ is always feasible, and pass through the matching gadgets of Průša and Werner [23, Section 5.2], which keep the values $1/3$ and $2/3$. Together with Proposition 2.1, Theorem 1.3 places the tractability boundary exactly between $(2, 2, 2)$ and $(3, 2, 2)$.

For linear optimization the same slim class is universal. De Loera and Onn [7, Theorem 1.1] showed that every rational polytope can be represented, in polynomial time, as a slim transportation polytope of format $(r, c, 3)$, through a coordinate projection that is bijective on real and integer points. Theorem 1.2(2) keeps the slim format while bounding the union support in every direction and restricting every line sum to $\{0, 1, 2\}$. Průša and Werner [23] showed that fractional perfect matching polytopes of degree at most three are universal for linear programming; combined with Lemma 3.1, this gives Corollary 4.4.

With all line sums equal to one, $P(b)$ is the linear relaxation of planar three-index assignment, an optimization problem that is NP-hard [12]; its integer points are Latin squares. The convex hull of the Latin squares, whose linear relaxation is Δ_N , has been studied through its dimension and facets [10, 1, 20]. Colbourn [4] proved that completing a partial Latin square is NP-complete. Prescribed entries define a face of Δ_N , so his theorem already gives hardness for faces, with no local restriction. Section 5 shows that hardness and universality for faces persist with line bounds $(3, 3, 3)$.

The vertex structure of Δ_N has also received considerable attention. Linial and Luria [18] denote it by $\Omega_N^{(2)}$, and Taranenko [26] by Ω_N^3 . We follow the notation of Linial, Luria and Trakhtman [19], who prove that Δ_N has at least $L_N^{2-o(1)}$ vertices, where L_N is the number of Latin squares of order N . Potapov and Taranenko [22] study further asymptotic bounds. Taranenko's enumeration for order four contains

denominators 1 through 8, and her product constructions give additional vertices. For the axial assignment polytope, in which every sum over a coordinate plane equals one, Gromova showed that every positive rational number below one occurs as an entry of some vertex [5, Theorem 3.15]. Theorem 1.4 asks how large an array must be to realize a prescribed denominator, and determines the answer up to constant factors, both for the number of fractional entries and for the array order.

2. The case of at most two entries per line. We begin with the graph description that explains both the geometric and algorithmic behavior below the universality thresholds.

PROPOSITION 2.1. *Let $Q = P_S(b)$ be nonempty, and suppose that its union support has line bounds $(2, 2, 2)$. Then Q is affinely isomorphic to a product of intervals, allowing degenerate factors. If the line sums are integral, every vertex is half-integral, and integer feasibility and linear optimization over the integer points of Q can be solved in polynomial time.*

Proof. Delete every coordinate outside the union support. Each equation with a nonzero line sum now contains one or two variables. Form a graph whose vertices are the remaining coordinates and whose edges correspond to the equations with two variables (Figure 4).

Choose a root coordinate t in a connected component. Along a spanning tree, each equation $x_u + x_v = b_e$ expresses one variable in terms of the other. Hence every coordinate in the component has the form

$$x_v = \varepsilon_v t + c_v, \quad \varepsilon_v \in \{-1, 1\}.$$

A remaining equation either gives a consistency condition independent of t , or fixes t ; an equation of the second kind arises from an odd cycle or from an equation with a single variable. Otherwise t remains free and nonnegativity restricts it to an interval. This interval is bounded because the transportation polytope is bounded. Different components involve disjoint sets of variables, which proves the product description.

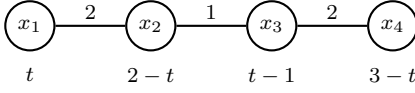
Suppose now that the line sums are integral. Then every c_v is integral. An equation with a single variable fixes t to an integer, while an odd cycle fixes $2t$ to an integer. If t remains free, the inequalities $\varepsilon_v t + c_v \geq 0$ give integer interval endpoints. Thus every vertex of Q has coordinates in $\frac{1}{2}\mathbb{Z}$.

The same description gives an algorithm. In each fixed component, check whether the root value is integral. In a free component, choose an integer in its feasible interval. These conditions are necessary and sufficient because the root coordinate is t itself and all other coordinates differ from t or $-t$ by integers. A linear objective becomes a linear function of the free parameters, so integer optimization amounts to choosing the appropriate endpoint in each free component. Propagation, consistency checks, and interval computations take polynomial time in the input bit length.

If the union support is not given, it can first be found by testing real feasibility and maximizing each allowed coordinate by linear programming. This requires only polynomially many linear programs in the explicit input encoding. If the allowed set already has line bounds $(2, 2, 2)$, the graph algorithm can instead be applied directly, including its feasibility checks. $\square$

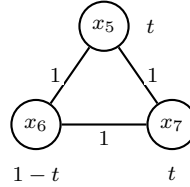
Unit line sums make the threshold higher because a bound of two in just one direction is already enough. To see this, choose a relative interior point x , which is positive on the union support. Let s be the support size, $w = \sum_f x_f$ the total mass, and q the number of entries equal to one. In each direction there are exactly w active

(a) A path



nonnegativity: $1 \leq t \leq 2$

(b) An odd cycle



$2t = 1$, so $t = 1/2$

FIG. 4. The graph in the proof of Proposition 2.1. Vertices are coordinates, edges are equations with two variables, and edge labels are line sums. (a) Along a path the values alternate between $t+c$ and $-t+c$, and nonnegativity leaves the interval $1 \leq t \leq 2$. (b) An odd cycle fixes $2t$; here $t = 1/2$.

lines, since each has sum one. There are exactly q lines of degree one: such a line contains an entry equal to one, and every entry equal to one is alone on all three of its lines.

Let n_d be the number of active lines of degree d in a chosen direction. Counting entries and active lines gives

$$\sum_{d \geq 3} (d-2)n_d = s - 2w + q.$$

The right-hand side is independent of the direction. If one direction has no line of degree at least three, this sum is zero in every direction. All three line bounds are therefore at most two, and Proposition 2.1 applies.

A bound of one in any direction fixes every coordinate directly. Finally, a triangle is not affinely isomorphic to a product of intervals. These observations prove the remaining parts of Theorem 1.1. They also show why a line sum of two is necessary in the $(3, 2, 2)$ construction: if all nonzero line sums were one, either direction with line bound two would force a product of intervals.

Remark (lines with three or more cells). The proof of Proposition 2.1 also measures how much of an instance lies above the threshold. A line with one cell fixes that cell, and lines with two cells join the cells into paths and cycles on which every coordinate has the form $\pm t + c$, with one parameter t per component. Each remaining line has at least three cells and gives a linear equation in these parameters with coefficients ± 1 . With integral line sums, integer feasibility therefore reduces to an integer program whose number of variables is at most the number k of cells on lines with at least three cells. Lenstra's algorithm [17] solves it in time $f(k)$ times a polynomial in the input size. All the difficulty in Theorem 1.3 lies on the lines with three cells.

3. Integer feasibility at line bounds $(3, 2, 2)$. This section proves Theorem 1.3. A positive 1-in-3 formula becomes a system of equations $t_i + t_j + t_k = 1$; the gadgets of Section 3.2 turn it into a fractional perfect matching polytope, and the cycles of six cells of Section 3.1 turn that polytope into a slim transportation polytope.

3.1. Cycles of six cells. For a tripartite three-uniform hypergraph $H = (A \sqcup B \sqcup C, E)$, write

$$\mathcal{M}(H) = \left\{ t \in \mathbb{R}_{\geq 0}^E : \sum_{e \ni v} t_e = 1 \text{ for every } v \in A \sqcup B \sqcup C \right\}$$

for its fractional perfect matching polytope.

LEMMA 3.1 (Cycles of six cells). *Let $H = (A \sqcup B \sqcup C, E)$ have no isolated vertices and maximum vertex degree at most three. Then $\mathcal{M}(H)$ is isomorphic, by an integral affine map, to an entire planar transportation polytope of format $3 \times |A \sqcup B \sqcup C| \times |E|$, with line sums in $\{0, 1, 2\}$ and line bounds $(3, 2, 2)$. The line sums in the last two directions are zero or one.*

Proof. Set $I = [3]$ and $V = A \sqcup B \sqcup C$, and use array indices $(i, v, e) \in I \times V \times E$. For $e = (a, b, c)$, put

$$\begin{pmatrix} X_{1,a,e} & X_{1,b,e} & X_{1,c,e} \\ X_{2,a,e} & X_{2,b,e} & X_{2,c,e} \\ X_{3,a,e} & X_{3,b,e} & X_{3,c,e} \end{pmatrix} = \begin{pmatrix} 1-t_e & 0 & t_e \\ 0 & t_e & 1-t_e \\ t_e & 1-t_e & 0 \end{pmatrix},$$

with all other entries zero. Prescribe $b_{ie}^{IE} = 1$ for every (i, e) , $b_{ve}^{VE} = 1$ for $v \in e$ and zero otherwise, and

$$(b_{1v}^{IV}, b_{2v}^{IV}, b_{3v}^{IV}) = \begin{cases} (d(v) - 1, 0, 1), & v \in A, \\ (0, 1, d(v) - 1), & v \in B, \\ (1, d(v) - 1, 0), & v \in C. \end{cases}$$

Since $1 \leq d(v) \leq 3$, all line sums have the claimed values. The zero line sums exclude every cell outside the six cells of each hyperedge. If $d(v) = 1$, a further line sum is zero; this forces the corresponding $1 - t_e$ coordinate to vanish, consistently with the matching equation $t_e = 1$.

Conversely, for a fixed e , the IE and VE equations with line sum one on its cycle of six cells force its entries to alternate between a single t_e and $1 - t_e$ (Figure 2(a)), with $0 \leq t_e \leq 1$. The remaining IV equations are exactly $\sum_{e \ni v} t_e = 1$ and its complement $\sum_{e \ni v} (1 - t_e) = d(v) - 1$. Hence every feasible array comes from a unique point of $\mathcal{M}(H)$. The map is integral affine, and its inverse reads $t_e = X_{3,a,e}$ for $e = (a, b, c)$.

An IV line contains at most $d(v) \leq 3$ of these cells (Figure 2(b)). Every IE or VE line contains at most two. Thus $(d_{IV}, d_{IE}, d_{VE}) \leq (3, 2, 2)$ on the union support. The construction uses $6|E|$ cells and polynomially many line sums. $\square$

Remark (cycles of six cells as flags). Read the slice index i as a type of edge of the triangle $e = (a, b, c)$: $i = 1$ for $\{a, c\}$, $i = 2$ for $\{b, c\}$, and $i = 3$ for $\{a, b\}$. The six cells of Lemma 3.1 are then exactly the triples (i, v, e) with v in the edge of type i , that is, the flags vertex $\subset$ edge $\subset$ triangle. The line families IE , VE , and IV correspond to the pairs of a triangle and an edge, of a triangle and a vertex, and of a vertex and an edge. The flag encoding in Section 5 uses the same six flags with genuine edges. Lemma 3.1 replaces each edge by one of three types, and this compression is what makes the array slim.

3.2. Matching gadgets.

LEMMA 3.2 (Matching gadgets). *Let a system of equations $t_i + t_j + t_k = 1$ in variables $t_1, \dots, t_r \geq 0$ be given, in which every variable occurs. Its solution set is isomorphic to $\mathcal{M}(H)$ for a tripartite three-uniform hypergraph H whose vertex degrees are two or three, by an integral affine map whose inverse is a coordinate projection. The construction has size linear in the number of equations, and the assignment $t_i = 1/3$ maps to edge weights in $\{1/3, 2/3\}$.*

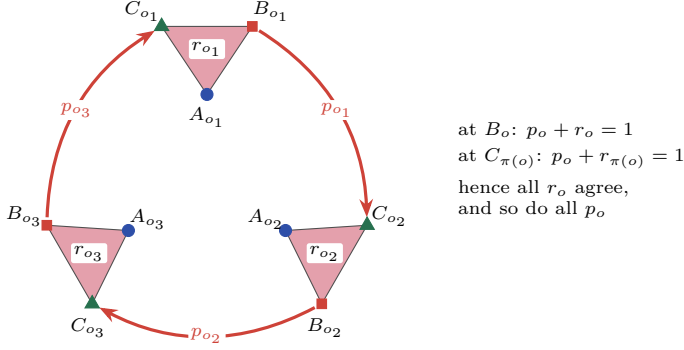


FIG. 5. The matching gadget of Lemma 3.2 for one variable with three occurrences o_1, o_2, o_3 and $\pi = (o_1 o_2 o_3)$. The edge $R_o = (A_o, B_o, C_o)$ is shaded, and the edge P_o joins B_o to $C_{\pi(o)}$; its third vertex $A_{s,0}$ is not drawn. The equations at B_o and $C_{\pi(o)}$ force the weights to agree around the cycle.

Proof. We use the construction of Průša and Werner [23, Section 5.2] and record its unique extension property. Index the equations by s and their occurrences by $o = (s, j)$, $1 \leq j \leq 3$. Choose a permutation π that cycles through the occurrences of each variable. Create vertices $A_{s,j}, B_{s,j}, C_{s,j}$ for $0 \leq j \leq 3$, and edges

$$P_o = (A_{s,0}, B_o, C_{\pi(o)}), \quad Q_o = (A_o, B_{s,0}, C_{s,0}), \quad R_o = (A_o, B_o, C_o).$$

Write p_o, q_o, r_o for their weights. The equations at A_o, B_o, C_o give

$$q_o + r_o = p_o + r_o = p_{\pi^{-1}(o)} + r_o = 1.$$

Thus all occurrences of a variable t_i carry a common weight, with $p_o = q_o = t_i$ and $r_o = 1 - t_i$ (Figure 5). The three vertices indexed by $(s, 0)$ impose three copies of equation s . Every vertex has degree two or three, these identities give an integral affine bijection with a coordinate projection as inverse, and at $t_i = 1/3$ the weights are $p_o = q_o = 1/3$ and $r_o = 2/3$. $\square$

3.3. Proof of Theorem 1.3. We start from a system that is always feasible over the reals, while its integer feasibility is NP-complete.

The source problem is Positive 1-in-3 SAT: each clause consists of three unnegated Boolean variables and requires exactly one of them to be true. Its NP-completeness is classical [24]; stronger restrictions are treated in [14]. Replacing Boolean variables by variables in $[0, 1]$ turns each clause into an equation $t_i + t_j + t_k = 1$. The constant assignment $t_i = 1/3$ satisfies every such equation.

Proof of Theorem 1.3. Let the source formula have q clauses, and discard variables that do not occur. Apply Lemma 3.2 to its equations, and then apply Lemma 3.1. The result is an entire $P(b)$ of format $3 \times c \times h$ with the stated line sums and line bounds. Both maps are integral affine isomorphisms, so integer points correspond exactly to satisfying Boolean assignments.

At the constant assignment $t_i = 1/3$, the matching weights belong to $\{1/3, 2/3\}$. Lemma 3.1 takes copies and complements of these weights. It therefore supplies a feasible point y^* with the required two values, positive on all these cells. These cells are consequently exactly the union support. They can also be read from the line sums as

$$U(b) = \{(i, v, e) : b_{iv}^{IV} > 0, b_{ie}^{IE} > 0, b_{ve}^{VE} > 0\} :$$

all matching degrees are two or three, so this set consists precisely of the six cells of each hyperedge.

The matching construction has $12q$ vertices and $9q$ edges. Thus the array format is $3 \times 12q \times 9q$, and even a dense listing of the line sums and the witness has polynomial size. This proves NP-hardness with all the stated restrictions.

Every feasible coordinate is bounded by a line sum of at most one in either of the last two directions. An integer feasible point is therefore an array with entries zero and one of polynomial encoding length. Its line sums can be checked in polynomial time. The line bounds can be checked by the linear programs described in Proposition 2.1; the supplied fractional point and its positivity on that support can then also be checked. Hence the restricted decision problem belongs to NP. $\square$

A small example illustrates why the supplied fractional point does not settle integer feasibility. Consider

$$\begin{aligned} t_1 + t_2 + t_3 &= 1, & t_1 + t_2 + t_4 &= 1, \\ t_1 + t_3 + t_4 &= 1, & t_2 + t_3 + t_4 &= 1. \end{aligned}$$

Subtracting equations shows that all four variables are equal, so the only solution is $t_i = 1/3$. Lemmas 3.2 and 3.1 therefore give a nonempty slim transportation polytope with the restrictions in Theorem 1.3, but with no integer point. This is an analogue, with denominator three, of Vlach's $3 \times 4 \times 6$ example, which has real but no integer feasible points, recovered by De Loera and Onn [6, Example 2.2]. Disjoint copies of this system of four equations give a family of singleton polytopes with the same restrictions on line sums and line bounds. The general reduction allows arbitrary collections of these constraints with three variables.

4. Every rational polytope as a slim transportation polytope. This section proves Theorem 1.2(2). A normalization step writes an arbitrary rational polytope as a literal system. The gadgets of Section 3.2, extended to complements and fixed values, turn it into a fractional perfect matching polytope, and Lemma 3.1 does the rest.

4.1. Literal systems. A *literal* is a variable t_i or its complement $1 - t_i$. A *literal system* has variables $t_1, \dots, t_r \in [0, 1]$ and equations

$$(4.1) \quad \ell_{j1} + \dots + \ell_{jq_j} = 1, \quad q_j \in \{2, 3\}.$$

Repeated occurrences are allowed. We assume that each variable occurs at least once; if necessary, we add the tautology $t_i + (1 - t_i) = 1$. Let k_i be the number of occurrences of variable i , counting repetitions. On expanding the complements and moving constants to the right-hand side, write the system as $Rt = a$.

Every rational polytope can be put into this form by introducing variables whose values are uniquely determined. This is the only stage at which the construction may change the integer points.

LEMMA 4.1. *Every nonempty rational polytope is rationally affinely isomorphic to a literal system.*

Proof. Translate and scale the polytope rationally so that it lies in $[0, 1]^d$. Clear denominators in its defining inequalities, and write each as $a \cdot u \leq b$ with integral coefficients. We will encode each inequality by literal equations, while keeping every auxiliary variable in $[0, 1]$.

The following equations perform the required elementary operations:

$$\begin{aligned} u + (1 - v) &= 1 & (u = v), \\ u + v + (1 - w) &= 1 & (w = u + v), \\ v + v + (1 - u) &= 1 & (v = u/2), \\ c + (1 - c) + (1 - c) &= 1 & (c = 1). \end{aligned}$$

Repeated halving gives dyadic multiples of each u_i and of the constant $c = 1$. Adding the terms selected by a binary expansion then gives the required integer multiples of these dyadic quantities.

For a fixed inequality, choose a power of two $D > \sum_i |a_i| + |b|$ and form

$$p(u) = \frac{\sum_{a_i > 0} a_i u_i + \max(-b, 0)}{D}, \quad q(u) = \frac{\sum_{a_i < 0} (-a_i) u_i + \max(b, 0)}{D}.$$

Every term is nonnegative. The choice of D ensures that both expressions, and every partial sum used to compute them, are less than one throughout the cube. Thus all additions are valid within $[0, 1]$. An empty sum can be represented by a zero variable, fixed by $z + z + (1 - z) = 1$.

The inequality $a \cdot u \leq b$ is equivalent to $p(u) \leq q(u)$. Introduce a slack variable $s \in [0, 1]$ and impose the literal equation

$$p + s + (1 - q) = 1.$$

For every feasible u , this uniquely fixes $s = q(u) - p(u) \in [0, 1]$. Conversely, nonnegativity of s implies the original inequality. The variables used to compute p and q are also uniquely determined by u .

Do this for each defining inequality. Projection onto the u coordinates is a bijection, and its inverse is rational affine because every auxiliary variable is rational affine in u . A variable that does not otherwise occur can be included through a tautological equation with two literals. This gives the required literal system. The vertices of a rational polytope have encoding length polynomial in that of its defining inequalities [25, Section 10.2], so the translation and scaling can be computed in polynomial time. The number $\log_2 D$ is polynomial in the input size, and each inequality uses $O(d \log D)$ auxiliary variables and equations. The literal system is therefore computed in polynomial time. $\square$

4.2. The general matching encoding.

LEMMA 4.2 (Matching encoding). *Every literal system is isomorphic, by an integral affine map whose inverse is a coordinate projection, to $\mathcal{M}(H)$ for a tripartite three-uniform hypergraph whose vertex degrees belong to $\{1, 2, 3\}$. The construction has size linear in the number of literal occurrences.*

Proof. First rewrite the literal system so that every equation has the form $y_1 + \dots + y_q = 1$ with distinct variables. Introduce $u_i = t_i$ and $v_i = 1 - t_i$, with $u_i + v_i = 1$. For each literal occurrence introduce a separate variable w , and impose $w + v_i = 1$ for a positive occurrence or $w + u_i = 1$ for a negative one. Use these occurrence variables in the original equations. Introduce constants p, z, z' with $p = 1$ and $p + z + z' = 1$; nonnegativity fixes $z = z' = 0$. Adding z to every equation with two variables gives it three. Every new variable is uniquely determined by the original ones. The resulting system has only equations $y_i + y_j + y_k = 1$ and pins $y_i = 1$, and every variable occurs in an equation with three variables.

Apply the gadgets of Lemma 3.2 to the equations with three variables, and omit the edge R_o at every occurrence of a pinned variable. A missing edge forces $r_o = 0$, so the equations at A_o, B_o, C_o give weight one to every occurrence of that variable, and these three vertices have degree one. $\square$

Proof of Theorem 1.2(2). Apply Lemma 4.1 to obtain a literal system, and then Lemmas 4.2 and 3.1. Each step runs in polynomial time. $\square$

The integer behavior of these maps is worth recording separately, since the next application depends on it.

PROPOSITION 4.3 (Integral affine equivalence). *For a polytope already given as a literal system, the composition of Lemmas 4.2 and 3.1 is an integral affine isomorphism onto its image. It gives a bijection on integer points and preserves the denominator of every rational point.*

Proof. Lemmas 4.2 and 3.1 use only copies, complements, and fixed integral coordinates, with coordinate projections as inverses. Thus integer points correspond in both directions. The same argument shows that a positive integer clears the input denominators if and only if it clears the output denominators.

After adjoining fixed zero coordinates, these maps extend to unimodular affine transformations: retain one copy of each input coordinate and obtain every additional coordinate by an integer shear and translation. The normalization in Lemma 4.1, including its halving equations, is the only part of the construction that may change integer points. $\square$

Integer points cannot be preserved for every polytope. Every coordinate of a point of $P(b)$ is at most the largest line sum, so with line sums in $\{0, 1, 2\}$ every integer point lies in $\{0, 1, 2\}^n$. An affine injection that maps the integer points $0, 1, 2, 3$ of the segment $[0, 3]$ to integer points sends k to $\varphi(0) + kd$ for a nonzero integer vector d , so some coordinate takes four distinct values. The same argument with the segment $[0, M + 1]$ applies to any bound M on the line sums. The slim representation of De Loera and Onn [7] preserves integer points, with line sums that grow with the input.

Composing the reduction of Průša and Werner [23] with Lemma 3.1 transfers linear optimization to the slim class.

COROLLARY 4.4 (Linear optimization in the slim class). *Every rational linear program reduces in nearly linear time, in the sparse encoding of [23], to at most two linear programs over entire planar transportation polytopes of format $3 \times c \times h$ with line sums in $\{0, 1, 2\}$ and line bounds $(3, 2, 2)$. If the original program has an optimal solution, the first instance has the same optimal value, and an optimal solution of the original program can be recovered from one of the first instance.*

Proof. Průša and Werner [23, Corollaries 2.5 and 2.9, Section 5.2] reduce a rational LP to optimization over a fractional perfect matching polytope $\mathcal{M}(H)$, with degrees at most three as in Lemma 4.2. Their output is feasible exactly when the input has a finite optimum; in that case, it preserves the optimal value and gives an explicit recovery map. If γ_e are the resulting edge costs, Lemma 3.1 transfers the objective by assigning cost γ_e to $X_{3,a,e}$ for $e = (a, b, c)$ and zero to the other coordinates:

$$\sum_{e \in E} \gamma_e t_e = \sum_{e = (a, b, c) \in E} \gamma_e X_{3,a,e}.$$

The inverse projection in that lemma therefore also recovers an optimal matching point.

If the output is infeasible, apply the reductions for linear feasibility in [23, Theorems 2.1, 2.4 and 2.7], followed by Lemma 3.1, to the original constraint system alone. This second instance, with zero objective, is feasible exactly when the original LP is feasible. It distinguishes unboundedness from infeasibility.

The bounds in [23] are nearly linear in a sparse encoding that lists indices with their nonzero values. In the same encoding the conversion of Lemma 3.1 has linear size: for a matching instance with v vertices and h edges it has $6h$ allowed cells, $3h$ nonzero line sums in each of the directions IE and VE , at most $2v$ nonzero line sums in the direction IV , and h nonzero costs. The composition is therefore a nearly linear reduction. Listed as explicit arrays, the output has $3vh$ entries and $3v + 3h + vh$ line sums, which is still polynomial. Line sums in the last two directions are zero or one, and all rational data are encoded in binary. $\square$

5. Unit line sums: faces of Δ_N . In the language of the opening paragraph, a face of Δ_N is a timetable with forbidden combinations. With all line sums equal to one, the construction of Lemma 3.1 is unavailable, since it needs line sums equal to two. Section 2 shows that a line bound of two in any one direction then forces a product of intervals, so lines in every direction must be allowed three cells. This section builds such supports from triangulated surfaces and proves Theorem 1.2(1).

A simplicial complex is *balanced* here if its vertices are colored with three colors A, B, C and every triangle has one vertex of each color. A triangle can then be identified with a triple (a, b, c) . Its three edges correspond to the three pairs indexing the equations for the line sums. An edge contained in d triangles is then a line with d allowed cells, and the equation for its line sum says that the weights of those triangles add up to one. To obtain lines with three cells, we glue triangular holes in spheres.

5.1. A surface for one variable. For each variable of a literal system (Section 4.1) we construct a sphere with one triangular hole for each occurrence. We call these boundary triangles *ports*. The weights next to a port will specify whether that occurrence is t_i or $1 - t_i$. The third property below ensures that gluing identifies only the boundary vertices of the ports, and no other edges or triangles.

LEMMA 5.1 (Triangular ports). *For any $k \geq 1$ and any prescribed list of k literal signs, there is a balanced triangulated sphere with k pairwise disjoint open triangles removed and $71k - 24$ triangles remaining, with the following properties:*

1. *Every interior edge has degree two. Its equation is part of a system that expresses all triangle weights as t or $1 - t$, with both types present and no further restriction on $t \in [0, 1]$.*
2. *At each boundary component, all three adjacent triangle weights equal the prescribed literal.*
3. *There are no edges between different boundary components. Each interior vertex has at most one boundary neighbor of each color.*

Proof. Begin with $2k$ triangular rings $U_j = (u_{j,0}, u_{j,1}, u_{j,2})$, where $0 \leq j < 2k$. For $0 \leq j < 2k - 1$, join consecutive rings by the triangles

$$(u_{j,a}, u_{j,a+1}, u_{j+1,a+1}), \quad (u_{j,a}, u_{j+1,a+1}, u_{j+1,a}), \quad a \pmod{3},$$

and cap the two end rings (Figure 6(a)). The resulting sphere has $6k$ vertices, $18k - 6$ edges, and $12k - 4$ triangles. Select the k vertex-disjoint triangles

$$f_i = (u_{2i,0}, u_{2i,1}, u_{2i+1,1}), \quad 0 \leq i < k,$$

and mark the vertex $v_i = u_{2i,0}$ in each. No two marked vertices are adjacent.

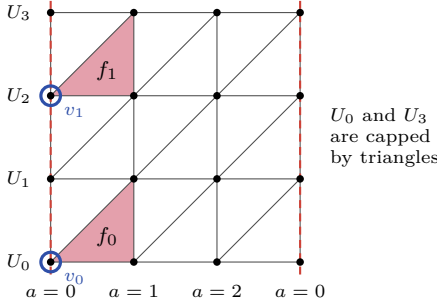
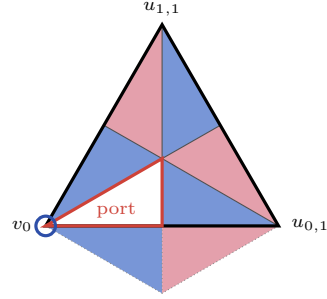
(a) Rings $U_0, \dots, U_3$, unrolled(b) Subdividing f_0 

FIG. 6. The sphere in the proof of Lemma 5.1 for $k = 2$. (a) The rings $U_0, \dots, U_3$, unrolled; the dashed sides are identified, and the end rings are capped by triangles. The selected faces f_0, f_1 are shaded, and the marked vertices v_0, v_1 are circled. (b) The barycentric subdivision of f_0 , with the two signs of flags in two colors; the flags of the neighboring face across the edge $v_0 u_{0,1}$ are shown below it. Removing the flag at v_0 on that edge leaves a port whose three neighbors have the same sign.

Barycentrically subdivide the sphere. A vertex of the subdivision represents an original vertex, edge, or face; color it by that original dimension, 0, 1, or 2. Each new triangle is a flag $v \in e \subset f$. Give it a sign according to whether its color order agrees with a coherent orientation of the sphere. Adjacent triangles have opposite signs: their induced orientations on the shared edge are opposite, while the color order on that edge is the same.

In each selected face f_i , exactly two flags contain v_i , and they have opposite signs. Remove one of these two triangles (Figure 6(b)). Its three neighbors all have the opposite sign, so the choice of the removed flag determines the sign at that port. This realizes any prescribed list of literal signs. The number of retained triangles is

$$6(12k - 4) - k = 71k - 24.$$

The removed triangles are pairwise vertex-disjoint. Each vertex link therefore remains a circle or becomes an interval, and the remaining complex is a connected sphere with k open disks removed. Its triangle adjacency graph, using only interior edges, is connected: a path between triangle interiors can be perturbed to avoid vertices and to cross only interior edges. The inherited signs make this graph bipartite. Hence the equations of its edges propagate a weight t on one sign class and $1 - t$ on the other. Every interior equation is then satisfied, and nonnegativity is exactly $0 \leq t \leq 1$.

It remains to check the separation properties needed for gluing. A port has one boundary vertex of each color: a marked v_i , a selected original edge, and the selected face f_i . Every vertex of a port is a simplex of its selected face, and an edge of the barycentric subdivision joins two simplices one of which contains the other. The selected faces are vertex-disjoint, so there is no edge between different ports. Since the marked vertices are pairwise nonadjacent, an original edge or face contains at most one marked vertex. An original vertex or face is incident with at most one selected edge, since the selected faces are vertex-disjoint. Similarly, an original vertex or edge belongs to at most one selected face. These incidence relations are exactly the possible adjacencies in the barycentric subdivision. Thus an interior vertex has at most one boundary neighbor of each color. $\square$

5.2. Gluing the variable surfaces. We now identify the ports belonging to a single equation. The boundary edges then become lines whose entries are the literals in that equation.

LEMMA 5.2 (Literal compiler). *Every nonempty literal system is affinely isomorphic to $P_S(\mathbf{1})$ for a balanced simplicial complex whose triangles form S and whose edge degrees are two or three. Here $\mathbf{1}$ means one on every edge of the complex and zero on all other pairs. The number of triangles is*

$$|S| = \sum_i (71k_i - 24).$$

Proof. For variable t_i , take the surface in Lemma 5.1 with one port for each occurrence and with the required literal sign at that port. For each equation, identify its two or three port boundaries vertex by vertex, matching colors. The common triangular hole is left empty. Each of its three edges is now incident with the triangles carrying the literals in that equation, and so imposes (4.1). Every other edge still imposes an equation $t_i + (1 - t_i) = 1$. Figure 7 shows this local operation.

We must check that the identifications do not create unintended edges or triangles. Only boundary vertices are identified, and each port is used in exactly one equation. The ports are disjoint and have no edges between them. Every nonboundary edge has an interior endpoint, whose boundary neighbors of a given color are unique by Lemma 5.1. Two such edges therefore cannot merge. Likewise, every retained triangle has an interior vertex. If two triangles merged, they would share that interior vertex and their corresponding boundary neighbors would be identified. Uniqueness of those neighbors forces the triangles to have been identical. These arguments also cover equations in which a variable occurs more than once.

Within each variable surface, the interior equations determine all coordinates uniquely as t_i or $1 - t_i$. Since both types occur, their nonnegativity is equivalent to $0 \leq t_i \leq 1$. The boundary equations are three copies of the input equations. We have therefore obtained an affine bijection on the full feasible sets, including their boundaries. No retained triangles were identified, so their number is the stated sum. $\square$

If the literal system has $K = \sum_i k_i$ occurrences, the compiled complex has $71K - 24r$ triangles and $O(K)$ vertices and edges. Its pieces and gluing instructions can be listed in polynomial time. This size bound is used for the hardness statement at the end of this section. Figure 7 illustrates the gluing operation.

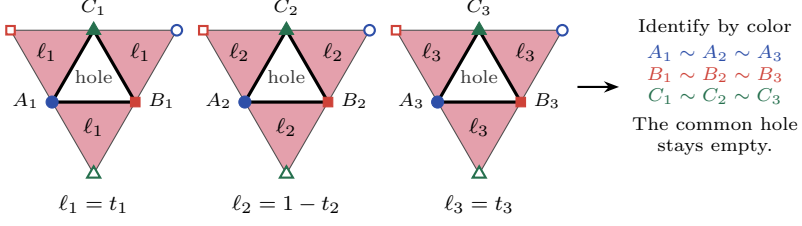
5.3. Completion to a face of Δ_N . We use the embedding of a partial Latin square into a complete one from Colbourn's reduction [4]; see also Bartlett [3, Section 3.3]. The argument below records its effect on fractional arrays and on the constraint matrix M_S , whose columns are indexed by S , whose rows are indexed by the three families of pairs, and in which the column of (a, b, c) has ones in rows (a, b) , (a, c) , (b, c) and zeros elsewhere.

LEMMA 5.3 (Completion). *Let $P_S(\mathbf{1})$ be nonempty with $|A|, |B|, |C| \leq n$, where the line sums are one on the pairs occurring in S and zero elsewhere. Let G be the tripartite graph of pairs occurring in S . Let $n' \geq n$. Pad each class to size n , and then add n' new vertices to each class. With $N = n + n'$, the graph $K_{N,N,N} \setminus G$ has a triangle decomposition T for which*

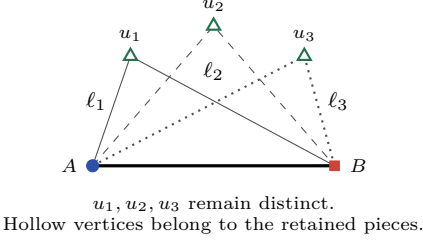
$$x \mapsto x + \mathbf{1}_T$$

is an affine isomorphism from $P_S(\mathbf{1})$ onto a face of Δ_N . The added coordinates are fixed to one. The map preserves the denominator of every rational point.

(a) Three separate ports



(b) One glued edge



(c) Three copies of one equation

$$\begin{aligned} AB: \quad & \ell_1 + \ell_2 + \ell_3 = 1 \\ AC: \quad & \ell_1 + \ell_2 + \ell_3 = 1 \\ BC: \quad & \ell_1 + \ell_2 + \ell_3 = 1 \end{aligned}$$

Only boundary vertices are identified.
 Separated ports and unique boundary
 neighbors of each color prevent
 extra identifications.

FIG. 7. Gluing three ports to impose one literal equation. (a) The triangles next to port i all carry the same literal ℓ_i . (b) After identifying boundary vertices of the same color, the common edge AB has three incident triangles and imposes $\ell_1 + \ell_2 + \ell_3 = 1$. Their interior vertices u_1, u_2, u_3 remain distinct. (c) The other two boundary edges impose the same equation. The separation properties in Lemma 5.1 prevent additional identifications.

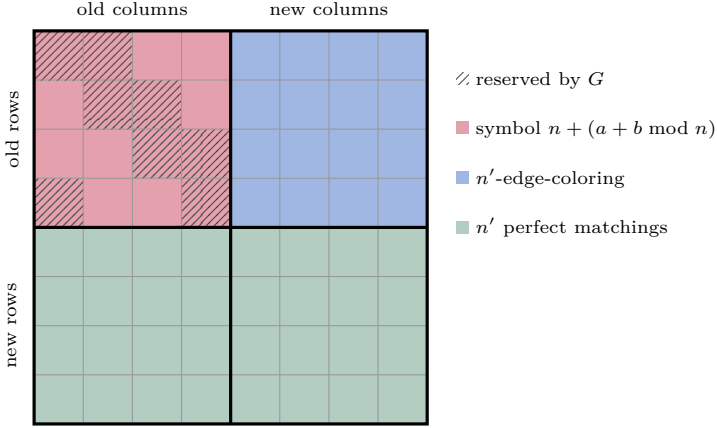


FIG. 8. The completion in Lemma 5.3, with rows, columns, and symbols as the three classes. Cells whose row and column form a pair of G are reserved. The other old cells receive new symbols $n + (a + b \bmod n)$. An n' -edge-coloring fills the cells in old rows and new columns, and n' perfect matchings fill the new rows.

Proof. First pad the original classes to size n . Feasibility gives a useful balance condition on G : each vertex has the same number of neighbors in the other two classes. For example, summing the AB line sums at a and summing the AC line sums at a both give $\sum_{b,c} x_{abc}$.

Add n' new vertices to each class, and view the classes as rows, columns, and

symbols. We fill exactly those cells, pairs of a row and a symbol, and pairs of a column and a symbol that are not reserved by G (Figure 8). The entries used in this filling will be the triangles of T .

Index the old rows and columns by $0, \dots, n-1$. In each old cell (a, b) outside G_{AB} , place the new symbol $n + (a + b \bmod n)$. In a fixed old row, the symbols still allowed are the old symbols outside its G_{AC} neighborhood and the new symbols not yet used in that row. There are exactly n' such symbols, by the balance condition. The bipartite graph of old rows and their allowed symbols has maximum degree at most n' , since there are n old rows, and every row has degree n' . A proper n' -edge-coloring assigns these pairs to the n' new columns. This fills every cell in an old row and a new column without repeating a symbol in a row or column.

Now consider a column. Exclude the symbols already used in it and, for an old column, the old symbols reserved by G_{BC} . The remaining pairs of a column and a symbol form an n' -regular bipartite graph on the N columns and N symbols. Each column meets the n old rows in n distinct used or reserved symbols, by balance, so its degree is n' . An old symbol has been placed in the old rows that do not reserve it and is reserved in as many old columns as old rows reserve it; a new symbol has been placed once in each old row. In both cases n distinct columns are excluded, so every symbol has degree n' . Decompose this bipartite graph into n' perfect matchings and use them to fill the n' new rows. The completed entries give the desired triangle decomposition of the complement of G .

Restrict Δ_N to the support $S \cup T$. Each pair outside G belongs to exactly one triangle of T and to no triangle of S . Its unit line sum forces the added coordinate to one. The equations on the remaining coordinates are exactly those of $P_S(\mathbf{1})$, so the map is an affine isomorphism onto this face.

After ordering rows and columns, the constraint matrix consists of the original block and separate blocks of one column each for the added coordinates. Each added block has a unit Smith factor. Adjoining these fixed integral coordinates preserves denominators. $\square$

Proof of Theorem 1.2(1). Apply Lemma 4.1, Lemma 5.2, and Lemma 5.3. Each step runs in polynomial time; the completion uses edge-colorings and perfect matchings of bipartite graphs. $\square$

Once the input is a literal system, the surface construction also preserves integer points: Lemma 5.2 replaces each input coordinate by copies of t_i and $1 - t_i$, and completion adjoins coordinates equal to one.

Remark (entire polytopes from surfaces). The prescribed zeros of $P_S(\mathbf{1})$ can also be enforced by line sums. Index a new array by the vertices V , edges E , and triangles F of the complex, and allow only the flags $v \subset e \subset f$. The six flags of each triangle alternate between x_f and $1 - x_f$. Prescribe line sum one on the VF and EF pairs of flags, line sums one and $d_e - 1$ on the two VE pairs of each edge, and zero elsewhere. These line sums make $P_S(\mathbf{1})$ affinely isomorphic to an entire transportation polytope with line sums in $\{0, 1, 2\}$ and line bounds $(3, 2, 2)$. This is the encoding of Lemma 3.1 with genuine edges in place of the three edge types, so the array is not slim.

Colbourn's hardness for faces of Δ_N [4] persists with line bounds $(3, 3, 3)$, together with a fractional witness as in Theorem 1.3.

COROLLARY 5.4 (Integer feasibility for faces at line bounds $(3, 3, 3)$). *It is NP-complete to decide whether a specified face of Δ_N contains an integer point. This remains true when the union support of the face has line bounds $(3, 3, 3)$ and a feasible*

point, positive on the union support and taking only the values $1/3$, $2/3$, and 1 there, is supplied with the input.

Proof. Apply Lemma 5.2 to the equations of a positive 1-in-3 formula, and then Lemma 5.3. Both maps preserve integer points, so the integer points of the face correspond to satisfying assignments. At $t_i = 1/3$ every triangle weight is t_i or $1 - t_i$, hence $1/3$ or $2/3$, and completion adds coordinates equal to one. The construction has polynomial size, and the edge-coloring and matching steps of the completion run in polynomial time. Membership in NP follows as in the proof of Theorem 1.3. The face is specified by its allowed support; Δ_N itself always contains integer points, namely the Latin squares. $\square$

Proof of Theorem 1.1. Line bounds are upper bounds, and the three directions can be permuted, so a representation or reduction with line bounds $(3, 2, 2)$ or $(3, 3, 3)$ also serves every profile that is at least as large after a permutation of the directions. The positive parts therefore follow from Theorems 1.2 and 1.3 and Corollary 5.4. The remaining parts are Proposition 2.1 and the observations after it. The statements about denominators follow from Proposition 2.1, Corollary 6.2, and Theorem 1.4. $\square$

6. Vertices with prescribed denominators. We now construct the vertices in Theorem 1.4. A literal system will force one variable to equal $1/m$.

6.1. Vertices and column independence. A feasible point positive on S is a vertex if and only if the columns of M_S are independent. A nonzero kernel vector would give a small feasible perturbation in both directions, and column independence makes the feasible solution on that support unique.

PROPOSITION 6.1 (Denominators in the topology of the support). *Let x be a vertex of Δ_N of denominator m , let S be its support, and let $K(S)$ be the two-dimensional complex whose triangles are the triples of S . Then $\text{Tor } H_1(K(S); \mathbb{Z})$ contains an element of order m .*

Proof. Let $L \subseteq \mathbb{Z}^R$ be the lattice spanned by the columns of M_S , and let $L' = \mathbb{Q}L \cap \mathbb{Z}^R$. Then $\text{Tor coker } M_S = L'/L$. The all-ones vector equals $M_S x$, so it lies in L' . If $k\mathbf{1} = M_S z$ with z integral, column independence gives $z = kx$, so k is a multiple of m ; conversely $m\mathbf{1} = M_S(mx) \in L$. Hence $\mathbf{1}$ has order exactly m in $\text{Tor coker } M_S$. Ordering the vertices of $K(S)$ by color gives $\partial_2[a, b, c] = [b, c] - [a, c] + [a, b]$, so changing the signs of the AC rows turns M_S into the boundary matrix ∂_2 of $K(S)$, and $\text{Tor } H_1(K(S); \mathbb{Z}) \cong \text{Tor coker } \partial_2$ because $\text{im } \partial_1$ is free abelian. $\square$

6.2. A short literal system for $1/m$.

Proof of the construction in Theorem 1.4. Start with a variable t , which the system will force to be $1/m$. Read the binary expansion of m after its leading one. At each stage retain a variable representing the current prefix times t . For a nonfinal bit, first double the current value by imposing

$$u + u + (1 - v) = 1,$$

so that $v = 2u$. If the bit is one, also impose $v + t + (1 - w) = 1$, giving $w = 2u + t$. At the final bit, close the system with $u + u = 1$ if that bit is zero, or with $u + u + t = 1$ if it is one. The closing equation is exactly $mt = 1$ after substitution.

Every equation except the last defines a new variable, so the resulting literal system is square and has a unique solution. Each variable is a proper positive prefix of m , or an intermediate doubled prefix, divided by m . All values are therefore strictly between zero and one.

Write $L = \lfloor \log_2 m \rfloor$ and let h be the number of ones in the binary expansion of m . The number of variables and equations is

$$r = L + h - 1 - (m \bmod 2) \leq 2L - 1.$$

The defining equations have coefficient -1 on their new variables. Integer elimination along these equations leaves the single relation $mt = 1$. Hence the coefficient matrix has Smith factors $1, \dots, 1, m$.

Lemma 5.2 now produces a unique feasible point positive on every retained triangle. Its support columns are independent, and its coordinates have denominator exactly m : they are all integer multiples of $1/m$, and one of them is $t = 1/m$.

We next count the size. The total number of occurrences is $k = 3r - 1$ for even m , because the last equation has two literals, and $k = 3r$ for odd m . The number of fractional coordinates before completion is therefore

$$s = 71k - 24r = \begin{cases} 189r - 71, & m \text{ even}, \\ 189r, & m \text{ odd}. \end{cases}$$

Before gluing the ports, the three color classes have sizes $6k$, $18k - 6r$, and $12k - 4r$. Gluing the k ports into r equations removes $k - r$ vertices from each class. The final class sizes are

$$5k + r, \quad 17k - 5r, \quad 11k - 3r.$$

The largest is

$$n = \begin{cases} 46r - 17, & m \text{ even}, \\ 46r, & m \text{ odd}. \end{cases}$$

Apply Lemma 5.3 with $n' = n$. The completed point belongs to Δ_{2n} , and the additional coordinates are fixed to one. Its denominator is unchanged, and its support columns remain independent. Thus it is a vertex with $N = 2n$ and exactly s fractional entries. Substituting $r \leq 2L - 1$ gives $N \leq 184 \lfloor \log_2 m \rfloor - 92$ and at most $378 \lfloor \log_2 m \rfloor - 189$ fractional entries. $\square$

COROLLARY 6.2 (Denominators in the slim class). *For every integer $m \geq 2$, there is an entire planar transportation polytope of format $3 \times c \times h$ with line sums in $\{0, 1, 2\}$ and line bounds $(3, 2, 2)$ that consists of a single point of denominator m .*

Proof. The literal system of Section 6.2 has a unique solution, whose coordinates have denominator m . Lemmas 4.2 and 3.1 represent it as an entire polytope of the stated kind, and Proposition 4.3 shows that this representation preserves denominators. $\square$

6.3. The lower bound on fractional entries.

Proof of the lower bound in Theorem 1.4. Let a tristochastic vertex have denominator m and f fractional entries. Remove its entries equal to one. Each such entry is alone on all three of its lines, so the remaining fractional coordinates satisfy equations with integral line sums and still have independent support columns.

Choose a nonsingular $f \times f$ submatrix B of this support matrix. Its entries are zero or one, and it has at most $3f$ ones because each column of the full matrix has three. Cramer's rule shows that m divides $|\det B|$. The sparse determinant bound of Araujo, Balogh and Wang [2] gives

$$m \leq |\det B| \leq 2^{2f/3}.$$

Taking logarithms to base two yields $f \geq \frac{3}{2} \log_2 m$. Since column independence also gives $f \leq 3N^2$, we obtain $N \geq \sqrt{\frac{1}{2} \log_2 m}$. $\square$

The constant $\frac{3}{2}$ is close to the limit of this method. The incidence matrix of the Fano plane has three ones in each column and determinant ± 24 , so block-diagonal copies give 0–1 matrices of order f with three ones in each column and determinant $\pm 24^{f/7}$. A bound on such determinants alone cannot give a constant larger than $7/\log_2 24 \approx 1.53$, so narrowing the gap to the upper bound 378 requires better constructions.

6.4. Merging vertices.

LEMMA 6.3 (Merging vertices). *Let $S \subseteq A \times B \times C$, and let $\alpha : A \rightarrow A'$, $\beta : B \rightarrow B'$, $\gamma : C \rightarrow C'$ be maps that are injective on the pairs occurring in S in each of the three directions. Then $S' = \{(\alpha a, \beta b, \gamma c) : (a, b, c) \in S\}$ has the same constraint matrix as S after relabeling rows and columns. In particular, $P_{S'}(\mathbf{1})$ and $P_S(\mathbf{1})$ are related by a relabeling of coordinates, which preserves line bounds, vertices, denominators, and Tor coker M_S .*

Proof. Two distinct triples of S differ in at least one coordinate, hence in at least two of their three pairs, and their images are distinct by injectivity on pairs. A triple contains a pair exactly when its image contains the image of that pair, by injectivity on pairs. Hence rows and columns correspond bijectively and incidences are preserved. $\square$

LEMMA 6.4 (Bounded degree). *Every vertex of the complex in Lemma 5.2 has at most 32 neighbors.*

Proof. In the sphere of Lemma 5.1 before subdivision, every vertex has two neighbors in its own ring and at most two in each adjacent ring, so its degree is at most six. After barycentric subdivision, a vertex coming from an original vertex of degree d has $2d \leq 12$ neighbors, one coming from an edge has 4, and one coming from a face has 6. Removing the port triangles deletes no edges. Ports on one sphere are vertex-disjoint, and every equation has at most three literals, so gluing identifies at most three boundary vertices. Their two boundary edges are shared by all copies, and the other edges stay distinct by the proof of Lemma 5.2. Hence a glued vertex has at most $3 \cdot 12 - 2 \cdot 2 = 32$ neighbors. $\square$

LEMMA 6.5 (Greedy coloring). *Let G be a tripartite graph with classes A, B, C , n vertices, and maximum degree Δ , and let $K \geq 4\Delta\sqrt{n} + 8\Delta^2$. Maps $\alpha : A \rightarrow [K]$, $\beta : B \rightarrow [K]$, $\gamma : C \rightarrow [K]$ that are injective on the edges of each of the three directions can be computed in polynomial time.*

Proof. Color the vertices one at a time, giving each vertex a color in $[K]$ through the map of its class. When v is colored, forbid two kinds of colors: (i) the colors of colored vertices of the class of v that share a neighbor with v ; (ii) for every colored neighbor u of v and every other colored vertex u' of the class of u with the color of u , the colors of the colored neighbors of u' in the class of v . Among the remaining colors choose one used least often in the class of v .

Let s be the largest number of vertices of one class with the same color. There are at most Δ^2 colors of kind (i) and at most $\Delta^2 s$ of kind (ii). At most $K/2$ colors are used $2n/K$ times or more in a class. Hence, if fewer than $K/2$ colors are forbidden, the chosen color has been used fewer than $2n/K$ times, and $s \leq 2n/K + 1$ remains true. The forbidden colors then number at most $\Delta^2(2n/K + 2) \leq \Delta^2\sqrt{n}/2 + 2\Delta^2 \leq 3K/8$, so the induction closes.

Now take two distinct edges in one direction. If they share an endpoint, their other endpoints lie in one class and share a neighbor, and (i) gives them different colors. Otherwise, let v be the last of the four endpoints to be colored, u its neighbor on the same edge, and $u'w$ the other edge with u' in the class of u . If u' has the color of u , then (ii) forbids the color of w at v . In either case the two edges carry different pairs of colors. $\square$

Proof of the bound $N = O(\sqrt{\log m})$ in Theorem 1.4. Let S be the support produced by Lemma 5.2 for the literal system of Section 6.2, before completion. Its 1-skeleton has $n = O(\log m)$ vertices and maximum degree at most 32 by Lemma 6.4. Lemma 6.5 with $K = \lceil 128\sqrt{n} \rceil + 8192$ gives maps that are injective on the edges of each direction, so Lemma 6.3 applies. The merged support keeps the denominator, the line bounds, and the independence of the support columns, and each class has at most $K = O(\sqrt{\log m})$ elements. Lemma 5.3 then gives a vertex of Δ_N with $N = O(\sqrt{\log m})$, and every step runs in polynomial time. Harmonious colorings satisfy the same condition, and Edwards [9] showed that graphs of bounded degree with e edges have them with $(1+\varepsilon)\sqrt{2e}$ colors when e is large; this improves the constant. $\square$

Proof of Corollary 1.5. A vertex of Δ_N with f fractional entries has $f \leq 3N^2$ by column independence, and its denominator is at most $2^{2f/3}$ by the proof of the lower bound in Theorem 1.4; hence $\log_2 D(N) \leq 2N^2$. For the lower bound, take $m = 2^k$. The merged support of the previous proof has classes of size at most $C\sqrt{k}$, and Lemma 5.3 completes it to Δ_N for every $N \geq 2C\sqrt{k}$. Choose $k = \lfloor (N/2C)^2 \rfloor$. $\square$

Remark (support torsion). As in the proof of Proposition 6.1, M_S is the boundary matrix of $K(S)$ up to signs, so column independence is equivalent to $H_2(K; \mathbb{Q}) = 0$. For the complex of Lemma 5.2, eliminating the interior equations along spanning trees uses pivots ± 1 and leaves three copies of each row of R . Hence M_S and R have the same nonunit Smith factors, and completion adds only unit factors. For the vertex of Theorem 1.4, R has Smith factors $1, \dots, 1, m$, so $\text{Tor } H_1(K; \mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z}$: the denominator is recorded in the topology of the support. A small instance is the balanced triangulation of the Klein bottle in [27] with weight $1/2$ on each of its 22 triangles (Figure 9 in Appendix A), a vertex of denominator two whose support has $H_1 = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Lemma 6.3 preserves the constraint matrix, so the vertices obtained by merging have the same support torsion.

7. Open questions. With unit line sums, Theorem 1.1 concerns faces of Δ_N . The holes left by gluing cannot be closed by line sums: a cell in a hole lies on three lines of sum one, and none of them forces it to zero. An entire polytope $P(b)$ with line sums in $\{0, 1\}$ is the polytope of fractional triangle decompositions of a tripartite graph: its coordinates are the triangles, and every edge has total weight one. Is every rational polytope affinely isomorphic to such a polytope in which every edge lies in at most three triangles? Integer feasibility is already NP-complete in this class, even for slim arrays of format $3 \times c \times h$. A proper 3-edge-coloring of a cubic bipartite graph that extends a given precoloring is an integer point of such an array, and deciding whether one exists is NP-complete [11, 8].

A sharper form of the question asks whether the slim format can be kept. With line sums in $\{0, 1\}$, Section 2 shows that universality requires lines with three cells in every direction. An affirmative answer would tighten the last row of Table 1 from $\{0, 1, 2\}$ to $\{0, 1\}$. A negative answer would reveal hidden structure in these arrays, for instance a uniform bound on vertex denominators.

Appendix A. Examples and exact certificates. The construction code and

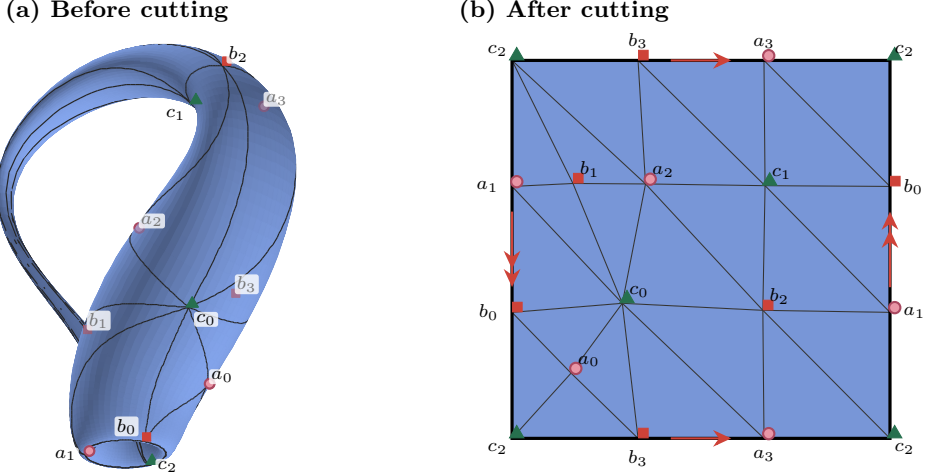


FIG. 9. The balanced triangulation of the Klein bottle from [27]. (a) The triangulation drawn on an immersion of the Klein bottle in $\mathbb{R}^3$; vertices on the back of the surface are drawn faded. (b) The same triangulation cut open along two closed edge paths through c_2 , which gives a square. Sides with the same arrows are identified in the direction of the arrows; the vertical sides are glued with a reversal, which makes the surface nonorientable. Vertices of the classes A, B, C are drawn as circles, squares, and triangles. Weight $1/2$ on each of the 22 triangles gives unit line sums on every edge and a vertex of denominator two.

exact certificates are available at

<https://github.com/PillowSoprano/local-transportation-universality>

The repository's README.md gives the construction and replay commands. The code checks line sums over the rationals. A separate verifier reconstructs, from the stored arrays, the graph formed by the lines with two entries, eliminates it over the integers, and computes the reduced Smith form; it does not import the compiler.

For Theorem 1.3, the repository also contains the slim array of format $3 \times 48 \times 36$ produced from the four equations of Section 3, with its witness, and a satisfiable instance with two clauses. A separate script checks the line sums, the line bounds $(3, 2, 2)$, the witness, the dimension of each polytope, and the integer points obtained from satisfying assignments.

The twelve tested denominators

$$2, 3, 4, 5, 7, 8, 11, 13, 31, 97, 257, 65537$$

have the predicted exact rank, unit line sums on the active pairs, and cyclic torsion. Full matrix Smith computations also agree for $m = 2, 3, 4, 5$. Table 2 records selected construction sizes.

For $m = 97$, the full completion adds 414078 entries equal to one. The resulting array has 415401 positive entries and exact column rank 415401; all $3 \cdot 644^2$ line sums were checked. For $m = 65537$, only the partial array was constructed and verified. Its stated completed order, 1472, follows from Lemma 5.3.

Examples of positive dimension test the correspondence of faces and vertices. A triangle, a square, and the pentagon $[0, 1]^2 \cap \{u + v \leq 3/2\}$ give two-dimensional polytopes with unit line sums with support sizes 141, 236, and 1015. Each listed polygon vertex maps to a vertex, as checked by the exact rank of its positive support. Flag encoding preserves dimension and makes all required zeros consequences of zero

TABLE 2

Sizes from the binary construction. The last column is the order given by Lemma 5.3; among these examples, the archive contains the full completed array only for $m = 97$.

m	Partial grid	Fractional entries	Completed order N
3	$16 \times 46 \times 30$	189	92
4	$27 \times 75 \times 49$	307	150
97	$112 \times 322 \times 210$	1323	644
65537	$256 \times 736 \times 480$	3024	1472

line sums. The completed pentagon face in Δ_{494} has 244546 allowed coordinates and rank 244544.

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